Probability Theroy based on Measure Theory

STAT 517

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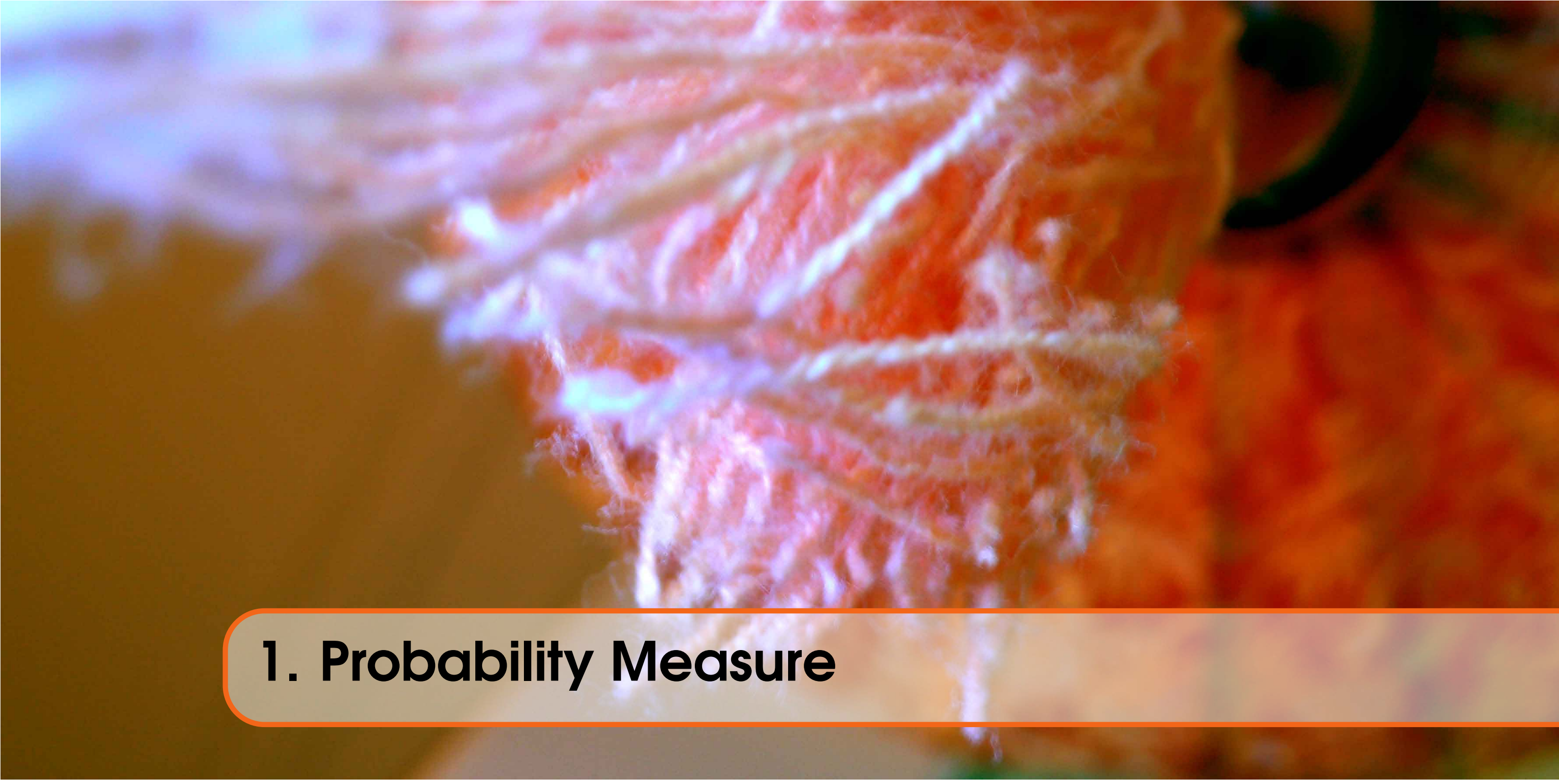
Part One



1. Probability Measure

1.1 Overview

1.2 Probability on a Field

Definition 1.2.1 — Ω. Non emtpy set.

Definition 1.2.2 — Paving. A collection of a subset of Ω is a paving.

Definition 1.2.3 — Field. A **field** $\mathscr{F}$ is a paving satisfying

(i) $\Omega \in \mathscr{F}$

(ii) $A \in \mathscr{F} \Rightarrow A^C \in \mathscr{F}$

(iii) $A, B, \in \mathscr{F}, \Rightarrow A \bigcup B \in \mathscr{F}$

Derived Properties about a Field

- $\emptyset \in \mathscr{F}$ (by (i) and (ii):

$$\begin{aligned}\Omega \in \mathscr{F} &\Rightarrow \Omega^C \in \mathscr{F} \\ &\Rightarrow \emptyset \in \mathscr{F})\end{aligned}$$

- (i) can be replaced by "$\mathscr{R}$ is nonempty" because,
  Let $A \in \mathscr{F}$,

$$\begin{aligned}&\Rightarrow A^c \in \mathscr{F} \\ &\Rightarrow A^C \bigcup A \in \mathscr{F} \\ &\Rightarrow \Omega \in \mathscr{F}\end{aligned}$$

- $A \in \mathscr{F}, B \in \mathscr{F}, \Rightarrow, A \bigcap B \in \mathscr{F}$ because,

$$\begin{aligned}&(A \bigcap B)^C = A^C \bigcup B^C (DeMorgan's Law) \\ &A \bigcap B = (A^C \bigcup B^C)^C\end{aligned}$$

- $A_1, \dots, A_m \in \mathscr{F} \Rightarrow A_1 \bigcup, \dots, \bigcup A_m \in \mathscr{F}$ (mathematical induction)
- $A_1, \dots, A_m \in \mathscr{F} \Rightarrow A_1 \bigcap, \dots, \bigcap A_m \in \mathscr{F}$

Definition 1.2.4 — σ-Field. Similar to the definition of a field except for (iii). A paving satisfying

(i) $\Omega \in \mathscr{F}$

(ii) $A \in \mathscr{F} \Rightarrow A^C \in \mathscr{F}$

(iii) $A_1 \in \mathscr{F}, \dots, A_m \in \mathscr{F}$

$\bigcup_{k=1}^{m} A_k \in \mathscr{F}$ (finite additvity)

If we replace (iii) from before by (iii') here:
For $A_1 \in \mathscr{F}, \dots, A_m \in \mathscr{F}$

$$\bigcup_{k=1}^{\infty} A_k \in \mathscr{F}$$

then $\mathscr{F}$ is called a **σ-field**.

Derived Facts

- Again, (i) can be repalced by $\mathscr{F}$ no empty, (iii) can be replaced $A_1 \in \mathscr{F}, \dots, A_m \in \mathscr{F}$

■ **Example 1.1** $\Omega = (0, 1]$ (from now on all intervals are left open, right closed)

R Recall that σ-fields are generated by fields. Fancy scripts denote a σ-field. Fancy scripts with a zero subscript denote a field.

$\mathscr{B}_0$ is the collection of all finite union of disjoint intervals. Aside: something that we can consider easily, but computers cannot. (e.g. think of all stars, etc)

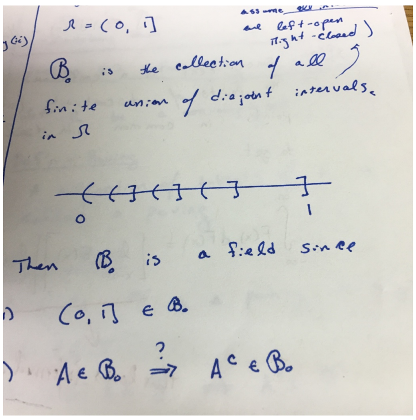


Figure 1.1: Finite unioin of three disjoint intervals.

Then $\mathscr{B}_0$ is a field.

(i) (0, 1] $\in \mathscr{B}_0$

(ii) $A \in \mathscr{B}_0 \Rightarrow A^c \in \mathscr{B}_0$

(iii) $A \in \mathscr{B}_o, B \in \mathscr{B}_o \Rightarrow A \bigcup B \in \mathscr{B}_o$

■

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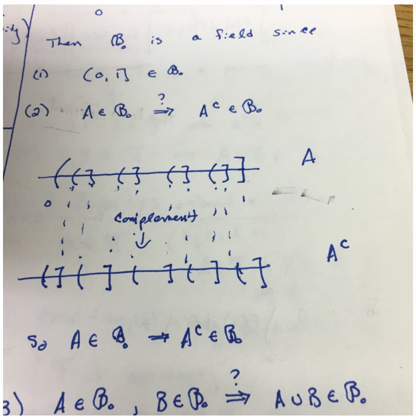

Figure 1.2: A and complement of A.

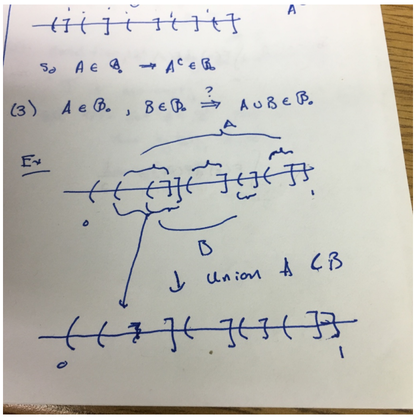

Figure 1.3: Union of A and B is still in $\mathscr{B}_o$

$\mathscr{B}_0$ = collection of finite unions of disjoin subintervals of (0, 1]. Is a field.

Definition 1.2.5 — Power Set. A σ-field is generated by a paving of power set. Let Ω be a set. The collection of all subsets of Ω is the power set written as 2^{Ω}.

R Where does this notation come from?
Consider ths case where Ω is finite

$$\Omega = \{\omega_1, \dots, \omega_n\}$$

Total number of subsets of Ω.
$\emptyset$, 1 element sets, 2-element sets, ..., n-element ests.

$$() + () + \cdots + = (1+1)^n$$

$\#(\mathscr{F}) = 2^{\#\Omega}$, so it seems reasonable to denote $\mathscr{F} = 2^{\Omega}$.
It is also easy to show that 2^{Ω} is a σ-field. (The largest, even. The smallest: $\{\emptyset, \Omega\}$ which is also a σ-field.)

$$\{\emptyset, \Omega\} \subseteq \sigma\text{-field} \subseteq 2^{\Omega}$$

It turns out we can extend notion of lenght from $\mathscr{B}_0$ to σ-field generated by $\mathscr{B}_o$.

Now, let $\mathscr{A}$ be a nonempty paving of Ω. We define

$$\sigma(\mathscr{A}) = \bigcap\{\mathscr{B} \subseteq 2^{\Omega} : \mathscr{B} \text{ is a } \sigma\text{-field}, \mathscr{A} \subseteq \mathscr{B}\}$$

OR rather, the *intersection* of all σ-fields that contains $\mathscr{A}$.

Let

$$\mathbb{F}(\mathscr{A}) = \{\mathscr{B} \subseteq 2^{\Omega} : \mathscr{B} \text{ is a } \sigma\text{-field, } \mathscr{B} \supseteq \mathscr{A}\}$$

Then,

$$\sigma(\mathscr{A}) = \bigcap \mathscr{B}$$
$$\mathscr{B} \in \mathbb{F}(\mathscr{A})$$

Derived Facts
$\mathbb{F}(\mathscr{A})$ is nonempty. For example, 2^{Ω} is a σ-field and $2^{\Omega} \supseteq \mathscr{A}$.
$\bigcap B$ is a σ-field. $(B \in \mathbb{F}(\mathscr{A}))$

R Get notes about notation/levels.

Proof. We will prove that indeed $\sigma(\mathscr{A})$ is a σ -field. Recall that we have three conditions above for σ-field.

(i)

$$\Omega \in \sigma(\mathscr{A})$$

$$\Omega \in \cap_{B\in\mathbb{F}(\mathscr{A})} B$$

Because: B is σ-field, $A \in B$, $\forall B \in \mathbb{F}(\mathscr{A})$.
$A \in \bigcap B \Rightarrow A \in B \forall B \in \mathbb{F}(\mathscr{A})$

(ii) $\Rightarrow A^C \in B, \forall B \in \mathbb{F}(\mathscr{A})$
$\Rightarrow A^C \in \cap_{B\in\mathbb{F}}(\mathscr{A}) B$

(iii) $A_1, \ldots, \in \cap_{B\in\mathbb{F}(\mathscr{A})} B, \forall B \in \mathbb{F}(\mathscr{A})$
$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in B, \forall B \in \mathbb{F}(\mathscr{A})$

So, $\sigma(\mathscr{A}$ is a σ-field, we call it the σ-field, generated by $\mathscr{B}_o$. We know how tot assign lenth to members of $\mathscr{B}_o$, we now show the assignment can be extended to $\sigma(\mathscr{B}_o)$ ■

■ **Example 1.2** Let $\mathscr{I}$ be the collection of *all* subintervals of (0,1].

Note that $\mathscr{I}$ is a smaller collection than $\mathscr{B}_0$ since $\mathscr{B}_0$ can have numerous different combinations of the sets.

Let

$$\mathscr{B} = \sigma(\mathscr{I})$$

.

This is a Borel-σ-field. (a member of $\mathscr{B}$ in Borel set.)
It turns out

$$\sigma(\mathscr{I}) = \sigma(\mathscr{B}_o)$$

This is because $\sigma(\mathscr{I})$ is a σ-field.
So,

$$\sigma(\mathscr{I}) \supseteq \mathscr{B}_o$$
$$\sigma(\mathscr{I}) \supseteq \sigma(\mathscr{B}_o)$$

Also,

$$\mathscr{I} \subseteq \mathscr{B}_o$$
$$\sigma(\mathscr{I}) \subseteq \sigma(\mathscr{B}_o)$$

Thus,

$$\sigma(\mathscr{I}) = \sigma(\mathscr{B}_o)$$

■

Definition 1.2.6 — Probability Measure. Probablity measures on field. Suppose $\mathscr{F}$ is a field on a nonempty set Ω. A probability measure is a function $P : \mathscr{F} \to \mathbb{R}$.
(i) $0 \leq P(A) \leq 1, \forall A \in \mathscr{F}$
(ii) $P(\emptyset) = 0$, $P(\Omega) = 1$
(iii) If $A_1, \ldots$ are disjoint emembers of $\mathscr{F}$ and $\bigcup A_n \in \mathscr{F}$ then we have countable additivity:

$$P(\bigcup A_n) = \sum_{n=1}^{\infty} P(A_N)$$

R Note that (iii) also implies finite additivity. Prove by adding infinite empty sets on end.

If Ω is nonempty set. And $\mathscr{F}$ is a σ-field on Ω. And P is a probability measure on $\mathscr{F}$.
Then $(\Omega, \mathscr{F}, P)$ is called a **probability space.**
And $(\Omega, \mathscr{F})$ is called a **measurable space.**

R If $A \subseteq B$, then $P(A) \leq P(B)$. This is because we may write B as

$$B = A \bigcup (B \setminus A)$$

R

$$P(A) + P(B) = P(A \bigcup B) + P(A \bigcap B)$$

Friday August 26

Recall,
Probability measure on a field, $\mathscr{F}_0$.
- $P(A) + P(B) = P(A \bigcup B) + P(A \bigcap B)$
  - $P(A) = P(AB^C) + P(AB)$
  - $P(B) = P(BA^C) + P(AB)$
  - $P(A) + P(B) = P(AB^C) + P(BA^C) + 2P(AB)$

  - $P(A\bigcup B) = P(AB^C) + P(BA^C) + P(AB)$
- $P(A\bigcup B) = P(A) + P(B) - P(AB)$ By induction, we can prove if $A_1,\dots A_n$,

$$P(\bigcup_{k=1}^{n} A_k) = \sum_{k=1}^{n} P(A_k) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} A_iA_j) + \dots + (-1)^{n+1}P(A_1,\dots,A_n)$$

  Inclusion- Exclusion Formula
- If $A_1,\dots A_n \in \mathscr{F}$,

$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1$$
$$\vdots$$

  Then,

$$\bigcup_{k=1}^{n} A_k = \bigcup_{k=1}^{n} B_k$$

  but the B_i are disjoint. Also $A_K \subseteq B_k \forall k = 1,\dots,n$.

$$P(\bigcup_{k=1}^{n} A_k) = P(\bigcup_{k=1}^{n} B_k) = \sum_{k=1}^{n} B_k \leq \sum_{k=1}^{n} A_k$$

  Thus, $P(\bigcup_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} A_k$. Finite subadditivity.

