Week 8: Supplemental slides on Vector spaces and inner products

Amath 301

TA Session

Today

> A computation is a temptation that should be resisted as long as possible

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0. Algebra (it's much more interesting than you learned in high school)
1. What's an inner product?
2. Projections

Vector space (algebra!) V: A generalization of $\mathbb{R}$ (think $\mathbb{R}^n$) where you have "the usual" operations and properties: add and multiply, but you separate "numbers" and "vectors".

Let $a, b \in \mathbb{R}$ and $f, g, h \in V$.
The "usual" properties:

- V contains a "zero" (Sometimes denoted $\vec{0}$) such that $f + \vec{0} = f$
- Associativity $f + (g + h) = (f + g) + h$
- Commutativity $f + g = g + f$
- Distribution: $a(f + g) = af + ag$, $af, gf \in V$ too.
- Inverses exist, if $f \in V$, then $-f$ exists such that $f + (-f) = \vec{0}$
- Scalar associativity: $a(bf) = (ab)f$.
- Scalar distribution: $(a + b)f = af + bf$
- Scalar identity: there exists a 1 such that $1f = f$.

Examples

(You have seen these before)

1. Vectors $\mathbb{R}^2$ (You can check that these make sense)
2. Vectors $\mathbb{R}^n$ (Same idea!)
3. Space of continuous functions on an interval, denoted $C([a,b])$ usually. (Use limits to prove these identities)
4. Space of differentiable functions. (More limits)
5. Space of polynomials of degree n.
6. Space of sums of cosine and sine curves with differing periods (Fourier idea)
7. Space of complex valued vectors (Use coefficients in $\mathbb{C}$ instead)

Why bother with this? These spaces come with special structure we can leverage.

(Definition) Real-valued inner product: a map (function) $(\cdot, \cdot) : V \times V \to \mathbb{R}$ (sometimes denoted $\langle \cdot, \cdot \rangle$) that satisfies the following

1. **Linearity** in the first argument

$$(a+b, c) = (a, c) + (b, c), \quad (ka, c) = k(a, c)$$

where $k \in \mathbb{R}$ and $a, b, c \in V$.

2. **Symmetry** (in the real case):

$$(x, y) = (y, x)$$

3. Positive definite-ness. If x is not 0 (or $\vec{0}$), then

$$(x, x) > 0, \quad (0, 0) = 0$$

This definition extends to complex spaces too! Replace symmetry with conjugate symmetry.

A vector space V with an inner product $(\cdot, \cdot)$ is called an Inner Product Space (IPS).

Examples:

On $\mathbb{R}^n$: (Dot product!)

$$(x, y) = \sum_{j=1}^{n} x_i y_i$$

On $\mathbb{R}^n$ for a matrix M (you can check that this gives a # back)

$$(x, y)_M = x^T M y$$

On the space of (real) continuous functions:

$$(f, g)_w = \int_a^b f(x) g(x) w(x) dx$$

w is called a weight function.

The integral inner product is the function the dot products are trying to be.

In an application: multiple choices for inner product.

- Can choose the weights so a set of functions are orthogonal:
- Fourier problems: $w(x) = 1$ (sin and cos functions)
- Generalize the notion of **"linear independence"** to more complicated (abstract) spaces and problems.

Definition: Two elements f, g in a vector space V (can be vectors or functions) are said to be **orthogonal** with respect to an inner product $(\cdot, \cdot)_w$ if $(f, g)_w = 0$.

Definition A collection of vectors $\{f_i\}$ (can be functions) are said to be **Orthonormal** $(f_i, f_i)_w = 1$ if

$$(f_i, f_j)_w = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

This one of the top 3 **most useful** concept in applied mathematics.

Inner products are imensely useful

Really useful for understanding functions.

Can project functions onto other functions (not just vectors onto vectors)

Projection of f onto g:

$$Proj_g(f) = \frac{(f,g)_w}{(g,g)_w}\, g(x)$$

If we have an orthonormal system, $(g,g)_w = 1$. Then we have

$$Proj_g(f) = (f,g)_w\, g(x)$$

Very clean!
We can find a representation (or approximation) of a function in terms of polynomials (did somebody say Taylor series?)

In practice

Projections and inner products have a strong geometric interpretation and solve a minimization problem. A key problem is finding polynomial approximations of functions.

While we have discussed discretization and interpolation (through splines and other curve-fitting tools), projection onto an orthonormal basis gives the **optimal** polynomial (as measured by the norm) representation of that function (when compared to all the other possible polynomials you could try of the same degree).

What about projecting onto a basis? It follows directly from the properties of the inner product.

Consider a basis $\{\psi_1, \psi_2, \psi_3\}$ that are orthogonal with respect to a weight function w on $[-1, 1]$. Then the projection of f onto the space of all quadratic polynomials is

$$proj(f) = \tilde{f} = (f, \phi_1)_w \phi_1 + (f, \phi_2)_w \phi_2 + (f, \phi_3)\phi_3$$

The drawback of this approach is that we not need to evaluate 3 functions (and do 3 multiplications) to obtain a value from $\tilde{f} \approx f$. Many bases are simply *orthogonal* and not orthonormal. Instead, we might re-label

$$\tilde{f} = \sum_{j=0}^{2} a_j \phi_j$$

and absorb both the inner product (f, ϕ_j) and the normalization into a single constant. This is still stable (generally) for computation and works very well in practice. Numerical integration becomes a very useful tool here.