Probability Theroy based on Measure Theory

STAT 517

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Part One

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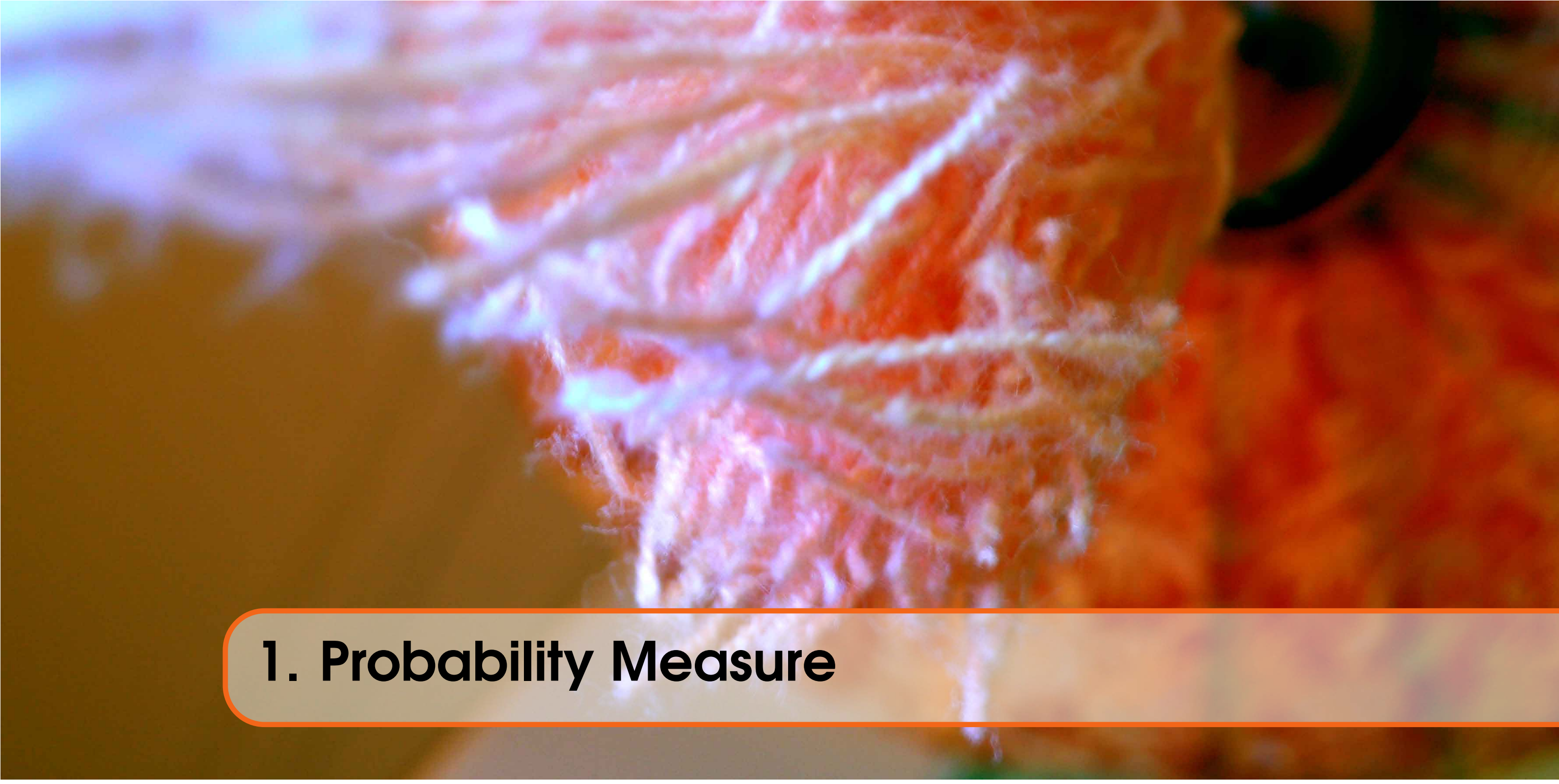

1. Probability Measure

1.1 Overview

1.2 Probability on a Field

Definition 1.2.1 — Ω. Non emtpy set.

Definition 1.2.2 — Paving. A collection of a subset of Ω is a paving.

Definition 1.2.3 — Field. A **field** $\mathscr{F}$ is a paving satisfying

(i) $\Omega \in \mathscr{F}$

(ii) $A \in \mathscr{F} \Rightarrow A^C \in \mathscr{F}$

(iii) $A, B, \in \mathscr{F}, \Rightarrow A \bigcup B \in \mathscr{F}$

Derived Properties about a Field

- $\emptyset \in \mathscr{F}$ (by (i) and (ii):

$$\begin{aligned}\Omega \in \mathscr{F} &\Rightarrow \Omega^C \in \mathscr{F} \\ &\Rightarrow \emptyset \in \mathscr{F})\end{aligned}$$

- (i) can be replaced by "$\mathscr{R}$ is nonempty" because,
  Let $A \in \mathscr{F}$,

$$\begin{aligned}&\Rightarrow A^c \in \mathscr{F} \\ &\Rightarrow A^C \bigcup A \in \mathscr{F} \\ &\Rightarrow \Omega \in \mathscr{F}\end{aligned}$$

- $A \in \mathscr{F}, B \in \mathscr{F}, \Rightarrow, A \bigcap B \in \mathscr{F}$ because,

$$\begin{aligned}&(A \bigcap B)^C = A^C \bigcup B^C (DeMorgan's Law) \\ &A \bigcap B = (A^C \bigcup B^C)^C\end{aligned}$$

- $A_1,\dots,A_m \in \mathscr{F} \Rightarrow A_1 \bigcup,\dots,\bigcup A_m \in \mathscr{F}$ (mathematical induction)
- $A_1,\dots,A_m \in \mathscr{F} \Rightarrow A_1 \bigcap,\dots,\bigcap A_m \in \mathscr{F}$

R A field is a *set of sets*. Suppose we are flipping a coin twice, so the sample space is $\Omega = \{\text{HH, HT, TH, TT}\}$. We observe an event $A \subset \Omega$, say $A = \{\text{TT}\}$. If we consider the set of sets $\mathscr{G} = \{\varnothing, \Omega, A, A^c\}$, then we *cannot* say $\text{TT} \in \mathscr{G}$. Rather, we should write $\{\text{TT}\} \in \mathscr{G}$ or $A \in \mathscr{G}$.

The following are some examples of fields.

- $\{\varnothing, \Omega\}$
- $\{\varnothing, \Omega, A, A^c\}$

Definition 1.2.4 — σ-Field. Similar to the definition of a field except for (iii). A paving satisfying

(i) $\Omega \in \mathscr{F}$
(ii) $A \in \mathscr{F} \Rightarrow A^C \in \mathscr{F}$
(iii) For $A_1 \in \mathscr{F},\dots,\in \mathscr{F}$

$$\bigcup_{k=1}^{\infty} A_k \in \mathscr{F}$$

Derived Facts

- Again, (i) can be repalced by $\mathscr{F}$ no empty, (iii) can be replaced $A_1 \in \mathscr{F},\dots,A_m \in \mathscr{F}$

The above examples of a field are very easy to verify using the three properties of Definition 1.2.3. The next three take a little more thought.

■ **Example 1.1 — Finite-cofinite field.** Define the set $\mathscr{F} = \{A: \text{ either } A \text{ is finite or } A^c \text{ is finite}\}$. The term "finite-cofinite" comes from this definition. Every element is either finite, or its complement is. Suppose our sample space is the natural numbers, $\Omega = \{1,2,3,\dots\}$. Clearly $\varnothing \in \mathscr{F}$ and $\Omega \in \mathscr{F}$, satisfying property 1.

For property 2, suppose $A \in \mathscr{F}$. Again, this means that either A is finite or A^c is. If A is finite, then $A^c \in \mathscr{F}$ because $(A^c)^c = A$ is finite. If A^c is finite, then $A^c \in \mathscr{F}$ by our construction of $\mathscr{F}$. Either way, we see that $A^c \in \mathscr{F}$ and therefore property 2 holds.

For property 3, there are four cases in which $A \in \mathscr{F}$ and $B \in \mathscr{F}$: A and B are finite, A^c and B are finite, A and B^c are finite, or A^c and B^c are finite. Obviously if A and B are finite then $A \cup B$ is also finite and $A \cup B \in \mathscr{F}$. If A^c and B are finite, then $(A\cup B)^c = A^c \cap B^c \subset A^c$, so $(A \cup B)^c$ is finite and $A \cup B \in \mathscr{F}$. This argument shows that $A \cup B \in \mathscr{F}$ in the third and fourth cases as well. Therefore property 3 holds, and we can conclude that $\mathscr{F}$ is a field.

We might also ask whether $\mathscr{F}$ is a σ-field. Take $A_1 = \{2\}, A_2 = \{4\}, \dots, A_n = \{2n\}$. Then $\bigcup_{n=1}^{\infty} A_n = \{2,4,6,8,\dots\}$, the set of all even integers. This is an infinite set, as is its complement, the set of all odd integers. Thus $\bigcup_{n=1}^{\infty} A_n \notin \mathscr{F}$, meaning that $\mathscr{F}$ is a field but *not* a σ-field. ■

■ **Example 1.2 — Countable-cocountable field.** Define the set $\mathscr{F} = \{A: \text{ either } A \text{ is countable or } A^c \text{ is countable}\}$. Suppose the sample space is some infinite set. Again, property 1 is satisfied because $\varnothing \in \mathscr{F}$ and $\Omega \in \mathscr{F}$. The arguments for why properties 2 and 3 are satisfied follow the same reasoning in Example 1.1.

Is $\mathscr{F}$ a σ-field? If Ω is a countable set, then $\mathscr{F} = \mathscr{P}(\Omega)$, the *power set* of Ω (simply the set of all possible subsets of Ω). $\mathscr{F}$ is a σ-field, because any $\bigcup_{n=1}^{\infty} A_n$ is a subset of Ω, and therefore is contained in $\mathscr{F}$. If Ω is uncountable, then $\mathscr{F}$ is still a σ-field. ■

The previous two examples show two sets that appear similar and are both fields, but the one in Example 1.2 is a σ-field whereas the one in Example 1.1 is not.

■ **Example 1.3** Define $\mathscr{I} = \{(a,b] : 0 \leq a \leq b \leq 1\}$. Then if we take $\mathscr{B}_0 = \{\varnothing,$ all finite disjoint unions of sets from $\mathscr{I}\}$, it turns out that $\mathscr{B}_0$ is a field on $(0,1]$. Again, it is easy to verify that properties 1 and 2 hold. Next, take $A = \bigcup_{i=1}^{k} I_i$ and $B = \bigcup_{j=1}^{m} I_j$. Then $A^c = \bigcap_{i=1}^{k} I_i^c$, which is still a finite union of disjoint intervals. So $A^c \in \mathscr{B}_0$, and $\mathscr{B}_0$ is indeed a field.

To determine whether it is a σ-field, take $A_n = (0, 1 - \frac{1}{n}]$. Then $\bigcup_{n=1}^{\infty} A_n = (0,1)$. This open interval cannot be written as a finite disjoint union of intervals, so $\mathscr{B}_0$ is *not* a σ-field. ■

Definition 1.2.5 The field *generated* by $\mathscr{A}$ is the smallest field containing $\mathscr{A}$:

$$f(\mathscr{A}) = \bigcap_{\text{field } \mathscr{G} \supset \mathscr{A}} \mathscr{G}.$$

The σ-field *generated* by $\mathscr{A}$ is the smallest σ-field containing $\mathscr{A}$:

$$\sigma(\mathscr{A}) = \bigcap_{\sigma-\text{field } \mathscr{G} \supset \mathscr{A}} \mathscr{G}.$$

A very useful σ-field is the *Borel σ-field.*

Definition 1.2.6

$$\mathscr{B} = \sigma(\mathscr{I}) = \sigma(\mathscr{B}_0) = \sigma(\mathscr{I}_0) = \sigma(\text{open sets}) = \sigma(\text{open intervals})$$

is the *Borel σ-field* on $(0,1]$, where $\mathscr{I}_0 = \{(a,b] \in \mathscr{I} : a,b \text{ rationals}\}$. Sets in $\mathscr{B}$ are called *Borel sets*.

$\mathscr{B}_0$ is the collection of all finite union of disjoint intervals. Aside: something that we can consider easily, but computers cannot. (e.g. think of all stars, etc)

■ **Example 1.4** $\Omega = (0,1]$ (from now on all intervals are left open, right closed)

R Recall that σ-fields are generated by fields. Fancy scripts denote a σ-field. Fancy scripts with a zero subscript denote a field.

In the following definitions, let $\mathscr{A}$ be a class of subsets of a non-empty set Ω.

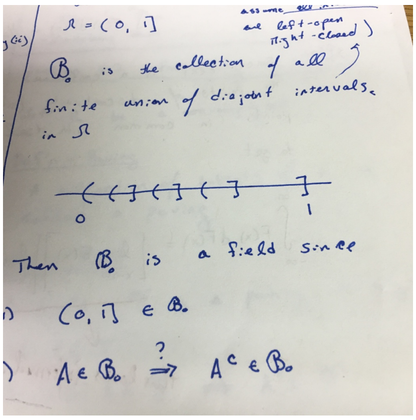

Figure 1.1: Finite unioin of three disjoint intervals.

We can show that $\mathscr{B}_0$ is a field.

(i) (0, 1] $\in \mathscr{B}_0$

(ii) $A \in \mathscr{B}_0 \Rightarrow A^c \in \mathscr{B}_0$

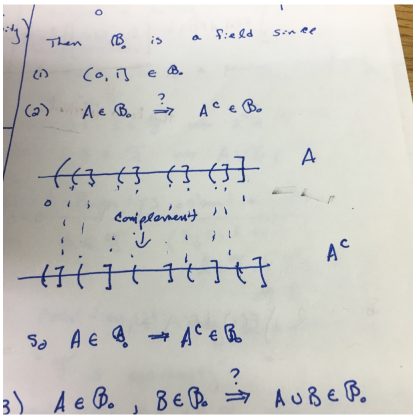

Figure 1.2: A and complement of A.

(iii) $A \in \mathscr{B}_o, B \in \mathscr{B}_o \Rightarrow A \bigcup B \in \mathscr{B}_o$

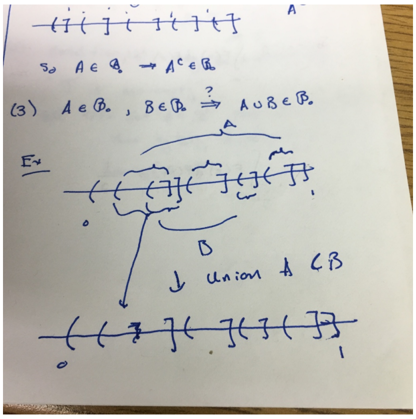

Figure 1.3: Union of A and B is still in $\mathscr{B}_o$

■

Wednesday August 24
$\mathscr{B}_0$ = collection of finite unions of disjoin subintervals of (0, 1]. Is a field.

Definition 1.2.7 — Power Set. A σ-field is generated by a paving of power set. Let Ω be a set. The collection of all subsets of Ω is the power set written as 2^{Ω}.

R Where does this notation come from?
Consider ths case where Ω is finite

$$\Omega = \{\omega_1, \ldots, \omega_n\}$$

Total number of subsets of Ω.
$\emptyset$, 1 element sets, 2-element sets, ..., n-element ests.

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = (1+1)^n$$

$\#(\mathscr{F}) = 2^{\#\Omega}$, so it seems reasonable to denote $\mathscr{F} = 2^{\Omega}$.

It is also easy to show that 2^{Ω} is a σ-field. (The largest, even. The smallest: $\{\emptyset, \Omega\}$ which is also a σ-field.)

$$\{\emptyset, \Omega\} \subseteq \sigma\text{-field} \subseteq 2^{\Omega}$$

It turns out we can extend notion of lenght from $\mathscr{B}_0$ to σ-field generated by $\mathscr{B}_o$.

Now, let $\mathscr{A}$ be a nonempty paving of Ω. We define

$$\sigma(\mathscr{A}) = \bigcap \{\mathscr{B} \subseteq 2^{\Omega} : \mathscr{B} \text{ is a } \sigma\text{-field}, \mathscr{A} \subseteq \mathscr{B}\}$$

OR rather, the *intersection* of all σ-fields that contains $\mathscr{A}$.

Let

$$\mathbb{F}(\mathscr{A}) = \{\mathscr{B} \subseteq 2^{\Omega} : \mathscr{B} \text{ is a } \sigma\text{-field, } \mathscr{B} \supseteq \mathscr{A}\}$$

Then,

$$\sigma(\mathscr{A}) = \bigcap \mathscr{B}$$

$$\mathscr{B} \in \mathbb{F}(\mathscr{A})$$

Derived Facts

$\mathbb{F}(\mathscr{A})$ is nonempty. For example, 2^{Ω} is a σ-field and $2^{\Omega} \supseteq \mathscr{A}$.

$\bigcap B$ is a σ-field. ($B \in \mathbb{F}(\mathscr{A})$)

R Get notes about notation/levels.

Proof. We will prove that indeed $\sigma(\mathscr{A})$ is a σ -field. Recall that we have three conditions above for σ-field.

(i)

$$\Omega \in \sigma(\mathscr{A})$$

$$\Omega \in \bigcap_{B \in \mathbb{F}(\mathscr{A})} B$$

Because: B is σ-field, $A \in B$, $\forall B \in \mathbb{F}(\mathscr{A})$.

$A \in \bigcap B \Rightarrow A \in B \forall B \in \mathbb{F}(\mathscr{A})$

(ii)
$$\Rightarrow A^C \in B, \forall B \in \mathbb{F}(\mathscr{A})$$
$$\Rightarrow A^C \in \bigcap_{B \in \mathbb{F}} (\mathscr{A}) B$$

(iii) $A_1, \ldots, \in \bigcap_{B \in \mathbb{F}(\mathscr{A})} B, \forall B \in \mathbb{F}(\mathscr{A})$

$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in B, \forall B \in \mathbb{F}(\mathscr{A})$

So, $\sigma(\mathscr{A}$ is a σ-field, we call it the σ-field, generated by $\mathscr{B}_o$. We know how tot assign lenth to members of $\mathscr{B}_o$, we now show the assignment can be extended to $\sigma(\mathscr{B}_o)$

■

■ **Example 1.5** Let $\mathscr{I}$ be the collection of *all* subintervals of (0,1].

Note that $\mathscr{I}$ is a smaller collection than $\mathscr{B}_0$ since $\mathscr{B}_0$ can have numerous different combinations of the sets.

Let

$$\mathscr{B} = \sigma(\mathscr{I})$$

.

This is a Borel-σ-field. (a member of $\mathscr{B}$ in Borel set.)

It turns out

$$\sigma(\mathscr{I}) = \sigma(\mathscr{B}_o)$$

This is because $\sigma(\mathscr{I})$ is a σ-field.

So,

$$\sigma(\mathscr{I}) \supseteq \mathscr{B}_o$$
$$\sigma(\mathscr{I}) \supseteq \sigma(\mathscr{B}_o)$$

Also,

$$\mathscr{I} \subseteq \mathscr{B}_o$$
$$\sigma(\mathscr{I}) \subseteq \sigma(\mathscr{B}_o)$$

Thus,

$$\sigma(\mathscr{I}) = \sigma(\mathscr{B}_o)$$

■

Definition 1.2.8 — Probability Measure. Probablity measures on field. Suppose $\mathscr{F}$ is a field on a nonempy set Ω. A probability measure is a function $P : \mathscr{F} \to \mathbb{R}$.

(i) $0 \leq P(A) \leq 1, \forall A \in \mathscr{F}$

(ii) $P(\emptyset) = 0$, $P(\Omega) = 1$

(iii) If $A_1, \ldots$ are disjoint emembers of $\mathscr{F}$ and $\bigcup A_n \in \mathscr{F}$ then we have countable additivity:

$$P(\bigcup A_n) = \sum_{n=1}^{\infty} P(A_N)$$

R Note that (iii) also implies finite additivity. Prove by adding infinite empty sets on end.

If Ω is nonempty set. And $\mathscr{F}$ is a σ-field on Ω. And P is a probability measure on $\mathscr{F}$.

Then $(\Omega, \mathscr{F}, P)$ is called a **probability space.**

And $(\Omega, \mathscr{F})$ is called a **measurable space.**

R If $A \subseteq B$, then $P(A) \leq P(B)$. This is because we may write B as

$$B = A \bigcup (B \setminus A)$$

R

$$P(A)+P(B)=P(A\bigcup B)+P(A\bigcap B)$$

Friday August 26

Recall,
Probability measure on a field, $\mathscr{F}_0$.

- $P(A)+P(B)=P(A\bigcup B)+P(A\bigcap B)$
  - $P(A)=P(AB^C)+P(AB)$
  - $P(B)=P(BA^C)+P(AB)$
  - $P(A)+P(B)=P(AB^C)+P(BA^C)+2P(AB)$
  - $P(A\bigcup B)=P(AB^C)+P(BA^C)+P(AB)$
- $P(A\bigcup B)=P(A)+P(B)-P(AB)$ By induction, we can prove if $A_1,\dots A_n$,

$$P(\bigcup_{k=1}^{n} A_k)=\sum_{k=1}^{n} P(A_k)-\sum_{i<j} P(A_iA_j)+\sum_{i<j<k} A_iA_j)+\cdots+(-1)^{n+1}P(A_1,\dots,A_n)$$

  Inclusion- Exclusion Formula
- If $A_1,\dots A_n \in \mathscr{F}$,

$$B_1=A_1$$
$$B_2=A_2\setminus A_1$$
$$\vdots$$

Then,

$$\bigcup_{k=1}^{n} A_k=\bigcup_{k=1}^{n} B_k$$

but the B_i are disjoint. Also $A_K\subseteq B_k \forall k=1,\dots,n$.

$$P(\bigcup_{k=1}^{n} A_k)=P(\bigcup_{k=1}^{n} B_k)=\sum_{k=1}^{n} B_k\leq \sum_{k=1}^{n} A_k$$

Thus, $P(\bigcup_{k=1}^{n} A_k)\leq \sum_{k=1}^{n} A_k$. Finite subadditivity.

Some conventions,
If $A_1,\dots$ is a sequence of sets, we say $A_n\uparrow A$ if

1. $A_1\subseteq A_2\subseteq \dots$
2. $\bigcup_{k=1}^{\infty} A_k=A$

If $A_1,\dots$ is a sequence of sets, we say $A_n\downarrow A$ if

1. $A_1\supseteq A_2\supseteq \dots$
2. $\bigcap_{k=1}^{\infty} A_k=A$

Theorem 1.2.1 If P is a probability measure on a field $\mathscr{F}$ Then,

1. Continuity from below.
   If $A_n\in\mathscr{F} \quad \forall n, A\in\mathscr{F}$

$$A_n\uparrow A$$

then

$$P(A_n) \uparrow P(A)$$

2. Continuity from above.
   If $A_n \in \mathscr{F} \quad \forall n. A \in \mathscr{F}$

$$A_n \downarrow A$$

   then

$$P(A_n) \downarrow P(A)$$

3. Countable subadditivity.
   If $A_n \in \mathscr{F} \quad \forall n. \bigcup_{k=1}^{\infty} A_k \in \mathscr{F}$ then

$$P(\bigcup_{n=1}^{\infty} A_k) \leq \sum_{n=1}^{\infty} P(A_k)$$

Proof. 1. If $A_1, \ldots A_n \in \mathscr{F}$,

$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1$$
$$B_3 = A_3 \setminus A_2$$
$$\vdots$$

then, $B_1, \ldots$ are disjoint.

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

$$\begin{aligned} P(A) &= P(\bigcup_{n=1}^{\infty} A_n) \\ &= P(\bigcup_{n=1}^{\infty} B_n) \\ &= \sum_{n=1}^{\infty} P(B_n) \\ &= \lim_{n\to\infty} \sum_{n=1}^{\infty} P(B_n) \\ &= \lim_{n\to\infty} P(A_n) \end{aligned}$$

2. $A_n \downarrow A \Leftrightarrow A_n^C \uparrow A^C$
   But by (1),

$$P(A_n^C) \uparrow P(A^C)$$
$$1 - P(A_n) \uparrow 1 - P(A)$$
$$P(A_n) \downarrow P(A)$$

3. By finite subadditivity,

$$P(\bigcup_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} P(A_k) \leq \sum_{n=1}^{\infty} P(A_n)$$

But since, by (1), because

$$\bigcup_{k=1}^{n} A_k \uparrow \bigcup_{n=1}^{\infty} A_n$$

$$P(\bigcup_{k=1}^{n} A_k) \uparrow P(\bigcup_{n=1}^{\infty} A_n)$$

So,

$$P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)$$

■

R $A \in \mathscr{F} =$ "A is F-set".

1.3 Extention of Probability Measure to a σ-field

Let f be a function $f : D \to R$.

Let $\tilde{D}$ be another set such that

$$D \subseteq \tilde{D}$$

An extantion of f onto $\tilde{D}$ is

$$\tilde{f} : \tilde{D} \to R$$

Such that $f(x) = \tilde{f}(x) \forall x \in D$

$\tilde{f}$ is an extention of f on D.

We say f has unique extention, $\tilde{f}$ onto $\tilde{D}$ if

1. $\tilde{f}$ is an extension of f to $\tilde{D}$.
2. if g is another extension of f to $\tilde{D}$ then $\tilde{f} = g$ on D.

Theorem 1.3.1 A probability measure on a field has a unique extension on the σ-field generated by this field.

This means that if $\mathscr{F}_0$ is a field, and P is a probability measure on $\mathscr{F}_0$, then there exists a probability measure, Q on $\sigma(\mathscr{F})$ such that

$$Q(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Moreover, if $\tilde{Q}$ is another probability measure on $\sigma(\mathscr{F}_0)$ such that $\tilde{Q} = P(A) \quad \forall A \in \mathscr{F}$ then

$$\tilde{Q} = Q$$

.

R The proof of this theorem will come after several definitions and lemmas.

Outer Measure $P^*: 2^\Omega \to \mathbb{R}$
For any $A \in 2^\Omega$ $(A \subseteq \Omega)$

$$P^*(A) = \inf\{\sum_{n=1}^{\infty} P(A_n) : A_1, A_2, \dots \text{is a sequence of } \mathscr{F}_0 \text{ sets, } A \subseteq \bigcup_{n=1}^{\infty} A_n\}$$

P^* is a measure out until $\mathscr{M}$, but it is only a function beyond that on 2^Ω.

Inner Measure
$P_*(A) = 1 - P^*(A)$

Define the paving $\mathscr{M}$ as followes

$$\mathscr{M} = \{A \in 2^\Omega : E \in 2^\Omega, P^*(E) = P^*(E \bigcap A) + P^*(E \bigcap A^C)\}$$

Idea: we came up with this $\mathscr{M}$ such that P^* behaves as a measure. It will turn out to be that $\mathscr{M}$ is a σ-field that contains $\sigma(\mathscr{F}_0)$.

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P^* satisfies the following probabilities:
(i) $P^*(\emptyset) = 0$
(ii) $P^*(A) \geq 0 \quad \forall A \in 2^\Omega$
(iii) $A \subseteq B \Rightarrow P^*(A) \subseteq P^*(B)$
(iv) $P^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P^*(A_n))$

Proof. (i) Take $\{\emptyset, \emptyset, \dots\}$.

$$\emptyset \in \mathscr{F}_0, \quad \emptyset \bigcup_{n=1}^{\infty} \emptyset$$

So,

$$P^*(\emptyset) \leq \sum_{n=1}^{\infty} P(\emptyset) = 0$$

Note,

$$P(A) \geq 0 \quad \forall A$$

So,

$$P^*(\emptyset) \geq \emptyset$$

Thus,

$$P^*(\emptyset) = \emptyset$$

(ii) Already done as part of (i).

(iii) Let $A \subseteq B$

$$P^*(A) = \inf\{\sum_{n=1}^{\infty} P(A_n), A_n \in \mathscr{F}_0, A \subseteq \bigcup A_n\}$$

Now, if $B_1, \cdots \in \mathscr{F}_0 \subseteq \bigcup B_n$
Then,

$$A \subseteq B \subseteq \bigcup_n B_n$$

If $\{\{B_n\}_{n=1}^{\infty} : B_n \in \mathscr{F}_0, B \subseteq \bigcup_n B_n\} \subseteq \{\{A_n\}_{n=1}^{\infty} : A_n \in \mathscr{F}_0, A \subseteq \bigcup_n A_n\}$
Or in short, Collection 1 $\subseteq$ Collection 2.

So the inf of a larger set is smaller than (or equal to) the inf of a smaller set.

So,
$P^*(A) = \inf\{\sum_{n=1}^{\infty} P(A_n), A_n \in \text{ collection \#1}\} \leq P^*(B) = \inf\{\sum_{n=1}^{\infty} P(B_n), A_n \in \text{ collection \#2}\} = P^*(B)$

(iv) Want

$$P^*(\bigcup_n A_n) \leq \sum_n P^*(A_n)$$

$P^*(A_n) = \inf\{\sum_{n=1}^{\infty} P(A_n) : A_{nk} \in \mathscr{F}_0, A \subseteq \bigcup_k A_{nk}\}$

Let $\varepsilon > 0$, by defnition of there exists,

$$\{B_n\}_{n=1}^{\infty}$$

such that

$$\sum_{k=1}^{\infty} P(B_{nk}) \leq P^*(A_n) + \frac{\varepsilon}{2^n}$$

So,

$$\bigcup_n A_n \subseteq \bigcup_{n,k} B_{nk}$$

and,

$$\begin{aligned} P^*(\bigcup_n A_n) &\leq \sum_{n,k} P(B_{nk}) \\ &< \sum_n P^*(A_n) + \sum_n (\varepsilon 2^{-n}) \end{aligned}$$

$P^*(\bigcup_n A_n) < \sum_n P^*(A_n) + \varepsilon \quad \forall \varepsilon > 0$

Simply put,
b
So,

$$P^*(\bigcup_n A_n) \leq \sum_n P^*(A_n)$$

■

By definition, $A \in \mathscr{M}$ if and only if $P^*(EA) + P^*(EA^C) = P^*(E)$.

We know that P^* is subadditive.

So, by subadditivity we know,

$$P^*(E) \leq P^*(AE) + P^*(A^C E)$$

Therefore, to show $A \in \mathscr{M}$ we only need to show

$$P^*(E) \geq P^*(AE) + P^*(A^C E)$$

$\mathscr{M}$ is defined by P^* and P^* is defined using $\mathscr{F}_0$ so $\mathscr{M}$ is indireclty tied to $\mathscr{F}_0$.