Some conventions,
If $A_1,\dots$ is a sequence of sets, we say $A_n \uparrow A$ if
1. $A_1 \subseteq A_2 \subseteq \dots$
2. $\bigcup_{k=1}^{\infty} A_k = A$

If $A_1,\dots$ is a sequence of sets, we say $A_n \downarrow A$ if
1. $A_1 \supseteq A_2 \supseteq \dots$
2. $\cap_{k=1}^{\infty} A_k = A$

Theorem 1.2.1 If P is a probability measure on a field $\mathscr{F}$ Then,
1. Continuity from below.
   If $A_n \in \mathscr{F} \quad \forall n, A \in \mathscr{F}$
   $$A_n \uparrow A$$
   then
   $$P(A_n) \uparrow P(A)$$
2. Continuity from above.
   If $A_n \in \mathscr{F} \quad \forall n.A \in \mathscr{F}$
   $$A_n \downarrow A$$
   then
   $$P(A_n) \downarrow P(A)$$
3. Countable subadditivity.
   If $A_n \in \mathscr{F} \quad \forall n. \bigcup_{k=1}^{\infty} A_k \in \mathscr{F}$ then

$$P(\bigcup_{n=1}^{\infty} A_k) \le \sum_{n=1}^{\infty} P(A_k)$$

Proof. 1. If $A_1, \ldots A_n \in \mathscr{F}$,

$$B_1 = A_1$$
$$B_2 = A_2 \setminus A_1$$
$$B_3 = A_3 \setminus A_2$$
$$\vdots$$

then, $B_1, \ldots$ are disjoint.

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

$$\begin{aligned} P(A) &= P(\bigcup_{n=1}^{\infty} A_n) \\ &= P(\bigcup_{n=1}^{\infty} B_n) \\ &= \sum_{n=1}^{\infty} P(B_n) \\ &= \lim_{n\to\infty} \sum_{n=1}^{\infty} P(B_n) \\ &= \lim_{n\to\infty} P(A_n) \end{aligned}$$

2. $A_n \downarrow A \Leftrightarrow A_n^C \uparrow A^C$
But by (1),

$$P(A_n^C) \uparrow P(A^C)$$
$$1 - P(A_n) \uparrow 1 - P(A)$$
$$P(A_n) \downarrow P(A)$$

3. By finite subadditivity,

$$P(\bigcup^{n} k = 1 A_k) \le \sum^{n} k = 1 P(A_k) \le \sum_{n=1}^{\infty} P(A_n)$$

But since, by (1), because

$$\bigcup_{k=1}^{n} A_k \uparrow \bigcup_{n=1}^{\infty} A_n$$

$$P(\bigcup_{k=1}^{n} A_k) \uparrow P(\bigcup_{n=1}^{\infty} A_n)$$

So,

$$P(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} P(A_n)$$

■

R $A \in \mathscr{F} =$ "A is F-set".

1.3 Extention of Probability Measure to a σ-field

Let f be a function $f : D \to R$.

Let $\tilde{D}$ be another set such that

$$D \subseteq \tilde{D}$$

An extantion of f onto $\tilde{D}$ is

$$\tilde{f} : \tilde{D} \to R$$

Such that $f(x) = \tilde{f}(x) \forall x \in D$

$\tilde{f}$ is an extention of f on D.

We say f has unique extention, $\tilde{f}$ onto $\tilde{D}$ if

1. $\tilde{f}$ is an extension of f to $\tilde{D}$.
2. if g is another extension of f to $\tilde{D}$ then $\tilde{f} = g$ on D.

Theorem 1.3.1 A probability measure on a field has a unique extension on the σ-field generated by this field.

This means that if $\mathscr{F}_0$ is a field, and P is a probability measure on $\mathscr{F}_0$, then there exists a probability measure, Q on $\sigma(\mathscr{F})$ such that

$$Q(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Moreover, if $\tilde{Q}$ is another probability measure on $\sigma(\mathscr{F}_0)$ such that $\tilde{Q} = P(A) \quad \forall A \in \mathscr{F}$ then

$$\tilde{Q} = Q$$

.

R The proof of this theorem will come after several definitions and lemmas.

Outer Measure $P^* : 2^{\Omega} \to \mathbb{R}$

For any $A \in 2^{\Omega}$ $(A \subseteq \Omega)$

$$P^*(A) = \inf\{\sum_{n=1}^{\infty} P(A_n) : A_1, A_2, \ldots \text{is a sequence of } \mathscr{F}_0 \text{ sets}, A \subseteq \bigcup_{n=1}^{\infty} A_n\}$$

P^* is a measure out until $\mathscr{M}$, but it is only a function beyond that on 2^{Ω}.

Inner Measure

$P_*(A) = 1 - P^*(A)$

Define the paving $\mathscr{M}$ as followes

$$\mathscr{M} = \{A \in 2^{\Omega} : E \in 2^{\Omega}, P^*(E) = P^*(E \bigcap A) + P^*(E \bigcap A^C)\}$$

Idea: we came up with this $\mathscr{M}$ such that P^* behaves as a measure. It will turn out to be that $\mathscr{M}$ is a σ-field that contains $\sigma(\mathscr{F}_0)$.

Monday August 29

P^* satisfies the following probabilities:

(i) $P^*(\emptyset) = 0$
(ii) $P^*(A) \geq 0 \quad \forall A \in 2^{\Omega}$
(iii) $A \subseteq B \Rightarrow P^*(A) \subseteq P^*(B)$
(iv) $P^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P^*(A_n))$

Proof. (i) Take $\{\emptyset, \emptyset, \dots\}$.

$$\emptyset \in \mathscr{F}_0, \quad \emptyset \bigcup_{n=1}^{\infty} \emptyset$$

So,

$$P^*(\emptyset) \leq \sum_{n=1}^{\infty} P(\emptyset) = 0$$

Note,

$$P(A) \geq 0 \quad \forall A$$

So,

$$P^*(\emptyset) \geq \emptyset$$

Thus,

$$P^*(\emptyset) = \emptyset$$

(ii) Already done as part of (i).
(iii) Let $A \subseteq B$

$$P^*(A) = \inf\{\sum_{n=1}^{\infty} P(A_n), A_n \in \mathscr{F}_0, A \subseteq \bigcup A_n\}$$

Now, if $B_1, \dots \in \mathscr{F}_0 \subseteq \bigcup B_n$
Then,

$$A \subseteq B \subseteq \bigcup_n B_n$$

If $\{\{B_n\}_{n=1}^{\infty} : B_n \in \mathscr{F}_0, B \subseteq \bigcup_n B_n\} \subseteq \{\{A_n\}_{n=1}^{\infty} : A_n \in \mathscr{F}_0, A \subseteq \bigcup_n A_n\}$
Or in short, Collection 1 $\subseteq$ Collection 2.

So the inf of a larger set is smaller than (or equal to) the inf of a smaller set.

So,

$P^*(A) = \inf\{\sum_{n=1}^{\infty} P(A_n), A_n \in \text{ collection \#1}\} \leq P^*(B) = \inf\{\sum_{n=1}^{\infty} P(B_n), A_n \in \text{ collection \#2}\} = P^*(B)$

(iv) Want

$$P^*(\bigcup_n A_n) \leq \sum_n P^*(A_n)$$

$P^*(A_n) = \inf\{\sum_{n=1}^{\infty} P(A_n) : A_{nk} \in \mathscr{F}_0, A \subseteq \bigcup_k A_{nk}\}$

Let $\varepsilon > 0$, by defnition of there exists,

$$\{B_n\}_{n=1}^{\infty}$$

such that

$$\sum_{k=1}^{\infty} P(B_{nk}) \leq P^*(A_n) + \frac{\varepsilon}{2^n}$$

So,

$$\bigcup_n A_n \subseteq \bigcup_{n,k} B_{nk}$$

and,

$$\begin{aligned} P^*(\bigcup_n A_n) &\leq \sum_{n,k} P(B_{nk}) \\ &< \sum_n P^*(A_n) + \sum_n (\varepsilon 2^{-n}) \end{aligned}$$

$P^*(\bigcup A_n) < \sum_n P^*(A_n) + \varepsilon \quad \forall \varepsilon > 0$

Simply put,

b

So,

$$P^*(\bigcup_n A_n) \leq \sum_n P^*(A_n)$$

■

By definition, $A \in \mathscr{M}$ if and only if $P^*(EA) + P^*(EA^C) = P^*(E)$.

We know that P^* is subadditive.

So, by subadditivity we know,

$$P^*(E) \leq P^*(AE) + P^*(A^C E)$$

Therefore, to show $A \in \mathscr{M}$ we only need to show

$$P^*(E) \geq P^*(AE) + P^*(A^C E)$$

$\mathscr{M}$ is defined by P^* and P^* is defined using $\mathscr{F}_0$ so $\mathscr{M}$ is indireclty tied to $\mathscr{F}_0$.

Lemma 1. $\mathscr{M}$ is a field.

Proof. (i) $\Omega \in \mathscr{M}$

$$A = \Omega$$
$$P^*(\emptyset) = 0$$
$$P^*(E) + P^*(\emptyset) = P^*(E)$$

(ii) $A \in \mathscr{M} = A^C \in \mathscr{M}$

$$\begin{aligned} P^*(E) &= P^*(EA) + P^*(A^C E) \\ &= P^*(EA^C) + P^*(AE) \\ &= P^*(EA^C) + P^*((A^C)^C E) \end{aligned}$$

(iii) $A, B \in \mathscr{M} \to A \bigcap B \in \mathscr{M}$

$B \in \mathscr{M} \Rightarrow P^*(E) = P^*(Eb) + P^*(B^C E) \quad \forall E$
$A \in \mathscr{M} \Rightarrow P^*(BE) = P^*((BE)A) + P^*(A^C(BE))$
$A \in \mathscr{M} \Rightarrow P^*(B^C E) = P^*((B^C E)A) + P^*(A^C(B^C E))$

Hence,

$$P^*(BE) + P^*(B^C E) = P^*((BE)A) + P^*(A^C(BE)) + P^*((B^C E)A) + P^*(A^C(B^C E))$$

$$\begin{aligned} P^*(A^C(BE)) + P^*((B^C E)A) + P^*(A^C(B^C E)) &\geq P^*((A^C BE) \bigcup (AB^C E) \bigcup (A^C BE)) \\ &= P^*(E \cap [A^C B \bigcup AB^C \bigcup A^C B^C]) \\ &= P^*(E \bigcap (AB)^C) \end{aligned}$$

$$\begin{aligned} P^*(E) &= P^*(BE) + P^*(B^C E) \\ &= P^*((BE)A) + (P^*(A^C(BE)) + P^*((B^C E)A) + P^*(A^C(B^C E))) \\ &\geq P^*(ABE) + P^*(E(AB)^C) \end{aligned}$$

So, $A, B \in \mathscr{M}$

■

Lemma 2. If $A_1, A_2, \ldots$ is a sequence of disjoint $\mathscr{M}$-sets then for each $E \subseteq \Omega$,

$$P^*(E \cap (\bigcup_k A_k)) = \sum_k P^*(E \bigcap A_k)$$

Proof. First, prove this statement for finite sequence.

$$A_1, \ldots, A_n$$

by mathematical induction.

If $n = 1$ this is 'trivial',

$$P^*(E\bigcap A_1) = P^*(E\bigcap A_1)$$

If $n = 2$ we need to show,

$$P^*(E(A_1\bigcup A_n)) = P^*(EA_1) + P^*(EA_2)$$

Because $A_1 \in \mathscr{M}$,

$$P^*(E(A_1\bigcup A_2)) = P^*(E(A_1\bigcup A_2))A_1 + P^*(E(A_1\bigcup A_2)A_1^2)$$

$$E(A_1\bigcup A_2) = E(A_1A_2\bigcup A_1A_2 = EA_1$$

$$E(A_1\bigcup A_2)A_1^C = E(A_1A_1^C\bigcup A_2A_2^C)$$

So,

$$P^*(E(A_1\bigcup A_2)) = P^*(EA_1) + P^*(EA_2)$$

Suppose true for n = k. (induction hypothesis)

Now we must show for n = k + 1.

$$P^*(E\bigcap(\bigcup_{n=1}^{k+1} A_n)) = P^*([E\bigcap(\bigcup_{n=1}^{k} A_n)]\bigcup A_{k+1})$$

$(\bigcup_{n=1}^{k} A_n), A_{k+1}$ are two disjoint sets. Using the n=2 case,

$$= \sum_{n=1}^{k} P^*(E\bigcap A_n) + P(E\bigcap A_{k+1}) = \sum_{n=1}^{k+1} P^*(E\bigcap A_n)$$

So this is now shown to be true for $\{A_1,\ldots,A_n\}$. Next, showtrue for $A_1,\ldots in \mathscr{M}$ (disjoint). Want:

$$P^*(E\bigcap(\bigcup_{n=1}^{\infty} A_n)) = \sum_{n=1}^{\infty} P^*(E\bigcap A_n)$$

Using countable subadditivity,

$$P^*(E\bigcap(\bigcup_{n=1}^{\infty} A_n)) = P^*(\bigcup_{n=1}^{\infty} E\bigcap A_n) \leq \sum_{n=1}^{\infty} P^*(E\bigcap A_n)$$

In the meantime, by the monotonicity of P^*

$$P^*(E\bigcap(\bigcup_{n=1}^{\infty} A_n)) \geq P^*(E\bigcap(\bigcup_{n=1}^{m} A_n)) = \sum_{n=1}^{\infty} P^*(E\bigcap A_n)$$

So,

$$P^*(E\bigcap(\bigcup_{n=1}^{\infty} A_n)) \geq \lim \sum_{n=1}^{m} P^*(E\bigcap A_n)$$

(*), (**) gives us,

$$P^*(E\bigcap(\bigcup_{n=1}^{\infty}A_n)) = \sum_{n=1}^{\infty}P^*(E\bigcap A_n)$$

■

Wednesday August 31

(finished proof)

Lemma 3.

1. $\mathscr{M}$ is a σ-field
2. P^* restricted on $\mathscr{M}$ is countably additive.

Proof. First we show if

1. $\mathscr{M}$ is a fieldd
2. $\mathscr{M}$ is closed under countable disjoint union.

then $\mathscr{M}$ is a σ-field.

Let's create disjoints sets,

$A_n \in \mathscr{M}, n = 1,2,\dots$ $B_1 = A_1$ $B_2 = A_2A_1^C \vdots B_n = A_nA_1^C\dots A_{n-1}^C$

$B_1,\dots,B_n \in \mathscr{M}$ (disjoint)

$$\bigcup_{n=1}^{\infty}B_n = \bigcup_{n=1}^{\infty}A_n$$

But we know that $\bigcup_{n=1}^{\infty}B_n \in \mathscr{M}$ so $\bigcup_{n=1}^{\infty}A_n \in \mathscr{M}$ and thus $\mathscr{M}$ is a σ-field.

So it suffices to show that $\mathscr{M}$ is closed under disjoint countable unions.

Let $A_1,A_2,\dots$ are disjoins $\mathscr{M}$-sets.

Let $A = \bigcup_{n=1}^{\infty}A_n$.

Let $F_n = \bigcup^n k = 1A_k$.

Then $F_n \in \mathscr{M}$.

So, $\forall E \in 2^{\Omega}$,

$$P^*(E) = P^*(EF_n) + P^*(EF_n^C)$$

$$
\begin{aligned}
P^*(EF_n) &= P^*(E(\bigcup_{k=1}^{n} A_k)) \\
&= \sum_{k=1}^{n} P^*(EA_k) \\
P^*(EF_n^C) &\geq P^*(EA^C)(F_n \subseteq A, F_n^C \supseteq A^C) \\
\Rightarrow P^*(E) \geq \lim_{n\to\infty} P^*(EA_k) + P^*(EA^C) \\
&= \sum_{k=1}^{n} P^*(EA_k) + P^*(EA^C) \\
&= P^*(EA) + P^*(EA^C)
\end{aligned}
$$

■

So $A \in \mathscr{M}$ and $\mathscr{M}$ is a σ-field.

Now, let's show P^* is countably additive.
Let $A_1, A_2, \dots$ be disjoint members of $\mathscr{M}$. Then $\forall E \in 2^\Omega$,

$$P^*(E(\bigcup_{n=1}^{\infty} A_n)) = \sum_{n=1}^{\infty} P(EA_n)$$

Take $E = \Omega$.

$$P^*(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

Lemma 4. $\mathscr{F}_0 \subseteq \mathscr{M}$

Proof. Let $A \in \mathscr{F}$.

Want:

$$A \in \mathscr{M}$$

$$P^*(E) = P^*(EA) + P^*(EA^C)$$

By definition, there exists $E_n \in \mathscr{F}_0$ such that

$$\sum_{n=1}^{\infty} P^*(E_n) \leq P^*(E) + \varepsilon$$

$$\begin{aligned}
P^*(EA) &\le P^*((\bigcup_{n=1}^{\infty} E_n)A) \text{ (monotonocity)}\\
&= P^*(\bigcup^{i} nfty_{n=1}(E_nA))\\
&\le \sum^{i} nfty_{n=1} P^*((E_nA)) \text{ (countibly subadd)}\\
P^*(EA^C) &\le \sum_{n=1}^{\infty} P^*(E_nA^C)\\
P^*(EA)+P^*(EA^C) &\le \sum_{n=1}^{\infty} P^*(E_nA)+P^*(E_nA^C)\\
&= \sum_{n=1}^{\infty} P^*(E_n)
\end{aligned}$$

Recall, $A, E_n \in \mathscr{F}_0$

$$\begin{aligned}
&\le P^*(E)+\varepsilon\\
P^*(EA)+P^*(EA^C) &\le P^*(E)+\varepsilon \quad \forall\varepsilon\\
\Rightarrow P^*(EA)+P^*(EA^C) &= P^*(E)\\
\Rightarrow A &\in \mathscr{M}\\
\mathscr{F}_0 &\in \mathscr{M}
\end{aligned}$$

■

Lemma 5.

$$P^*(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Proof. Let $A \in \mathscr{F}_0$.

