Theory of Statistics I

Take Two

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Contents

Part One

1. Real Analysis Review

1.1 The Real Number System

1.1.1 Rationals

Start with integers as given.

Definition 1.1.1 — Rational Numbers. Rationals are numbers of the form $\frac{m}{n}$, for m,n integers, $n \neq 0$ such that:

PR 1: The sum, difference, product, and ratio (division by 0 excluded) of any two rationals is a rational.

PR 2: $p+q=q+p, pq=qp$ (Commutative Property)

PR 3: $(p+q)+r=p+(q+r), (pq)r=p(qr),$ (Associative Property)

PR 4: $(p+q)r=pr+qr$ (Distributive Property)

PR 5: $\forall$ two rationals p and q we have either p=q, p<q, or q<p (Ordering Property)

PR 6: If $p<q$ and $q<r$, then $p<r$ (Transitivity of <)

PR 7: If $p>0$ and $q>0$, then $p+q>0$ and $pq>0$

PR 8: If $p<q$, then $p+r<q+r$ $\forall$ r

The rational number system is inadequate.

■ **Example 1.1** There is no rational number p that satisfies $p^2=2$ ■

Proof. Suppose such a p existed, and so $p=\frac{m}{n}$. Note that m,n can be chosen so not both are even. Then we have,

$$m^2=2n^2$$

Thus, m^2 is even, and hence m is even. (The square of an odd number is odd). Hence, m^2 is divided by 4. So, $2n^2$ is divisible by 4, or n^2 is even which implies that n is even - **contradiction**. ■

This example can be used to show that we can have a set of rational numbers bounded from above but has no supremum.

■ **Example 1.2** Let A be the set of < 0 rationals p, such that $p^2 < 2$. Let B be the set of > 0 rationals p, such that $p^2 > 2$. Then A contains no largest number and B contains no smallest number. ■

Proof. If $p \in A$, choose a rational h such that, $0 < h < 1$ and $h < \frac{2-p^2}{2p+1}$ and set $q = p + h$. Then q is rational and

$$\begin{aligned} q^2 &= p^2 + (2p+h)h \\ &< p^2 + (2p+1)h \\ &< p^2 + (2-p^2) \\ &= 2 \end{aligned}$$

If $p \in B$, set

$$q = p - \frac{p^2-2}{2p} = \frac{p}{2} + \frac{1}{p}$$

and

$$\begin{aligned} q^2 &= p^2 - (p^2-2) + (\frac{p^2-2}{2p})^2 \\ &> p^2 - (p^2-2) \\ &= 2 \end{aligned}$$

■

R An axiomatic treatment of the real number system uses PR1 - PR8 as axioms together with the "completeness axiom". The non-axiomatics treatment is due to Dedekind.

1.1.2 Sets and Subsets

If A is any set, $\mathbf{x} \in \mathbf{A}$ means that x is a member of A, and $\mathbf{x} \notin \mathbf{A}$ means x is not a member of A. A set B is a **subset** of A if for every $x \in B$ we have $x \in A$, and we write $A \subseteq B$. B is a **proper subset** of A, $B \subset A$, if there $\exists\, x \in A$ with $x \notin B$. The **empty set** is denoted by $\emptyset$, and $\emptyset \in A$, $\forall$ other set A.

$A \cup B = B \cup A$ - union with commutative property

$A \cap B = B \cap A$ - intersection with commutative property

$(A \cup B) \cup C = A \cup (B \cup C)$
$(A \cap B) \cap C = A \cap (B \cap C)$ - associative property

$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ - distributive property
$(\cup A_i)^c = \cap A_i^c$
$(\cap A_i)^c = \cup A_i^c$

Definition 1.1.2 — Dedekind Cuts. A set α of rational numbers is said to be a **cut** if

1. α is a proper, but non-empty, subset of the rational numbers.
2. If $p \in \alpha$ (p is rational), and $q < p$ (q is rational) then $q \in \alpha$
3. It contains no largest rational.

A cut of the form $\alpha = \{$p: p is rational and $p < r\}$ where r is rational are called **rational cuts** and are denoted by r^*.

The development of the real number system proceeds as follows:
First, the set of cuts is equipped with an order relation, and the operation of addition and multiplication an it will show that the resulting arethmatic satisfies PR 1 - PR 8.