Lemma 1. $\mathscr{M}$ is a field.

Proof. (i) $\Omega \in \mathscr{M}$

$$\begin{aligned} A &= \Omega \\ P^*(\emptyset) &= 0 \\ P^*(E) + P^*(\emptyset) &= P^*(E) \end{aligned}$$

(ii) $A \in \mathscr{M} = A^C \in \mathscr{M}$

$$\begin{aligned} P^*(E) &= P^*(EA) + P^*(A^C E) \\ &= P^*(EA^C) + P^*(AE) \\ &= P^*(EA^C) + P^*((A^C)^C E) \end{aligned}$$

(iii) $A, B \in \mathscr{M} \to A \bigcap B \in \mathscr{M}$

$$\begin{aligned} &B \in \mathscr{M} \Rightarrow P^*(E) = P^*(Eb) + P^*(B^C E) \quad \forall E \\ &A \in \mathscr{M} \Rightarrow P^*(BE) = P^*((BE)A) + P^*(A^C(BE)) \\ &A \in \mathscr{M} \Rightarrow P^*(B^C E) = P^*((B^C E)A) + P^*(A^C(B^C E)) \end{aligned}$$

Hence,

$$P^*(BE) + P^*(B^C E) = P^*((BE)A) + P^*(A^C(BE)) + P^*((B^C E)A) + P^*(A^C(B^C E))$$

$$\begin{aligned} P^*(A^C(BE)) + P^*((B^C E)A) + P^*(A^C(B^C E)) &\geq P^*((A^C BE) \bigcup (AB^C E) \bigcup (A^C BE)) \\ &= P^*(E \bigcap [A^C B \bigcup AB^C \bigcup A^C B^C]) \\ &= P^*(E \bigcap (AB)^C) \end{aligned}$$

$$\begin{aligned} P^*(E) &= P^*(BE) + P^*(B^C E) \\ &= P^*((BE)A) + (P^*(A^C(BE)) + P^*((B^C E)A) + P^*(A^C(B^C E))) \\ &\geq P^*(ABE) + P^*(E(AB)^C) \end{aligned}$$

So, $A, B \in \mathscr{M}$

■

Lemma 2. If $A_1, A_2, \ldots$ is a sequence of disjoint $\mathscr{M}$-sets then for each $E \subseteq \Omega$,

$$P^*(E \bigcap (\bigcup_k A_k)) = \sum_k P^*(E \bigcap A_k)$$

Proof. First, prove this statement for finite sequence.

$$A_1, \ldots, A_n$$

by mathematical induction.

If $n = 1$ this is 'trivial',

$$P^*(E \bigcap A_1) = P^*(E \bigcap A_1)$$

If $n = 2$ we need to show,

$$P^*(E(A_1 \bigcup A_n)) = P^*(EA_1) + P^*(EA_2)$$

Because $A_1 \in \mathscr{M}$,

$$P^*(E(A_1 \bigcup A_2)) = P^*(E(A_1 \bigcup A_2))A_1 + P^*(E(A_1 \bigcup A_2)A_1^2)$$

$$E(A_1 \bigcup A_2) = E(A_1 A_2 \bigcup A_1 A_2 = EA_1$$

$$E(A_1 \bigcup A_2)A_1^C = E(A_1 A_1^C \bigcup A_2 A_2^C)$$

So,

$$P^*(E(A_1 \bigcup A_2)) = P^*(EA_1) + P^*(EA_2)$$

Suppose true for n = k. (induction hypothesis)

Now we must show for n = k + 1.

$$P^*(E \bigcap (\bigcup_{n=1}^{k+1} A_n)) = P^*([E \bigcap (\bigcup_{n=1}^{k} A_n)] \bigcup A_{k+1})$$

$(\bigcup_{n=1}^{k} A_n), A_{k+1}$ are two disjoint sets. Using the n=2 case,

$$= \sum_{n=1}^{k} P^*(E \bigcap A_n) + P(E \bigcap A_{k+1}) = \sum_{n=1}^{k+1} P^*(E \bigcap A_n)$$

So this is now shown to be true for $\{A_1, \ldots, A_n\}$. Next, showtrue for $A_1, \ldots in \mathscr{M}$ (disjoint). Want:

$$P^*(E \bigcap (\bigcup_{n=1}^{\infty} A_n)) = \sum_{n=1}^{\infty} P^*(E \bigcap A_n)$$

Using countable subadditivity,

$$P^*(E \bigcap (\bigcup_{n=1}^{\infty} A_n)) = P^*(\bigcup_{n=1}^{\infty} E \bigcap A_n) \leq \sum_{n=1}^{\infty} P^*(E \bigcap A_n)$$

In the meantime, by the monotonicity of P^*

$$P^*(E\bigcap(\bigcup_{n=1}^{\infty}A_n)) \geq P^*(E\bigcap(\bigcup_{n=1}^{m}A_n)) = \sum_{n=1}^{\infty}P^*(E\bigcap A_n)$$

So,

$$P^*(E\bigcap(\bigcup_{n=1}^{\infty}A_n)) \geq \lim\sum_{n=1}^{m}P^*(E\bigcap A_n)$$

(*), (**) gives us,

$$P^*(E\bigcap(\bigcup_{n=1}^{\infty}A_n)) = \sum_{n=1}^{\infty}P^*(E\bigcap A_n)$$

■

Wednesday August 31
(finished proof)

Lemma 3.
1. $\mathscr{M}$ is a σ-field
2. P^* restricted on $\mathscr{M}$ is countably additive.

Proof. First we show if

1. $\mathscr{M}$ is a fieldd
2. $\mathscr{M}$ is closed under countable disjoint union.

then $\mathscr{M}$ is a σ-field.
Let's create disjoints sets,

$A_n \in \mathscr{M}, n = 1,2,\ldots$ $B_1 = A_1$ $B_2 = A_2A_1^C \vdots B_n = A_nA_1^C\ldots A_{n-1}^C$
$B_1,\ldots,B_n \in \mathscr{M}$ (disjoint)

$$\bigcup_{n=1}^{\infty}B_n = \bigcup_{n=1}^{\infty}A_n$$

But we know that $\bigcup_{n=1}^{\infty}B_n \in \mathscr{M}$ so $\bigcup_{n=1}^{\infty}A_n \in \mathscr{M}$ and thus $\mathscr{M}$ is a σ-field.
So it suffices to show that $\mathscr{M}$ is closed under disjoint countable unions.

Let $A_1, A_2, \ldots$ are disjoins $\mathscr{M}$-sets.

Let $A = \bigcup_{n=1}^{\infty}A_n$.

Let $F_n = \bigcup^n k = 1A_k$.

Then $F_n \in \mathscr{M}$.

So, $\forall E \in 2^{\Omega}$,

$$P^*(E) = P^*(EF_n) + P^*(EF_n^C)$$

$$
\begin{aligned}
P^*(EF_n) &= P^*(E(\bigcup_{k=1}^{n} A_k)) \\
&= \sum_{k=1}^{n} P^*(EA_k) \\
P^*(EF_n^C) &\geq P^*(EA^C)(F_n \subseteq A, F_n^C \supseteq A^C) \\
\Rightarrow P^*(E) \geq \lim_{n\to\infty} P^*(EA_k) + P^*(EA^C) \\
&= \sum_{k=1}^{n} P^*(EA_k) + P^*(EA^C) \\
&= P^*(EA) + P^*(EA^C)
\end{aligned}
$$

■

So $A \in \mathscr{M}$ and $\mathscr{M}$ is a σ-field.

Now, let's show P^* is countably additive.
Let $A_1, A_2, \ldots$ be disjoint members of $\mathscr{M}$. Then $\forall E \in 2^\Omega$,

$$P^*(E(\bigcup_{n=1}^{\infty} A_n)) = \sum_{n=1}^{\infty} P(EA_n)$$

Take $E = \Omega$.

$$P^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

Lemma 4. $\mathscr{F}_0 \subseteq \mathscr{M}$

Proof. Let $A \in \mathscr{F}$.

Want:

$$A \in \mathscr{M}$$

$$P^*(E) = P^*(EA) + P^*(EA^C)$$

By definition, there exists $E_n \in \mathscr{F}_0$ such that

$$\sum_{n=1}^{\infty} P^*(E_n) \leq P^*(E) + \varepsilon$$

$$
\begin{aligned}
P^*(EA) &\le P^*((\bigcup_{n=1}^{\infty} E_n)A) \text{ (monotonocity)}\\
&= P^*(\bigcup^{i} nfty_{n=1}(E_nA))\\
&\le \sum^{i} nfty_{n=1} P^*((E_nA)) \text{ (countibly subadd)}\\
P^*(EA^C) &\le \sum_{n=1}^{\infty} P^*(E_nA^C)\\
P^*(EA)+P^*(EA^C) &\le \sum_{n=1}^{\infty} P^*(E_nA)+P^*(E_nA^C)\\
&= \sum_{n=1}^{\infty} P^*(E_n)
\end{aligned}
$$

Recall, $A, E_n \in \mathscr{F}_0$

$$
\begin{aligned}
&\le P^*(E)+\varepsilon\\
P^*(EA)+P^*(EA^C) &\le P^*(E)+\varepsilon \quad \forall \varepsilon\\
\Rightarrow P^*(EA)+P^*(EA^C) &= P^*(E)\\
\Rightarrow A &\in \mathscr{M}\\
\mathscr{F}_0 &\in \mathscr{M}
\end{aligned}
$$

■

Lemma 5.

$$P^*(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Proof. Let $A \in \mathscr{F}_0$.
Because, $A, \emptyset, \emptyset, \ldots, \in \mathscr{F}_0$.

$$A \subseteq A \bigcup \emptyset \bigcup \emptyset \ldots$$

$$P^*(A) \le P(A) + P(\emptyset) + \ldots$$

But, if

$$A_n \in \mathscr{F}_0$$

$$A \subseteq \bigcup_{n=1}^{\infty} A_n$$

$$P^*(A) \le \sum_{n=1}^{\infty} P(A_n)$$

$$\Rightarrow P^*(A) \le \inf \sum_{n=1}^{\infty} P(A_n)$$

$$= P^*(A)$$

■

Friday September 2

R **5 Lemma Recap**

Lemma 1. $\mathscr{M}$ is a field.
Lemma 2. If $A_1, A_2, \ldots$ is a sequence of disjoint $\mathscr{M}$-sets then for each $E \subseteq \Omega$,

$$P^*(E \bigcap (\bigcup_k A_k)) = \sum_k P^*(E \bigcap A_k)$$

Lemma 3.
1. $\mathscr{M}$ is a σ-field
2. P^* restricted on $\mathscr{M}$ is countably additive.

Lemma 4.

$$\mathscr{F}_0 \subseteq \mathscr{M}$$

Lemma 5.

$$P^*(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Recall, Extension Theorem. That is, If $\mathscr{F}$ is a field and P is a probability measure, then there exists a measure, Q such that

$$Q(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Proof. By Lemma 5,

$P^*(\Omega) = P(\Omega) = 1$
$P^*(\emptyset) = P(\emptyset) = 0$

Outline of what we need to show:

- $0 \leq M(A) \leq 1$
- $M(\emptyset) = 0, \quad M(\Omega) = 1$
- $M(\bigcup_n A_n) = \sum_n M(A_n)$

Since $\forall A \in \mathscr{M}$,

$$\emptyset \subseteq A \subset \Omega$$

then

$$0 \leq P^*(\emptyset) \leq P^*(A) \leq P^*(\Omega) \leq 1$$

But, by Lemma 3, P^* is contably additive on $\mathscr{M}$. So P^* is probability measure on $\mathscr{M}$ (which is a σ-field, by Lemma 3).

By Lemma 4, $\mathscr{F}_0 \subset \mathscr{M} \Rightarrow \sigma(\mathscr{F}_0 \subseteq \mathscr{M}$. So P^* is also probabliity measure on $\sigma(\mathscr{F}_0)$.

Finally, by Lemma 5, again $P^*(A) = P(A)$, P^* is an extention of P form $\mathscr{F}_0$ to $\sigma(\mathscr{F}_0)$.

■

Uniqueness of of the extention, $\pi - \lambda$ Theorem

Paving - $\{\pi$-system and λ-system.$\}$ (?)

Definition 1.3.1 — π-System. A class of subsets $\mathscr{P}$ of Ω is a π system, if

$$A, B \in \mathscr{P} \Rightarrow AB \in \mathscr{P}$$

Definition 1.3.2 — λ-System. A class $\mathscr{L}$ is a λ-system if
λ(i) $\Omega \in \mathscr{L}$
λ(ii) $A \in \mathscr{L} \Rightarrow A^C \in \mathscr{L}$
λ(iii) If $A_1, \cdots \in \mathscr{L}$ are disjoint then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{L}$

So, the only difference is "disjoint". Weaker than a σ-field (i.e. A σ-field is always a λ-system). Note that (λ_2) can be replace by $(\lambda_{2\prime})$ wherein

$$A, B \in \mathscr{F}, A \subseteq B, \Rightarrow B \setminus A \in \mathscr{L}$$

That is $\lambda_1, \lambda_2, \lambda_3 \Leftrightarrow \lambda_1, \lambda_{2\prime}, \lambda_3$

Lemma 6. A class of sets that is both π-systema and λ-system is a σ-field.

Proof. Suppose $\mathscr{F}$ is both π-system and λ-system.
By definition,
1. $\Omega \in \mathscr{F}$
2. $A \in \mathscr{F} \Rightarrow A^C \in \mathscr{F}$

Let $A_1, A_2, \ldots$ be $\mathscr{F}$ sets.
Let's constructs disjoints sets, B

$$B_1 = A_1$$
$$B_2 = A_1 A_2^C$$
$$\vdots$$

Then B_n are $\mathscr{F}$-sets (by $\lambda_{2\prime} - A_2^C = \Omega A)2^C \in \mathscr{F}$, by π-system, $A_1 A_2^C \in \mathscr{F}$).

By λ_3,

$$\bigcup_n^{\infty} B_n \in \mathscr{F}$$

So,

$$\bigcup_n^{\infty} A_n \in \mathscr{F}$$

■

Theorem 1.3.2 — π-λ Theorem. If $\mathscr{P}$ is in a π-system, $\mathscr{L}$ is in a λ-system, then

$$\mathscr{P} \subseteq \mathscr{L} \Rightarrow \sigma(\mathscr{P} \subseteq \mathscr{L})$$

Proof. Let $\lambda(\mathscr{P})$ be the intersection of all λ-system that contains $\mathscr{P}$.

$$\lambda(\mathscr{P}) = \bigcap\{\mathscr{L}' : \mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}' \text{ is } \lambda\text{-set }\}$$

$\lambda(\mathscr{P})$ is a λ-system.

Goal: prove $\lambda(\mathscr{P})$ is a σ-field. So we want to show that $\lambda(\mathscr{P})$ is a π-system.

1. $\Omega \in \lambda(\mathscr{P})$?

$$\Omega \in \mathscr{L}' \quad \forall \mathscr{L}'$$
$$\Omega \in \lambda(\mathscr{P})$$

2. $A \in \lambda(\mathscr{P}) \Rightarrow A^C \in \lambda(\mathscr{P})$?

$$A \in \lambda(\mathscr{P}) \Rightarrow A \in \bigcap\{\mathscr{L}' : \mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}' \text{ is } \lambda\text{-set }\}$$

Then
$A \in \mathscr{L}'$ for any $\mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}'$ is λ-system.

$$\begin{aligned} &\Rightarrow A^C \in \mathscr{L}' \\ &\Rightarrow A^C \in \bigcap\{\mathscr{L}' : \mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}' \text{ is } \lambda\text{-set }\} \\ &= \lambda(\mathscr{P}) \end{aligned}$$

3. $A_1, A_2, \cdots \in \lambda(\mathscr{P})$ are disjoint then $A_1, A_2, \cdots \in \mathscr{L}' \quad \forall \mathscr{L}'$.
Then $\bigcup A_n \in \mathscr{L}' (\mathscr{L}' \lambda\text{-system})$
So $\bigcup_n A_n \in \lambda(\mathscr{P})$.
We call $\lambda(\mathscr{P})$ the λ-system generated by $\mathscr{P}$.

If we can say that $\lambda(\mathscr{P})$ is also a σ-field, then $\sigma(\mathscr{P}) \subseteq \lambda(\mathscr{P})$ because $\sigma(\mathscr{P})$ is smallest.
So then, $\sigma(\mathscr{P}) \subseteq \mathscr{L}$ because $\lambda(\mathscr{P})$ is the small λ-system.
So it suffices to show that $\lambda(\mathscr{P})$ is a σ-field. But we know if $\lambda(\mathscr{P})$ is a system then $\lambda(\mathscr{P})$ is σ-field. So it suffices to show that $\lambda(\mathscr{P})$ is a π-system.

Construct again for any $A \in 2^\Omega \quad (A \subseteq \Omega)$, let

$$\mathscr{L}_A = \{B : AB \in \lambda(\mathscr{P})\}$$

Claim: If $A \in \lambda(\mathscr{P})$ then $\mathscr{L}_A$ is λ-system.

(a) $\Omega \in \mathscr{L}_A$?

$$A\Omega = A \in \mathscr{L}_A$$

(b) $(\lambda_2') : B_1, B_2 \in \mathscr{L}_A, B_1 \subseteq B_2 \Rightarrow B_2 B_1^C \in \mathscr{L}_A$?

$$B_1 \in \mathscr{L}_A \Rightarrow AB_1 \in \lambda(\mathscr{P})$$
$$B_2 \in \mathscr{L}_A \Rightarrow AB_2 \in \lambda(\mathscr{P})$$

Since $AB_1 \subseteq AB_2$, $\lambda(\mathscr{P})$ is λ-system by (λ_2') for $\lambda(\mathscr{P})$

$$
\begin{aligned}
(B_2A)(B_1A)^C \in \lambda(\mathscr{P}) \\
&= B_2(A)(B_1^C \bigcup A^C) \\
&= (B_2AB_1^C) \bigcup (B_2AA^C) \\
&= (B_2AB_1^C) \\
&= A(B_2B_1^C)
\end{aligned}
$$

So we have that $A(B_2B_1^C) \in \lambda(\mathscr{P})$.

(c) If B_n is disjoint, $\mathscr{L}_A$-sets.
Want $\bigcup_n B_n$ because

$$B_n \in \mathscr{L}_A$$

$$B_nA \in \lambda(\mathscr{P})$$

Because B_n disjoint we know that B_nA is also disjoint.
Hence,

$$\bigcup_n (B_nA) \in \lambda(\mathscr{P})$$

Claim: $\lambda(\mathscr{P})$ is π-sytem.

(a) If $A \in \mathscr{P}$, then $\mathscr{P} \subseteq \mathscr{L}_A$
Suppose $A \in \mathscr{P}$.
Let $B \in \mathscr{P}$, then $AB \in \mathscr{P}$ (π-system), and $AB \in \lambda(\mathscr{P}) \Rightarrow B \in \mathscr{L}_A$

(b) If $A \in \mathscr{P}$ then $\lambda(\mathscr{P}) \subset \mathscr{L}_A$.

(c) If $A \in \lambda(\mathscr{P})$, then $\mathscr{P} \in \mathscr{L}_A$
Suppose, $A \in \lambda(\mathscr{P})$ and let $B \in \mathscr{P}$.
By step 2,
$A \in \mathscr{L}_A$
$\Rightarrow AB \in \lambda(\mathscr{P})$
$\Rightarrow B \in \mathscr{L}_A$

(d) If $A \in \lambda(\mathscr{P})$, then $\lambda(\mathscr{P}) \subseteq \mathscr{L}_A$. This is because $\lambda(\mathscr{P})$ is the smallest λ-system, $\mathscr{L}_A$ is λ-system containing $\mathscr{P}$ (by step 3).

Now show that $\lambda(\mathscr{P})$ is π-system.

$A, B \in \lambda(\mathscr{P})$ because $A \in \lambda(\mathscr{P})$. We have that $\lambda(\mathscr{P}) \in \mathscr{L}_A$.

So

$$B \in \mathscr{L}_A$$

$$BA \in \lambda(\mathscr{P})$$

Thus $\lambda(\mathscr{P})$ is π-system.

■

Wednesday September 7

Theorem 1.3.3 Suppose P_1 and P_2 are probability measures on $\sigma(\mathscr{P})$ where $\mathscr{P}$ is a π-system. If P_1 and P_2 agree on $\mathscr{P}$ (that is, $P_1(A) = P_2(A) \quad \forall A \in \mathscr{P}$) then they agree on $\sigma(\mathscr{P})$.

Proof. Let

$$\mathscr{L} = \{A \in \sigma(\mathscr{P}) : P_1(A) = P_2(A)\}$$

Then $\mathscr{P} \subseteq \mathscr{L}$.

It suffices to show that $\mathscr{L}$ is a λ-system (because if so, then $\sigma(\mathscr{P}) \subseteq \mathscr{L}$ - in fact, $\sigma(\mathscr{P}) = \mathscr{L}$).

Show $\mathscr{L}$ is a λ-system.

1. $\Omega \in \mathscr{L}$?
$$P_1(\Omega) = P_2(\Omega) = 1, \quad \Omega \in \mathscr{P}$$
2. $A \in \mathscr{L}$
$$P_1(A) = P_2(A) \Rightarrow P_1(A^C) = P_2(A^C), \quad A^C \in \mathscr{L}$$
3. $A \in \mathscr{L}$. A_n disjoint. Want $\bigcup_{n=1}^{\infty} A_n \in \mathscr{L}$.
Since
$$A_n \in \mathscr{L}$$
$$P_1(A_n) = P_2(A_n) \quad \forall n$$
$$\sum_{n=1}^{\infty} P_1(A_n) = \sum_{n=1}^{\infty} P_2(A_n)$$
$$P_1 \bigcup_{n=1}^{\infty} (A_n) = P_2 \bigcup_{n=1}^{\infty} (A_n)$$
So, $\bigcup A_n \in \mathscr{L}$.

■

So our extention of (and uniqueness of the extention of) P on $\mathscr{F}_0$ to $\sigma(\mathscr{F}_0)$ is complete. We have shown the existance of Q on $\mathscr{M}$.

Since Q agrees with P on $\mathscr{F}_0$ and $\mathscr{F}_0$ is a field, this implies that this is a π-system.
If you have another extention, say $\tilde{Q}$, then $\tilde{Q} = P$ on $\mathscr{F}_0$. That is, $\tilde{Q} = Q$ on $\mathscr{M}$, where $\mathscr{M}$ is a σ-field, which is a π -system.

So by Theorem 1.3.3,
$\tilde{Q} = Q$ on $\sigma(\mathscr{P})$.

So, Theorem 1.3.1, is completely proved.

Lemma 1 - Lemma 5 imply the extention.
$\pi - \lambda$ Theorem and Theorem 1.3.3 implies uniqueness.
This wraps up Theorem 1.3.1.

Lebesque measure on (0,1]

$$\Omega = (0,1]$$

Recall, $\mathscr{B}_0$ is the finite disjoint unions of intervals in (0,1] and that $\mathscr{B}_0$ is a field.

Let $\mathscr{B} = \sigma(\mathscr{B}_0)$.

For each $A \in \mathscr{B}_0$,

$$A = \bigcup_{i=1}^{n}(a_i, b_i]$$

Let $\lambda(A) = \sum_{i=1}^{n}(b_i - a_i)$.

Question: Is λ a probability measure on $\mathscr{B}_0$?

Theorem 1.3.4 — Theorem 2.2 in Billingsly. The set function λ on $\mathscr{B}_0$ is a probability measure on $\mathscr{B}_0$.

Proof. 1. $0 \le \lambda(A) \le 1$

2.

$$\lambda(\Omega) = \lambda((0,1]) = 1 - 0 = 1$$

$$\lambda(\emptyset) = \lambda((0,0]) = 0$$

3. This one requires Theorem 1.3 in Billingsly (proof omitted - calculus, blah).
Theorem 1.3 - If I is an interval in (0,1] and $\{I_k : k = 1,2,\ldots\}$ are disjoint intervas in (0,1] such that

$$I = \bigcup_{k=1}^{\infty} I_k$$

then,

$$|I| = \sum_{k=1}^{\infty} |I_k|$$

where |a| means length of interval a.
Since $\bigcup_{j=1}^{m_k} I_{kj} \in \mathscr{B}_0$ and $\bigcup_{i=1}^{m} I_i = \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{m_k} I_{kj}$.

Then

$$\lambda(A)\lambda(\bigcup_{i=1}^{m} I_i) = \sum_{i=1}^{m} |I_i|$$

Since, $I_i \subset \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{m_k} I_{kj}$, then

$$I_i = I_i(\bigcup_{k=1}^{\infty}\bigcup_{j=1}^{m_k} I_{kj}) = \bigcup_{k=1}^{\infty}\bigcup_{j=1}^{m_k} I_i I_{kj}$$

By Theorem 1.3,

$$|I| = \sum_{k=1}^{\infty}\sum_{j=1}^{m_k} |I_i I_{jk}|$$

$$\lambda(A) = \sum_{i=1}^{m}\sum_{k=1}^{\infty}\sum_{j=1}^{m_k} |I_i I_{jk}| = \sum_{k=1}^{\infty}\sum_{j=1}^{m_k}\sum_{i=1}^{m} |I_i I_{jk}|$$

Because $I_{jk} \subseteq \bigcup_{i=1}^{m} I_i$, we have that

$$I_{kj} = \bigcup_{i=1}^{m} I_{kj} I_i$$

Again by Theorem 1.3, (note that $I_{kj}I_i$ are disjoint intervals)

$$|I_{kj} = \sum_{i=1}^{m} |I_i I_{jk}|$$

So, $\lambda(A) = \sum_{k=1}^{\infty}\sum_{j=1}^{m_k} |I_{kj}| = \sum_{k=1}^{\infty} \lambda(A_k)$

■

Friday September 9

Finished above proof.

So λ is a probability on $\mathscr{B}_0$. By Theorem 3.1, there exists a unique measure τ on $\sigma(\mathscr{B}_0) = \mathscr{B}$ such that

$$\tau(A) = \lambda(A) \quad \forall A \in \mathscr{B}_0$$

τ is called **Lebesgue Measure** on (0,1]. We may still write it as λ.

1.4 Probabilities Concerning Sequences of Events

Set Limit

Let $(\Omega, \mathscr{F})$ be a measureable space (i.e. Ω is nonempty set and $\mathscr{F}$ is σ-field).

let $A_1, \cdots \in \mathscr{F}$. We define

$$\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \limsup_{n\to\infty} A_n$$

It is trivial to show that $\limsup_{n\to\infty} A_n \in \mathscr{F}$.

$$\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k = \liminf_{n\to\infty} A_n$$

We swapped intersection/union...what we are doing here?
ω (means outcome) $\in \Omega$

$$\omega \in \limsup_{n\to\infty} A_n \Leftrightarrow \omega \in \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$$

$$\Leftrightarrow \omega \in \bigcup_{k=1}^{\infty} A_k \quad \forall n = 1,2,\dots$$

$$\Leftrightarrow \omega \in A_k \quad \text{for some } k \geq n, \quad \forall n = 1,2,\dots$$

$\Leftrightarrow \omega$ is in infinitely many k.
Similarly,

$$\omega \in \liminf_{n\to\infty} A_n \Leftrightarrow \omega \in \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k$$

$$\Leftrightarrow \omega \in \bigcap_{k=1}^{\infty} A_k \quad \text{for some } n = 1,2,\dots$$

$$\Leftrightarrow \omega \in A_k \quad \forall k \geq n, \quad \text{for some } n$$

$$\Leftrightarrow \omega \in \text{ all but finitely many } A_k$$

So this is a much stronger requirement. Intuitively, if ω is in all but finitely many A_k, then it must be in infinitely many A_k (i.e. $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$).

For i > max(n,m),

$$\begin{aligned}
&\bigcap_{k=m}^{\infty} A_k \subseteq A_i \subseteq \bigcup_{k=n}^{\infty} A_k \\
&\Rightarrow \bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty} A_k \subseteq \bigcup_{k=n}^{\infty} A_k \\
&\Rightarrow \bigcup_{m=1}^{\infty}\bigcap_{k=m}^{\infty} A_k \subseteq \bigcap_{n=1}^{\infty}\bigcup_{k=n} A_k \\
&\Rightarrow \liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n
\end{aligned}$$

$$\bigcap_{k=n}^{\infty} A_k \uparrow \liminf_{n\to\infty} A_n$$

$$\bigcup_{k=n}^{\infty} A_k \downarrow \limsup_{n\to\infty} A_n$$

If $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$, then we say that the sequences $\{A_n\}$ has a limit,

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

$$\lim_{n\to\infty} A_n \in \mathscr{F}$$

Sometimes we write,

$$\limsup_{n\to\infty} A_n = [A_n \text{ i.o. }]$$

Theorem 1.4.1 Suppose $(\Omega, \mathscr{F}, P)$ is a probability space and $A_n \in \mathscr{F} \quad n = 1, 2, \ldots$.

(i)

$$\limsup_{n\to\infty} P(A_n) \leq P(\limsup_{n\to\infty} A_n)$$

$$\liminf_{n\to\infty} P(A_n) \geq P(\liminf_{n\to\infty} A_n)$$

(ii) $A_n \to A (A = \lim_{n\to\infty} A_n)$, then we have continuity of probability of a set funciton:

$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n)$$

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Proof. (i) Let $B_n = \bigcap_{k=n}^{\infty} A_k$.

$$B_n \uparrow \liminf_n A_n$$

By Theorem 2.1,

$$P(B_n) \uparrow P(\liminf_n A_n)$$

So,

$$P(B_n) \leq P(\liminf_n A_n) \quad \forall n$$

$$\lim_{n\to\infty} P(B_n) = P(\liminf_n A_n)$$

$$P(A_n) \geq P(B_n) \to P(\liminf_n A_n)$$

$$\liminf_n P(A_n) \geq P(\liminf_n A_n)$$

Similarly,
Let $C_n = \bigcup_{k=n}^{\infty} A_k$.
Then,

$$C_n \downarrow \bigcup_{k=n}^{\infty} A_k$$

$$P(A_n) \leq P(C_n) \to P(\limsup_n A_n)$$

$$\limsup_n P(A_n) \leq P(\limsup_n A_n)$$

(ii) If A_n has a limit (i.e. $\limsup_n A_n = \limsup_n A_n = \lim A$) then,

$$\liminf_n P(A_n) \geq P(\liminf_n A_n) = P(\limsup_n A_n) \geq \limsup_n P(A_n)$$

So, $\liminf_n P(A_n) = \limsup_n P(A_n)$, thus

$$\lim_n P(A_n) = P(\lim_n A_n)$$

■

Independent Events

$(\Omega, \mathscr{F}, P)$

Let $A, B \in \mathscr{F}$. They are independent if and only iff:

$$P(AB) = P(A)P(B)$$

$$A \perp\!\!\!\perp B$$

$A_1, \ldots A_n$ are independent if and only if for any $\{k_1, \ldots, k_j\} \subseteq \{1, \ldots, n\}$,

$$P(A_{k_1} \ldots A_{k_j}) = P(A_{k_1}) \ldots P(A_{k_i})$$

In this case we write: $A_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp A_n$.