Because, $A, \emptyset, \emptyset, \dots, \in \mathscr{F}_0$.

$$A \subseteq A \bigcup \emptyset \bigcup \emptyset \dots$$

$$P^*(A) \le P(A)+P(\emptyset)+\dots$$

But, if

$$A_n \in \mathscr{F}_0$$

$$A \subseteq \bigcup_{n=1}^{\infty} A_n$$

$$P^*(A) \le \sum_{n=1}^{\infty} P(A_n)$$

$$\Rightarrow P^*(A) \le \inf \sum_{n=1}^{\infty} P(A_n)$$

$$= P^*(A)$$

■

Friday September 2

R **5 Lemma Recap**

Lemma 1. $\mathscr{M}$ is a field.
Lemma 2. If $A_1, A_2, \ldots$ is a sequence of disjoint $\mathscr{M}$-sets then for each $E \subseteq \Omega$,

$$P^*(E \cap (\bigcup_k A_k)) = \sum_k P^*(E \bigcap A_k)$$

Lemma 3.
1. $\mathscr{M}$ is a σ-field
2. P^* restricted on $\mathscr{M}$ is countably additive.

Lemma 4.

$$\mathscr{F}_0 \subseteq \mathscr{M}$$

Lemma 5.

$$P^*(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Recall, Extension Theorem. That is, If $\mathscr{F}$ is a field and P is a probability measure, then there exists a measure, Q such that

$$Q(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Proof. By Lemma 5,

$P^*(\Omega) = P(\Omega) = 1$
$P^*(\emptyset) = P(\emptyset) = 0$

Outline of what we need to show:

- $0 \leq M(A) \leq 1$
- $M(\emptyset) = 0, \quad M(\Omega) = 1$
- $M(\bigcup_n A_n) = \sum_n M(A_n)$

Since $\forall A \in \mathscr{M}$,

$$\emptyset \subseteq A \subset \Omega$$

then

$$0 \leq P^*(\emptyset) \leq P^*(A) \leq P^*(\Omega) \leq 1$$

But, by Lemma 3, P^* is contably additive on $\mathscr{M}$. So P^* is probability measure on $\mathscr{M}$ (which is a σ-field, by Lemma 3).

By Lemma 4, $\mathscr{F}_0 \subset \mathscr{M} \Rightarrow \sigma(\mathscr{F}_0 \subseteq \mathscr{M}$. So P^* is also probabliity measure on $\sigma(\mathscr{F}_0)$.

Finally, by Lemma 5, again $P^*(A) = P(A)$, P^* is an extention of P form $\mathscr{F}_0$ to $\sigma(\mathscr{F}_0)$.

∎

Uniqueness of of the extention, $\pi - \lambda$ Theorem

Paving - $\{\pi$-system and λ-system.$\}$ (?)

Definition 1.3.1 — π-System. A class of subsets $\mathscr{P}$ of Ω is a π system, if

$$A, B \in \mathscr{P} \Rightarrow AB \in \mathscr{P}$$

Definition 1.3.2 — λ-System. A class $\mathscr{L}$ is a λ-system if

λ(i) $\Omega \in \mathscr{L}$

λ(ii) $A \in \mathscr{L} \Rightarrow A^C \in \mathscr{L}$

λ(iii) If $A_1, \cdots \in \mathscr{L}$ are disjoint then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{L}$

So, the only difference is "disjoint". Weaker than a σ-field (i.e. A σ-field is always a λ-system). Note that (λ_2) can be replace by $(\lambda_{2\prime})$ wherein

$$A, B \in \mathscr{F}, A \subseteq B, \Rightarrow B \setminus A \in \mathscr{L}$$

That is $\lambda_1, \lambda_2, \lambda_3 \Leftrightarrow \lambda_1, \lambda_{2\prime}, \lambda_3$

Lemma 6. A class of sets that is both π-systema and λ-system is a σ-field.

Proof. Suppose $\mathscr{F}$ is both π-system and λ-system.

By definition,

1. $\Omega \in \mathscr{F}$
2. $A \in \mathscr{F} \Rightarrow A^C \in \mathscr{F}$

Let $A_1, A_2, \ldots$ be $\mathscr{F}$ sets.

Let's constructs disjoints sets, B

$$B_1 = A_1$$
$$B_2 = A_1 A_2^C$$
$$\vdots$$

Then B_n are $\mathscr{F}$-sets (by $\lambda_{2\prime} - A_2^C = \Omega A)2^C \in \mathscr{F}$, by π-system, $A_1 A_2^C \in \mathscr{F}$).

By λ_3,

$$\bigcup_n^{\infty} B_n \in \mathscr{F}$$

So,

$$\bigcup_n^{\infty} A_n \in \mathscr{F}$$

■

Theorem 1.3.2 — π-λ Theorem. If $\mathscr{P}$ is in a π-system, $\mathscr{L}$ is in a λ-system, then

$$\mathscr{P} \subseteq \mathscr{L} \Rightarrow \sigma(\mathscr{P} \subseteq \mathscr{L})$$

Proof. Let $\lambda(\mathscr{P})$ be the intersection of all λ-system that contains $\mathscr{P}$.

$$\lambda(\mathscr{P}) = \cap\{\mathscr{L}' : \mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}' \text{ is } \lambda\text{-set }\}$$

$\lambda(\mathscr{P})$ is a λ-system.

Goal: prove $\lambda(\mathscr{P})$ is a σ-field. So we want to show that $\lambda(\mathscr{P})$ is a π-system.

1. $\Omega \in \lambda(\mathscr{P})$?

$$\Omega \in \mathscr{L}' \quad \forall \mathscr{L}'$$

$$\Omega \in \lambda(\mathscr{P})$$

2. $A \in \lambda(\mathscr{P}) \Rightarrow A^C \in \lambda(\mathscr{P})$?

$$A \in \lambda(\mathscr{P}) \Rightarrow A \in \cap\{\mathscr{L}' : \mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}' \text{ is } \lambda\text{-set }\}$$

Then
$A \in \mathscr{L}'$ for any $\mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}'$ is λ-system.

$$\Rightarrow A^C \in \mathscr{L}'$$

$$\Rightarrow A^C \in \cap\{\mathscr{L}' : \mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}' \text{ is } \lambda\text{-set }\} = \lambda(\mathscr{P})$$

3. $A_1, A_2, \cdots \in \lambda(\mathscr{P})$ are disjoint then $A_1, A_2, \cdots \in \mathscr{L}' \quad \forall \mathscr{L}'$.
Then $\bigcup A_n \in \mathscr{L}'(\mathscr{L}' \lambda\text{-system})$
So $\bigcup_n A_n \in \lambda(\mathscr{P})$.
We call $\lambda(\mathscr{P})$ the λ-system generated by $\mathscr{P}$.

If we can say that $\lambda(\mathscr{P})$ is also a σ-field, then $\sigma(\mathscr{P}) \subseteq \lambda(\mathscr{P})$ because $\sigma(\mathscr{P})$ is smallest.
So then, $\sigma(\mathscr{P}) \subseteq \mathscr{L}$ because $\lambda(\mathscr{P})$ is the small λ-system.
So it suffices to show that $\lambda(\mathscr{P})$ is a σ-field. But we know if $\lambda(\mathscr{P})$ is a system then $\lambda(\mathscr{P})$ is σ-field. So it suffices to show that $\lambda(\mathscr{P})$ is a π-system.

Construct again for any $A \in 2^\Omega \quad (A \subseteq \Omega)$, let

$$\mathscr{L}_A = \{B : AB \in \lambda(\mathscr{P})\}$$

Claim: If $A \in \lambda(\mathscr{P})$ then $\mathscr{L}_A$ is λ-system.

(a) $\Omega \in \mathscr{L}_A$?

$$A\Omega = A \in \mathscr{L}_A$$

(b) $(\lambda_2') : B_1, B_2 \in \mathscr{L}_A, B_1 \subseteq B_2 \Rightarrow B_2 B_1^C \in \mathscr{L}_A$?

$$B_1 \in \mathscr{L}_A \Rightarrow AB_1 \in \lambda(\mathscr{P})$$

$$B_2 \in \mathscr{L}_A \Rightarrow AB_2 \in \lambda(\mathscr{P})$$

Since $AB_1 \subseteq AB_2$, $\lambda(\mathscr{P})$ is λ-system by (λ_2') for $\lambda(\mathscr{P})$

(c) If B_n is disjoint, $\mathscr{L}_A$-sets.
Want $\bigcup_n B_n$ because

$$B_n \in \mathscr{L}_A$$

$$B_n A \in \lambda(\mathscr{P})$$

Because B_n disjoint we know that $B_n A$ is also disjoint.
Hence,

$$\bigcup_n (B_n A) \in \lambda(\mathscr{P})$$

Claim: $\lambda(\mathscr{P})$ is π-sytem.

(a) If $A \in \mathscr{P}$, then $\mathscr{P} \subseteq \mathscr{L}_A$
Suppose $A \in \mathscr{P}$.
Let $B \in \mathscr{P}$, then $AB \in \mathscr{P}$ (π-system), and $AB \in \lambda(\mathscr{P}) \Rightarrow B \in \mathscr{L}_A$

(b) If $A \in \mathscr{P}$ then $\lambda(\mathscr{P}) \subset \mathscr{L}_A$.

(c) If $A \in \lambda(\mathscr{P})$, then $\mathscr{P} \in \mathscr{L}_A$
Suppose, $A \in \lambda(\mathscr{P})$ and let $B \in \mathscr{P}$.
By step 2,
$A \in \mathscr{L}_A$
$\Rightarrow AB \in \lambda(\mathscr{P})$
$\Rightarrow B \in \mathscr{L}_A$

(d) If $A \in \lambda(\mathscr{P})$, then $\lambda(\mathscr{P}) \subseteq \mathscr{L}_A$. This is because $\lambda(\mathscr{P})$ is the smallest λ-system, $\mathscr{L}_A$ is λ-system containing $\mathscr{P}$ (by step 3).

Now show that $\lambda(\mathscr{P})$ is π-system.

$A, B \in \lambda(\mathscr{P})$ because $A \in \lambda(\mathscr{P})$. We have that $\lambda(\mathscr{P}) \in \mathscr{L}_A$.

So

$$B \in \mathscr{L}_A$$

$$BA \in \lambda(\mathscr{P})$$

Thus $\lambda(\mathscr{P})$ is π-system.

■

Wednesday September 7

Theorem 1.3.3 Suppose P_1 and P_2 are probability measures on $\sigma(\mathscr{P})$ where $\mathscr{P}$ is a π-system. If P_1 and P_2 agree on $\mathscr{P}$ (that is, $P_1(A) = P_2(A) \quad \forall A \in \mathscr{P}$) then they agree on $\sigma(\mathscr{P})$.

Proof. Let

$$\mathscr{L} = \{A \in \sigma(\mathscr{P}) : P_1(A) = P_2(A)\}$$

Then $\mathscr{P} \subseteq \mathscr{L}$.

It suffices to show that $\mathscr{L}$ is a λ-system (because if so, then $\sigma(\mathscr{P}) \subseteq \mathscr{L}$ - in fact, $\sigma(\mathscr{P}) = \mathscr{L}$).

Show $\mathscr{L}$ is a λ-system.

1. $\Omega \in \mathscr{L}$?
$$P_1(\Omega) = P_2(\Omega) = 1, \quad \Omega \in \mathscr{P}$$
2. $A \in \mathscr{L}$
$$P_1(A) = P_2(A) \Rightarrow P_1(A^C) = P_2(A^C), \quad A^C \in \mathscr{L}$$
3. $A \in \mathscr{L}$. A_n disjoint. Want $\bigcup_{n=1}^{\infty} A_n \in \mathscr{L}$.
   Since
$$A_n \in \mathscr{L}$$
$$P_1(A_n) = P_2(A_n) \quad \forall n$$
$$\sum_{n=1}^{\infty} P_1(A_n) = \sum_{n=1}^{\infty} P_2(A_n)$$
$$P_1 \bigcup_{n=1}^{\infty}(A_n) = P_2 \bigcup_{n=1}^{\infty}(A_n)$$
   So, $\bigcup A_n \in \mathscr{L}$.

■

So our extention of (and uniqueness of the extention of) P on $\mathscr{F}_0$ to $\sigma(\mathscr{F}_0)$ is complete. We have shown the existance of Q on $\mathscr{M}$.

Since Q agrees with P on $\mathscr{F}_0$ and $\mathscr{F}_0$ is a field, this implies that this is a π-system.
If you have another extention, say $\tilde{Q}$, then $\tilde{Q} = P$ on $\mathscr{F}_0$. That is, $\tilde{Q} = Q$ on $\mathscr{M}$, where $\mathscr{M}$ is a σ-field, which is a π -system.

So by Theorem 1.3.3,
$\tilde{Q} = Q$ on $\sigma(\mathscr{P})$.

So, Theorem 1.3.1, is completely proved.

Lemma 1 - Lemma 5 imply the extention.
$\pi - \lambda$ Theorem and Theorem 1.3.3 implies uniqueness.
This wraps up Theorem 1.3.1.

Lebesque measure on (0,1]

$$\Omega = (0, 1]$$

Recall, $\mathscr{B}_0$ is the finite disjoint unions of intervals in (0,1] and that $\mathscr{B}_0$ is a field.

Let $\mathscr{B} = \sigma(\mathscr{B}_0)$.

For each $A \in \mathscr{B}_0$,

$$A = \bigcup_{i=1}^{n}(a_i, b_i]$$

Let $\lambda(A) = \sum_{i=1}^{n}(b_i - a_i)$.

Question: Is λ a probability measure on $\mathscr{B}_0$?

Theorem 1.3.4 — Theorem 2.2 in Billingsly. The set function λ on $\mathscr{B}_0$ is a probability measure on $\mathscr{B}_0$.

Proof. 1. $0 \leq \lambda(A) \leq 1$

2.

$$\lambda(\Omega) = \lambda((0,1]) = 1 - 0 = 1$$

$$\lambda(\emptyset) = \lambda((0,0]) = 0$$

3. This one requires Theorem 1.3 in Billingsly (proof omitted - calculus, blah).
Theorem 1.3 - If I is an interval in (0,1] and $\{I_k : k = 1, 2, \dots\}$ are disjoint intervas in (0,1] such that

$$I = \bigcup_{k=1}^{\infty} I_k$$

then,

$$|I| = \sum_{k=1}^{\infty} |I_k|$$

where |a| means length of interval a.
Since $\bigcup_{j=1}^{m_k} I_{kj} \in \mathscr{B}_0$ and $\bigcup_{i=1}^{m} I_i = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} I_{kj}$.

Then

$$\lambda(A)\lambda(\bigcup_{i=1}^{m} I_i) = \sum_{i=1}^{m} |I_i|$$

Since, $I_i \subset \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} I_{kj}$, then

$$I_i = I_i(\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} I_{kj}) = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} I_i I_{kj}$$

By Theorem 1.3,

$$|I| = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |I_i I_{jk}|$$

$$\lambda(A) = \sum_{i=1}^{m} \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |I_i I_{jk}| = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \sum_{i=1}^{m} |I_i I_{jk}|$$

Because $I_{jk} \subseteq \bigcup_{i=1}^{m} I_i$, we have that

$$I_{kj} = \bigcup_{i=1}^{m} I_{kj} I_i$$

Again by Theorem 1.3, (note that $I_{kj} I_i$ are disjoint intervals)

$$|I_{kj} = \sum_{i=1}^{m} |I_i I_{jk}|$$

So, $\lambda(A) = \sum_{k=1}^{\infty}\sum_{j=1}^{m_k}|I_{kj}| = \sum_{k=1}^{\infty}\lambda(A_k)$

■

Friday September 9

Finished above proof.

So λ is a probability on $\mathscr{B}_0$. By Theorem 3.1, there exists a unique measure τ on $\sigma(\mathscr{B}_0) = \mathscr{B}$ such that

$$\tau(A) = \lambda(A) \quad \forall A \in \mathscr{B}_0$$

τ is called **Lebesgue Measure** on (0,1]. We may still write it as λ.