If α, β are cuts then,

$$\begin{aligned} \alpha < \beta &\text{ if } \alpha \subset \beta \text{ and} \\ \alpha \leq \beta &\text{ if } \alpha \subseteq \beta \\ \alpha + \beta &= \{r : r = p + q \text{ for some } p \in \alpha, q \in \beta\} \\ (\alpha + 0^* &= \alpha) \end{aligned}$$

If $\alpha + \beta = 0^*$, write $\beta = -\alpha$. (It can be shown that $\forall \alpha$ there is one and only one β such that $\alpha + \beta = 0^*$.)

$$|\alpha| = \begin{cases} \alpha, & \text{if } \alpha \geq 0^*, \\ -\alpha, & \text{if } \alpha < 0^*. \end{cases}$$

For $\alpha \geq 0^*$ and $\beta \geq 0^*$,

$$\alpha\beta = \{\text{p:p rational such that either } p < 0 \text{ or } p = pq, \text{ for } q \in \alpha, r \in \beta \text{ with } q \geq 0 \text{ and } r \geq 0.\}$$

For general α, β,

$$\alpha\beta = \begin{cases} -(|\alpha||\beta|), & \text{if } \alpha < 0^*, \text{and } \beta \geq 0^* \\ & \text{or if } \alpha \geq 0^* \text{and } \beta < 0^* \\ |\alpha||\beta|, & \text{if } \alpha < 0^*, \text{and } \beta < 0^* \end{cases}$$

If $\alpha \neq 0^*$, then $\forall \beta$ there is one and only one γ such that $\alpha\gamma = \beta$, and this γ is denoted by $\frac{\beta}{\alpha}$. (In technical terms, we made the set of cuts an **ordered field**.)

Second, it is shown that replacing the set of rational numbers by the corresponding cuts preserves sums, products and order, ie,

1. $p^* + q^* = (p+q)^*$
2. $p^* q^* = (pq)^*$
3. $p^* < q^*$ iff $p < q$

In technical terms, the ordered field of rational numbers is **isomorphic** to that of rational events.

Theorem 1.1.1 — Dedekind. Let A, B be $\subset \mathbb{R}$ such that,

(a) $A \cap B = \emptyset$
(b) $A \cup B = \mathbb{R}$
(c) neither A nor B is empty
(d) if $\alpha \in A$, $\beta \in B$, then $\alpha < \beta$

Then there $\exists\, \gamma \in \mathbb{R}$ such that $\alpha \leq \gamma$, $\forall \alpha \in A$ and $\gamma \leq \beta$, $\forall \beta \in B$.

Proof. First, suppose there are 2 γ, say $\gamma_1 < \gamma_2$. Take γ_3 such that $\gamma_1 < \gamma_3 < \gamma_2$.

$$\gamma_3 < \gamma_2 \text{ implies that } \gamma_3 \in A$$

$$\gamma_1 < \gamma_3 \text{ implies that } \gamma_3 \in B$$

However, these implications contradict the disjointness (part (a)). Define $\gamma - \{$p: p rational such that p $\in \alpha$ for some $\alpha \in A\}$. The proof proceeds by showing that γ is a cut, and hense a real number that satisfies $\alpha \leq \gamma$ for $\alpha \in A$ and $\gamma \leq \beta \ \forall \ \beta \in B$. ■

Corollary 1.1.2 If A, B are as in the theorem, then either A contains a largest number or B contains a smallest number.

Corollary 1.1.3 Let $E \neq \emptyset$ be a subset of $\mathbb{R}$. Then, if E is bounded above a supremum (least upper bound) exists.

Proof. Define
$A = \{\alpha : \alpha < x \text{ for some } x \in E\}$
$B = A^c$
Clearly, all members of B are upper bounds of E. It is sufficient to prove that B contains a smallest nubmer, or, by Corollary 1, that A does not contain a largest number (and thus prove by contradiction). Indeed if $\alpha \in A$ $\exists$ an $x \in E$ such that $\alpha < x$. But, by Property 1 (???) there $\exists$ an α' such that $\alpha < \alpha' < x$ where $\alpha' \in A$ (i.e. we can always find a larger α so, since there is no largest α, there MUST be a smallest β). ■

Theorem 1.1.4 Any real number admits a decimal expansion.

Proof. Let $x > 0$, $x \in \mathbb{R}$. Let $n_0 = [x]$ (n largest integer < x). Let n_1 be the largest integer such that $n_0 + \frac{n_1}{10} < x$. Having defined $n_0 \ldots n_{k-1}$, define n_k as the largest integer such that $n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \cdots + \frac{n_k}{1-k} \leq x$. Let E bet he set of resluting numbers for $k = 1, 2, \ldots$. Then x is the supremum of E and $n_0, n_1, \ldots$ is its **decimal expansion**. Conversely, and set of integers $n_0, n_1, \ldots$ defines a set of numbers, E, bounded above by $n_0 + 1$. ■

Definition 1.1.3 — Extended Real Number System.

