Theory of Statistics I

Take Two

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Part One

1. Real Analysis Review

1.1 The Real Number System

1.1.1 Rationals

Start with integers as given.

Definition 1.1.1 — Rational Numbers. Rationals are numbers of the form $\frac{m}{n}$, for m,n integers, $n \neq 0$ such that:

PR 1: The sum, difference, product, and ratio (division by 0 excluded) of any two rationals is a rational.

PR 2: $p+q = q+p, pq = qp$ (Commutative Property)

PR 3: $(p+q)+r = p+(q+r), (pq)r = p(qr),$ (Associative Property)

PR 4: $(p+q)r = pr+qr$ (Distributive Property)

PR 5: $\forall$ two rationals p and q we have either p=q, p<q, or q<p (Ordering Property)

PR 6: If $p < q$ and $q < r$, then $p < r$ (Transitivity of <)

PR 7: If $p > 0$ and $q > 0$, then $p+q > 0$ and $pq > 0$

PR 8: If $p < q$, then $p+r < q+r \; \forall$ r

The rational number system is inadequate.

■ **Example 1.1** There is no rational number p that satisfies $p^2 = 2$ ■

Proof. Suppose such a p existed, and so $p = \frac{m}{n}$. Note that m,n can be chosen so not both are even. Then we have,

$$m^2 = 2n^2$$

Thus, m^2 is even, and hence m is even. (The square of an odd number is odd). Hence, m^2 is divided by 4. So, $2n^2$ is divisible by 4, or n^2 is even which implies that n is even - **contradiction**. ■

This example can be used to show that we can have a set of rational numbers bounded from above but has no supremum.

■ **Example 1.2** Let A be the set of < 0 rationals p, such that $p^2 < 2$. Let B be the set of > 0 rationals p, such that $p^2 > 2$. Then A contains no largest number and B contains no smallest number.
■

Proof. If $p \in A$, choose a rational h such that, $0 < h < 1$ and $h < \frac{2-p^2}{2p+1}$ and set $q = p+h$. Then q is rational and

$$\begin{aligned} q^2 &= p^2 + (2p+h)h \\ &< p^2 + (2p+1)h \\ &< p^2 + (2-p^2) \\ &= 2 \end{aligned}$$

If $p \in B$, set

$$q = p - \frac{p^2-2}{2p} = \frac{p}{2} + \frac{1}{p}$$

and

$$\begin{aligned} q^2 &= p^2 - (p^2-2) + (\frac{p^2-2}{2p})^2 \\ &> p^2 - (p^2-2) \\ &= 2 \end{aligned}$$

■

R An axiomatic treatment of the real number system uses PR1 - PR8 as axioms together with the "completeness axiom". The non-axiomatics treatment is due to Dedekind.

1.1.2 Sets and Subsets

If A is any set, $\mathbf{x} \in \mathbf{A}$ means that x is a member of A, and $\mathbf{x} \notin \mathbf{A}$ means x is not a member of A. A set B is a **subset** of A if for every $x \in B$ we have $x \in A$, and we write $A \subseteq B$. B is a **proper subset** of A, $B \subset A$, if there $\exists\, x \in A$ with $x \notin B$. The **empty set** is denoted by $\emptyset$, and $\emptyset \in A$, $\forall$ other set A.

$$A \cup B = B \cup A \text{ - union with commutative property}$$

$$A \cap B = B \cap A \text{ - intersection with commutative property}$$

$$\begin{aligned} &(A \cup B) \cup C = A \cup (B \cup C) \\ &(A \cap B) \cap C = A \cap (B \cap C) \text{ - associative property} \end{aligned}$$

$$\begin{aligned} &(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \text{ - distributive property} \\ &(\cup A_i)^c = \cap A_i^c \\ &(\cap A_i)^c = \cup A_i^c \end{aligned}$$

Definition 1.1.2 — Dedekind Cuts. A set α of rational numbers is said to be a **cut** if

a) α is a proper, but non-empty, subset of the rational numbers.
b) If $p \in \alpha$ (p is rational), and $q < p$ (q is rational) then $q \in \alpha$
c) It contains no largest rational.

A cut of the form $\alpha = \{$p: p is rational and $p < r\}$ where r is rational are called **rational cuts** and are denoted by r^*.

The development of the real number system proceeds as follows:
First, the set of cuts is equipped with an order relation, and the operation of addition and multiplication an it will show that the resulting arethmatic satisfies PR 1 - PR 8.

If α, β are cuts then,

$$\alpha < \beta \text{ if } \alpha \subset \beta \text{ and}$$
$$\alpha \leq \beta \text{ if } \alpha \subseteq \beta$$
$$\alpha + \beta = \{r : r = p + q \text{ for some } p \in \alpha, q \in \beta\}$$
$$(\alpha + 0^* = \alpha)$$

If $\alpha + \beta = 0^*$, write $\beta = -\alpha$. (It can be shown that $\forall \alpha$ there is one and only one β such that $\alpha + \beta = 0^*$.)

$$|\alpha| = \begin{cases} \alpha, & \text{if } \alpha \geq 0^*, \\ -\alpha, & \text{if } \alpha < 0^*. \end{cases}$$

For $\alpha \geq 0^*$ and $\beta \geq 0^*$,

$$\alpha\beta = \{\text{p:p rational such that either } p < 0 \text{ or } p = pq, \text{ for } q \in \alpha,\ r \in \beta \text{ with } q \geq 0 \text{ and } r \geq 0.\}$$

For general α, β,

$$\alpha\beta = \begin{cases} -(|\alpha||\beta|), & \text{if } \alpha < 0^*, \text{and } \beta \geq 0^* \\ & \text{or if } \alpha \geq 0^* \text{and } \beta < 0^* \\ |\alpha||\beta|, & \text{if } \alpha < 0^*, \text{and } \beta < 0^* \end{cases}$$

If $\alpha \neq 0^*$, then $\forall \beta$ there is one and only one γ such that $\alpha\gamma = \beta$, and this γ is denoted by $\frac{\beta}{\alpha}$. (In technical terms, we made the set of cuts an **ordered field**.)

