Probability Theroy based on Measure Theory

STAT 517

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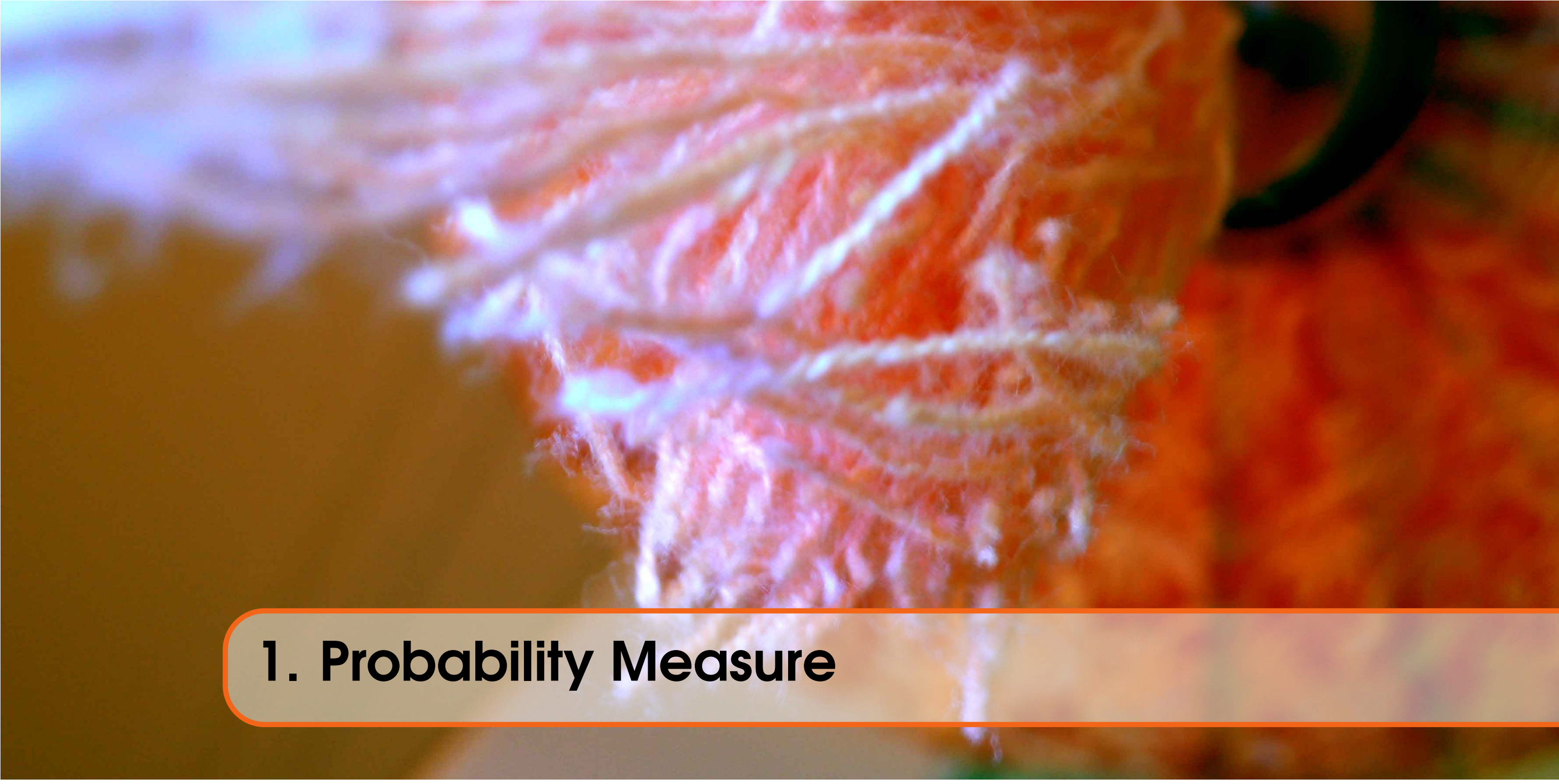
Part One



1. Probability Measure

1.1 Overview

1.2 Probability on a Field

Definition 1.2.1 — Ω. Non emtpy set.

Definition 1.2.2 — Paving. A collection of a subset of Ω is a paving.

Definition 1.2.3 — Field. A **field** $\mathscr{F}$ is a paving satisfying

(i) $\Omega \in \mathscr{F}$

(ii) $A \in \mathscr{F} \Rightarrow A^C \in \mathscr{F}$

(iii) $A, B, \in \mathscr{F}, \Rightarrow A \cup B \in \mathscr{F}$

Derived Properties about a Field

- $\emptyset \in \mathscr{F}$ (by (i) and (ii):

$$\begin{aligned} \Omega \in \mathscr{F} &\Rightarrow \Omega^C \in \mathscr{F} \\ &\Rightarrow \emptyset \in \mathscr{F}) \end{aligned}$$

- (i) can be replaced by "$\mathscr{R}$ is nonempty" because,
  Let $A \in \mathscr{F}$,

$$\begin{aligned} &\Rightarrow A^c \in \mathscr{F} \\ &\Rightarrow A^C \cup A \in \mathscr{F} \\ &\Rightarrow \Omega \in \mathscr{F} \end{aligned}$$

- $A \in \mathscr{F}, B \in \mathscr{F}, \Rightarrow, A \cap B \in \mathscr{F}$ because,

$$\begin{aligned} &(A \cap B)^C = A^C \cup B^C (DeMorgan's Law) \\ &A \cap B = (A^C \cup B^C)^C \end{aligned}$$

- $A_1, \dots, A_m \in \mathscr{F} \Rightarrow A_1 \cup, \dots, \cup A_m \in \mathscr{F}$ (mathematical induction)
- $A_1, \dots, A_m \in \mathscr{F} \Rightarrow A_1 \cap, \dots, \cap A_m \in \mathscr{F}$

Definition 1.2.4 — σ-Field. Similar to the definition of a field except for (iii). A paving satisfying

(i) $\Omega \in \mathscr{F}$
(ii) $A \in \mathscr{F} \Rightarrow A^C \in \mathscr{F}$
(iii) $A_1 \in \mathscr{F}, \dots, A_m \in \mathscr{F}$
$\cup_{k=1}^{m} A_k \in \mathscr{F}$ (finite additvity)

If we replace (iii) from before by (iii') here:
For $A_1 \in \mathscr{F}, \dots, A_m \in \mathscr{F}$

$$\cup_{k=1}^{\infty} A_k \in \mathscr{F}$$

then $\mathscr{F}$ is called a **σ-field**.

Derived Facts

- Again, (i) can be repalced by $\mathscr{F}$ no empty, (iii) can be replaced $A_1 \in \mathscr{F}, \dots, A_m \in \mathscr{F}$

Example 1.1 $\Omega = (0,1]$ (from now on all intervals are left open, right closed)

R Recall that σ-fields are generated by fields. Fancy scripts denote a σ-field. Fancy scripts with a zero subscript denote a field.

$\mathscr{B}_0$ is the collection of all finite union of disjoint intervals. Aside: something that we can consider easily, but computers cannot. (e.g. think of all stars, etc)

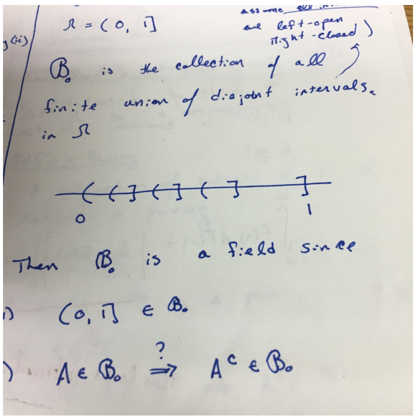


Figure 1.1: Finite unioin of three disjoint intervals.

Then $\mathscr{B}_0$ is a field.
(i) (0, 1] $\in \mathscr{B}_0$
(ii) $A \in \mathscr{B}_0 \Rightarrow A^c \in \mathscr{B}_0$
(iii) $A \in \mathscr{B}_o, B \in \mathscr{B}_o \Rightarrow A \cup B \in \mathscr{B}_o$

■

Wednesday August 24
$\mathscr{B}_0 =$ collection of finite unions of disjoin subintervals of (0, 1]. Is a field.

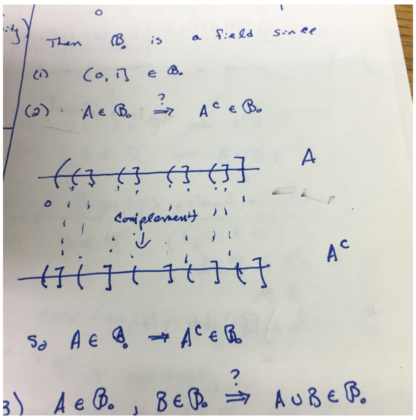


Figure 1.2: A and complement of A.

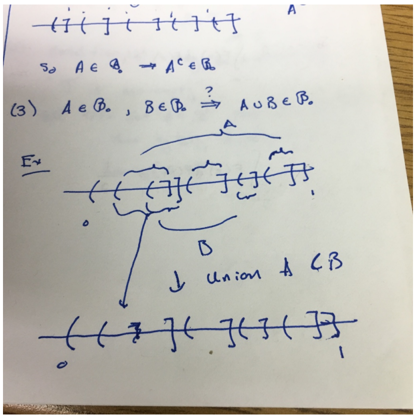


Figure 1.3: Union of A and B is still in $\mathscr{B}_o$

Definition 1.2.5 — Power Set. A σ-field is generated by a paving of power set. Let Ω be a set. The collection of all subsets of Ω is the power set written as 2^{Ω}.