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

1.1.3 Euclidean Space

Definition 1.1.4 — Vector Space. For any $k \in \mathbb{Z}^+$. Let $\mathbb{R}^K$ be the set of ordered k-tuples.

$$\begin{aligned}
\underline{x} &= (x_1, \ldots, x_k) \text{ with } x_i \in \mathbb{R} \\
\underline{y} &= (y_1, \ldots, y_k) \alpha \in \mathbb{R} \\
\underline{x} + \underline{y} &= (x_1 + y_1, \ldots, x_k + y_k) \\
\alpha \underline{x} &= (\alpha x_1, \ldots, \alpha x_k)
\end{aligned}$$

Which makes $\mathbb{R}^k$ a **vector space** over the **real field**.

Definition 1.1.5 — Inner/Scalar/Dot Product.

$$\underline{x} \cdot \underline{y} = \sum_{i=1}^{k} x_i y_i$$

Definition 1.1.6 — Norm/Length.

$$|\underline{x}| = (\underline{x}\underline{x})^{\frac{1}{2}} = \sqrt{\sum_{i=1}^{k} x_i^2}$$

Definition 1.1.7 — Euclidean K-space. The vector space $\mathbb{R}^k$ with the inner product and norm is called **Euclidean k-space**.

Theorem 1.1.5 For $\underline{x}, \underline{y} \in \mathbb{R}^k, \alpha \in \mathbb{R}$

a) $|\underline{x}| \geq 0, |\underline{x}| = 0$ iff $\underline{x} = \underline{0}$
   $|\alpha \underline{x}| = |\underline{\alpha}||\underline{x}|$
b) **Cauchy-Schwarz Inequality** $|\underline{x} \cdot \underline{y}| \leq |\underline{x}||\underline{y}|$
c) **Triangle Inequality** $|\underline{x} + \underline{y}| \leq |\underline{x}| + |\underline{y}|$

1.2 Elements of Set Theory

Definition 1.2.1 Let A, B be sets and suppose that to each $x \in A$ there corresponds an elements of B denoted by $f(x)$. Then f is a **function** (or in more general space, mapping) from A (in)to B.

A is called the **domain** of f. $f(x)$ is the **value** of f at x, $R(f) = \{f(x) : x \in A\}$ is the **range** of f.

Definition 1.2.2 — Image. If f is a function from A to B ($A \to B$) and $E \subseteq A$ we write $f(E) = \{f(x) : x \in E\}$ and call it the **image** of E under f. If $f(A) = B$, then we say f maps A **onto** B.

Definition 1.2.3 — Inverse Image. Let $f : A \to B$ and $E \subseteq B$. We write $f^{-1}(E) = \{x]inA : f(x) \in E\}$ and call it the **inverse image** of E **under** f. NB: If $E = \{y\}, y \in B$ we also write $f^{-1}(y)$ (versus $f^{-1}(\{y\})$). If $\forall y \in B$ $f^{-1}(y)$ consists of at most one element, then f is one to one mapping of A **into** B.

Theorem 1.2.1 a) $f^{-1}(A \cup B) = F^{-1}(A) \cup f^{-1}(B)$
b) $f(A \cup B) = f(A) \cup f(B)$
c) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

Actually, these may be extended to arbitrary unions and intersections.

$$f^{-1}(\bigcup_{\alpha} A_\alpha) = \bigcup_{\alpha} f^{-1}(A_\alpha)$$
$$f^{-1}(\bigcap_{\alpha} A_\alpha) = \bigcap_{\alpha} f^{-1}(A_\alpha)$$
$$f(\bigcup_{\alpha} A_\alpha) = \bigcup_{\alpha} f(A_\alpha)$$

Note: $f(A \cap B)$ is not necessarily equal to $f(A) \cap f(B)$ (see notes for example and sketch)

Definition 1.2.4 — Cardinal Number. If $\exists$ a one-to-one mapping of A onto B, we say that A and B have the same **cardinal number**, or that they are **equivalent** $A \sim B$.

a) $A \sim A$ (reflective)
b) If $A \sim B$, then $B \sim A$ (symmetric)
c) If $A \sim B$ and $B \sim C$, then $A \sim C$. (transitive)

Definition 1.2.5 — (In)finite/(Un)Countable. Let $\mathbb{Z}^+ = \{1,2,\dots\}$ and $\mathbb{Z}_n^+ = \{1,2,\dots,n\}$ and let A be a set.

a) We say A is **finite** if $A \sim \mathbb{Z}_n^+$ for some n or if $A = \emptyset$
b) A is **infinite** if it is not finite
c) A is **countable** if $A \sim \mathbb{Z}^+$
d) A is **uncountable** if A is not finite and countable.