Now let, $\mathscr{A}_1, \ldots, \mathscr{A}_n$ be pavings in $\mathscr{F}$ (i.e. $\mathscr{A}_k \subseteq \mathscr{F}$).

We say $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are independent if and only if for any $A_1 \in \mathscr{A}_1, \ldots, A_n \in \mathscr{A}_n$ we have

$$A_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp A_n$$

.

In this case we write: $\mathscr{A}_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp \mathscr{A}_n$.

Theorem 1.4.2 Suppose for $(\Omega, \mathscr{F}, P)$ is a probability space if,

$$\mathscr{A}_1 \subseteq \mathscr{F} \ldots \mathscr{A}_n \subseteq \mathscr{F}$$

are π-systems. Then,

$$\mathscr{A}_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp \mathscr{A}_n \Rightarrow \sigma(\mathscr{A}_1) \perp\!\!\!\perp \ldots \perp\!\!\!\perp \sigma(\mathscr{A}_n)$$

Proof. Let $\mathscr{B}_i = \mathscr{A}_i \bigcup \{\Omega\}$.

It is easy to show (in homework)

1. $\mathscr{B}_i$ is still a π-system
2. $\mathscr{B}_i$ are still independent

$$\mathscr{B}_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp \mathscr{B}_n$$

For $B_2 \in \mathscr{B}_n, \ldots, B_n \in \mathscr{B}_n$ define,

$$\mathscr{L}(B_2, \ldots, B_n) = \{B \in \mathscr{F} : B \perp\!\!\!\perp B_2 \perp\!\!\!\perp \ldots \perp\!\!\!\perp B_N\}$$

1. First we show $\mathscr{L}(B_2, \ldots, B_n)$ is λ-system.

$$\Omega \in \mathscr{L}(B_2, \ldots, B_n)$$

$$\Omega \perp\!\!\!\perp B_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp B_n$$

This is true because $P(\Omega B_2 \ldots B_n) = P(B_2 \ldots B_n) = P(B_2) \ldots P(B_n) = P(\Omega)P(B_2) \ldots P(B_n)$

2. Now $A \in \mathscr{L}(B_2, \dots, B_n) \Rightarrow A^C \in \mathscr{L}(B_2, \dots, B_n)$
   $A\&in\mathscr{L}(B_2, \dots, B_n)$

$$
\begin{aligned}
&\Rightarrow A \perp\!\!\!\perp B_2 \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n \\
&\Rightarrow P(AB_2 \dots B_n) = P(A)P(B_2)\dots P(B_n) \\
&\Rightarrow P(A^C B_2 \dots B_n) \\
&\quad P(B_2 \dots B_n) \setminus AB_2 \dots B_n) \\
&\quad P(B_2 \dots B_n) - P(AB_2 \dots B_n) \\
&\quad P(B_2)\dots P(B_n) - P(A)P(B_2)\dots P(B_n) \\
&\quad (1 - P(A))P(B_2)\dots P(B_n)\ P(A^C)P(B_2)\dots P(B_n)
\end{aligned}
$$

   Then we run this through all subadditives of $A, B_2, \dots, B_n$.
   $A^C \perp\!\!\!\perp B_2 \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n$
3. If $C_1, C_2, \dots, \in \mathscr{L}(B_2, \dots, B_n)$ they are disjoint. Want to show

$$\bigcup^{i} nfty_{m=1} C_m \in \mathscr{L}(B_2, \dots, B_n)$$

   $C_m \in \mathscr{L}(B_2, \dots, B_n)$

$$
\begin{aligned}
&\Rightarrow C_m \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n \\
&\Rightarrow P(C_m B_2 \dots B_n) = P(C_m)\dots P(B_n) \quad \forall m = 1, 2 \dots
\end{aligned}
$$

$$\sum_{m=1}^{\infty} P(C_m B_2 \dots B_n) = (\sum_{m=1}^{\infty} P(C_m))P(B_2)\dots P(B_n)$$

   But $\{C_m, B_2, \dots, B_n, m = 1, 2 \dots\}$

So $\bigcup_m C_m \in \mathscr{L}(B_2, \dots, B_n)$.

And $\mathscr{L}(B_2, \dots, B_n)$.

Also, $B_1 \in \mathscr{L}(B_2, \dots, B_n) \quad \forall B_1 \in \mathscr{B}_1$ therefore by definition,

$$\mathscr{B}_1 \subseteq \mathscr{L}(B_2, \dots, B_n)$$

So, $\sigma(\mathscr{B}_1) \subseteq \mathscr{L}(B_2, \dots, B_n)$ and we have our $\lambda - \pi$-theorem.
This means that for all $B_1 \in \sigma(\mathscr{B}_1)$

$$B_1 \perp\!\!\!\perp B_2 \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n$$

Recall that B_i are arbitrary members of,

$$\sigma(\mathscr{B}_1) \perp\!\!\!\perp B_2 \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n \Leftrightarrow \mathscr{B}_2 \perp\!\!\!\perp \sigma(\mathscr{B}_1) \perp\!\!\!\perp \dots \perp\!\!\!\perp \mathscr{B}_n$$

Run the previous argument repeatedly.
So

$$\sigma(\mathscr{B}_1) \perp\!\!\!\perp \sigma(\mathscr{B}_2) \perp\!\!\!\perp \dots \perp\!\!\!\perp \sigma(\mathscr{B}_n)$$

■

■ **Example 1.6** Let $\mathscr{I}$ be the collection of all intervals, then its π-system. When we want to check $X \perp\!\!\!\perp Y$, we only need to check

$$P(X \in \text{interval}, Y \in \text{interval}) = P(X \in \text{interval})P(Y \in \text{interval})$$

■

Wednesday September 14

Indepencence of Infinite Classes

Let $\{\mathscr{A}_\theta : \theta \in \Theta\}$ where θ is any infinite set (need not be countable) if and only if any (infinite) $\{A_\theta : \theta \in \Theta\}$ where $A_\theta \in \mathscr{A}_\theta$ are independent.

We alraedy define independence of $\{A_\theta : \theta \in \Theta\}$; that is for an infinite collection of sets is independent if and only if any finite subcollection $\{A_{\theta_1}, \dots A_{\theta_n}\}$ is independent.

With this device, we may make claims such as

$$\{X_t : t \in (0,1]\}$$

are independent. Useful for stochastic process, functional data analysis.

It follows trivially, $\{\mathscr{A}_\theta : \theta \in \Theta\}$ are independent if and only if any finite collection, say $\{\mathscr{A}_{\theta_1}, \dots, \mathscr{A}_{\theta_n}\}$ are independent.

Corollary 1.4.3 — To Theorem 4.2. If $(\Omega, \mathscr{F}, P), \mathscr{A}_\theta \subset \mathscr{F}$, $\{\mathscr{A}_\theta : \theta \in \Theta\}$ is independent and each $\mathscr{A}_\theta$ is a π-system, then

$$\{\sigma(\mathscr{A}_\theta) : \theta \in \Theta\}$$

are independent.

Proof.

$$\begin{aligned}\{\mathscr{A}_\theta : \theta \in \Theta\} \perp\!\!\!\perp &\Leftrightarrow \{\mathscr{A}_{\theta_1}, \dots, \mathscr{A}_{\theta_n}\} \perp\!\!\!\perp \\ &\Leftrightarrow \{\sigma(\mathscr{A}_{\theta_1}), \dots, \sigma(\mathscr{A}_{\theta_n})\} \perp\!\!\!\perp\end{aligned}$$

■

Corollary 1.4.4 Suppose we have an array of sets,

$$\begin{matrix} A_{11} & A_{12} & \dots & \dots \\ A_{21} & A_{22} & \dots & \dots \\ \vdots & \vdots & & \\ \vdots & \vdots & & \end{matrix} = \{A_{ij} : i, j = 1, \dots\} \subset \mathscr{F}$$

and this array is independent.

And let $\mathscr{F}_i = \sigma(A_{i1}, A_{i2}, \dots)$.

Then $\mathscr{F}_1 \perp\!\!\!\perp \mathscr{F}_2$

Proof. Let $\mathscr{A}_i$ be the class of all the finite intersections of

$$A_{i1}, A_{i2}, \dots$$

then $\mathscr{A}_i$ is a π-system.

So,

$$\sigma(\mathscr{A}_i) = \mathscr{F}_i$$

because $\{A_{i1}, A_{i2}, \dots\}$ are contained in $\mathscr{A}_i$ which implies $\mathscr{F}_i \subset \sigma(\mathscr{A}_i)$ and also $\mathscr{A}_i \subset \mathscr{F}_i \Rightarrow \sigma(\mathscr{A}_i \leq \mathscr{F}_i$.

By Corollary 1, it suffices to show that $\mathscr{A}_1, \mathscr{A}_2, \dots$ are independent. Further, it suffices to show that any finite subcollection is independent.

Let $\{i_1, \dots, i_n\} \subseteq \{1, 2, \dots\}$. We want to show that $\mathscr{A}_{i_1} \perp\!\!\!\perp \dots \perp\!\!\!\perp \mathscr{A}_{i_n}$. This would be implied by the following

$$\forall C_{i_1} \in \mathscr{A}_{i_1}, \dots, C_{i_n} \in \mathscr{A}_{i_n}$$

$$C_{i_1} \perp\!\!\!\perp \dots \perp\!\!\!\perp C_{i_n}$$

Because, and watch out with this notation here, for

$$C_{i_\alpha} \in \mathscr{A}_{i_\alpha}$$

there exists

$$A_{i_\alpha j_1}, A_{i_\alpha j_2}, \dots, A_{i_\alpha j_{m_\alpha}}$$

such that

$$C_{i_\alpha} = A_{i_\alpha j_1}, A_{i_\alpha j_2}, \dots, A_{i_\alpha j_{m_\alpha}}$$

We have

$$P(\bigcap_{\alpha=1}^{n} C_{i_\alpha}) = P(\bigcap_{\alpha=1}^{n} \bigcup_{\beta=1}^{m_\alpha} A_{i_\alpha j_\beta})$$

because

$$\{A_{i_\alpha j_\beta} : \alpha, = 1, 2, \dots, n, \beta = 1, 2, \dots m_\alpha\} \subseteq \{A_{ij} : i, j = 1, 2, \dots\}$$

$$\begin{aligned} P(\bigcap_{\alpha=1}^{n} \bigcup_{\beta=1}^{m_\alpha} P(A_{i_\alpha j_\beta}) &= \prod_{\alpha=1}^{n} \prod_{\beta=1}^{m_\alpha} P(A_{i_\alpha j_\beta}) \\ &= \prod_{\alpha=1}^{n} P(C_{i_\alpha}) \end{aligned}$$

■

Borel-Cantelli Lemmas (that are actually Theorems)

Theorem 1.4.5 — BC1. For $(\Omega, \mathscr{F}, P)$ probability space,

$$A_n \in \mathscr{F}, \quad n = 1, 2, \dots$$

If $\sum_{n=1}^{\infty} P(A_n) < +\infty$ then

$$P(\limsup_{n\to\infty} A_n) = 0$$

Proof. $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \subset \bigcup_{k=n}^{\infty} P(A_k) \quad \forall n$
So, using monotonicity and then subadditivity, we have

$$P(\limsup_{n\to\infty} A_n) \leq P(\bigcup_{k=n}^{\infty} P(A_k)) \leq \sum_{k=n}^{\infty} P(A_n)$$

$$a \leq \sum_{k=1}^{\infty} P(A_k)\forall n$$

By taking limits we get

$$a \leq lim_n \sum_{k=n}^{\infty} P(A_k) = 0$$

■

Theorem 1.4.6 — BC2. If $\{A_n\}$ are independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then

$$P(\limsup_n A_n) = 1$$

Proof. $P(\limsup_n A_n) = 1$

$$\Leftrightarrow P(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k) = 1$$

$$\Leftrightarrow P(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k^C) = 0 \quad (*)$$

because,

$$\Leftrightarrow P(\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k^C) \leq \sum_{k=n}^{\infty} P(\bigcap_{k=n}^{\infty} A_k^C)$$

$$\Leftarrow P(\bigcap_{k=n}^{\infty} A_k^C) = 0 \quad \forall n = 1, 2, \ldots$$

but we need to prove this to imply (*).
Shit, calculus.

$$1 - x \leq e^{-1} \quad \forall x \in \mathbb{R}$$

For any $j = 1, 2, \ldots,$

$$\begin{aligned} P\left(\bigcap_{k=n}^{n+j} A_k^C\right) &= \prod_{k=n}^{n+j}(1 - P(A_k)) \\ &\leq \prod_{k=n}^{n+j} e^{-P(A_k)} \\ &= e^{-\sum_{k=n}^{n+j} P(A_k)} \end{aligned}$$

Now, $\sum_{k=1}^{\infty} P(A_k) = \infty$ and also

$$\sum_{k=n}^{\infty} P(A_k) \quad \forall n$$

So,

$$\lim \sum_{k=n}^{n+j} P(A_k) \to \infty \quad \forall n$$

$$\lim_{j\to\infty} P\left(\bigcap_{k=n}^{n+j} A_k^C\right) = 0$$

Because,

$$\bigcap_{k=n}^{n+j} A_k^C \downarrow \bigcap_{k=n}^{infty} A_k^C \quad j \to \infty$$

By continuity of probability,

$$P\left(\bigcap_{k=n}^{n+j} A_k^C\right) \downarrow P\left(\bigcap_{k=n}^{infty} A_k^C\right) \quad j \to \infty$$

So,

$$P\left(\bigcap_{k=n}^{infty} A_k^C\right) = 0$$

■

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finished proof.

BC1 and BC2 say that $P\left(\limsup_n A_n\right)$ is either 0 or 1.

This is a special case of a general phonemonon, the 0-1 Law.
Take σ-field, $(\Omega, \mathscr{F}, P)$,

$$A_1, \dots \in \mathscr{F}$$

For each n,

$$\sigma(A_n, A_{n+1}, dots)$$

We have another σ-field called "tail of σ-field",

$$\mathscr{T} = \bigcap_{n=1}^{\infty} \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots)$$

Example 1.7 — 4.18 in Billingsly. $\limsup A_n \in \mathscr{T}$?

$$\bigcap_n^\infty = 1 \bigcup^i nfty_{n=k} A_k$$

$$A_n, A_{n+1}, \dots \in \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots) \Rightarrow \bigcup_{k=n}^\infty A_k \in \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots)$$

$$\bigcap_n^\infty = 1 \bigcup_{n=k}^\infty A_k \in \bigcap_{n=1}^\infty \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots)$$

$$\begin{aligned}
\liminf A_n &= \left[(\bigcup_n^\infty = 1 \bigcap_{n=k}^\infty A_k)^C \right]^C \\
&= \left[\bigcup_n^\infty = 1 \bigcap_{n=k}^\infty A_k^C \right]^C \\
&= \left[\limsup_n A_k^C \right]^C \in \mathscr{T}
\end{aligned}$$

■

Theorem 1.4.7 If $A_1, A_2, \dots$ are independent, then for each $A \in \mathscr{T}$ we have $P(A) = 0$ or 1.

Proof. By Corollary 2,

$\sigma(A_1)$
$\sigma(A_2)$
$\vdots$
$\vdots$
$\sigma(A_{n-1})$
$\sigma(A_n, A_{n+1}, \dots)$

$$\sigma(A_1) \perp\!\!\!\perp \dots \perp\!\!\!\perp \sigma(A_n, A_{n+1}, \dots)$$

Let $A \in \mathscr{T}$, then,

$$A \in \sigma(A_n, A_{n+1}, \dots) \quad \forall n$$

So, $A_1, \dots, A_{n-1}, A$ are independent. By taking nlarge enough, this implies that any finite subcollectoin of $A, A_1, A_2, \dots$ is also independent.

This implies that the sequence $\{A, A_1, A_2, \dots\}$ are independent.

But $A \in \sigma(A_1, A_2, \dots)$ so,

$$A \perp\!\!\!\perp A$$

$$P(AA) = P(A)P(A) = P(A)^2$$

So $P(A)$ must be zero or 1!

■

R We are now skipping a few sections (5 -9) in Billingsly. These are about special random variables, random walks, etc...

1.5 Omitted from Course

1.6 Omitted from Course

1.7 Omitted from Course

1.8 Omitted from Course

1.9 Omitted from Course

1.10 General Measure on a Field

Borel Sets in $\mathbb{R}^k$

Two jumps, from $(0,1] \to \mathbb{R} \to \mathbb{R}^k$.
$\mathscr{B}$ on $(0,1]$ is a σ-field generated by $\mathscr{J}$ = collection of all intervals in (0,1].

$$\sigma(\mathscr{I}) = \mathscr{B} \text{ on } (0,1]$$

When we work with $\mathbb{R}$,

$$\mathscr{I}' = \text{collection of all intervals in } \mathbb{R}, (a,b)$$

$$\sigma(\mathscr{I}') = \mathscr{R}' \quad \text{linear Borel } \sigma\text{-field}$$

$\mathscr{I}^k$ is the collection of all rectangles in $\mathbb{R}^k$.

$$\mathscr{I}^k = \{(a_1,b_1]x\ldots x(a_k,b_k] : a_1,b_1,\ldots,a_k,b_k \in \mathbb{R}\}$$

$$\sigma(\mathscr{I}^k) = \mathscr{R}^k \quad \text{Borel } \sigma\text{-field in } \mathbb{R}^k$$

Properties of $\mathbb{R}^k$

(*) Any open set are in $\mathscr{R}^k$.

Let $\mathbb{Q}$ be the set of all rational numbers. This is countable and dense subset of $\mathbb{R}$.

Definition 1.10.1 — Dense. Look up definition!

Class of rational rectangles:

$$\mathscr{I}^k_{\mathbb{Q}} = \{(a_1,b_1]x\ldots x(a_k,b_k] : a_1,b_1,\ldots,a_k,b_k \in \mathbb{Q}\}$$

Let G be an open set in $\mathbb{R}^k$ and $y \in G$, then there exists

$$A_y \in \mathscr{I}^k_Q$$

such that

$$y \in A_y \subset G$$

because $\mathbb{Q}$ is dense in $\mathbb{R}$.

Note that, $\bigcup_{y\in G} A_y = G$.

But, $\{A_y : y \in G\} \subseteq \mathscr{I}^k_{\mathbb{Q}}$.

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Note that the above union of A_y is countable.
Also, $G = \bigcup_{y\in G} A_y$ and so $G \in \sigma(\mathscr{I}^k_Q) \subseteq sigma(\mathscr{I}_Q) = \mathscr{R}^k$.
Immediately, we see that

(*) All closed sets F are in $\mathscr{R}^k$.

All sets we commonly see are in $\mathscr{R}^k$.

(*) $\mathscr{R}^k$ is in fact also the σ-field gnerated by the class of all open sets in $\mathbb{R}^k$, $\mathscr{G}^k$.

Proof. Let

$$\mathscr{J}^k = \{(a_1,b_1)x\dots x(a_k,b_k) : a_1,b_1,\dots,a_k,b_k \in \mathbb{R}, k = 1,2,\dots\}$$

Claim: $\sigma(\mathscr{J}^k) \in \mathscr{R}^k)$

Note that any

$$(a_1,b_1]x\dots x(a_k,b_k] \in \mathscr{I}^k$$

can be written as

$$\bigcap_{n=1}^{\infty}(a_1,b_1+n^{-1}]x\dots x(a_k,b_k+n^{-1}]$$

The above statement is within $\mathscr{I}^k$ with the intersection, otherwise it'd be in $\mathscr{J}^k$.

That means each $A \in \mathscr{I}^k$ is in $\sigma(\mathscr{J}^k)$.

$$\mathscr{I}^k \subseteq \sigma(\mathscr{J}^k)$$

$$\sigma(\mathscr{I}^k) \subseteq \sigma(\mathscr{J}^k)$$

$$\mathscr{R}^k \subset \sigma(\mathscr{J}^k)$$

But since $\mathscr{R}^k$ contains all open sets (previous (*)), we have that $\mathscr{R}^k \supseteq \mathscr{J}^k$ and $\mathscr{R}^k \supseteq \sigma(\mathscr{J}^k)$. So,

$$\mathscr{R}^k = \sigma(\mathscr{J}^k)$$

Our claim is proved.
Because $\mathscr{R}^k \supseteq \mathscr{G}^k$,

$$\mathscr{R}^k \supseteq \sigma(\mathscr{G}^k)$$

But, $\mathscr{J}^k \subset \mathscr{G}^k$

$$\mathscr{R}^k = \sigma(\mathscr{J}^k) \subset \sigma(\mathscr{G}^k)$$

So,

$$\mathscr{R}^k = \sigma(\mathscr{G}^k)$$

And this is the general definition of Borel σ-field, because open sets exists much more generally than rectangles.

In fact, Borel Sets in Hilbert spaces, Banach...etc. whereever you can define open sets, you can define Borel Sets.

■

Borel Sets in Topological Space

Definition 1.10.2 — Topology. A paving $\mathscr{T}$ is a **topology** on Ω if it is closed under arbitrary union of finite intersections. That is, if you have an arbitrary index,

1. If $\{A_\theta : \theta \in \Theta\} \subseteq \mathscr{T}$ then the

$$\bigcup_{\theta\in\Theta} A_\theta \in \mathscr{T}$$

2. If $A, B \in \mathscr{T}, AB \in \mathscr{T}$ then any set $A \in \mathscr{T}$ is called an open set with respect to $\mathscr{T}$.

$(\Omega, \mathscr{T}) \leftarrow$ Topological Space

$\sigma(\mathscr{T}) \leftarrow$ Borell σ-field generated by $\mathscr{T}$-open sets.

$(\Omega, \sigma(\mathscr{T}))$ is measureable.

Here is another question:

$$\mathscr{B}, \mathscr{R}', \mathscr{R}^\star$$

Is it reasonalbe to conjecture to following?

$$\{A \subseteq \mathscr{R}' : A \subseteq (0,1]\} = \mathscr{B}$$

σ-field Restricted on a Set

Let $(\Omega, \mathscr{F})$ be a measure space.

$$\Omega_0 \subseteq \Omega$$

(otherwise arbitrary, especially Ω_0 need not be in $\mathscr{F}$)

Define (with some "lazy" notation),

$$\mathscr{F}\bigcap\Omega_0 = \{A\Omega_0 : A \in \mathscr{F}\}$$

Theorem 1.10.1 (i) $\mathscr{F}\bigcap\Omega_0$ is a σ-field in Ω_0
(ii) If $\mathscr{A}$ generates $\mathscr{F}$ then, $A\bigcap\Omega_0$ generates $\mathscr{F}\bigcap\Omega_0$

Proof. (i) *$\mathscr{F}\bigcap\Omega_0$ is a σ-field in Ω_0*

Want to show: $\Omega_0 \in \mathscr{F}\bigcap\Omega_0$

$$\Omega \in \mathscr{F}$$
$$\Omega_0 = \Omega\bigcap\Omega_0 \in \mathscr{F}\bigcap\Omega_0$$

Want to show: $A \in \mathscr{F}\bigcap\Omega_0 \to A \in A^C \in \mathscr{F}\bigcap\Omega_0$

$$A \in \mathscr{F}\bigcap\Omega_0 \Rightarrow A = B\Omega_0, B \in \mathscr{F}$$
$$B \in \mathscr{F} \Rightarrow B^c \in \mathscr{F}$$

So,

$$B^C\Omega_0 \in \mathscr{F}\Omega_0$$

But,
$\Omega_0 \setminus A$
$= \Omega_0 \setminus (B\Omega_0)$
$= \Omega_0(B\Omega_0)^C$
$= \Omega_0(B^C\bigcup\Omega_0^C)$
$= (\Omega_0 B^C)\bigcup(\Omega_0\Omega_0^C)$
$= (\Omega_0 B^C)\bigcup\emptyset$
$= (\Omega_0 B^C) \in \mathscr{F}\bigcap\Omega_0$

Want to show: $A_1, \cdots \in \mathscr{F}\bigcap\Omega_0 \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{F}\bigcap\Omega_0$

$$A_n \in \mathscr{F}\bigcap\Omega_0 \Rightarrow A_n = B_n\Omega_0, B_n \in \mathscr{F}$$

But,
$$\bigcup_n B_n \in \mathscr{F} \Rightarrow (\bigcup_n B_n)\Omega_0 \in \mathscr{F}\bigcap\Omega_0$$
$$\Rightarrow \bigcup_n (B_n\Omega_0) \in \mathscr{F}\bigcap\Omega_0$$
$$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{F}\bigcap\Omega_0$$

So the three things we've just shown gives us that $\mathscr{F}\bigcap\Omega_0$ is in deed a σ-field.

(ii) *If $\mathscr{A}$ generates $\mathscr{F}$ then, $A\bigcap\Omega_0$ generates $\mathscr{F}\bigcap\Omega_0$*
Let $\mathscr{A} \subseteq \mathscr{F}, \sigma(\mathscr{A}) = \mathscr{F}$.
Let $\mathscr{F}_0 = \sigma(\mathscr{A}\bigcap\Omega_0)$

Our goal: $\mathscr{F} \bigcap \Omega_0 = \sigma(\mathscr{A})$

Step 1: $\mathscr{F}_0 \subseteq \mathscr{F} \bigcap \Omega_0$

$$\mathscr{A} \bigcap \Omega_0 \subset \mathscr{F} \bigcap \Omega_0$$

$$\mathscr{F} \bigcap \Omega_0 \text{ is } \sigma\text{-field}$$

Step 2: $\mathscr{F} \bigcap \Omega_0 \subset \mathscr{F}_0$ holds if $[A \in \mathscr{F} \Rightarrow A\Omega_0 \in \mathscr{F}]$

$$\Rightarrow \bigcap \Omega_0 \subset \mathscr{F}_0$$

If bracket statement is true, then $\mathscr{F} \bigcap \Omega_0 \subset \mathscr{F}$.
Let $A \in \mathscr{F} \bigcap \Omega_0$ then

$$A = B\Omega_0, B \in \mathscr{F}$$

But by bracket statement,

$$B \in \mathscr{F} \Rightarrow B\Omega_0 \in \mathscr{F} \Rightarrow A \in \mathscr{F}_0$$

So, $\mathscr{F} \bigcap \Omega_0 \subset \mathscr{F}_0$.

Step 3: Let $\mathscr{G} = \{A \subset \Omega : A\Omega_0 \in \mathscr{F}_0\}$. Then,

$$\mathscr{A} \subseteq \mathscr{G}$$

Pick $A \in \mathscr{A}$.

$$\Rightarrow A\Omega_0 \in \mathscr{A} \bigcap \Omega_0 \subset \mathscr{F}_0$$

$$\Rightarrow A \in \mathscr{G}$$

So, $\mathscr{A} \subset \mathscr{G}$.

Step 4: $\mathscr{G}$ is σ-field in Ω.
(a)

$$\Omega \in \mathscr{G} : \Omega\Omega_0 = \Omega_0 \in \mathscr{F}_0 = \sigma(\mathscr{A} \bigcap \Omega_0)$$

Generally, if Ω is a set $\mathscr{B}$ is a pvaing on Ω then $\sigma(\mathscr{B}) = \bigcap\{\mathscr{B}' : \mathscr{B}' \supseteq \mathscr{B}, \mathscr{B}' \text{ is a } \sigma\text{-field on } \Omega\}$.
Which is true because $\mathscr{A}$ generates $\mathscr{F}, \Omega \in \mathscr{A}$.
This means that

$$\sigma(\mathscr{A} \bigcap \Omega_0) = \bigcap\{\mathscr{B} : \mathscr{B} \supset \mathscr{A} \bigcap \Omega_0, \mathscr{B} \text{ is a } \sigma\text{-field on } \Omega\}$$

So, $\Omega_0 \in \mathscr{F}_0$
(b) $A \in \mathscr{G} \Rightarrow A^C \in \mathscr{G}$.

$A \in \mathscr{G} \Rightarrow A\Omega_0 \in \mathscr{F}_0$
$\Rightarrow \Omega_0 \setminus (A\Omega_0) \in \mathscr{F}_0$
$\Rightarrow \Omega_0 \bigcap (A\Omega_0)^C \in \mathscr{F}_0$
$\Rightarrow \Omega_0 \bigcap (A^C \Omega_0^C) \in \mathscr{F}_0$
$\Rightarrow (\Omega_0 A^C) \cup (\Omega_0 \Omega_0^C)$
$= \Omega_0 A^C \in \mathscr{F}_0$
So, $A^C \in \mathscr{F}$.

(c) $A_1, A_2, \cdots \in \mathscr{G}$ are disjoint means $A_n\Omega_0 \in \mathscr{F}_0$ and $A_n\Omega_0$ disjoint.

$$\bigcup_{n=1}^{\infty} (A_n\Omega_0) \in \mathscr{F}$$

$$(\bigcup_{n=1}^{\infty} A_n)\Omega_0 \in \mathscr{F}$$

$$\bigcup_{n=1}^{\infty} A_n \in \mathscr{G}$$

Step 5: $\mathscr{F} \bigcap \Omega_0 \subseteq \mathscr{F}_0$
By Step 3, we know that $\mathscr{A} \subset \mathscr{G}$.

By Step 4, $\mathscr{G}$ is σ-field.

Together, $\sigma(\mathscr{A}) = \mathscr{G}$.

$$\sigma(\mathscr{A}) = \mathscr{F} \in \mathscr{G}$$

$$\Rightarrow [A \in \mathscr{F} \Rightarrow A\Omega_0 \in \mathscr{F}]$$

$$\Rightarrow \bigcap \Omega_0 \subset \mathscr{F}_0$$

■

Wednesday September 21

Worked on proof, but will need to go back to it in the future.

