Day 3: Functions, Polynomials, and Arithmetic

Places to visit!

- Hershey: Park, chocolate factory, hotel and spa
- Lancaster county: Smorgasbords, Amish goods, the central market. I recommend staying in a bed and breakfast and visiting Kitchen Kettle
- Falling Water and Polymath Park
- The Grange Fair: It’s not great, but it’s quintessential Centre county PA. Starts Friday.

More unsolicited advice

- Asking good questions is a function of reading the literature and experience. But mostly reading.
- When you have good research questions, write them down.
- Get two monitors.

Odds and ends

- Order of operations: PEMDAS
- Transitive property: If A = B and B = 4 then A = 4
- Commutative property: An expression is commutative if changing the order of the operation does not change the result. Addition and multiplication are commutative. Subtraction and division are not.
- Associative property: An expression is associative when rearranging the parentheses does not change the result.

Review: Hats!

Why don't *x* and *c* have a hat in this equation?

$$\hat{y}_i = \widehat{\beta_0} + \widehat{\beta_1} x_i + \widehat{\beta_2} c_i$$

Review: Betas and Coefficients

- Conceptualize each "term" with respect to its relationship to Y

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- Assume Y is the probability that someone votes and C is cost. If C = 100 and a value of 100 reduces the probability that someone votes by .1, then:
- $\beta_2 \times 100 = -0.1$
- $\beta_2 = \ -\frac{0.1}{100} = -.001$

Functions

Functions

A mathematical mapping of one input to exactly one output

$$x \rightarrow \boxed{f()} \rightarrow f(x)$$

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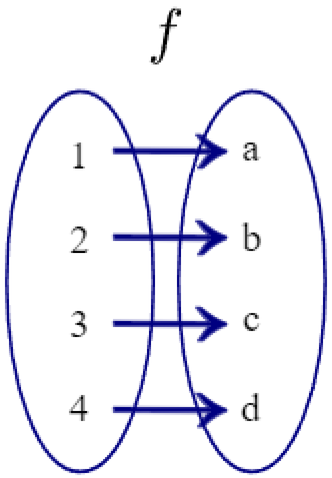
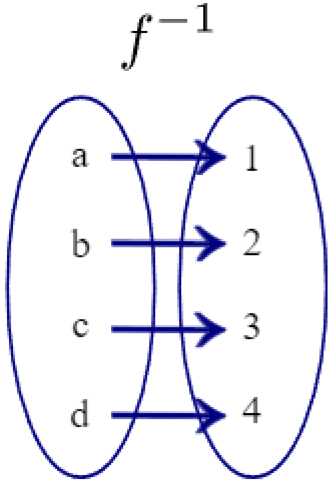
$$f(x) = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

Inverse functions





Inverse functions

Guideline for Computing Inverses.

1. Write down $y = f(x)$.
2. Solve for x in terms of y.
3. Switch the x's and y's.
4. The result is $y = f^{-1}(x)$.

Example2.20. Finding the Inverse Function.

We find the inverse of the function $f(x) = 2x^3 + 1$.

▼ Solution

Starting with $y = 2x^3 + 1$ we solve for x as follows:

$$y - 1 = 2x^3 \quad \rightarrow \quad \frac{y-1}{2} = x^3 \quad \rightarrow \quad x = \sqrt[3]{\frac{y-1}{2}}.$$

Therefore, $f^{-1}(x) = \sqrt[3]{\frac{x-1}{2}}$.

Some special functions…

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$$\mathrm{logit}^{-1}(\alpha) = \mathrm{logistic}(\alpha) = \frac{1}{1+\exp(-\alpha)} = \frac{\exp(\alpha)}{\exp(\alpha)+1}$$

