考试科目：《高等代数》（A 卷）期末考试参考答案

一、填空题（共 2 小题，第1小题 15 分，第2小题 6 分，共 21 分）

1. (a) $\{\vec{0}\}$, (b) 2, (c) $\begin{bmatrix}0\\1\\0\end{bmatrix}$, (d) $\begin{bmatrix}0\\1\\0\end{bmatrix}$, (e) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix}$ (每小题各 3 分)

2. (a) −6, (b) −6, (c) 0 (每小题各 2 分)

二、选择题（共 2 小题，每小题 5 分，共 10 分）

1. C, 2. D

三、计算题（共 4 小题，第1、3小题各 10 分，第2、4小题各 15 分，共 50 分）

1. (a) $\begin{bmatrix}1&1&0&3\\1&1&b&2\\a&b&b-a&1+3a\end{bmatrix}\sim\begin{bmatrix}1&1&0&3\\0&0&b&-1\\0&b-a&b-a&1\end{bmatrix}\sim\begin{bmatrix}1&1&0&3\\0&b-a&b-a&1\\0&0&b&-1\end{bmatrix}$ (3分)

   i) $b=0$ or $a=b$, *ii*) $b\neq 0$ and $a\neq b$, *iii*) impossible (3 分)

   (b) $\begin{bmatrix}1&1&0&3\\0&-1&-1&1\\0&0&1&-1\end{bmatrix}\sim\begin{bmatrix}1&1&0&3\\0&1&1&-1\\0&0&1&-1\end{bmatrix}\sim\begin{bmatrix}1&0&0&3\\0&1&0&0\\0&0&1&-1\end{bmatrix}$, $\begin{bmatrix}3\\0\\-1\end{bmatrix}$ (4 分)

2. (a) By linear property of differential operator, $T(af+bg)=aT(f)+bT(g)$ (3分)

   (b) $[T]_{\mathcal{B}}=\left[\left[T(\vec{b}_1)\right]_{\mathcal{B}},\left[T(\vec{b}_2)\right]_{\mathcal{B}},\left[T(\vec{b}_3)\right]_{\mathcal{B}},\left[T(\vec{b}_4)\right]_{\mathcal{B}}\right]=\begin{bmatrix}0&2&2&0\\0&0&4&6\\0&0&0&6\\0&0&0&0\end{bmatrix}$ (3 分)

   $[T]_{\mathcal{C}}=[[T(\vec{c}_1)]_{\mathcal{C}},[T(\vec{c}_2)]_{\mathcal{C}},[T(\vec{c}_3)]_{\mathcal{C}},[T(\vec{c}_4)]_{\mathcal{C}}]=\begin{bmatrix}0&2&0&-6\\0&0&4&4\\0&0&0&6\\0&0&0&0\end{bmatrix}$ (3 分)

   (c) $P_{\mathcal{B}\leftarrow\mathcal{C}}=[[\vec{c}_1]_{\mathcal{B}},[\vec{c}_2]_{\mathcal{B}},[\vec{c}_3]_{\mathcal{B}},[\vec{c}_4]_{\mathcal{B}}]=\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$ (2 分)

   $P_{\mathcal{C}\leftarrow\mathcal{B}}=P_{\mathcal{B}\leftarrow\mathcal{C}}^{-1}=\begin{bmatrix}1&-1&0&0\\0&1&-1&0\\0&0&1&-1\\0&0&0&1\end{bmatrix}$ (2 分)

$$[T]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[T]_{\mathcal{B}}P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 6 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 2 & -2 & -6 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & -6 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (2 分)}$$

3. (a) Inconsistent (2分), $A^TA\hat{x} = A^T\vec{b}$ (2分), the least-squares solution is $\hat{x} = \begin{bmatrix} -\frac{17}{15} \\ \frac{2}{3} \end{bmatrix}$ (2分)

(b) $\text{proj}_{\text{Col}A}\vec{b} = A\hat{x} = \begin{bmatrix} \frac{16}{15} \\ \frac{17}{15} \\ \frac{2}{3} \end{bmatrix}$ (2分), $\left\|\vec{b} - \text{proj}_{\text{Col}A}\vec{b}\right\| = \left\|\begin{bmatrix} -\frac{1}{15} \\ -\frac{2}{15} \\ \frac{1}{3} \end{bmatrix}\right\| = \frac{\sqrt{30}}{15}$ (2分)

4. (a) Since $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, 3 is an eigenvalue of B. (1分)

rank B = 1, dim Nul B = 3 – rank B = 2. (1分)

0 is an eigenvalue of B with multiplicity 2. (1分)

If $B\vec{x} = \lambda_B\vec{x}$, then $A\vec{x} = (\lambda_B + 1)\vec{x}$. (1分)

On the other hand, $\dim \text{Nul}[A - (\lambda_B + 1)I_3] = \dim \text{Nul}[B - \lambda_B I_3]$. (1分)

Thus, $\lambda_B + 1$ is the eigenvalue of A with the same multiplicity as λ_B for B. (1分)

As a reulst, $\lambda_B = 0,0,3$, $\lambda_A = \lambda_B + 1 = 1,1,4$. (1分)

(b) Eigenvector basis: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (2分)

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \text{ (1分)}$$

$$D = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 4 \end{bmatrix} \text{ (1分)}$$

(c) $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (2分)

$$A^{100}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = A^{100}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + A^{100}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1^{100}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 4^{100}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4^{100} + 1 \\ 4^{100} - 1 \\ 4^{100} \end{bmatrix} \text{ (2分)}$$

四、证明题（共 2 小题，第1小题 10 分，第2小题 9 分，共 19 分）

1. (a) Col $AB \subseteq$ Col A, Row $AB \subseteq$ Row B (2分)

rank AB = dim Col $AB \leq$ dim Col A = rank A, rank AB = dim Row $AB \leq$ dim Row B = rank B (3分)

(b) $\Rightarrow$: Col AB = Col A, so $AB\vec{x} = \vec{a}_j$ is consistent for any column $\vec{a}_j$. (3分)

$\Leftarrow$: rank A = rank $ABX \leq$ rank AB by (a). On the other hand, rank $A \geq$ rank AB by (a). (2分)

2. Since rank $A = n-1$, there are $n-1$ linear independent columns in A. Denote A_1 as the matrix of the $n-1$ linear independent columns in A, rank $A = n-1$. So there are $n-1$ linear independent rows in A_1 and the determinant of the square matrix of these rows is non-zero. This means at least one cofactor of A is non-zero, i.e., A^* is not a zero matrix and rank $A^* \geq 1$. (5分)

det $A = 0$, so $A A^* = O$. Thus, Col $A^* \subseteq$ Nul A. While rank A^* = dim Col $A^* \leq$ dim Nul A = $n - (n-1) = 1$, rank A^* = dim Nul A = 1. So Nul A = Col A^*. (4分)