So we have now that $\sigma(\mathscr{A} \bigcap \Omega_0) = \sigma(\mathscr{A} \setminus \bigcap \Omega)$
Lemma 1. If $\Omega_0 \in \mathscr{F}$ then

$$\mathscr{F} \bigcap \Omega_0 = \{B \in \mathscr{F} : B \subset \Omega_0\}$$

Proof. Need to show that $\{A\Omega_0 : A \in \mathscr{F}\} = \{B \in \mathscr{F} : B \subset \Omega_0\}$
Let $C \in LHS$.

$$C = A\Omega_0, A \in \mathscr{F}$$

then $C \in \mathscr{F}, C \subset \Omega_0$.

So $C \in RHS$.

Let $B \in RHS$,

$$B \subset \Omega_0, B \in \mathscr{F}$$

$$B = B \bigcap \Omega_0, B \in \mathscr{F}$$

So, $B \in LHS$. ■

Corollary 1.10.2 $\mathscr{B} = \{A \subset (0,1] : A \in \mathscr{R}'\} = \{A \in \mathscr{R}' : A \in (0,])\}$

Proof. Let $\Omega \in \mathbb{R}$.

$$\Omega_0 = (0,1]$$

$$\mathscr{F} = \mathscr{R}'$$

$$\dashv = \mathscr{I}'$$

By Theorem 10.1,

$$\sigma(\mathscr{A}) \bigcap \Omega_0 = \sigma(\mathscr{A} \bigcap \Omega)$$

$$\Leftrightarrow \sigma((\mathscr{I}') \bigcap (0,1]) = \sigma(\mathscr{I}' \bigcap (0,1]) = \sigma(\mathscr{I}) = \mathscr{B}$$

Now $(0,1] \in \mathscr{R}' \Leftrightarrow \Omega_0 \in \mathscr{F}$.

By Lemma 1,

$$\Rightarrow \mathscr{F} \bigcap \Omega_0 = \{A \in \mathscr{F} : A \subset \Omega_0\}$$

$$\Rightarrow \mathscr{R} \bigcap (0,1] = \{A \in \mathscr{R} : A \subset (0,1]\}$$

So,

$$\mathscr{B} = \{A \in \mathscr{R} : A \subset (0,1]\}$$

■

For general measure, neet infinity convention.

For

$$x, y \in [0,\infty] = [0,\infty) \bigcup \{\infty\}$$

$x \le y$ means that (either/or)
1. $y = \infty$
2. $y < \infty, x < \infty, x \le y$

$x < y$ means that (either/or)
1. $y = \infty, x < \infty$

2. $x, y < \infty, x < y$

For a finite or infinite sequence, $x, x_1, \cdots \in [0, \infty]$,

$x = \sum_{k=1}^{\infty} x_k$ means that (either/or)

1. $x = \infty, x_k = \infty$ for some k
2. $x = \infty, x < \infty \forall k$
   $\sum_{k=1}^{n} x_k : n = 1, 2, \ldots$ diverges.
3. $x < \infty, x_k < \infty$

   $\lim_{n\to\infty} \sum_{k=1}^{n} x_k = x$

Friday September 23

For any infinte sequence, $x_1, x_2, \cdots \in [0, \infty]$ and $x \in [0, \infty]$

$x_k \uparrow x$ is true if and only if

1. $x_k \leq x_{k+1} \leq x \quad \forall k$
2. either
   (a) $x < \infty$, and $x_k \uparrow x$ in usual sense.
   (b) $x_k = \infty$ for $\forall k \geq m, x = \infty$
   (c) $x = \infty, x_k < \infty, x_k \uparrow \infty$

Measures on Field

Let $\Omega \leftarrow$ nonempty and $\mathscr{F}$ be a field on Ω.

Definition 1.10.3 A measure, μ is a function on $\mathscr{F}$ such that

1. $\mu(A) \in [0, \infty]$
2. $\mu(\emptyset) = 0$
3. If $A_n \in \mathscr{F}$ are disjoint then

$$\mu(\bigcup_{n-1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

A measure if finite if $\mu(\Omega) < \infty$.

A measure is a probability measure or simply probability if $\mu(\Omega) = 1$.

A measure is τ-finite if there exists $\mathscr{F}$-sets A_n such that

$$\bigcup_{n=1}^{\infty} A_n = \Omega, \mu(A_n) < \infty$$

If $\mathscr{F}$ is a σ-field then $(\Omega, \mathscr{F})$ is a measureable space.

If μ is a measure on $\mathscr{F}$ then $(\Omega, \mathscr{F}, \mu)$ is a measure space.

Definition 1.10.4 — Support. If $A \in \mathscr{F}, \mu(A^C) = 0$ then A is a support of μ.

A measure, μ on $(\Omega, \mathscr{F})$ has the following properties (proof similar to probability cass omitted).

Finite additivity: $A_1, \ldots, A_n$ are disjoint $\mathscr{F}$-sets implies

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

Monotonicity: If $A, B \in \mathscr{F}, A \subseteq B$ then

$$\mu(A) \leq \mu(B)$$

Inclusion-Exclusion Formula: $A_1, \ldots, A_n \in \mathscr{F}_1$

$$\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k) - \sum_{i=k<l=n} \mu(A_k A_l) + \cdots + (-1)^{n+1} \mu(A_1 \ldots A_k)$$

Countable or Finite Subadditivity: $A_1, \cdots \in \mathscr{F}$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Theorem 1.10.3 Let μ be a measure on a σ-field, $\mathscr{F}$.

1. Continuity from below.

$$A_n, A \in \mathscr{F}, A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$$

2. Continuity from above.

$$A_n, A \in \mathscr{F}, \mu(A_1) < \infty, A_n \downarrow A \Rightarrow \mu(A_n) \rightarrow \mu(A)$$

3. Countable subadditivity.

$$A_n \in \mathscr{F}, \bigcup_n \in \mathscr{F} \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

4. If μ is σ-finite on $\mathscr{F}$ then $\mathscr{F}$ contain an uncountable, disjoint collection of sets with positive μ-measure.

Proof. (i) and (iii) exactly the same as in probability case.

(ii) takes a little extra work.

If $\mu(A_1) < \infty$ then

$$A_1 \setminus A_n \uparrow A_1 \setminus A$$

then by (i) we have

$$\mu(A_1 \setminus A_n) \uparrow \mu(A_1 \setminus A)$$

$$\mu(A_1) - \mu(A_n) \uparrow \mu(A_1) - \mu(A)$$

$$\mu(A_n) \downarrow \mu(A)$$

(iv) Let $\{B_\theta : \theta \in \Theta\}$ be a disjoint collection of $\mathscr{F}$-sets such that $\mu(B_\theta) > 0$. We want to show it is countable.

Claim: If $A \in \mathscr{F}$ and $\mu(A) < \infty, \varepsilon > 0$. then,

$$\{\theta : \mu(AB_\theta) > \varepsilon\} \text{ is finite.}$$

Show: If $\{\theta : \mu(AB_\theta) > \varepsilon\}$ is infinite then there exists $\theta_1, \theta_2, \dots$ such that $\mu(AB_{\theta_i}) > \varepsilon \quad \forall \theta_i$. But, if this were so,

$$\sum_{i=1}^{n} \mu(AB_{\theta_i}) \geq n\varepsilon$$

$$\sum_{i=1}^{n} \mu(AB_{\theta_i}) \geq \mu(A)$$

for sufficiently large n. CONTRADICTION.
But, $\{\theta : \mu(AB_\theta) > 0\} = \bigcup_{r\in\mathbb{Q}} \{\theta : \mu(AB_\theta) > r\}$ is a countable union of a finite set.

So $\{\theta : \mu(AB_\theta) > 0\}$ is countable. Now we just need to show $\{\theta : \mu(B_\theta) > 0\}$ is also countable.

But we know that $\{\theta : \mu(\Omega B_\theta) > 0\}$ is countable because μ is σ-finite so there exists $A_n \in \mathscr{F}, n = 1,2,\dots$ such that $\mu(A_n) < \infty, \bigcup_{n=1}^{\infty} A_n = \Omega$.

Want to show $\{\theta : \mu(\bigcup_{n=1}^{\infty} A_n B_\theta) > 0\}$ is countable.
Note that

$$\mu(\bigcup_{n=1}^{\infty} A_n B_\theta) \leq \mu(\sum_{n=1}^{\infty} A_n B_\theta)$$

So,

$$\{\theta : \mu(\bigcup_{n=1}^{\infty} A_n B_\theta) > 0\} \subseteq \{\theta : \mu(\sum_{n=1}^{\infty} A_n B_\theta) > 0\}$$

The RHS may be rewritten as

$$\bigcup_{n=1}^{\infty} \{\theta : \mu(\bigcup_{n=1}^{\infty} A_n B_\theta) > 0\}$$

■

Monday September 26

Uniqueness of Extension

That is, if the extension exists (we'll explore later), then is it unique?

Theorem 1.10.4 Suppose $\mu_1, \mu_2$are measures on $\sigma(\mathscr{P})$, where $\mathscr{P}$ is a π -system of subsets of Ω. Suppose μ_1, μ_2 are σ-finite on $\mathscr{P}$. If μ_1, μ_2 agree on $\mathscr{P}$ (meanig $\mu_1(A) = \mu_2(A) \forall A \in \mathscr{P}$) then they agree on $\sigma(\mathscr{P}).\mathscr{P}$

σ-Finiate on $\mathscr{P}$

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, $\mathscr{A} \subseteq \mathscr{F}$. We say μ is σ-finite on $\mathscr{A}$ if there exists $\{A)n\}_{n-1}^{\infty}, A_n \in \mathscr{A}$ such that

$$\mu(A_n) < \infty, \Omega = \bigcup_{n=1}^{\infty} A_n$$

Proof. Idea: extend the parallel theorem in probability case.

Step 1: Prove that if $B \in \mathscr{P}$, $\mu_1(B) = \mu_2(B) < \infty$ then

$$\mu_1(BA) = \mu_2(BA) \quad \forall A \in \sigma(\mathscr{P})$$

Let $\mathscr{L}_B = \{A \in \sigma(\mathscr{P}) : \mu_1(BA) = \mu_2(BA) < \infty\}$.

Need to prove $\mathscr{L}_B$ is a λ-system.

1. $\Omega \in \mathscr{L}_B$.

   We know that $\Omega \in \sigma(\mathscr{P})$, because $\sigma(\mathscr{P})$ is itself a σ-field. Thus,

$$\mu_1(B\Omega) = \mu_1(B) = \mu_2(B) = \mu_2(B\Omega)$$

   and so we have that $\Omega \in \mathscr{L}_B$.

2. Want $A \in \mathscr{L}_B \Rightarrow A^C \in \mathscr{L}_B$.

$$\begin{aligned} A \in \mathscr{L}_B &\Rightarrow \mu_1(AB) = \mu_2(AB) \\ &\Rightarrow \mu_1(B) - \mu_1(AB) = \mu_2(B) - \mu_2(AB) \\ &\Rightarrow \mu_1(B \setminus (AB)) = \mu_2(B \setminus (AB)) \\ &\Rightarrow \mu_1(BA^C) = \mu_2(BA^C) \end{aligned}$$

   We have that $A^C \in \mathscr{L}_B$.

3. Want that if A_n are disjoint $\mathscr{L}_B$-sets $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{L}_B$.

$$A_n \in \mathscr{L}_B : \mu_1(A_nB) = \mu_2(A_nB)$$

$$\sum_{n=1}^{\infty} \mu_1(A_nB) = \sum_{n=1}^{\infty} \mu_2(A_nB)$$

   Because the A_n are disjoint, we have that BA_n are also disjoint.

$$\mu_1\left(\bigcup_{n=1}^{\infty}(A_nB)\right) = \mu_2\left(\bigcup_{n=1}^{\infty}(A_nB)\right)$$

$$\mu_1\left((\bigcup_{n=1}^{\infty} A_n)B\right) = \mu_2\left((\bigcup_{n=1}^{\infty} A_n)B\right)$$

   So, $\bigcup_n A_n \in \mathscr{L}_B$. And thus $\mathscr{L}_B$ is a λ-system.

But, $\mathscr{L}_B \supseteq \mathscr{P}$ by definition. So $\mathscr{L}_B \supseteq \sigma(\mathscr{P})$ by the $\pi - \lambda$ Theorem. Se we have

$$\mu_1(AB) = \mu_2(AB) \quad \forall A \in \sigma(\mathscr{P})$$

and step 1 is finished.

Step 2: Want $\mu_1(A) = \mu_2(A) \forall A \in \sigma(\mathscr{P})$. Or rather $\mu_1(A\Omega) = \mu_2(A\Omega) \forall A \in \sigma(\mathscr{P})$, but Ω, unlike B, may not have $\mu(\Omega) < \infty$.

Because μ is σ-finite on $\mathscr{P}$ there exists $B_n \in \mathscr{P}, \mu(B_n) < \infty$ such that $\bigcup_n B_n = \Omega$.

So we need to show that

$$\mu_1\left((\bigcup_{i=1}^{n} B_i)A\right) = \mu_2\left((\bigcup_{i=1}^{n} B_i)A\right)$$

$$\begin{aligned}\mu_1\left((\bigcup_{i=1}^{n} B_i)A\right) &= \mu_1(\bigcup_{i=1}^{n}(B_iA)) \\ &= \sum_{1\le i\le n}\mu_1(B_iA) - \sum_{1\le i<j\le n}\mu_1(B_iB_jA) + \cdots + (-1)^{n+1}\mu_1(B_1\ldots B_nA) \\ &= \mu_2((\bigcup_{i=1}^{n} B_i)A)\end{aligned}$$

by the continuity shown below,

$$mu_1((\bigcup_{i=1}^{n} B_i)A) \uparrow mu_1((\bigcup_{n} B_n)A)$$

$$mu_2((\bigcup_{i=1}^{n} B_i)A) \uparrow mu_2((\bigcup_{n} B_n)A)$$

So, $mu_1(\bigcup_n B_nA) = mu_2(\bigcup_n B_nA)$, and since the union of all of the B_n is Ω we have,

$$\mu_1(A) = \mu_2(A)$$

■

This means, that if we can extend μ from a field, $\mathscr{F}_)$ to a σ-field, $\sigma(\mathscr{F}_0)$ and we know that μ is σ-finite on $\mathscr{F}_0$, then the extension is unique.

Theorem 1.10.5 Suppose μ_1 and μ_2 are finite meaures on $\sigma(\mathscr{P})$ where $\mathscr{P}$ is a π-system and Ω is a countable union of $\mathscr{P}$-sets. then if μ_1, μ_2 agree on $\mathscr{P}$ then they will agree on $\sigma(\mathscr{P})$.

Proof. Be assumption, there exists $B_1, B_2, \cdots \in \mathscr{P}$ such that $\Omega = \bigcup_n B_n$. Because $\mu_1(B_n) \le \mu_1(\Omega) < \infty \forall n$ and $\mu_2(B_n) \le \mu_2(\Omega) < \infty \forall n$ then they are σ-finite on $\mathscr{P}$. Then the theorem follows from the previous theorem. ■

1.11 Extension of General Measure to σ-Field

Outer Measure

Definition 1.11.1 — Outer Measure. Ω nonempty set

An outer measure is a function on 2^Ω such that

1. $\mu^\star(A) \in [0, \infty] \forall A \subseteq \Omega$
2. $\mu^\star(\emptyset) = 0$
3. $A \subseteq B \Rightarrow \mu^\star(A) \le \mu^\star(B)$
4. $\forall \{A_n\} \subseteq 2^\Omega, \mu^\star(\bigcup_n A_n) \le \sum_n \mu^\star(A_n)$

■ **Example 1.8 — 11.1 in Billingsly.** Ω-set.

$\mathscr{A}$ is class of subsets of $\Omega, \emptyset \in \mathscr{A}$

$\rho : \mathscr{A} \to [0,\infty], \rho(\emptyset) = 0$

We say that $\{A_n\}$ is an $\mathscr{A}$ covering of $A \subseteq \Omega$ if $A \subseteq \bigcup_n A_n, A_n \in \mathscr{A}$.

∎

Wednesday September 28

Proof. 1. Want $\mu^\star(A) \in [0,\infty]$

$$\mu^\star(A) = \inf\{\sum_n \rho(A_n), \{A_n\} \text{ is a } \mathscr{A}\text{-covering}\}$$

Here, we know that the probability is between $[0,\infty]$ so the sum must be too, and thus the entire equation must be.

2. Want $\mu^\star(\emptyset) = 0$

Because,

$$\emptyset \in \mathscr{A}$$

So,

$$\emptyset \subseteq \emptyset \cup \emptyset \cup \emptyset \cup \dots$$
$$\mu^\star(\emptyset) \leq \sum_n \rho(\emptyset) = 0$$

3. Want that if $A \subseteq B$ and if $\{A_n\}$ is a $\mathscr{A}$-covering of A then the collection of all $\mathscr{A}$-coverings of B is contained within the collection of all $\mathscr{A}$-coverings of $\mathscr{A}$.

So we can see that the inf{statement 1} $\geq$ inf{statement 2}.

4. Let A_n be sets in 2^Ω, we want that $\mu^\star(\bigcup_n A_n) \leq \sum_n \mu^\star(A_n)$.

For each A_n let $\{B_{nk}\}_{k=1}^\infty$ be an $\mathscr{A}$-covering of A_n such that

$$\sum_{k-1}^{n} \rho(B_{nk}) \leq \mu^\star(A_n) + \frac{\varepsilon}{2^n}$$

Also, we have $\{B_{nk} : n = 1,2\dots, k = 1,2,\dots\}$ this is an $\mathscr{A}$-covering, $\bigcup_n A_n$.

$$\mu^\star(\bigcup_n A_n) \leq \sum_{n=1}^\infty \sum_{k=1}^\infty \rho(B_{nk}) \leq \mu^\star(A_n) + \frac{\varepsilon}{2^n}$$

$$\begin{aligned}\mu^\star(A_n) + \frac{\varepsilon}{2^n} &= \sum_{n=1}^\infty \mu^\star(A_n) + \varepsilon \sum_{n=1}^\infty \frac{1}{2^n} \\ &= \sum_{n=1}^\infty \mu^\star(A_n) + \varepsilon\end{aligned}$$

So $\mu^\star(\bigcup_n A_n) \le \sum_n \mu^\star(A_n) + \varepsilon \Rightarrow \mu^\star(\bigcup_n A_n) \le \sum_n \mu^\star(A_n) \quad \forall \varepsilon > 0$.

■

Now let $\mathscr{M}(\mu^\star)$ to be the paving,

$$\{A : \mu^\star(E) = \mu^\star(EA) + \mu^\star(EA^C), \forall E \in 2^\rceil\Omega\}$$

Theorem 1.11.1 If $\mu^\star$ is an outer measure, then $\mathscr{M}(\mu^\star)$ is a σ-field and $\mu^\star$ restricted on $\mathscr{M}(\mu^\star)$ is a measure.

Proof. The proof is the exact same as Lemma 3 of Section 3, except $P^\star$ is replace by $\mu^\star$. It turns out that the ∞ value of $\mu^\star$ does not cause any change in the proof once we invoke the three infinity conventions.

Thus, proof omitted. ■

Extension of (General) Measure from a Field to a σ-Field

Theorem 1.11.2 A measure on a field has an extension to the generated σ-field.

We will use a different proof from section 3 (even though we could use it). We are going to prove a more general result. But first, we need to introduce the notion of Semiring.

Definition 1.11.2 — Semiring. A class of subsets of pavings on Ω is called a **semiring** if

1. $\emptyset \in \mathscr{A}$
2. $A \in \mathscr{S}, B \in \mathscr{A} \Rightarrow AB \in \mathscr{S}$
3. If $A \subset B$ then,

$$B \setminus A = \bigcup_{k=1}^{n} C_k$$

where $C_1, \ldots, C_k$ are disjoint intervals.

A semiring is also a π-system.

■ **Example 1.9** Let $\mathscr{A} = \{(a,b] : a, b \in \mathscr{R}\}$

1. $\emptyset \in \mathscr{A}$? $(a,a] = \emptyset$
2. See photo.
3. See photo.

■

Proof. Suppose $\mathscr{A}$ is a semiring.
μ is a set function such that $\mu : \mathscr{A} \to [0, \infty)$.

Assume μ is finitely additive and countable subadditive.

Then μ extends to a meassure on $\sigma(\mathscr{A})$ ■

More genreal than the previous theorem,

1. $\mathscr{A}$ needs not be a field.
2. μ need not be a measure.

Proof. 1. $\mu^\star(\emptyset) = 0$.

2. By monotonicty, let $A, B \in \mathscr{A}, A \subseteq B$.
   So, because $\mathscr{A}$ is a semiring,

$$\mu(B \setminus A) = \mu(\bigcup_{k=1}^{n} C_k)$$
$$\mu(B) - \mu(A) = \sum_{k=1}^{n} \mu(C_k) \quad \text{finite additivity}$$
$$\mu(B) = \mu(A) + \sum_{k=1}^{n} \mu(C_k)$$
$$\geq \mu(A)$$

   First, let us show that

$$\mathscr{A} \subseteq \mathscr{M}(\mu^\star)$$

   So we need to show that $A \in \mathscr{A}$.

$$\mu^\star(E) = \mu^\star(EA) + \mu^\star(EA^C) \quad \forall E \in 2^\Omega$$

   But we know $\leq$, so we need to show $\geq$.

   If $\mu^\star(E) = \infty$ this is certainly true by ∞-convention. So let's do $\mu^\star(E) < \infty$.

   Fix a $\varepsilon > 0$. Let A_n be a $\mathscr{A}$-coving of E such that

$$]sum_n \mu(A_n) leq \mu^\star(E) + \varepsilon$$

   Let $A \in \mathscr{A}, A \setminus A_n$, then we have

$$A = AA_n \cup AA_n^C = \bigcup_{k=1}^{m_n} C_n k$$

   Note the C's are disjoint $\mathscr{A}$ sets.
   So, for all $A \in \mathscr{A}$ we may write A as the disjoint union of $\mathscr{A}$ sets:

$$A = B_n \cup \left(\bigcup_{k=1}^{m_n} c_{nk} \right)$$

   Want that $\mu^\star(EA) + \mu^\star(EA^C) \leq \mu^\star(E)$

$$\mu^\star(EA) = \mu^\star \left(E(B_n \cup \bigcup_{k=1}^{m_n} c_{nk}) \right)$$
$$= \mu^\star \left(EB_n \cup (E \bigcup_{k=1}^{m_n} c_{nk}) \right)$$

■

Monday October 3

Corollary 1.11.3 — To Theorem 11.1 & 11.3. Suppose that $\mathscr{A}$ is a semiring and that we have μ such that $\mathscr{A} \to [0, \infty]$ is set function such that:

1. $\mu(\emptyset) = 0$
2. μ is finitely additive on $\mathscr{A}$
3. μ is countably subadditive on $\mathscr{A}$
4. μ is σ-finite on $\mathscr{A}$

Then, μ is a unique extension on $\sigma(\mathscr{A})$.

■ **Example 1.10** Let $\mathscr{A}$ be the collection of all in $\mathbb{R}$, that is,

$$\mathscr{A} = \{(a,b] : a,b, \in \mathbb{R}\}$$

Let $\lambda_1 : \mathscr{A} \to \mathbb{R}, (a,b] = b - a$.

By Theorem 1.3, λ_1 is finitely additive, and indeed also countably subadditive. So it can be extended to $\sigma(\mathscr{A}) = \mathscr{R}^1$. But also we have that λ_1 is σ-finite on $\mathscr{A}$.

$$\Omega = \mathbb{R} = \bigcup_{n=1}^{\infty}(-n,n], \lambda_1(-n,n] = 2n$$

Therefore the extension is unique. This is defined to be the Lebesgue measure on $\mathscr{R}^1$ ■

Approximation Theorem

Approximate $\mu(A), A \in \sigma(\mathscr{A})$ by $\mu(B), B \in \mathscr{A}$.

Lemma 1. If $\mathscr{A}$ is a semiring, and $A, A_1, \ldots, A_n \in \mathscr{A}$, then we may write,

$$AA_1^C \ldots A_n$$

as finite disjoint unions of $\mathscr{A}$-sets. That is there exists

$$C_1, \ldots, C_m \in \mathscr{A}$$

that are disjoints such that

$$AA_1^C \ldots A_n^C = C_1 \cup \cdots \cup C_m$$

Think of this as a generalization of (iii) of Semiring.

Proof. Proof by Induction.

n = 1 Case

Want: $AA_1^C \ldots A_n^C = C_1 \cup \cdots \cup C_m$

$$AA_1^C = A \setminus (AA_1)$$
$$(AA_1) \subseteq A$$

By (iii) of Semiring, the above is equal to $C_1 \cup \cdots \cup C_m$.

Assume statement true for n.

n + 1 Case

Induction hypothesis gives us that

$$(AA_1^C \ldots A_n^C)A_{n+1}^C = (C_1 \cup \cdots \cup C_m)A_{n+1}^C$$

But when we use the n=1 case we have that the following inner term are finitely disjoint unions of $\mathscr{A}$-sets,

$$\bigcup_{k=1}^{\infty} C_k A_{n+1}^C$$

So we have that all together (unioned) is a finite disjoints union of $\mathscr{A}$-sets.

■

Symmetric Difference of Sets A, B

Notation 1.1.

$$A \triangle B = AB^C \cup BA^C$$

Theorem 1.11.4 Suppose $\mathscr{A}$ is a semiring, μ is a measure on $\sigma(\mathscr{A}) = \mathscr{F}$ and μ is σ-finite on $\mathscr{A}$.

1. For $B \in \mathscr{F}$, $\varepsilon > 0$, there exists a disjoins $mathcalA$-sequence $A_1, A_2, \dots$ such that

$$B \subseteq \bigcup_k A_k, \mu(\bigcup_k A_k \setminus B) < \varepsilon$$

2. If $B \in \mathscr{F}, \mu(B) < \infty$ then for any $\varepsilon > 0$, there exists a finite disjoint $\mathscr{A}$-sequence, $A_1, \dots, A_n$ such that

$$\mu(B \triangle \bigcup_{k=1}^{n} A_k) < \varepsilon$$

Proof. 1. Let $\mu^\star$ be the outer measure,

$$\mu^\star(A) = \inf\{\sum_n \mu(A_n) : \{A_n\} is a \mathscr{A} - covering\}$$

then $\mathscr{M}(\mu^\star)$ is a σ-field, $\mathscr{F} \subseteq \mathscr{M}(\mu^\star)$, $\mu^\star$ is a measure, $\mathscr{M}(\mu^\star)$, $\mu = \mu^\star$ on $\sigma(\mathscr{A}) = \mathscr{F}$.

$$\mu(A) = \inf\{\sum_n \mu(A_n) : \{A_n\} is a \mathscr{A} - covering\}$$

Let $B \in \mathscr{F}, \mu(B) < \infty$. Then there exists an $\mathscr{A}$-covering $\{A_k\}$ of B such that

$$\sum_n \mu(A_k) \le \mu(B) + \varepsilon$$

So $\mu(\cup_k A_k) \le \sum_k \mu(A_k) \le \mu(B) + \varepsilon$.

And $\mu(\cup_k A_k) - \mu(B) \le \varepsilon$.

$$\mu(\cup_k A_k \setminus B) \le \varepsilon$$

Let

$$
\begin{aligned}
B_1 &= A_1 \\
B_2 &= A_2 A_1^C \\
&\vdots \\
B_k = A_k A_1^C \ldots A_{k-1}^C \\
&\vdots
\end{aligned}
$$

So by Lemma 1, the B_k are finite dijoint union of $\mathscr{A}$-sets. Also,

$$\bigcup_k A_k = \bigcup_k B_k$$

so we have that there exists $C_1, C_2, \ldots$ that are also disjoint $\mathscr{A}$-sets such that

$$\mu(\cup_k C_k \setminus B) \leq \varepsilon$$

Now suppose that $B \in \mathscr{F}, \mu(B) = \infty$. Because μ is σ-finite on $\mathscr{A}$ there exists $C_1, \cdots \in \mathscr{A}$ such that $\mu(C)m \leq \infty$.

$$\Omega = \bigcup_m C_M$$

So then

$$
\begin{aligned}
B &= B\Omega \\
&= B(\cup_m C_m) \\
&= \cup_m BC_m
\end{aligned}
$$

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But $\mu(BC_m) \leq \mu(C_m) < \infty$ so by the finite case there exists a disjoint $\mathscr{A}$-sequence $\{A_{mk} : k = 1, 2, \ldots\}$ such that

$$\bigcup_k A_{mk} \supseteq BC_m$$

$$\mu(\bigcup_k A_{mk} \setminus (BC_m) < \frac{\varepsilon}{2^k}$$

So now,

$$B = \cup_m BC_m \subseteq \bigcup_m \bigcup_k A_{mk}$$

$$
\begin{aligned}
\mu(\bigcup_m \bigcup_k A_{mk} \setminus B) &= \mu((\bigcup_m \bigcup_k A_{mk})(\cup_m BC_m)) \\
&= \mu((\bigcup_m \bigcup_k A_{mk})(\bigcap_m (BC_m)^C)) \\
&= \mu(\bigcup_m (\bigcup_k A_{mk})(BC_m)^C) \\
&\leq \sum_m \mu(\bigcup_k A_{mk} \setminus (BC_m)) \\
&\leq \varepsilon
\end{aligned}
$$

Since $\bigcup_m (\bigcup_k A_{mk}$ is a countable union of $\mathscr{A}$-sets, we can write as

$$\bigcup_k D_k$$

and make them disjoint as before,

$$E_1 = D_1$$

$$E_2 = D_2 D_1^C$$

$$\vdots$$

By Lemma 1, each E_m is finite disjoint union of $\mathscr{A}$-sets.

$$\bigcup_m E_m = \bigcup_m F_m$$

where F_m are disjoin $\mathscr{A}$-sets.
Hense, part (i) is proved.

2. Recall for any $B \in \mathscr{F}, \mu(B) < \infty$ and $\varepsilon > 0$, there exists a finite $\mathscr{A}$-sequence, $A_1, \ldots, A_n$, such that

$$\mu(B \triangle \bigcup_{k=1}^{n} A_k) < \varepsilon$$

$$
\begin{aligned}
\mu\left[(\bigcup_{k=1}^{n} A_k) \triangle B\right] &= \mu((\bigcup_{k=1}^{n} A_k)^C) B \cup (\bigcup_{k=1}^{n} A_k) B^C) \\
&\leq \mu((\bigcup_{k=1}^{n} A_k)^C) B) + \mu((\bigcup_{k=1}^{n} A_k) B^C)
\end{aligned}
$$

By (i) there exists disjoint $\mathscr{A}$-sets $\{A_n\}$ such that

$$\mu(\cup_n A_n \setminus B) < \varepsilon$$

Let $A = \bigcup_n A_n$.