Note: If A and B are finite, then $A \sim B$ if and only if they have the same number of elements. This is not true if they are infinite.

Example 1.3 Equivalent Infinite Sets

1. The set $\mathbb{Z}^+$ of all integers is countable. Then take

$$f(x) = \begin{cases} \frac{n}{2}, & \text{if n is even} \\ -(\frac{n-1}{2}), & \text{if n is odd} \end{cases}$$

1	$\to$	0
2	$\to$	1
3	$\to$	-1
4	$\to$	2
5	$\to$	-2
6	$\to$	3
7	$\to$	-3

Table 1.1: Corresponding Integers

The set of positive, even integers is countable. Take

$$f(x) = 2n$$

■

Theorem 1.2.2 The countable union of countable sets is countable.

Proof. Let $A_1, A_2, \dots$ be countable and assume that they are disjoint (for if not, you can consider the sequences of countable sets that are disjoint - $A_1, A_2 - A_1, \dots$), which are countable and have the same union. Let $A_k = \{a_{k1}, a_{k2}, \dots\}$ and consider the arrangement of $\bigcup_{k=1}^{\infty} A_k$.

a_{11}	a_{12}	a_{13}	a_{14}	$\dots$	1	2	6	7	$\dots$
a_{21}	a_{22}	a_{23}	a_{24}	$\dots$	3	5	8	$\dots$	$\dots$
a_{31}	a_{32}	a_{33}	a_{34}	$\dots$	4	9	13	$\dots$	$\dots$
a_{41}	a_{42}	a_{43}	a_{44}	$\dots$	10	12	$\dots$	$\dots$	$\dots$

Table 1.2: Reassigning new values to counting integers.

■

Theorem 1.2.3 Every infinite set has a countable subset.

Proof. Let a_1 be any element of A. Since A is infinite, it contains an $a_2 \neq a_1 \dots$. So it contains a countable subset. ■

Theorem 1.2.4 Every infinite set, A, is equivalent to at least one of its proper subsets.

Proof. Let $E - \{a_1, a_2, \dots\}$ be a countable subset of A (which exists by previous Theorem). Write,

$$E = E_1 \cup E_2$$
$$E_1 = \{a_{odd}\}$$
$$E_2 = \{a_{even}\}$$

Then, $E \sim E_2$
Define,

$$g : E \to E_2$$
$$g(a_i) = a_{2i}$$

$$f(a) = \begin{cases} a, & \text{if } a \notin E, \\ g(a), & \text{if } a \in E. \end{cases}$$

So, we can also say that $A - E_1 \subset A$ and thus, $A \sim (A - E_1)$ ■

Theorem 1.2.5 The set of real numbers in [0,1] is uncountable.

Proof. Suppose all numbers in [0,1] are countable, $\{a_1, a_2, \dots\}$.
Write them in decimal expansion form. So, we can say

$$a_1 = 0.a_{11}a_{12}\dots a_{1n}\dots$$
$$a_2 = 0.a_{21}a_{22}\dots a_{2n}\dots$$

Recall,

$$0 = 0.000000000\dots$$
$$1 = 0.999999999\dots$$

Now, consider the number, β with decimal expansion $\beta = 0.b_1 b_2 \dots$ where

$$b_n = \begin{cases} 1, & \text{if } a_{nn} = 1, \\ 2, & \text{if } a_{nn} \neq 1. \end{cases}$$

There will always be 1 element difference. ALWAYS. ■

Theorem 1.2.6 If A is countable, then so is A^n, where

$$A^n = \{(a_1, \dots, a_n); a_i \in A\}$$

Proof. Statement is true for any n=1 since $A^1 = A$. Assume true for n=k. To show A^{k+1} is countable, write an element $(a_1, a_2, \dots, a_k, a_{k+1}) = (\underline{a}, a_{k+1}), \underline{a} \in A^k$. Thus, $A^{k+1} = \bigcup_{\underline{a} \in A^k} \{\underline{a}, a_{k+1}); a_k \in A\}$ (see previous Theorem). ■

Bibliography

Index