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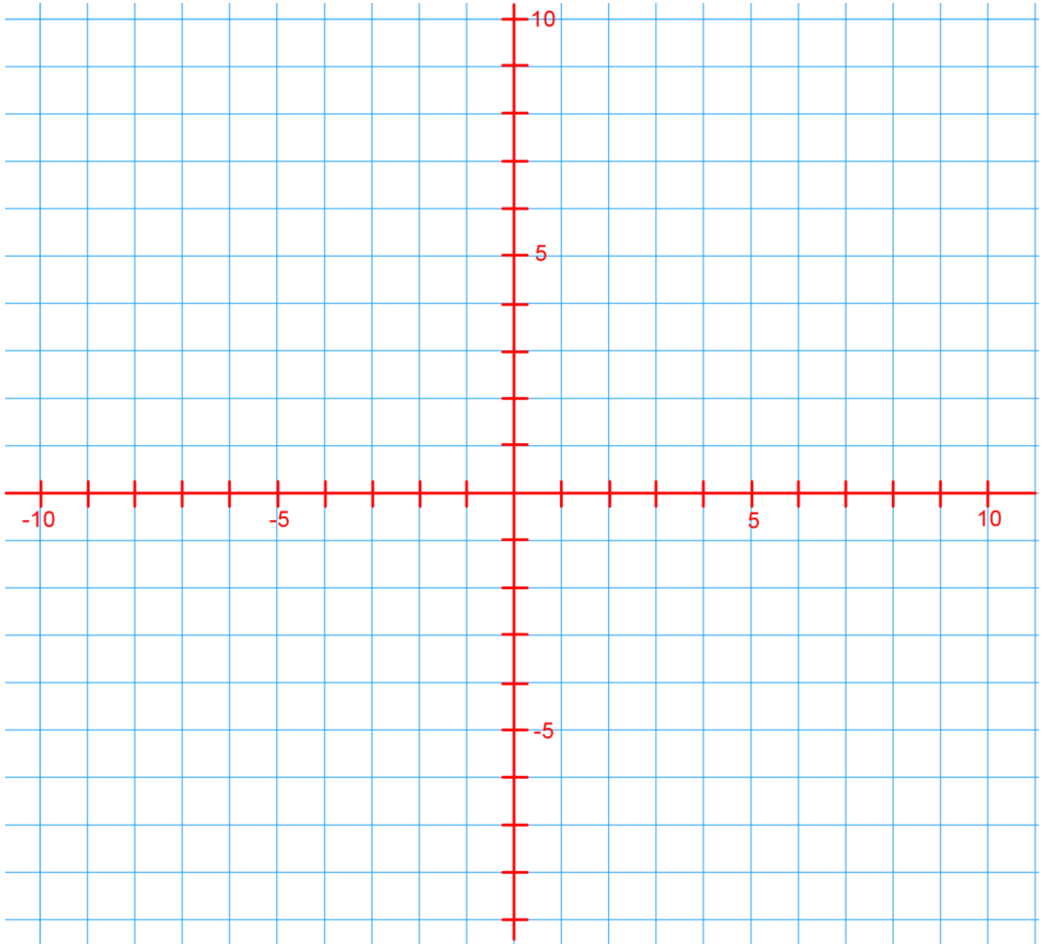
	Logarithm	=	Exponent
	$\log_a N = x$	$\longleftrightarrow$	$N = a^x$
(Common Log)	$\log N = x$	$\longleftrightarrow$	$N = 10^x$
(Natural Log)	$\ln N = x$	$\longleftrightarrow$	$N = e^x$