Then,

$$A \setminus \bigcup_{k=1}^{n} A_k \downarrow \emptyset$$

Need, $\mu(A \setminus A_1) < \infty$ and $\mu(A) < \infty$.

By continuity from above, we have $\mu(A \setminus \bigcup_{k=1}^{n} A_K) \downarrow 0$.

So, for sufficiently large n we have,

$$\mu(A \setminus \bigcup_{k=1}^{n} A_K) < \varepsilon$$

Now first take a look at $\mu((\bigcup_{k-1}^{n} A_k)B^C)$.

$$\begin{aligned}\mu((\bigcup_{k-1}^{n} A_k)B^C) &\le \mu(AB^C) \\ &= \mu(A \setminus B) < \varepsilon\end{aligned}$$

$$\begin{aligned}\mu((\bigcup_{k-1}^{n} A_k)^C A) &\le \\ &= \mu(A \setminus \bigcup_{k-1}^{n} A_k) < \varepsilon\end{aligned}$$

So we have that $\mu(\bigcup_{k=1}^{n} A_k \triangle B) < 2\varepsilon$

■

The next lemma will be used in the next section, this is an extension of Theorem 1.3 in the textbook.

Lemma 2. Suppose $\mathscr{A}$ is a semiring, $A_1, \ldots, A_n, A$ be $\mathscr{A}$ -sets, and μ is a non-negative, finitely additive set function on $\mathscr{A}$. Then

1. If $\bigcup_{k=1}^{n} A_k \subset A$ and A_k are disjoint, then

$$\sum_{k=1}^{n} \mu(A_k) \le mu(A)$$

2. If $A \subset \bigcup_{k=1}^{n} A_k$ (A_k don't have to be disjoint) then

$$\mu(A) \le \sum_{k=1}^{n} \mu(A_k)$$

Proof. 1. By Lemma 1,

$$\begin{aligned}A \setminus \bigcup_{k=1}^{n} A_k &= A(\bigcup_{k=1}^{n} A_k)^C \\ &= AA_1^C \ldots A_n^C \\ &= C_1 \cup \cdots \cup C_n\end{aligned}$$

where the C_k are disjoint $\mathscr{A}$-sets.

So,

$$A = ((\bigcup_{k=1}^{n} A_k) \cup (\bigcup_{l=1}^{m} C_l))$$

And thus we now have $A_1, \ldots, A_n, C_1, \ldots, C_m$ disjoint $\mathscr{A}$-sets. By finite additivity of μ,

$$\mu(A) = \mu(A_1) + \cdots + \mu(A_n) + \mu(C_1) + \cdots + \mu(C_m) \geq \mu(A_1) + \cdots + \mu(A_n)$$

2. Want $A \subseteq \bigcup_{k=1}^{n} A_k \Rightarrow \mu(A) \leq \sum_{k=1}^{n} \mu(A_k)$.

Let
$B_1 = A_1$
$B_2 = A_2 A_1^C$
$\vdots$
$B_n = A_n A_1^C \ldots A_{n-1}^C$
Let
$C_1 = AB_1$
$\vdots$
$C_n = AB_n$

Then, $C_1, \ldots, C_n$ are disjoint.

$$A = \bigcup_{i=1}^{n} C_i$$

By Lemma 1,

$$C_i = AA_i A_1^C \ldots A_{i-1}^C$$

which are finite disjoint union of $\mathscr{A}$-sets (AA_i are π-system $\mathscr{A}$-sets).

$$C_i = \bigcup_{j=1}^{mi} D_{ij}$$

In the meantime,

$$C_i = AA_i A_1^C \ldots A_{i-1}^C \subseteq A_i$$

Therefore,

$$\bigcup_{j=1}^{mi} D_{ij} \subseteq A_i$$

By part (i)

$$\sum_{j=1}^{mi} \mu(D_{ij}) \leq mu(A_i)$$

$$\sum_{i=1}^{n.} \sum_{j=1}^{mi} \mu(D_{ij}) \leq \sum_{i=1}^{n.} mu(A_i)$$

But the D_{ij} are disjoint $\mathscr{A}$-sets.

$$\bigcup_{i=1}^{m}\bigcup_{j=1}^{mi} D_{ij} = A$$

By finite additivity of μ on $\mathscr{A}$,

$$\mu(A) = \sum_{j=1}^{mi} \mu(D_{ij})$$

Hence,

$$\mu(A) \leq \sum_{i=1}^{n} \mu(A_i)$$

■

1.12 Measure in Euclidean

Extend Measure to $\mathscr{R}^k$

What we have done is to extned μ from intervals to $\mathscr{R}'$.

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Lebesgue Measure in $\mathbb{R}^k$

Let $\mathscr{A} = \{(a_1, b_1]x\dots x(a_k, b_k] : a_1, b_1, dotsa_k, b_k \in \mathbb{R}\}$.

$\mathscr{A}$ is called the collection of bounded rectangles in $\mathbb{R}^k$.

First, we want to see that this is a semiring. There is a rigorious proff in Example 11.4, but here we will just use pictures.

Key Property of Semiring

$$B|A = C_1 \bigcup \dots \bigcup C_m$$

where $B|A$ are $\mathscr{A}$-sets, $A \subseteq B$ and the C_i are finite disjoint unions of $\mathscr{A}$-sets.

$\mathscr{A}$ 2 - line rectangle $b \in \mathscr{A}$.
So $B \setminus A = C_1 \bigcup \dots \bigcup C_8$, which are disjoint $\mathscr{A}$-sets.
Similar argument for three dimensions with 26 cubes in rectangle.

Define λ_k on $\mathscr{A}$ as

$$\lambda_k : \mathscr{A} \to \mathbb{R}$$

$$(a_1, b_b x \dots x(a_k, b_k] \to (b_1 - a_1) \dots (b_k - a_k)$$

For μ to have an extension to $\sigma(\mathscr{A})$, μ has to be nonnegative, finitely additive, countably subadditive. We are not going to prove this now but we'll prove it later in a more general setting. For now, assume the extension exists.

Because μ is σ-finite on $\mathscr{A}$ and $\mathscr{A}$ is a π-system, this extension is unique and we call it the Lebesgue Measure in $\mathbb{R}^k$.

But why is $\mu\sigma$-finite on $\mathscr{A}$?

$$\bigcup_{n=1}^{\infty}(-n,n]x\ldots x(-n,n]$$

$$\mu((-n,n]x\ldots x(-n,n]) = (2n)^k$$

Characterizing Measures in $\mathbb{R}$

The only measure we know so far is the Lebesgue measure. Let μ be any measure that has finite value on bounded sets, $\mu(A) < \infty$.

A is ?.

Bounded set: $\sup_{x\in A}||x|| < \infty$

Then we can define

$$F(x) = \begin{matrix} \mu(0,x] & x \geq 0 \\ -\mu(x,0] & x < 0 \end{matrix}$$

This is obviously nondecreasing.

For $0 < a < b, a < 0 < b, a < b < 0$ we can show that $F(b) \geq F(a)$.

Also it is right continuous for $x \geq 0$. So if $x_n \downarrow x$,

$$(0,x_n] \downarrow (0,x]$$

implies

$$\mu(0,x_n] \downarrow \mu(0,x]$$

and

$$F(x_n) \to F(x)$$

for $x < 0$, where $x_n \downarrow x$,

$$(x_n,0] \downarrow (x,0]$$

implies

$$\mu(x_n,0] \downarrow \mu(x,0]$$

So,

$$-\mu(x_n,0] \to -\mu(x,0]$$

and

$$F(x_n) \to F(x)$$

So F is right continuous.

Theorem 1.12.1 — Theorem 12.4. If F is
1. nondecreasing
2. right continuous

then there exists a unique measure, μ on $\mathscr{R}$ such that

$$\mu(a,b] = F(b) - F(a) \quad \forall a,b \in \mathbb{R}$$

Characterizing Measures in $\mathbb{R}^k$

Let $\mathscr{R}^k$ be the Borel σ-field on $\mathbb{R}^k$. Note that $\sigma(\mathscr{A}) = \mathscr{R}'$.

So $\mathscr{R}^k = \sigma\{(a_1,b_1]x\dots x(a_k,b_k] : a_1,b_1,\dots a_k,b_k \in \mathbb{R}\}$.
For each $x \in \mathscr{R}^k$, let

$$S_x = \text{"Southwest"} = (-\infty,x_1]x\dots x(-\infty,x_k]$$

Then we can show that

$$\mathscr{R}^k = \sigma\{S_x : X \in \mathbb{R}^K\}_{rs}$$

To see this, for any bounded rectangle $A = (a_1,b_1]x\dots x(a_k,b_k]$ let V_A be the collection of all vertices of A, that is $V_A = \{a_1,b_1\}x\dots x\{a_k,b_k\}$. So we have that $\#(V_A) = 2^k$.

Now we can express

$$(a_1,b_1]x\dots x(a_k,b_k] = S_{b_1,\dots,b_k} \setminus \bigcup_{V_A\setminus\{b_1,\dots,b_k\}S(x_1,\dots,x_k}$$

So, $\sigma\{\mathscr{A}\} = \mathscr{R}^k \subseteq \sigma\{S_x : x \in \mathbb{R}^k\}$.

In the other direction, we have any

$$S_x = \bigcup_{n=1}^{\infty}(x_i - n, x_i]$$

Then $\sigma\{S_x : x \in \mathbb{R}^k\} \subseteq \mathscr{R}^k$.

So,
$$\sigma\{S_x : x \in \mathbb{R}^k\} = \mathscr{R}^k$$

Notation 1.2 (Signum)**.** *$sgn_A(x)$ is a signum of a vertex in a rectangle. Signum means "sign" in Latin.*

$$sgn_A(x) = \begin{matrix} 1 & \{i : x_i = a_1\} \text{ is even} \\ 0 & \text{else} \end{matrix}$$

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Let $F : \mathbb{R}^k \to \mathbb{R}$. Also $A \in \mathscr{I}^k, \mathscr{I}^k$ is the collection of all bounded rectangles. That is, $\{(a_1,b_1]x\dots x(a_k,b_k] : a_1,b_1,\dots,a_k,b_k \in \mathbb{R}\}$.

Let

$$\triangle_A F = \sum_{x \in V_a} sgn_A(x) F(x)$$

where V_a is the collection of all vertices of A.

For illustration, assume μ is a finite measure. We won't need this assumption.

Let $F(x) = \mu(S_x)$, where S_x is the southwest of x.

$$S_x = (-\infty, x_1] x \dots x (-\infty, x_k]$$

Then we show for any $A \in \mathscr{I}^k$ we have

$$\mu(A) = \triangle_A F$$

To see this,

$$A = \bigcup_{x \in V_A \setminus \{b_1, \dots, b_k\}} S_{(x_1, \dots, x_k)}$$

For $k = 2$,

$$\mu(\bigcup_{x \in V_A \setminus \{b_1, \dots, b_k\}} S_{(x_1, \dots, x_k)}) = \mu(S_{b_1 \dots b_k}) - \bigcup_{x \in V_A \setminus \{b_1, \dots, b_k\}} S_k$$

There are $2^k - 1$ number of values in this range.

$$\bigcup_{i=1}^{m} B_i \quad m = 2^k - 1$$

$$\mu(\bigcup_{i=1}^{m} B_i) = \sum_{i=1}^{m} \mu(B_i) - \sum_{i<j} \mu(B_i B_j) + \dots + (-1)^{m+1} \mu(B_1 \dots B_m)$$

For k = 2,

$$\sum_{i=1}^{m} \mu(B_i) = \mu(S_{a_1 a_2}) + \mu(S_{a_a b_2}) + \mu(S_{b_1 a_2})$$

$$\sum_{i<j} \mu(B_i B_j) = \mu(B_1 B_2) + \mu(B_2 B_3) + \mu(B_1 B_3) = 3\mu(B_1)$$

$$\mu(B_1 B_2 B_3) = \mu(B_1)$$

All together,

$$\begin{aligned}
\mu(\bigcup_{i=1}^{m} B_i) &= \mu(S_{B_1 B_2}) - \mu(B_1) - \mu(B_2) - \mu(B_3) + 3\mu(B_1) - \mu(B_1) \\
&= \mu(S_{B_1 B_2}) + \mu(B_1) - \mu(B_2) - \mu(B_3) \\
&= \sum_{x \in V_A} sqn_A(X) F(x) \\
&= \triangle_A F
\end{aligned}$$

By induction, we may show that $\mu(A) = \triangle_A F$.

So,

$$\triangle_A F \geq 0 \quad \forall A \in \Omega^k$$

Also, if $x^{(n)} \downarrow x$, in the sense that

$$x_1^{(n)} \downarrow x, \dots, x_k^{(n)} \downarrow x$$

then, $S_{x^{(n)}} \downarrow S_x$.

So $\mu(S_{x^{(n)}}) \downarrow \mu(S_x)$.

$$F(x^{(n)}) \to F(x)$$

So, $F(x)$ is continuous from above in the sense that $x^{(n)} \downarrow x$.

$$F(x^{(n)}) \downarrow F(x)$$

This shows that $\mu \Rightarrow F$ such that $\triangle_A F \geq 0$ continuous from above.

In fact, such an F, also uniquely determines a measure.

Theorem 1.12.2 Suppose $F : \mathbb{R}^k \to \mathbb{R}$ is continuous from above. That is,

$$\lim_{x^{(n)} \downarrow x} F(x^{(n)}) = F(x)$$

Also suppose that for all $A \in \mathscr{I}^k$,

$$\triangle_A F \geq 0$$

corresponding to right continuous and nondecreasing in $\mathbb{R}^k$.
Then, there exists a unique measure μ on $(\mathbb{R}^k, \mathscr{R}^k)$ such that for all $A \in \mathscr{I}^k$,

$$\mu(A) = \triangle_A F$$

The most important special case of this theorem is the case

$$F(x) = x_1 \dots x_k$$

$$\sum_{x \in V_A} sgn_A(x) F(x) = (b_1 - a_1) \dots (b_k - a_k)$$

So the μ corresponding to this F is the Lebesgue measure.

This characterizes all measures in $\mathbb{R}^k$.

Proof. Note that μ is defined on $\mathscr{I}^k$. So we need to show μ can be uniquely extended to $\sigma(\mathscr{I}^k) = \mathscr{R}^k$.

Uniqueness

Want $\mu\sigma$-finite on $\mathscr{I}^k$.

$$A_n = (-n,n]x\ldots x(-n,n]$$

$$\cup_n A_n = \mathbb{R}^k$$

$$\mu(A_n) = \sum_{x\in V_A} sqn_{A_n}(x)F(x)$$

But $F : \mathbb{R}^k \to \mathbb{R}$, $F(x) < \infty \forall x$, and $\mu(A_n) < \infty$.

So μ is σ-finite on $\mathscr{I}^k$.

Still need existence of extension finitely additive, countably subadditive.

Step 1 Finitely additive.

Step 1 (a)

Finitely additive on regular partition of $A \in \mathscr{I}^k$.

What's a regular partition? Irregular partition? Think disjoints vs overlapping.

It's easy to turn an irregular partition into a regular one.

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More explicitly,

$$A = I_1 x\ldots x I_k$$

where, $I_i = (a_i, b_i]$ for $i = 1,\ldots,k$.

For each i let,

$$J_{i1},\ldots,J_{ik}$$

be a partition of I_i into subintervals.

For each $(j_1,\ldots,j_k) \in \{1,\ldots,n_1\}x\ldots x\{1,\ldots,n_k\}$

Write

$$B_{j_1,\ldots,j_k} = J_{1j_1}x\ldots xJ_{kj_k}$$

So we have

$$\mathscr{B} = \{B_{j_1\ldots j_k} : j_i \in \{1,\ldots,n_i\}\forall i = 1,\ldots,k\}$$

$$\#\mathscr{B} = n_1 \dots n_k$$

So $\mathscr{B}$ is called a **regular decomposition of A.**

Obviously $A = \bigcup_{B\in\mathscr{B}} B$ and $\{B : B \in \mathscr{B}\}$ are disjoint and $B \in \mathscr{I}^k$.

Overall, Step 1 (a) claims that $\mu(A) = \sum_{B\in\mathscr{B}} \mu(B)$.

Recall that $\mu(B) = \triangle_B F = \sum_{x\in V_B} sgn_B(x)F(x)$.
Here we have,

$$\sum_{B\in\mathscr{B}} \mu(B) = \sum_{B\in\mathscr{B}} \sum_{x\in V_B} sgn_B(x)F(x)$$

We may change the order of summations,

$$\sum_{x\in V} \sum_{B\in W_x} sgn_B(x)F(x)$$

where $W_x = \{B \in \mathscr{B} : x \in V_B\}$ and $V = \bigcup_{B\in\mathscr{B}} V_B$.

Now be separating into values of x in and not in V_A, we have,

$$\sum_{x\in V_A} \sum_{B\in W_x} sgn_B(x)F(x) + \sum_{x\notin V_A} \sum_{B\in W_x} sgn_B(x)F(x)$$

So the idea is that when $x \notin V_1$ then $\sum_{B\in W_x} sgn_B(x) = 0$ which means the second term above is also zero. But, if $x \in V_A$, then W_x is singleton and $B \in W_x$ has same sign as A.

So ultimately,

$$\sum_{B\in\mathscr{B}} \mu(B) = \sum_{x\in V_A} sgn_a(x)F(x0 = \mu(A)$$

Step 1 (b)

Now, consider, general situation. If we let $A \in \mathscr{I}^k$ and suppose that $A = \bigcup_{u=1}^n A_u, A_u \in \mathscr{I}^k$. Because $A_n \in \mathscr{I}^k$, and $A_u = I_{1u}x \dots xI_{ku} \in \mathscr{I}^1$ we have,

$$A = \bigcup_{u=1}^{n} (I_{1u}x \dots xI_{ku})$$

Meanwhile, $A \in \mathscr{I}^k$, so $A = I_1x \dots xI_k$.
Claim:

$$I_1x \dots xI_k = \bigcup_{u=1}^{n} (I_{1u}x \dots xI_{ku}) = \bigcup_{u=1}^{n} (I_{1u})x \dots x \bigcup_{u=1}^{n} I_{ku})$$

But if $(a_1, \dots, a)k) \in \bigcup_{u=1}^n (I_{1u}x \dots xI_{ku})$, then

$$(a_1, \dots, a)k) \in I_{1u}x \dots xI_{ku}$$

for some u.

Then we have $a_i \in I_{iu} \subseteq \bigcup_{u=1}^{n} I_{iu}$ which leads to

$$(a_1,\ldots,a)k) \subseteq (\bigcup_{u=1}^{n} I_{1u})x\ldots x(\bigcup_{u=1}^{n} I_{ku})$$

So,

$$\bigcup_{u=1}^{n} (I_{1u}x\ldots xI_{ku}) \subset (\bigcup_{u=1}^{n} I_{1u})x\ldots x(\bigcup_{u=1}^{n} I_{ku})$$

On the other hand,

$$I_{iu} \subset I_u$$

$$(\bigcup_{u=1}^{n} I_{iu}) \subseteq I_u$$

$$(\bigcup_{u=1}^{n} I_{1u})x\ldots x(\bigcup_{u=1}^{n} I_{ku}) \subseteq I_1x\ldots xI_k$$

Thus, claim is proved. Hence,

$$(\bigcup_{u=1}^{n} I_{1u})x\ldots x(\bigcup_{u=1}^{n} I_{ku}) = I_1x\ldots xI_k$$

The union, $(\bigcup_{u=1}^{n} I_{iu})$, needs not be disjoint, but we may use all the endpoints of I_{iu} to make disjoints partitions.

So then

$$A = (\bigcup_{v=1}^{m_1} \tilde{I}_{1v})x\ldots x(\bigcup_{v=1}^{m_k} \tilde{I}_{kv})$$

which by Step 1 (a) is a regular partition.

$$\mu(A) = \sum_{v=1}^{m_1}\ldots\sum_{v=1}^{m_k} \mu(\tilde{I}_{1v_1})x\ldots x\tilde{I}_{kv_k}))$$

Let $\tilde{\mathscr{B}} = \{\tilde{I}_{1v_1})x\ldots x\tilde{I}_{kv_k} : v_i = 1,\ldots,m_i \forall i = 1,\ldots,k\}$.
So,

$$\mu(A) = \sum_{\tilde{B}\in\tilde{\mathscr{B}}} \mu(\tilde{B}) = \sum_{u=1}^{n} \sum_{\tilde{B}\subseteq A_u} \mu(\tilde{B})$$

Thus fnite additivity done.
Step 2 Countably subadditive.

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First we want to show finite subadditive.

That is if $A_1,\ldots,A_n,A \in mathcalI^k$ and $A \subseteq \bigcup *n_{i=1}A_i$ then,

$$\mu(A) \leq \sum_{i=1}^{n} \mu(A_i)$$

But this is implied by Lemma 2 at end of section 11.

We need to extend this to countably subadditivity.

Recall compact set, in $\mathbb{R}^N$ (definition applies to any toplogical space). A set is **compact** if any open covering has a finite subcovering. That is, if $A \subseteq \bigcup_{G \in \mathscr{G}} G$ where G is open, then there exists a subcollection

$$\{G_1, \ldots, G_n\} \subset \mathscr{G}$$

so that $A \subset \bigcup_{i=1}^{n} G_i$ and we may call A compact. In the appendix of Billingsly, there is the Heine - Borel Theorem: A bounded and closed set in $\mathbb{R}^k$ is compact.

Now we want to show that if $A_1, \ldots \& A \in \mathscr{I}^k$ and $A \subseteq \bigcup_n A_n$ then

$$\mu(A) \leq \sum_n \mu(A_n)$$

Because $A < \mathscr{I}^k$, then $A = (a_1, b_1]x \ldots x(a_k, b_k]$. Let

$$B(\delta) = (a_1 + \delta, b_1]x \ldots x(a_k + \delta, b_k]$$

Then

$$\mu(B(\delta)) = \sum_{x \in V_{B(\delta)}} sgn_{B_\delta}(x) F(x)$$

Note that $x(\delta) \in V_{B_\delta}$.

So $x(\delta)$ may be written as

$$x(\delta) = x + \begin{pmatrix} \delta \\ 0 \\ \delta \\ \vdots \\ \delta \end{pmatrix} = x + \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} \delta \\ \delta \\ \delta \\ \vdots \\ \delta \end{pmatrix}$$

So as $\delta \downarrow 0$, $x(\delta) \downarrow x$.

Thus, $F(x(\delta)) \downarrow F(x)$ as $\delta \downarrow 0$.

Also, for $\delta > 0$ small enough,

$$sgn_B(x) = sgn_{B(\delta)}(x(\delta))$$

So,

$$\sum_{x \in V_{B(\delta)}} sgn_{B(\delta)}(x(\delta)) F(x(\delta)) \to \sum_{x \in V_A} sgn_B(x) F(x) = \mu(A)$$

$$\mu(B(\delta)) \uparrow \mu(A)$$

Also, for all $\varepsilon > 0$, there exists a $\delta > 0$ so that

$$\mu(A) < \mu(B(\delta)) + \varepsilon$$

Similarly, $A_1, \ldots, A_n, \ldots$ for each $A_u = (a_{1u}, b_{1u}]x \ldots x(a_{ku}, b_{ku}]$. Let,

$$B_u = (a_{1u}, b_{1u} + \delta_u]x \ldots x(a_{ku}, b_{ku} + \delta_u]$$

then, as argued before,

$$\mu(B(\delta)) \downarrow \mu(A) \quad \delta_u \to 0$$

So, for all ε there exists $\delta_u > 0$ such that

$$\mu(A_u) + \frac{\varepsilon}{2u} > \mu(B_u)$$

$$\mu(A_u) < \mu(B_u) - \frac{\varepsilon}{2u}$$

Let the closure of B be,

$$B^- = [a_{1u} + \delta, b_{1u}]x \ldots x[a_{ku} + \delta, b_{ku}]$$

And the interior of B be,

$$B_u^0 = (a_{1u}, b_{1u} + \delta_u)x \ldots x(a_{ku}, b_{ku} + \delta_u)$$

Then, by construction

$$B^- \subseteq A \subseteq \bigcup_{n=1}^{\infty} B_n^0$$

and there exists a finite set,

$$D \subseteq \{1, 2, \ldots\}$$

$$B^- \subseteq \bigcup_{u \in D} B_u^0$$

Take n large enough so that

$$B^- \subseteq \bigcup_{u=1} B_u^0$$

But we know that

$$\mu(A) - \varepsilon < \mu(B) \tag{1.1}$$
$$\mu(An) + \frac{\varepsilon}{2^n} > \mu(B_u) \tag{1.2}$$
$$\mu(B) \leq \mu(A) \tag{1.3}$$
$$\leq \mu(\bigcup_{u=1}^{n} B_u) \tag{1.4}$$
$$\leq \sum_{u=1}^{n} \mu(B_u) \tag{1.5}$$
$$\leq \sum_{u=1}^{\infty} \mu(B_u) \tag{1.6}$$
$$\leq \sum_{u=1}^{\infty} (\mu(B_u) + \frac{\varepsilon}{2^u}) \tag{1.7}$$
$$= \sum_{u=1}^{\infty} (\mu(A_u) + \varepsilon \tag{1.8}$$
$$\tag{1.9}$$

So we have that for all $\varepsilon > 0$,

$$\mu(A) \leq \sum_{u=1}^{\infty} \mu(A_u)$$

■

1.13 Measurable Functions

Measureable

Measureable spaces: $(\Omega, \mathscr{F}); (\Omega', \mathscr{F}')$

Let

$$T : \Omega \to \Omega'$$

We say that T is measureable. T measures $\mathscr{F}/\mathscr{F}'$ if $\forall A' \in \mathscr{F}', \{\omega : T(\omega) \in A'\} \in \mathscr{F}$

Notation 1.3. *Whenever we encounter the notation,*

$$T \text{ⓜ} \mathscr{F} \setminus \mathscr{F}'$$

we may interpret this as T is a measureable function that takes inputs from Ω and maps them to outputs in Ω' where $\mathscr{F}$ is the σ-field on Ω and $\mathscr{F}'$ is the σ-field on Ω'. The measurability implies that any set in $\mathscr{F}'$ has its preimage in $\mathscr{F}$.

To understand this we need set inverse for $A' \subseteq \Omega'$.

$$T^{-1}(A') = \{\omega \in \Omega : T(\omega) \in A'\}$$

So, in terms of set inverse, $T^{-1}(A\prime)$,

$$T$$

Don't have measure? $(\Omega', \mathscr{F}')$
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In particular

$$T^{-1}(\{x\}) \neq T(\{x\})$$

$$T^{-1}(\{x\}) = \{\omega \in \Omega : T(\omega) = x\}$$

Set inverse enjoys many nice properties.

- $\forall A' \in \Omega'$,

$$T^{-1}((A')^C) = T^{-1}((A'))^C$$

- If $\mathscr{B}$ is a collection of sets (could be uncountable),

$$T^{-1}(\bigcup_{B \in \mathscr{B}} B) = \bigcup_{B \in \mathscr{B}} T^{-1}(B)$$

-

Proof.
- Let $\omega \in T^{-1}((A')^C)$.
  $\Leftrightarrow T(\omega \in (A')^C)$
  $\Leftrightarrow T(\omega) \notin A'$
  $\Leftrightarrow \omega \notin T^{-1}(A')$
  $\Leftrightarrow \omega \in (T(A'))^C$
- Let $T(\omega) \in T^{-1}(\bigcup_{B \in \mathscr{B}})B$.
  $\Leftrightarrow T(\omega) \in \bigcup_{B \in \mathscr{B}} B$
  $\Leftrightarrow T(\omega) \in B \quad \text{for some } B \in \mathscr{B}$
  $\Leftrightarrow \omega \in T^{-1}(B) \quad \text{for some } B \in \mathscr{B}$
  $\Leftrightarrow \omega \in \bigcup_{B \in \mathscr{B}} T^{-1}(B)$
-

■

Composite Function

$(\Omega, \mathscr{F}), (\Omega', \mathscr{F}'), (\Omega'', \mathscr{F}'')$

$$T : \Omega \to \Omega', T' : \Omega' \to \Omega''$$

$$T' \circ T \Omega \to \Omega'', \omega \mapsto T'(T(\omega))$$

Then,

$$(T' \circ T)^{-1} = T' \circ (T)^{-1}$$

Proof. Want to show that for all $A'' \subset \Omega''$

$$(T' \circ T)^-1(A'') = T^{-1} \circ (T')^{-1}(A''), \omega \in (T' \circ T)^-1(A'')$$

$$\Leftrightarrow (T' \circ T)^-1(\omega) \in A'' \Leftrightarrow T'(T(\omega)) \in A''$$

■

Theorem 1.13.1 1. If $\mathscr{A}'$ generates $\mathscr{F}'$ and $T^{-1}(\mathscr{A}') \subseteq \mathscr{F}$ then T measure $\mathscr{F}/\mathscr{F}'$.
2. T measure $\mathscr{F} \setminus \mathscr{F}'$, T' measure $\mathscr{F}' \setminus \mathscr{F}''$
What's the omtivation of (i)? In order to check T measures $\mathscr{F} \setminus \mathscr{F}'$ we don't have to check $T^{-1}(A') \in \mathscr{F} \forall A' \in \mathscr{F}'$.

We only need to check

$$T^{-1}(A') \in \mathscr{F} \forall A' \in \mathscr{A}'$$

So in order to check wither a subset

$$f : \Omega \to \mathbb{R} \text{ is measure on } \mathscr{F} \setminus \mathscr{R}$$

all we need to check is

$$f^{-1}((-\infty, x]) \in \mathscr{F} \Leftrightarrow \{\omega : f(\omega) \leq +x\} \in \mathscr{F} \forall x \in \mathbb{R}$$

Proof. 1. Define paving in Ω'.

$$\mathscr{G}' = \{A' \subseteq \Omega' : T^{-1}(A') \in \mathscr{F}\}$$

Interestingly, this is sometimes written as,

$$T(\mathscr{F})$$

which is not $\{T(A) : A \in \mathscr{F}\}$.

