



# Theory of Statistics I

Take Two

**Meridith L Bartley**



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# Part One

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# 1. Real Analysis Review

## 1.1 The Real Number System

### 1.1.1 Rationals

Start with integers as given.

**Definition 1.1.1 — Rational Numbers.** Rationals are numbers of the form  $\frac{m}{n}$ , for  $m, n$  integers,  $n \neq 0$  such that:

PR 1: The sum, difference, product, and ratio (division by 0 excluded) of any two rationals is a rational.

PR 2:  $p + q = q + p, pq = qp$  (Commutative Property)

PR 3:  $(p + q) + r = p + (q + r), (pq)r = p(qr)$ , (Associative Property)

PR 4:  $(p + q)r = pr + qr$  (Distributive Property)

PR 5:  $\forall$  two rationals  $p$  and  $q$  we have either  $p=q$ ,  $p < q$ , or  $q < p$  (Ordering Property)

PR 6: If  $p < q$  and  $q < r$ , then  $p < r$  (Transitivity of  $<$ )

PR 7: If  $p > 0$  and  $q > 0$ , then  $p + q > 0$  and  $pq > 0$

PR 8: If  $p < q$ , then  $p + r < q + r \forall r$

The rational number system is inadequate.

■ **Example 1.1** There is no rational number  $p$  that satisfies  $p^2 = 2$  ■

*Proof.* Suppose such a  $p$  existed, and so  $p = \frac{m}{n}$ . Note that  $m, n$  can be chosen so not both are even. Then we have,

$$m^2 = 2n^2$$

Thus,  $m^2$  is even, and hence  $m$  is even. (The square of an odd number is odd). Hence,  $m^2$  is divided by 4. So,  $2n^2$  is divisible by 4, or  $n^2$  is even which implies that  $n$  is even - **contradiction**. ■

This example can be used to show that we can have a set of rational numbers bounded from above but has no supremum.

■ **Example 1.2** Let  $A$  be the set of  $< 0$  rationals  $p$ , such that  $p^2 < 2$ . Let  $B$  be the set of  $> 0$  rationals  $p$ , such that  $p^2 > 2$ . Then  $A$  contains no largest number and  $B$  contains no smallest number.

■

*Proof.* If  $p \in A$ , choose a rational  $h$  such that,  $0 < h < 1$  and  $h < \frac{2-p^2}{2p+1}$  and set  $q = p + h$ . Then  $q$  is rational and

$$\begin{aligned} q^2 &= p^2 + (2p+h)h \\ &< p^2 + (2p+1)h \\ &< p^2 + (2-p^2) \\ &= 2 \end{aligned}$$

If  $p \in B$ , set

$$q = p - \frac{p^2 - 2}{2p} = \frac{p}{2} + \frac{1}{p}$$

and

$$\begin{aligned} q^2 &= p^2 - (p^2 - 2) + \left(\frac{p^2 - 2}{2p}\right)^2 \\ &> p^2 - (p^2 - 2) \\ &= 2 \end{aligned}$$

■



An axiomatic treatment of the real number system uses PR1 - PR8 as axioms together with the "completeness axiom". The non-axiomatics treatment is due to Dedekind.

### 1.1.2 Sets and Subsets

If  $A$  is any set,  $x \in A$  means that  $x$  is a member of  $A$ , and  $x \notin A$  means  $x$  is not a member of  $A$ . A set  $B$  is a **subset** of  $A$  if for every  $x \in B$  we have  $x \in A$ , and we write  $A \subseteq B$ .  $B$  is a **proper subset** of  $A$ ,  $B \subset A$ , if there  $\exists x \in A$  with  $x \notin B$ . The **empty set** is denoted by  $\emptyset$ , and  $\emptyset \in A$ ,  $\forall$  other set  $A$ .

$A \cup B = B \cup A$  - union with commutative property

$A \cap B = B \cap A$  - intersection with commutative property

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$(A \cap B) \cap C = A \cap (B \cap C)$  - associative property

$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$  - distributive property

$$(\cup A_i)^c = \cap A_i^c$$

$$(\cap A_i)^c = \cup A_i^c$$



**Definition 1.1.2 — Dedekind Cuts.** A set  $\alpha$  of rational numbers is said to be a **cut** if

- a)  $\alpha$  is a proper, but non-empty, subset of the rational numbers.
- b) If  $p \in \alpha$  ( $p$  is rational), and  $q < p$  ( $q$  is rational) then  $q \in \alpha$
- c) It contains no largest rational.

A cut of the form  $\alpha = \{p: p \text{ is rational and } p < r\}$  where  $r$  is rational are called **rational cuts** and are denoted by  $r^*$ .

The development of the real number system proceeds as follows:

First, the set of cuts is equipped with an order relation, and the operation of addition and multiplication and it will show that the resulting arithmetic satisfies PR 1 - PR 8.

If  $\alpha, \beta$  are cuts then,

$$\begin{aligned} \alpha < \beta & \text{ if } \alpha \subset \beta \text{ and} \\ \alpha & \leq \beta \text{ if } \alpha \subseteq \beta \\ \alpha + \beta & = \{r : r = p + q \text{ for some } p \in \alpha, q \in \beta\} \\ (\alpha + 0^*) & = \alpha \end{aligned}$$

If  $\alpha + \beta = 0^*$ , write  $\beta = -\alpha$ . (It can be shown that  $\forall \alpha$  there is one and only one  $\beta$  such that  $\alpha + \beta = 0^*$ .)

$$|\alpha| = \begin{cases} \alpha, & \text{if } \alpha \geq 0^*, \\ -\alpha, & \text{if } \alpha < 0^*. \end{cases}$$

For  $\alpha \geq 0^*$  and  $\beta \geq 0^*$ ,

$$\alpha\beta = \{p: p \text{ rational such that either } p < 0 \text{ or } p = pq, \text{ for } q \in \alpha, r \in \beta \text{ with } q \geq 0 \text{ and } r \geq 0.\}$$

For general  $\alpha, \beta$ ,

$$\alpha\beta = \begin{cases} -(|\alpha||\beta|), & \text{if } \alpha < 0^*, \text{ and } \beta \geq 0^* \\ & \text{or if } \alpha \geq 0^* \text{ and } \beta < 0^* \\ |\alpha||\beta|, & \text{if } \alpha < 0^*, \text{ and } \beta < 0^* \end{cases}$$

If  $\alpha \neq 0^*$ , then  $\forall \beta$  there is one and only one  $\gamma$  such that  $\alpha\gamma = \beta$ , and this  $\gamma$  is denoted by  $\frac{\beta}{\alpha}$ . (In technical terms, we made the set of cuts an **ordered field**.)