R Where does this notation come from?
Consider ths case where Ω is finite

$$\Omega = \{\omega_1, \ldots, \omega_n\}$$

Total number of subsets of Ω.
$\emptyset$, 1 element sets, 2-element sets, ..., n-element ests.

$$() + () + \cdots + = (1+1)^n$$

$\#(\mathscr{F}) = 2^{\#\Omega}$, so it seems reasonable to denote $\mathscr{F} = 2^{\Omega}$.
It is also easy to show that 2^{Ω} is a σ-field. (The largest, even. The smallest: $\{\emptyset, \Omega\}$ which is also a σ-field.)

$$\{\emptyset, \Omega\} \subseteq \sigma\text{-field} \subseteq 2^{\Omega}$$

It turns out we can extend notion of lenght from $\mathscr{B}_0$ to σ-field generated by $\mathscr{B}_o$.

Now, let $\mathscr{A}$ be a nonempty paving of Ω. We define

$$\sigma(\mathscr{A}) = \cap\{\mathscr{B} \subseteq 2^{\Omega} : \mathscr{B} \text{ is a } \sigma\text{-field}, \mathscr{A} \subseteq \mathscr{B}\}$$

OR rather, the *intersection* of all σ-fields that contains $\mathscr{A}$.

Let

$$\mathbb{F}(\mathscr{A}) = \{\mathscr{B} \subseteq 2^{\Omega} : \mathscr{B} \text{ is a } \sigma\text{-field}, \mathscr{B} \supseteq \mathscr{A}\}$$

Then,

$$\sigma(\mathscr{A}) = \cap\mathscr{B}$$

$$\mathscr{B} \in \mathbb{F}(\mathscr{A})$$

Derived Facts
$\mathbb{F}(\mathscr{A})$ is nonempty. For example, 2^{Ω} is a σ-field and $2^{\Omega} \supseteq \mathscr{A}$.
$\cap B$ is a σ-field. ($B \in \mathbb{F}(\mathscr{A})$)

R Get notes about notation/levels.

Proof. We will prove that indeed $\sigma(\mathscr{A})$ is a σ -field. Recall that we have three conditions above for σ-field.

(i)

$$\Omega \in \sigma(\mathscr{A})$$

$$\Omega \in \cap_{B\in\mathbb{F}(\mathscr{A})} B$$

Because: B is σ-field, $A \in B$, $\forall B \in \mathbb{F}(\mathscr{A})$.

(ii)

(iii) $A_1, \ldots, \in \cap_{B\in\mathbb{F}(\mathscr{A})} B, \forall B \in \mathbb{F}(\mathscr{A})$
$\Rightarrow \cup_{n=1}^{\infty} A_n \in B, \forall B \in \mathbb{F}(\mathscr{A})$

So, $\sigma(\mathscr{A}$ is a σ-field, we call it the σ-field, generated by $\mathscr{B}_o$. We know how tot assign lenth to members of $\mathscr{B}_o$, we now show the assignment can be extended to $\sigma(\mathscr{B}_o)$

■

■ **Example 1.2** Let $\mathscr{I}$ be the collection of *all* subintervals of (0,1].

Note that $\mathscr{I}$ is a smaller collection than $\mathscr{B}_0$ since $\mathscr{B}_0$ can have numerous different combinations of the sets.

Let

$$\mathscr{B} = \sigma(\mathscr{I})$$

.

This is a Borel-σ-field. (a member of $\mathscr{B}$ in Borel set.)
It turns out

$$\sigma(\mathscr{I}) = \sigma(\mathscr{B}_o)$$

This is because $\sigma(\mathscr{I})$ is a σ-field.

So,

$$\sigma(\mathscr{I}) \supseteq \mathscr{B}_o$$
$$\sigma(\mathscr{I}) \supseteq \sigma(\mathscr{B}_o)$$

Also,

$$\mathscr{I} \subseteq \mathscr{B}_o$$
$$\sigma(\mathscr{I}) \subseteq \sigma(\mathscr{B}_o)$$

Thus,

$$\sigma(\mathscr{I}) = \sigma(\mathscr{B}_o)$$

■

Definition 1.2.6 — Probability Measure. Probablity measures on field. Suppose $\mathscr{F}$ is a field on a nonempy set Ω. A probability measure is a function $P : \mathscr{F} \to \mathbb{R}$.
(i) $0 \leq P(A) \leq 1, \forall A \in \mathscr{F}$
(ii) $P(\emptyset) = 0$, $P(\Omega) = 1$
(iii) If $A_1, \ldots$ are disjoint emembers of $\mathscr{F}$ and $\cup A_n \in \mathscr{F}$ then we have countable additivity:

$$P(\cup A_n) = \sum_{n=1}^{\infty} P(A_N)$$

R Note that (iii) also implies finite additivity. Prove by adding infinite empty sets on end.

If Ω is nonempty set. And $\mathscr{F}$ is a σ-field on Ω. And P is a probability measure on $\mathscr{F}$.
Then $(\Omega, \mathscr{F}, P)$ is called a **probability space.**
And $(\Omega, \mathscr{F})$ is called a **measurable space.**

R If $A \subseteq B$, then $P(A) \leq P(B)$. This is because we may write B as

$$B = A \cup (B \setminus A)$$

R

$$P(A) + P(B) = P(A \cup B) + P(A \cap B)$$

Friday August 26

Recall,
Probability measure on a field, $\mathscr{F}_0$.

- $P(A) + P(B) = P(A \cup B) + P(A \cap B)$
  - $P(A) = P(AB^C) + P(AB)$
  - $P(B) = P(BA^C) + P(AB)$
  - $P(A) + P(B) = P(AB^C) + P(BA^C) + 2P(AB)$
  - $P(A \cup B) = P(AB^C) + P(BA^C) + P(AB)$

- $P(A \cup B) = P(A) + P(B) - P(AB)$ By induction, we can prove if $A_1, \dots A_n$,

$$P(\cup_{k=1}^{n} A_k) = \sum_{k=1}^{n} P(A_k) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k} A_i A_j) + \dots + (-1)^{n+1} P(A_1, \dots, A_n)$$

  Inclusion- Exclusion Formula
- If $A_1, \dots A_n \in \mathscr{F}$,

$$B_1 = A_1$$

$$B_2 = A_2 \setminus A_1$$

$$\vdots$$

  Then,

$$\cup_{k=1}^{n} A_k = \cup_{k=1}^{n} B_k$$

  but the B_i are disjoint. Also $A_K \subseteq B_k \forall k = 1, \dots, n$.

$$P(\cup_{k=1}^{n} A_k) = P(\cup_{k=1}^{n} B_k) = \sum_{k=1}^{n} B_k \leq \sum_{k=1}^{n} A_k$$

  Thus, $P(\cup_{k=1}^{n} A_k) \leq \sum_{k=1}^{n} A_k$. Finite subadditivity.

Some conventions,
If $A_1, \dots$ is a sequence of sets, we say $A_n \uparrow A$ if
1. $A_1 \subseteq A_2 \subseteq \dots$
2. $\cup_{k=1}^{\infty} A_k = A$

If $A_1, \dots$ is a sequence of sets, we say $A_n \downarrow A$ if
1. $A_1 \supseteq A_2 \supseteq \dots$
2. $\cap_{k=1}^{\infty} A_k = A$

Theorem 1.2.1 If P is a probability measure on a field $\mathscr{F}$ Then,
1. Continuity from below.
   If $A_n \in \mathscr{F} \quad \forall n, A \in \mathscr{F}$

$$A_n \uparrow A$$

   then

$$P(A_n) \uparrow P(A)$$

2. Continuity from above.
   If $A_n \in \mathscr{F} \quad \forall n. A \in \mathscr{F}$

$$A_n \downarrow A$$

   then

$$P(A_n) \downarrow P(A)$$

3. Countable subadditivity.
   If $A_n \in \mathscr{F} \quad \forall n. \cup_{k=1}^{\infty} A_k \in \mathscr{F}$ then

$$P(\cup_{n=1}^{\infty} A_k) \leq \sum_{n=1}^{\infty} P(A_k)$$

Proof. 1. If $A_1, \dots A_n \in \mathscr{F}$,

$$B_1 = A_1$$

$$B_2 = A_2 \setminus A_1$$

$$B_3 = A_3 \setminus A_2$$

$$\vdots$$

then, $B_1, \dots$ are disjoint.

$$\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$$

$$\begin{aligned} P(A) &= P(\cup_{n=1}^{\infty} A_n) \\ &= P(\cup_{n=1}^{\infty} B_n) \\ &= \sum_{n=1}^{\infty} P(B_n) \\ &= \lim_{n\to\infty} \sum_{n=1}^{\infty} P(B_n) \\ &= \lim_{n\to\infty} P(A_n) \end{aligned}$$

2. $A_n \downarrow A \Leftrightarrow A_n^C \uparrow A^C$
But by (1),

$$P(A_n^C) \uparrow P(A^C)$$

$$1 - P(A_n) \uparrow 1 - P(A)$$

$$P(A_n) \downarrow P(A)$$

3. By finite subadditivity,

$$P(\cup^n k = 1 A_k) \leq \sum^n k = 1 P(A_k) \leq \sum_{n=1}^{\infty} P(A_n)$$

But since, by (1), because

$$\cup_{k=1}^{n} A_k \uparrow \cup_{n=1}^{\infty} A_n$$

$$P(\cup_{k=1}^{n} A_k) \uparrow P(\cup_{n=1}^{\infty} A_n)$$

So,

$$P(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n)$$

■

R $A \in \mathscr{F} =$ "A is F-set".