1.4 Probabilities Concerning Sequences of Events

Set Limit

Let $(\Omega, \mathscr{F})$ be a measureable space (i.e. Ω is nonempty set and $\mathscr{F}$ is σ-field).

let $A_1, \cdots \in \mathscr{F}$. We define

$$\cap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k = \limsup_{n\to\infty} A_n$$

It is trivial to show that $\limsup_{n\to\infty} A_n \in \mathscr{F}$.

$$\bigcup_{n=1}^{\infty}\cap_{k=n}^{\infty} A_k = \liminf_{n\to\infty} A_n$$

We swapped intersection/union...what we are doing here?
ω (means outcome) $\in \Omega$

$$\omega \in \limsup_{n\to\infty} A_n \Leftrightarrow \omega \in \cap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k$$

$$\Leftrightarrow \omega \in \bigcup_{k=1}^{\infty} A_k \quad \forall n = 1, 2, \ldots$$

$$\Leftrightarrow \omega \in A_k \quad \text{for some } k \geq n, \quad \forall n = 1, 2, \ldots$$

$\Leftrightarrow$ ω is in infinitely many k.
Similarly,

$$\omega \in \liminf_{n\to\infty} A_n \Leftrightarrow \omega \in \bigcup_{n=1}^{\infty}\cap_{k=n}^{\infty} A_k$$

$$\Leftrightarrow \omega \in \cap_{k=1}^{\infty} A_k \quad \text{for some } n = 1, 2, \ldots$$

$$\Leftrightarrow \omega \in A_k \quad \forall k \geq n, \quad \text{for some } n$$

$$\Leftrightarrow \omega \in \text{ all but finitely many } A_k$$

So this is a much stronger requirement. Intuitively, if ω is in all but finitely many A_k, then it must be in infinitely many A_k (i.e. $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$).

For i > max(n,m),

$$\cap_{k=m}^{\infty} A_k \subseteq A_i \subseteq \bigcup_{k=n}^{\infty} A_k$$
$$\Rightarrow \bigcup_{m=1}^{\infty} \cap_{k=m}^{\infty} A_k \subseteq \bigcup_{k=n}^{\infty} A_k$$
$$\Rightarrow \bigcup_{m=1}^{\infty} \cap_{k=m}^{\infty} A_k \subseteq \cap_{n=1}^{\infty} \bigcup_{k=n} A_k$$
$$\Rightarrow \liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$$

$$\cap_{k=n}^{\infty} A_k \uparrow \liminf_{n\to\infty} A_n$$

$$\bigcup_{k=n}^{\infty} A_k \downarrow \limsup_{n\to\infty} A_n$$

If $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$, then we say that the sequences $\{A_n\}$ has a limit,

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

$$\lim_{n\to\infty} A_n \in \mathscr{F}$$

Sometimes we write,

$$\limsup_{n\to\infty} A_n = [A_n \text{ i.o. }]$$

Theorem 1.4.1 Suppose $(\Omega, \mathscr{F}, P)$ is a probability space and $A_n \in \mathscr{F} \quad n = 1, 2, \ldots.$

(i)

$$\limsup_{n\to\infty} P(A_n) \leq P(\limsup_{n\to\infty} A_n)$$

$$\liminf_{n\to\infty} P(A_n) \geq P(\liminf_{n\to\infty} A_n)$$

(ii) $A_n \to A(A = \lim_{n\to\infty} A_n)$, then we have continuity of probability of a set funciton:

$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n)$$

Monday September 12

Proof. (i) Let $B_n = \cap_{k=n}^{\infty} A_k$.

$$B_n \uparrow \liminf_n A_n$$

By Theorem 2.1,

$$P(B_n) \uparrow P(\liminf_n A_n)$$

So,

$$P(B_n) \leq P(\liminf_n A_n) \quad \forall n$$

$$\lim_{n\to\infty} P(B_n) = P(\liminf_n A_n)$$

$$P(A_n) \geq P(B_n) \to P(\liminf_n A_n)$$

$$\liminf_n P(A_n) \geq P(\liminf_n A_n)$$

Similarly,
Let $C_n = \bigcup_{k=n}^{\infty} A_k$.
Then,

$$C_n \downarrow \bigcup_{k=n}^{\infty} A_k$$

$$P(A_n) \leq P(C_n) \to P(\limsup_n A_n)$$

$$\limsup_n P(A_n) \leq P(\limsup_n A_n)$$

(ii) If A_n has a limit (i.e. $\limsup_n A_n = \limsup_n A_n = \lim A$) then,

$$\liminf_n P(A_n) \geq P(\liminf_n A_n) = P(\limsup_n A_n) \geq \limsup_n P(A_n)$$

So, $\liminf_n P(A_n) = \limsup_n P(A_n)$, thus

$$\lim_n P(A_n) = P(\lim_n A_n)$$

■

Independent Events

$(\Omega, \mathscr{F}, P)$

Let $A, B \in \mathscr{F}$. They are independent if and only iff:

$$P(AB) = P(A)P(B)$$

$$A \perp\!\!\!\perp B$$

$A_1, \ldots A_n$ are independent if and only if for any $\{k_1, \ldots, k_j\} \subseteq \{1, \ldots, n\}$,

$$P(A_{k_1} \ldots A_{k_j}) = P(A_{k_1}) \ldots P(A_{k_i})$$

In this case we write: $A_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp A_n$.

Now let, $\mathscr{A}_1, \ldots, \mathscr{A}_n$ be pavings in $\mathscr{F}$ (i.e. $\mathscr{A}_k \subseteq \mathscr{F}$).

We say $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are independent if and only if for any $A_1 \in \mathscr{A}_1, \ldots, A_n \in \mathscr{A}_n$ we have

$$A_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp A_n$$

.

In this case we write: $\mathscr{A}_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp \mathscr{A}_n$.

Theorem 1.4.2 Suppose for $(\Omega, \mathscr{F}, P)$ is a probability space if,

$$\mathscr{A}_1 \subseteq \mathscr{F} \ldots \mathscr{A}_n \subseteq \mathscr{F}$$

are π-systems. Then,

$$\mathscr{A}_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp \mathscr{A}_n \Rightarrow \sigma(\mathscr{A}_1) \perp\!\!\!\perp \ldots \perp\!\!\!\perp \sigma(\mathscr{A}_n)$$

Proof. Let $\mathscr{B}_i = \mathscr{A}_i \bigcup \{\Omega\}$.

It is easy to show (in homework)

1. $\mathscr{B}_i$ is still a π-system
2. $\mathscr{B}_i$ are still independent

$$\mathscr{B}_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp \mathscr{B}_n$$

For $B_2 \in \mathscr{B}_n, \ldots, B_n \in \mathscr{B}_n$ define,

$$\mathscr{L}(B_2, \ldots, B_n) = \{B \in \mathscr{F} : B \perp\!\!\!\perp B_2 \perp\!\!\!\perp \ldots \perp\!\!\!\perp B_N\}$$

1. First we show $\mathscr{L}(B_2, \ldots, B_n)$ is λ-system.

$$\Omega \in \mathscr{L}(B_2, \ldots, B_n)$$
$$\Omega \perp\!\!\!\perp B_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp B_n$$

This is true because $P(\Omega B_2 \ldots B_n) = P(B_2 \ldots B_n) = P(B_2) \ldots P(B_n) = P(\Omega) P(B_2) \ldots P(B_n)$

2. Now $A \in \mathscr{L}(B_2, \ldots, B_n) \Rightarrow A^C \in \mathscr{L}(B_2, \ldots, B_n)$
$A \& in \mathscr{L}(B_2, \ldots, B_n)$

$$\begin{aligned}
&\Rightarrow A \perp\!\!\!\perp B_2 \perp\!\!\!\perp \ldots \perp\!\!\!\perp B_n \\
&\Rightarrow P(AB_2 \ldots B_n) = P(A) P(B_2) \ldots P(B_n) \\
&\Rightarrow P(A^C B_2 \ldots B_n) \\
&\quad P(B_2 \ldots B_n) \setminus AB_2 \ldots B_n) \\
&\quad P(B_2 \ldots B_n) - P(AB_2 \ldots B_n) \\
&\quad P(B_2) \ldots P(B_n) - P(A) P(B_2) \ldots P(B_n) \\
&\quad (1 - P(A)) P(B_2) \ldots P(B_n) \; P(A^C) P(B_2) \ldots P(B_n)
\end{aligned}$$

Then we run this through all subadditives of $A, B_2, \ldots, B_n$.
$A^C \perp\!\!\!\perp B_2 \perp\!\!\!\perp \ldots \perp\!\!\!\perp B_n$

3. If $C_1, C_2, \ldots, \in \mathscr{L}(B_2, \ldots, B_n)$ they are disjoint. Want to show

$$\bigcup^{i} nfty_{m=1} C_m \in \mathscr{L}(B_2, \ldots, B_n)$$

$C_m \in \mathscr{L}(B_2, \ldots, B_n)$

$$\begin{aligned}
&\Rightarrow C_m \perp\!\!\!\perp \ldots \perp\!\!\!\perp B_n \\
&\Rightarrow P(C_m B_2 \ldots B_n) = P(C_m) \ldots P(B_n) \quad \forall m = 1, 2 \ldots
\end{aligned}$$

$$\sum_{m=1}^{\infty} P(C_m B_2 \ldots B_n) = (\sum_{m=1}^{\infty} P(C_m)) P(B_2) \ldots P(B_n)$$

But $\{C_m, B_2, \ldots, B_n, m = 1, 2 \ldots\}$

So $\bigcup_m C_m \in \mathscr{L}(B_2, \ldots, B_n)$.

And $\mathscr{L}(B_2, \ldots, B_n)$.

Also, $B_1 \in \mathscr{L}(B_2, \ldots, B_n) \quad \forall B_1 \in \mathscr{B}_1$ therefore by definition,

$$\mathscr{B}_1 \subseteq \mathscr{L}(B_2, \ldots, B_n)$$

So, $\sigma(\mathscr{B}_1) \subseteq \mathscr{L}(B_2, \ldots, B_n)$ and we have our $\lambda - \pi$-theorem.

This means that for all $B_1 \in \sigma(\mathscr{B}_1)$

$$B_1 \perp\!\!\!\perp B_2 \perp\!\!\!\perp \ldots \perp\!\!\!\perp B_n$$

Recall that B_i are arbitrary members of,

$$\sigma(\mathscr{B}_1) \perp\!\!\!\perp B_2 \perp\!\!\!\perp \ldots \perp\!\!\!\perp B_n \Leftrightarrow \mathscr{B}_2 \perp\!\!\!\perp \sigma(\mathscr{B}_1) \perp\!\!\!\perp \ldots \perp\!\!\!\perp \mathscr{B}_n$$

Run the previous argument repeatedly.

So

$$\sigma(\mathscr{B}_1) \perp\!\!\!\perp \sigma(\mathscr{B}_2) \perp\!\!\!\perp \ldots \perp\!\!\!\perp \sigma(\mathscr{B}_n)$$

■

■ **Example 1.3** Let $\mathscr{I}$ be the collection of all intervals, then its π-system. When we want to check $X \perp\!\!\!\perp Y$, we only need to check

$$P(X \in \text{interval}, Y \in \text{interval}) = P(X \in \text{interval}) P(Y \in \text{interval})$$

■

Wednesday September 14

Indepencence of Infinite Classes

Let $\{\mathscr{A}_\theta : \theta \in \Theta\}$ where θ is any infinite set (need not be countable) if and only if any (infinite) $\{A_\theta : \theta \in \Theta\}$ where $A_\theta \in \mathscr{A}_\theta$ are independent.

We alraedy define independence of $\{A_\theta : \theta \in \Theta\}$; that is for an infinite collection of sets is independent if and only if any finite subcollection $\{A_{\theta_1}, \ldots A_{\theta_n}\}$ is independent.

With this device, we may make claims such as

$$\{X_t : t \in (0,1]\}$$

are independent. Useful for stochastic process, functional data analysis.

It follows trivially, $\{\mathscr{A}_\theta : \theta \in \Theta\}$ are independent if and only if any finite collection, say $\{\mathscr{A}_{\theta_1}, \ldots, \mathscr{A}_{\theta_n}\}$ are independent.

Corollary 1.4.3 — To Theorem 4.2. If $(\Omega, \mathscr{F}, P), \mathscr{A}_\theta \subset \mathscr{F}$, $\{\mathscr{A}_\theta : \theta \in \Theta\}$ is independent and each $\mathscr{A}_\theta$ is a π-system, then

$$\{\sigma(\mathscr{A}_\theta) : \theta \in \Theta\}$$

are independent.

Proof.

$$
\begin{aligned}
\{\mathscr{A}_\theta : \theta \in \Theta\} \perp\!\!\!\perp &\Leftrightarrow \{\mathscr{A}_{\theta_1}, \dots, \mathscr{A}_{\theta_n}\} \perp\!\!\!\perp \\
&\Leftrightarrow \{\sigma(\mathscr{A}_{\theta_1}), \dots, \sigma(\mathscr{A}_{\theta_n})\} \perp\!\!\!\perp
\end{aligned}
$$

■

Corollary 1.4.4 Suppose we have an array of sets,

$$
\begin{array}{cccc}
A_{11} & A_{12} & \dots & \dots \\
A_{21} & A_{22} & \dots & \dots \\
\vdots & \vdots & & \\
\vdots & \vdots & &
\end{array}
= \{A_{ij} : i, j = 1, \dots\} \subset \mathscr{F}
$$

and this array is independent.

And let $\mathscr{F}_i = \sigma(A_{i1}, A_{i2}, \dots)$.

Then $\mathscr{F}_1 \perp\!\!\!\perp \mathscr{F}_2$

Proof. Let $\mathscr{A}_i$ be the class of all the finite intersections of

$$A_{i1}, A_{i2}, \dots$$

then $\mathscr{A}_i$ is a π-system.

So,

$$\sigma(\mathscr{A}_i) = \mathscr{F}_i$$

because $\{A_{i1}, A_{i2}, \dots\}$ are contained in $\mathscr{A}_i$ which implies $\mathscr{F}_i \subset \sigma(\mathscr{A}_i)$ and also $\mathscr{A}_i \subset \mathscr{F}_i \Rightarrow \sigma(\mathscr{A}_i \leq \mathscr{F}_i$.

By Corollary 1, it suffices to show that $\mathscr{A}_1, \mathscr{A}_2, \dots$ are independent. Further, it suffices to show that any finite subcollection is independent.

Let $\{i_1, \dots, i_n\} \subseteq \{1, 2, \dots\}$. We want to show that $\mathscr{A}_{i_1} \perp\!\!\!\perp \dots \perp\!\!\!\perp \mathscr{A}_{i_n}$. This would be implied by the following

$$\forall C_{i_1} \in \mathscr{A}_{i_1}, \dots, C_{i_n} \in \mathscr{A}_{i_n}$$

$$C_{i_1} \perp\!\!\!\perp \dots \perp\!\!\!\perp C_{i_n}$$

Because, and watch out with this notation here, for

$$C_{i_\alpha} \in \mathscr{A}_{i_\alpha}$$

there exists

$$A_{i_\alpha j_1}, A_{i_\alpha j_2}, \dots, A_{i_\alpha j_{m\alpha}}$$

such that

$$C_{i_\alpha} = A_{i_\alpha j_1}, A_{i_\alpha j_2}, \dots, A_{i_\alpha j_{m\alpha}}$$

We have

$$P(\cap_{\alpha=1}^{n} C_{i_\alpha}) = P(\cap_{\alpha=1}^{n} \bigcup_{\beta=1}^{m_\alpha} A_{i_\alpha j_\beta})$$

because

$$\{A_{i_\alpha j_\beta} : \alpha, = 1,2,\dots,n, \beta = 1,2,\dots m_\alpha\} \subseteq \{A_{ij} : i,j = 1,2,\dots\}$$

$$\begin{aligned} P(\cap_{\alpha=1}^{n} \bigcup_{\beta=1}^{m_\alpha} P(A_{i_\alpha j_\beta}) &= \prod_{\alpha=1}^{n} \prod_{\beta=1}^{m_\alpha} P(A_{i_\alpha j_\beta}) \\ &= \prod_{\alpha=1}^{n} P(C_{i_\alpha}) \end{aligned}$$

■

Borel-Cantelli Lemmas (that are actually Theorems)

Theorem 1.4.5 — BC1. For $(\Omega, \mathscr{F}, P)$ probability space,

$$A_n \in \mathscr{F}, \quad n = 1,2,\dots$$

If $\sum_{n=1}^{\infty} P(A_n) < +\infty$ then

$$P(\limsup_{n\to\infty} A_n) = 0$$

Proof. $\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \subset \bigcup_{k=n}^{\infty} P(A_k) \quad \forall n$
So,

$$P(\limsup_{n\to\infty} A_n) \le P(\bigcup_{k=n}^{\infty} P(A_k)) \le \sum_{k=1}^{\infty} P(A_k)$$

■

Theorem 1.4.6 — BC2. If $\{A_n\}$ are independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then

$$P(\limsup_{n} A_n) = 1$$

Proof. $P(\limsup_n A_n) = 1$

$$\Leftrightarrow P(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k) = 1$$

$$\Leftrightarrow P(\bigcup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k^C) = 0 \quad (*)$$

because,

$$\Leftrightarrow P(\bigcup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k^C) \leq \sum_{k=n}^{\infty} P(\cap_{k=n}^{\infty} A_k^C)$$

$$\Leftarrow P(\cap_{k=n}^{\infty} A_k^C) = 0 \quad \forall n = 1, 2, \ldots$$

but we need to prove this to imply (*).
Shit, calculus.

$$1 - x \leq e^{-1} \quad \forall x \in \mathbb{R}$$

For any $j = 1, 2, \ldots,$

$$\begin{aligned} P\left(\bigcap_{k=n}^{n+j} A_k^C\right) &= \prod_{k=n}^{n+j} (1 - P(A_k)) \\ &\leq \prod_{k=n}^{n+j} e^{-P(A_k)} \\ &= e^{-\sum_{k=n}^{n+j} P(A_k)} \end{aligned}$$

Now, $\sum_{k=1}^{\infty} P(A_k) = \infty$ and also

$$\sum_{k=n}^{\infty} P(A_k) \quad \forall n$$

So,

$$\lim \sum_{k=n}^{n+j} P(A_k) \to \infty \quad \forall n$$

$$\lim_{j \to \infty} P\left(\bigcap_{k=n}^{n+j} A_k^C\right) = 0$$

Because,

$$\bigcap_{k=n}^{n+j} A_k^C \downarrow \bigcap_{k=n}^{infty} A_k^C \quad j \to \infty$$

By continuity of probability,

$$P\left(\bigcap_{k=n}^{n+j} A_k^C\right) \downarrow P\left(\bigcap_{k=n}^{infty} A_k^C\right) \quad j \to \infty$$

So,

$$P\left(\bigcap_{k=n}^{infty} A_k^C\right) = 0$$

■

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finished proof.