Claim: $\mathscr{G}'$ is σ-field.
(a) $\Omega' \in \mathscr{G}' \Leftrightarrow T^{-1}(\Omega') \in \mathscr{F}$

$$\{\omega \in \Omega : T(\omega) \in \Omega'\}$$

because T is a mapping.
(b) Show $A' \in \mathscr{G}' \Rightarrow (A')^C \in \mathscr{G}'$.

$$(A')^C \in \mathscr{G}' \Leftrightarrow T'((a')^C) \in \mathscr{F} \Leftrightarrow (T^{-1}(A'))^C \in \mathscr{F}$$

$$T^{-1}() \in \mathscr{F} \text{ because } A' \in \mathscr{G}'$$

$$(T^{-1}(A'))^C \in \mathscr{F}$$

because $\mathscr{F}$ is a σ-field.

(c) $A'_1, \cdots \in \mathscr{G}'$. Want to show that $\bigcup_n A'_n \in \mathscr{G}'$

$$\Leftrightarrow T^{-1}(\bigcup_n A'_n) \in \mathscr{F}$$

$$\Leftrightarrow \bigcup_n (T^{-1}(A'_n)) \in \mathscr{F}$$

Now $\mathscr{A}'$ generates $\mathscr{F}'$ and $\mathscr{G}'$ is a σ-field.

$$\mathscr{A}' \subseteq \mathscr{G}'$$

beacuse

2. Want $T' \circ T$ is a measure $\mathscr{F} \setminus \mathscr{F}''$, $A'' \in \mathscr{F}''$.

   Want $(T' \circ T)^{-1}(A'') \in \mathscr{F}$.

$$\Leftrightarrow T^{-1} \circ (T')^{-1}(A'') \in \mathscr{F}$$

$$\Leftrightarrow T^{-1}((T^prime)^{-1}(A''))$$

   But we know that

■

Mapping into $\mathbb{R}^k$

$(\Omega, \mathscr{F}), (\mathbb{R}^k, \mathscr{R}^k)$

$$f : \Omega \to \mathbb{R}^k$$

Since the set of the form

$$\mathscr{A} = \{(-\infty, x_1]x \ldots x(-\infty, x_k] : x_1, \ldots, x_k \in \mathbb{R}\}$$

generates $\mathscr{R}^k$, for f to be a measre on $\mathscr{F} \setminus \mathscr{R}'$ we need

$$f^{-1}((-\infty, x_1]x \ldots x(-\infty, x_k]) \in \mathscr{F}$$

$$= \{\omega \in \Omega : f_1(\omega) \leq x_1, \ldots, f_k(\omega)\}$$

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$$= \bigcap_{i=1}^{k} \{\omega \in \Omega : f_i(-\infty, x_i]\}$$

if f_i is a measure on $\mathscr{F} \setminus \mathscr{R}$. Then the f_i are in $\mathscr{F}$. So,

$$\bigcap_{i=1}^{k} \{\omega \in \Omega : f_i(-\infty, x_i]\} \in \mathscr{F}$$

And we have that f is a measure on $\mathscr{F} \setminus \mathscr{A}$ which implies f is a measure on $\mathscr{F} \setminus \mathscr{R}^k$.

So if f_i is a measure on $\mathscr{F} \setminus \mathscr{R}$ for $i = 1, \dots, k$, then f is a measure on $\mathscr{F} \setminus \mathscr{R}^k$.

Conversely, we may also show that if f is a measure on $\mathscr{F} \setminus \mathscr{R}^k$ this implies that f_i is also a measure on $\mathscr{F} \setminus \mathscr{R}$ for $i = 1, \dots, k$.

Let $x \in \mathbb{R}$.

Let $A_n = (\bigcap_{i \neq j} \{\omega_i f_j(\omega) \leq n\}) \bigcap \{\omega_i f_i(\omega) \leq x\}$. Then

$$A_n \uparrow \{\omega : f_i(\omega) \leq x\}$$

But since f is a measure on $\mathscr{F} \setminus \mathscr{R}^k$ we have that

$$A_n \in \mathscr{F}$$

$$\bigcup_n A_n \in \mathscr{F}$$

$$\{\omega : f_i(\omega) \leq x\} \in \mathscr{F}$$

Where f_i is a measure on $\mathscr{F} \setminus \mathscr{A}_0$.
And because $\mathscr{A}_0 = \{(-\infty, x] : x \in \mathbb{R}\}$ generates $\mathscr{R}$ we have that f_i is a measure on $\mathscr{F} \setminus \mathscr{R}$.

Conlusion: We have that f is a measure on $\mathscr{F} \setminus \mathscr{R}^k$ is exchangeable (implies/implied by) f_i being a measure on $\mathscr{F} \setminus \mathscr{R}$.

Theorem 1.13.2 If $f : \mathbb{R}^i \to \mathbb{R}^k$ is continuous then f is a measure on $\mathscr{R}^i \setminus \mathscr{R}^k$.

Proof. Let $\mathscr{G}^k$ be the collection of open sets in $\mathscr{R}^k$. Because $f : \mathbb{R}^i \to \mathbb{R}^k$ is continous we have that

$$f^{-1}(\mathscr{G}^k) \subset \mathscr{G}^i$$

But $\mathscr{G}^i generates \mathscr{R}^i$.

So, $f^{-1}(\mathscr{G}^k) \subset \mathscr{R}^i$ and f is a measure on $\mathscr{R}^i \setminus \mathscr{G}^k$.

But $\mathscr{G}^k generates \mathscr{R}^k$.

So, similarly,f is a meausre on $\mathscr{R}^i \setminus \mathscr{R}^k$.

■

More generally, suppose

$$(\mathbb{X}, \mathscr{X}), (\mathbb{Y}, \mathscr{Y})$$

are topologicaly spaces where $f : \mathbb{X} \to \mathbb{Y}$ is continuous. Then we have that f is a meausre on $\mathscr{B}_X \setminus \mathscr{B}_Y$ where

$$\mathscr{B}_X = \sigma(\mathscr{X})$$
$$\mathscr{B}_Y = \sigma(\mathscr{Y})$$

Proof. Because f is continuous,

$$f^{-1}(\mathscr{Y}) \subseteq \mathscr{X} \subseteq \sigma(\mathscr{X}) = \mathscr{B}_X$$

So f is measure on $\mathscr{B}_X \setminus \mathscr{Y}$.

But $\mathscr{Y}$ generates $\sigma(\mathscr{Y} = \mathscr{B}_Y)$. So f is measure on $\mathscr{B}_X \setminus \mathscr{B}_Y$. ■

Theorem 1.13.3 If $f_j : \Omega \to \mathbb{R}$ is a measure on $\mathscr{F} \setminus \mathscr{R}$ for $j = 1, \dots, k$ and $\mathscr{Y} : \mathbb{R}^k \to \mathbb{R}$ is a measure on $\mathscr{F} \setminus \mathscr{R}$ then

$$\omega \mapsto \mathscr{G}(f_1(\omega), \dots, f_k(\omega)) \text{ is a meausre on } \mathscr{F} \setminus \mathscr{R}$$

Proof. ■

Measurability of Limits

Let $\bar{\mathbb{R}} = \{-\infty\} \bigcup \{\infty\} \bigcup \mathbb{R}$ and $\bar{\mathscr{R}}$ be the σ-field generated by

$$\{\{-\infty\}, \{\infty\}, \mathscr{R}\}$$

Let $(\Omega, \mathscr{F})$ be measureable space.

Let $f : \Omega \to \bar{\mathbb{R}}$ be measure on $\mathscr{F} \setminus \bar{\mathscr{R}}$.

Here f being a measure on $\mathscr{F} \setminus \bar{\mathscr{R}}$ implies
1. $f^{-1}(\mathscr{R}) \in \mathscr{F}$
2. $f^{-1}(\{-\infty\}) \in \mathscr{F}$
3. $f^{-1}(\{+\infty\}) \in \mathscr{F}$

Theorem 1.13.4 Suppose $f_1, f_2, \dots,$ are function $\Omega \to \bar{\mathbb{R}}$ which are measure on $\mathscr{F} \setminus \bar{\mathscr{R}}$. Then
1. (a) $\sup_n f_n$ is a measure on $\mathscr{F} \setminus \bar{\mathscr{R}}$
   (b) $\inf_n f_n$ is a measure on $\mathscr{F} \setminus \bar{\mathscr{R}}$
   (c) $\limsup_n f_n$ is a measure on $\mathscr{F} \setminus \bar{\mathscr{R}}$
   (d) $\liminf_n f_n$ is a measure on $\mathscr{F} \setminus \bar{\mathscr{R}}$
2. If $\limsup_n f_n(\omega)$ exists for all $\omega \in \Omega$ then $\lim_n f_n$ is a measure on $\mathscr{F} \setminus \bar{\mathscr{R}}$.
3. $\{\omega : \exists \lim_n f_n(\omega)\} \in \mathscr{F}$
4. If f is a measure on $\mathscr{F} \setminus \bar{\mathscr{R}}$ then $\{\omega : f_n(\omega) \to f(\omega)\} \in \mathscr{F}$.

Proof. 1. (a)
   (b)
   (c)
   (d)
2. If $\lim_n f_n(\omega)$ exists for all ω, then it is trivial that

$$\lim_n f_n \text{ⓜ} \mathscr{F} \setminus \bar{\mathscr{R}}$$

If $\lim_n f_n$ exists this means that

$$\limsup_n f_n = \liminf_n f_n$$

Then,

$$\liminf_n f_n \equiv \limsup_n f_n$$

3. We want to show that $\{\omega : \lim_n f_n(\omega)\exists\} \in \mathscr{F}$.
   This may be rewritten as,

$$\{\omega : \liminf_n f_n(\omega) = \limsup_n f_n(\omega)\} \in \mathscr{F}$$

$$\{\omega : \liminf_n f_n(\omega) - \limsup_n f_n(\omega) = 0\} \in \mathscr{F}$$

   But we have that $\liminf_n f_n$ and $\limsup_n f_n$ are measurable $\mathscr{F} \setminus \bar{\mathscr{R}}$. Also that their difference is also within $\mathscr{F}$ because
   (a) $\{0\} \in \mathscr{R}$
   (b) x-y is continuous
4. Want to show that if f ⓜ $\mathscr{F} \setminus \bar{\mathscr{R}}$, then

$$\{\omega : \lim_n f_n(\omega) = f(\omega)\} \in \mathscr{F}$$

   Which may be rewritten as

$$\{\omega : \lim_n f_n(\omega) - f(\omega) = 0\} \in \mathscr{F}$$

   But we can show that both terms are measurable, as is their diffrence and since 0 is in Borel set, the entire statement is in $\mathscr{F}$.

■

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Continued proving last theorem.

Algorithm of Proving Thing: 3-step Argument

1. A statement is true for indicator functions.
2. Then the statement is true for simple functions.
3. The statement is true for all measurable functions.

The net result is that if something is true for indicator functions, then it is true for measureable function.

Definition 1.13.1 — Simple Function. $(\Omega, \mathscr{F})$ is measureable space.

If $\{A_1, \ldots, A_n\}$ are $\mathscr{F}$-sets if $\bigcup_{i=1}^n A_i = \Omega$ and they are disjoint then $\{A_1, \ldots, A_n\}$ is an $\mathscr{F}$-partition.

So a simple function on Ω is any function of the form

$$f(\omega) = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}(\omega)$$

where $a_1, \ldots, a_n \in \mathbb{R}, \{A_1 \ldots, A_n\}\mathscr{F}$- partitions.

Theorem 1.13.5 If $f : \Omega \to \bar{\mathbb{R}} : ⓜ\mathscr{F} \setminus \mathscr{R}$ then there exists sequence of simple functions $\{f_n\}$ such that

if $f(\omega) = 0$,

$$0 \leq f_n(\omega) \uparrow f(\omega)$$

and if $f(\omega) \leq 0$,

$$0 \geq f_n(\omega) \downarrow f(\omega)$$

Proof. Construct a coarse grid of length 1 for each n.

For each of these strips fold it n times and then spread the paper out.

We have 2^n fine grids.

This process as a formula:

$$f_n(\omega) = \begin{cases} -n & -\infty \leq f(\omega) \leq -n \\ -(k-1)2^{-n} & -k2^{-n} < f(\omega) < -(k-1)2^{-n}, k = 1, \dots, n2^n \\ (k-1)2^{-n} & (k-1)2^{-n} \leq f(\omega) < k2^{-n}, k = 1, \dots, n2^n \\ n & n \leq f(\omega) \leq \infty \end{cases}$$

See photo for working through different cases.

Similarly for the opposite (down) direction. ■

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Measured Induces by Measureable Transformation

We have measurable spaces, $(\Omega, \mathscr{F}), (\Omega', \mathscr{F}')$.

μ is a measure on $(\Omega, \mathscr{F})$,

$$T : \Omega \to \Omega' ⓜ\mathscr{F} \setminus \mathscr{F}'$$

Define the set function μ' on $(\Omega', \mathscr{F}')$,

$$\mu' = \mu(T^{-1}(A')), A' \in \mathscr{F}'$$

Claim: We want to say that μ' is a measure on $\mathscr{F}'$.

1. $\mu'(\emptyset') = 0$?

$$\mu'(\emptyset') = \mu(T^{-1}(\emptyset'))$$

But, by the definition of the function

$$T^{-1}(\emptyset') = \{\omega \in \Omega : T(\omega) \in \emptyset'\} = \emptyset$$

So,

$$\mu'(\emptyset') = 0$$

2. $\mu'(A') = \mu(T^{-1}(A')) \geq 0$
3. $A'_1, A'_2 \ldots$ disjoint. We want to know if $\mu'(\bigcup_n A'_n) = \sum_n \mu'(A'_n)$

$$\begin{aligned} \mu'(\bigcup_n A'_n) &= \mu(T^{-1}(\bigcup_n A'_n)) \\ &= \mu(\bigcup_n T^{-1}(A'_n)) \\ \text{Note:} T^{-1}(A'_n) \bigcap T^{-1}(A'_m) &= T^{-1}(A'_n \bigcap A'_m) \\ &= \emptyset \text{ So, above:} \qquad = \sum_n \mu(T^{-1}(A'_n)) \\ &= \sum_n \mu'(A'_n) \end{aligned}$$

μ' is ameasure on $(\Omega', \mathscr{F}')$, this is "induced" measure by T^{-1}. We may write μ' as

$$\mu' = \mu \circ T^{-1} = \mu T^{-1}$$

Also note that,

$$\mu'(\Omega') = \mu(T^{-1}(\Omega'))$$

and that,

$$T^{-1}(\Omega') = \{\omega \in \Omega : T(\omega) \in \Omega'\}$$

this is Ω be ... so we have that

$$\mu'(\Omega') = \mu(\Omega)$$

So if μ is a finite measure, then μ' is finite measure; if μ is a probability measure, then so in μ'.

1.14 Distribution of Random Elements

We have measurable spaces, $(\Omega, \mathscr{F}), (\Omega', \mathscr{F}')$.

Definition 1.14.1 — Random Variable. $X : \Omega \to \Omega', ⓜ \mathscr{F} \setminus \mathscr{F}'$
if $\Omega = \mathbb{R}$ and $\mathscr{F}' = \mathscr{R}$, then X is called a **random variable**.

Definition 1.14.2 — Random Vector. X as above, if $\Omega = \mathbb{R}^k$ and $\mathscr{F}' = \mathscr{R}^k$, then X is called a **random vector**.

If Ω' is a general space, Hilber space, Bunach spaces, etc then $\mathscr{F}'$ is a σ-field on these spaces.

In these cases, X is called a **random element** or **random function/proccess/field** depending on the context.

If P is a probability measure on $(\Omega, \mathscr{F})$ then the induce measure, PX^{-1} on $\mathscr{F}'$ is called the **distribution of X**. Sometimes PX^{-1} is denoted as P_X.

Note that random variable comes first then the distribution.

R For $A' \in \mathscr{F}'$, $\mu(T^{-1}(A'))$ is sometimes written as $\mu(T \in A')$. Because

$$\begin{aligned}\mu(T^{-1}(A') &= \mu(\omega \in \Omega : T(\omega) \in A') \\ &= \mu(\omega : T(\omega) \in A') \\ &= \mu(T \in A')\end{aligned}$$

Now consider the case where $\Omega' = \mathbb{R}, \mathscr{F}' = \mathscr{R}$. Let

$$F(x) = P(X^{-1}(-\infty, x]) = P(X \in (-\infty, x]) = P(X \leq x)$$

This is called the distribution function of X.

Some properties of distribution functions, F

1. F is right continuous.

   $$x_n \downarrow x, (-\infty, x_n] \downarrow (-\infty, x]$$

   So,

   $$PX^{-1}((-\infty, x_n]) \to PX^{-1}((-\infty, x])$$

   Which may be written as,

   $$F(x_n) \to F(x)$$

2. Left limit: $x_n \uparrow x$.

   $$(-\infty, x_n] \uparrow (-\infty, x]$$

   $$PX^{-1}((-\infty, x_n]) \to PX^{-1}((-\infty, x)$$

   Which may be written as,

   $$F(x_n) \to P(X \leq x)$$

   That is to say,

   $$\lim_{x' \uparrow x} F(x') = P(X < x)$$

   $$F(x-) = P(X < x)$$

3. Jump at x.

   $$F(x) - F(x-) = P(X \leq x) - P(X < x) = P(X = x)$$

4. 2 end points.

   $$\lim_{x \to \{-\infty, \infty\}} F(x) = ?$$

   For $x_n \to -\infty, (-\infty, x_n] \downarrow \emptyset$.

$$PX^{-1}((-\infty, x_n]) \to PX^{-1}(\emptyset) = 0$$

So $F(x_n) \to 0, F(-\infty) = 0$.

For $x_n \to \infty, (-\infty, x_n] \to \mathbb{R}$.

$$PX^{-1}((-\infty, x_n]) \to PX^{-1}(\mathbb{R}) =$$

So $F(x_n) \to 1, F(\infty) = 1$.

5. F can only have countably many jumps. Let

$$D_F = \{x : F(x) - F(x-) > 0\}$$

Then D_F is countable.

Recall that Thm 10.2(iv), which supposes that μ is a measure on a field, $\mathscr{F}$ and that μ is σ-finite, then $\mathscr{F}$ cannot contain an uncountable disjoint collecition of sets with positive μ-measure.

Therefore D_F must be countable.

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Theorem 1.14.1 If F is nondecreasing and right continuous and we have that,

$$\lim_{x \to -\infty} F(x) = 0, \lim_{x \to \infty} F(x) = 1$$

then there exists, on some probability space $(\Omega, \mathscr{F})$, a random variable $X : \Omega \to \mathbb{R}$ measure on $\mathscr{F} \setminus \mathscr{R}$, such that

$$P(X \leq x) = F(x)$$

Proof. Recall that Theorem 12.4 (proved more general result in $\mathbb{R}^k$).
It says, if F is nondecreasing and right continuous then there exists on $\mathscr{R}$ a unique measure μ such that fora all a and b in $\mathbb{R}$,

$$\mu((a, b]) = F(b) - F(a)$$

So we have a measure on $(\mathbb{R}, \mathscr{R})$. To prove this theorem we need that

- μ is a probabililty measure ($\mu(\mathbb{R}) = 1$)
- X on $(\mathbb{R}, \mathscr{R})$ such that $\mu X^{-1}((-\infty, x]) = F(x)$

1. Want that $\mu(\mathbb{R}) = 1$. Note that $\forall b \leftarrow \mathbb{R}$, $(-n, b] \uparrow (-\infty, b]$. By continuity of μ from below we have that

$$\mu(-n, b]) \uparrow \mu(-\infty, b]$$

which gives us that

$$\mu(-n, b]) = F(b) - F(-n) = F(b) - 0$$

Now $(-\infty, n] \uparrow \mathbb{R}$ by continuity of μ from below.

■

2. Integration with Respect to a Measure

2.1 Integration

We have measure space $(\Omega, \mathscr{F}, \mu)$.

Let $f : \Omega \to \bar{\mathbb{R}} \textcircled{m} \mathscr{F} \setminus \bar{\mathscr{R}}$. We will keep these assumptions throughout this section.

First, assume that $f \geq 0$, and let $\mathscr{P}$ be the collection of all finite $\mathscr{F}$-partitions of Ω. That is, a memeber of $\mathscr{P}$ is $\{A_1, \ldots, A_k\}$ where $A_i \in \mathscr{F}$, disjoint, $\bigcup_{i=1}^{k} A_i = \Omega$.

$$\sup_{\{A_1,\ldots,A_k\}} \sum_{i=1}^{k} [\inf_{\omega \in A_i} f(\omega)] \mu(A_i) = \int f d\mu$$

which is the integral of f with respect to μ.

This is also written as

$$\int f(\omega) d\mu(\omega)$$

or

$$\int f(\omega) \mu(d\omega)$$

In this definition we adopt the following regarding positive infinity (∞). We pretend ∞ to be a finite positive number such that,

$$\infty * 0 = 0$$

$$0 * \infty = 0$$

$$\infty * x = \infty$$

$$x * \infty = \infty$$

$$\infty * \infty = \infty$$

Now, arbitrary function,

positive part

$$f^+(\omega) = \begin{matrix} f(\omega) & f(\omega) \geq 0 \\ 0 & f(\omega) \leq 0 \end{matrix}$$

negative part

$$f^-(\omega) = \begin{matrix} -f(\omega) & f(\omega) \leq 0 \\ 0 & f(\omega) \geq 0 \end{matrix}$$

So we have that

$$f = f^+ - f^-$$

$$|f| = f^+ + f^-$$

The integral of f is defined according to the follwoing 4 scenarios

1. For $\int f^+ d\mu < \infty, \int f^- d\mu < 0$,

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

2. For $\int f^+ d\mu < \infty, \int f^- d\mu = \infty$, we say that f is not integrable but has intagral

$$\int f d\mu = -\infty$$

3. For $\int f^+ d\mu = \infty, \int f^- d\mu < \infty$, we say that f is not integrable with repsect to μbut has integral

$$\int f d\mu = \infty$$

4. For $\int f^+ d\mu = \infty, \int f^- d\mu = \infty$, we say that f is not integrable with respect to μ.

Properties of Integrals for Nonnegative Functions

In the follwoing theroem,

$$f : \Omega \to \bar{\mathbb{R}} ⓜ \mathscr{F} \setminus \bar{\mathscr{R}}$$
$$g : \Omega \to \bar{\mathbb{R}} ⓜ \mathscr{F} \setminus \bar{\mathscr{R}}$$

Theorem 2.1.1 1. If $\{A_1, \ldots, A_k\}$ is an $\mathscr{F}$-partition of Ω and $f(\omega) = \sum_{i=1}^k x)kI_{A_i}$ then

$$\int f d\mu = \sum_{i=1}^k x_i \mu(A_i)$$

2. If $0 \leq f(\omega) \leq g(\omega) \forall \omega \in \Omega$ then

$$\int f d\mu \leq \int g d\mu$$

3. If $0 \leq f_n(\omega) \uparrow f(\omega) \forall \omega \in \Omega$ then

$$\int f_n d\mu \uparrow \int f d\mu$$

4. If $f \geq 0, g \geq 0, \alpha \geq 0, \beta \geq 0$ then

$$\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu$$

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Proof. 1. Let $\{B_j : j = 1, \ldots, m\} \in \mathscr{P}$.

$$\sum_{j=1}^{m} [\inf_{B_j} f(\omega)] \mu(B_j)$$

Recall that $f = \sum_{i=1}^{k} x_i I_{A_i}$

$$\begin{aligned}
\sum_{j=1}^{m} [\inf_{B_j} f(\omega)] \mu(B_j) &= \sum_{j=1}^{m} [\inf_{B_j} f(\omega)] \sum_{i=1}^{k} \mu(B_j A_i) \\
&= \sum_{j=1}^{m} \sum_{i=1}^{k} [\inf_{B_j} f(\omega)] \mu(B_j A_i) \\
&= \sum_{A_i B_j \neq \emptyset} [\inf_{B_j} f(\omega)] \mu(B_j A_i) \\
&\leq \sum_{A_i B_j \neq \emptyset} x_i \mu(B_j A_i) \\
&\leq \sum_{j=1}^{m} \sum_{i=1}^{k} x_i \mu(B_j A_i) \\
&= \sum_{i=1}^{k} x_i \sum_{j=1}^{m} \mu(B_j A_i) \\
&= \sum_{i=1}^{k} x_i \mu(A_i)
\end{aligned}$$

So for all $\{B_j : j = 1, \ldots, m\} \in \mathscr{P}$,

$$\sum_{j=1}^{m} [\inf_{B_j} f(\omega)] \mu(B_j) \leq \sum_{i=1}^{k} x_i \mu(A_i)$$

Similarly the supremum of the LHS is also less than or equal to the RHS so,

$$\int f d\mu \leq \sum_{i=1}^{k} x_i \mu(A_i)$$

Now for $\{A_1, \ldots, A_k\} \in \mathscr{P}$,

$$\inf_{A_i} f(\omega) = x_i$$

So we have

$$\sum_{\{B_j\} \in \mathscr{P}} \sum_{j=1}^{m} [\inf_{B_j} f(\omega)] \mu(B_j) \geq \sum_{i=1}^{k} x_i \mu(A_i)$$

Thus,

$$\int f d\mu \geq \sum_{i=1}^{k} x_i \mu(A_i)$$

2. Want $f \leq g \Rightarrow \int f d\mu \leq \int g d\mu$.

$$f = \sum_{\{A_1,\ldots,A)k\}\in\mathscr{P}} \sum_{i=1}^{k} [\inf_{A_i} f(\omega)]\mu(A_i)$$

$$\leq \sum_{\mathscr{P}} \sum_{i=1}^{k} [\inf_{A_i} g(\omega)]\mu(A_i) \leq \int g d\mu$$

3. Want $0 \leq f_n \uparrow f \Rightarrow \int f_n d\mu \uparrow \int f d\mu$.
Since we have that $f_n \leq f$ for all ω by (ii) we have,

$$\int f_n d\mu \leq \int f d\mu$$

$$\limsup_n \int f_n d\mu \leq \int f d\mu$$

Because $\int f_n d\mu \uparrow$ it has limit $\lim_n \int f_n d\mu \leq \int f d\mu$.
Now we must show

$$\lim_n \int f_n d\mu \geq \int f d\mu$$

This is implied by

$$\lim_n \int f_n d\mu \geq \sum_i [\inf_{A_i} f(\omega)]\mu_(A_i)$$

for all $A_i \in \mathscr{P}$.
Case 1 Supose that $\sum_i [\inf_{A_i} f(\omega)]\mu(A_i) < \infty$.

Assume that $[\inf_{A_i} f(\omega)] > 0$, $\mu(A_i) > 0, i = 1,\ldots,k$.

Hence, $\mu(A_i) < \infty, \forall i = 1,\ldots,k$.

For sufficiently small $\varepsilon > 0$, we may define

$$A_{ij} = \{\omega \in A_i : f_n(\omega) > \inf_{A_i} f(\omega) - \varepsilon\}$$

Because $f_n(\omega) \uparrow f(\omega)$, we have that $A_{in} \uparrow A_i$. Note that $\{A_{1n},\ldots,A_{kn},(\bigcup_{k=1}^{k} A_{in})^C\} \in \mathscr{P}$.

So now we have that

$$\begin{aligned}\int f_n d\mu &\geq \sum_i [\inf_{A_i} f_n(\omega)]\mu(A_i) + [\inf_{\bigcup_{i=1}^k A_{in}} f_n(\omega)]\mu(\bigcup_{i=1}^k A_{in}) \\ &\geq \sum_i [\inf_{A_{in}} f_n(\omega)]\mu(\bigcup_{i=1}^k A_{in}) \\ &\geq \sum_i [\inf_{A_i} f(\omega) - \varepsilon]\mu(A_{in}) \\ &\to \sum_i [\inf_{A_i} f(\omega) - \varepsilon]\mu(A_i)\end{aligned}$$

So

$$\lim_n \int f_n d\mu \geq \sum_i [\inf_{A_i} f(\omega) - \varepsilon]\mu(A_i) = \sum_i [\inf_{A_i} f(\omega)]\mu(A_i) - \varepsilon \sum_i \mu(A_i)$$

Because $\mu(A_i) < \infty$ for $i = 1, \ldots, k$ and we have that the above term is true for all ε we have that,

$$\lim_n \int f_n d\mu \geq \sum_i [\inf_{A_i} f(\omega)]\mu(A_i)$$

Case 2 Here,

$$\sum_i [\inf_{A_i} f(\omega)]\mu(A_i) < \infty$$

We don't assume that

$$[\inf_{A_i} f(\omega)] > 0, \mu(A_i) > 0, i = 1, \ldots, k$$

But any way we have

$$[\inf_{A_i} f(\omega)]\mu(A_i) < \infty$$

Without loss of generality we may assume for $i = 1, \ldots, k_0$ that

$$[\inf_{A_i} f(\omega)]\mu(A_i) > 0$$

and for $i = k_{0+1}, \ldots k$

$$[\inf_{A_i} f(\omega)]\mu(A_i) = 0$$

Frome here we may just apply argument for Case 1 to the first set to show the same inequality.

Case 3 $\sum_i [\inf_{A_i} f]\mu(A_i) = \infty$, $\{A_i\} \in \mathscr{P}$

Then

$$[\inf_{A_{i_0}} f]\mu(A_{i_0}) = \infty$$

for some $i_0 \in \{1,\dots,k\}$.

Supose that $0 < x < \inf_{\omega \in A_{i_0}} f(\omega) \le \infty$

$$- < y < \mu(A_{i_0}) \le \infty$$

Let $A_{i_{0n}} = \{\omega \in A_{i_0} : f_n(\omega) > x\}$ and note that $f_n \uparrow f, f_n(\omega) > x$ for sufficiently large n then,

$$A_{i_{0n}} \uparrow A_{i_0}$$

So we have that (by continuity),

$$\mu(A_{i_{0n}}) \to \mu(A_{i_0})$$

and that $\mu(A_{i_{0n}}) > y$ for large n.

Since $\{A_{i_{0n}}, A^C_{i_{0n}}\} \in \mathscr{P}$, we have that

$$\begin{aligned}\int f_n d\mu &\ge [\inf_{A_{i_{0n}}} f_n]\mu(A_{i_{0n}}) + [\inf_{A^C_{i_{0n}}} f_n]\mu(A^C_{i_{0n}})\\ &\ge [\inf_{A_{i_{0n}}} f_n]\mu(A_{i_{0n}})\\ &\ge xy\end{aligned}$$

So we have that

$$\lim_n \int f_n d\mu \ge xy$$

But since $[\inf_{A_{i_0}} f]\mu(A_{i_0}) = \infty$ we have that either

$$\begin{gathered}\inf_{A_{i_0}} f = \infty\\ \mu(A_{i_0}) = \infty\end{gathered}$$

If the first is equal to infinity we may choose x arbitrarily large. If the second is equal to infinity, we may choose y arbitrarily large. So either way (iii) holds.