1.3 Extention of Probability Measure to a σ-field

Let f be a function $f : D \to R$.

Let $\tilde{D}$ be another set such that

$$D \subseteq \tilde{D}$$

An extantion of f onto $\tilde{D}$ is

$$\tilde{f} : \tilde{D} \to R$$

Such that $f(x) = \tilde{f}(x) \forall x \in D$
$\tilde{f}$ is an extention of f on D.
We say f has unique extention, $\tilde{f}$ onto $\tilde{D}$ if

1. $\tilde{f}$ is an extension of f to $\tilde{D}$.
2. if g is another extension of f to $\tilde{D}$ then $\tilde{f} = g$ on D.

Theorem 1.3.1 A probability measure on a field has a unique extension on the σ-field generated by this field.

This means that if $\mathscr{F}_0$ is a field, and P is a probability measure on $\mathscr{F}_0$, then there exists a probability measure, Q on $\sigma(\mathscr{F})$ such that

$$Q(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Moreover, if $\tilde{Q}$ is another probability measure on $\sigma(\mathscr{F}_0)$ such that $\tilde{Q} = P(A) \quad \forall A \in \mathscr{F}$ then

$$\tilde{Q} = Q$$

.

R The proof of this theorem will come after several definitions and lemmas.

Outer Measure $P^* : 2^{\Omega} \to \mathbb{R}$
For any $A \in 2^{\Omega}$ $(A \subseteq \Omega)$

$$P^*(A) = \inf\{\sum_{n=1}^{\infty} P(A_n) : A_1, A_2, \ldots \text{is a sequence of } \mathscr{F}_0 \text{ sets, } A \subseteq \cup_{n=1}^{\infty} A_n\}$$

P^* is a measure out until $\mathscr{M}$, but it is only a function beyond that on 2^{Ω}.

Inner Measure
$P_*(A) = 1 - P^*(A)$

Define the paving $\mathscr{M}$ as followes

$$\mathscr{M} = \{A \in 2^{\Omega} : E \in 2^{\Omega}, P^*(E) = P^*(E \cap A) + P^*(E \cap A^C)\}$$

Idea: we came up with this $\mathscr{M}$ such that P^* behaves as a measure. It will turn out to be that $\mathscr{M}$ is a σ-field that contains $\sigma(\mathscr{F}_0)$.

Monday August 29

P^* satisfies the following probabilities:

(i) $P^*(\emptyset) = 0$
(ii) $P^*(A) \geq 0 \quad \forall A \in 2^\Omega$
(iii) $A \subseteq B \Rightarrow P^*(A) \subseteq P^*(B)$
(iv) $P^*(\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P^*(A_n))$

Proof. (i) Take $\{\emptyset, \emptyset, \dots\}$.

$$\emptyset \in \mathscr{F}_0, \quad \emptyset \cup_{n=1}^{\infty} \emptyset$$

So,

$$P^*(\emptyset) \leq \sum_{n=1}^{\infty} P(\emptyset) = 0$$

Note,

$$P(A) \geq 0 \quad \forall A$$

So,

$$P^*(\emptyset) \geq \emptyset$$

Thus,

$$P^*(\emptyset) = \emptyset$$

(ii) Already done as part of (i).
(iii) Let $A \subseteq B$

$$P^*(A) = \inf\{\sum_{n=1}^{\infty} P(A_n), A_n \in \mathscr{F}_0, A \subseteq \cup A_n\}$$

Now, if $B_1, \dots \in \mathscr{F}_0 \subseteq \cup B_n$
Then,

$$A \subseteq B \subseteq \cup_n B_n$$

If $\{\{B_n\}_{n=1}^{\infty} : B_n \in \mathscr{F}_0, B \subseteq \cup_n B_n\} \subseteq \{\{A_n\}_{n=1}^{\infty} : A_n \in \mathscr{F}_0, A \subseteq \cup_n A_n\}$
Or in short, Collection 1 $\subseteq$ Collection 2.

So the inf of a larger set is smaller than (or equal to) the inf of a smaller set.

So,
$P^*(A) = \inf\{\sum_{n=1}^{\infty} P(A_n), A_n \in \text{ collection \#1}\} \leq P^*(B) = \inf\{\sum_{n=1}^{\infty} P(B_n), A_n \in \text{ collection \#2}\} = P^*(B)$

(iv) Want

$$P^*(\cup_n A_n) \leq \sum_n P^*(A_n)$$

$P^*(A_n) = \inf\{\sum_{n=1}^{\infty} P(A_n) : A_{nk} \in \mathscr{F}_0, A \subseteq \cup_k A_{nk}\}$

Let $\varepsilon > 0$, by defnition of there exists,

$$\{B_n\}_{n=1}^{\infty}$$

such that

$$\sum_{k=1}^{\infty} P(B_{nk}) \leq P^*(A_n) + \frac{\varepsilon}{2^n}$$

So,

$$\cup_n A_n \subseteq \cup_{n,k} B_{nk}$$

and,

$$\begin{aligned} P^*(\cup_n A_n) &\leq \sum_{n,k} P(B_{nk}) \\ &< \sum_n P^*(A_n) + \sum_n (\varepsilon 2^{-n}) \\ P^*(\cup A_n) &< \sum_n P^*(A_n) + \varepsilon \quad \forall \varepsilon > 0 \end{aligned}$$

Simply put,
b
So,

$$P^*(\cup_n A_n) \leq \sum_n P^*(A_n)$$

■

By definition, $A \in \mathscr{M}$ if and only if $P^*(EA) + P^*(EA^C) = P^*(E)$.

We know that P^* is subadditive.

So, by subadditivity we know,

$$P^*(E) \leq P^*(AE) + P^*(A^C E)$$

Therefore, to show $A \in \mathscr{M}$ we only need to show

$$P^*(E) \geq P^*(AE) + P^*(A^C E)$$

$\mathscr{M}$ is defined by P^* and P^* is defined using $\mathscr{F}_0$ so $\mathscr{M}$ is indireclty tied to $\mathscr{F}_0$.

Lemma 1. $\mathscr{M}$ is a field.

Proof. (i) $\Omega \in \mathscr{M}$

$$\begin{aligned} A &= \Omega \\ P^*(\emptyset) &= 0 \\ P^*(E) + P^*(\emptyset) &= P^*(E) \end{aligned}$$

(ii) $A \in \mathscr{M} = A^C \in \mathscr{M}$

$$\begin{aligned} P^*(E) &= P^*(EA) + P^*(A^C E) \\ &= P^*(EA^C) + P^*(AE) \\ &= P^*(EA^C) + P^*((A^C)^C E) \end{aligned}$$

(iii) $A, B \in \mathscr{M} \to A \cap B \in \mathscr{M}$

$B \in \mathscr{M} \Rightarrow P^*(E) = P^*(Eb) + P^*(B^C E) \quad \forall E$
$A \in \mathscr{M} \Rightarrow P^*(BE) = P^*((BE)A) + P^*(A^C(BE))$
$A \in \mathscr{M} \Rightarrow P^*(B^C E) = P^*((B^C E)A) + P^*(A^C(B^C E))$

Hence,

$$P^*(BE) + P^*(B^C E) = P^*((BE)A) + P^*(A^C(BE)) + P^*((B^C E)A) + P^*(A^C(B^C E))$$

$$\begin{aligned} P^*(A^C(BE)) + P^*((B^C E)A) + P^*(A^C(B^C E)) &\geq P^*((A^C BE) \cup (AB^C E) \cup (A^C BE)) \\ &= P^*(E \cap [A^C B \cup AB^C \cup A^C B^C]) \\ &= P^*(E \cap (AB)^C) \end{aligned}$$

$$\begin{aligned} P^*(E) &= P^*(BE) + P^*(B^C E) \\ &= P^*((BE)A) + (P^*(A^C(BE)) + P^*((B^C E)A) + P^*(A^C(B^C E))) \\ &\geq P^*(ABE) + P^*(E(AB)^C) \end{aligned}$$

So, $A, B \in \mathscr{M}$

■

Lemma 2. If $A_1, A_2, \ldots$ is a sequence of disjoint $\mathscr{M}$-sets then for each $E \subseteq \Omega$,

$$P^*(E \cap (\cup_k A_k)) = \sum_k P^*(E \cap A_k)$$

Proof. First, prove this statement for finite sequence.

$$A_1, \ldots, A_n$$

by mathematical induction.

