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Short communication

Exponential formula for computing effective viscosity

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Abstract

An exponential model is proposed for evaluating the effective viscosity of a particle–fluid mixture. First, theoretical consideration is restricted to the dilute condition without effects of dynamic particle interactions and fluid turbulence. This leads to a power series expressed in terms of particle concentration, which can be viewed as an extension of the Einstein's formula. The derivation is then extended using an exponential model for the condition of high particle concentrations. The exponential formula obtained, which is not subjected to the maximum particle concentration, is found comparable to various empirical formulas available in the literature.

Keywords: Effective viscosity; Concentration; Suspension; Two-phase flow; Particle interaction

1. Introduction

The phenomenon of solid particles suspended in fluid is of key interest in hydraulic engineering such as sediment transport in rivers and slurry transport in pipelines. Similar two-phase suspensions can also be found in industrial applications such as the processing of cement, composite materials and foodstuffs. As an important macroscopic property of these suspensions, the effective viscosity is usually required in the analysis of the related transport processes.

It is well known that the earliest theoretical work on the effective viscosity was due to Einstein [1] whose derivation led to the effective viscosity to be linearly related to the particle concentration as follows:

$$\mu_r = 1 + 2.5\phi_p \tag{1}$$

where $\mu_r = \mu_m/\mu_f$ is the relative viscosity defined as the ratio of the effective viscosity of the particle–fluid mixture, μ_m, to the viscosity of fluid, μ_f, and ϕ_p the volumetric concentration of the particles. This expression is exact when viscous effect is dominant so that the creeping flow equations can be applied at the particle level. Since the influence of particle interaction is not considered, Eq. (1) is only applicable to suspensions with low particle concentrations, say less than 2% [2].

Following Einstein's work, numerous expressions have been proposed to extend the range of validity to higher concentrations. They are either theoretical expansions of Eq. (1) to higher order in ϕ_p, or empirical expressions that were obtained based on experimental data. The theoretical expansions are usually expressed in the form of a power series,

$$\mu_r = 1 + k_1\phi_p + k_2\phi_p^2 + k_3\phi_p^3 + \cdots \tag{2}$$

where k_1, k_2 and k_3 are coefficients. Evaluation of these coefficients other than k_1 requires the formulation of particle interactions, which is rather difficult. This is one of the reasons that only the theoretical values of the coefficients related to lower order in ϕ_p, like k_2 and k_3, can be found in the literature so far. Even for these lower order coefficients, the calculations are available only for idealized cases when the particle arrangement or its statistic properties are simple. An example for determining the k_2 value was presented by Batchelor and Green [3], who reported that the coefficient k_2 was equal to 7.6 for a suspension undergoing a pure straining motion. A recalculation for the same condition suggests the k_2 value to be revised to 6.95 [4]. Other k_2 values have been reviewed by Happel and Brenner [5]. Furthermore, Thomas and Muthukumar [6] found the third order coefficient, k_3, to be 6.40 by applying the multiple scattering theory to the evaluation of the hydrodynamic interaction of three spheres.

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Following these theoretical attempts, it can be expected that more complicated configurations of the particle arrangements will lead to further difficulties in the mathematical formulation. On the other hand, particles suspended in fluid should in fact be distributed in a random manner so that their arrangement cannot be fitted to a simple configuration or described by a simple distribution function. Therefore, although an idealized particle arrangement may enable a theoretical evaluation of the k coefficients in higher order, the applicability of the effective viscosity so derived is limited in practice. Comparisons with experimental results show that the existing theoretical expressions are only applicable to those conditions with low concentration up to 10% [7].

In addition to the theoretical expressions, various empirical relationships have been proposed to evaluate the effective viscosity of a suspension with higher concentration. An early survey of such relationships showed that large discrepancies exist among the different relationships [8]. These discrepancies could partially be attributed to the viscometric measurements that were normally subjected to shear-induced particle diffusion [2].