BC1 and BC@ say that $P(\limsup_n A_n)$ is either 0 or 1.

This is a special case of a general phonemonon, the 0-1 Law.
Take σ-field, $(\Omega, \mathscr{F}, P)$,

$$A_1, \dots \in \mathscr{F}$$

For each n,

$$\sigma(A_n, A_{n+1}, dots)$$

We have another σ-field called "tail of σ-field",

$$\mathscr{T} = \bigcap_{n=1}^{\infty} \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots)$$

■ **Example 1.4 — 4.18 in Billingsly.** $\limsup A_n \in \mathscr{T}$?

$$\bigcap_n^{\infty} = 1 \bigcup^{i} nfty_{n=k} A_k$$

$$A_n, A_{n+1}, \dots \in \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots) \Rightarrow \bigcup_{k=n}^{\infty} A_k \in \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots)$$

$$\bigcap_n^{\infty} = 1 \bigcup_{n=k}^{\infty} A_k \in \bigcap_{n=1}^{\infty} \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots)$$

$$\begin{aligned}
\liminf A_n &= \left[(\bigcup_n^{\infty} = 1 \bigcap_{n=k}^{\infty} A_k)^C \right]^C \\
&= \left[\bigcup_n^{\infty} = 1 \bigcap_{n=k}^{\infty} A_k^C \right]^C \\
&= \left[\limsup_n A_k^C \right]^C \in \mathscr{T}
\end{aligned}$$

■

Theorem 1.4.7 If $A_1, A_2, \dots$ are independent, then for each $A \in \mathscr{T}$ we have $P(A) = 0$ or 1.

Proof. By Corollary 2,

$\sigma(A_1)$
$\sigma(A_2)$
$\vdots$
$\vdots$
$\sigma(A_{n-1})$
$\sigma(A_n, A_{n+1}, \dots)$

$$\sigma(A_1) \perp\!\!\!\perp \dots \perp\!\!\!\perp \sigma(A_n, A_{n+1}, \dots)$$

Let $A \in \mathscr{T}$, then,

$$A \in \sigma(A_n, A_{n+1}, \dots) \quad \forall n$$

So, $A_1, \dots, A_{n-1}, A$ are independent. By taking nlarge enough, this implies that any finite subcollectoin of $A, A_1, A_2, \dots$ is also independent.

This implies that the sequence $\{A, A_1, A_2, \dots\}$ are independent.

But $A \in \sigma(A_1, A_2, \dots)$ so,

$$A \perp\!\!\!\perp A$$

$$P(AA) = P(A)P(A) = P(A)^2$$

So $P(A)$ must be zero or 1!

■

R We are now skipping a few sections (5 -9) in Billingsly. These are about special random variables, random walks, etc...

1.5 General Measure on a Field

Borel Sets in $\mathbb{R}^k$

Two jumps, from $(0,1] \rightarrow \mathbb{R} \rightarrow \mathbb{R}^k$.
$\mathscr{B}$ on $(0,1]$ is a σ-field generated by $\mathscr{J}$ = collection of all intervals in (0,1].

$$\sigma(\mathscr{I}) = \mathscr{B} \text{ on } (0,1]$$

When we work with $\mathbb{R}$,

$$\mathscr{I}' = \text{collection of all intervals in } \mathbb{R}, (a,b)$$

$$\sigma(\mathscr{I}') = \mathscr{R}' \quad \text{linear Borel } \sigma\text{-field}$$

$\mathscr{I}^k$ is the collection of all rectangles in $\mathbb{R}^k$.

$$\mathscr{I}^k = \{(a_1, b_1]x \dots x(a_k, b_k] : a_1, b_1, \dots, a_k, b_k \in \mathbb{R}\}$$

$$\sigma(\mathscr{I}^k) = \mathscr{R}^k \quad \text{Borel } \sigma\text{-field in } \mathbb{R}^k$$

Properties of $\mathbb{R}^k$

(*) Any open set are in $\mathscr{R}^k$.

Let $\mathbb{Q}$ be the set of all rational numbers. This is countable and dense subset of $\mathbb{R}$.

Definition 1.5.1 — Dense. Look up definition!

Class of rational rectangles:

$$\mathscr{I}_{\mathbb{Q}}^k = \{(a_1,b_1]x\dots x(a_k,b_k] : a_1,b_1,\dots,a_k,b_k \in \mathbb{Q}\}$$

Let G be an open set in $\mathbb{R}^k$ and $y \in G$, then there exists

$$A_y \in \mathscr{I}_Q^k$$

such that

$$y \in A_y \subset G$$

because $\mathbb{Q}$ is dense in $\mathbb{R}$.

Note that, $\bigcup_{y \in G} A_y = G$.

But, $\{A_y : y \in G\} \subseteq \mathscr{I}_{\mathbb{Q}}^k$.

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Note that the above union of A_y is countable.
Also, $G = \bigcup_{y \in G} A_y$ and so $G \in \sigma(\mathscr{I}_Q^k) \subseteq sigma(\mathscr{I}_Q) = \mathscr{R}^k$.
Immediately, we see that

(*) All closed sets F are in $\mathscr{R}^k$.

All sets we commonly see are in $\mathscr{R}^k$.

(*) $\mathscr{R}^k$ is in fact also the σ-field gnerated by the class of all open sets in $\mathbb{R}^k$, $\mathscr{G}^k$.

Proof. Let

$$\mathscr{J}^k = \{(a_1,b_1)x\dots x(a_k,b_k) : a_1,b_1,\dots,a_k,b_k \in \mathbb{R}, k = 1,2,\dots\}$$

Claim: $\sigma(\mathscr{J}^k) \in \mathscr{R}^k)$

Note that any

$$(a_1,b_1]x\dots x(a_k,b_k] \in \mathscr{I}^k$$

can be written as

$$\bigcap_{n=1}^{\infty}(a_1,b_1+n^{-1}]x\dots x(a_k,b_k+n^{-1}]$$

The above statement is within $\mathscr{I}^k$ with the intersection, otherwise it'd be in $\mathscr{J}^k$.

That means each $A \in \mathscr{I}^k$ is in $\sigma(\mathscr{J}^k)$.

$$\mathscr{I}^k \subseteq \sigma(\mathscr{J}^k)$$

$$\sigma(\mathscr{I}^k) \subseteq \sigma(\mathscr{J}^k)$$

$$\mathscr{R}^k \subset \sigma(\mathscr{J}^k)$$

But since $\mathscr{R}^k$ contains all open sets (previous (*)), we have that $\mathscr{R}^k \supseteq \mathscr{J}^k$ and $\mathscr{R}^k \supseteq \sigma(\mathscr{J}^k)$. So,

$$\mathscr{R}^k = \sigma(\mathscr{J}^k)$$

Our claim is proved.
Because $\mathscr{R}^k \supseteq \mathscr{G}^k$,

$$\mathscr{R}^k \supseteq \sigma(\mathscr{G}^k)$$

But, $\mathscr{J}^k \subset \mathscr{G}^k$

$$\mathscr{R}^k = \sigma(\mathscr{J}^k) \subset \sigma(\mathscr{G}^k)$$

So,

$$\mathscr{R}^k = \sigma(\mathscr{G}^k)$$

And this is the general definition of Borel σ-field, because open sets exists much more generally than rectangles.

In fact, Borel Sets in Hilbert spaces, Banach...etc. whereever you can define open sets, you can define Borel Sets.

■

Borel Sets in Topological Space

Definition 1.5.2 — Topology. A paving $\mathscr{T}$ is a **topology** on Ω if it is closed under arbitrary union of finite intersections. That is, if you have an arbitrary index,

1. If $\{A_\theta : \theta \in \Theta\} \subseteq \mathscr{T}$ then the

$$\bigcup_{\theta \in \Theta} A_\theta \in \mathscr{T}$$

2. If $A, B \in \mathscr{T}, AB \in \mathscr{T}$ then any set $A \in \mathscr{T}$ is called an open set with respect to $\mathscr{T}$.

$(\Omega, \mathscr{T}) \leftarrow$ Topological Space

$\sigma(\mathscr{T}) \leftarrow$ Borell σ-field generated by $\mathscr{T}$-open sets.

$(\Omega, \sigma(\mathscr{T}))$ is measureable.

Here is another question:

$$\mathscr{B}, \mathscr{R}', \mathscr{R}^{\star}$$

Is it reasonalbe to conjecture to following?

$$\{A \subseteq \mathscr{R}' : A \subseteq (0,1]\} = \mathscr{B}$$

σ-field Restricted on a Set

Let $(\Omega, \mathscr{F})$ be a measure space.

$$\Omega_0 \subseteq \Omega$$

(otherwise arbitrary, especially Ω_0 need not be in $\mathscr{F}$)

Define (with some "lazy" notation),

$$\mathscr{F} \bigcap \Omega_0 = \{A\Omega_0 : A \in \mathscr{F}\}$$

Theorem 1.5.1 (i) $\mathscr{F} \bigcap \Omega_0$ is a σ-field in Ω_0
(ii) If $\mathscr{A}$ generates $\mathscr{F}$ then, $A \bigcap \Omega_0$ generates $\mathscr{F} \bigcap \Omega_0$

Proof. (i) *$\mathscr{F} \bigcap \Omega_0$ is a σ-field in Ω_0*

Want to show: $\Omega_0 \in \mathscr{F} \bigcap \Omega_0$

$$\Omega \in \mathscr{F}$$
$$\Omega_0 = \Omega \bigcap \Omega_0 \in \mathscr{F} \bigcap \Omega_0$$

Want to show: $A \in \mathscr{F} \bigcap \Omega_0 \to A \in A^C \in \mathscr{F} \bigcap \Omega_0$

$$A \in \mathscr{F} \bigcap \Omega_0 \Rightarrow A = B\Omega_0, B \in \mathscr{F}$$
$$B \in \mathscr{F} \Rightarrow B^c \in \mathscr{F}$$

So,

$$B^C \Omega_0 \in \mathscr{F} \Omega_0$$

But,
$\Omega_0 \setminus A$
$= \Omega_0 \setminus (B\Omega_0)$
$= \Omega_0 (B\Omega_0)^C$
$= \Omega_0 (B^C \bigcup \Omega_0^C)$
$= (\Omega_0 B^C) \bigcup (\Omega_0 \Omega_0^C)$
$= (\Omega_0 B^C) \bigcup \emptyset$
$= (\Omega_0 B^C) \in \mathscr{F} \bigcap \Omega_0$

Want to show: $A_1, \dots \in \mathscr{F} \bigcap \Omega_0 \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{F} \bigcap \Omega_0$

$$A_n \in \mathscr{F} \bigcap \Omega_0 \Rightarrow A_n = B_n \Omega_0, B_n \in \mathscr{F}$$

$$\bigcup_n B_n \in \mathscr{F} \Rightarrow (\bigcup_n B_n)\Omega_0 \in \mathscr{F} \bigcap \Omega_0$$

But, $$\Rightarrow \bigcup_n (B_n\Omega_0) \in \mathscr{F} \bigcap \Omega_0$$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{F} \bigcap \Omega_0$$

So the three things we've just shown gives us that $\mathscr{F} \bigcap \Omega_0$ is in deed a σ-field.

(ii) *If $\mathscr{A}$ generates $\mathscr{F}$ then, $A \bigcap \Omega_0$ generates $\mathscr{F} \bigcap \Omega_0$*

Let $\mathscr{A} \subseteq \mathscr{F}, \sigma(\mathscr{A}) = \mathscr{F}$.
Let $\mathscr{F}_0 = \sigma(\mathscr{A} \bigcap \Omega_0)$
Our goal: $\mathscr{F} \bigcap \Omega_0 = \sigma(\mathscr{A})$

Step 1: $\mathscr{F}_0 \subseteq \mathscr{F} \bigcap \Omega_0$

$$\mathscr{A} \bigcap \Omega_0 \subset \mathscr{F} \bigcap \Omega_0$$

$$\mathscr{F} \bigcap \Omega_0 \text{ is } \sigma\text{-field}$$

Step 2: $\mathscr{F} \bigcap \Omega_0 \subset \mathscr{F}_0$ holds if $[A \in \mathscr{F} \Rightarrow A\Omega_0 \in \mathscr{F}]$

$$\Rightarrow \bigcap \Omega_0 \subset \mathscr{F}_0$$

If bracket statement is true, then $\mathscr{F} \bigcap \Omega_0 \subset \mathscr{F}$.
Let $A \in \mathscr{F} \bigcap \Omega_0$ then

$$A = B\Omega_0, B \in \mathscr{F}$$

But by bracket statement,

$$B \in \mathscr{F} \Rightarrow B\Omega_0 \in \mathscr{F} \Rightarrow A \in \mathscr{F}_0$$

So, $\mathscr{F} \bigcap \Omega_0 \subset \mathscr{F}_0$.

Step 3: Let $\mathscr{G} = \{A \subset \Omega : A\Omega_0 \in \mathscr{F}_0\}$. Then,

$$\mathscr{A} \subseteq \mathscr{G}$$

Pick $A \in \mathscr{A}$.

$$\Rightarrow A\Omega_0 \in \mathscr{A} \bigcap \Omega_0 \subset \mathscr{F}_0$$

$$\Rightarrow A \in \mathscr{G}$$

So, $\mathscr{A} \subset \mathscr{G}$.

Step 4: $\mathscr{G}$ is σ-field in Ω.

(a)

$$\Omega \in \mathscr{G} : \Omega\Omega_0 = \Omega_0 \in \mathscr{F}_0 = \sigma(\mathscr{A} \bigcap \Omega_0)$$

Generally, if Ω is a set $\mathscr{B}$ is a pvaing on Ω then $\sigma(\mathscr{B}) = \bigcap\{\mathscr{B}' : \mathscr{B}' \supseteq \mathscr{B}, \mathscr{B}'$ is a σ-field on $\Omega\}$.
Which is true because $\mathscr{A}$ generates $\mathscr{F}, \Omega \in \mathscr{A}$.

This means that

$$\sigma(\mathscr{A}\bigcap\Omega_0) = \bigcap\{\mathscr{B} : \mathscr{B} \supset \mathscr{A} \cap \Omega_0, \mathscr{B} \text{ is a } \sigma\text{-field on } \Omega\}$$

So, $\Omega_0 \in \mathscr{F}_0$

(b) $A \in \mathscr{G} \Rightarrow A^C \in \mathscr{G}$.

$$\begin{aligned} A \in \mathscr{G} &\Rightarrow A\Omega_0 \in \mathscr{F}_0 \\ &\Rightarrow \Omega_0 \setminus (A\Omega_0) \in \mathscr{F}_0 \\ &\Rightarrow \Omega_0 \cap (A\Omega_0)^C \in \mathscr{F}_0 \\ &\Rightarrow \Omega_0 \cap (A^C\Omega_0^C) \in \mathscr{F}_0 \\ &\Rightarrow (\Omega_0 A^C) \cup (\Omega_0\Omega_0^C) \\ &= \Omega_0 A^C \in \mathscr{F}_0 \end{aligned}$$

So, $A^C \in \mathscr{F}$.

(c) $A_1, A_2, \cdots \in \mathscr{G}$ are disjoint means $A_n\Omega_0 \in \mathscr{F}_0$ and $A_n\Omega_0$ disjoint.

$$\bigcup_{n=1}^{\infty}(A_n\Omega_0) \in \mathscr{F}$$

$$(\bigcup_{n=1}^{\infty} A_n)\Omega_0 \in \mathscr{F}$$

$$\bigcup_{n=1}^{\infty} A_n \in \mathscr{G}$$

Step 5: $\mathscr{F}\bigcap\Omega_0 \subseteq \mathscr{F}_0$

By Step 3, we know that $\mathscr{A} \subset \mathscr{G}$.

By Step 4, $\mathscr{G}$ is σ-field.

Together, $\sigma(\mathscr{A}) = \mathscr{G}$.

$$\sigma(\mathscr{A}) = \mathscr{F} \in \mathscr{G}$$

$$\Rightarrow [A \in \mathscr{F} \Rightarrow A\Omega_0 \in \mathscr{F}]$$

$$\Rightarrow \bigcap\Omega_0 \subset \mathscr{F}_0$$

■

Wednesday September 21

Worked on proof, but will need to go back to it in the future.