Second, it is shown that replacing the set of rational numbers by the corresponding cuts preserves sums, products and order, ie,

1.  $p^* + q^* = (p + q)^*$
2.  $p^* q^* = (pq)^*$
3.  $p^* < q^*$  iff  $p < q$

In technical terms, the ordered field of rational numbers is **isomorphic** to that of rational events.

**Theorem 1.1.1 — Dedekind.** Let  $A, B$  be  $\subset \mathbb{R}$  such that,

- (a)  $A \cap B = \emptyset$
- (b)  $A \cup B = \mathbb{R}$
- (c) neither  $A$  nor  $B$  is empty
- (d) if  $\alpha \in A, \beta \in B$ , then  $\alpha < \beta$

Then there  $\exists \gamma \in \mathbb{R}$  such that  $\alpha \leq \gamma, \forall \alpha \in A$  and  $\gamma \leq \beta, \forall \beta \in B$ .

*Proof.* First, suppose there are 2  $\gamma$ , say  $\gamma_1 < \gamma_2$ . Take  $\gamma_3$  such that  $\gamma_1 < \gamma_3 < \gamma_2$ .

$$\gamma_3 < \gamma_2 \text{ implies that } \gamma_3 \in A$$

$\gamma_1 < \gamma_3$  implies that  $\gamma_3 \in B$

However, these implications contradict the disjointness (part (a)). Define  $\gamma = \{p: p \text{ rational such that } p \in A \text{ for some } \alpha \in A\}$ . The proof proceeds by showing that  $\gamma$  is a cut, and hence a real number that satisfies  $\alpha \leq \gamma$  for  $\alpha \in A$  and  $\gamma \leq \beta \forall \beta \in B$ . ■

**Corollary 1.1.2** If  $A, B$  are as in the theorem, then either  $A$  contains a largest number or  $B$  contains a smallest number.

**Corollary 1.1.3** Let  $E \neq \emptyset$  be a subset of  $\mathbb{R}$ . Then, if  $E$  is bounded above a supremum (least upper bound) exists.

*Proof.* Define

$$A = \{\alpha : \alpha < x \text{ for some } x \in E\}$$

$$B = A^c$$

Clearly, all members of  $B$  are upper bounds of  $E$ . It is sufficient to prove that  $B$  contains a smallest number, or, by Corollary 1, that  $A$  does not contain a largest number (and thus prove by contradiction). Indeed if  $\alpha \in A \exists$  an  $x \in E$  such that  $\alpha < x$ . But, by Property 1 (???) there  $\exists$  an  $\alpha'$  such that  $\alpha < \alpha' < x$  where  $\alpha' \in A$  (i.e. we can always find a larger  $\alpha$  so, since there is no largest  $\alpha$ , there MUST be a smallest  $\beta$ ). ■

**Theorem 1.1.4** Any real number admits a decimal expansion.

*Proof.* Let  $x > 0, x \in \mathbb{R}$ . Let  $n_0 = [x]$  ( $n$  largest integer  $< x$ ). Let  $n_1$  be the largest integer such that  $n_0 + \frac{n_1}{10} < x$ . Having defined  $n_0 \dots n_{k-1}$ , define  $n_k$  as the largest integer such that  $n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \dots + \frac{n_k}{10^{k+1}} \leq x$ . Let  $E$  be the set of resulting numbers for  $k = 1, 2, \dots$ . Then  $x$  is the supremum of  $E$  and  $n_0, n_1, \dots$  is its **decimal expansion**. Conversely, any set of integers  $n_0, n_1, \dots$  defines a set of numbers,  $E$ , bounded above by  $n_0 + 1$ . ■

**Definition 1.1.3 — Extended Real Number System.**

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

### 1.1.3 Euclidean Space

**Definition 1.1.4 — Vector Space.** For any  $k \in \mathbb{Z}^+$ . Let  $\mathbb{R}^k$  be the set of ordered  $k$ -tuples.

$$\underline{x} = (x_1, \dots, x_k) \text{ with } x_i \in \mathbb{R}$$

$$\underline{y} = (y_1, \dots, y_k) \alpha \in \mathbb{R}$$

$$\underline{x} + \underline{y} = (x_1 + y_1, \dots, x_k + y_k)$$

$$\alpha \underline{x} = (\alpha x_1, \dots, \alpha x_k)$$

Which makes  $\mathbb{R}^k$  a **vector space** over the **real field**.

**Definition 1.1.5 — Inner/Scalar/Dot Product.**

$$\underline{x} \cdot \underline{y} = \sum_{i=1}^k x_i y_i$$

**Definition 1.1.6 — Norm/Length.**

$$|\underline{x}| = (\underline{x}\underline{x})^{\frac{1}{2}} = \sqrt{\sum_{i=1}^k x_i^2}$$

**Definition 1.1.7 — Euclidean K-space.** The vector space  $\mathbb{R}^k$  with the inner product and norm is called **Euclidean k-space**.

**Theorem 1.1.5** For  $\underline{x}, \underline{y} \in \mathbb{R}^k, \alpha \in \mathbb{R}$

- a)  $|\underline{x}| \geq 0, |\underline{x}| = 0$  iff  $\underline{x} = \underline{0}$   
 $|\alpha \underline{x}| = |\alpha| |\underline{x}|$
- b) **Cauchy-Schwarz Inequality**  $|\underline{x} \cdot \underline{y}| \leq |\underline{x}| |\underline{y}|$
- c) **Triangle Inequality**  $|\underline{x} + \underline{y}| \leq |\underline{x}| + |\underline{y}|$

## 1.2 Elements of Set Theory

**Definition 1.2.1** Let  $A, B$  be sets and suppose that to each  $x \in A$  there corresponds an elements of  $B$  denoted by  $f(x)$ . Then  $f$  is a **function** (or in more general space, mapping) from  $A$  (in)to  $B$ .

$A$  is called the **domain** of  $f$ .  $f(x)$  is the **value** of  $f$  at  $x$ ,  $R(f) = \{f(x) : x \in A\}$  is the **range** of  $f$ .

**Definition 1.2.2 — Image.** If  $f$  is a function from  $A$  to  $B$  ( $A \rightarrow B$ ) and  $E \subseteq A$  we write  $f(E) = \{f(x) : x \in E\}$  and call it the **image** of  $E$  under  $f$ . If  $f(A) = B$ , then we say  $f$  maps  $A$  **onto**  $B$ .