Case 4 If $g \ge 0, f \ge 0, \alpha \ge 0, \beta \ge 0$ then

$$\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu$$

Suppose that f, g are simple, that is there eists $\{A_i\}, \{B_j\} \in \mathscr{P}$).

$$f = \sum_i x_i I_{A_i}$$

$$g = \sum_j y_j I_{B_j}$$

$$A_i = \bigcup_j A_i B_j$$

$$B_j = \bigcup_i A_i B_j$$

$$I_{A_i} = \sum_j I_{A_iB_j}$$

$$I_{B_j} = \sum_i I_{A_iB_j}$$

$$\begin{aligned}
\int (\alpha f + \beta g) d\mu &= \int (\alpha \sum_i x_i I_{A_i} + \beta \sum_j y_j I_{B_j}) d\mu \\
&= \int (\alpha \sum_i x_i \sum_j I_{A_iB_j} + \beta \sum_j y_j \sum_i I_{A_iB_j}) d\mu \\
&= \int (\sum_i \sum_j (\alpha x_i + \beta y_j) I_{A_iB_j}) d\mu \\
&= \sum_i \sum_j (\alpha x_i + \beta y_j) \mu(A_iB_j)) \\
&= \sum_i \sum_j (\alpha x_i \mu(A_iB_j) + \sum_i \sum_j (\beta y_j \mu(A_iB_j) \\
&= \alpha \sum_i x_i \mu(A_i) + \beta \sum_j y_j \mu(B_j) \\
&= \alpha \int f d\mu + \beta \int g d\mu
\end{aligned}$$

Note that we've used the first two steps in the three step argument collapsed into one. So now step three of the three step argument:

$$0 \le f ⓜ \mathscr{F} \setminus \bar{\mathscr{R}}$$

$$0 \le g ⓜ \mathscr{F} \setminus \bar{\mathscr{R}}$$

Then by Theorem 13.1 there exists $0 \le f_n \uparrow f, 0 \le g_n \uparrow g$ and by (iii) we have that

$$\int f_n \mu \to \int f d\mu, \int g_n \mu \to \int g d\mu$$

So

$$\alpha \int f_n d\mu + \beta \int g_n d\mu \to \alpha \int f d\mu + \beta \int g d\mu$$

But by step one and two,

$$\alpha \int f_n d\mu + \beta \int g_n d\mu = \int (\alpha f_n + \beta g_n) d\mu \to \int (\alpha f + \beta g) d\mu$$

■

Exercise 2.1 — Application of (i). Consider $(\mathbb{R}, \mathscr{R}, \lambda)$.

Suppose

$$-\infty < a_0 \le a_1 \le \cdots \le a_m < \infty$$

Let

$$f = \sum_{i=1}^{m} x_i I_{(a_{i-1},a_i]} + 0I_{(-\infty,a_0]} + 0I_{(a_m,\infty)}$$

But by (i) of the above Theorem, we have that

$$\begin{aligned}\int f d\mu &= \sum_{i=1}^{m} x_i \lambda(a_{i-1},a_i] + 0\lambda(-\infty,a_0] + 0\lambda(a_m,\infty) \\ &= \sum_{i=1}^{m} x_i(a_i - a_{i-1}) + 0 + 0\end{aligned}$$

Consider

$$f = \begin{cases} \infty & x = a \\ 0 & x \neq 0 \end{cases}$$

So then we have that

$$\inf f d\lambda = \infty * \lambda(\{a\}) * 0 * \lambda(\mathbb{R} \setminus \{a\}) = \infty * 0 + 0 * \infty = 0$$

$$f = I_{(a,\infty)}$$

By (i),

$$\int f d\lambda = \infty$$

Almost Surely

Now we move on to the concept of almost surely with respect to the measure μ.

$(\Omega, \mathscr{F}, \mu)$ is our measure space.

$$F : \Omega \to \bar{\mathbb{R}}, ⓜ\mathscr{F} \setminus \bar{\mathscr{R}}$$

Definition 2.1.1 — Almost Surely. Let $B \in \bar{\mathscr{R}}$.

We say that $f \in B a.s. \mu$ if

$$\mu\{\omega : f(\omega) \notin B\} = 0$$

Where we me rewrite above as

$$\mu f^{-1}(B^C) = 0$$

So the we may define almost surely with

$$f \in B a.s. \mu \Leftrightarrow \mu f^{-1}(B^C) = 0$$

Recall that in the definition of integral we requred f to be measureable, but we did not explicitely use it. Actually we don't always need it!

The original definition of an integral involves a 'lower' and 'upper' integral. But when f is measureable, these are equal.

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Theorem 2.1.2 Suppose $f, g, \geq 0$ and are measruable on $\mathscr{F} \setminus \bar{\mathscr{R}}$.

1. If f = 0 almost everywhere (a. e.) μ then
$$\int f d\mu = 0$$
2. If $\mu\{\omega : f(\omega) > 0\} > 0$, then
$$\int f d\mu > 0$$
3. If $\int f d\mu < \infty$, then
$$f < \infty a.e.\mu$$
4. If $f \leq g a.e.\mu$ then,
$$\int f d\mu \leq \int g d\mu$$
5. If $f = g a.e.$ then
$$\int f d\mu = \int g d\mu$$

Proof. 1. Want f = 0 a.e. $\mu \Rightarrow \int f d\mu = 0$.

Let $\mathscr{A}_0 = \{\omega : f(\omega) = 0\}$. For and $\{A_i\} \in \mathscr{P}$

$$\begin{aligned}\sum_i (\inf_{A_i} f(\omega))\mu(A_i) = \sum_i (\inf_{A_i} f(\omega))(\mu(A_iA_iA_0) + \mu(A_iA_0^C)) \\ \leq \sum_i (\int_{A_iA_0} f(\omega))\mu(A_iA_0) \\ = 0\end{aligned}$$

So we have that

$$\int f d\mu = \sup_{\{A_i\}\in\mathscr{P}} \sum_i (\int_{A_iA_0} f(\omega))\mu(A_iA_0) = 0$$

2. Want that $\mu(\{\omega : f(\omega) > 0\} > 0 \Rightarrow \int f d\mu > 0$

Let $A_\varepsilon = \{\omega : f(\omega) \geq \varepsilon\}$ as $\varepsilon \downarrow 0$ such that

$$A_\varepsilon \uparrow \{\omega : f(\omega) > 0\}$$

By the continuity of probability,

$$\mu(A_\varepsilon) \to \mu(\omega : f > 0)$$

where $\mu(A_\varepsilon) > 0$ for sufficiently small ε.
So we have that

$$\{A_\varepsilon, A_\varepsilon^C\} \in \mathscr{P}$$

$$\int f d\mu \geq \{\inf_{A_\varepsilon} f\}\mu(A_\varepsilon) + (\inf_{A_\varepsilon^C} f)\mu(A_\varepsilon^C) > 0$$

Not that the first part of the first term is ε while the second part is greater than 0 and the second term is equal to zero.

3. Want that $\int f d\mu < \infty \Rightarrow f < \infty a.e.$. And that this is interchangeable with $(f < \infty a.e.)^C \Rightarrow \int f d\mu = \infty$.

   Also that $f < \infty a.e.$ not true implies that $\mu(f = \infty) > 0$.

   Let $A = \{\omega : f(\omega)\}, \{A, A^C\} \in \mathscr{P}$.

$$\int f d\mu \geq (\inf_A f)\mu(A) = \infty * (> 0) = \infty$$

4. If f < g a.e. then $\int f d\mu \leq \int g d\mu$.

   Let $G = \{\omega : f(\omega) \geq g(\omega)\}, \mu(G^C) = 0$.

   Also, let $\{A_i\} \in \mathscr{P}$

$$\begin{aligned}
\sum_i (\inf_{A_i})\mu(A_i) &= \sum(\inf_{A_i} f)\mu(A_i G) \\
&\leq \sum(\inf_{A_i G} f)\mu(A_i G) \\
&\leq (\sum \inf_{A_i} g)\mu(A_i G) \\
&\leq \sum(\inf_{A_i} g)\mu(A_i G) + (\inf_{G^C} g)\mu(G^C) \\
&\leq \int g d\mu
\end{aligned}$$

   But note that $\{A_i G, \ldots, A_m G, G^C\} \in \mathscr{P}$ and the above is true for all $\{A_i\} \in \mathscr{P}$ and altogehter this implies that

$$\int f d\mu \leq \int g d\mu$$

5. Want to show that $f = g a.e. \Rightarrow \int f d\mu = \int g d\mu$.

$$f = g a.e. \Rightarrow \begin{array}{c} f \leq g a.e. \to \int f d\mu \leq \int g d\mu \\ g \leq g a.e. \to \int g d\mu \leq \int f d\mu \end{array} \Rightarrow \int f d\mu = \int g d\mu$$

■

2.2 Properties of Integral in General Case

General case refers to g and f not being negative.

If f has a definite integral (has an integral) and f = g a.e. μ, then $\int f d\mu = \int g d\mu$ (that is, not negative).

Proof. We note that f = g a.e. $\mu \Rightarrow \begin{cases} f^+ = g^+ \\ f^- = g^- \end{cases}$
So, by part 5 of the previous theorem,

$$\int f^+ d\mu = \int g^+ d\mu$$

$$\int f^- d\mu = \int g^- d\mu$$

By having definite integral, we can take the difference.

$$\int f^+ d\mu - \int f^- d\mu = \int g^+ d\mu - \int g^- d\mu$$

$$\int f d\mu = \int g d\mu$$

■

Equalities and Inequalities

First note that $\int |f| d\mu = \int f^+ d\mu + \int f^- d\mu =$ by linearity and nonnegative case.

Also note that f integrable with respect to μ (f ⓢ μ) means that $\int f^+ d\mu < \infty, \int f^- d\mu < \infty$.

$$f \text{ ⓢ } \mu \Leftrightarrow \int |f| d\mu < \infty$$

So if f is integrable and $|g| < |f|$ a.e. then g is integrable because

$$\int |g| d\mu \leq \int |f| d\mu < \infty$$

In particular if f is bounded ($|f| \leq M \forall \omega \in \Omega$), $\mu(\Omega) < \infty$ then f ⓢ μ.

$$\int |f| d\mu \leq \int M d\mu = M\mu(\Omega) \leq \infty$$

Theorem 2.2.1
1. Monotonicity

If f, g ⓜ μ and $f \leq g$ then

$$\int f d\mu \leq \int g d\mu$$

2. *Linearity*
If $\alpha \in \mathbb{R}, \beta \in \mathbb{R}$; f, g ⓜ μ and we have that $\alpha f + \beta g$ ⓜ μ then

$$\int (\alpha f + \beta g) d\mu = \alpha \int f d\mu + \beta \int g d\mu$$

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Proof. 1. Monotonicity

We know that

$$f = f^+ - f^-$$
$$g = g^+ - g^-$$

so if we know that $f \leq g$ we have that

$$f^+ - f^- \leq g^+ - g^- \Leftrightarrow (f^+ - g^+) - (f^- - g^-) \leq 0$$

We also can easily see that both $(f^+ - g^+), (f^- - g^-) \geq 0$. Other results include

$$f^+ \leq g^+ a.e.$$
$$f^- \geq g^- a.e.$$

and

$$\int f^+ d\mu \leq \int g^+ d\mu$$
$$-\int f^- d\mu \leq -\int g^- d\mu$$
$$\int f^+ d\mu - \int f^- d\mu \leq \int g^+ d\mu - \int g^- d\mu$$

so we have our result that

$$\int f \leq \int g$$

2. *Monotonicity*

First we need that $\alpha f + \beta g ⓜ \mu$.

$$\begin{aligned}\int |\alpha f + \beta g| d\mu &\leq \int |\alpha||f| + |\beta||g| d\mu \\ &\leq |\alpha| \int |f| d\mu + |\beta| \int |g| d\mu \\ &< \infty\end{aligned}$$

Claim: $\int \alpha f d\mu = \alpha \int f d\mu$

$$\int \alpha f d\mu = \int (\alpha f)^+ d\mu - \int (\alpha f)^- d\mu$$

When $\alpha \geq 0$,

$$(\alpha f)^+ = \alpha f^+$$
$$(\alpha f)^- = \alpha f^-$$

So we have that

$$\begin{aligned}\int(\alpha f)^{+}d\mu-\int(\alpha f)^{-}d\mu&=\int\alpha f^{+}d\mu-\int\alpha f^{-}d\mu\\&=\alpha\int f^{+}d\mu-\alpha\int f^{-}d\mu\\&=\alpha(\int f^{+}d\mu-\int f^{-}d\mu)\\&=\alpha\int fd\mu\end{aligned}$$

But, when $\alpha<0$,

$$(\alpha f^{+})=\begin{matrix}\alpha f(\omega) & \alpha f(\omega)\geq 0\\ 0 & \alpha f(\omega)<0\end{matrix}$$

When $\alpha f(\omega)\geq 0$ we have that $f(\omega)\leq 0$

$$f(\omega)=-f^{-}(\omega)$$

$$\alpha f(\omega)=-\alpha f^{-}(\omega)$$

If $\alpha f(\omega)<0$, then $f(\omega)\geq 0$ so we have

$$f^{-}(\omega)=0$$

So

$$(\alpha f)^{-}=0=-\alpha f^{-}(\omega)$$

$$(\alpha f)=-\alpha f^{-}$$

Similarly, when $\alpha<0$,

$$(\alpha f)^{-}=-\alpha f^{+}$$

So

$$\begin{aligned}\int\alpha fd\mu=\int(\alpha f)^{+}d\mu=\int(\alpha f)^{-}d\mu&\\&=\int(-\alpha)f^{-}d\mu-\int(-\alpha)f^{+}d\mu\\&=\alpha\int f^{+}d\mu-\alpha\int(f)^{-}d\mu\\&=\alpha\int fd\mu\end{aligned}$$

Now we have that

$$\alpha f+\beta g=(\alpha f+\beta g)^{+}-(\alpha f+\beta g)^{-}$$

and also that

$$\alpha f+\beta g=(\alpha f)^{+}-(\alpha f)^{-}+(\beta g)^{+}-(\beta g)^{-}$$

Which means that

$$(\alpha f+\beta g)^{+}-(\alpha f+\beta g)^{-}=(\alpha f)^{+}-(\alpha f)^{-}+(\beta g)^{+}-(\beta g)^{-}$$

Rewriting and using linearity in nonnegative cases we get

$$\int(\alpha f+\beta)^{+}d\mu+\int(\alpha f)^{-}d\mu+\int(\beta g)^{-}d\mu=\int(\alpha f+\beta)^{-}+\int(\alpha f)^{+}d\mu+\int(\beta g)^{+}$$

■

A special case is necessary because $-|f|\leq f\leq|f|$ so we have that

$$-\int|f|d\mu\leq\int fd\mu\leq\int|f|d\mu$$

So we have that $|\int fd\mu|\leq\int|f|d\mu$ if f is measureable on μ.

A second special case is neede for f and g measureable on μ for

$$|\int f-gd\mu|\leq\int|f-g|d\mu$$

or when

$$|\int fd\mu-\int gd\mu|\leq\int|f-g|d\mu$$

For this next theorem, recall Theorem 15.1 (iii) where if

$$0\leq f_n\uparrow f\forall\omega\in\Omega$$

then this gives us that

$$\int f_n d\mu\uparrow\int fd\mu$$

Now we relax this theotem to almost everywhere.

Theorem 2.2.2 — Monotone Convergence Theorem. If $0\leq f_n\uparrow f$ a.e. μ then we have that

$$\int f_n d\mu\uparrow\int fd\mu$$

Proof. Let $A=\{\omega\in\Omega:0\leq f_n(\omega)\uparrow f(\omega)\}$ and we have that $\mu(A^C)=0$.
Then $f_nI_A\uparrow fI_A$ everywhere. So by Theorem 15.1 (iii),

$$\int f_nI_Ad\mu\uparrow\int fI_Ad\mu$$

But $f_nI_A=f_n$ a.e. μ. Similarly for $fI_A=f$ which implies that

$$\int f_n d\mu\uparrow\int fd\mu$$

■

Fatou's Lemma

Theorem 2.2.3 If $f_n \geq 0, f_n$ is measureable $\mathscr{F} \setminus \mathscr{R}$ then

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu$$

Proof. This is a direct consequence of MCT.

Let $g_n = \inf_{k \geq n} f_n$.

$$g = \lim_{n\to\infty} \inf_{k\geq n} f_n = \liminf_n f_n$$

By definition,

$$0 \leq g_n \uparrow g$$

$$\int g_n d\mu \to \int g d\mu = \int \liminf_n f_n d\mu$$

We also have that $f_n \geq \inf_{k\geq n} f_n = g_n$ so

$$\int f_n d\mu \geq \int g_n d\mu$$

$$\liminf_n \int f_n d\mu \geq \liminf_n \int g_n d\mu$$
$$\liminf_n \int f_n d\mu \geq \int \liminf_n f_n d\mu$$

■

You may wonder if

$$\liminf_n \int f_n d\mu \geq \liminf_n \int g_n d\mu$$

but not generally grue. But true if LHS is finite.
Lebesgue's Dominated Convergence Theorem

■ **Example 2.1** Consider $(\mathbb{R}, \mathscr{R}, \lambda)$.

$$f_n = n^2 I_{(0,\frac{1}{n})}$$

For all $\omega \in \mathbb{R}$, $f_n(\omega) \to 0$.

So we have that $\lim f_n = 0$ for all ω.

$$f(\omega) = \lim_n f_n(\omega)$$
$$\int f(\omega) d\lambda = \int 0 d\lambda = 0$$
$$\lim_n \int f(\omega) d\lambda = \lim_n \int_0^{1/n} n^2 d\lambda$$
$$= \lim_n \frac{1}{n} n^2 = \lim_n n = \infty$$

■

Wednesday November 14

Theorem 2.2.4 If $|f_n| \leq g$, a.e. μ, g is measurable on μ, and if $f_n \to f$ a.e. μ then both f and f_n are measurable μ and

$$\int f_n d\mu \to \int f d\mu$$

Proof. Since $|f_n| \leq g$ a.e. μ we have that

1. f_n measureable on μ for all n because

$$\int |f_n| d\mu \leq \int g d\mu < \infty$$

2. $\limsup_n f_n$ is measureable μ because we have that

$$|\limsup_n f_n| \leq \limsup_n |f_n| \leq g$$

So we have that

$$\int |\limsup_n f_n| d\mu \leq \int g d\mu$$

3. $\liminf_n$ is measurable μ because

$$\begin{aligned} \liminf_n f_n \textcircled{m} \mu &\Leftrightarrow -\liminf_n -f_n \textcircled{m} \mu \\ &\Leftrightarrow \limsup_n (-f_n) \textcircled{m} \mu \end{aligned}$$

But

$$|\limsup_n (-f_n)| \leq \limsup_n |-f_n| = \limsup_n |f_n| \leq g$$

Because we have that $|f_n| \leq g$ (this is wehre we used dominating condition) we have that

$$g - f_n \geq 0$$
$$g + f_n \geq 0$$

Applying Fatou's Theorem to both,

$$\int \liminf_n (g + f_n) d\mu \leq \liminf_n \int (g + f_n) d\mu$$

With the LHS we get that

$$\int g + \liminf_n f_n d\mu = \int g d\mu + \int \liminf_n f_n d\mu$$

by linearity of integrable version, this is where we use $\liminf_n f$ measurabe μ.

With the RHS this term is equal to

$$\liminf_n (\int g d\mu + \int f_n d\mu)$$

by the measurability version of linearity as we know both f_n and g are measurable so this is equal to

$$\int g d\mu + \liminf_n \int f_n d\mu$$

Equating the two results we have that

$$\int g d\mu + \int \liminf_n f_n d\mu = \int g d\mu + \liminf_n \int f_n d\mu$$

because y is also measurable on μ you conclude that

$$\liminf_n f_n d\mu = \liminf_n \int f_n d\mu$$

Apply Fatou in this same manner to $g - f_n$ and through the same argument we get that

$$\liminf_n f_n d\mu = \liminf_n \int f_n d\mu$$

$$\begin{aligned} \int \liminf_n f_n d\mu &\leq \liminf_n \int f_n d\mu \\ &\leq \limsup_n \int f_n d\mu \\ &\leq \int \limsup_n f_n d\mu \end{aligned}$$

But we assume that $f_n \to f$ a. e. μ. Therefor this means that

$$\limsup_n f_n = \liminf_n f_n \text{ a.e. } \mu$$

and the above inequalities are all now equal.
Which may be extended to

$$\int \limsup_n f_n d\mu = \int \limsup_n f_n d\mu = \int \lim_n f_n d\mu$$

Therefore,

$$\lim_n \int f_n d\mu = \int \lim_n f_n d\mu$$

■

■ **Example 2.2** Let Ω be a countable set and without loss of generality assume that

$$\Omega = \{1, 2, \dots\}$$

Let $\mathscr{F} = 2^\Omega$ and $\kappa : 2^\Omega \to \mathbb{N}$.

$$A \in 2^\Omega \mapsto \#(A)$$

Where # means that if A is finite $\#(A)$ is equal to the number of elements in A, otherwise it is equal to ∞.

It can be shown (not hard) that κ is a measure and it is called the counting measure.

For

$$(\Omega, 2^{\Omega}, \kappa)$$

what does $\int f d\kappa$ mean?

A function on Ω can be written as a sequence

$$f(1), f(2), \cdots = \{x_1, x_2, \ldots x_m, \ldots\}$$

We may define a truncated sequence,

$$x_{nm} = \begin{cases} x_m & m \leq n \\ 0 & m > 0 \end{cases}$$

then $f_n(m) = x_{nm}$ is a simple function because $\{\{1\}, \ldots, \{n\}, \ldots\}$ is a finite $\mathscr{F}$-partition (in $\mathscr{P}$).

So we have that

$$\int f_n d\mu = \sum_{m=1}^{n} x_{nm}\kappa(\{m\}) + 0 * k(\{n+1, \ldots\}) = \sum_{m=1}^{n} x_{nm}$$

Now assume that $x_m \geq 0$ first. Then by MCT we have

$$0 \leq f_n \uparrow f$$

where $f = \sum_{m=1}^{\infty} x_m$ and this implies that

$$\lim_n \int f_n d\kappa = \int f d\kappa$$

So we have that

$$\lim_n \sum_{m=1}^{n} x_m \triangleq \sum_{m=1}^{\infty} x_m$$

So, not suprisingly,

$$\int f d\mu = \sum_{m=1}^{\infty} x_m$$

Now, suppose that f is measurable on μ so that $\int |f| d\kappa < \infty$ so we have that

$$\sum^{i} nfty_{m=1} |x_m| < \infty$$

Then

$$\int f d\kappa = \int f^{+} d\kappa - \int f^{-} d\kappa = \sum^{\infty} x_m^{+} - \sum^{\infty} x_m^{-}$$

So this is how to integrate with respect to counting measure.
f ⓜ μ if and only if $\sum x_m^{+} < \infty, \sum x_m^{-} < \infty$
It has a definite integrable if at least one is finite. And it does not have definite integral and if both are infinite.

When can you exchange integral and limit? That is, what is the special form of DCT in this setting?

Suppose for each $n = 1,2,\dots, \{x_{nm} : m = 1,2,\dots\}$ is a sequence (not necessarily a function, so we have a sequence of sequences) such that

1. $|x_{nm}| \le M_m$

$$\sum_{m=1}^{\infty} M_m < \infty$$

2. $\lim_{n\to\infty} x_{nm} = x_m$ for $m = 1,\dots,M$

Let

$$f : \Omega \to \mathbb{R}, m \mapsto x_m$$
$$f_n : \Omega \to \mathbb{R}, m \mapsto x_{nm}$$
$$g : \Omega \to \mathbb{R}, m \mapsto M_m$$

Then, the first item above becomes

$$|f_n| \le g, \int g d\kappa < \infty$$

and the second becomes

$$f_n \to f$$

So by DCT,

$$\int f_n d\kappa \to \int f d\kappa$$

that is that we have

$$\lim_n \sum^{\infty} x_{nm} = \sum^{\infty} x - m = \sum^{\infty} \lim_n x_{nm}$$

This is called Weierstass M-test in calculus.

■

Wednesday November 16

Theorem 2.2.5 Suppose $(\Omega, \mathscr{F}, \mu)$ is a measure space.

$f : \Omega \to \mathbb{R} ⓜ \mathscr{F} \setminus \mathscr{R}$ is bounded, that is $|f(\omega)| \le M$ a.e. μ.

Suppose $\mu(\Omega) < \infty$ and that $f_n \to f$ a.e. μ for some $f : \Omega \to \mathbb{R}$ measurable $\mathscr{F} \setminus \mathscr{R}$. Then we have that

$$\int f_n d\mu \to \int f d\mu$$

Proof. In the DCT, take $g \equiv M$, then $|f| \le g$a.s. μ and $f_n \to f$ a.e. μ.

By DCT

$$\int f_n d\mu \to \int f d\mu$$

■

These theorems (MCT, DCT) can also be stated in series form.

Theorem 2.2.6 — 16.6 Series Version of MCT. If $f_n \geq 0(f_n ⓜ \mathscr{F} \setminus \bar{\mathscr{R}}$ then we have that

$$\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu$$

Proof. Let $g_n = \sum_{m=1}^{n} f_m, g = \sum_{m=1}^{\infty} f_m$. Then we have that $0 \leq g_n \uparrow g$. Using the MCT,

$$\lim_n \int g_n d\mu = \int \lim_n g_n d\mu$$

But we know that

$$\begin{aligned} \lim_n g_n &= \lim_n \sum_{m=1}^{n} f_m \\ &= \sum_{m=1}^{\infty} f \end{aligned}$$

And also that

$$\begin{aligned} \lim_n \int g_n d\mu &= \lim_n \int \sum_{m-1}^{n} f_m d\mu \\ &= \lim_n \sum_{m=1}^{n} \int f_m d\mu \\ &= \sum_{m=1}^{\infty} \int f_m d\mu \end{aligned}$$

■

Theorem 2.2.7 — 16.7 Series version of DLT. If $\sum_{n=1}^{\infty} f_n$ converges almost everywhere μ (in calculus, this means that the sequence $\{\sum_{m=1}^{n} f_m : n = 1,2,\dots\}$ converges) and $|\sum_{m=1}^{n} f_m| \leq g$ almost everywhere μ and also g integrable with respect to μ then, both f_n and $\sum_{n=1}^{\infty} f_n$ are integrable with respect to μ and we have that

$$\int \sum_n f_n d\mu = \sum_n \int f_n d\mu$$

Proof. Let $\mathscr{L}_n = \sum_{m=1}^{n} f_m$ and $\mathscr{L} = \sum_{m=1}^{\infty} f_m$. Then just need to use DCT. ■

Corollary 2.2.8 If $\sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$ then $\sum_{n=1}^{\infty} f_n$ converges absolutely, almost everywhere μ and is integrable with respect to μ. Also,

$$\sum_{n=1}^{\infty} \int f_n d\mu = \int \sum_{n=1}^{\infty} f_n d\mu$$

Point of this corollary is that it gives you a dominating function.

Proof. Let $g = \sum_{n=1}^{\infty} |f_n|$. Then by Theorem 16.6 (MCT for series) we have that

$$\int \sum_{n=1}^{\infty} |f_n| d\mu = \sum_{n=1}^{\infty} \int |f_n| d\mu < \infty$$

so therefor g is integrable with respect to μ.

Since $\sum_n |f_n| \geq 0$ we know that

$$\int \sum_n |f_n| d\mu < \infty$$

Which means that the integrand $(\sum_n |f_n|)$ is less than ∞ almost everywhere μ and so we know that

$$g < \infty a.e. \mu$$

Because (see remark below) $\sum_n |f_n| < \infty$ a.e. μ we know that $\sum_n f_n$ converges a. e. μ. So we have that

1. $\sum_n f_n$ converges a.e. mu
2. $|\sum_{k=1}^{n} f_k| \leq g$ a.e. μ, thus g is integrable with respect to μ.

By Theorem 16.7

$$\int \sum_n f_n d\mu = \sum_n \int f_n d\mu$$

■

R [Absolute Convergence] A series, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n| < \infty$.

Claim: if $\sum_{n=1}^{\infty} |a_n| < \infty$ then $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof. It suffices to show that $\{\sum_{m=1}^{n} a_m : n = 1, \dots\}$ is a Cauchy sequence. That is for all $\varepsilon < 0$ there exists n_0 such that for all $n \geq m \geq n_0$

$$|\sum_{k=m}^{n} a_m| < \varepsilon$$

but since $\sum |a_n| < \infty$ we have that

$$\{\sum_{m=1}^{n} a_m : n = 1, dots\}$$

is indeed Cauchy.

So we have that

$$|\sum_{k=m}^{n} a_m| \leq \sum_{k=m}^{n} |a_m| < \varepsilon$$

So $\{\sum_{k=m}^{n} a_m\}$ is Cauchy. ■

Now consider the case where

$$\{f_t(\omega) : t > 0\}$$

$$f_t : \Omega \to \bar{\mathbb{R}}, \mathscr{F} \setminus \bar{\mathscr{R}}$$

$$f : \Omega \to \bar{\mathbb{R}} ⓜ \mathscr{F} \setminus \bar{\mathscr{R}}$$

Suppose

$$\lim_{t \to t_0} f_t(\omega) = f(\omega)$$

This means that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $|t - t_0| < \delta$ we have that $|f(t) - f(t_0)| < \varepsilon$.

Consider the set

$$\begin{aligned}\{\omega : f_t(\omega) \to f_{t_0}(\omega)\} &= \{\omega : \forall \varepsilon > 0, \exists \delta > 0, s.t. \forall |t - t_0| < \delta, |f_t(\omega) - f_{t_0}(\omega)| < \varepsilon\} \\ &= \bigcap_{\varepsilon > 0} \bigcup_{\delta > 0} \bigcap_{|t - t_0| < \delta} \{\omega : |f_t(\omega) - f_{t_0}(\omega)| < \varepsilon\}\end{aligned}$$

All of this is just to say that the inner most intersection above need not be measureable.

So this means that extra care ahs to be taken when we say that

$$f_t \to f_{t_0} a.e. \mu$$

because $\{\omega : f_t(\omega) - f_{t_0}(\omega)\}$ may not be in $\mathscr{F}$.