Second, it is shown that replacing the set of rational numbers by the corresponding cuts preserves sums, products and order, ie,

1. $p^* + q^* = (p+q)^*$
2. $p^* q^* = (pq)^*$
3. $p^* < q^*$ iff $p < q$

In technical terms, the ordered field of rational numbers is **isomorphic** to that of rational events.

Theorem 1.1.1 — Dedekind. Let A, B be $\subset \mathbb{R}$ such that,

(a) $A \cap B = \emptyset$
(b) $A \cup B = \mathbb{R}$
(c) neither A nor B is empty
(d) if $\alpha \in A$, $\beta \in B$, then $\alpha < \beta$

Then there $\exists\ \gamma \in \mathbb{R}$ such that $\alpha \leq \gamma$, $\forall \alpha \in A$ and $\gamma \leq \beta$, $\forall \beta \in B$.

Proof. First, suppose there are 2 γ, say $\gamma_1 < \gamma_2$. Take γ_3 such that $\gamma_1 < \gamma_3 < \gamma_2$.

$$\gamma_3 < \gamma_2 \text{ implies that } \gamma_3 \in A$$

$$\gamma_1 < \gamma_3 \text{ implies that } \gamma_3 \in B$$

However, these implications contradict the disjointness (part (a)). Define $\gamma - \{$p: p rational such that p $\in \alpha$ for some $\alpha \in A\}$. The proof proceeds by showing that γ is a cut, and hense a real number that satisfies $\alpha \leq \gamma$ for $\alpha \in A$ and $\gamma \leq \beta \; \forall \; \beta \in B$. ■

Corollary 1.1.2 If A, B are as in the theorem, then either A contains a largest number or B contains a smallest number.

Corollary 1.1.3 Let $E \neq \emptyset$ be a subset of $\mathbb{R}$. Then, if E is bounded above a supremum (least upper bound) exists.

Proof. Define
$A = \{\alpha : \alpha < x \text{ for some } x \in E\}$
$B = A^c$
Clearly, all members of B are upper bounds of E. It is sufficient to prove that B contains a smallest nubmer, or, by Corollary 1, that A does not contain a largest number (and thus prove by contradiction). Indeed if $\alpha \in A \; \exists$ an $x \in E$ such that $\alpha < x$. But, by Property 1 (???) there $\exists$ an α' such that $\alpha < \alpha' < x$ where $\alpha' \in A$ (i.e. we can always find a larger α so, since there is no largest α, there MUST be a smallest β). ■

Theorem 1.1.4 Any real number admits a decimal expansion.

Proof. Let $x > 0$, $x \in \mathbb{R}$. Let $n_0 = [x]$ (n largest integer < x). Let n_1 be the largest integer such that $n_0 + \frac{n_1}{10} < x$. Having defined $n_0 \dots n_{k-1}$, define n_k as the largest integer such that $n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \cdots + \frac{n_k}{1-k} \leq x$. Let E bet he set of resluting numbers for $k = 1, 2, \dots$. Then x is the supremum of E and $n_0, n_1, \dots$ is its **decimal expansion**. Conversely, and set of integers $n_0, n_1, \dots$ defines a set of numbers, E, bounded above by $n_0 + 1$. ■

Definition 1.1.3 — Extended Real Number System.

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

1.1.3 Euclidean Space

Definition 1.1.4 — Vector Space. For any $k \in \mathbb{Z}^+$. Let $\mathbb{R}^K$ be the set of ordered k-tuples.

$$\begin{aligned} \underline{x} &= (x_1, \dots, x_k) \text{ with } x_i \in \mathbb{R} \\ \underline{y} &= (y_1, \dots, y_k) \alpha \in \mathbb{R} \\ \underline{x} + \underline{y} &= (x_1 + y_1, \dots, x_k + y_k) \\ \alpha \underline{x} &= (\alpha x_1, \dots, \alpha x_k) \end{aligned}$$

Which makes $\mathbb{R}^k$ a **vector space** over the **real field**.

Definition 1.1.5 — Inner/Scalar/Dot Product.

$$\underline{x} \cdot \underline{y} = \sum_{i=1}^{k} x_i y_i$$

Definition 1.1.6 — Norm/Length.

$$|\underline{x}| = (\underline{x}\underline{x})^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{k} x_i^2}$$

Definition 1.1.7 — Euclidean K-space. The vector space $\mathbb{R}^k$ with the inner product and norm is called **Euclidean k-space**.

Theorem 1.1.5 For $\underline{x}, \underline{y} \in \mathbb{R}^k, \alpha \in \mathbb{R}$

a) $|\underline{x}| \geq 0, |\underline{x}| = 0$ iff $\underline{x} = \underline{0}$
   $|\alpha\underline{x}| = |\underline{\alpha}||\underline{x}|$

b) **Cauchy-Schwarz Inequality** $|\underline{x} \cdot \underline{y}| \leq |\underline{x}||\underline{y}|$

c) **Triangle Inequality** $|\underline{x} + \underline{y}| \leq |\underline{x}| + |\underline{y}|$

1.2 Elements of Set Theory

Definition 1.2.1 Let A, B be sets and suppose that to each $x \in A$ there corresponds an elements of B denoted by $f(x)$. Then f is a **function** (or in more general space, mapping) from A (in)to B.

A is called the **domain** of f. $f(x)$ is the **value** of f at x, $R(f) = \{f(x) : x \in A\}$ is the **range** of f.

Definition 1.2.2 — Image. If f is a function from A to B ($A \to B$) and $E \subseteq A$ we write $f(E) = \{f(x) : x \in E\}$ and call it the **image** of E under f. If $f(A) = B$, then we say f maps A **onto** B.

Definition 1.2.3 — Inverse Image. Let $f : A \to B$ and $E \subseteq B$. We write $f^{-1}(E) = \{x]inA : f(x) \in E\}$ and call it the **inverse image** of E **under** f. NB: If $E = \{y\}, y \in B$ we also write $f^{-1}(y)$ (versus $f^{-1}(\{y\})$). If $\forall y \in B$ $f^{-1}(y)$ consists of at most one element, then f is one to one mapping of A **into** B.

