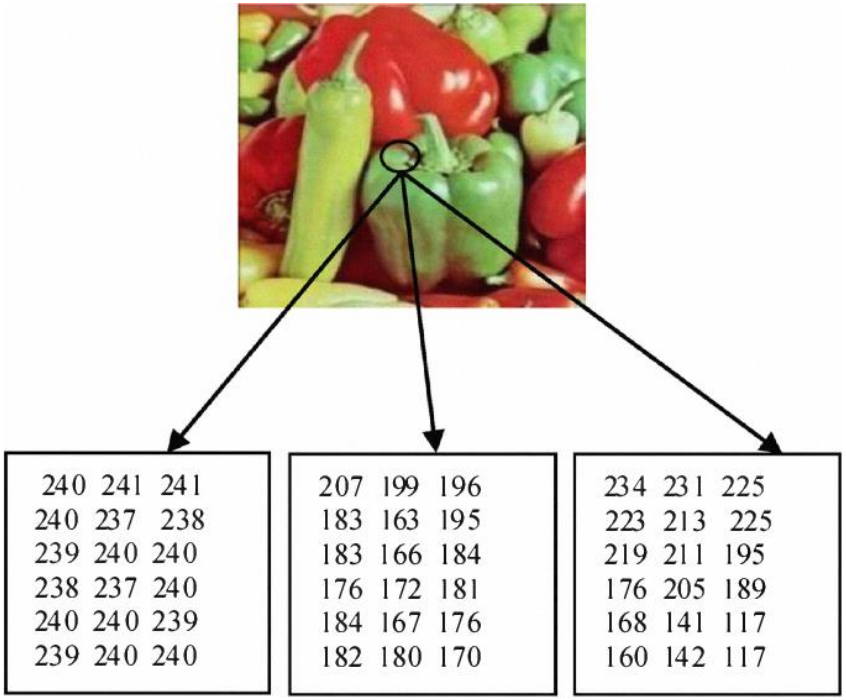
Linear Algebra 101

Agenda

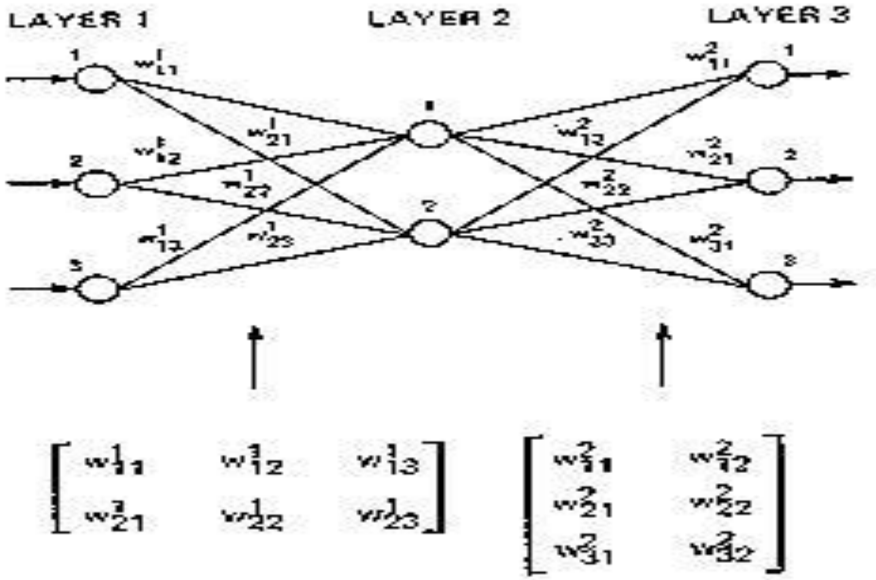
- ❖Why learn linear algebra?
- ❖Solution of a system of equation
- ❖Matrix
- ❖Matrix Math
- ❖ Eigenvalue & Eigen vectors
- ❖Dimensionality Reduction
- ❖Singular Value Decomposition

Motivation to learn Linear Algebra



- ❖ What you see when you look at this image?
- ❖ How computer process this image?