If $n = 1$ this is 'trivial',

$$P^*(E \cap A_1) = P^*(E \cap A_1)$$

If $n = 2$ we need to show,

$$P^*(E(A_1 \cup A_n)) = P^*(EA_1) + P^*(EA_2)$$

Because $A_1 \in \mathscr{M}$,

$$P^*(E(A_1 \cup A_2)) = P^*(E(A_1 \cup A_2))A_1 + P^*(E(A_1 \cup A_2)A_1^2)$$

$$E(A_1 \cup A_2) = E(A_1 A_2 \cup A_1 A_2 = EA_1$$

$$E(A_1 \cup A_2)A_1^C = E(A_1 A_1^C \cup A_2 A_2^C)$$

So,

$$P^*(E(A_1 \cup A_2)) = P^*(EA_1) + P^*(EA_2)$$

Suppose true for n = k. (induction hypothesis)

Now we must show for n = k + 1.

$$P^*(E \cap (\cup_{n=1}^{k+1} A_n)) = P^*([E \cap (\cup_{n=1}^{k} A_n)] \cup A_{k+1})$$

$(\cup_{n=1}^{k} A_n), A_{k+1}$ are two disjoint sets. Using the n=2 case,

$$= \sum_{n=1}^{k} P^*(E \cap A_n) + P(E \cap A_{k+1}) = \sum_{n=1}^{k+1} P^*(E \cap A_n)$$

So this is now shown to be true for $\{A_1, \ldots, A_n\}$. Next, showtrue for $A_1, \ldots in \mathscr{M}$ (disjoint). Want:

$$P^*(E \cap (\cup_{n=1}^{\infty} A_n)) = \sum_{n=1}^{\infty} P^*(E \cap A_n)$$

Using countable subadditivity,

$$P^*(E \cap (\cup_{n=1}^{\infty} A_n)) = P^*(\cup_{n=1}^{\infty} E \cap A_n) \leq \sum_{n=1}^{\infty} P^*(E \cap A_n)$$

In the meantime, by the monotonicity of P^*

$$P^*(E \cap (\cup_{n=1}^{\infty} A_n)) \geq P^*(E \cap (\cup_{n=1}^{m} A_n)) = \sum_{n=1}^{\infty} P^*(E \cap A_n)$$

So,

$$P^*(E \cap (\cup_{n=1}^{\infty} A_n)) \geq \lim \sum_{n=1}^{m} P^*(E \cap A_n)$$

(*), (**) gives us,

$$P^*(E \cap (\cup_{n=1}^{\infty} A_n)) = \sum_{n=1}^{\infty} P^*(E \cap A_n)$$

■

Wednesday August 31
(finished proof)

Lemma 3.
1. $\mathscr{M}$ is a σ-field
2. P^* restricted on $\mathscr{M}$ is countably additive.

Proof. First we show if

1. $\mathscr{M}$ is a fieldd
2. $\mathscr{M}$ is closed under countable disjoint union.

then $\mathscr{M}$ is a σ-field.
Let's create disjoints sets,

$A_n \in \mathscr{M}, n = 1,2,\ldots$ $B_1 = A_1$ $B_2 = A_2A_1^C \vdots B_n = A_nA_1^C \ldots A_{n-1}^C$
$B_1, \ldots, B_n \in \mathscr{M}$ (disjoint)

$$\cup_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} A_n$$

But we know that $\cup_{n=1}^{\infty} B_n \in \mathscr{M}$ so $\cup_{n=1}^{\infty} A_n \in \mathscr{M}$ and thus $\mathscr{M}$ is a σ-field.
So it suffices to show that $\mathscr{M}$ is closed under disjoint countable unions.

Let $A_1, A_2, \ldots$ are disjoins $\mathscr{M}$-sets.

Let $A = \cup_{n=1}^{\infty} A_n$.

Let $F_n = \cup^n k = 1A_k$.

Then $F_n \in \mathscr{M}$.

So, $\forall E \in 2^{\Omega}$,

$$P^*(E) = P^*(EF_n) + P^*(EF_n^C)$$

$$\begin{aligned}
P^*(EF_n) &= P^*(E(\cup_{k=1}^{n} A_k)) \\
&= \sum_{k=1}^{n} P^*(EA_k) \\
P^*(EF_n^C) &\geq P^*(EA^C)(F_n \subseteq A, F_n^C \supseteq A^C) \\
\Rightarrow P^*(E) \geq \lim_{n\to\infty} P^*(EA_k) + P^*(EA^C) \\
&= \sum_{k=1}^{n} P^*(EA_k) + P^*(EA^C) \\
&= P^*(EA) + P^*(EA^C)
\end{aligned}$$

■

So $A \in \mathscr{M}$ and $\mathscr{M}$ is a σ-field.

Now, let's show P^* is countably additive.
Let $A_1, A_2, \ldots$ be disjoint members of $\mathscr{M}$. Then $\forall E \in 2^{\Omega}$,

$$P^*(E(\cup_{n=1}^{\infty} A_n)) = \sum_{n=1}^{\infty} P(EA_n)$$

Take $E = \Omega$.

$$P^*(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} P(A_n)$$

Lemma 4. $\mathscr{F}_0 \subseteq \mathscr{M}$

Proof. Let $A \in \mathscr{F}$.

Want:

$$A \in \mathscr{M}$$

$$P^*(E) = P^*(EA) + P^*(EA^C)$$

By definition, there exists $E_n \in \mathscr{F}_0$ such that

$$\sum_{n=1}^{\infty} P^*(E_n) \leq P^*(E) + \varepsilon$$

$$\begin{aligned}
P^*(EA) &\leq P^*((\cup_{n=1}^{\infty} E_n)A) \text{ (monotonocity)} \\
&= P^*(\cup^i nfty_{n=1}(E_nA)) \\
&\leq \sum^i nfty_{n=1} P^*((E_nA)) \text{ (countibly subadd)} \\
P^*(EA^C) &\leq \sum_{n=1}^{\infty} P^*(E_nA^C) \\
P^*(EA) + P^*(EA^C) &\leq \sum_{n=1}^{\infty} P^*(E_nA) + P^*(E_nA^C) \\
&= \sum_{n=1}^{\infty} P^*(E_n)
\end{aligned}$$

Recall, $A, E_n \in \mathscr{F}_0$

$$\begin{aligned}
&\leq P^*(E) + \varepsilon \\
P^*(EA) + P^*(EA^C) &\leq P^*(E) + \varepsilon \quad \forall \varepsilon \\
\Rightarrow P^*(EA) + P^*(EA^C) &= P^*(E) \\
\Rightarrow A &\in \mathscr{M} \\
\mathscr{F}_0 &\in \mathscr{M}
\end{aligned}$$

■

Lemma 5.

$$P^*(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Proof. Let $A \in \mathscr{F}_0$.
Because, $A, \emptyset, \emptyset, \ldots, \in \mathscr{F}_0$.

$$A \subseteq A \cup \emptyset \cup \emptyset \ldots$$

$$P^*(A) \leq P(A) + P(\emptyset) + \ldots$$

But, if

$$A_n \in \mathscr{F}_0$$

$$A \subseteq \cup_{n=1}^{\infty} A_n$$

$$P^*(A) \leq \sum_{n=1}^{\infty} P(A_n)$$

$$\Rightarrow P^*(A) \leq \inf \sum_{n=1}^{\infty} P(A_n)$$

$$= P^*(A)$$

■

Friday September 2

R **5 Lemma Recap**

Lemma 1. $\mathscr{M}$ is a field.
Lemma 2. If $A_1, A_2, \dots$ is a sequence of disjoint $\mathscr{M}$-sets then for each $E \subseteq \Omega$,

$$P^*(E \cap (\cup_k A_k)) = \sum_k P^*(E \cap A_k)$$

Lemma 3.
1. $\mathscr{M}$ is a σ-field
2. P^* restricted on $\mathscr{M}$ is countably additive.

Lemma 4.

$$\mathscr{F}_0 \subseteq \mathscr{M}$$

Lemma 5.

$$P^*(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Recall, Extension Theorem. That is, If $\mathscr{F}$ is a field and P is a probability measure, then there exists a measure, Q such that

$$Q(A) = P(A) \quad \forall A \in \mathscr{F}_0$$

Proof. By Lemma 5,

$P^*(\Omega) = P(\Omega) = 1$
$P^*(\emptyset) = P(\emptyset) = 0$

Outline of what we need to show:

- $0 \leq M(A) \leq 1$
- $M(\emptyset) = 0, \quad M(\Omega) = 1$
- $M(\cup_n A_n) = \sum_n M(A_n)$

Since $\forall A \in \mathscr{M}$,

$$\emptyset \subseteq A \subset \Omega$$

then

$$0 \le P^*(\emptyset) \le P^*(A) \le P^*(\Omega) \le 1$$

But, by Lemma 3, P^* is contably additive on $\mathscr{M}$. So P^* is probability measure on $\mathscr{M}$ (which is a σ-field, by Lemma 3).

By Lemma 4, $\mathscr{F}_0 \subset \mathscr{M} \Rightarrow \sigma(\mathscr{F}_0 \subseteq \mathscr{M}$. So P^* is also probabliity measure on $\sigma(\mathscr{F}_0)$.

Finally, by Lemma 5, again $P^*(A) = P(A)$, P^* is an extention of P form $\mathscr{F}_0$ to $\sigma(\mathscr{F}_0)$.

■

Uniqueness of of the extention, $\pi - \lambda$ Theorem

Paving - {π-system and λ-system.} (?)