Theorem 1.2.1 a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

b) $f(A \cup B) = f(A) \cup f(B)$

c) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Actually, these may be extended to arbitrary unions and intersections.

$$f^{-1}(\bigcup_{\alpha} A_\alpha) = \bigcup_{\alpha} f^{-1}(A_\alpha)$$

$$f^{-1}(\bigcap_{\alpha} A_\alpha) = \bigcap_{\alpha} f^{-1}(A_\alpha)$$

$$f(\bigcup_{\alpha} A_\alpha) = \bigcup_{\alpha} f(A_\alpha)$$

Note: $f(A \cap B)$ is not necessarily equal to $f(A) \cap f(B)$ (see notes for example and sketch)

Definition 1.2.4 — Cardinal Number. If $\exists$ a one-to-one mapping of A onto B, we say that A and B have the same **cardinal number**, or that they are **equivalent** $A \sim B$.

a) $A \sim A$ (reflective)

b) If $A \sim B$, then $B \sim A$ (symmetric)

c) If $A \sim B$ and $B \sim C$, then $A \sim C$. (transitive)

Definition 1.2.5 — (In)finite/(Un)Countable. Let $\mathbb{Z}^+ = \{1,2,\ldots\}$ and $\mathbb{Z}_n^+ = \{1,2,\ldots,n\}$ and let A be a set.

a) We say A is **finite** if $A \sim \mathbb{Z}_n^+$ for some n or if $A = \emptyset$
b) A is **infinite** if it is not finite
c) A is **countable** if $A \sim \mathbb{Z}^+$
d) A is **uncountable** if A is not finite and countable.

Note: If A and B are finite, then $A \sim B$ if and only if they have the same number of elements. This is not true if they are infinite.

■ **Example 1.3 Equivalent Infinite Sets**

1. The set $\mathbb{Z}^+$ of all integers is countable. Then take

$$f(x) = \begin{cases} \frac{n}{2}, & \text{if n is even} \\ -(\frac{n-1}{2}), & \text{if n is odd} \end{cases}$$

$$\begin{array}{ccc} 1 & \to & 0 \\ 2 & \to & 1 \\ 3 & \to & -1 \\ 4 & \to & 2 \\ 5 & \to & -2 \\ 6 & \to & 3 \\ 7 & \to & -3 \end{array}$$

Table 1.1: Corresponding Integers

The set of positive, even integers is countable. Take

$$f(x) = 2n$$

■

Theorem 1.2.2 The countable union of countable sets is countable.

Proof. Let $A_1, A_2, \ldots$ be countable and assume that they are disjoint (for if not, you can consider the sequences of countable sets that are disjoint - $A_1, A_2 - A_1, \ldots$), which are countable and have the same union. Let $A_k = \{a_{k1}, a_{k2}, \ldots\}$ and consider the arrangement of $\bigcup_{k=1}^{\infty} A_k$.

$$\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{14} & \ldots \\ a_{21} & a_{22} & a_{23} & a_{24} & \ldots \\ a_{31} & a_{32} & a_{33} & a_{34} & \ldots \\ a_{41} & a_{42} & a_{43} & a_{44} & \ldots \end{array} \qquad \begin{array}{ccccc} 1 & 2 & 6 & 7 & \ldots \\ 3 & 5 & 8 & \ldots & \ldots \\ 4 & 9 & 13 & \ldots & \ldots \\ 10 & 12 & \ldots & \ldots & \ldots \end{array}$$

Table 1.2: Reassigning new values to counting integers.

■

Theorem 1.2.3 Every infinite set has a countable subset.

Proof. Let a_1 be any element of A. Since A is infinite, it contains an $a_2 \neq a_1 \ldots$. So it contains a countable subset. ■

Theorem 1.2.4 Every infinite set, A, is equivalent to at least one of its proper subsets.

Proof. Let $E - \{a_1, a_2, \dots\}$ be a countable subset of A (which exists by previous Theorem). Write,

$$
\begin{aligned}
E &= E_1 \cup E_2 \\
E_1 &= \{a_{odd}\} \\
E_2 &= \{a_{even}\}
\end{aligned}
$$

Then, $E \sim E_2$
Define,

$$
\begin{aligned}
&g : E \to E_2 \\
&g(a_i) = a_{2i}
\end{aligned}
$$

$$
f(a) = \begin{cases} a, & \text{if } a \notin E, \\ g(a), & \text{if } a \in E. \end{cases}
$$

So, we can also say that $A - E_1 \subset A$ and thus, $A \sim (A - E_1)$ ■

Theorem 1.2.5 The set of real numbers in [0,1] is uncountable.

Proof. Suppose all numbers in [0,1] are countable, $\{a_1, a_2, \dots\}$.
Write them in decimal expansion form. So, we can say

$$
\begin{aligned}
a_1 &= 0.a_{11}a_{12}\dots a_{1n}\dots \\
a_2 &= 0.a_{21}a_{22}\dots a_{2n}\dots
\end{aligned}
$$

Recall,

$$
\begin{aligned}
0 &= 0.000000000\dots \\
1 &= 0.999999999\dots
\end{aligned}
$$

Now, consider the number, β with decimal expansion $\beta = 0.b_1 b_2 \dots$ where

$$
b_n = \begin{cases} 1, & \text{if } a_{nn} = 1, \\ 2, & \text{if } a_{nn} \neq 1. \end{cases}
$$

There will always be 1 element difference. ALWAYS. ■

Theorem 1.2.6 If A is countable, then so is A^n, where

$$
A^n = \{(a_1, \dots, a_n); a_i \in A\}
$$

Proof. Statement is true for any n=1 since $A^1 = A$. Assume true for n=k. To show A^{k+1} is countable, write an element $(a_1, a_2, \dots, a_k, a_{k+1}) = (\underline{a}, a_{k+1}), \underline{a} \in A^k$. Thus, $A^{k+1} = \bigcup_{\underline{a} \in A^k} \{\underline{a}, a_{k+1}); a_k \in A\}$ (see previous Theorem). ■