Neural Network/Deep Learning



Text Mining

- Document Term Matrix (DTM)

	Doc 1	Doc 2	Doc 3
abbey	2	3	5
spinning	1	0	1
soil	3	4	1
stunned	2	1	3
wrath	1	1	4

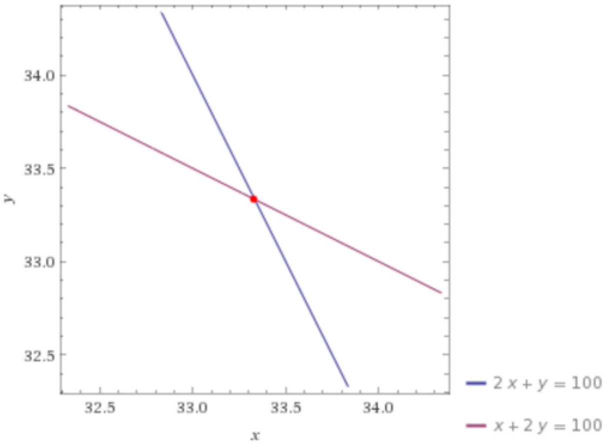
Representation of Problems in Linear Algebra

Suppose that price of 1 ball & 2 bat or 2 bat and 1 ball is 100 units. We need to find price of a ball and a bat.

How do we define this problem?

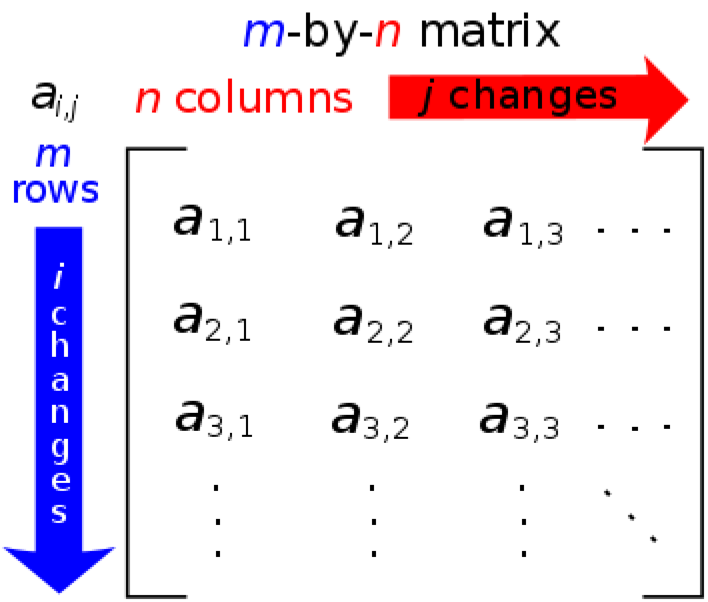
- Let's translate this in mathematical form –
- $2x + y = 100$ ....(1)
- Similarly, for the second condition-
- $x + 2y = 100$ ....(2)

Solving: $x=100/3$, $y=100/3$



What is a Matrix?

Matrix is a rectangular *array* of numbers, symbols, or expressions, arranged in *rows* and *columns*.



$$\begin{aligned} a_0 + a_1 + a_2 &= -1 \\ a_0 + 2a_1 + 4a_2 &= 3 \\ a_0 + 3a_3 + 9a_2 &= 3 \\ a_0 + 4a_1 + 16a_2 &= 5 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 3 \\ 1 & 4 & 16 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Terms related to Matrix

- **Order of matrix** – If a matrix has 3 rows and 4 columns, order of the matrix is 3*4 i.e. row*column.

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & 3 & 0 \\ -2 & 0 & 0 & 4 \\ 0 & 6 & -4 & -3 \end{pmatrix}$$

- **What is the order of the matrix?**
- **What is** A_{23} ?
- **Square matrix** – The matrix in which the number of rows is equal to the number of columns.

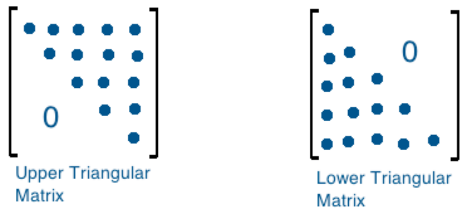
4 x 4 Square Matix

$$\begin{bmatrix} -2 & 4 & 7 & 31 \\ 6 & 9 & 12 & 6 \\ 12 & 11 & 0 & 1 \\ 9 & 10 & 2 & 3 \end{bmatrix}$$

- **Diagonal matrix** – A matrix with all the non-diagonal elements equal to 0 is called a diagonal matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

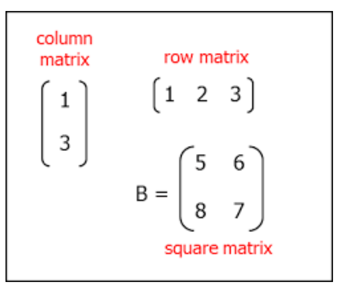
- **Upper triangular matrix** – Square matrix with all the elements below diagonal equal to 0.
- **Lower triangular matrix** – Square matrix with all the elements above the diagonal equal to 0.