Definition 1.3.1 — π-System. A class of subsets $\mathscr{P}$ of Ω is a π system, if

$$A, B \in \mathscr{P} \Rightarrow AB \in \mathscr{P}$$

Definition 1.3.2 — λ-System. A class $\mathscr{L}$ is a λ-system if

λ(i) $\Omega \in \mathscr{L}$

λ(ii) $A \in \mathscr{L} \Rightarrow A^C \in \mathscr{L}$

λ(iii) If $A_1, \cdots \in \mathscr{L}$ are disjoint then $\cup_{n=1}^{\infty} A_n \in \mathscr{L}$

So, the only difference is "disjoint". Weaker than a σ-field (i.e. A σ-field is always a λ-system). Note that (λ_2) can be replace by $(\lambda_{2\prime})$ wherein

$$A, B \in \mathscr{F}, A \subseteq B, \Rightarrow B \setminus A \in \mathscr{L}$$

That is $\lambda_1, \lambda_2, \lambda_3 \Leftrightarrow \lambda_1, \lambda_{2\prime}, \lambda_3$

Lemma 6. A class of sets that is both π-systema and λ-system is a σ-field.

Proof. Suppose $\mathscr{F}$ is both π-system and λ-system.

By definition,

1. $\Omega \in \mathscr{F}$
2. $A \in \mathscr{F} \Rightarrow A^C \in \mathscr{F}$

Let $A_1, A_2, \ldots$ be $\mathscr{F}$ sets.

Let's constructs disjoints sets, B

$$B_1 = A_1$$
$$B_2 = A_1 A_2^C$$
$$\vdots$$

Then B_n are $\mathscr{F}$-sets (by $\lambda_{2\prime} - A_2^C = \Omega A)2^C \in \mathscr{F}$, by π-system, $A_1 A_2^C \in \mathscr{F}$).

By λ_3,

$$\cup_n^{\infty} B_n \in \mathscr{F}$$

So,

$$\cup_n^\infty A_n \in \mathscr{F}$$

■

Theorem 1.3.2 — π-λ Theorem. If $\mathscr{P}$ is in a π-system, $\mathscr{L}$ is in a λ-system, then

$$\mathscr{P} \subseteq \mathscr{L} \Rightarrow \sigma(\mathscr{P} \subseteq \mathscr{L})$$

Proof. Let $\lambda(\mathscr{P})$ be the intersection of all λ-system that contains $\mathscr{P}$.

$$\lambda(\mathscr{P}) = \cap\{\mathscr{L}' : \mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}' \text{ is } \lambda\text{-set }\}$$

$\lambda(\mathscr{P})$ is a λ-system.

Goal: prove $\lambda(\mathscr{P})$ is a σ-field. So we want to show that $\lambda(\mathscr{P})$ is a π-system.

1. $\Omega \in \lambda(\mathscr{P})$?

$$\Omega \in \mathscr{L}' \quad \forall \mathscr{L}'$$

$$\Omega \in \lambda(\mathscr{P})$$

2. $A \in \lambda(\mathscr{P}) \Rightarrow A^C \in \lambda(\mathscr{P})$?

$$A \in \lambda(\mathscr{P}) \Rightarrow A \in \cap\{\mathscr{L}' : \mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}' \text{ is } \lambda\text{-set }\}$$

Then
$A \in \mathscr{L}'$ for any $\mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}'$ is λ-system.

$$\Rightarrow A^C \in \mathscr{L}'$$

$$\Rightarrow A^C \in \cap\{\mathscr{L}' : \mathscr{L}' \supseteq \mathscr{P}, \mathscr{L}' \text{ is } \lambda\text{-set }\} = \lambda(\mathscr{P})$$

3. $A_1, A_2, \cdots \in \lambda(\mathscr{P})$ are disjoint then $A_1, A_2, \cdots \in \mathscr{L}' \quad \forall \mathscr{L}'$.
Then $\cup A_n \in \mathscr{L}'(\mathscr{L}'\lambda\text{-system})$
So $\cup_n A_n \in \lambda(\mathscr{P})$.
We call $\lambda(\mathscr{P})$ the λ-system generated by $\mathscr{P}$.

If we can say that $\lambda(\mathscr{P})$ is also a σ-field, then $\sigma(\mathscr{P}) \subseteq \lambda(\mathscr{P})$ because $\sigma(\mathscr{P})$ is smallest.
So then, $\sigma(\mathscr{P}) \subseteq \mathscr{L}$ because $\lambda(\mathscr{P})$ is the small λ-system.
So it suffices to show that $\lambda(\mathscr{P})$ is a σ-field. But we know if $\lambda(\mathscr{P})$ is a system then $\lambda(\mathscr{P})$ is σ-field. So it suffices to show that $\lambda(\mathscr{P})$ is a π-system.

Construct again for any $A \in 2^\Omega \quad (A \subseteq \Omega)$, let

$$\mathscr{L}_A = \{B : AB \in \lambda(\mathscr{P})\}$$

Claim: If $A \in \lambda(\mathscr{P})$ then $\mathscr{L}_A$ is λ-system.

(a) $\Omega \in \mathscr{L}_A$?

$$A\Omega = A \in \mathscr{L}_A$$

(b) $(\lambda_2') : B_1, B_2 \in \mathscr{L}_A, B_1 \subseteq B_2 \Rightarrow B_2 B_1^C \in \mathscr{L}_A$?

$$B_1 \in \mathscr{L}_A \Rightarrow AB_1 \in \lambda(\mathscr{P})$$

$$B_2 \in \mathscr{L}_A \Rightarrow AB_2 \in \lambda(\mathscr{P})$$

Since $AB_1 \subseteq AB_2$, $\lambda(\mathscr{P})$ is λ-system by (λ_2') for $\lambda(\mathscr{P})$

(c) If B_n is disjoint, $\mathscr{L}_A$-sets.
Want $\cup_n B_n$ because

$$B_n \in \mathscr{L}_A$$

$$B_n A \in \lambda(\mathscr{P})$$

Because B_n disjoint we know that $B_n A$ is also disjoint.
Hence,

$$\cup_n (B_n A) \in \lambda(\mathscr{P})$$

Claim: $\lambda(\mathscr{P})$ is π-sytem.

(a) If $A \in \mathscr{P}$, then $\mathscr{P} \subseteq \mathscr{L}_A$
Suppose $A \in \mathscr{P}$.
Let $B \in \mathscr{P}$, then $AB \in \mathscr{P}$ (π-system), and $AB \in \lambda(\mathscr{P}) \Rightarrow B \in \mathscr{L}_A$

(b) If $A \in \mathscr{P}$ then $\lambda(\mathscr{P}) \subset \mathscr{L}_A$.

(c) If $A \in \lambda(\mathscr{P})$, then $\mathscr{P} \in \mathscr{L}_A$
Suppose, $A \in \lambda(\mathscr{P})$ and let $B \in \mathscr{P}$.
By step 2,
$A \in \mathscr{L}_A$
$\Rightarrow AB \in \lambda(\mathscr{P})$
$\Rightarrow B \in \mathscr{L}_A$

(d) If $A \in \lambda(\mathscr{P})$, then $\lambda(\mathscr{P}) \subseteq \mathscr{L}_A$. This is because $\lambda(\mathscr{P})$ is the smallest λ-system, $\mathscr{L}_A$ is λ-system containing $\mathscr{P}$ (by step 3).

Now show that $\lambda(\mathscr{P})$ is π-system.

$A, B \in \lambda(\mathscr{P})$ because $A \in \lambda(\mathscr{P})$. We have that $\lambda(\mathscr{P}) \in \mathscr{L}_A$.

So

$$B \in \mathscr{L}_A$$

$$BA \in \lambda(\mathscr{P})$$

Thus $\lambda(\mathscr{P})$ is π-system.

■

Wednesday September 7

Theorem 1.3.3 Suppose P_1 and P_2 are probability measures on $\sigma(\mathscr{P})$ where $\mathscr{P}$ is a π-system. If P_1 and P_2 agree on $\mathscr{P}$ (that is, $P_1(A) = P_2(A) \quad \forall A \in \mathscr{P}$) then they agree on $\sigma(\mathscr{P})$.

Proof. Let

$$\mathscr{L} = \{A \in \sigma(\mathscr{P}) : P_1(A) = P_2(A)\}$$

Then $\mathscr{P} \subseteq \mathscr{L}$.

It suffices to show that $\mathscr{L}$ is a λ-system (because if so, then $\sigma(\mathscr{P}) \subseteq \mathscr{L}$ - in fact, $\sigma(\mathscr{P}) = \mathscr{L}$).

Show $\mathscr{L}$ is a λ-system.

1. $\Omega \in \mathscr{L}$?
$$P_1(\Omega) = P_2(\Omega) = 1, \quad \Omega \in \mathscr{P}$$
2. $A \in \mathscr{L}$
$$P_1(A) = P_2(A) \Rightarrow P_1(A^C) = P_2(A^C), \quad A^C \in \mathscr{L}$$
3. $A \in \mathscr{L}$. A_n disjoint. Want $\cup_{n=1}^{\infty} A_n \in \mathscr{L}$.
Since
$$A_n \in \mathscr{L}$$
$$P_1(A_n) = P_2(A_n) \quad \forall n$$
$$\sum_{n=1}^{\infty} P_1(A_n) = \sum_{n=1}^{\infty} P_2(A_n)$$
$$P_1 \cup_{n=1}^{\infty} (A_n) = P_2 \cup_{n=1}^{\infty} (A_n)$$
So, $\cup A_n \in \mathscr{L}$.