Table 1 shows some typical examples of the existing empirical formulas, where ϕ_{pmax} is the maximum particle concentration and μ_{in} the intrinsic viscosity. It is noted that for those relationships where the so-called maximum concentration is included as a parameter, the effective viscosity approaches infinity when the concentration is equal to the maximum value. This may not be physically reasonable. Strictly speaking, there are only two extreme conditions that are meaningful for the effective viscosity. The first condition is a suspension without particles, implying that the effective viscosity is the same as the fluid viscosity. The other is a suspension without fluid, which would then theoretically behave as a solid with infinite viscosity. On the other hand, only for particles with regular shapes can the maximum concentration be mathematically determined for a given packing arrangement. For example, the maximum concentration can reach 1.0 for cubes packed face-to-face and 0.74 for spheres with the closest hexagonal packing arrangement. With irregular particles, the maximum value varies markedly even though the particle size is uniform. As a matter of fact, the maximum concentration and intrinsic viscosity were often used as two empirical parameters in fitting experimental data to an empirical formula (e.g., Refs. [7,9]).

Table 1
Empirical formulas for effective viscosity

Researchers	μ_{m}/μ	μ_{in}	ϕ_{pmax}
Mooney [14]	$\exp\left(\frac{2.5\phi_{\mathrm{p}}}{1-\phi_{\mathrm{p}}/\phi_{\mathrm{pmax}}}\right)$		0.52–0.74
Thomas and Muthukumar [12]	$1+2.5\phi_{\mathrm{p}}+10.05\phi_{\mathrm{p}}^2+0.00273\exp(16.6\phi_{\mathrm{p}})$		
Metzner [16]	$\left(1-\frac{\phi_{\mathrm{p}}}{\phi_{\mathrm{pmax}}}\right)^{-2}$		0.68
Leighton and Acrivos [9]	$\left(1+\frac{0.5\mu_{\mathrm{in}}\phi_{\mathrm{p}}}{1-\phi_{\mathrm{p}}/\phi_{\mathrm{pmax}}}\right)^2$	3.0	0.58
Barnes et al. [7]	$\left(1-\frac{\phi_{\mathrm{p}}}{\phi_{\mathrm{pmax}}}\right)^{-\mu_{\mathrm{in}}\phi_{\mathrm{pmax}}}$	2.71–3.13	0.63–0.71

This study starts with a derivation of the effective viscosity for the dilute condition without dynamic particle interaction. The expression of the effective viscosity so obtained recovers the Einstein's formula. The expression is then modified to be applicable for high particle concentrations. Comparisons of the present study with other empirical relationships are finally provided.

2. Derivation

2.1. Effective viscosity for dilute conditions

The following approach illustrates an iterative concept, based on the Einstein's formula, which is simple in implementation yet retains the basic physical representation of the situation. The objective is to derive the effective viscosity in the dilute condition but with the particle concentration beyond the range of validity of the Einstein formula. Note that, the interparticle collision and effects of the fluid turbulence and particle random motion are assumed to be insignificant.

Consider a suspension made of clean fluid with uniformly distributed particles, whose volumes are v_{f} and v_{p}, respectively. First, divide v_{p} into n subvolumes, i.e.,

$$v_{\mathrm{p}} = \Delta v_{\mathrm{p},1} + \Delta v_{\mathrm{p},2} + \Delta v_{\mathrm{p},3} + \cdots + \Delta v_{\mathrm{p},n} \tag{3}$$

With this division, a series of slightly different suspension samples can be constructed with progressively increasing particle concentration, as sketched in Fig. 1. It is noted that although the particle volume varies in the different samples, the fluid volume remains constant. For simplicity, all the particle volume increments can be set to be equal, i.e., $\Delta v_{\mathrm{p},i} = \Delta v_{\mathrm{p}}$ for $i = 1, 2, \ldots, I$. Hence, the sample volumes are $v_{\mathrm{f}} + \Delta v_{\mathrm{p}}$, $v_{\mathrm{f}} + 2\Delta v_{\mathrm{p}}$, …, $v_{\mathrm{f}} + n\Delta v_{\mathrm{p}}$, respectively.