**Definition 1.2.3 — Inverse Image.** Let  $f : A \rightarrow B$  and  $E \subseteq B$ . We write  $f^{-1}(E) = \{x \in A : f(x) \in E\}$  and call it the **inverse image** of  $E$  **under**  $f$ . NB: If  $E = \{y\}, y \in B$  we also write  $f^{-1}(y)$  (versus  $f^{-1}(\{y\})$ ). If  $\forall y \in B$   $f^{-1}(y)$  consists of at most one element, then  $f$  is one to one mapping of  $A$  **into**  $B$ .

**Theorem 1.2.1** a)  $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

b)  $f(A \cup B) = f(A) \cup f(B)$

c)  $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Actually, these may be extended to arbitrary unions and intersections.

$$f^{-1}\left(\bigcup_{\alpha} A_{\alpha}\right) = \bigcup_{\alpha} f^{-1}(A_{\alpha})$$

$$f^{-1}\left(\bigcap_{\alpha} A_{\alpha}\right) = \bigcap_{\alpha} f^{-1}(A_{\alpha})$$

$$f\left(\bigcup_{\alpha} A_{\alpha}\right) = \bigcup_{\alpha} f(A_{\alpha})$$

Note:  $f(A \cap B)$  is not necessarily equal to  $f(A) \cap f(B)$  (see notes for example and sketch)

**Definition 1.2.4 — Cardinal Number.** If  $\exists$  a one-to-one mapping of  $A$  onto  $B$ , we say that  $A$  and  $B$  have the same **cardinal number**, or that they are **equivalent**  $A \sim B$ .

- a)  $A \sim A$  (reflective)
- b) If  $A \sim B$ , then  $B \sim A$  (symmetric)
- c) If  $A \sim B$  and  $B \sim C$ , then  $A \sim C$ . (transitive)

**Definition 1.2.5 — (In)finite/(Un)Countable.** Let  $\mathbb{Z}^+ = \{1, 2, \dots\}$  and  $\mathbb{Z}_n^+ = \{1, 2, \dots, n\}$  and let  $A$  be a set.

- a) We say  $A$  is **finite** if  $A \sim \mathbb{Z}_n^+$  for some  $n$  or if  $A = \emptyset$
- b)  $A$  is **infinite** if it is not finite
- c)  $A$  is **countable** if  $A \sim \mathbb{Z}^+$
- d)  $A$  is **uncountable** if  $A$  is not finite and countable.

Note: If  $A$  and  $B$  are finite, then  $A \sim B$  if and only if they have the same number of elements. This is not true if they are infinite.

■ **Example 1.3 Equivalent Infinite Sets**

1. The set  $\mathbb{Z}^+$  of all integers is countable. Then take

$$f(x) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ -(\frac{n-1}{2}), & \text{if } n \text{ is odd} \end{cases}$$

|   |   |    |
|---|---|----|
| 1 | → | 0  |
| 2 | → | 1  |
| 3 | → | -1 |
| 4 | → | 2  |
| 5 | → | -2 |
| 6 | → | 3  |
| 7 | → | -3 |

Table 1.1: Corresponding Integers

The set of positive, even integers is countable. Take

$$f(x) = 2n$$

■

**Theorem 1.2.2** The countable union of countable sets is countable.

*Proof.* Let  $A_1, A_2, \dots$  be countable and assume that they are disjoint (for if not, you can consider the sequences of countable sets that are disjoint -  $A_1, A_2 - A_1, \dots$ ), which are countable and have the same union. Let  $A_k = \{a_{k1}, a_{k2}, \dots\}$  and consider the arrangement of  $\bigcup_{k=1}^{\infty} A_k$ .

|          |          |          |          |     |    |    |     |     |     |
|----------|----------|----------|----------|-----|----|----|-----|-----|-----|
| $a_{11}$ | $a_{12}$ | $a_{13}$ | $a_{14}$ | ... | 1  | 2  | 6   | 7   | ... |
| $a_{21}$ | $a_{22}$ | $a_{23}$ | $a_{24}$ | ... | 3  | 5  | 8   | ... | ... |
| $a_{31}$ | $a_{32}$ | $a_{33}$ | $a_{34}$ | ... | 4  | 9  | 13  | ... | ... |
| $a_{41}$ | $a_{42}$ | $a_{43}$ | $a_{44}$ | ... | 10 | 12 | ... | ... | ... |

Table 1.2: Reassigning new values to counting integers.

■

**Theorem 1.2.3** Every infinite set has a countable subset.

*Proof.* Let  $a_1$  be any element of  $A$ . Since  $A$  is infinite, it contains an  $a_2 \neq a_1$ . So it contains a countable subset. ■

**Theorem 1.2.4** Every infinite set,  $A$ , is equivalent to at least one of its proper subsets.

*Proof.* Let  $E = \{a_1, a_2, \dots\}$  be a countable subset of  $A$  (which exists by previous Theorem). Write,

$$E = E_1 \cup E_2$$

$$E_1 = \{a_{odd}\}$$

$$E_2 = \{a_{even}\}$$

Then,  $E \sim E_2$

Define,

$$g : E \rightarrow E_2$$

$$g(a_i) = a_{2i}$$

$$f(a) = \begin{cases} a, & \text{if } a \notin E, \\ g(a), & \text{if } a \in E. \end{cases}$$

So, we can also say that  $A - E_1 \subset A$  and thus,  $A \sim (A - E_1)$  ■

**Theorem 1.2.5** The set of real numbers in  $[0,1]$  is uncountable.

*Proof.* Suppose all numbers in  $[0,1]$  are countable,  $\{a_1, a_2, \dots\}$ .