BUT the following is true:

$$f(t) \to f(t_0)$$

if and only iff for any sequence $t_n \to t_0$ we have that $f(t_n) \to f(t_0)$.

Theorem 2.2.9 — 16.8. Suppose that

$$f(\cdot, t) : \Omega \to \bar{\mathbb{R}} ⓜ \mathscr{F} \setminus \bar{\mathscr{R}}$$

and that f is integrable with respect to μ for all $t \in (a, b)$.

Let $\phi(t) = \int f(\omega, t) \mu(d\omega)$.

1. Suppose that there exists $A \in \mathscr{F}$ with $\mu(A^C) = 0$ such that for all $\omega \in A$ we have that $f(\omega, t)$ is continuous at t_0.

   Also suppose that there exists $\delta > 0$ such that

   $$g : \Omega \to \bar{\mathbb{R}} ⓜ \mathscr{F} \setminus \bar{\mathscr{R}}$$

   is integrable w.r.t. μ such that

$$|f(\omega,t)| \le g(\omega)$$

for $\omega \in A$ and $|t-t_0| < \delta$ then $\phi(t)$ is continuous at t_0.

Friday November 18

Proof. 1. **Monday November 28**

Since we also know (by assumption) that

$$\int_{B_nC_n} f - g d\mu \le 0$$

We have that

$$\int I_{B_nC_n}(f-g)d\mu = 0$$

But we know that

$$I_{B_nC_n}(f-g) \ge 0 \Rightarrow I_{B_nC_n}(f-g) = 0(a.e.)$$

Claim: if $h \ne 0$ on A and $I_A h = 0$ a.e. μ then $\mu(A) = 0$.
Proof of Claim:
$I_A h = 0$ a.e. $\mu \Rightarrow \mu(I_A h \ne 0) = 0$.

Since $h \ne 0$ on A then $\omega \in A \Rightarrow h(\omega) \ne 0$.

So

$$\omega \in A \Rightarrow I_A h = \ne 0$$

$$I_A \ne 0 \Rightarrow I_A h \ne 0$$

$$\{I_A \ne 0\} \subseteq \{I_A h \ne 0\}$$

$$mu(I_A h \ne 0) = 0 \Rightarrow \mu(I_A \ne 0) = 0 \Rightarrow \mu(A) = 0$$

Claim is proved.
We know that $f < g$ on B_nC_n. So if we let "A" above be B_nC_n and "h" be (f-g) we have that

$$I_{B_nC_n}(f-g) = 0(a.e.)$$
$$\mu(B_nC_n) = 0$$

So

$$\begin{aligned}\mu(0 \le g < f) &= \mu(0 \le g < f, g < \infty)\\ &= \mu(\bigcup_n \{0 \le g < f, g \le n\})\\ &\le \sum_n \mu(C_n) = 0\end{aligned}$$

Which show that $f \leq g$ a.e. μ.

Thus we have shown that

$$\int_A f d\mu \leq \int_a g d\mu \forall A \in \mathscr{F}$$

which implies that $f \leq q$ a.e. μ.

So if

$$\int_A f d\mu = \int_A g d\mu \forall A \in \mathscr{F}$$

Then this implies that

$$\left\{ \begin{array}{l} \int_A f d\mu \leq \int_A g d\mu \forall A \Rightarrow f \leq g a.e. \\ \int_A f d\mu \geq \int_A g d\mu \forall A \Rightarrow g \leq f a.e. \end{array} \right\} \Rightarrow f = g a.e.$$

2. If f and g are integrable w.r.t μ and $\int_A f d\mu = \int_A g d\mu \forall A$ then f = g a. e. μ. NB: No σ-finite on μ.

   Again, suffices to show that

$$\int_A f d\mu \leq \int_A g d\mu \forall A \Rightarrow f \leq g a.e.\mu$$

$$\Rightarrow \int_A f d\mu - \int_A g d\mu \leq 0$$

   Note that we can only do this because both are finite.

   By nonegative version of linearity

$$\int_A f - g d\mu \leq 0 \forall A \in \mathscr{F} \Rightarrow \int_{f>g} f - g d\mu \leq 0$$

$$\Rightarrow \int I_{f>g}(f-g) d\mu \leq 0$$

   So we have that

$$\int I_{f>g}(f-g) d\mu = 0 \Rightarrow I_{f-g}(f-g) = 0 a.e.\mu$$

   By the claim proved above (here we will let h = f - g and A = {f > g}) we have that

$$\mu(f > g) = 0 \Rightarrow f \leq g a.e.\mu$$

   Hence:

$$\int_A f d\mu = \int_A g d\mu \forall A \in \mathscr{F} \Rightarrow f = g a.e.\mu$$

3. Here in this generalization of (ii), we want to show that if f, g are integrable on μ and

$$\int_A f d\mu = \int_A g d\mu \forall A \in \mathscr{P}$$

   where $\mathscr{P}$ is a π-system generated by $\mathscr{F}$ and Ω is a finte or countalbe union of $\mathscr{P}$-sets. Then we have that f = g a. e. μ

Step 1. Suppose $f \geq 0, g \geq 0$. Recall Theorem 10.4, suppose that μ_1, μ_2 are finite measures on $\sigma(\mathscr{P})$ and Ω is finte on countable union of σ-sets. If μ_1, μ_2 agree on $\mathscr{P}$ then they agree on $\sigma(\mathscr{P})$.

Let,

$\nu_1(A) = \int_A f d\mu$
$\nu_2(A_= \int_A g d\mu$

Claim: ν_1 (and also ν_2) is a finite measure on $(\Omega, \mathscr{F})$.

(a) $\nu_1(\emptyset) = \int_\emptyset f d\mu = \int f I_\emptyset d\mu = 0$
(b) $\nu_1(A) = \int_A f d\mu \geq 0$
(c) Let $A_1, \ldots$ be disjoint $\mathscr{F}$-sets.

$$\begin{aligned} \nu_1(\bigcup_n A_n) &= \int_{\bigcup_n A_n} d\mu \\ &= \sum_n \int_{A_n} d\mu \\ &= \sum_n \nu_1(A_n) \end{aligned}$$

(d) $\nu_a(\Omega) = \int_\Omega f d\mu < \infty$
So ν_1, ν_2 are finite meausures on $(\Omega, \mathscr{F})$ and by Theorem 10. 4 we know that $\nu_1 = \nu_2 \forall A \in \mathscr{F}$.

Step 2. Suppose only that f integrable, g integrable w.r.t. μ.
Suppose that

$$\int_A f d\mu = \int_A g d\mu \forall A$$

Because every term (can split into $f^+ f^-$) is finite we can move them around within this equation.

$$\int_A f^+ d\mu + \int_A g^- d\mu = \int_A g^+ d\mu + \int_A f^- d\mu$$

By linearity,

$$\int_A f^+ + g^- d\mu = \int_A g^+ + f^- d\mu$$

Using Step 1 and some more reordering of the equation, we may conclude that

$$f^+ - f^- = g^+ - g^- a.e. \mu$$

■

R Central takeaway of last proof are the following statements

$$\int_A f d\mu \leq \int_A g d\mu$$

$$\int_A f - g d\mu \leq 0 \forall A$$

$$\int_{f>g} f - g d\mu \leq 0$$

Density

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space where $\delta \geq 0$ is a measureable funciton.

By the proof of the last theorem we know that

$$A \mapsto \int_A \delta d\mu$$

is a measure. Alost that if δ' is another funciton such that

$$\int_A \delta' d\mu = \int_A d d\mu \forall A$$

then $\delta = \delta'$ a.e. μ.

If μ, ν are measurable on $(\Omega, \mathscr{F})$ then there exists $\delta \geq 0$ such that

$$\nu(A) = \int_A \delta d\mu$$

then δ is called the density of ν with respect to μ.
Wednesday November 30

Note that for the equation

$$\int_A d\nu = \int_A \delta d\mu$$

we may use the notation,

$$d\nu = \delta d\mu$$

.

Theorem 2.2.10 Suppose that ν has density δ with respect to μ. Then

1. (Nonnegative Version) For any $f \geq 0$,

$$\int f d\nu = \int f \delta d\mu$$

2. (Integrable Version)

$$f Ⓢ \nu \Leftrightarrow f \delta Ⓢ \mu$$

in which case,

$$\int f d\nu = \int f \delta d\mu$$

Proof. Proof is three step argument. "If an equation holds for all indecator functions then it holds for all functions".

1. Step 1.

   Let $f = I_A, A \in \mathscr{F}$.

   Then
   $$\begin{aligned}\int f d\nu &= \int I_A d\nu \\ &= \int_A d\nu \\ &= \nu(A) \\ &= \int_A \delta d\mu \\ &= \int I_A \delta d\mu \\ &= \int f \delta d\mu\end{aligned}$$
   Step 2. $f \geq 0$ f is simple. Then

   $$f = \sum_{i=1}^{k} x_i I_{A_i}$$

   where
   $A_1, \ldots, A_k \in \mathscr{P}$
   $x_i \geq 0$

   $$\begin{aligned}\int f d\nu &= \int (\sum_i x_i I_{A_i}) d\nu \\ &= \sum_i x_i \int_{A_i} d\nu \\ &= \sum_i x_i \int_{A_i} \delta d\mu \\ &= \int \sum_i x_i I_{A_i} \delta d\mu \\ &= \int f \delta d\mu\end{aligned}$$
   Step 3. Let $f \geq 0$. Then there exists $0 \leq f_n \uparrow f$ where f_n is simple. Using the MCT,
   $$\begin{aligned}\int f_n d\nu &\to \int f d\nu \\ &= \int f_n \delta d\mu \\ &= \int \delta d\mu\end{aligned}$$
2. f is integrable on ν is interchangable with $f\delta$ integrable on μ. In this case,

   $$\int f d\nu = \int f \delta d\mu$$

   By (i),
   $$\begin{aligned}\int |f| d\nu &= \int |f| \delta d\mu \\ &= \int |f\delta| d\mu\end{aligned}$$
   Since the RHS is finite $\Leftrightarrow$ the LHS is finite.

Since f integrable on ν,

$$\int f d\nu = \int f^+ d\nu - \int f^- d\nu$$
$$= \int f^+ \delta d\mu - \int f^- \delta d\mu$$
$$= \int (f^+ \delta - f^- \delta) d\mu$$
$$= \int (f^+ - f^-) \delta d\mu$$
$$= \int f \delta d\mu$$

■

Scheffe's Theorem

Lemma. If $f_n \to 0 a.e. \mu$ then $f_n^+ \to 0 a.e. \mu$ and $f_n^- \to 0 a.e. \mu$.

Proof. Suffices to show that

$$\mu(f_n^+ \not\to 0) = 0$$

$$\mu(f_n^+ \not\to 0) = \mu(f_n \not\to 0, f_n \geq 0) + \mu(f_n^+ \not\to 0), f_n < 0)$$

But when $f_n < 0$, $f_n^+ = 0$, then $f_n^+ \not\to 0$ will not happen.

■

Theorem 2.2.11 — 16.2 Sheffe's Theorem. Suppose $(\Omega, \mathscr{F}, \mu)$ is a measure space. $\delta_n \geq 0, \delta \geq 0$ are measurable and $\delta_n \to \delta$ a.e μ. For all $A \in \mathscr{F}$ let

$$\nu_n(A) = \int_A \delta_n d\mu$$

$$\nu_n(A) = \int_A \delta d\mu$$

and suppose that

$$\nu_n(\Omega) = \nu(\Omega) < \infty$$

which is true if ν_n, ν are probability measures.

Then

$$\sup_{A \in \mathscr{F}} |\nu_n(A) - \nu(A)| \leq \int |\delta_n - \delta| d\mu \to 0$$

Note that the first term is total variation distance between two measures, and the second is the L_1 distance between two densities.

Proof.

$$\begin{aligned}
\sup_{A\in\mathscr{F}} |\nu(A)-\nu_n(A)| &= \sup_{A\in\mathscr{F}} \left|\int I_A\delta d\mu - \int I_A\delta_n d\mu\right| \\
&= \sup_{A\in\mathscr{F}} \left|\int I_A(\delta-\delta_n)d\mu\right| \\
&\le \sup_{A\in\mathscr{F}} \int I_A|\delta-\delta_n|d\mu \\
&\le \sup_{A\in\mathscr{F}} \int_A |\delta-\delta_n|d\mu \\
&\le \int |\delta-\delta_n|d\mu
\end{aligned}$$

Let $g_n = \delta - \delta_n$. Since $\delta - \delta_n \to 0 a.e.\mu$ we have that $g_n^+ \to 0 a.e.\mu$ by lemma. Since $0 \le g_n^+ \le \delta$ and we know that δ is integrable, then by the DCT

$$\int g_n^+ d\mu \to 0$$

Also,

$$\int g_n d\mu = \int \delta - \delta_n d\mu = \nu(\Omega) - \nu_n(\Omega) = 0$$

$$\int_{g_n\ge 0} g_n d\mu + \int_{g_n<0} g_n d\mu = 0$$

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$$\begin{aligned}
\int |g_n|d\mu &= \int_{g_n\ge 0} g_n d\mu - \int_{g_n<0} g_n d\mu \\
&= 2\int_{g_n\ge 0} g_n d\mu \\
&= 2\int_{g_n\ge 0} g_n^+ d\mu \qquad \le 2\int g_n d\mu \to 0
\end{aligned}$$

■

Change of Variable Theorem

This result is very similar to the density theorem. That is, if $d\nu = \delta d\mu \Rightarrow \int f d\nu = \int f\delta d\mu$. But the scenario is different. Here $(\Omega, \mathscr{F}, \mu)$ is a measure space, while $(\Omega', \mathscr{F}')$ is a measureable space (no measure).

$$T : \Omega \to \Omega' ⓜ \mathscr{F} \setminus \mathscr{F}'$$

$\mu \cdot T^{-1}$ is the induced measure.

$$f : \Omega' \to \bar{\mathbb{R}} ⓜ \mathscr{F}' \setminus \bar{\mathscr{R}}$$

Theorem 2.2.12 — 16.3 Change of Variable Theorem. 1. Nonnegative Case.

If $f \geq 0$ then for all $A' \in \mathscr{F}'$

$$\int_{A'} fd(\mu T^{-1}) = \int_{T^{-1}A'} f \circ T d\mu$$

2. A function f integrable $\mu T^{-1} \Leftrightarrow f \circ T$ is integrable on μ in which case (again) for all $A' \in \mathscr{F}'$

$$\int_{A'} fd(\mu T^{-1}) = \int_{T^{-1}A'} f \circ T d\mu$$

Proof. 1. Want to again use Three Step Argument.

Step 1.

$$f = I_{A'}, A' \in \mathscr{F}'$$

$$f \circ T = I_{A'} = I_{A'} \circ T$$

Claim: $I_{A'} \circ T = I_{T^{-1}}(A')$

$$(I_{A'} \circ T)(\omega) = I_{A'}(T(\omega)) = \begin{cases} 1 & T(\omega) \in A' \\ 0 & T(\omega) \notin A' \end{cases} = \begin{cases} 1 & \omega \in T^{-1}(A') \\ 0 & \omega \notin T^{-1}(A') \end{cases} = I_{T^{-1}(A')}$$

$$\begin{aligned} \int f \circ T d\mu &= \int I_{T^{-1}(A')} d\mu \\ &= \mu(T^{-1}(A')) \\ &= \mu T^{-1}(A') \\ &= \int I_{A'} d\mu T^{-1} \end{aligned}$$

Step 2.

$f \geq 0$ f simple.

$$\begin{aligned} \int f \circ T d\mu &= \int (\sum_k x_k I_{A'_k})(T(\omega)) d\mu(\omega) \\ &= \int \sum_i x_k (I_{A'_k}(T(\omega))) d\mu(\omega) \\ &= \sum_i x_k \int I_{A'_k} T d\mu \\ &= \sum_i x_k \int I_{A'_k} d(\mu T^{-1}) \\ &= \int \sum_k x_k I_{A'_k} d(\mu T^{-1}) \\ &= \int f d\mu T^{-1} \end{aligned}$$

Step 3.

$f \geq 0$ measureable on $\mathscr{F}' \setminus \bar{\mathscr{R}}$. Then there exists $0 \leq f_n \uparrow f$, f_n simple.

MCT:

$$\int f_n d\mu T^{-1} \to \int f d(\mu T^{-1}) = \int f_n \circ T d\mu$$

But $0 \leq f_n \circ T \uparrow f \circ T$. By MCT,

$$\int f_n \circ T d\mu \to \int f \circ T d\mu$$

Take 2 limints of same sequnce, so

$$\int f d\mu T^{-1} = \int f T d\mu$$

Now consider the gerneral case,

$$\begin{aligned}\int_{A'} f d(\mu \circ T^{-1}) &= \int f * I_{A'} d\mu (T^{-1}) \\ &= \int (f * I_{A'}) \circ T d\mu\end{aligned}$$

But we claim that

$$\begin{aligned}(\omega) &= (f * I_{A'}) T(\omega) \\ &= f(T(\omega)) * I_{A'}) T(\omega) \\ &= f(T(\omega)) * I_{T^{-1}(A')})(\omega) \\ &= (f \circ T) * I_{T^{-1}(A')})(\omega)\end{aligned}$$

Returning to above equations,

$$\begin{aligned}\int_{A'} f d(\mu \circ T^{-1}) &= \int f * I_{A'} d\mu (T^{-1}) \\ &= \int (f * I_{A'}) \circ T d\mu \\ &= \int (f \circ T) * I_{T^{-1}(A')}) d\mu \\ &= \int_{T^{-1}(A')} f \circ T d\mu\end{aligned}$$

2. Since $|f| \geq 0$

$$\int |f| d\mu T^{-1} = \int |f| \circ T d\mu$$

Claim: $|f| \circ T = |f \circ T|$.

$$\begin{aligned}(|f| \circ t)(\omega) &= |f|(T(\omega)) \\ &= |f(T(\omega))|\end{aligned}$$

CIP
Thus,

$$\int |f| d\mu T^{-1} = \int |f| \circ T d\mu$$
$$= \int |f \circ T| d\mu$$

So f is integrable w.r.t $\mu T^{-1} \Leftrightarrow f \circ T$ integrable mu.

Now since f is measurabe on μ,

$$\int f d\mu T^{-1} = \int f^+ d\mu T^{-1} - \int f^- d\mu T^{-1}$$

Claim: $f^+ \circ T = (f \circ T)^+$ and same for f^-.
Claim left for you to prove.

■

Monday December 5

Uniform Integrability

This is an alternative condition for,

$$\lim_n \int f_n d\mu = \int \lim_n f_n d\mu$$

which applies to DCT, MCT, BCT. Here, u_i is a weaker condition than dominated condition but requres $\mu(\Omega) < \infty$. So it's neight weaker or stronger than DCT. (?)

Motivation (want to investigate a single function): If $(\Omega, \mathscr{F}, \mu)$ is a measure space,

$$f : \mathscr{F} \to \bar{\mathbb{R}}, ⓜ \mathscr{F} \setminus \bar{\mathscr{R}}$$

$$f ⓢ \mu \Rightarrow \lim_{\alpha \to \infty} \int_{|f| \geq 0} |f| d\mu = 0$$

Proof. Since f integrable on μ,

$$\int |f| d\mu < \infty \Rightarrow \mu(|f| = \infty) = 0$$

Let's show that $|f| I(|f| \geq \alpha) \to 0 a.e.\mu$. That is,

$$\mu(|f| I(|f| \geq \alpha) \not\to 0) \to 0$$

$$|f| I(|f| \geq \alpha) \not\to 0 = \limsup_{\alpha \to \infty} |f| I(|f| \geq \alpha) > 0$$

So there exists α_n where $\lim(\alpha_n) \to \infty$ such that

$$|f| I(|f| \geq \alpha_n) > 0 \forall n$$

Which implies that

$$I(|f| \geq \alpha_n) > 0 \forall n \Rightarrow |f| = \infty$$

But note that $\mu(|f| = \infty) = 0$ which implies that

$$\mu(|f|I(|f| \geq \alpha) \not\to 0) = 0$$

■

Now, let f_n be a sequence of functions

$$\Omega \to \bar{\mathbb{R}}, ⓜ\mathscr{F} \setminus \bar{\mathscr{R}}$$

We say that $\{f_n\}$ is uniformly integrable if

$$\lim_{\alpha\to\infty} \sup_n \int_{|f_n|\geq\alpha} |f_n| d\mu = 0$$

in which case we write $\{f_n\}ⓤ\mu$

Notation 2.1. *Uniformly Integrable Here we shall use ⓤ to denote uniform integrability.*

Fact 1.

If $\{f_n\}$ is uniformly integrable on $\mu, \mu(\Omega) < \infty$ then each f_n is integrable on μ.

$$\begin{aligned}
\int |f_n| d\mu &\leq \int_{|f_n|\leq\alpha} |f_n| d\mu + \int_{|f_n|>\alpha} |f_n| d\mu \\
&\leq \alpha\mu(\Omega) + \int_{|f_n|>\alpha} |f_n| d\mu \\
&\leq \alpha\mu(\Omega) + \sup_n \int_{|f_n|>\alpha} |f_n| d\mu
\end{aligned}$$

Take α to be sufficiently large then $\int |f_n| d\mu < \alpha\mu(\Omega) + 1 < \infty$.

Fact 2.

If f_n and g_n are both uniformly integrable on μ then $f_n + g_n$ is uniformly integrable on μ.

Proof. Let $h_n = \max(|f_n|, |g_n|)$.

Consider

$$\int_{|f_n+g_n|\geq\alpha} |f_n + g_n| d\mu$$

By the triangle inequality,

$$\begin{aligned}
|f_n + g_n| &\leq |f_n| + |g_n| \\
&\leq 2h_n \\
&\leq \int_{h_n\geq\frac{\alpha}{2}} |f_n + g_n| d\mu \\
&\leq \int_{h_n\geq\frac{\alpha}{2}, |f_n|\geq|g_n|} |f_n + g_n| d\mu + \int_{h_n\geq\frac{\alpha}{2}, |f_n|<|g_n|} |f_n + g_n| d\mu \\
&\leq 2\int_{h_n\geq\frac{\alpha}{2}, |f_n|\geq|g_n|} h_n^{|f_n|} d\mu + 2\int_{h_n\geq\frac{\alpha}{2}, |f_n|<|g_n|} h_n^{|g_n|} d\mu \\
&\leq 2\int_{|f_n|\geq\frac{\alpha}{2}} |f_n| d\mu + 2\int_{|g_n|\geq\frac{\alpha}{2}} |g_n| d\mu \\
&\leq
\end{aligned}$$

■

Can think of this as integrability of a sequence of function.

Theorem 2.2.13 — 16.14. Suppose that $\mu(\Omega) < \infty$ and $f_n \to f$ a. e. μ.

1. If f_n uniformly integrable on μ then f is uniformly integrable on μ as well. Also,
$$\int f_n d\mu \to \int f d\mu$$
2. If $f \geq 0, f_n \geq 0$, f integrable on μ, and f_n integrable on μ then
$$\int f_n d\mu \to \int f d\mu \Rightarrow f_n ⓤ \mu$$

Proof.

$$\limsup_n \left| \int f_n d\mu - \int f \mu \right| \leq \limsup_n \int_{|f_n| \geq \alpha} |f_n| d\mu + \int_{|f| \geq \alpha} |f| d\mu$$
$$\leq \sup_n \int_{|f_n| \geq \alpha} |f_n| d\mu + \int_{|f| \geq \alpha} |f| d\mu$$

So, this remember, is true under the assumption that $\mu\{|f| = \alpha\} = 0$. Recall we used this to prove $f_n^{(\alpha)} \to f^{(\alpha)} a.e. \mu$. There can only be countable many opositive measure sets for σ-finite μ.

so this means that $\{\alpha : \mu\{|f| = \alpha\} > 0\}$ is countable. Thus, there exists sequence $\{\alpha_m\}$ such that

$$\lim_m \alpha_m = \infty, \mu\{|f| = \alpha_m\} = 0$$

Thus,

$$\lim_m \limsup_n \left| \int f_n d\mu - \int f d\mu \right| \leq \lim_m \sup_n \int_{|f_n| \geq \alpha_m} |f_n| d\mu + \lim_m \int_{|f| \geq \alpha_m} |f| d\mu$$

And ultimately we get that f_n is uniformly integrable on μ.
Suppose $\mu(|f| = \alpha) = 0$. Then we have shown that

$$\int f_n^{(\alpha)} d\mu \to \int f^{(\alpha)} d\mu$$

$$\int_{|f_n| \geq \alpha_m} f_n d\mu - \int_{|f| \geq \alpha_m} f d\mu = \int f_n d\mu - \int f d\mu - \left(\int f_n^{(\alpha)} d\mu - \int f^{(\alpha)} d\mu \right)$$

So we have that

$$\left| \int_{|f_n| \geq \alpha_m} f_n d\mu - \int_{|f| \geq \alpha_m} f d\mu \right| \leq \left| \int f_n d\mu - \int f d\mu \right| + \left| \left(\int f_n^{(\alpha)} d\mu - \int f^{(\alpha)} d\mu \right) \right|$$

Bue we've shown that both terms in the RHS go to zero (by assumption or previous proof). Thus

$$\int_{|f_n| \geq \alpha_m} f_n d\mu \to \int_{|f| \geq \alpha_m} f d\mu$$

for all α such that $\mu(|f| = \alpha) = 0$.
Now, let $\varepsilon > 0$. then there exists α such that

$$\mu(|f| = \alpha) = 0, \int_{|f|]geq\alpha} f d\mu < \varepsilon$$

So then

$$\int_{|f_n| \geq \alpha} f_n d\mu < \varepsilon$$

Hence,

$$\sup_{n \geq n_0} \int_{|f_n| \geq \alpha} f_m d\mu < \varepsilon$$

But $f_1, \dots, f_{n_0}$ are integrable on μ.
So there exists some $\alpha' > \alpha$ such that

$$\int_{|f_1| \geq \alpha} f_1 d\mu < \varepsilon, \dots, \int_{|f_{n_0}| \geq \alpha} f_{n_0} d\mu < \varepsilon$$

So there exists α' such that

$$\sup_n \int_{|f_n| \geq \alpha} f_n d\mu < \varepsilon$$

But then for all $\alpha'' > \alpha'$ we have that

$$\sup_n \int_{|f_n| \geq \alpha''} f_n d\mu < \varepsilon$$

That is for all $\varepsilon > 0$ there exists $\alpha' > 0$ such that for all $\alpha'' > \alpha'$ we have that

$$\sup_n \int_{f_n \geq \alpha} f_n d\mu < \varepsilon$$

Hense,

$$\lim_{\alpha \to \infty} \sup \int_{|f_n| \geq \alpha} f_n d\mu < \varepsilon = 0$$

Therefor f_n is uniformly integrable on μ.

∎

Wednesday December 7

Finished proof.

Corollary Suppose that $\mu(\Omega) < \infty$. If f, f_n are integrable on μ and $f_n \to f$ a. e. μ then the following conditions are equivalent.

1. f_n uniformly integrable on μ
2. $\int |f - f_n| d\mu \to 0$
3. $\int |f_n| d\mu \to \int |f| d\mu$

Proof. Note that (i) $\Rightarrow$ (ii).

∎

A sufficient condition for uniformly integrable is

$$\sup_n \int |f_n|^{1+\varepsilon} d\mu < \infty$$

for some $\varepsilon > 0$

Proof. ■

Wednesday December 7 - Evening Make Up Course

2.3 Radon-Nikodgm Derivative

$(\Omega, \mathscr{F})$ is measurable space.
Additive Set Function

Also known as signed measure.

$$\Psi : \mathscr{F} \to \mathbb{R}$$

such that if $A_n \in \mathscr{F}$ is disjoint then

$$\Psi(\bigcup_n A_n) = \sum_n \Psi(A_n)$$

Two differences between Ψ and measure
1. $\Psi : \mathscr{F} \to \mathbb{R}$ (not $\bar{\mathbb{R}}$)
2. $\Psi(A)$ can be negative.
Lemma 1. If $E_n \uparrow E$ or $E_n \downarrow E$ then

$$\Psi(E_n) \to \Psi(E)$$

Proof. Suppose that $E_n \uparrow E$.

$$E = E_1 \cup \bigcup_{n=1}^{\infty} (E_{n+1} \setminus E_n)$$

$$\Psi(E) = \Psi(E_1) + \Psi(E_2 \setminus E_1) + \cdots = \Psi(E_1) + \Psi(E_2) - \Psi(E_2) + \cdots = \lim_{\nu \to \infty} \Psi(E_\nu)$$

Suppose that $E_n \downarrow E$ then ${}_n^C \uparrow E^C$. Use above proof.

■

Hahn-decomposition of Additive Set Function (ASF)

Theorem 2.3.1 Let Ψ be ASF then there exists A^+, A^- such that

$$A^+ \bigcup A^- = \Omega$$

and

$$A^+ A^- = \emptyset$$

and also that $\Psi(E) > 0$ for all $E \in A^+$, and similarly $\Psi(E) \leq 0$ for all $E \in A^-$.

Here, A^+ is called the positive set and A^- is the negative set.

$\{A^+, A^-\}$ is called the Hahn decomposition of Φ.

Proof. In photos. ∎

Jordon Decomposition of ASF

Let $\Psi^+(A) = \Psi(A \cap A^+), \forall A \in \mathscr{F}$.

Then Ψ^+ is greater than zero, additive, and a finite measure on $(\Omega, \mathscr{F})$.

$$\Psi^-(A) = -\Psi(A \cap A^-)$$

then Ψ^- is a finite measure on $(\Omega, \mathscr{F})$.

Definition 2.3.1 If μ is a measure on $(\Omega, \mathscr{F})$ then $S_\mu \in \mathscr{F}$ is a **support** of μ if

$$\mu(S_\mu^C) = 0$$

Now, $\Psi^+(A^+)^C) = \Psi((A^+)^C A^+) = \Psi(\emptyset) = 0$. So A^+ supports Ψ^+ and similarly A^- supports Ψ^-.

Since $A^+A^- = \emptyset$ we say that Ψ^+, Ψ^- are mutually singular.

In general, if μ, v are measures on $(\Omega, \mathscr{F})$ then they are mutually singular if they have disjoint support.

Now, for all $A \in \mathscr{F}$,

$$\begin{aligned}\Psi(A) &= \Psi(AA^+) + \Psi(AA^-) \\ &= \Psi(AA^+) - (-\Psi(AA^{)}) \\ &= \Psi^+(A) - (\Psi^-(A))\end{aligned}$$