The effective viscosity values of the samples are denoted as $\mu_{\mathrm{m},1}$, $\mu_{\mathrm{m},2}, \ldots, \mu_{\mathrm{m},n}$, respectively. Their relationship can be conceptualized as follows. Consider the $(i-1)$th sample with the effective viscosity, $\mu_{\mathrm{m},i-1}$. If an additional volume of particles of Δv_{p} is further dispersed in this sample, it will then be changed to the ith sample with the effective viscosity increased to $\mu_{\mathrm{m},i}$. Since Δv_{p} is arbitrary, it can be chosen in such a manner that $\Delta v_{\mathrm{p}}/(v_{\mathrm{f}} + i\,\Delta v_{\mathrm{p}})$, for $i = 1, 2, \ldots, n$, is small within the range of validity of the Einstein's equation. With these considerations, it is expected that Eq. (1) is applicable to the ith suspension in the form

$$\mu_{\mathrm{m},i} = \mu_{\mathrm{m},i-1}(1 + 2.5\phi_{\mathrm{pr},i}) \tag{4}$$

where $\phi_{\mathrm{pr},i}$ is the ith relative volumetric concentration defined as

$$\phi_{\mathrm{pr},i} = \frac{\Delta v_{\mathrm{p}}}{v_{\mathrm{f}} + i\Delta v_{\mathrm{p}}} \tag{5}$$

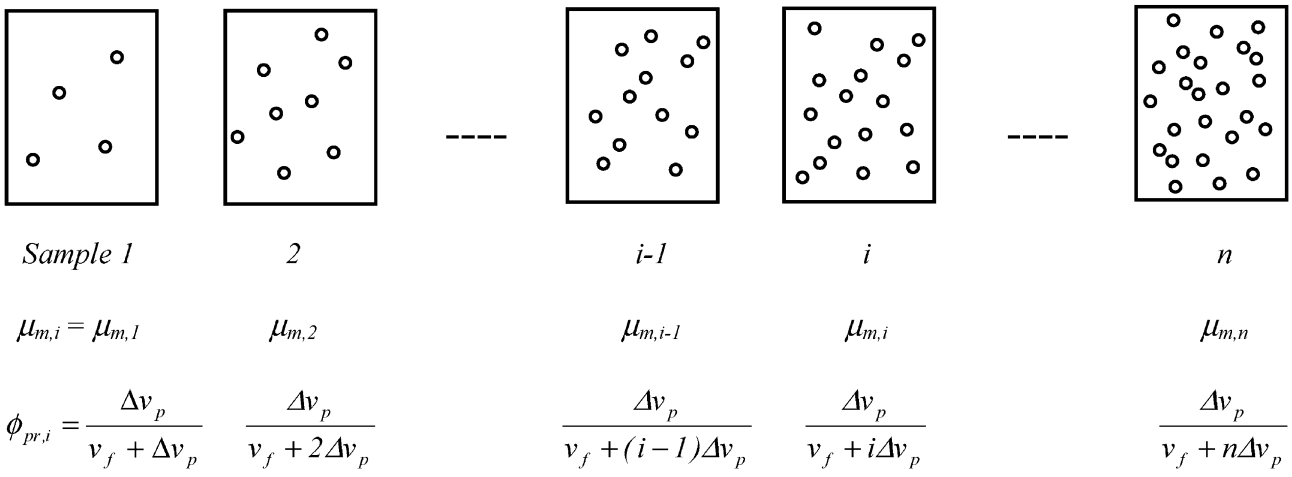

Fig. 1. Sketch of suspension series.

Substituting Eq. (5) into Eq. (4) yields

$$\mu_{\mathrm{m},i} = \mu_{\mathrm{m},i-1}\left(1 + 2.5\frac{\Delta v_{\mathrm{p}}}{v_{\mathrm{f}} + i\Delta v_{\mathrm{p}}}\right) \tag{6}$$

Note that $\mu_{\mathrm{m},0} = \mu_{\mathrm{f}}$. Applying Eq. (6) repeatedly from $i = 1$ to $i = n$, we can get

$$\mu_{\mathrm{m},n} = \mu_{\mathrm{f}}\prod_{i=1}^{n}\left(1 + \frac{2.5\Delta v_{\mathrm{p}}}{v_{\mathrm{f}} + i\Delta v_{\mathrm{p}}}\right) \tag{7}$$