Write them in decimal expansion form. So, we can say

$$a_1 = 0.a_{11}a_{12} \dots a_{1n} \dots$$

$$a_2 = 0.a_{21}a_{22} \dots a_{2n} \dots$$

Recall,

$$0 = 0.000000000 \dots$$

$$1 = 0.999999999 \dots$$

Now, consider the number,  $\beta$  with decimal expansion  $\beta = 0.b_1b_2 \dots$  where

$$b_n = \begin{cases} 1, & \text{if } a_{nn} = 1, \\ 2, & \text{if } a_{nn} \neq 1. \end{cases}$$

There will always be 1 element difference. ALWAYS. ■

**Theorem 1.2.6** If  $A$  is countable, then so is  $A^n$ , where

$$A^n = \{(a_1, \dots, a_n); a_i \in A\}$$

*Proof.* Statement is true for any  $n=1$  since  $A^1 = A$ . Assume true for  $n=k$ . To show  $A^{k+1}$  is countable, write an element  $(a_1, a_2, \dots, a_k, a_{k+1}) = (\underline{a}, a_{k+1}), \underline{a} \in A^k$ . Thus,  $A^{k+1} = \bigcup_{\underline{a} \in A^k} \{\underline{a}, a_{k+1}\}; a_{k+1} \in A$  (see previous Theorem). ■

### 1.2.1 Metric Spaces

**Definition 1.2.6** A set  $X$  is a **metric space** if  $\forall x, y \in X$  there is a **real** number,  $d(x, y)$  called the **distance** between  $x$  and  $y$  such that,

- a)  $d(x, y) > 0$  if  $x \neq y$  and  $d(x, x) = 0$
- b)  $d(x, y) = d(y, x)$
- c)  $d(x, y) \leq d(x, z) + d(z, y), \forall z \in X$

- **Example 1.4**
- a) Euclidean spaces  $\mathbb{R}^k$  are metric spaces with  $d(x, y) = |x - y|$
  - b) Any subset of a metric space is a metric space with same distance.
  - c) The set  $\mathbb{R}^k$  can also be metrized with

$$d_1(x, y) = \sum_{i=1}^k |x_i - y_i|$$

or with

$$d_2(x, y) = \left( \sum_{i=1}^k |x_i - y_i|^p \right)^{\frac{1}{p}}$$

- d) The set  $C_{[a, b]}$  of all continuous functions on  $[a, b]$  with

$$d_1(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)|$$

or with

$$d_2(f, g) = \left( \int_a^b [f(t) - g(t)]^2 dt \right)^{\frac{1}{2}}$$

- e) The set  $l_p$  of all infinite sequences  $x = (x_1, x_2, \dots)$  satisfying  $\sum_{i=1}^{\infty} |x_i|^p < \infty$  for  $p \geq 1$  with

$$d(x, y) = \left( \sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

■

**Definition 1.2.7** Let  $X$  be a metric space. All sets and points mentioned are sets and elements of  $X$ .

- a) An **open ball** of radius  $r$  and center  $x$  is

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

The **closed ball** is

$$B[x, r] = \{y \in X : d(x, y) \leq r\}$$

Open ball with center  $x$  are also called **neighborhoods** of  $x$  and  $B(x, y)$  is denoted by  $N_r(x)$ .

- b) A point  $x$  is a **limit point** of a set  $E$  if  $\forall r > 0$   $E \cap N_r(x)$  contains a point  $\neq x$ . If  $x$  is not a limit point it is called an **isolated point**.
- c) A point  $x$  is an **interior point** of  $E$  if there  $\exists r$  such that  $N_r(x) \subseteq E$ .
- d)  $E$  is **open** if every point of  $E$  is an interior point.
- e)  $E$  is **closed** if every point of  $E$  belongs in  $E$ .
- f)  $E$  is **dense** in  $X$  if every point of  $X$  is a limit point of  $E$ , or a point of  $E$ , or both. (e.g. rationals with in real numbers)
- g)  $E$  is **bounded** if for some  $r > 0$ , and  $x \in X$ ,  $E \subseteq N_r(x)$ .

**Theorem 1.2.7** Every neighborhood is an open set.

**Theorem 1.2.8** If  $x$  is limit point of  $E$ , then every neighborhood of  $x$  contains infinitely many points of  $E$ .

- **Example 1.5**  $X = \mathbb{R}$ , then  $(a, b)$  is open,  $[a, b]$  is close,  $(a, b]$  and  $[a, b)$  are neither open nor closed.

■

■ **Example 1.6**  $X = \mathbb{R}^2$  (see sketch in notes.) ■

**Theorem 1.2.9** Suppose  $Y \subset X$  (a metric space) and take  $E \subseteq Y$ , then  $E$  is open relative to  $Y$  if and only if  $E = Y \cap G$  for some open set  $G$  of  $X$ .

**Theorem 1.2.10**  $E$  is open if and only if its complement is closed.

**Corollary 1.2.11**

- a) Both  $X$  and  $\emptyset$  are closed.
- b) The union of finite numbers of closed sets is closed.
- c) Arbitrary intersections of closed sets is closed.

**Theorem 1.2.12** For any metric space  $X$ , we have

- a)  $X$  and  $\emptyset$  are open.
- b) The intersection of a finite number of open sets is open. (Note: must be finite.  $E_n = (-\frac{1}{n}, \frac{1}{n})$ , then  $\bigcap_{n=1}^{\infty} E_n = \{0\}$ )
- c) The union of every collection of open sets is open.

## 1.2.2 Compact Sets

**Definition 1.2.8** A subset  $K$  of a metric space  $X$  is **compact** if every open cover of  $K$  contains a finite subcover. That is for all collections  $G_\alpha, \alpha \in A$  of open sets such that  $\bigcup_A G_\alpha \supset K$  there exists a finite collection  $G_{\alpha_i}, i = 1, 2, \dots, n$  such that  $K \subset \bigcup G_{\alpha_i}$

**R** To visualize an open cover that is not compact, think of  $K = [0, 1]$  and  $G_\alpha = (-1, 1 - \frac{1}{\alpha})$ .  $\bigcup_1^\infty G_\alpha$  will cover  $K$ , but  $\bigcup_1^{999} G_\alpha$  will not.

■ **Example 1.7** a)  $X = \mathbb{R}, E = (0, 1)$

Let  $G_\alpha = (\frac{1}{\alpha}, 1), \alpha = 1, 2, \dots$

Clearly,  $\bigcup_{\alpha=1}^\infty G_\alpha \subset (0, 1)$ , but also,

$K \not\subset \bigcup_{\alpha=1}^\infty G_\alpha$ .

b)  $X = \mathbb{R}, E = [0, \infty)$ , let  $G_\alpha = (-1, \alpha), \alpha \geq 1$ . Then  $E \subset \bigcup_{\alpha=1}^\infty G_\alpha$ , but  $E \not\subset \bigcup_{\alpha=1}^n G_\alpha, \forall n$ .

■

**Theorem 1.2.13** Suppose  $K \subset Y \subset X$ , ( $X$  is a metric space). Then  $K$  is a compact space with respect to  $Y$  if and only if  $K$  is a compact space of  $X$ .