- **Identity matrix** – Square matrix with all the diagonal elements equal to 1 and all the non-diagonal elements equal to 0.

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

- **Column matrix** – The matrix which consists of only 1 column. Sometimes, it is used to represent a vector.
- **Row matrix** – A matrix consisting only of row.



Trace of a Matrix

- Trace of a matrix is equal to adding the diagonal elements of the matrix

$tr(A+B)= tr(A) +tr(B)$

$tr(cAB) = c*tr(A)$

$tr(A)=Tr(A^T)$

Basic Mathematical Operation on Matrix

Addition:

$$\mathbf{G}=\begin{bmatrix}1 & 2\\ 3 & 4\\ 6 & 7\end{bmatrix},\quad \mathbf{F}=\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}$$

$$\mathbf{F+G}=\begin{bmatrix}1 & 2\\ 3 & 4\\ 6 & 7\end{bmatrix}+\begin{bmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1+1 & 2+0\\ 3+1 & 4+1\\ 6+0 & 7+1\end{bmatrix}=\begin{bmatrix}2 & 2\\ 4 & 5\\ 6 & 8\end{bmatrix},$$

Subtraction:

$$\begin{bmatrix}-1 & 2 & 0\\ 4 & 1 & 10\end{bmatrix}-\begin{bmatrix}3 & 2 & 4\\ 0 & 8 & 16\end{bmatrix}=\begin{bmatrix}-1-3 & 2-2 & 0-4\\ 4-0 & 1-8 & 10-16\end{bmatrix}$$

$$=\boxed{\begin{bmatrix}-4 & 0 & -4\\ 4 & -7 & -6\end{bmatrix}}$$

final answer

Multiplication:

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\times\begin{bmatrix}7 & 8\\ 9 & 10\\ 11 & 12\end{bmatrix}=\begin{bmatrix}58 & \\ & \end{bmatrix}\qquad 1\times7+2\times9+3\times11=58$$

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\times\begin{bmatrix}7 & 8\\ 9 & 10\\ 11 & 12\end{bmatrix}=\begin{bmatrix}58 & 64\\ & \end{bmatrix}\qquad 1\times8+2\times10+3\times12=64$$

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\times\begin{bmatrix}7 & 8\\ 9 & 10\\ 11 & 12\end{bmatrix}=\begin{bmatrix}58 & 64\\ 139 & \end{bmatrix}\qquad 4\times7+5\times9+6\times11=139$$

$$\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}\times\begin{bmatrix}7 & 8\\ 9 & 10\\ 11 & 12\end{bmatrix}=\begin{bmatrix}58 & 64\\ 139 & 154\end{bmatrix}\qquad 4\times8+5\times10+6\times12=154$$

Basic Mathematical Operation on Matrix

- ❖ The multiplication of two matrices of orders i*j and j*k results into a matrix of order i*k. Just keep the outer indices in order to get the indices of the final matrix.

- ❖ Two matrices will be compatible for multiplication only if the number of columns of the first matrix and the number of rows of the second one are same.

- ❖ The third point is that order of multiplication matters.

- **Scalar Multiplication**

$$2 \cdot \begin{bmatrix} 10 & 6 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot 10 & 2 \cdot 6 \\ 2 \cdot 4 & 2 \cdot 3 \end{bmatrix}$$

- **Transposition**

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

Representing Equations in matrix Form

$$\overbrace{\begin{matrix}1 & 1 & 1\\ 2 & 1 & 0\\ 5 & 3 & 2\end{matrix}}^{A}\quad\overbrace{\begin{matrix}x\\ y\\ z\end{matrix}}^{X} = \overbrace{\begin{matrix}1\\ 1\\ 4\end{matrix}}^{Z}$$

How to solve AX=Z?

$X = A^{-1} Z$

For this you will have to know how to take inverse of a matrix. We will revisit this problem soon.