Furthermore, since $\Delta v_{\mathrm{p}} = v_{\mathrm{p}}/n$ and $v_{\mathrm{p}}/(v_{\mathrm{p}} + v_{\mathrm{f}}) = \phi_{\mathrm{p}}$, Eq. (7) can be changed to

$$\mu_{\mathrm{m},n} = \mu_{\mathrm{f}}\prod_{i=1}^{n}\left(1 + \frac{2.5\phi_{\mathrm{p}}}{n(1-\phi_{\mathrm{p}}) + i\phi_{\mathrm{p}}}\right) \tag{8}$$

If n is large, the effective viscosity of the suspension with the volumetric particle concentration ϕ_{p} can be expressed as

$$\mu_{\mathrm{m}} = \mu_{\mathrm{f}}\lim_{n\to\infty}\left[\prod_{i=1}^{n}\left(1 + \frac{2.5\phi_{\mathrm{p}}}{n(1-\phi_{\mathrm{p}}) + i\phi_{\mathrm{p}}}\right)\right] \tag{9}$$

Using the Gamma function and its recursive property, Eq. (9) can be converted to

$$\mu_{\mathrm{r}} = \lim_{n\to\infty}\left[\frac{\Gamma\left(\frac{n}{\phi_{\mathrm{p}}} + 3.5\right)}{\Gamma\left(\frac{n}{\phi_{\mathrm{p}}} + 1\right)}\frac{\Gamma\left(n\frac{1-\phi_{\mathrm{p}}}{\phi_{\mathrm{p}}} + 1\right)}{\Gamma\left(n\frac{1-\phi_{\mathrm{p}}}{\phi_{\mathrm{p}}} + 3.5\right)}\right] \tag{10}$$

Furthermore, using the following relationship [11]

$$\lim_{x\to\infty}\left[\frac{\Gamma(x+a)}{\Gamma(x+b)}\right] = \lim_{x\to\infty} x^{a-b} \tag{11}$$

where a and b are parameters, Eq. (10) can finally be simplified to

$$\mu_{\mathrm{r}} = \lim_{n\to\infty}\left[\left(\frac{n}{\phi_{\mathrm{p}}}\right)^{2.5}\left(n\frac{1-\phi_{\mathrm{p}}}{\phi_{\mathrm{p}}}\right)^{-2.5}\right] = (1-\phi_{\mathrm{p}})^{-2.5} = \phi_{\mathrm{f}}^{-2.5} \tag{12}$$

where $\phi_{\mathrm{f}} = (1 - \phi_{\mathrm{p}})$ is the volumetric fraction of fluid. The above derivation is based on the iterative concept without any empiricism, hence the resulting formula Eq. (12) should be considered as analytically vigorous and a direct extension of Eq. (1). The formula can further be expanded in a power series as

$$\mu_{\mathrm{r}} = 1 + \frac{5}{2}\phi_{\mathrm{p}} + \frac{35}{8}\phi_{\mathrm{p}}^2 + \frac{105}{16}\phi_{\mathrm{p}}^3 + \frac{1155}{128}\phi_{\mathrm{p}}^4 + \frac{3003}{256}\phi_{\mathrm{p}}^5 + \cdots \tag{13}$$

This gives $k_2 = 4.38$ and $k_3 = 6.56$, which are very close to the coefficients computed by Thomas and Muthukumar [6,12].

As the dynamic effects of particles and fluid are not included, Eq. (12) can only be applied to dilute suspensions. If the particle concentration is increased, interparticle collisions and random motion of particles are unavoidable, and thus the effective viscosity should also increase. This phenomenon can be illustrated effectively by introducing different stress components for the fluid and solid phases, respectively, as detailed in Ref. [13]. Theoretically, the effective viscosity subjected to the dynamic effects for nondilute conditions can be obtained if the different stresses can be formulated. Unfortunately, this is almost impossible as the information related to the particle collision and kinetic stress is still very limited in the literature. Alternatively, an exponential model is proposed in the following to extend Eq. (12) for the case of higher particle concentrations.