1.2.1 Metric Spaces

Definition 1.2.6 A set X is a **metric space** is $\forall x, x \in X$ there is a **real** number, $d(x_1, x_2)$ called the **distance** between x_1 and x_2 such that,

a) $d(x_1, x_2) > 0$ if $x_1 \neq x_2$ and $d(x_1, x_1) = 0$
b) $d(x_1, x_2) = d(x_2, x_1)$
c) $d(x_1, x_2) \leq d(x_1, x_3) + d(x_2, x_3), \forall x_3$

■ **Example 1.4** a) Euclidean spaces $\mathbb{R}^k$ are metric spaces with $d(x_1, x_2) = |x_1 - x_2|$
b) Any subset of a metric space is a metric space with same distance.
c) The set $\mathbb{R}^k$ can also be metrized with
$d_1(x_1, x_2) = \sum_{i=1}^{k} |x_{1i} - x_{2i}|$
or with
$d_2(x_1, x_2) = (\sum_{i=1}^{k} |x_{1i} - x_{2i}|^p)^{\frac{1}{p}}$
d) The set $C_{[a,b]}$ of all continuous functions on $[a, b]$ with
$d_1(f, g) = \max_{a \leq t \leq b} |f(t) - g(t)|$
or with
$d_2(f, g) = (\int_a^b [f(t) - g(t)]^2)^{\frac{1}{2}}$
e) The set l_p of all infinite sequences $x = (x_1 x_2, \dots)$ satisfying $\sum_{i=1}^{\infty} |x_i|^p < \infty$ for $p \geq 1$ with
$d(x_1, x_2) = \sum_{i=1}^{\infty} |x_{1i} - x_{2i}|^p)^{\frac{1}{p}}$

■

Definition 1.2.7 Let X be a metric space. All sets and points mentioned are sets and elements of X.

a) An **open ball** of radius r and center x is

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

The **closed ball** is

$$B[x, r] = \{y \in X : d(x, y) \leq r\}$$

Open ball with center x are also called **neighborhoods** of x and $B(x, y)$ is denoted by $N_r(x)$.
b) A point x is a **limit point** of a set E if $\forall r > 0$ $E \cap N_r(x)$ contains a point $\neq$ x. If x is not a limit point it is called an **isolated point**.
c) A point x is an **interior point** of E if there $\exists$ r such that $N_r(x) \leq E$.
d) E is **open** if every point of E is an interior point.
e) E is **closed** if every point of E belongs in E.
f) E is **dense** in X if every point of X is a limit point of E, or a point of E, or both. (e.g. rationals with in real numbers)
g) E is **bounded** if for some $r > 0$, and $x \in X$, $E \subseteq N_r(x)$.

Theorem 1.2.7 Every neighborhood is an open set.

Theorem 1.2.8 If X is limit point of E, then every neighborhood of X contains infinitely many points of E.

■ **Example 1.5** X = $\mathbb{R}$, then (a,b) is open, [a,b] is close, (a,b] and [a,b) are neither open nor closed.

■

■ **Example 1.6** $X = \mathbb{R}^2$ (see sketch in notes.) ■

Theorem 1.2.9 Suppose $Y \subset X$ (a metric space) and take $E \subseteq Y$, then E is open relative to Y if and ony if $E = Y \cap G$ for some open set G of X.

Theorem 1.2.10 E is open if and only if its compliment is closed.

Corollary 1.2.11 a) Both X and $\emptyset$ are closed.
b) The union of finite numbers of closed sets is closed.
c) Arbitrary intersections of closed sets is closed.

Theorem 1.2.12 For any metric space X, we have
a) X and $\emptyset$ are open.
b) The intersection of a finite nuber of open sets is open. (Note: must be finite. $E_n = (-\frac{1}{n}, \frac{1}{n})$, then $\bigcap^{\infty} E_n = \{0\}$)
c) The union of every collection of open sets is open.

1.2.2 Compact Sets

Definition 1.2.8 A subset K of a metric space X is **compact** if every open cover of k contains a finite subcover. That is for all collections $G_\alpha, \alpha \in A$ of open sets such that $\bigcup_A G_\alpha \supset K$ there exists a finite collection $G_{\alpha_i}, i = 1, 2, \ldots, n$ such that $K \subset \bigcup G_\alpha$

R To visualize an open cover that is not compact, think of K = [0,1] and $G_\alpha = (-1, 1 - \frac{1}{\alpha})$. $\bigcup_1^\infty G_\alpha$ will cover K, but $\bigcup_1^{999} G_\alpha$ will not.

■ **Example 1.7** a) X = ℝ, E = (0,1)
Let $G_\alpha = (\frac{1}{\alpha}, 1), \alpha = 1, 2, \ldots$
Clearly , $\bigcup_{\alpha=1}^{\infty} G_\alpha \subset (0, 1)$, but also,
$K \not\subset \bigcup_{\alpha=1}^{\infty} G_\alpha$.
b) $X = \mathbb{R}, E = [0, \infty)$, let $G_\alpha(-1, \alpha), \alpha \geq 1$. Then $E \subset \bigcup^{\infty} G_\alpha$, but $E \not\subset \bigcup_{\alpha=1}^{\infty}, \forall n$.
■

Theorem 1.2.13 Suppose $K < Y < X$, (X is a metric space). Then K is a compact space with respect to Y if and only if k is a compact space of X.