So we have now that $\sigma(\mathscr{A}\bigcap\Omega_0) = \sigma(\mathscr{A}\setminus\bigcap\Omega)$

Lemma 1. If $\Omega_0 \in \mathscr{F}$ then

$$\mathscr{F}\bigcap\Omega_0 = \{B \in \mathscr{F} : B \subset \Omega_0\}$$

Proof. Need to show that $\{A\Omega_0 : A \in \mathscr{F}\} = \{B \in \mathscr{F} : B \subset \Omega_0\}$
Let $C \in LHS$.

$$C = A\Omega_0, A \in \mathscr{F}$$

then $C \in \mathscr{F}, C \subset \Omega_0$.

So $C \in RHS$.

Let $B \in RHS$,

$$B \subset \Omega_0, B \in \mathscr{F}$$

$$B = B \bigcap \Omega_0, B \in \mathscr{F}$$

So, $B \in LHS$. ■

Corollary 1.5.2 $\mathscr{B} = \{A \subset (0,1] : A \in \mathscr{R}'\} = \{A \in \mathscr{R}' : A \in (0,])\}$

Proof. Let $\Omega \in \mathbb{R}$.

$$\Omega_0 = (0,1]$$

$$\mathscr{F} = \mathscr{R}'$$

$$\dashv = \mathscr{I}'$$

By Theorem 10.1,

$$\sigma(\mathscr{A}) \bigcap \Omega_0 = \sigma(\mathscr{A} \bigcap \Omega)$$

$$\Leftrightarrow \sigma((\mathscr{I}') \bigcap (0,1]) = \sigma(\mathscr{I}' \bigcap (0,1]) = \sigma(\mathscr{I}) = \mathscr{B}$$

Now $(0,1] \in \mathscr{R}' \Leftrightarrow \Omega_0 \in \mathscr{F}$.

By Lemma 1,

$$\Rightarrow \mathscr{F} \bigcap \Omega_0 = \{A \in \mathscr{F} : A \subset \Omega_0\}$$

$$\Rightarrow \mathscr{R} \bigcap (0,1] = \{A \in \mathscr{R} : A \subset (0,1]\}$$

So,

$$\mathscr{B} = \{A \in \mathscr{R} : A \subset (0,1]\}$$

■

For general measure, neet infinity convention.

For

$$x, y \in [0, \infty] = [0, \infty) \bigcup \{\infty\}$$

$x \le y$ means that (either/or)
1. $y = \infty$
2. $y < \infty, x < \infty, x \le y$

$x < y$ means that (either/or)
1. $y = \infty, x < \infty$
2. $x, y < \infty, x < y$

For a finite or infinite sequence, $x, x_1, \cdots \in [0, \infty]$,
$x = \sum_{k=1}^{\infty} x_k$ means that (either/or)
1. $x = \infty, x_k = \infty$ for some k
2. $x = \infty, x < \infty \forall k$
   $\sum_{k=1}^{n} x_k : n = 1, 2, \ldots$ diverges.
3. $x < \infty, x_k < \infty$

   $\lim_{n\to\infty} \sum_{k=1}^{n} x_k = x$

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For any infinte sequence, $x_1, x_2, \cdots \in [0, \infty]$ and $x \in [0, \infty]$
$x_k \uparrow x$ is true if and only if
1. $x_k \le x_{k+1} \le x \quad \forall k$
2. either
   (a) $x < \infty$, and $x_k \uparrow x$ in usual sense.
   (b) $x_k = \infty$ for $\forall k \ge m, x = \infty$
   (c) $x = \infty, x_k < \infty, x_k \uparrow \infty$

Measures on Field

Let $\Omega \leftarrow$ nonempty and $\mathscr{F}$ be a field on Ω.

Definition 1.5.3 A measure, μ is a function on $\mathscr{F}$ such that
1. $\mu(A) \in [0, \infty]$
2. $\mu(\emptyset) = 0$
3. If $A_n \in \mathscr{F}$ are disjoint then

$$\mu(\bigcup_{n-1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

A measure if finite if $\mu(\Omega) < \infty$.

A measure is a probability measure or simply probability if $\mu(\Omega) = 1$.

A measure is τ-finite if there exists $\mathscr{F}$-sets A_n such that

$$\bigcup_{n=1}^{\infty} A_n = \Omega, \mu(A_n) < \infty$$

If $\mathscr{F}$ is a σ-field then $(\Omega, \mathscr{F})$ is a measureable space.

If μ is a measure on $\mathscr{F}$ then $(\Omega, \mathscr{F}, \mu)$ is a measure space.

Definition 1.5.4 — Support. If $A \in \mathscr{F}, \mu(A^C) = 0$ then A is a support of μ.

A measure, μ on $(\Omega, \mathscr{F})$ has the following properties (proof similar to probability cass omitted).

Finite additivity: $A_1, \ldots, A_n$ are disjoint $\mathscr{F}$-sets implies

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

Monotonicity: If $A, B \in \mathscr{F}, A \subseteq B$ then

$$\mu(A) \leq \mu(B)$$

Inclusion-Exclusion Formula: $A_1, \ldots, A_n \in \mathscr{F}_1$

$$\mu(\bigcup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k) - \sum_{i=k<l=n} \mu(A_k A_l) + \cdots + (-1)^{n+1} \mu(A_1 \ldots A_k)$$

Countable or Finite Subadditivity: $A_1, \cdots \in \mathscr{F}$, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

Theorem 1.5.3 Let μ be a measure on a σ-field, $\mathscr{F}$.

1. Continuity from below.

$$A_n, A \in \mathscr{F}, A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A)$$

2. Continuity from above.

$$A_n, A \in \mathscr{F}, \mu(A_1) < \infty, A_n \downarrow A \Rightarrow \mu(A_n) \to \mu(A)$$

3. Countable subadditivity.

$$A_n \in \mathscr{F}, \bigcup_n \in \mathscr{F} \Rightarrow \mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

4. If μ is σ-finite on $\mathscr{F}$ then $\mathscr{F}$ contain an uncountable, disjoint collection of sets with positive μ-measure.

Proof. (i) and (iii) exactly the same as in probability case.

(ii) takes a little extra work.

If $\mu(A_1) < \infty$ then

$$A_1 \setminus A_n \uparrow A_1 \setminus A$$

then by (i) we have

$$\mu(A_1 \setminus A_n) \uparrow \mu(A_1 \setminus A)$$

$$\mu(A_1) - \mu(A_n) \uparrow \mu(A_1) - \mu(A)$$

$$\mu(A_n) \downarrow \mu(A)$$

(iv) Let $\{B_\theta : \theta \in \Theta\}$ be a disjoint collection of $\mathscr{F}$-sets such that $\mu(B_\theta) > 0$. We want to show it is countable.

Claim: If $A \in \mathscr{F}$ and $\mu(A) < \infty, \varepsilon > 0$. then,

$$\{\theta : \mu(AB_\theta) > \varepsilon\} \text{ is finite.}$$

Show: If $\{\theta : \mu(AB_\theta) > \varepsilon\}$ is infinite then there exists $\theta_1, \theta_2, \ldots$ such that $\mu(AB_{\theta_i}) > \varepsilon \quad \forall \theta_i$.
But, if this were so,

$$\sum_{i=1}^{n} \mu(AB_{\theta_i}) \geq n\varepsilon$$

$$\sum_{i=1}^{n} \mu(AB_{\theta_i}) \geq \mu(A)$$

for sufficiently large n. CONTRADICTION.
But, $\{\theta : \mu(AB_\theta) > 0\} = \bigcup_{r \in \mathbb{Q}} \{\theta : \mu(AB_\theta) > r\}$ is a countable union of a finite set.

So $\{\theta : \mu(AB_\theta) > 0\}$ is countable. Now we just need to show $\{\theta : \mu(B_\theta) > 0\}$ is also countable.

But we know that $\{\theta : \mu(\Omega B_\theta) > 0\}$ is countable because μ is σ-finite so there exists $A_n \in \mathscr{F}, n = 1, 2, \ldots$ such that $\mu(A_n) < \infty, \bigcup_{n=1}^{\infty} A_n = \Omega$.

Want to show $\{\theta : \mu(\bigcup_{n=1}^{\infty} A_n B_\theta) > 0\}$ is countable.
Note that

$$\mu(\bigcup_{n=1}^{\infty} A_n B_\theta) \leq \mu(\sum_{n=1}^{\infty} A_n B_\theta)$$

So,

$$\{\theta : \mu(\bigcup_{n=1}^{\infty} A_n B_\theta) > 0\} \subseteq \{\theta : \mu(\sum_{n=1}^{\infty} A_n B_\theta) > 0\}$$

The RHS may be rewritten as

$$\bigcup_{n=1}^{\infty} \{\theta : \mu(\bigcup_{n=1}^{\infty} A_n B_\theta) > 0\}$$

■

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Uniqueness of Extension

That is, if the extension exists (we'll explore later), then is it unique?

Theorem 1.5.4 Suppose $\mu_1, \mu_2$are measures on $\sigma(\mathscr{P})$, where $\mathscr{P}$ is a π -system of subsets of Ω. Suppose μ_1, μ_2 are σ-finite on $\mathscr{P}$. If μ_1, μ_2 agree on $\mathscr{P}$ (meanig $\mu_1(A) = \mu_2(A) \forall A \in \mathscr{P}$) then they agree on $\sigma(\mathscr{P}).\mathscr{P}$

σ-Finiate on $\mathscr{P}$

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, $\mathscr{A} \subseteq \mathscr{F}$. We say μ is σ-finite on $\mathscr{A}$ if there exists $\{A)n\}_{n-1}^{\infty}, A_n \in \mathscr{A}$ such that

$$\mu(A_n) < \infty, \Omega = \bigcup_{n=1}^{\infty} A_n$$

Proof. Idea: extend the parallel theorem in probability case.

Step 1: Prove that if $B \in \mathscr{P}$, $\mu_1(B) = \mu_2(B) < \infty$ then

$$\mu_1(BA) = \mu_2(BA) \quad \forall A \in \sigma(\mathscr{P})$$

Let $\mathscr{L}_B = \{A \in \sigma(\mathscr{P}) : \mu_1(BA) = \mu_2(BA) < \infty\}$.

Need to prove $\mathscr{L}_B$ is a λ-system.

1. $\Omega \in \mathscr{L}_B$.

   We know that $\Omega \in \sigma(\mathscr{P})$, because $\sigma(\mathscr{P})$ is itself a σ-field. Thus,

   $$\mu_1(B\Omega) = \mu_1(B) = \mu_2(B) = \mu_2(B\Omega)$$

   and so we have that $\Omega \in \mathscr{L}_B$.

2. Want $A \in \mathscr{L}_B \Rightarrow A^C \in \mathscr{L}_B$.

   $$\begin{aligned} A \in \mathscr{L}_B &\Rightarrow \mu_1(AB) = \mu_2(AB) \\ &\Rightarrow \mu_1(B) - \mu_1(AB) = \mu_2(B) - \mu_2(AB) \\ &\Rightarrow \mu_1(B \setminus (AB)) = \mu_2(B \setminus (AB)) \\ &\Rightarrow \mu_1(BA^C) = \mu_2(BA^C) \end{aligned}$$

   We have that $A^C \in \mathscr{L}_B$.

3. Want that if A_n are disjoint $\mathscr{L}_B$-sets $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{L}_B$.

   $$A_n \in \mathscr{L}_B : \mu_1(A_nB) = \mu_2(A_nB)$$

   $$\sum_{n=1}^{\infty} \mu_1(A_nB) = \sum_{n=1}^{\infty} \mu_2(A_nB)$$

   Because the A_n are disjoint, we have that BA_n are also disjoint.

   $$\mu_1\left(\bigcup_{n=1}^{\infty}(A_nB)\right) = \mu_2\left(\bigcup_{n=1}^{\infty}(A_nB)\right)$$

   $$\mu_1\left((\bigcup_{n=1}^{\infty}A_n)B\right) = \mu_2\left((\bigcup_{n=1}^{\infty}A_n)B\right)$$

   So, $\bigcup_n A_n \in \mathscr{L}_B$. And thus $\mathscr{L}_B$ is a λ-system.

But, $\mathscr{L}_B \supseteq \mathscr{P}$ by definition. So $\mathscr{L}_B \supseteq \sigma(\mathscr{P})$ by the $\pi - \lambda$ Theorem. Se we have

$$\mu_1(AB) = \mu_2(AB) \quad \forall A \in \sigma(\mathscr{P})$$

and step 1 is finished.

Step 2: Want $\mu_1(A) = \mu_2(A) \forall A \in \sigma(\mathscr{P})$. Or rather $\mu_1(A\Omega) = \mu_2(A\Omega) \forall A \in \sigma(\mathscr{P})$, but Ω, unlike B, may not have $\mu(\Omega) < \infty$.

Because μ is σ-finite on $\mathscr{P}$ there exists $B_n \in \mathscr{P}, \mu(B_n) < \infty$ such that $\bigcup_n B_n = \Omega$.

So we need to show that

$$\mu_1\left((\bigcup_{i=1}^{n} B_i)A\right) = \mu_2\left((\bigcup_{i=1}^{n} B_i)A\right)$$

$$\begin{aligned}
\mu_1\left((\bigcup_{i=1}^{n} B_i)A\right) &= \mu_1(\bigcup_{i=1}^{n}(B_iA)) \\
&= \sum_{1\le i\le n} \mu_1(B_iA) - \sum_{1\le i<j\le n} \mu_1(B_iB_jA) + \cdots + (-1)^{n+1}\mu_1(B_1 \ldots B_nA) \\
&= \mu_2((\bigcup_{i=1}^{n} B_i)A)
\end{aligned}$$

by the continuity shown below,

$$mu_1((\bigcup_{i=1}^{n} B_i)A) \uparrow mu_1((\bigcup_{n} B_n)A)$$

$$mu_2((\bigcup_{i=1}^{n} B_i)A) \uparrow mu_2((\bigcup_{n} B_n)A)$$

So, $mu_1(\bigcup_n B_nA) = mu_2(\bigcup_n B_nA)$, and since the union of all of the B_n is Ω we have,

$$\mu_1(A) = \mu_2(A)$$

■

This means, that if we can extend μ from a field, $\mathscr{F}_)$ to a σ-field, $\sigma(\mathscr{F}_0)$ and we know that μ is σ-finite on $\mathscr{F}_0$, then the extension is unique.

Theorem 1.5.5 Suppose μ_1 and μ_2 are finite meaures on $\sigma(\mathscr{P})$ where $\mathscr{P}$ is a π-system and Ω is a countable union of $\mathscr{P}$-sets. then if μ_1, μ_2 agree on $\mathscr{P}$ then they will agree on $\sigma(\mathscr{P})$.

Proof. Be assumption, there exists $B_1, B_2, \cdots \in \mathscr{P}$ such that $\Omega = \bigcup_n B_n$. Because $\mu_1(B_n) \le \mu_1(\Omega) < \infty \forall n$ and $\mu_2(B_n) \le \mu_2(\Omega) < \infty \forall n$ then they are σ-finite on $\mathscr{P}$. Then the theorem follows from the previous theorem. ■

1.6 Extension of General Measure to σ-Field

Outer Measure

Definition 1.6.1 — Outer Measure. Ω nonempty set

An outer measure is a function on 2^{Ω} such that

1. $\mu^\star(A) \in [0,\infty] \forall A \subseteq \Omega$
2. $\mu^\star(\emptyset) = 0$
3. $A \subseteq B \Rightarrow \mu^\star(A) \leq \mu^\star(B)$
4. $\forall \{A_n\} \subseteq 2^\Omega, \mu^\star(\bigcup_n A_n) \leq \sum_n \mu^\star(A_n)$

Example 1.5 — 11.1 in Billingsly. Ω-set.

$\mathscr{A}$ is class of subsets of $\Omega, \emptyset \in \mathscr{A}$

$$\rho : \mathscr{A} \to [0,\infty], \rho(\emptyset) = 0$$

We say that $\{A_n\}$ is an $\mathscr{A}$ covering of $A \subseteq \Omega$ if $A \subseteq \bigcup_n A_n, A_n \in \mathscr{A}$.

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Proof. 1. Want $\mu^\star(A) \in [0,\infty]$

$$\mu^\star(A) = \inf\{\sum_n \rho(A_n), \{A_n\} \text{ is a } \mathscr{A}\text{-covering}\}$$

Here, we know that the probability is between $[0,\infty]$ so the sum must be too, and thus the entire equation must be.

2. Want $\mu^\star(\emptyset) = 0$

Because,

$$\emptyset \in \mathscr{A}$$

So,

$$\emptyset \subseteq \emptyset \cup \emptyset \cup \emptyset \cup \ldots$$
$$\mu^\star(\emptyset) \leq \sum_n \rho(\emptyset) = 0$$

3. Want that if $A \subseteq B$ and if $\{A_n\}$ is a $\mathscr{A}$-covering of A then the collection of all $\mathscr{A}$-coverings of B is contained within the collection of all $\mathscr{A}$-coverings of $\mathscr{A}$.

So we can see that the $\inf\{\text{statement } 1\} \geq \inf\{\text{statement } 2\}$.