*Proof.* " $\Leftarrow$ " Suppose  $K$  is compact relative to  $X$  and let  $V_\alpha, \alpha \in A$  be open sets relative to  $Y$ , such that  $K \subset \bigcup_{\alpha \in A} V_\alpha$ . By Theorem 1.2.12 (13 in notes),  $V_\alpha = Y \cap G_\alpha$ , some  $G_\alpha$  open relative to  $X$ . (Note:

$k \subset \cup G_\alpha$ .) Thus, there exists a finite subcover,  $k \subset \bigcup_{i=1}^n G_{\alpha_i}$ . But then,

$$\begin{aligned} k \subset Y \cap \left( \bigcup_{i=1}^n G_{\alpha_i} \right) &= \bigcup_{i=1}^n (Y \cap G_{\alpha_i}) \\ &= \bigcup_{i=1}^n V_{\alpha_i} \end{aligned}$$

" $\Rightarrow$ " Suppose  $k$  is compact relative to  $Y$ , and let  $G_\alpha, \alpha \in A$  be open relative to  $X$ , so  $k \subset \bigcup_{\alpha \in A} G_\alpha$ . But then,

$$\begin{aligned} k \subset Y \cap \left( \bigcup_{i=1}^n G_{\alpha_i} \right) &= \bigcup_{i=1}^n (Y \cap G_{\alpha_i}) \\ &= \bigcup_{i=1}^n V_{\alpha_i}, V_{\alpha_i} \text{ open with respect to } Y. \end{aligned}$$

$$\text{Thus, } k \subset \bigcup_{i=1}^n V_{\alpha_i} = Y \cap \left( \bigcup_{i=1}^n G_{\alpha_i} \right)$$

So,  $k \subset \bigcup_{i=1}^n G_{\alpha_i}$  ■

**Theorem 1.2.14** If  $k$  is a compact subset of a metric space,  $X$ , then  $k$  is closed and bounded.

*Proof.* We'll show  $k$  is closed by showing  $k^c$  is open. Let  $p \in k^c$ . For each  $q \in k$  we will consider  $N_{r_q}(q)$  where  $r_q = \frac{1}{2}d(p, q)$ . Since  $k$  is compact, there exists  $(q_1, q_2, \dots, q_n) \in k$  such that  $k \subset \bigcup_{i=1}^n N_{r_{q_i}}(q_i)$ . Let  $G = \bigcap_{i=1}^n N_{r_{q_i}}(p)$ . (Note:  $(\bigcup_{i=1}^n N_{r_{q_i}}(q_i)) \cap G = \emptyset$ .) ■

**Theorem 1.2.15** Closed (with respect to  $X$ ) subsets of compact sets are compact.

*Proof.* Let  $F \subseteq k \subseteq X$ , where  $X$  is a metric space,  $k$  is compact, and  $F$  is closed with respect to  $X$ . Let  $G_\alpha, \alpha \in A$ , be open such that  $F \subset \bigcup_{\alpha \in A} G_\alpha$  ( $F$  is "covered" by  $\cup G_\alpha$ ).  $F$  closed implies  $F^c$  is open. Then the collection  $\{F^c, G_\alpha\}$  covers  $k$ . Let  $k \subset F^c \cup G_\alpha$  which implies  $F \subset \cup G_\alpha$ . ■

**Theorem 1.2.16** If  $E$  is an infinite subset of a compact set  $k$ , then  $E$  has a limit point in  $k$ . ("Countable compactness.")

*Proof.* If no point in  $K$  is a limit point of  $E$ , then each  $q \in K$  will have a neighborhood,  $N(q)$ , which contains at most point point of  $E$  (which is  $q$  if  $q \in E$ ). Thus, no finite subcollection of  $\{N(q), q \in K\}$  for which no finite collection covers  $k$ . Contradiction. ■

■ **Example 1.8** Let  $X$  be the space of rational numbers, with  $d(p, d) = |p - d|$ . Show that  $E = \{p \in X; 2 < p^2 < 3\}$  is closed, bounded, but not compact. ■



**Theorem 1.2.17** If  $K_\alpha \subseteq X$ ,  $X$  a metric space,  $\alpha \in A$ , are compact such that the intersection of every finite collection is empty, then entire  $\bigcap K_\alpha \neq \emptyset$

*Proof.* Let  $K_1$  be a member of  $\{K_\alpha, \alpha \in A\}$  such that no point of  $K$  belongs to all  $K_\alpha$ . Then  $G_\alpha = K_\alpha^c$  are open and cover  $K_1$ . Thus,  $\exists \alpha_1, \alpha_2, \dots, \alpha_n$  such that  $K_1 \subseteq \bigcup_{i=1}^n G_{\alpha_i}$ . This implies that  $K_1 \cap (\bigcup_{i=1}^n G_{\alpha_i})^c$  or  $K_1 \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \emptyset$ . Contradiction. ■

■ **Example 1.9**  $X$  = space of rational numbers,  $d(p, q) = |p - q|$ ,  $E = \{p \in X : \sqrt{2} < p < \sqrt{3}\}$ . Then  $E$  is closed, bounded but not compact.

*Proof.*  $E = X \cap [\sqrt{2}, \sqrt{3}]$ , and since  $X \subseteq \mathbb{R}$  and  $[\sqrt{2}, \sqrt{3}]$  is closed in  $\mathbb{R}$ ,  $E$  is closed (and bounded but not compact). ■

In Euclidean spaces, if a set is closed and bounded, then it is compact. The main step of showing this is ...

**Theorem 1.2.18** Every  $k$ -cell in  $\mathbb{R}^k$  is compact, where  $k$ -cells are of the form:

$$I = \{x \in \mathbb{R}, a_i \leq x_i \leq b_i, i = 1, \dots, k\}$$

To prove, we must first state the following lemma and corollary.

**Lemma 1** If  $I_n$  is a sequence of  $k$ -cells such that  $I_n \subseteq I_{n+1}$ , then  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**Corollary 1.2.19** As stated above, if  $E \subseteq \mathbb{R}^k$  is closed and bounded then it is compact.

**Theorem 1.2.20** If  $K \subset \mathbb{R}^k$  is countably compact, then it is compact.

**Theorem 1.2.21 — Bolzano-Weierstrauss.** Every bounded infinite subset of  $\mathbb{R}^k$  has a limit point in  $\mathbb{R}^k$ .

*Proof.* Begin bounded, it is a subset of a  $k$ -cell,  $I$ . Since  $k$ -cells are compact, each infinite subset of  $I$  has a limit point by Theorem 1.2.16. ■

## 1.3 Sequences and Sets

**R** Reading: definition of convergent sequences in metric space, and result of limit of convergence unique/bounded in Rudin text.