∎

So our extention of (and uniqueness of the extention of) P on $\mathscr{F}_0$ to $\sigma(\mathscr{F}_0)$ is complete. We have shown the existance of Q on $\mathscr{M}$.

Since Q agrees with P on $\mathscr{F}_0$ and $\mathscr{F}_0$ is a field, this implies that this is a π-system.
If you have another extention, say $\tilde{Q}$, then $\tilde{Q} = P$ on $\mathscr{F}_0$. That is, $\tilde{Q} = Q$ on $\mathscr{M}$, where $\mathscr{M}$ is a σ-field, which is a π -system.

So by Theorem 1.3.3,
$\tilde{Q} = Q$ on $\sigma(\mathscr{P})$.

So, Theorem 1.3.1, is completely proved.

Lemma 1 - Lemma 5 imply the extention.
$\pi - \lambda$ Theorem and Theorem 1.3.3 implies uniqueness.
This wraps up Theorem 1.3.1.

Lebesque measure on (0,1]

$$\Omega = (0,1]$$

Recall, $\mathscr{B}_0$ is the finite disjoint unions of intervals in (0,1] and that $\mathscr{B}_0$ is a field.

Let $\mathscr{B} = \sigma(\mathscr{B}_0)$.

For each $A \in \mathscr{B}_0$,

$$A = \cup_{i=1}^{n}(a_i, b_i]$$

Let $\lambda(A) = \sum_{i=1}^{n}(b_i - a_i)$.

Question: Is λ a probability measure on $\mathscr{B}_0$?

Theorem 1.3.4 — Theorem 2.2 in Billingsly. The set function λ on $\mathscr{B}_0$ is a probability measure on $\mathscr{B}_0$.

Proof. 1. $0 \leq \lambda(A) \leq 1$

2. $$\lambda(\Omega) = \lambda((0,1]) = 1 - 0 = 1$$
$$\lambda(\emptyset) = \lambda((0,0]) = 0$$

3. This one requires Theorem 1.3 in Billingsly (proof omitted - calculus, blah).
Theorem 1.3 - If I is an interval in (0,1] and $\{I_k : k = 1,2,\dots\}$ are disjoint intervas in (0,1] such that

$$I = \cup_{k=1}^{\infty} I_k$$

then,

$$|I| = \sum_{k=1}^{\infty} |I_k|$$

where |a| means length of interval a.
Since $\cup_{j=1}^{m_k} I_{kj} \in \mathscr{B}_0$ and $\cup_{i=1}^{m} I_i = \cup_{k=1}^{\infty} \cup_{j=1}^{m_k} I_{kj}$.

Then

$$\lambda(A)\lambda(\cup_{i=1}^{m} I_i) = \sum_{i=1}^{m} |I_i|$$

Since, $I_i \subset \cup_{k=1}^{\infty} \cup_{j=1}^{m_k} I_{kj}$, then

$$I_i = I_i(\cup_{k=1}^{\infty} \cup_{j=1}^{m_k} I_{kj}) = \cup_{k=1}^{\infty} \cup_{j=1}^{m_k} I_i I_{kj}$$

By Theorem 1.3,

$$|I| = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |I_i I_{jk}|$$

$$\lambda(A) = \sum_{i=1}^{m} \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |I_i I_{jk}| = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \sum_{i=1}^{m} |I_i I_{jk}|$$

Because $I_{jk} \subseteq \cup_{i=1}^{m} I_i$, we have that

$$I_{kj} = \cup_{i=1}^{m} I_{kj} I_i$$

Again by Theorem 1.3, (note that $I_{kj}I_i$ are disjoint intervals)

$$|I_{kj} = \sum_{i=1}^{m} |I_i I_{jk}|$$

So, $\lambda(A) = \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} |I_{kj}| = \sum_{k=1}^{\infty} \lambda(A_k)$

■

Friday September 9

Finished above proof.

So λ is a probability on $\mathscr{B}_0$. By Theorem 3.1, there exists a unique measure τ on $\sigma(\mathscr{B}_0) = \mathscr{B}$ such that

$$\tau(A) = \lambda(A) \quad \forall A \in \mathscr{B}_0$$

τ is called **Lebesgue Measure** on (0,1]. We may still write it as λ.

1.4 Probabilities Concerning Sequences of Events

Set Limit

Let $(\Omega, \mathscr{F})$ be a measureable space (i.e. Ω is nonempty set and $\mathscr{F}$ is σ-field).

let $A_1, \cdots \in \mathscr{F}$. We define

$$\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \limsup_{n\to\infty} A_n$$

It is trivial to show that $\limsup_{n\to\infty} A_n \in \mathscr{F}$.

$$\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k = \liminf_{n\to\infty} A_n$$

We swapped intersection/union...what we are doing here?
ω (means outcome) $\in \Omega$

$$\omega \in \limsup_{n\to\infty} A_n \Leftrightarrow \omega \in \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$$

$$\Leftrightarrow \omega \in \cup_{k=1}^{\infty} A_k \quad \forall n = 1, 2, \ldots$$

$$\Leftrightarrow \omega \in A_k \quad \text{for some } k \geq n, \quad \forall n = 1, 2, \ldots$$

$\Leftrightarrow \omega$ is in infinitely many k.
Similarly,

$$\omega \in \liminf_{n\to\infty} A_n \Leftrightarrow \omega \in \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$$

$$\Leftrightarrow \omega \in \cap_{k=1}^{\infty} A_k \quad \text{for some } n = 1, 2, \ldots$$

$$\Leftrightarrow \omega \in A_k \quad \forall k \geq n, \quad \text{for some } n$$

$$\Leftrightarrow \omega \in \text{ all but finitely many } A_k$$

So this is a much stronger requirement. Intuitively, if ω is in all but finitely many A_k, then it must be in infinitely many A_k (i.e. $\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n$).

For i > max(n,m),

$$\begin{aligned} &\cap_{k=m}^{\infty} A_k \subseteq A_i \subseteq \cup_{k=n}^{\infty} A_k \\ \Rightarrow &\cup_{m=1}^{\infty} \cap_{k=m}^{\infty} A_k \subseteq \cup_{k=n}^{\infty} A_k \\ \Rightarrow &\cup_{m=1}^{\infty} \cap_{k=m}^{\infty} A_k \subseteq \cap_{n=1}^{\infty} \cup_{k=n} A_k \\ &\Rightarrow \liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n \end{aligned}$$

$$\cap_{k=n}^{\infty} A_k \uparrow \liminf_{n\to\infty} A_n$$

$$\cup_{k=n}^{\infty} A_k \downarrow \limsup_{n\to\infty} A_n$$

If $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$, then we say that the sequences $\{A_n\}$ has a limit,

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

$$\lim_{n\to\infty} A_n \in \mathscr{F}$$

Sometimes we write,

$$\limsup_{n\to\infty} A_n = [A_n \text{ i.o. }]$$

Theorem 1.4.1 Suppose $(\Omega, \mathscr{F}, P)$ is a probability space and $A_n \in \mathscr{F} \quad n = 1, 2, \ldots$.

(i)

$$\limsup_{n\to\infty} P(A_n) \leq P(\limsup_{n\to\infty} A_n)$$

$$\liminf_{n\to\infty} P(A_n) \geq P(\liminf_{n\to\infty} A_n)$$

(ii) $A_n \to A (A = \lim_{n\to\infty} A_n)$, then we have continuity of probability of a set funciton:

$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n)$$

Monday September 12

Proof. (i) Let $B_n = \cap_{k=n}^{\infty} A_k$.

$$B_n \uparrow \liminf_n A_n$$

By Theorem 2.1,

$$P(B_n) \uparrow P(\liminf_n A_n)$$

So,

$$P(B_n) \le P(\liminf_n A_n) \quad \forall n$$

$$\lim_{n\to\infty} P(B_n) = P(\liminf_n A_n)$$

$$P(A_n) \ge P(B_n) \to P(\liminf_n A_n)$$

$$\liminf_n P(A_n) \ge P(\liminf_n A_n)$$

Similarly,
Let $C_n = \cup_{k=n}^{\infty} A_k$.
Then,

$$C_n \downarrow \cup_{k=n}^{\infty} A_k$$

$$P(A_n) \le P(C_n) \to P(\limsup_n A_n)$$

$$\limsup_n P(A_n) \le P(\limsup_n A_n)$$

(ii) If A_n has a limit (i.e. $\limsup_n A_n = \limsup_n A_n = \lim A$) then,

$$\liminf_n P(A_n) \ge P(\liminf_n A_n) = P(\limsup_n A_n) \ge \limsup_n P(A_n)$$

So, $\liminf_n P(A_n) = \limsup_n P(A_n)$, thus

$$\lim_n P(A_n) = P(\lim_n A_n)$$

■

Independent Events

$(\Omega, \mathscr{F}, P)$

Let $A, B \in \mathscr{F}$. They are independent if and only iff:

$$P(AB) = P(A)P(B)$$
$$A \perp\!\!\!\perp B$$

$A_1, \ldots A_n$ are independent if and only if for any $\{k_1, \ldots, k_j\} \subseteq \{1, \ldots, n\}$,

$$P(A_{k_1} \ldots A_{k_j}) = P(A_{k_1}) \ldots P(A_{k_i})$$

In this case we write: $A_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp A_n$.