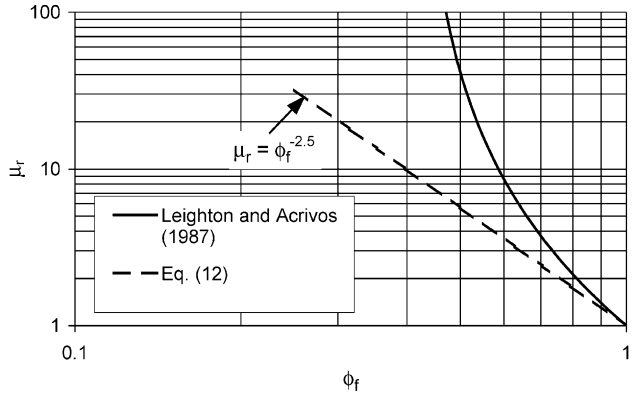

Fig. 2. Effective viscosity decreases with increasing fluid fractions.

2.2. Exponential model

It is known that the effective viscosity has been extensively measured, and different data sets reported by different investigators usually exhibit a considerable discrepancy at higher particle fractions [2,10]. This discrepancy may depend on the property of particles, shear rate and mechanism of the viscometer. For a specific suspension, however, the relative viscosity generally decreases with increasing fluid fraction described as follows. First, the relative viscosity declines rapidly with increasing fluid fraction, ϕ_f, for small ϕ_f values. This is then followed by a gentle reduction if ϕ_f is further increased. For the limiting condition of $\phi_f = 1$, the relative viscosity then reduces to unity. Such variations can be illustrated in Fig. 2, for example, by an empirical relationship which was obtained for a suspension comprised of 46-μm polystyrene spheres [9]. Also superimposed in the figure is Eq. (12), which serves as an asymptote to the measurements for very high ϕ_f values but deviates from the empirical relationship, as expected, if ϕ_f is reduced.

Fig. 2 shows that the slope of the trendline fitted to the experimental data, when plotted on the logarithmic scale, always decreases with increasing fluid fraction, approaching 2.5 for the dilute condition. Mathematically, as a first approximation, this can be expressed as

$$\frac{d\ln\mu_r}{d\ln\phi_f} = -\frac{2.5}{\phi_f^{\beta}} \tag{14}$$

where β is an exponent. Integration of Eq. (14) with respect to ϕ_f leads to

$$\mu_r = \alpha\exp\left(\frac{2.5}{\beta\phi_f^{\beta}}\right) \tag{15}$$

Since $\mu_r = 1$ for $\phi_f = 1$, $\alpha = \exp(-2.5/\beta)$. Substituting into Eq. (15) yields

$$\mu_r = \exp\left[\frac{2.5}{\beta}\left(\frac{1}{\phi_f^{\beta}} - 1\right)\right] \tag{16}$$

or

$$\mu_r = \exp\left[\frac{2.5}{\beta}\left(\frac{1}{(1-\phi_p)^{\beta}} - 1\right)\right] \tag{17}$$

From Eq. (16), it follows that with increasing β values, the $\mu_r - \phi_f$ relationship deviates gradually from $\mu_r = \phi_f^{-2.5}$,

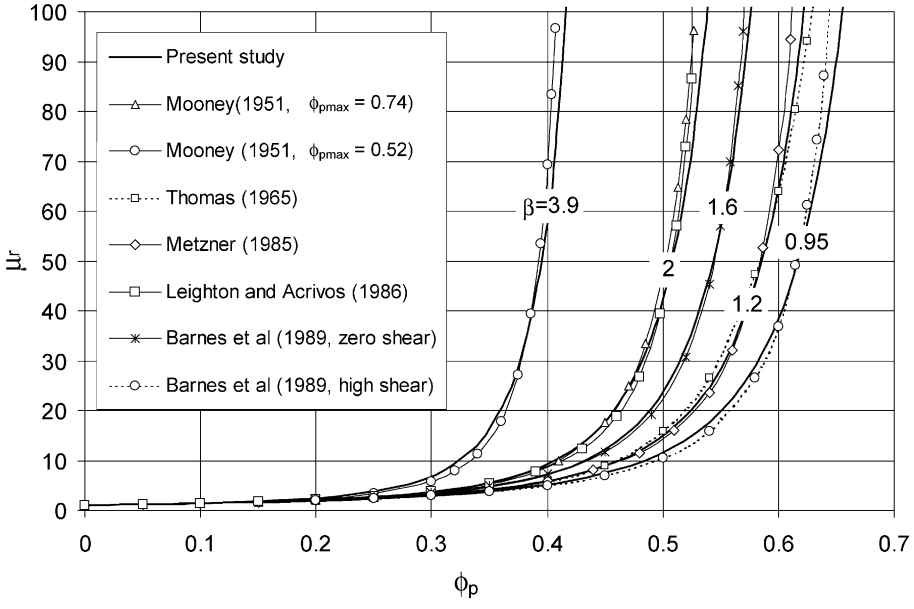

Fig. 3. Comparisons with previous empirical relationships.

and for the same fluid fraction, μ_r increases with increasing β values.