**Definition 1.3.1 — Converge.** A sequence  $\{p_n\}$  in a metric space  $X$  is said to **converge** if there is a point  $p \in X$  with the following property: For every  $\varepsilon > 0$  there is an integer  $N$  such that  $n \geq N$  implies that  $d(p_n, p) < \varepsilon$ . In this case we also say that  $\{p_n\}$  converges to  $p$ , or that  $p$  is the limit of  $\{p_n\}$  and we write  $p_n \rightarrow p$ . If  $\{p_n\}$  does not converge, it is said to **diverge**.

**Theorem 1.3.1** The limit of a convergent sequence is uniquely defined.

**Theorem 1.3.2** A convergent sequence  $\{x_n\}$  is bounded.

**Theorem 1.3.3** Let  $\{x_n\}$  be a sequence in a metric space  $X$ . Then

- a)  $\{x_n\} \rightarrow p$  if and only if every neighborhood of  $p$  contains all but finite many elements of  $\{x_n\}$
- b) If  $p$  is a limit point of a set  $E \subseteq X$ , there  $\exists$  a sequences  $\{x_n\}$  of points in  $E$  such that  $x_n \rightarrow p$ .

*Proof.* a) Let  $V$  be a neighborhood of  $p$ . Then for some  $\varepsilon > 0$ ,  $d(q, p) < \varepsilon$  implies  $q \in V$ . But corresponding to this  $\varepsilon$  there exists  $N$  such that  $n \geq N$  implies  $d(x_n, p) < \varepsilon$  or  $x_n \in V$ .  
 b) For each  $n$ , there exists  $x_n \in E$  such that  $d(x_n, p) < \frac{1}{n}$ . This defines a sequence  $\{x_n\}$  such that  $x_n \rightarrow p$ . ■

**Definition 1.3.2 — Subsequence.** Given a sequence  $\{x_n\}$ ,  $n \geq 1$ , and a sequence  $\{n_k\}$  of positive integers, such that  $n_1 < n_2, \dots$ , the sequence  $\{x_{n_k}\}$ ,  $k \geq 1$ , is called a **subsequence** of  $\{x_n\}$ . If the subsequence converges its limit is called a **subsequential limit** of  $\{x_n\}$ .

**Theorem 1.3.4**  $X_n \rightarrow X$  if and only if  $X_{n_k} \rightarrow X$  for all subsequences.

*Proof.* Left as a potential exercises. ■

**Theorem 1.3.5** Let  $X = \mathbb{R}^k$ . Then

- a)  $\underline{x}_n \rightarrow \underline{x}$  if and only if  $x_{n_i} \rightarrow x_i \forall i = 1, \dots, k$ .
- b) If  $\{\underline{x}_n\}$  is bounded then it contains a convergent subsequences.

*Proof.* a) Left as an exercise.  
 b) Suppose that  $\{\underline{x}_n\}$  is an infinite set. Then the Bolzano-Weierstrauss Theorem implies that there is a limit point of  $\{\underline{x}_n\}$  in  $\mathbb{R}^k$ . By theorem 3.b (in notes), we are done. ■

**Theorem 1.3.6** The subsequential limits of a sequence  $\{p_n\}$  in a vector space  $X$  form a closed set.

*Proof.* Let  $E$  be the set of subsequential limits of  $\{p_n\}$  and let  $q$  be a limit point of  $E$ . We need to show that  $q \in E$ . Let  $q_k \in E, k \geq 1$  be a subsequence converging to  $q$ . We can choose  $q_k$  such that  $0 \leq d(q_k, q) \leq \frac{1}{2k} = \frac{\varepsilon_k}{2}$ . Since  $q_k \in E$  there is a  $p_{n_k} \in \{p_n, n \geq 1\}$  such that  $d(p_{n_k}, q_k) < d(q_k, q) < \frac{\varepsilon_k}{2}$ . Thus,  $p_{n_k} \neq 1$  and  $0 < d(p_{n_k}, q) < d(p_{n_k}, q_k) + d(q_k, q)$  ■

**Definition 1.3.3** a) A sequence  $\{p_n\}$  in a metric space,  $X$ , is said to be a **Cauchy sequence** if for all  $\varepsilon > 0$  there exists  $N$  such that  $n \geq N$  and  $m \geq N$  we have that  $d(p_n, p_m) < \varepsilon$ .  
 b) A sequence  $\{x_n\}, x_n \in \mathbb{R}$  such that  $\forall m > 0$  there exists  $N > 0$  such that  $n \geq N$  implies that  $x_n \geq M$  then  $x_n \rightarrow \infty$ . Similarly, for  $x_n \rightarrow -\infty$ .

**Definition 1.3.4** For a subset of a metric space,  $E \subseteq X$ , the **diameter** of  $E$  is

$$\text{diam}(E) = \sup\{d(p, q), p \in E, q \in E\}$$

- Lemma 2** a) Let  $\bar{E} = E \cup \{\text{limit points of } E\}$  be the **closure** of  $E$ . Then  $\text{diam}(E) = \text{diam}(\bar{E})$ . So, the limit points do not increase the diameter.  
 b) The closure,  $\bar{E}$  is a closed set.

**Theorem 1.3.7** a) For a sequence  $\{p_n\}$  in a metric space  $X$ , set  $E_N = \{p_N, p_{N+1}, \dots\}$ . Then  $\{p_n\}$  is a Cauchy sequence if and only if the  $\text{diam}(E_N) \rightarrow 0$ .  
 b) Every convergent sequence in a metric space is a Cauchy sequence.  
 c) Every Cauchy sequence in  $\mathbb{R}^k$  converges.

*Proof.* a) Left as exercises

- b) If  $p_n \rightarrow p, \forall \varepsilon > 0$  there exists  $N$  such that  $n \geq N$  implies  $d(p_n, p) < \frac{\varepsilon}{2}$ . Hence if  $n \geq N$ , and  $m \geq N$ ,  $d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \varepsilon$ .  
 c) Suppose  $\{x_n\}$  is a Cauchy sequence in  $\mathbb{R}^k$ , let  $E_N = \{x_N, x_{N+1}, \dots\}$  and  $\bar{E}_N$  be its closure. By the lemma (part a)) and part a of this theorem, we have  $\text{diam}(\bar{E}_N) \rightarrow 0$ . This implies that  $\bar{E}_N$  are bounded sets. By lemma (part b)),  $\bar{E}_N$  is closed. Thus  $\bar{E}_N$  are compact sets. Thus, by Theorem 2.17b (by notes numbering),  $\bigcap_{N=1}^{\infty} \bar{E}_N \neq \emptyset$  (i.e. must be single point which is the limit). ■

**Definition 1.3.5 — Complete.** A metric space,  $X$ , for which every Cauchy sequence converges is called **complete**.