Proof. "$\Leftarrow$" Suppose k is compact relative to X and let $V_\alpha, \alpha \in A$ be open sets relative to Y, such that $k \subset \bigcup_{\alpha \in A}$. By Theorem 1.2.12 (13 in notes), $V_\alpha = Y \cap G_\alpha$, some G_α open relative to X. (Note:

$k \subset \cup G_\alpha$.) Thus, there exists a finite subcover, $k \subset \bigcup_{i=1}^{n} G_{\alpha_i}$. But then,

$$\begin{aligned} k \subset Y \cap (\bigcup_{i=1}^{n} G_{\alpha_i}) &= \bigcup_{i=1}^{n} (Y \cap G_{\alpha_i}) \\ &= \bigcup_{i=1}^{n} V_{\alpha_i} \end{aligned}$$

"$\Rightarrow$" Suppose k is compact relative to Y, and let $G_\alpha, \alpha \in A$ be open relative to X, so $k \subset \bigcup_{\alpha \in A}$. But then,

$$\begin{aligned} k \subset Y \cap (\bigcup_{i=1}^{n} G_{\alpha_i}) &= \bigcup_{i=1}^{n} (Y \cap G_\alpha) \\ &= \bigcup_{i=1}^{n} V_\alpha, V_\alpha \text{ open with respect to Y.} \end{aligned}$$

$$\text{Thus, } k \subset \bigcup_{i=1}^{n} V_{\alpha_i}) = Y \cap (\bigcup_{i=1}^{n} G_\alpha)$$

So, $k \subset \bigcup_{i=1}^{n} G_\alpha$ ■

Theorem 1.2.14 If k is a compact subset of a metric space, X, then k is closed and bounded.

Proof. We'll show k is closed by showing k^c is open. Let $p \in k^c$. For each $q \in k$ we will consider $N_{r_q}(q)$ where $r_q = \frac{1}{2}d(p,q)$. Since k is compact, there exists $(q_1, q_2, \ldots, q_n) \in k$ such that $k \subset \bigcup_{i=1}^{n} N_{r_{q_i}}(q_i)$. Let $G = \bigcap_{i=1}^{n} N_{r_{q_i}}(p)$. (Note: $(\bigcup^{n} N_{r_{q_i}}(q_i)) \cap G = \emptyset$.) ■

Theorem 1.2.15 Closed (with respect to X) subsets of compact sets are compact.

Proof. Let $F \subseteq k \subseteq X$, where X is a metric space, k is compact, and F is closed with respect to X. Let $G_\alpha, \alpha \in A$, be open such that $F \subset \bigcup_{\alpha \in A} G_\alpha$ (F is "covered" by $\cup G_\alpha$). F close implies F^c is open. Then the collection $\{F^c, G_\alpha\}$ covers k. Let $k \subset F^c \cup G_\alpha$ which implies $F \subset \cup G_\alpha$. ■

Theorem 1.2.16 If E is an infinite subset of a compact set k, then E has a limit point in k. ("Countable compactness.")

Proof. If no point in K is a limit point of E, then each $q \in K$ will have a neighborhood, $N(q)$, which contains at most point point of E (which is q if $q \in E$). Thus, no finite subcollection of $\{N(q), q \in K\}$ for which no finite collection covers k. Contradiciton. ■

■ **Example 1.8** Let X be the space of rational numbers, with $d(p,d) = |p-d|$. Show that $E = \{p \in X; 2 < p^2 < 3\}$ is closed, bounded, but not compact. ■

Theorem 1.2.17 If $K_\alpha \subseteq X$, X a metric space, $\alpha \in A$, are compact such that the intersection of every finite collection is empty, then entire $\bigcap K_\alpha \neq \emptyset$

Proof. Let K_1 be a member of $\{k_\alpha, \alpha \in A\}$ such that no point of K belongs to all K_α. Then $G_\alpha = K_\alpha^c$ are open and cover K_1. Thus, $\exists\ \alpha_1, \alpha_2, \dots, \alpha_n$ such that $K_1 \subseteq \bigcup_{i=i}^{n} G_{\alpha_i}$. This implies that $K_1 \bigcap (\bigcup^n G_{\alpha_i})^c$ or $K_1 \bigcap K_{\alpha_1} \bigcap \dots \bigcap K_{\alpha_n} = \emptyset$. Contradiction. ■

■ **Example 1.9** X = space of rational numbers, $d(p,q) = |p-q|$, $E = \{p \in X : \sqrt{2} < p < \sqrt{3}\}$. Then E is closed, bounded but not compact.

Proof. $E = X \bigcap [\sqrt{2}, \sqrt{3}]$, and since $X \subseteq \mathbb{R}$ and $[\sqrt{2}, \sqrt{3}$ is closed in $\mathbb{R}$, E is closed (and bounded but not compact). ■

■

In Euclidean spaces, if a set is closed and bounded, then it **is** compact. The main step of showing this is ...

Theorem 1.2.18 Ever k-cell in $\mathbb{R}^k$ is compact, where k-cells are of the form:

$$I = \{x \in \mathbb{R}, a_i \leq x_i \leq b_i, i = 1, \dots, k\}$$

To prove, we must first state the following lemma and corollary.

Lemma 1 If I_n is a sequence of k-cells such that $I_n \subseteq I_{n+1}$, then $\bigcap_{1}^{\infty} I_n \neq \emptyset$.

Corollary 1.2.19 As stated above, if $E \subseteq \mathbb{R}^k$ is closed and bounded then it is compact.

Theorem 1.2.20 If $K \subset \mathbb{R}^k$ is countably compact, then it is compact.

Theorem 1.2.21 — Bolzano-Weierstrauss. Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

Proof. Begin bounded, it is a subset of a k-cell, I. Since k-cells are compact, each infinite subset of I has a limit point by Theorem 1.2.16. ■

1.3 Sequences and Sets

R Reading: definition of convergent sequences in metric space, and result of limit of convergence unique/bounded in Rudin text.

Definition 1.3.1 — Converge. A sequence $\{p_n\}$ in a metric space X is said to **converge** if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$. In this case we also say that $\{p_n\}$ converges to p, or that p is the limit of $\{p_n\}$ and we write $p_n \to p$. If $\{p_n\}$ does not converge, it is said to **diverge**.

Theorem 1.3.1 The limit of a convergent sequence is uniquely defined.

Theorem 1.3.2 A convergent sequence $\{x_n\}$ is bounded.