4. Let A_n be sets in 2^Ω, we want that $\mu^\star(\bigcup_n A_n) \leq \sum_n \mu^\star(A_n)$.

For each A_n let $\{B_{nk}\}_{k=1}^\infty$ be an $\mathscr{A}$-covering of A_n such that

$$\sum_{k-1}^{n} \rho(B_{nk}) \leq \mu^\star(A_n) + \frac{\varepsilon}{2^n}$$

Also, we have $\{B_{nk} : n = 1,2\ldots, k = 1,2,\ldots\}$ this is an $\mathscr{A}$-covering, $\bigcup_n A_n$.

$$\mu^\star(\bigcup_n A_n) \le \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \rho(B_{nk}) \le \mu^\star(A_n) + \frac{\varepsilon}{2^n}$$

$$\begin{aligned} \mu^\star(A_n) + \frac{\varepsilon}{2^n} &= \sum_{n=1}^{\infty} \mu^\star(A_n) + \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} \\ &= \sum_{n=1}^{\infty} \mu^\star(A_n) + \varepsilon \end{aligned}$$

So $\mu^\star(\bigcup_n A_n) \le \sum_n \mu^\star(A_n) + \varepsilon \Rightarrow \mu^\star(\bigcup_n A_n) \le \sum_n \mu^\star(A_n) \quad \forall \varepsilon > 0$.

■

Now let $\mathscr{M}(\mu^\star)$ to be the paving,

$$\{A : \mu^\star(E) = \mu^\star(EA) + \mu^\star(EA^C), \forall E \in 2^\rceil\Omega\}$$

Theorem 1.6.1 If $\mu^\star$ is an outer measure, then $\mathscr{M}(\mu^\star)$ is a σ-field and $\mu^\star$ restricted on $\mathscr{M}(\mu^\star)$ is a measure.

Proof. The proof is the exact same as Lemma 3 of Section 3, except $P^\star$ is replace by $\mu^\star$. It turns out that the ∞ value of $\mu^\star$ does not cause any change in the proof once we invoke the three infinity conventions.

Thus, proof omitted. ■

Extension of (General) Measure from a Field to a σ-Field

Theorem 1.6.2 A measure on a field has an extension to the generated σ-field.

We will use a different proof from section 3 (even though we could use it). We are going to prove a more general result. But first, we need to introduce the notion of Semiring.

Definition 1.6.2 — Semiring. A class of subsets of pavings on Ω is called a **semiring** if

1. $\emptyset \in \mathscr{A}$
2. $A \in \mathscr{S}, B \in \mathscr{A} \Rightarrow AB \in \mathscr{S}$
3. If $A \subset B$ then,

$$B \setminus A = \bigcup_{k=1}^{n} C_k$$

where $C_1, \dots, C_k$ are disjoint intervals.

A semiring is also a π-system.

■ **Example 1.6** Let $\mathscr{A} = \{(a,b] : a, b \in \mathscr{R}\}$

1. $\emptyset \in \mathscr{A}$? $(a,a] = \emptyset$
2. See photo.
3. See photo.

■

Proof. Suppose $\mathscr{A}$ is a semiring.
μ is a set function such that $\mu : \mathscr{A} \to [0, \infty)$.

Assume μ is finitely additive and countable subadditive.

Then μ extends to a meassure on $\sigma(\mathscr{A})$ ■

More genreal than the previous theorem,

1. $\mathscr{A}$ needs not be a field.
2. μ need not be a measure.

Proof. 1. $\mu^\star(\emptyset) = 0$.

2. By monotonicty, let $A, B \in \mathscr{A}, A \subseteq B$.
   So, because $\mathscr{A}$ is a semiring,

$$\mu(B \setminus A) = \mu(\bigcup_{k=1}^{n} C_k)$$

$$\mu(B) - \mu(A) = \sum_{k=1}^{n} \mu(C_k) \quad \text{finite additivity}$$

$$\mu(B) = \mu(A) + \sum_{k=1}^{n} \mu(C_k)$$

$$\geq \mu(A)$$

First, let us show that

$$\mathscr{A} \subseteq \mathscr{M}(\mu^\star)$$

So we need to show that $A \in \mathscr{A}$.

$$\mu^\star(E) = \mu^\star(EA) + \mu^\star(EA^C) \quad \forall E \in 2^\Omega$$

But we know $\leq$, so we need to show $\geq$.

If $\mu^\star(E) = \infty$ this is certainly true by ∞-convention. So let's do $\mu^\star(E) < \infty$.

Fix a $\varepsilon > 0$. Let A_n be a $\mathscr{A}$-coving of E such that

$$]sum_n \mu(A_n) leq \mu^\star(E) + \varepsilon$$

Let $A \in \mathscr{A}, A \setminus A_n$, then we have

$$A = AA_n \cup AA_n^C = \bigcup_{k=1}^{m_n} C_n k$$

Note the C's are disjoint $\mathscr{A}$ sets.
So, for all $A \in \mathscr{A}$ we may write A as the disjoint union of $\mathscr{A}$ sets:

$$A = B_n \cup \left(\bigcup_{k=1}^{m_n} c_{nk} \right)$$

Want that $\mu^\star(EA) + \mu^\star(EA^C) \leq \mu^\star(E)$

$$\mu^\star(EA) = \mu^\star\left(E(B_n \cup \bigcup_{k=1}^{m_n} c_{nk}) \right)$$

$$= \mu^\star\left(EB_n \cup (E \bigcup_{k=1}^{m_n} c_{nk}) \right)$$

■

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Corollary 1.6.3 — To Theorem 11.1 & 11.3. Suppose that $\mathscr{A}$ is a semiring and that we have μ such that $\mathscr{A} \to [0,\infty]$ is set function such that:

1. $\mu(\emptyset) = 0$
2. μ is finitely additive on $\mathscr{A}$
3. μ is countably subadditive on $\mathscr{A}$
4. μ is σ-finite on $\mathscr{A}$

Then, μ is a unique extension on $\sigma(\mathscr{A})$.

■ **Example 1.7** Let $\mathscr{A}$ be the collection of all in $\mathbb{R}$, that is,

$$\mathscr{A} = \{(a,b] : a,b, \in \mathbb{R}\}$$

Let $\lambda_1 : \mathscr{A} \to \mathbb{R}, (a,b] = b-a$.

By Theorem 1.3, λ_1 is finitely additive, and indeed also countably subadditive. So it can be extended to $\sigma(\mathscr{A}) = \mathscr{R}^1$. But also we have that λ_1 is σ-finite on $\mathscr{A}$.

$$\Omega = \mathbb{R} = \bigcup_{n=1}^{\infty} (-n,n], \lambda_1(-n,n] = 2n$$

Therefore the extension is unique. This is defined to be the Lebesgue measure on $\mathscr{R}^1$ ■

Approximation Theorem

Approximate $\mu(A), A \in \sigma(\mathscr{A})$ by $\mu(B), B \in \mathscr{A}$.

Lemma 1. If $\mathscr{A}$ is a semiring, and $A, A_1, \ldots, A_n \in \mathscr{A}$, then we may write,

$$AA_1^C \ldots A_n$$

as finite disjoint unions of $\mathscr{A}$-sets. That is there exists

$$C_1, \ldots, C_m \in \mathscr{A}$$

that are disjoints such that

$$AA_1^C \ldots A_n^C = C_1 \cup \cdots \cup C_m$$

Think of this as a generalization of (iii) of Semiring.

Proof. Proof by Induction.

n = 1 Case

Want: $AA_1^C \ldots A_n^C = C_1 \cup \cdots \cup C_m$

$$AA_1^C = A \setminus (AA_1)$$

$$(AA_1) \subseteq A$$

By (iii) of Semiring, the above is equal to $C_1 \cup \cdots \cup C_m$.

Assume statement true for n.

n + 1 Case

Induction hypothesis gives us that

$$(AA_1^C \ldots A_n^C)A_{n+1}^C = (C_1 \cup \cdots \cup C_m)A_{n+1}^C$$

But when we use the n=1 case we have that the following inner term are finitely disjoint unions of $\mathscr{A}$-sets,

$$\bigcup_{k=1}^{\infty} C_k A_{n+1}^C$$

So we have that all together (unioned) is a finite disjoints union of $\mathscr{A}$-sets.

■

Symmetric Difference of Sets A, B

Notation 1.1.

$$A \triangle B = AB^C \cup BA^C$$

Theorem 1.6.4 Suppose $\mathscr{A}$ is a semiring, μ is a measure on $\sigma(\mathscr{A}) = \mathscr{F}$ and μ is σ-finite on $\mathscr{A}$.

1. For $B \in \mathscr{F}$, $\varepsilon > 0$, there exists a disjoins $mathcalA$-sequence $A_1, A_2, \ldots$ such that

$$B \subseteq \bigcup_k A_k, \mu(\bigcup_k A_k \setminus B) < \varepsilon$$

2. If $B \in \mathscr{F}, \mu(B) < \infty$ then for any $\varepsilon > 0$, there exists a finite disjoint $\mathscr{A}$-sequence, $A_1, \ldots, A_n$ such that

$$\mu(B \triangle \bigcup_{k=1}^{n} A_k) < \varepsilon$$

Proof. 1. Let $\mu^\star$ be the outer measure,

$$\mu^\star(A) = \inf\{\sum_n \mu(A_n) : \{A_n\} is a \mathscr{A} - covering\}$$

then $\mathscr{M}(\mu^\star)$ is a σ-field, $\mathscr{F} \subseteq \mathscr{M}(\mu^\star)$, $\mu^\star$ is a measure, $\mathscr{M}(\mu^\star)$, $\mu = \mu^\star$ on $\sigma(\mathscr{A}) = \mathscr{F}$.

$$\mu(A) = \inf\{\sum_n \mu(A_n) : \{A_n\} is a \mathscr{A} - covering\}$$

Let $B \in \mathscr{F}, \mu(B) < \infty$. Then there exists an $\mathscr{A}$-covering $\{A_k\}$ of B such that

$$\sum_n \mu(A_k) \leq \mu(B) + \varepsilon$$

So $\mu(\cup_k A_k) \leq \sum_k \mu(A_k) \leq \mu(B) + \varepsilon$.

And $\mu(\cup_k A_k) - \mu(B) \leq \varepsilon$.

$$\mu(\cup_k A_k \setminus B) \leq \varepsilon$$

Let

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 A_1^C \\ &\vdots \\ B_k &= A_k A_1^C \dots A_{k-1}^C \\ &\vdots \end{aligned}$$

So by Lemma 1, the B_k are finite dijoint union of $\mathscr{A}$-sets. Also,

$$\bigcup_k A_k = \bigcup_k B_k$$

so we have that there exists $C_1, C_2, \dots$ that are also disjoint $\mathscr{A}$-sets such that

$$\mu(\cup_k C_k \setminus B) \leq \varepsilon$$

Now suppose that $B \in \mathscr{F}, \mu(B) = \infty$. Because μ is σ-finite on $\mathscr{A}$ there exists $C_1, \dots \in \mathscr{A}$ such that $\mu(C)m \leq \infty$.

$$\Omega = \bigcup_m C_M$$

So then

$$\begin{aligned} B &= B\Omega \\ &= B(\cup_m C_m) \\ &= \cup_m BC_m \end{aligned}$$

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But $\mu(BC_m) \leq \mu(C_m) < \infty$ so by the finite case there exists a disjoint $\mathscr{A}$-sequence $\{A_{mk} : k = 1, 2, \dots\}$ such that

$$\bigcup_k A_{mk} \supseteq BC_m$$

$$\mu(\bigcup_k A_{mk} \setminus (BC_m) < \frac{\varepsilon}{2^k}$$

So now,

$$B = \cup_m BC_m \subseteq \bigcup_m \bigcup_k A_{mk}$$

$$\begin{aligned}
\mu(\bigcup_m\bigcup_k A_{mk}\setminus B) &= \mu((\bigcup_m\bigcup_k A_{mk})(\cup_m BC_m)) \\
&= \mu((\bigcup_m\bigcup_k A_{mk})(\cap_m (BC_m)^C)) \\
&= \mu(\bigcup_m(\bigcup_k A_{mk})(BC_m)^C) \\
&\leq \sum_m \mu(\bigcup_k A_{mk}\setminus(BC_m)) \\
&\leq \varepsilon
\end{aligned}$$

Since $\bigcup_m(\bigcup_k A_{mk}$ is a countable union of $\mathscr{A}$-sets, we can write as

$$\bigcup_k D_k$$

and make them disjoint as before,

$$E_1 = D_1$$

$$E_2 = D_2 D_1^C$$

$$\vdots$$

By Lemma 1, each E_m is finite disjoint union of $\mathscr{A}$-sets.

$$\bigcup_m E_m = \bigcup_m F_m$$

where F_m are disjoin $\mathscr{A}$-sets.
Hense, part (i) is proved.

2. Recall for any $B \in \mathscr{F}, \mu(B) < \infty$ and $\varepsilon > 0$, there exists a finite $\mathscr{A}$-sequence, $A_1, \ldots, A_n$, such that

$$\mu(B \triangle \bigcup_{k=1}^n A_k) < \varepsilon$$

$$\begin{aligned}
\mu\left[(\bigcup_{k=1}^n A_k)\triangle B\right] &= \mu((\bigcup_{k=1}^n A_k)^C)B \cup (\bigcup_{k=1}^n A_k)B^C) \\
&\leq \mu((\bigcup_{k=1}^n A_k)^C)B) + \mu((\bigcup_{k=1}^n A_k)B^C)
\end{aligned}$$

By (i) there exists disjoint $\mathscr{A}$-sets $\{A_n\}$ such that

$$\mu(\cup_n A_n \setminus B) < \varepsilon$$

Let $A = \bigcup_n A_n$.

Then,

$$A \setminus \bigcup_{k=1}^{n} A_k \downarrow \emptyset$$

Need, $\mu(A \setminus A_1) < \infty$ and $\mu(A) < \infty$.

By continuity from above, we have $\mu(A \setminus \bigcup_{k=1}^{n} A_K) \downarrow 0$.

So, for sufficiently large n we have,

$$\mu(A \setminus \bigcup_{k=1}^{n} A_K) < \varepsilon$$

Now first take a look at $\mu((\bigcup_{k-1}^{n} A_k)B^C)$.

$$\begin{aligned}\mu((\bigcup_{k-1}^{n} A_k)B^C) &\leq \mu(AB^C) \\ &= \mu(A \setminus B) < \varepsilon\end{aligned}$$

$$\begin{aligned}\mu((\bigcup_{k-1}^{n} A_k)^C A) &\leq \\ &= \mu(A \setminus \bigcup_{k-1}^{n} A_k) < \varepsilon\end{aligned}$$

So we have that $\mu(\bigcup_{k=1}^{n} A_k \triangle B) < 2\varepsilon$

■

The next lemma will be used in the next section, this is an extension of Theorem 1.3 in the textbook.

Lemma 2. Suppose $\mathscr{A}$ is a semiring, $A_1, \ldots, A_n, A$ be $\mathscr{A}$ -sets, and μ is a non-negative, finitely additive set function on $\mathscr{A}$. Then

1. If $\bigcup_{k=1}^{n} A_k \subset A$ and A_k are disjoint, then

$$\sum_{k=1}^{n} \mu(A_k) \leq mu(A)$$

2. If $A \subset \bigcup_{k=1}^{n} A_k$ (A_k don't have to be disjoint) then

$$\mu(A) \leq \sum_{k=1}^{n} \mu(A_k)$$

Proof. 1. By Lemma 1,

$$\begin{aligned}A \setminus \bigcup_{k=1}^{n} A_k &= A(\bigcup_{k=1}^{n} A_k)^C \\ &= AA_1^C \ldots A_n^C \\ &= C_1 \cup \cdots \cup C_n\end{aligned}$$

where the C_k are disjoint $\mathscr{A}$-sets.

So,

$$A = ((\bigcup_{k=1}^{n} A_k) \cup (\bigcup_{l=1}^{m} C_l))$$

And thus we now have $A_1, \dots, A_n, C_1, \dots, C_m$ disjoint $\mathscr{A}$-sets. By finite additivity of μ,

$$\mu(A) = \mu(A_1) + \cdots + \mu(A_n) + \mu(C_1) + \cdots + \mu(C_m) \geq \mu(A_1) + \cdots + \mu(A_n)$$

2. Want $A \subseteq \bigcup_{k=1}^{n} A_k \Rightarrow \mu(A) \leq \sum_{k=1}^{n} \mu(A_k)$.

Let
$B_1 = A_1$
$B_2 = A_2 A_1^C$
$\vdots$
$B_n = A_n A_1^C \dots A_{n-1}^C$
Let
$C_1 = AB_1$
$\vdots$
$C_n = AB_n$

Then, $C_1, \dots, C_n$ are disjoint.

$$A = \bigcup_{i=1}^{n} C_i$$

By Lemma 1,

$$C_i = AA_i A_1^C \dots A_{i-1}^C$$

which are finite disjoint union of $\mathscr{A}$-sets (AA_i are π-system $\mathscr{A}$-sets).

$$C_i = \bigcup_{j=1}^{mi} D_{ij}$$

In the meantime,

$$C_i = AA_i A_1^C \dots A_{i-1}^C \subseteq A_i$$

Therefore,

$$\bigcup_{j=1}^{mi} D_{ij} \subseteq A_i$$

By part (i)

$$\sum_{j=1}^{mi} \mu(D_{ij}) \leq mu(A_i)$$

$$\sum_{i=1}^{n.} \sum_{j=1}^{mi} \mu(D_{ij}) \leq \sum_{i=1}^{n.} mu(A_i)$$

But the D_{ij} are disjoint $\mathscr{A}$-sets.

$$\bigcup_{i=1}^{m}\bigcup_{j=1}^{mi} D_{ij} = A$$

By finite additivity of μ on $\mathscr{A}$,

$$\mu(A) = \sum_{j=1}^{mi} \mu(D_{ij})$$

Hence,

$$\mu(A) \leq \sum_{i=1}^{n} \mu(A_i)$$

■

1.7 Measure in Euclidean

Extend Measure to $\mathscr{R}^k$

What we have done is to extned μ from intervals to $\mathscr{R}'$.