Now let, $\mathscr{A}_1, \ldots, \mathscr{A}_n$ be pavings in $\mathscr{F}$ (i.e. $\mathscr{A}_k \subseteq \mathscr{F}$).

We say $\mathscr{A}_1, \ldots, \mathscr{A}_n$ are independent if and only if for any $A_1 \in \mathscr{A}_1, \ldots, A_n \in \mathscr{A}_n$ we have

$$A_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp A_n$$

.

In this case we write: $\mathscr{A}_1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp \mathscr{A}_n$.

Theorem 1.4.2 Suppose for $(\Omega, \mathscr{F}, P)$ is a probability space if,

$$\mathscr{A}_1 \subseteq \mathscr{F} \dots \mathscr{A}_n \subseteq \mathscr{F}$$

are π-systems. Then,

$$\mathscr{A}_1 \perp\!\!\!\perp \dots \perp\!\!\!\perp \mathscr{A}_n \Rightarrow \sigma(\mathscr{A}_1) \perp\!\!\!\perp \dots \perp\!\!\!\perp \sigma(\mathscr{A}_n)$$

Proof. Let $\mathscr{B}_i = \mathscr{A}_i \cup \{\Omega\}$.

It is easy to show (in homework)

1. $\mathscr{B}_i$ is still a π-system
2. $\mathscr{B}_i$ are still independent

$$\mathscr{B}_1 \perp\!\!\!\perp \dots \perp\!\!\!\perp \mathscr{B}_n$$

For $B_2 \in \mathscr{B}_n, \dots, B_n \in \mathscr{B}_n$ define,

$$\mathscr{L}(B_2, \dots, B_n) = \{B \in \mathscr{F} : B \perp\!\!\!\perp B_2 \perp\!\!\!\perp \dots \perp\!\!\!\perp B_N\}$$

1. First we show $\mathscr{L}(B_2, \dots, B_n)$ is λ-system.

$$\Omega \in \mathscr{L}(B_2, \dots, B_n)$$

$$\Omega \perp\!\!\!\perp B_1 \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n$$

This is true because $P(\Omega B_2 \dots B_n) = P(B_2 \dots B_n) = P(B_2) \dots P(B_n) = P(\Omega)P(B_2) \dots P(B_n)$

2. Now $A \in \mathscr{L}(B_2, \dots, B_n) \Rightarrow A^C \in \mathscr{L}(B_2, \dots, B_n)$

$A\&in\mathscr{L}(B_2, \dots, B_n)$

$$\begin{aligned}
&\Rightarrow A \perp\!\!\!\perp B_2 \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n \\
&\Rightarrow P(AB_2 \dots B_n) = P(A)P(B_2) \dots P(B_n) \\
&\Rightarrow P(A^C B_2 \dots B_n) \\
&\quad P(B_2 \dots B_n) \setminus AB_2 \dots B_n) \\
&\quad P(B_2 \dots B_n) - P(AB_2 \dots B_n) \\
&\quad P(B_2) \dots P(B_n) - P(A)P(B_2) \dots P(B_n) \\
&\quad (1 - P(A))P(B_2) \dots P(B_n)\ P(A^C)P(B_2) \dots P(B_n)
\end{aligned}$$

Then we run this through all subadditives of $A, B_2, \dots, B_n$.

$A^C \perp\!\!\!\perp B_2 \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n$

3. If $C_1, C_2, \dots, \in \mathscr{L}(B_2, \dots, B_n)$ they are disjoint. Want to show

$$\cup^i nfty_{m=1} C_m \in \mathscr{L}(B_2, \dots, B_n)$$

$C_m \in \mathscr{L}(B_2, \dots, B_n)$

$$\begin{aligned}
&\Rightarrow C_m \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n \\
&\Rightarrow P(C_m B_2 \dots B_n) = P(C_m) \dots P(B_n) \quad \forall m = 1, 2 \dots
\end{aligned}$$

$$\sum_{m=1}^{\infty} P(C_m B_2 \dots B_n) = (\sum_{m=1}^{\infty} P(C_m))P(B_2) \dots P(B_n)$$

But $\{C_m, B_2, \dots, B_n, m = 1, 2 \dots\}$

So $\cup_m C_m \in \mathscr{L}(B_2, \dots, B_n)$.

And $\mathscr{L}(B_2, \dots, B_n)$.

Also, $B_1 \in \mathscr{L}(B_2, \dots, B_n) \quad \forall B_1 \in \mathscr{B}_1$ therefore by definition,

$$\mathscr{B}_1 \subseteq \mathscr{L}(B_2, \dots, B_n)$$

So, $\sigma(\mathscr{B}_1) \subseteq \mathscr{L}(B_2, \dots, B_n)$ and we have our $\lambda - \pi$-theorem.
This means that for all $B_1 \in \sigma(\mathscr{B}_1)$

$$B_1 \perp\!\!\!\perp B_2 \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n$$

Recall that B_i are arbitrary members of,

$$\sigma(\mathscr{B}_1) \perp\!\!\!\perp B_2 \perp\!\!\!\perp \dots \perp\!\!\!\perp B_n \Leftrightarrow \mathscr{B}_2 \perp\!\!\!\perp \sigma(\mathscr{B}_1) \perp\!\!\!\perp \dots \perp\!\!\!\perp \mathscr{B}_n$$

Run the previous argument repeatedly.
So

$$\sigma(\mathscr{B}_1) \perp\!\!\!\perp \sigma(\mathscr{B}_2) \perp\!\!\!\perp \dots \perp\!\!\!\perp \sigma(\mathscr{B}_n)$$

■

■ **Example 1.3** Let $\mathscr{I}$ be the collection of all intervals, then its π-system. When we want to check $X \perp\!\!\!\perp Y$, we only need to check

$$P(X \in \text{interval}, Y \in \text{interval}) = P(X \in \text{interval})P(Y \in \text{interval})$$

■

Wednesday September 14

Indepencence of Infinite Classes

Let $\{\mathscr{A}_\theta : \theta \in \Theta\}$ where θ is any infinite set (need not be countable) if and only if any (infinite) $\{A_\theta : \theta \in \Theta\}$ where $A_\theta \in \mathscr{A}_\theta$ are independent.

We alraedy define independence of $\{A_\theta : \theta \in \Theta\}$; that is for an infinite collection of sets is independent if and only if any finite subcollection $\{A_{\theta_1}, \dots A_{\theta_n}\}$ is independent.

With this device, we may make claims such as

$$\{X_t : t \in (0,1]\}$$

are independent. Useful for stochastic process, functional data analysis.

It follows trivially, $\{\mathscr{A}_\theta : \theta \in \Theta\}$ are independent if and only if any finite collection, say $\{\mathscr{A}_{\theta_1}, \dots, \mathscr{A}_{\theta_n}\}$ are independent.

Corollary 1.4.3 — To Theorem 4.2. If $(\Omega, \mathscr{F}, P), \mathscr{A}_\theta \subset \mathscr{F}$, $\{\mathscr{A}_\theta : \theta \in \Theta\}$ is independent and each $\mathscr{A}_\theta$ is a π-system, then

$$\{\sigma(\mathscr{A}_\theta) : \theta \in \Theta\}$$

are independent.

Proof.

$$\{\mathscr{A}_\theta : \theta \in \Theta\} \perp\!\!\!\perp \Leftrightarrow \{\mathscr{A}_{\theta_1}, \dots, \mathscr{A}_{\theta_n}\} \perp\!\!\!\perp$$
$$\Leftrightarrow \{\sigma(\mathscr{A}_{\theta_1}), \dots, \sigma(\mathscr{A}_{\theta_n})\} \perp\!\!\!\perp$$

■

Corollary 1.4.4 Suppose we have an array of sets,

$$\begin{matrix} A_{11} & A_{12} & \dots & \dots \\ A_{21} & A_{22} & \dots & \dots \\ \vdots & \vdots & & \\ \vdots & \vdots & & \end{matrix} = \{A_{ij} : i, j = 1, \dots\} \subset \mathscr{F}$$

and this array is independent.

And let $\mathscr{F}_i = \sigma(A_{i1}, A_{i2}, \dots)$.

Then $\mathscr{F}_1 \perp\!\!\!\perp \mathscr{F}_2$

Proof. Let $\mathscr{A}_i$ be the class of all the finite intersections of

$$A_{i1}, A_{i2}, \dots$$

then $\mathscr{A}_i$ is a π-system.

So,

$$\sigma(\mathscr{A}_i) = \mathscr{F}_i$$

because $\{A_{i1}, A_{i2}, \dots\}$ are contained in $\mathscr{A}_i$ which implies $\mathscr{F}_i \subset \sigma(\mathscr{A}_i)$ and also $\mathscr{A}_i \subset \mathscr{F}_i \Rightarrow \sigma(\mathscr{A}_i \leq \mathscr{F}_i$.

By Corollary 1, it suffices to show that $\mathscr{A}_1, \mathscr{A}_2, \dots$ are independent. Further, it suffices to show that any finite subcollection is independent.