3. Comparison with previous studies

With different β values, Eq. (17) can be used to represent various empirical relationships included in Table 1, which were derived largely based on experimental data. For example, the formula given by Mooney [14] for $\phi_{pmax}=0.74$ and that reported by Leighton and Acrivos [9] can be well approximated by Eq. (17) with $\beta=2$; while Barnes et al.'s [7] expression for the high shear condition is close to Eq. (17) for $\beta=0.95$. More comparisons are given in Fig. 3, where the relative viscosity is plotted against the concentration of particles, and the corresponding β values are found varying from 0.95 to 3.9.

Similar to Eq. (12), Eq. (17) can also be expanded as a power series with β as a parameter:

$$\mu_r = 1+\frac{5}{2}\phi_p+\left(\frac{35}{8}+\frac{5}{4}\beta\right)\phi_p^2 + \left(\frac{105}{16}+\frac{35}{8}\beta+\frac{5}{12}\beta^2\right)\phi_p^3 + \left(\frac{1155}{128}+\frac{935}{96}\beta+\frac{235}{96}\beta^3+\frac{5}{48}\beta^4\right)\phi_p^4 + \left(\frac{3003}{256}+\frac{1125}{64}\beta+\frac{1465}{192}\beta^2+\frac{95}{96}\beta^3+\frac{1}{48}\beta^4\right)\phi_p^5 + \cdots \quad (18)$$

Clearly, for $\beta=0$, Eq. (18) reduces to Eq. (13). Comparing Eq. (18) with Eq. (2) yields that all the k coefficients except for $k_1=2.5$ are functions of β. Further computations show that $k_2=5.63-9.38$, $k_3=11.35-30.73$, $k_4=21.32-93.82$ and $k_5=37.95-272.79$ for β varying from 1 to 4. Obviously, larger variations are associated with higher-order coefficients.

Finally, it is interesting to note that the computed coefficients, $k_2=6.88$, $k_3=16.98$, $k_4=39.13$ and $k_5=85.66$ for $\beta=2$ are very close to those reported by Ward [15] who suggested the following expression for spherical particles

$$\mu_r = 1+2.5\phi_p+(2.5\phi_p)^2+(2.5\phi_p)^3+(2.5\phi_p)^4+(2.5\phi_p)^5+\cdots \quad (19)$$

to be fitted to the experimental data for the concentration up to 35%.

4. Conclusions

An approach is developed for calculating the effective viscosity for a particle suspension that is simple to implement yet maintains the underlying physical representation of the situation. The results obtained indicate that it is not necessary to involve the so-called maximum particle concentration, which is closely related to the randomness of particle packing, in computing the effective viscosity. For the condition without dynamic particle interactions and turbulence effects, the derived formula serves as an extension of the well-known Einstein's expression. This formula, after further modification, is able to match well with the empirical relationships reported previously. The analytical results obtained can also be given in the form of power series for comparison purposes.

Nomenclature

a	parameter
b	parameter
k	coefficient
k_i	coefficient ($i=1, 2, 3,\ldots$)
n	total number of suspension samples
v_f	volume of fluid
v_p	volume of particles
α	coefficient
β	exponent
μ_f	viscosity of fluid
μ_{in}	intrinsic viscosity
μ_m	effective viscosity of particle–fluid mixture
$\mu_{m,i}$	effective viscosity of the ith suspension sample ($i=1, 2,\ldots, n$)
μ_r	relative viscosity $=\mu_m/\mu_f$
ϕ_f	volumetric fraction of fluid
ϕ_p	volumetric concentration of particles
ϕ_{pmax}	maximum volumetric concentration
$\phi_{pr,i}$	ith relative volumetric concentration
$\Delta v_{p,i}$	increment of volume of particles ($i=1, 2,\ldots, n$)

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