Friday October 5

Characterizing Measures in $\mathbb{R}$

The only measure we know so far is the Lebesgue measure. Let μ be any measure that has finite value on bounded sets, $\mu(A) < \infty$.
A is ?.

Bounded set: $\sup_{x\in A} ||x|| < \infty$

Then we can define
This is obviously nondecreasing.
For $0 < a < b, a < 0 < b, a < b < 0$ we can show that $F(b) \geq F(a)$.

Also it is right continuous for $x \geq 0$. So if $x_n \downarrow x$,

$$(0, x_n] \downarrow (0, x]$$

implies

$$\mu(0, x_n] \downarrow \mu(0, x]$$

and

$$F(x_n) \to F(x)$$

for $x < 0$, where $x_n \downarrow x$,

$$(x_n, 0] \downarrow (x, 0]$$

implies

$$\mu(x_n, 0] \downarrow \mu(x, 0]$$

So,

$$-\mu(x_n,0] \to -\mu(x,0]$$

and

$$F(x_n) \to F(x)$$

So F is right continuous.

Theorem 1.7.1 — Theorem 12.4. If F is
1. nondecreasing
2. right continuous

then there exists a unique measure, μ on $\mathscr{R}$ such that

$$\mu(a,b] = F(b) - F(a) \quad \forall a,b \in \mathbb{R}$$

Characterizing Measures in $\mathbb{R}^k$

Let $\mathscr{R}^k$ be the Borel σ-field on $\mathbb{R}^k$. Note that $\sigma(\mathscr{A}) = \mathscr{R}'$.

So $\mathscr{R}^k = \sigma\{(a_1,b_1]x\ldots x(a_k,b_k] : a_1,b_1,\ldots a_k,b_k \in \mathbb{R}\}$.
For each $x \in \mathscr{R}^k$, let

$$S_x = \text{"Southwest"} = (-\infty,x_1]x\ldots x(-\infty,x_k]$$

Then we can show that

$$\mathscr{R}^k = \sigma\{S_x : X \in \mathbb{R}^K\}_{rs}$$

To see this, for any bounded rectangle $A = (a_1,b_1]x\ldots x(a_k,b_k]$ let V_A be the collection of all vertices of A, that is $V_A = \{a_1,b_1\}x\ldots x\{a_k,b_k\}$. So we have that $\#(V_A) = 2^k$.

Now we can express

$$(a_1,b_1]x\ldots x(a_k,b_k] = S_{b_1,\ldots,b_k} \setminus \bigcup_{V_A\setminus\{b_1,\ldots,b_k\}S(x_1,\ldots,x_k}$$

So, $\sigma\{\mathscr{A}\} = \mathscr{R}^k \subseteq \sigma\{S_x : x \in \mathbb{R}^k\}$.

In the other direction, we have any

$$S_x = \bigcup_{n=1}^{\infty}(x_i - n, x_i]$$

Then $\sigma\{S_x : x \in \mathbb{R}^k\} \subseteq \mathscr{R}^k$.

So,

$$\sigma\{S_x : x \in \mathbb{R}^k\} = \mathscr{R}^k$$

Notation 1.2 (Signum). *$sgn_A(x)$ is a signum of a vertex in a rectangle. Signum means "sign" in Latin.*

Monday October 10

Let $F : \mathbb{R}^k \to \mathbb{R}$. Also $A \in \mathscr{I}^k, \mathscr{I}^k$ is the collection of all bounded rectangles. That is, $\{(a_1,b_1]x\ldots x(a_k,b_k] : a_1,b_1,\ldots,a_k,b_k \in \mathbb{R}\}$.

Let

$$\triangle_A F = \sum_{x \in V_a} sgn_A(x)F(x)$$

where V_a is the collection of all vertices of A.

For illustration, assume μ is a finite measure. We won't need this assumption.

Let $F(x) = \mu(S_x)$, where S_x is the southwest of x.

$$S_x = (-\infty, x_1]x\ldots x(-\infty, x_k]$$

Then we show for any $A \in \mathscr{I}^k$ we have

$$\mu(A) = \triangle_A F$$

To see this,

$$A = \bigcup_{x \in V_A \setminus \{b_1,\ldots,b_k\}} S_{(x_1,\ldots,x_k)}$$

For $k = 2$,

$$\mu(\bigcup_{x \in V_A \setminus \{b_1,\ldots,b_k\}} S_{(x_1,\ldots,x_k)}) = \mu(S_{b_1 \ldots b_k}) - \bigcup_{x \in V_A \setminus \{b_1,\ldots,b_k\}} S_k$$

There are $2^k - 1$ number of values in this range.

$$\bigcup_{i=1}^{m} B_i \quad m = 2^k - 1$$

$$\mu(\bigcup_{i=1}^{m} B_i) = \sum_{i=1}^{m} \mu(B_i) - \sum_{i<j} \mu(B_i B_j) + \cdots + (-1)^{m+1} \mu(B_1 \ldots B_m)$$

For k = 2,

$$\sum_{i=1}^{m} \mu(B_i) = \mu(S_{a_1 a_2}) + \mu(S_{a_a b_2}) + \mu(S_{b_1 a_2})$$

$$\sum_{i<j} \mu(B_i B_j) = \mu(B_1 B_2) + \mu(B_2 B_3) + \mu(B_1 B_3) = 3\mu(B_1)$$

$$\mu(B_1 B_2 B_3) = \mu(B_1)$$

All together,

$$\begin{aligned}\mu(\bigcup_{i=1}^{m} B_i) &= \mu(S_{B_1B_2}) - \mu(B_1) - \mu(B_2) - \mu(B_3) + 3\mu(B_1) - \mu(B_1) \\ &= \mu(S_{B_1B_2}) + \mu(B_1) - \mu(B_2) - \mu(B_3) \\ &= \sum_{x\in V_A} sqn_A(X)F(x) \\ &= \triangle_A F\end{aligned}$$

By induction, we may show that $\mu(A) = \triangle_A F$.

So,

$$\triangle_A F \geq 0 \quad \forall A \in \Omega^k$$

Also, if $x^{(n)} \downarrow x$, in the sense that

$$x_1^{(n)} \downarrow x, \ldots, x_k^{(n)} \downarrow x$$

then, $S_{x^{(n)}} \downarrow S_x$.

So $\mu(S_{x^{(n)}}) \downarrow \mu(S_x)$.

$$F(x^{(n)}) \to F(x)$$

So, $F(x)$ is continuous from above in the sense that $x^{(n)} \downarrow x$.

$$F(x^{(n)}) \downarrow F(x)$$

This shows that $\mu \Rightarrow F$ such that $\triangle_A F \geq 0$ continuous from above.

In fact, such an F, also uniquely determines a measure.

Theorem 1.7.2 Suppose $F : \mathbb{R}^k \to \mathbb{R}$ is continuous from above. That is,

$$\lim_{x^{(n)}\downarrow x} F(x^{(n)}) = F(x)$$

Also suppose that for all $A \in \mathscr{I}^k$,

$$\triangle_A F \geq 0$$

corresponding to right continuous and nondecreasing in $\mathbb{R}^k$.
Then, there exists a unique measure μ on $(\mathbb{R}^k, \mathscr{R}^k)$ such that for all $A \in \mathscr{I}^k$,

$$\mu(A) = \triangle_A F$$

The most important special case of this theorem is the case

$$F(x) = x_1 \ldots x_k$$

$$\sum_{x\in V_A} sgn_A(x)F(x) = (b_1 - a_1)\ldots(b_k - a_k)$$

So the μ corresponding to this F is the Lebesgue measure.

This characterizes all measures in $\mathbb{R}^k$.

Proof. Note that μ is defined on $\mathscr{I}^k$. So we need to show μ can be uniquely extended to $\sigma(\mathscr{I}^k) = \mathscr{R}^k$.

Uniqueness

Want $\mu \sigma$-finite on $\mathscr{I}^k$.

$$A_n = (-n,n]x \ldots x(-n,n]$$

$$\cup_n A_n = \mathbb{R}^k$$

$$\mu(A_n) = \sum_{x \in V_A} sqn_{A_n}(x)F(x)$$

But $F : \mathbb{R}^k \to \mathbb{R}$, $F(x) < \infty \forall x$, and $\mu(A_n) < \infty$.

So μ is σ-finite on $\mathscr{I}^k$.

Still need existence of extension finitely additive, countably subadditive.

Step 1 Finitely additive.

Step 1 (a)

Finitely additive on regular partition of $A \in \mathscr{I}^k$.

What's a regular partition? Irregular partition? Think disjoints vs overlapping.

It's easy to turn an irregular partition into a regular one.

Wednesday October 12

More explicitly,

$$A = I_1 x \ldots x I_k$$

where, $I_i = (a_i, b_i]$ for $i = 1, \ldots, k$.

For each i let,

$$J_{i1}, \ldots, J_{ik}$$

be a partition of I_i into subintervals.

For each $(j_1, \ldots, j_k) \in \{1, \ldots, n_1\} x \ldots x \{1, \ldots, n_k\}$

Write

$$B_{j_1,\ldots,j_k} = J_{1j_1}x \ldots xJ_{kj_k}$$

So we have

$$\mathscr{B} = \{B_{j_1 \ldots j_k} : j_i \in \{1,\ldots,n_i\} \forall i = 1,\ldots,k\}$$

$$\#\mathscr{B} = n_1 \ldots n_k$$

So $\mathscr{B}$ is called a **regular decomposition of A.**

Obviously $A = \bigcup_{B\in\mathscr{B}} B$ and $\{B : B \in \mathscr{B}\}$ are disjoint and $B \in \mathscr{I}^k$.

Overall, Step 1 (a) claims that $\mu(A) = \sum_{B\in\mathscr{B}} \mu(B)$.

Recall that $\mu(B) = \triangle_B F = \sum_{x\in V_B} sgn_B(x)F(x)$.
Here we have,

$$\sum_{B\in\mathscr{B}} \mu(B) = \sum_{B\in\mathscr{B}} \sum_{x\in V_B} sgn_B(x)F(x)$$

We may change the order of summations,

$$\sum_{x\in V} \sum_{B\in W_x} sgn_B(x)F(x)$$

where $W_x = \{B \in \mathscr{B} : x \in V_B\}$ and $V = \bigcup_{B\in\mathscr{B}} V_B$.

Now be separating into values of x in and not in V_A, we have,

$$\sum_{x\in V_A} \sum_{B\in W_x} sgn_B(x)F(x) + \sum_{x\notin V_A} \sum_{B\in W_x} sgn_B(x)F(x)$$

So the idea is that when $x \notin V_1$ then $\sum_{B\in W_x} sgn_B(x) = 0$ which means the second term above is also zero. But, if $x \in V_A$, then W_x is singleton and $B \in W_x$ has same sign as A.

So ultimately,

$$\sum_{B\in\mathscr{B}} \mu(B) = \sum_{x\in V_A} sgn_a(x)F(x0 = \mu(A)$$

Step 1 (b)

Now, consider, general situation. If we let $A \in \mathscr{I}^k$ and suppose that $A = \bigcup_{u=1}^n A_u, A_u \in \mathscr{I}^k$. Because $A_n \in \mathscr{I}^k$, and $A_u = I_{1u}x \ldots xI_{ku} \in \mathscr{I}^1$ we have,

$$A = \bigcup_{u=1}^{n} (I_{1u}x \ldots xI_{ku})$$

Meanwhile, $A \in \mathscr{I}^k$, so $A = I_1 x \ldots xI_k$.
Claim:

$$I_1 x \ldots xI_k = \bigcup_{u=1}^{n} (I_{1u}x \ldots xI_{ku}) = \bigcup_{u=1}^{n} (I_{1u})x \ldots x \bigcup_{u=1}^{n} I_{ku})$$

But if $(a_1,\dots,a)k) \in \bigcup_{u=1}^{n}(I_{1u}x\dots xI_{ku})$, then

$$(a_1,\dots,a)k) \in I_{1u}x\dots xI_{ku}$$

for some u.

Then we have $a_i \in I_{iu} \subseteq \bigcup_{u=1}^{n} I_{iu}$ which leads to

$$(a_1,\dots,a)k) \subseteq (\bigcup_{u=1}^{n} I_{1u})x\dots x(\bigcup_{u=1}^{n} I_{ku})$$

So,

$$\bigcup_{u=1}^{n}(I_{1u}x\dots xI_{ku}) \subset (\bigcup_{u=1}^{n} I_{1u})x\dots x(\bigcup_{u=1}^{n} I_{ku})$$

On the other hand,

$$I_{iu} \subset I_u$$

$$(\bigcup_{u=1}^{n} I_{iu}) \subseteq I_u$$

$$(\bigcup_{u=1}^{n} I_{1u})x\dots x(\bigcup_{u=1}^{n} I_{ku}) \subseteq I_1x\dots xI_k$$

Thus, claim is proved. Hence,

$$(\bigcup_{u=1}^{n} I_{1u})x\dots x(\bigcup_{u=1}^{n} I_{ku}) = I_1x\dots xI_k$$

The union, $(\bigcup_{u=1}^{n} I_{iu})$, needs not be disjoint, but we may use all the endpoints of I_{iu} to make disjoints partitions.

So then

$$A = (\bigcup_{v=1}^{m_1} \tilde{I}_{1v})x\dots x(\bigcup_{v=1}^{m_k} \tilde{I}_{kv})$$

which by Step 1 (a) is a regular partition.

$$\mu(A) = \sum_{v=1}^{m_1}\dots\sum_{v=1}^{m_k}\mu(\tilde{I}_{1v_1})x\dots x\tilde{I}_{kv_k}))$$

Let $\tilde{\mathscr{B}} = \{\tilde{I}_{1v_1})x\dots x\tilde{I}_{kv_k} : v_i = 1,\dots,m_i \forall i = 1,\dots,k\}$.
So,

$$\mu(A) = \sum_{\tilde{B}\in\tilde{\mathscr{B}}}\mu(\tilde{B}) = \sum_{u=1}^{n}\sum_{\tilde{B}\subseteq A_u}\mu(\tilde{B})$$

Thus fnite additivity done.
Step 2 Countably subadditive.

Friday October 14

First we want to show finite subadditive.

That is if $A_1,\ldots,A_n,A \in mathcalI^k$ and $A \subseteq \bigcup *n_{i=1}A_i$ then,

$$\mu(A) \leq \sum_{i=1}^{n} \mu(A_i)$$

But this is implied by Lemma 2 at end of section 11.

We need to extend this to countably subadditivity.

Recall compact set, in $\mathbb{R}^N$ (definition applies to any toplogical space). A set is **compact** if any open covering has a finite subcovering. That is, if $A \subseteq \bigcup_{G\in\mathscr{G}} G$ where G is open, then there exists a subcollection

$$\{G_1,\ldots,G_n\} \subset \mathscr{G}$$

so that $A \subset \bigcup_{i=1}^{n} G_i$ and we may call A compact. In the appendix of Billingsly, there is the Heine - Borel Theorem: A bounded and closed set in $\mathbb{R}^k$ is compact.

Now we want to show that if $A_1,\ldots \& A \in \mathscr{I}^k$ and $A \subseteq \bigcup_n A_n$ then

$$\mu(A) \leq \sum_n \mu(A_n)$$

Because $A < \mathscr{I}^k$, then $A = (a_1,b_1]x\ldots x(a_k,b_k]$. Let

$$B(\delta) = (a_1+\delta,b_1]x\ldots x(a_k+\delta,b_k]$$

Then

$$\mu(B(\delta)) = \sum_{x\in V_{B(\delta)}} sgn_{B_\delta}(x)F(x)$$

Note that $x(\delta) \in V_{B_\delta}$.

So $x(\delta)$ may be written as

$$x(\delta) = x + \begin{pmatrix}\delta\\0\\\delta\\\vdots\\\delta\end{pmatrix} = x + \begin{pmatrix}1\\0\\1\\\vdots\\1\end{pmatrix}\begin{pmatrix}\delta\\\delta\\\delta\\\vdots\\\delta\end{pmatrix}$$

■

II

Part Two

2. Integration with Respect to a Measure

3. Random Variable

4. Convergence in Probability/Limit Theorem

5. Radon-Nikodym Derivative Theorem

6. Special Topics

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