Theorem 1.3.3 Let $\{x_n\}$ be a sequence in a metric space X. Then

a) $\{x_n\} \to p$ if and only if every neighborhood of p contains all but finite many elements of $\{x_n\}$

b) If p is a limit point of a set $E \subseteq X$, there $\exists$ a sequences $\{x_n\}$ of points in E such that $x_n \to p$.

Proof. a) Let V be a neighborhood of p. Then for some $\varepsilon > 0$, $d(q,p) < \varepsilon$ implies $q \in V$. But corresponding to this ε there exists N such that $n \geq N$ implies $d(x_n,p) < \varepsilon$ or $x_n \in V$.

b) For each n, there exists $x_n \in E$ such that $d(x_n,p) < \frac{1}{n}$. This defines a sequence $\{x_n\}$ such that $x_n \to p$.

■

Definition 1.3.2 — Subsequence. Given a sequence $\{x_n\}$, $n \geq 1$, and a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2, dots$, the sequence $\{x_{n_k}\}$, $k \geq 1$, is called a **subsequence** of $\{x_n\}$. If the subsequnce converges its limit is called a **subsequential limit** of $\{x_n\}$.

Theorem 1.3.4 $X_n \to X$ if and only if $X_{n_k} \to X$ for all subsequences.

Proof. Left as a potential exercies. ■

Theorem 1.3.5 Let $X = \mathbb{R}^k$. Then

a) $\underline{x_n} \to \underline{x}$ if and only if $x_{n_i} \to x_i$ $\forall i = 1,\ldots,k$.

b) If $\{\underline{x_n}\}$ is bounded then it contains a convergent subsequences.

Proof. a) Left as an exersize.

b) Suppose that $\{\underline{x_n}\}$ is an infinite set. Then the Bolzano-Weierstauss Theorem implies that there is a limit point of $\{\underline{x_n}\}$ in $\mathbb{R}^k$. By theorem 3.b (in notes), we are done.

■

Theorem 1.3.6 The subsequential limits of a sequence $\{p_n\}$ in a vector space X form a closed set.

Proof. Let E be the set of subsequential limits of $\{p_n\}$ and let q be a limit point of E. We need to show that $q \in E$. Let $q_k \in E, k \geq 1$ be a subsequence converging to q. We can choose q_k such that $0 \leq d(q_k,q) \leq \frac{1}{2k} = \frac{\varepsilon_k}{2}$. Since $q_k \in E$ there is a $p_{n_k} \in \{p_{n_k}, n \geq 1\}$ such that $d(p_{n_k},q_k) < d(q_k,q) < \frac{\varepsilon_k}{2}$. Thus, $p_{n_k} \neq 1$ and $0 < d(p_{n_k},q) < d(p_{n_k},q_k) + d(q_k,q)$ ■

Definition 1.3.3 a) A sequence $\{p_n\}$ in a metric space, X, is said to be a **Cauchy sequence** if for all $\varepsilon > 0$ there exists N such that $n \geq N$ and $m \geq N$ we have that $d(p_n,p_m) < \varepsilon$.

b) A sequence $\{x_n\}, x_n \in \mathbb{R}$ such that $\forall m > 0$ there exists N > 0 such that $n \geq N$ implies that $x_n \geq M$ then $x_n \to \infty$. Similarly, for $x_n \to -\infty$.

Definition 1.3.4 For a subset of a metrix space, $E \in X$, the **diameter of E** is

$$diam(E) = \sup\{d(p,q), p \in E, q \in E\}$$

.

Lemma 2 a) Let $\bar{E} = E \bigcup \{limit points of E\}$ be the **closure** of E. Then $diam(E) = diam(\bar{E})$. So, the limit points do not increase the diameter.
b) The closure, $\bar{E}$ is a closed set.

Theorem 1.3.7 a) For a sequence $\{p_n\}$ in a matrix space X, set $E_N = \{p_N, p_{N+1}, \dots\}$. Then $\{p_n\}$ is a Cauchy sequence if and only if the $diam(E_N) \to 0$.
b) Every convergent sequence in a metric space is a Cauchy sequence.
c) Every Cauchy sequence in $\mathbb{R}^k$ converges.

Proof. a) Left as exercies
b) If $p_n \to p, \forall \varepsilon > 0$ there exists N such that $n \geq N$ implies $d(p_n, p) < \frac{\varepsilon}{2}$. Hense if $n \geq N$, and $m \geq N$, $d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \varepsilon$.
c) Suppose $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}^k$, let $E_N = \{x_N, x_{N+1}, \dots\}$ and $\bar{E_N}$ be its closure. By the lemma (part a)) and part a of this theorem, we have $diam(\bar{E_N} \to 0$. This implies that $\bar{E_N}$ are bounded sets. By lemma (part b)), $\bar{E_N}$ is closed. Thus $\bar{E_N}$ are compact sets. Thus, by Theorem 2.17b (by notes numbering), $\bigcap_{N=1}^{\infty} \bar{E_N} \neq \emptyset$ (i.e. must be single point which is the limit).

■

Definition 1.3.5 — Complete. A metric space, X, for which every Cauchy sequence converges is called **complete**.

■ **Example 1.10** X = sorted rationals.
X is complete. ■

Definition 1.3.6 $\{s_n\}, s_n \in \mathbb{R}$, is said to be
a) **increasing** if $s_n \leq s_{n+1}, \forall n$ ($\nearrow$)
b) **decreasing** if $s_n \geq s_{n+1}, \forall n$ ($\searrow$)
c) **monotonic** if either $\nearrow$ or $\searrow$

Theorem 1.3.8 If $\{s_n\}$ is monotonic, it converges if and only if it is bounded.

Proof. Read in Rudin text. ■

Lemma 3 If $E \subseteq \mathbb{R}$ is closed and bounded then, $\sup\{E\} \in E$.