Determinant of a matrix

A determinant is a function of a square matrix that reduces it to a single number. The determinant of a matrix *A* is denoted |*A*| or *det*(*A*). If *A* consists of one element *a*, then |*A*| = *a*; in other words if *A* = [6] then |*A*| = 6.

Of 2*2 matrix:

$$A = \begin{pmatrix} 3 & 8 \\ 4 & 6 \end{pmatrix}$$

Det A= |A| = 3*6-8*4= 18-32= -14

Of 3*3 Matrix:

$$A = \begin{pmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{pmatrix}$$

Det. A = |A| = 6*(-2*7-5*8)-1*(4*7-5*2)+(1)*(4*8-(-2*2)) = 6*(-54)-1*18+1*36=-306

Of 4*4 Matrix or higher:

$$A = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ I & j & k & l \\ m & n & o & p \end{pmatrix}$$

- The pattern continues for 4×4 matrices:
- **plus a** times the determinant of the matrix that is **not** in **a**'s row or column,
- **minus b** times the determinant of the matrix that is **not** in **b**'s row or column,
- **plus c** times the determinant of the matrix that is **not** in **c**'s row or column,
- **minus d** times the determinant of the matrix that is **not** in **d**'s row or column,

$$|A| = a \cdot \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \cdot \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \cdot \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \cdot \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$

Minor & Cofactor of a Matrix

- If A is a square matrix, then the **minor** of the entry in the *i*-th row and *j*-th column (also called the ***(i,j)*** **minor** is the determinant of the submatrix formed by deleting the i-th row and j-th column. This number is often denoted $M_{i,j}$. The (i,j) **cofactor** is obtained by multiplying the minor by (-1)^{i+j}.
- To illustrate these definitions, consider the following 3 by 3 matrix,

$$\begin{bmatrix} 1 & 4 & 7 \\ 3 & 0 & 5 \\ -1 & 9 & 11 \end{bmatrix}$$

What is Minor M2,3 and Cofactor C 2,3 of Matrix A as shown above?

M2,3 = 13

C2,3 = 13

How to take inverse of a matrix?

It involves several steps:

Step 1: calculating the Matrix of Minors,

Step 2: then turn that into the Matrix of Cofactors,

Step 3: then the Adjugate and Step 4: multiply that by 1/Determinant

Find Inverse of this matrix A:

$$\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 0\times1 - (-2)\times1 & 2\times1 - (-2)\times0 & 2\times1 - 0\times0 \\ 0\times1 - 2\times1 & 3\times1 - 2\times0 & 3\times1 - 0\times0 \\ 0\times(-2) - 2\times0 & 3\times(-2) - 2\times2 & 3\times0 - 0\times2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}}_{\text{Matrix of Minors}}$$

Step 1

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}}_{\text{Matrix of Minors}} \rightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 2 & -2 & 2 \\ +2 & 3 & -3 \\ 0 & +10 & 0 \end{bmatrix}}_{\text{Matrix of CoFactors}}$$

Step 2

$$\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

Adjugate or Adjoint

Step 3

Inverse of a Matrix

- Final Step: Multiply by 1/determinant
- Det of 2 2 2
- -2 3 3
- 0 -10 0
- =2*(-(-10*3)-2*0+2*(-2*-10)=60+40= 100
- 2 2 2
- -2 3 3
- 0 -10 0
- Inverse is (1/100) *
-

Eigen values & Eigen Vectors

- An eigenvector is a nonzero vector that satisfies the equation $A\vec{v} = \lambda\vec{v}$ where A is a square matrix, λ is a scalar, and $\vec{v}$ is the eigen vector.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad A\vec{v} = \lambda\vec{v} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + x_2 &= \lambda x_1 \\ x_1 + 2x_2 &= \lambda x_2 \end{aligned} \longrightarrow \begin{aligned} (2-\lambda)x_1 + x_2 &= 0 \\ x_1 + (2-\lambda)x_2 &= 0 \end{aligned}$$

A necessary and sufficient condition for this system to have a nonzero vector $[x_1, x_2]$ is that the determinant of the coefficient matrix

$$\begin{bmatrix} (2-\lambda) & 1 \\ 1 & (2-\lambda) \end{bmatrix}$$

be equal to zero. Accordingly,

$$\begin{vmatrix} (2-\lambda) & 1 \\ 1 & (2-\lambda) \end{vmatrix} = 0$$

$$(2-\lambda)(2-\lambda) - 1 \cdot 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

There are two values of λ that satisfy the last equation; thus there are two eigenvalues of the original matrix A and these are $\lambda_1 = 3, \lambda_2 = 1$.

