中山大学本科生 2019 期末考试

考试科目：《高等代数》（A 卷）

学年学期：2019 学年第 1 学期　　　　姓　　名：________________

学　院/系：数据科学与计算机学院　　学　　号：________________

考试方式：闭卷　　　　　　　　　　年级专业：________________

考试时长：120 分钟　　　　　　　　班　　别：________________

警示《中山大学授予学士学位工作细则》第八条：“考试作弊者，不授予学士学位。”

------------以下为试题区域，共 4 道大题，总分 100 分，考生请在答题纸上作答------------

Hint: we use lowercase letter (e.g. a, b, c) to represent scalar, lowercase letter with arrow above (e.g. $\vec{\alpha}, \vec{b}, \vec{c}$) to represent vector and uppercase letter (e.g. A, B, C) to represent matrix. $\operatorname{rank}(A)$ is the rank of matrix A, $A^* = \operatorname{adj} A$ is the adjugate matrix of A, $\det(A)$ is the determinant of A, $\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}$ is the trace of a square A, where a_{ii} is the entry of the i-th column and i-th row of matrix A; A^{-1} is the inverse of A, and A^T is the transpose of A.

一、填空题（共 4 小题，每小题 4 分，共 16 分）

(1). Given a matrix $A = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 3 & 2 \\ 5 & 0 & 0 \end{bmatrix}$, $\det(2AA^*) =$ __________.

(2). Given a vector $\vec{\alpha} = [1,2,-2]^T$, $\operatorname{tr}(\vec{\alpha}\vec{\alpha}^T) =$ ____________.

(3). If vectors $\vec{\alpha}_1 = [0,1,\lambda]^T, \vec{\alpha}_2 = [\lambda, 1, 0]^T, \vec{\alpha}_3 = [0, \lambda, 1]^T$ are linearly dependent, then $\lambda =$ ________.

(4). If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 1 \\ 0 & 1 & a \end{bmatrix}$ is a positive definite matrix, then the value range of a is ________.

二、选择题（共 4 小题，每小题 4 分，共 16 分）

(1). If $AB = C$, then _______.

(A) $\text{rank}\,(A+B) \geq \text{rank}\,(A) + \text{rank}\,(B)$. (B) $\text{rank}\,(A) + \text{rank}\,(B) = \text{rank}\,(C)$.

(C). $\text{rank}\,(C) \leq \text{rank}\,(A)$. (D) $\text{rank}\,(B) \leq \text{rank}\,(C)$.

(2). Given a square matrix $A = \begin{bmatrix} \vec{\alpha}_1 \\ \vec{\alpha}_2 \\ \vec{\alpha}_3 \end{bmatrix} \in \mathbb{R}^{3\times 3}$, each $\vec{\alpha}_i (i = 1,2,3)$ is a 3-dimensional row vector. Matrix $B = \begin{bmatrix} \vec{\alpha}_2 \\ \vec{\alpha}_1 \\ \vec{\alpha}_3 - 2\vec{\alpha}_1 \end{bmatrix}$, $P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, then______.

(A) $AP_1P_2 = B$. (B) $AP_2P_1 = B$.

(C) $P_1P_2A = B$. (D) $P_2P_1A = B$.

(3). Equation $Ax = \vec{0}$ has only the trivial solution if and only if______.

(A) The columns of A are linearly dependent. (B) the rows of A are linearly dependent.

(C) A has full row rank. (D) A has full column rank.

(4). If $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$, then $\left(\frac{1}{2}A^*\right)^{-1} =$_________.

(A) $\frac{1}{2}A$. (B) $\frac{1}{4}A$. (C) $\frac{1}{8}A$. (D) $\frac{1}{16}A$.

三、计算题（共 5 小题，其中第 3 小题 10 分，其余小题 12 分，共 58 分）

(1). Find matrix B satisfies $A^*B = 2I + 2B$, where $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

(2). Let matrix $A = \begin{bmatrix} 1 & 0 & 2 & a \\ 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & a+b \\ 1 & 1 & 1 & 2a \end{bmatrix}$ and vector $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 0 \\ b+1 \end{bmatrix}$.

(a) Please tell when there is no solution, unique solution and infinitely many solutions to the matrix equation $Ax = \vec{b}$.

(b) Find the solution set when there are infinitely many solutions.

(3). Given three vectors $\vec{x} = \begin{bmatrix} 4 \\ -1 \\ 9 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 11 \\ 1 \\ 17 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} 3 \\ 6 \\ -2 \end{bmatrix}$ and their coordinate vectors $[\vec{x}]_\beta =$

$\begin{bmatrix}1\\1\\0\end{bmatrix}, [\vec{y}]_\beta = \begin{bmatrix}1\\0\\1\end{bmatrix}, [\vec{z}]_\beta = \begin{bmatrix}0\\-3\\1\end{bmatrix}$ relative to the basis β.

(a) Find the basis β.

(b) Given another basis $\mathcal{C} = \left\{\begin{bmatrix}3\\0\\6\end{bmatrix}, \begin{bmatrix}2\\2\\4\end{bmatrix}, \begin{bmatrix}1\\-2\\3\end{bmatrix}\right\}$, please find the change-of-coordinates matrix from β to $\mathcal{C}$.

(4) Given a vector set $\alpha = \{\vec{\alpha}_1 = [2,2,2,1]^T, \vec{\alpha}_2 = [1,0,2,1]^T, \vec{\alpha}_3 = [1,2,0,1]^T\}$. Answer the following questions.

(a) Can vector set $\beta = \{\vec{b}_1 = [3,4,2,3]^T, \vec{b}_2 = [4,2,6,3]^T\}$ be linearly represented by vector set α?

(b) If β can be linearly represented by α, please provide the coefficients of the linear combination.

(5) Let matrix $A = \begin{bmatrix}2 & 3 & -1\\2 & 1 & 0\\0 & 4 & 3\end{bmatrix}$ and $AX - A = 3X$. Find the matrix X.

四、证明题（共 1 小题，共 10 分）

Prove the following theorem: A $n \times n$ real matrix A is positive definite if and only if there exist n linearly independent real vectors $\vec{\alpha}_i = (m_{i1}, m_{i2}, \cdots, m_{in}), i = 1,2,\cdots,n$ satisfying that $A = \vec{\alpha}_1^T\vec{\alpha}_1 + \vec{\alpha}_2^T\vec{\alpha}_2 + \cdots + \vec{\alpha}_n^T\vec{\alpha}_n$.