Definition 1.3.7 — Upper/Lower limits. Let $\{x_n\}, x_n \in \mathbb{R}$ be a sequence and $E \subseteq \mathbb{R}, \mathbb{R} \bigcup \{-\infty, \infty\}$ be the set of all the subsequential limits. Then,

$$\sup\{E\} = \limsup x_n$$

$$\inf\{E\} = \liminf x_n$$

Theorem 1.3.9 With $\{x_n\}$ and E as in Definition 1.3.7, we have
a) $\limsup x_n \in E$
b) If $x > \limsup x_n$, there exists N such that $\forall n > N$ implies $x_n < x$.
Moreover, $\limsup x_n$ i sthe only number that satisfies a) and b).

Proof. a) If $\bar{X} \in \mathbb{R}$ then E is bounded above and hense, by Theorem 1.3.6, is closed. By the previous lemma, $\bar{X} \in E$.

b) Suppose $x > \bar{s}$ such that $x_n > x$ for infiniately man n. These values of n define a subsequence. This subsequence is either bounded or unbounded. In either case, there is $y \in E$ such that $y \geq x > \bar{x}$, which contradicts the definition of $\bar{x}$. To show uniqueness, let p and q satisfy a) and b), and suppose p<q. Let x be such that p<x<q. Since p satisfies b), $x_n < x \forall n \geq N$. But then q cannot satisfy a).

■

Theorem 1.3.10 a) Set $a_n = \sup\{x_k, k \geq n\}$. Then $\lim a_n = \limsup x_k$.
b) Set $b_n = \inf\{x_k, k \geq n\}$. Then, $\lim b_n = \liminf x_n$.
Is it clear that $\bar{x} \leq a_n \forall n$? Then also, $\bar{x} \leq \lim a_n$. If you show $\lim a_n \in E$, then also show $\lim a_n \leq \bar{x}$.

Theorem 1.3.11 — Frequently Occuring Sequences. a) If $p > 0, n^{-p} \to 0$
b) If $p > 0$, $\sqrt[n]{n} \to 1$
c) If $p > 0, \sqrt[n]{n} \to 1$
d) If $p > 0$, $\frac{n*a}{(1+p)^n} \to 0, \forall a \in \mathbb{R}$
e) If $|x| < 1, x^n \to 0$

1.3.1 Series

Given a sequence $\{a_n\}$ define $\{s_n\}$ where $s_n = \sum\limits_{i=1}^{n} a_i$. This sequence $\{s_n\}$ is called the sequence of partial sums, or **series**. We denote a series also as $\sum\limits_{n=1}^{\infty} a_n$. We write $\sum\limits^{\infty} a_n = s$ if $s_n \to s$. If $\{s_n\}$ diverges we say $\sum\limits_{n-1}^{\infty} a_n$ diverges.

Theorem 1.3.12 — Cauchy Criterium. $\sum\limits^{\infty} a_n$ converges if and only if for all $\varepsilon > 0$ exists N such that $|\sum\limits_{i=n}^{m} a_i| \leq \varepsilon$ holds for all $m, n \geq N$.

Theorem 1.3.13 — Comparison Test. a) If $|a_n| \leq c_n$, and if $\sum\limits^{\infty} c_n$ converges, so does $\sum\limits^{\infty} a_n$.
b) If $0 \leq d_n \leq a_n$ and $\sum\limits^{\infty} d_n$ diverges, so does $\sum\limits^{\infty} a_n$.

Proof. a) $|\sum\limits^{\infty} a_i| \leq \sum\limits^{\infty} |a_i| \leq \sum\limits^{\infty} c_i < \varepsilon$, $\forall n$ such that $n \geq N$ so $\sum\limits^{\infty} a_n$ converges.
b) By contradiction, if $\sum\limits^{\infty} a_n$ converges then by (a) so would $\sum\limits^{\infty} d_n$.

■

Theorem 1.3.14 — Geometric Series. If $0 \leq x < 1$, $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$, which diverges if $x \geq 1$, and converges otherwise.

Proof. If $x \neq 1$, then $\frac{1-x^{n+1}}{1-x} = \sum\limits_{i=0}^{n} x^i = s_n$. The result follows. ■

Lemma 4 Suppose $\{a_n\}$ is such that $a_n \searrow, a_n \geq 0$. Then $\sum\limits_{1}^{n} a_n$ converges if and only if $\sum\limits_{k=0}^{\infty} 2^k a_{2^k}$ (i.e. elements of $\{a_n\}$ indexed by 2^k: $\{a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots\}$).

Proof. Read in Rudin (listed as Theorem). ■

Theorem 1.3.15 $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p \geq 1$ and diverges if $p \leq 1$.

Proof. If $p \leq 0$, then $\frac{1}{n^p} \not\to 0$.

Recall, if sequence does not converge to zero, series cannot converge. But sequences converging to zero does not indicate that series must converge.

If $p \geq 0$, $a_n = n^{-p} = \sum_{k=0}^{\infty} [2^{1-p}]^k$, which is a geometric (power) series. But $2^{1-p} < 1$ if and only if $1 - p < 0$. ■

R The function $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$, for $p > 1$ is called the Reimann Zeta Function.

Theorem 1.3.16 $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $p \leq 1$. Also, $\sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log\log n)^2}$ converges, but $\sum_{n=2}^{\infty} \frac{1}{n(\log n)(\log\log n)}$ diverges.

Theorem 1.3.17 Let $f : [1, \infty) \to (0, \infty)$ be decreasing and $f(x) \to 0$. For $n \geq 1$ define $s_n = \sum_{k=1}^{n} f(k)$, $t_n = \int_1^n f(x)dx$ and set $d_n = s_n - t_n$. Then

i) d_n is decreasing dequence of nonegative numbers $[\{d_n\} \searrow, 0 \leq d_{n+1} < d_n < \cdots < d_1 = f(1)]$ which implies d_n converges (decreasing and bounded below). Does not imply that s_n, t_n converges, but it's a necessary condition.

ii) $0 \leq d_1 - \lim d_* \leq f(1), \forall x : ***$

Proof. Note:

$$
\begin{aligned}
t_{t+1} &= \sum_{k=1}^{n} \int_k^{k+1} f(x)dx \\
&\leq \sum_{k=1}^{\infty} \int_k^{k+1} f(k)dx \\
&= \sum_{k=1}^{n} f(k) \\
&= s_n
\end{aligned}
$$

a)

$$\begin{aligned} d_n - d_{n+1} &= (t_{n+1} - t_n) - (s_{n+1} - s_n) \\ &= \int_n^{n+1} f(x)dx = f(n+1) \\ &\geq \int_n^{n+1} f(n+1)dx - f(n+1) \\ &= 0 \end{aligned}$$

Also, $f(n+1) = s_{n+1} - s_n < s_{n+1} - t_{n+1} = d_{n+1}$ which shows $d_n \geq 0$.

b) He might write up later?