Eigen vector

- To find associated Eigen vector with Eigen value of 3,

$$(2-\lambda)x_1 + x_2 = 0$$

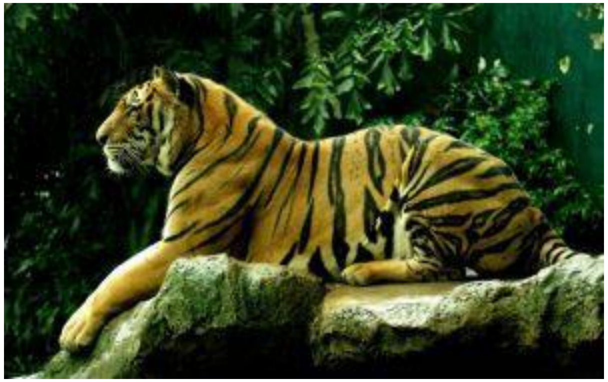
$$(2-3)x_1 + x_2 = 0$$

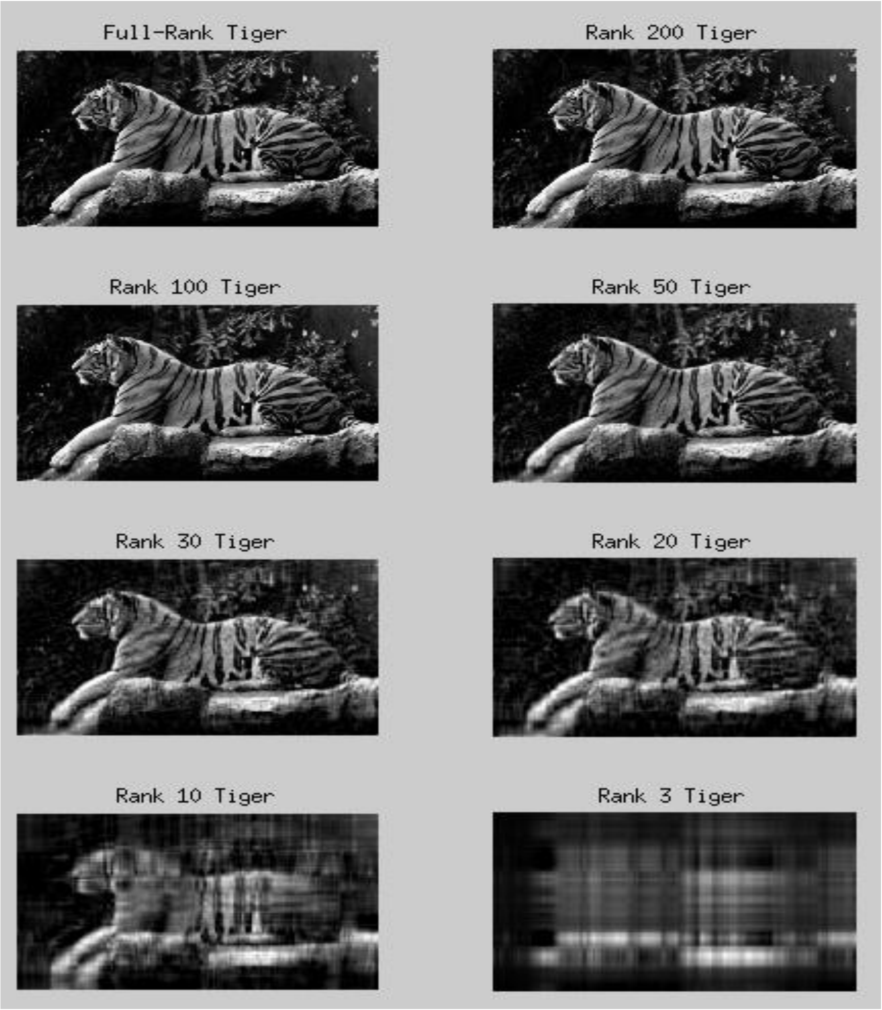
$x1 = x2$

There are an infinite number of values for $x1$ which satisfy this equation; the only restriction is that not all the components in an eigenvector can equal zero. So if $x1 = 1$, then $x2 = 1$ and an eigenvector corresponding to $\lambda = 3$ is [1, 1].