■ **Example 1.10**  $X =$  sorted rationals.  
 $X$  is complete. ■

**Definition 1.3.6**  $\{s_n\}, s_n \in \mathbb{R}$ , is said to be  
 a) **increasing** if  $s_n \leq s_{n+1}, \forall n$  ( $\nearrow$ )  
 b) **decreasing** if  $s_n \geq s_{n+1}, \forall n$  ( $\searrow$ )  
 c) **monotonic** if either  $\nearrow$  or  $\searrow$

**Theorem 1.3.8** If  $\{s_n\}$  is monotonic, it converges if and only if it is bounded.

*Proof.* Read in Rudin text. ■

**Lemma 3** If  $E \subseteq \mathbb{R}$  is closed and bounded then,  $\sup\{E\} \in E$ .

**Definition 1.3.7 — Upper/Lower limits.** Let  $\{x_n\}, x_n \in \mathbb{R}$  be a sequence and  $E \subseteq \mathbb{R}, \mathbb{R} \cup \{-\infty, \infty\}$  be the set of all the subsequential limits. Then,

$$\sup\{E\} = \limsup x_n$$

$$\inf\{E\} = \liminf x_n$$

**Theorem 1.3.9** With  $\{x_n\}$  and  $E$  as in Definition 1.3.7, we have

- a)  $\limsup x_n \in E$   
 b) If  $x > \limsup x_n$ , there exists  $N$  such that  $\forall n > N$  implies  $x_n < x$ .  
 Moreover,  $\limsup x_n$  is the only number that satisfies a) and b).

*Proof.* a) If  $\bar{X} \in \mathbb{R}$  then  $E$  is bounded above and hence, by Theorem 1.3.6, is closed. By the previous lemma,  $\bar{X} \in E$ .

- b) Suppose  $x > \bar{s}$  such that  $x_n > x$  for infinitely many  $n$ . These values of  $n$  define a subsequence. This subsequence is either bounded or unbounded. In either case, there is  $y \in E$  such that  $y \geq x > \bar{x}$ , which contradicts the definition of  $\bar{x}$ . To show uniqueness, let  $p$  and  $q$  satisfy a) and b), and suppose  $p < q$ . Let  $x$  be such that  $p < x < q$ . Since  $p$  satisfies b),  $x_n < x \forall n \geq N$ . But then  $q$  cannot satisfy a).

■

**Theorem 1.3.10** a) Set  $a_n = \sup\{x_k, k \geq n\}$ . Then  $\lim a_n = \limsup x_k$ .

b) Set  $b_n = \inf\{x_k, k \geq n\}$ . Then,  $\lim b_n = \liminf x_n$ .

Is it clear that  $\bar{x} \leq a_n \forall n$ ? Then also,  $\bar{x} \leq \lim a_n$ . If you show  $\lim a_n \in E$ , then also show  $\lim a_n \leq \bar{x}$ .

**Theorem 1.3.11 — Frequently Occurring Sequences.** a) If  $p > 0, n^{-p} \rightarrow 0$

b) If  $p > 0, \sqrt[p]{n} \rightarrow 1$

c) If  $p > 0, \sqrt[n]{n} \rightarrow 1$

d) If  $p > 0, \frac{n^a}{(1+p)^n} \rightarrow 0, \forall a \in \mathbb{R}$

e) If  $|x| < 1, x^n \rightarrow 0$

### 1.3.1 Series

Given a sequence  $\{a_n\}$  define  $\{s_n\}$  where  $s_n = \sum_{i=1}^n a_i$ . This sequence  $\{s_n\}$  is called the sequence of partial sums, or **series**. We denote a series also as  $\sum_{n=1}^{\infty} a_n$ . We write  $\sum_{n=1}^{\infty} a_n = s$  if  $s_n \rightarrow s$ . If  $\{s_n\}$  diverges we say  $\sum_{n=1}^{\infty} a_n$  diverges.

**Theorem 1.3.12 — Cauchy Criterion.**  $\sum_{n=1}^{\infty} a_n$  converges if and only if for all  $\varepsilon > 0$  exists  $N$  such that  $|\sum_{i=n}^m a_i| \leq \varepsilon$  holds for all  $m, n \geq N$ .

**Theorem 1.3.13 — Comparison Test.** a) If  $|a_n| \leq c_n$ , and if  $\sum_{n=1}^{\infty} c_n$  converges, so does  $\sum_{n=1}^{\infty} a_n$ .  
b) If  $0 \leq d_n \leq a_n$  and  $\sum_{n=1}^{\infty} d_n$  diverges, so does  $\sum_{n=1}^{\infty} a_n$ .

*Proof.* a)  $|\sum_{i=1}^{\infty} a_i| \leq \sum_{i=1}^{\infty} |a_i| \leq \sum_{i=1}^{\infty} c_i < \varepsilon, \forall n$  such that  $n \geq N$  so  $\sum_{n=1}^{\infty} a_n$  converges.

b) By contradiction, if  $\sum_{n=1}^{\infty} a_n$  converges then by (a) so would  $\sum_{n=1}^{\infty} d_n$ .

■

**Theorem 1.3.14 — Geometric Series.** If  $0 \leq x < 1, \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ , which diverges if  $x \geq 1$ , and converges otherwise.

*Proof.* If  $x \neq 1$ , then  $\frac{1-x^{n+1}}{1-x} = \sum_{i=0}^n x^i = s_n$ . The result follows.

■

**Lemma 4** Suppose  $\{a_n\}$  is such that  $a_n \searrow, a_n \geq 0$ . Then  $\sum_{n=1}^{\infty} a_n$  converges if and only if  $\sum_{k=0}^{\infty} 2^k a_{2^k}$  (i.e. elements of  $\{a_n\}$  indexed by  $2^k$ :  $\{a_1 + 2a_2 + 4a_4 + 8a_8 + \dots\}$ ).

*Proof.* Read in Rudin (listed as Theorem). ■

**Theorem 1.3.15**  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if  $p \geq 1$  and diverges if  $p \leq 1$ .

*Proof.* If  $p \leq 0$ , then  $\frac{1}{n^p} \not\rightarrow 0$ .