■

Corollary 1.3.18 — Integral Test. $\sum_1^\infty f(n)$ converges if and only if $\{t_n\}$ converges.

■ **Example 1.11** For $f(x) = \frac{1}{x}$,

$$s_n = \sum_{k=1}^{k} \frac{1}{k},$$

$$t_n = \int_1^n \frac{1}{x} dx = \log(n) \to \infty$$

Thus, $\{s_n\}$ diverges because $\{t_n\} \to \infty$. But, $d_n = \sum_{k=1}^{k} \frac{1}{k} - \log(n)$ converges. ■

The limit of of $\{d_n\}$ is known as **Eulers Constant,** γ. Theorem 1.3.17 (ii) gives the speed of convergence.

$$0 \leq d_n - \gamma \leq \frac{1}{n}$$

Lemma 5 $\sum_0^\infty \frac{1}{n!}$ converges.

Definition 1.3.8 — e. $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Theorem 1.3.19 $e = \lim_{n\to\infty} (1 + \frac{1}{n})^n$

Theorem 1.3.20 e is irrational.

Theorem 1.3.21 — Abel's Partial Summation Formulas. For say, $\{a_n\}$ and $\{b_n\}$,

$$\sum^n a_k b_k = \sum^n A_k(b_k - b_{k+1}) + A_n b_{n+1}$$

Where $A_n = a_1 + a_2 + \cdots + a_n$. Thus $\sum^\infty a_n b_n$ converges if both $\sum^n A_k(b_k - b_{k+1})$ and $A_n b_{n+1}$ converge.

Proof. Write $A_n = 0$, then,

$$\begin{aligned}\sum^{n} a_k b_k &= \sum^{n} (A_k - A_{k-1}) b_k \\ &= \sum^{n} A_k b_k - \sum_{k=1}^{n} A_k b_k + A_n b_{n+1} \\ &= \sum^{n} A_k (b_k - b_{k+1}) + A_n b_{n+1}\end{aligned}$$

■

Theorem 1.3.22 — Dirichlet's Test. Let $\sum_{1}^{\infty} a_n$ have partial sums, $A_n = a_1, \dots, a_n$ which are bounded and $\{b_n\}$ be such that $b_n \searrow 0$. Then $\sum^{\infty} a_n b_n$ converges.

Proof.

$$\begin{aligned}|\sum_{m}^{m+n} A_k (b_k - b_{k+1})| &\leq \sum_{m}^{m+n} |A_k| (b_k - b_{k+1}) \\ &\leq \sum_{m}^{m+n} M (b_k - b_{k+1}) \\ &= M(b_k - b_{k+1}) \underset{n\to\infty}{\to} 0\end{aligned}$$

■

Definition 1.3.9 — Cesarian Summability. a) If $\{s_n\}$ converges, and $t_n = \frac{s_1 + \dots + s_n}{n}$ then $\{t_n\}$ also converges. (Note: the opposite does not hold; i.e. $s_n = (-1)^n$)

b) Let $\{s_n\}$ be the sequence of partial sums of $\sum^{\infty} a_n$ and let $\{t_n\}$ be as above. If $\{t_n\}$ converges, we say that $\sum^{\infty} a_n$ is a **Cesaro Summable**.

■ **Example 1.12** a) $\sum_{1}^{\infty} (-1)^{n+1}$ does not converge, but $t_n \to \frac{1}{2}$.

b) $\sum_{1}^{\infty} (-1)^{n+1} n$ does not converge, but $\limsup t_n = \frac{1}{2} and \lim] inf t_n = 0$. Thus, we do not have a Cesaro Summable.

■

1.4 Continuity

Definition 1.4.1 Let X and Y be geometric spaces and $f : E \to Y$ for $E \subseteq X$. Let p be a limit point of E (not necessarily a member of E). We write

$$f(x) \underset{x\to p}{\to} q$$

if $\forall \varepsilon > 0$, there exists $\delta = \delta(\varepsilon, p)$ such that $d_y(f(x), q) < \varepsilon$ if $d_x(x, p) < \delta$.

Theorem 1.4.1 For X, Y, E, f, and p as in Definition 1.4.1,

$$f(x) \underset{x\to p}{\to} q \text{ iff } f(x_n) \underset{n\to\infty}{\to} q$$

for all sequences $\{x_n\}$ such that $x_n \neq p$ and $x_n \underset{n\to\infty}{\to} p$

Definition 1.4.2 X, Y, E, and f as in Definition 1.4.1. Let $p \in E$. Then,

a) If p is also a limit point of E, we say that f is **continuous at p** if $f(x_n) \underset{x\to p}{\to} f(p)$
b) If p is an isolated point of E, then f is **continuous at p**.
c) Alternatively to a) and b), if $\forall \varepsilon > 0$, there exists $\delta \equiv \delta(\varepsilon, p) > 0$ such that

$$d_x(x,p) < \delta \implies d_y(f(x), f(y)) < \varepsilon$$

d) If f is continuous at every point of E it is called **continuous on E**.

Theorem 1.4.2 For X, Y, Z matric spaces, and $E \subseteq X$,

$$f : E \to Y$$

$$g : f(E) \to Z$$

we define, $h : E \to Z$ by $h(x) = g(f(x))$. Then if f is continuous at p, and g is continuous at f(p) then h is continuous at p.

Theorem 1.4.3 $f : X \to Y$ is continuous if and only if $f^{-1}(V)$ is open in $X \forall V$ open in Y.

Proof. blah ■

Bibliography

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