Finding an eigenvector for $\lambda = 1$ works the same way.
$(2 - 1)x1 + x2 = 0$
$x1 = -x2$
So an eigenvector for $\lambda = 1$ is $[1, -1]$.

Single Value Decomposition (SVD): Motivation





SVD

SVD is based on a theorem from linear algebra which says that a rectangular matrix A can be broken down into the product of three matrices - an orthogonal matrix U, a diagonal matrix S, and the transpose of an orthogonal matrix V. The theorem is usually presented something like this:

$$A_{mn} = U_{mm} S_{mn} V_{nn}^T$$

where $U^T U = I, V^T V = I$; the columns of U are orthonormal eigenvectors of AA^T, the columns of V are orthonormal eigenvectors of $A^T A$, and S is a diagonal matrix containing the square roots of eigenvalues from U or V in descending order.

Singular Value Decomposition

$$A_{mn} = U_{mm} S_{mn} V_{nn}^T$$

$$A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

Next, we have to find the eigenvalues and corresponding eigenvectors of AA^T. We know that eigenvectors are defined by the equation $A\vec{v} = \lambda\vec{v}$, and applying this to AA^T gives us

$$\begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We rewrite this as the set of equations

$$11x_1 + x_2 = \lambda x_1$$

$$x_1 + 11x_2 = \lambda x_2$$

and rearrange to get

$$(11 - \lambda)x_1 + x_2 = 0$$

Finally, we find eigenvalue: $\lambda = 10$, $\lambda = 12$

Eigen vectors are respectively: [1,1] and [1,-1]

Now we need to find Ortho-normal basis for these using Gram-Schmidt Ortho Normalization process:

$$\vec{u_1} = \frac{\vec{v_1}}{|\vec{v_1}|} = \frac{[1,1]}{\sqrt{1^2+1^2}} = \frac{[1,1]}{\sqrt{2}} = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$$

$$\vec{w_2} = \vec{v_2} - \vec{u_1} \cdot \vec{v_2} * \vec{u_1} =$$

$$[1,-1] - [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] \cdot [1,-1] * [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] =$$

$$[1,-1] - 0 * [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}] = [1,-1] - [0,0] = [1,-1]$$

and normalize

$$\vec{u_2} = \frac{\vec{w_2}}{|\vec{w_2}|} = [\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}]$$

to give

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Find V

- Follow the same previous steps for $A^T A$,

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

Eigen values are: 0, 10, 12

Find S

- We take square root on non-negative eigen values. Non-zero eigen values are always the same for U & V.

$$S = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

These values indicate the variance of the linearly components along each dimension

$$A_{mn} = U_{mm} S_{mn} V_{nn}^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

An Example

$$A = \begin{bmatrix} 2 & 0 & 8 & 6 & 0 \\ 1 & 6 & 0 & 1 & 7 \\ 5 & 0 & 7 & 4 & 0 \\ 7 & 0 & 8 & 5 & 0 \\ 0 & 10 & 0 & 0 & 7 \end{bmatrix}$$

Consider A as the word document matrix

Non-zero eigen values are:

$$\lambda = 321.07, \lambda = 230.17, \lambda = 12.70, \lambda = 3.94, \lambda = 0.12$$

Keeping only 3 dimensions:

$$S = \begin{bmatrix} 17.92 & 0 & 0 \\ 0 & 15.17 & 0 \\ 0 & 0 & 3.56 \end{bmatrix}$$

$$\hat{A} =$$

$$\begin{bmatrix} -0.54 & 0.07 & 0.82 \\ -0.10 & -0.59 & -0.11 \\ -0.53 & 0.06 & -0.21 \\ -0.65 & 0.07 & -0.51 \\ -0.06 & -0.80 & 0.09 \end{bmatrix} \begin{bmatrix} 17.92 & 0 & 0 \\ 0 & 15.17 & 0 \\ 0 & 0 & 3.56 \end{bmatrix} \begin{bmatrix} -0.46 & 0.02 & -0.87 & -0.00 & 0.17 \\ -0.07 & -0.76 & 0.06 & 0.60 & 0.23 \\ -0.74 & 0.10 & 0.28 & 0.22 & -0.56 \end{bmatrix}$$

$$= \begin{bmatrix} 2.29 & -0.66 & 9.33 & 1.25 & -3.09 \\ 1.77 & 6.76 & 0.90 & -5.50 & -2.13 \\ 4.86 & -0.96 & 8.01 & 0.38 & -0.97 \\ 6.62 & -1.23 & 9.58 & 0.24 & -0.71 \\ 1.14 & 9.19 & 0.33 & -7.19 & -3.13 \end{bmatrix}$$

SVD Example: Cont’d

- Our idea was not to recreate the original matrix but to use the reduced dimensionality representation to identify similar word and documents.