Recall, if sequence does not converge to zero, series cannot converge. But sequences converging to zero does not indicate that series must converge.

If  $p \geq 0$ ,  $a_n = n^{-p} = \sum_{k=0}^{\infty} [2^{1-p}]^k$ , which is a geometric (power) series. But  $2^{1-p} < 1$  if and only if  $1 - p < 0$ . ■

**R** The function  $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$ , for  $p > 1$  is called the Reimann Zeta Function.

**Theorem 1.3.16**  $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ . Also,  $\sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log \log n)^2}$  converges, but  $\sum_{n=2}^{\infty} \frac{1}{n(\log n)(\log \log n)}$  diverges.

**Theorem 1.3.17** Let  $f : [1, \infty) \rightarrow (0, \infty)$  be decreasing and  $f(x) \rightarrow 0$ . For  $n \geq 1$  define  $s_n = \sum_{k=1}^n f(k)$ ,  $t_n = \int_1^n f(x)dx$  and set  $d_n = s_n - t_n$ . Then

- $d_n$  is decreasing dequence of nonegative numbers  $[\{d_n\} \searrow, 0 \leq d_{n+1} < d_n < \dots < d_1 = f(1)]$  which implies  $d_n$  converges (decreasing and bounded below). Does not imply that  $s_n, t_n$  converges, but it's a necessary condition.
- $0 \leq d_1 - \lim d_* \leq f(1), \forall x : ***$

*Proof.* Note:

$$\begin{aligned} t_{n+1} &= \sum_{k=1}^n \int_k^{k+1} f(x)dx \\ &\leq \sum_{k=1}^{\infty} \int_k^{k+1} f(k)dx \\ &= \sum_{k=1}^n f(k) \\ &= s_n \end{aligned}$$

a)

$$\begin{aligned}
d_n - d_{n+1} &= (t_{n+1} - t_n) - (s_{n+1} - s_n) \\
&= \int_n^{n+1} f(x) dx = f(n+1) \\
&\geq \int_n^{n+1} f(n+1) dx - f(n+1) \\
&= 0
\end{aligned}$$

Also,  $f(n+1) = s_{n+1} - s_n < s_{n+1} - t_{n+1} = d_{n+1}$  which shows  $d_n \geq 0$ .

b) He might write up later? ■

**Corollary 1.3.18 — Integral Test.**  $\sum_1^\infty f(n)$  converges if and only if  $\{t_n\}$  converges.

■ **Example 1.11** For  $f(x) = \frac{1}{x}$ ,

$$\begin{aligned}
s_n &= \sum_{k=1}^n \frac{1}{k}, \\
t_n &= \int_1^n \frac{1}{x} dx = \log(n) \rightarrow \infty
\end{aligned}$$

Thus,  $\{s_n\}$  diverges because  $\{t_n\} \rightarrow \infty$ . But,  $d_n = \sum_{k=1}^n \frac{1}{k} - \log(n)$  converges. ■

The limit of  $\{d_n\}$  is known as **Eulers Constant**,  $\gamma$ . Theorem 1.3.17 (ii) gives the speed of convergence.

$$0 \leq d_n - \gamma \leq \frac{1}{n}$$

**Lemma 5**  $\sum_0^\infty \frac{1}{n!}$  converges.

**Definition 1.3.8 — e.**  $e = \sum_{n=0}^\infty \frac{1}{n!}$ .

**Theorem 1.3.19**  $e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n$

**Theorem 1.3.20**  $e$  is irrational.

**Theorem 1.3.21 — Abel's Partial Summation Formulas.** For say,  $\{a_n\}$  and  $\{b_n\}$ ,

$$\sum_{k=1}^n a_k b_k = \sum_{k=1}^n A_k (b_k - b_{k+1}) + A_n b_{n+1}$$

Where  $A_n = a_1 + a_2 + \cdots + a_n$ . Thus  $\sum a_n b_n$  converges if both  $\sum_{k=1}^\infty A_k (b_k - b_{k+1})$  and  $A_n b_{n+1}$  converge.

*Proof.* Write  $A_n = 0$ , then,

$$\begin{aligned}\sum_{k=1}^n a_k b_k &= \sum_{k=1}^n (A_k - A_{k-1}) b_k \\ &= \sum_{k=1}^n A_k b_k - \sum_{k=1}^n A_{k-1} b_k + A_n b_{n+1} \\ &= \sum_{k=1}^n A_k (b_k - b_{k+1}) + A_n b_{n+1}\end{aligned}$$

■

**Theorem 1.3.22 — Dirichlet's Test.** Let  $\sum_{k=1}^{\infty} a_k$  have partial sums,  $A_n = a_1, \dots, a_n$  which are bounded and  $\{b_n\}$  be such that  $b_n \searrow 0$ . Then  $\sum_{k=1}^{\infty} a_k b_k$  converges.

*Proof.*

$$\begin{aligned}\left| \sum_{k=m}^{m+n} A_k (b_k - b_{k+1}) \right| &\leq \sum_{k=m}^{m+n} |A_k| (b_k - b_{k+1}) \\ &\leq \sum_{k=m}^{m+n} M (b_k - b_{k+1}) \\ &= M (b_m - b_{m+n+1}) \xrightarrow{n \rightarrow \infty} 0\end{aligned}$$

■

**Definition 1.3.9 — Cesarian Summability.** a) If  $\{s_n\}$  converges, and  $t_n = \frac{s_1 + \dots + s_n}{n}$  then  $\{t_n\}$  also converges. (Note: the opposite does not hold; i.e.  $s_n = (-1)^n$ )

b) Let  $\{s_n\}$  be the sequence of partial sums of  $\sum_{k=1}^{\infty} a_k$  and let  $\{t_n\}$  be as above. If  $\{t_n\}$  converges, we say that  $\sum_{k=1}^{\infty} a_k$  is a **Cesaro Summable**.

■ **Example 1.12** a)  $\sum_{k=1}^{\infty} (-1)^{k+1}$  does not converge, but  $t_n \rightarrow \frac{1}{2}$ .

b)  $\sum_{k=1}^{\infty} (-1)^{k+1} k$  does not converge, but  $\limsup t_n = \frac{1}{2}$  and  $\liminf t_n = 0$ . Thus, we do not have a Cesaro Summable.

■

## 1.4 Continuity

**Definition 1.4.1** Let  $X$  and  $Y$  be geometric spaces and  $f : E \rightarrow Y$  for  $E \subseteq X$ . Let  $p$  be a limit point of  $E$  (not necessarily a member of  $E$ ). We write







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