Let $\{i_1, \dots, i_n\} \subseteq \{1, 2, \dots\}$. We want to show that $\mathscr{A}_{i_1} \perp\!\!\!\perp \dots \perp\!\!\!\perp \mathscr{A}_{i_n}$. This would be implied by the following

$$\forall C_{i_1} \in \mathscr{A}_{i_1}, \dots, C_{i_n} \in \mathscr{A}_{i_n}$$

$$C_{i_1} \perp\!\!\!\perp \dots \perp\!\!\!\perp C_{i_n}$$

Because, and watch out with this notation here, for

$$C_{i_\alpha} \in \mathscr{A}_{i_\alpha}$$

there exists

$$A_{i_\alpha j_1}, A_{i_\alpha j_2}, \dots, A_{i_\alpha j_{m_\alpha}}$$

such that

$$C_{i_\alpha} = A_{i_\alpha j_1}, A_{i_\alpha j_2}, \dots, A_{i_\alpha j_{m_\alpha}}$$

We have

$$P(\cap_{\alpha=1}^{n} C_{i_\alpha}) = P(\cap_{\alpha=1}^{n} \cup_{\beta=1}^{m_\alpha} A_{i_\alpha j_\beta})$$

because

$$\{A_{i_\alpha j_\beta} : \alpha, = 1,2,\dots,n, \beta = 1,2,\dots m_\alpha\} \subseteq \{A_{ij} : i,j = 1,2,\dots\}$$

$$\begin{aligned} P(\cap_{\alpha=1}^{n} \cup_{\beta=1}^{m_\alpha} P(A_{i_\alpha j_\beta}) &= \prod_{\alpha=1}^{n} \prod_{\beta=1}^{m_\alpha} P(A_{i_\alpha j_\beta}) \\ &= \prod_{\alpha=1}^{n} P(C_{i_\alpha}) \end{aligned}$$

■

Borel-Cantelli Lemmas (that are actually Theorems)

Theorem 1.4.5 — BC1. For $(\Omega, \mathscr{F}, P)$ probability space,

$$A_n \in \mathscr{F}, \quad n = 1,2,\dots$$

If $\sum_{n=1}^{\infty} P(A_n) < +\infty$ then

$$P(\limsup_{n\to\infty} A_n) = 0$$

Proof. $\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k \subset \cup_{k=n}^{\infty} P(A_k) \quad \forall n$
So,

$$P(\limsup_{n\to\infty} A_n) \le P(\cup_{k=n}^{\infty} P(A_k)) \le \sum_{k=1}^{\infty} P(A_k)$$

■

Theorem 1.4.6 — BC2. If $\{A_n\}$ are independent and $\sum_{n=1}^{\infty} P(A_n) = \infty$, then

$$P(\limsup_{n} A_n) = 1$$

Proof. $P(\limsup_n A_n) = 1$

$$\Leftrightarrow P(\bigcap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k) = 1$$

$$\Leftrightarrow P(\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k^C) = 0 \quad (*)$$

because,

$$\Leftrightarrow P(\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k^C) \le \sum_{k=n}^{\infty} P(\cap_{k=n}^{\infty} A_k^C)$$

$$\Leftarrow P(\cap_{k=n}^{\infty} A_k^C) = 0 \quad \forall n = 1,2,\dots$$

but we need to prove this to imply (*).
Shit, calculus.

$$1 - x \leq e^{-1} \quad \forall x \in \mathbb{R}$$

For any $j = 1, 2, \ldots,$

$$\begin{aligned} P\left(\bigcap_{k=n}^{n+j} A_k^C\right) &= \prod_{k=n}^{n+j} (1 - P(A_k)) \\ &\leq \prod_{k=n}^{n+j} e^{-P(A_k)} \\ &= e^{-\sum_{k=n}^{n+j} P(A_k)} \end{aligned}$$

Now, $\sum_{k=1}^{\infty} P(A_k) = \infty$ and also

$$\sum_{k=n}^{\infty} P(A_k) \quad \forall n$$

So,

$$\lim \sum_{k=n}^{n+j} P(A_k) \to \infty \quad \forall n$$

$$\lim_{j\to\infty} P\left(\bigcap_{k=n}^{n+j} A_k^C\right) = 0$$

Because,

$$\bigcap_{k=n}^{n+j} A_k^C \downarrow \bigcap_{k=n}^{infty} A_k^C \quad j \to \infty$$

By continuity of probability,

$$P\left(\bigcap_{k=n}^{n+j} A_k^C\right) \downarrow P\left(\bigcap_{k=n}^{infty} A_k^C\right) \quad j \to \infty$$

So,

$$P\left(\bigcap_{k=n}^{infty} A_k^C\right) = 0$$

■

Friday September 16

finished proof.

BC1 and BC@ say that $P(\limsup_n A_n)$ is either 0 or 1.

This is a special case of a general phonemonon, the 0-1 Law.

Take σ-field, $(\Omega, \mathscr{F}, P)$,

$$A_1, \dots \in \mathscr{F}$$

For each n,

$$\sigma(A_n, A_{n+1}, dots)$$

We have another σ-field called "tail of σ-field",

$$\mathscr{T} = \bigcap_{n=1}^{\infty} \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots)$$

■ **Example 1.4 — 4.18 in Billingsly.** $\limsup A_n \in \mathscr{T}$?

$$\bigcap_{n}^{\infty} = 1 \bigcup^{i} nfty_{n=k} A_k$$

$$A_n, A_{n+1}, \dots \in \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots) \Rightarrow \bigcup_{k=n}^{\infty} A_k \in \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots)$$

$$\bigcap_{n}^{\infty} = 1 \bigcup_{n=k}^{\infty} A_k \in \bigcap_{n=1}^{\infty} \sigma(\mathscr{A}_n, \mathscr{A}_{n+1}, \dots)$$

$$\begin{aligned}
\liminf A_n &= \left[(\bigcup_{n}^{\infty} = 1 \bigcap_{n=k}^{\infty} A_k)^C \right]^C \\
&= \left[\bigcup_{n}^{\infty} = 1 \bigcap_{n=k}^{\infty} A_k^C \right]^C \\
&= \left[\limsup_{n} A_k^C \right]^C \in \mathscr{T}
\end{aligned}$$

■

Theorem 1.4.7 If $A_1, A_2, \dots$ are independent, then for each $A \in \mathscr{T}$ we have $P(A) = 0$ or 1.

Proof. By Corollary 2,

$\sigma(A_1)$
$\sigma(A_2)$
$\vdots$
$\vdots$
$\sigma(A_{n-1})$
$\sigma(A_n, A_{n+1}, \dots)$

$$\sigma(A_1) \perp\!\!\!\perp \dots \perp\!\!\!\perp \sigma(A_n, A_{n+1}, \dots)$$

Let $A \in \mathscr{T}$, then,

$$A \in \sigma(A_n, A_{n+1}, \dots) \quad \forall n$$

So, $A_1, \dots, A_{n-1}, A$ are independent. By taking nlarge enough, this implies that any finite subcollectoin of $A, A_1, A_2, \dots$ is also independent.

This implies that the sequence $\{A, A_1, A_2, \ldots\}$ are independent.

But $A \in \sigma(A_1, A_2, \ldots)$ so,

$$A \perp\!\!\!\perp A$$

$$P(AA) = P(A)P(A) = P(A)^2$$

So $P(A)$ must be zero or 1!

■

R We are now skipping a few sections (5 -9) in Billingsly. These are about special random variables, random walks, etc...

1.5 General Measure on a Field

Borel Sets in $\mathbb{R}^k$

Two jumps, from $(0,1] \to \mathbb{R} \to \mathbb{R}^k$.
$\mathscr{B}$ on $(0,1]$ is a σ-field generated by $\mathscr{J}$ = collection of all intervals in (0,1].

$$\sigma(\mathscr{I}) = \mathscr{B} \text{ on } (0,1]$$

When we work with $\mathbb{R}$,

$$\mathscr{I}' = \text{collection of all intervals in } \mathbb{R}, (a,b)$$

$$\sigma(\mathscr{I}') = \mathscr{R}' \quad \text{linear Borel } \sigma\text{-field}$$

$\mathscr{I}^k$ is the collection of all rectangles in $\mathbb{R}^k$.

$$\mathscr{I}^k = \{(a_1,b_1]x\ldots x(a_k,b_k] : a_1,b_1,\ldots,a_k,b_k \in \mathbb{R}\}$$

$$\sigma(\mathscr{I}^k) = \mathscr{R}^k \quad \text{Borel } \sigma\text{-field in } \mathbb{R}^k$$

Properties of $\mathbb{R}^k$

Any open set are in $\mathscr{R}^k$.

Let $\mathbb{Q}$ be the set of all rational numbers. This is countable and dense subset of $\mathbb{R}$.

Definition 1.5.1 — Dense. Look up definition!

Class of rational rectangles:

$$\mathscr{I}^k_{\mathbb{Q}} = \{(a_1,b_1]x\ldots x(a_k,b_k] : a_1,b_1,\ldots,a_k,b_k \in \mathbb{Q}\}$$

Let G be an open set in $\mathbb{R}^k$ and $y \in G$, then there exists

$$A_y \in \mathscr{I}^k_Q$$

such that

$$y \in A_y \subset G$$

because $\mathbb{Q}$ is dense in $\mathbb{R}$.

Note that, $\bigcup_{y \in G} A_y = G$. But, $\{A_y : y \in G\} \subseteq \mathscr{I}_{\mathbb{Q}}^k$.

2. General Measure

3. Integration with Respect to a Measure

4. Random Variable

5. Convergence in Probability/Limit Theorem

6. Radon-Nikodym Derivative Theorem

7. Special Topics

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