Documents are represented by row vectors in v and document similarity is obtained by comparing rows in VS.

Word similarity is obtained by comparing rows in US.

Inner product

The *inner product* of two vectors (also called the *dot product* or *scalar product*) defines multiplication of vectors. It is found by multiplying each component in $\vec{v_1}$ by the component in $\vec{v_2}$ in the same position and adding them all together to yield a scalar value. The inner product is only defined for vectors of the same dimension. The inner product of two vectors is denoted $(\vec{v_1}, \vec{v_2})$ or $\vec{v_1} \cdot \vec{v_2}$ (the dot product). Thus,

$$(\vec{x}, \vec{y}) = \vec{x} \cdot \vec{y} = \sum_{i=1}^{n} x_i y_i$$

For example, if $\vec{x} = [1, 6, 7, 4]$ and $\vec{y} = [3, 2, 8, 3]$, then

$$\vec{x} \cdot \vec{y} = 1(3) + 6(2) + 7(8) + 3(4) = 83$$

Two vectors are orthogonal to each other if their inner product is equal to zero.

Normal Vector

A *normal vector* (or *unit vector*) is a vector of length 1. Any vector with an initial length $>$ 0 can be normalized by dividing each component in it by the vector's length. For example, if $\vec{v} = [2, 4, 1, 2]$, then

$$|\vec{v}| = \sqrt{2^2 + 4^2 + 1^2 + 2^2} = \sqrt{25} = 5$$

Then $\vec{u} = [2/5, 4/5, 1/5, 1/5]$ is a normal vector because

$$|\vec{u}| = \sqrt{(2/5)^2 + (4/5)^2 + (1/5)^2 + (2/5)^2} = \sqrt{25/25} = 1$$

Orthonormal Vector

Vectors of unit length that are orthogonal to each other are said to be *orthonormal*. For example,

$$\vec{u} = [2/5, 1/5, -2/5, 4/5]$$

and

$$\vec{v} = [3/\sqrt{65}, -6/\sqrt{65}, 4/\sqrt{65}, 2/\sqrt{65}]$$

are orthonormal because

$$|\vec{u}| = \sqrt{(2/5)^2 + (1/5)^2 + (-2/5)^2 + (4/5)^2} = 1$$

$$|\vec{v}| = \sqrt{(3/\sqrt{65})^2 + (-6/\sqrt{65})^2 + (4/\sqrt{65})^2 + (2/\sqrt{65})^2} = 1$$

$$\vec{u} \cdot \vec{v} = \frac{6}{5\sqrt{65}} - \frac{6}{5\sqrt{65}} - \frac{8}{5\sqrt{65}} + \frac{8}{5\sqrt{65}} = 0$$

Gram-Schmidt Orthogonalization: Example

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\vec{v_1} = [1, 0, 2, 1] \qquad \vec{u_1} = [\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}].$$

$$\vec{w_2} = \vec{v_2} - \vec{u_1} \cdot \vec{v_2} * \vec{u_1} = [2, 2, 3, 1] - [\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}] \cdot [2, 2, 3, 1] * [\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}]$$

$$= [2, 2, 3, 1] - (\frac{9}{\sqrt{6}}) * [\frac{1}{\sqrt{6}}, 0, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}]$$

$$= [2, 2, 3, 1] - [\frac{3}{2}, 0, 3, \frac{3}{2}]$$

$$= [\frac{1}{2}, 2, 0, \frac{-1}{2}]$$

After normalizing: $\vec{u_2} = [\frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}, 0, \frac{-\sqrt{2}}{6}]$

$$\vec{w_3} = \vec{v_3} - \vec{u_1} \cdot \vec{v_3} * \vec{u_1} - \vec{u_2} \cdot \vec{v_3} * \vec{u_2}$$