The Cartesian Plane

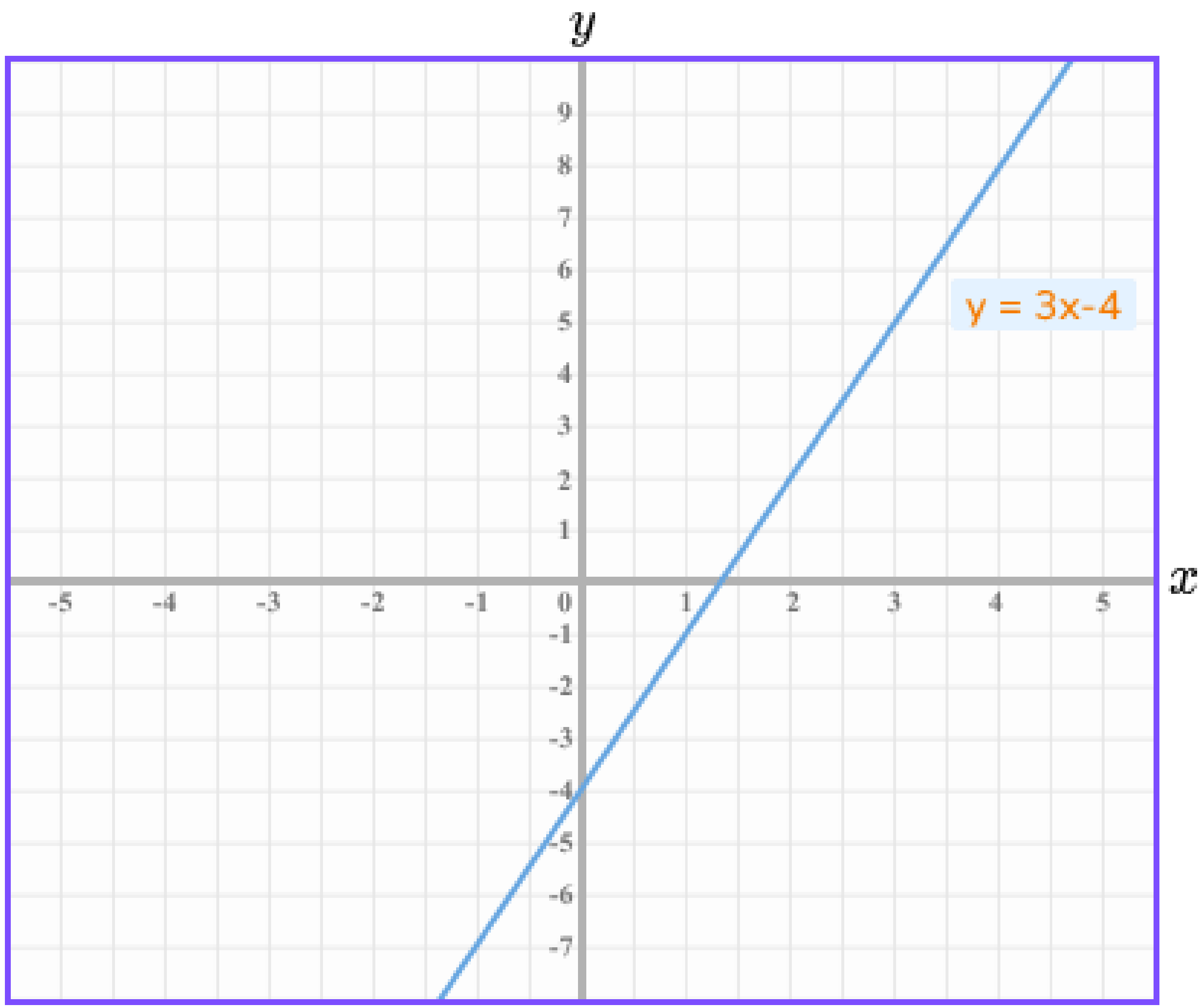
Fancy name for this thing:



Why is this important?

- Plotting allows us to visualize the relationship between variables.
- This makes relationships more intuitive and comprehensible

An example:



Key points:

- Relationships and probabilities can be conceptualized geometrically
- The geometry and the math are one and the same.
- You will often hear probabilities talked about as areas, densities, and mass.
- You will also hear coefficients and effect sizes referred to as “slopes”

Polynomials

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$$\text{Monomial : } g(x) = 3x$$

$$\text{Polynomial : } g(x) = 3x + 5$$

A polynomial is the sum of monomials

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- Algebraic expression consisting of variables, coefficients, and constants
- Defined by the largest exponent

$$g(x) = 3x + 1 \text{ (Linear)}$$

$$g(x) = 3x^2 + 1 \text{ (Quadratic)}$$

$$g(x) = 3x^3 + 1 \text{ (Cubic)}$$

$$g(x) = 3x^n + 1 \text{ (}n\text{th degree)}$$

Exponents, Roots, and Logarithms

What do exponents imply about a variable in a function?

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An exponent is a variable coefficient. It means the effect of X on Y changes as the value of X changes.

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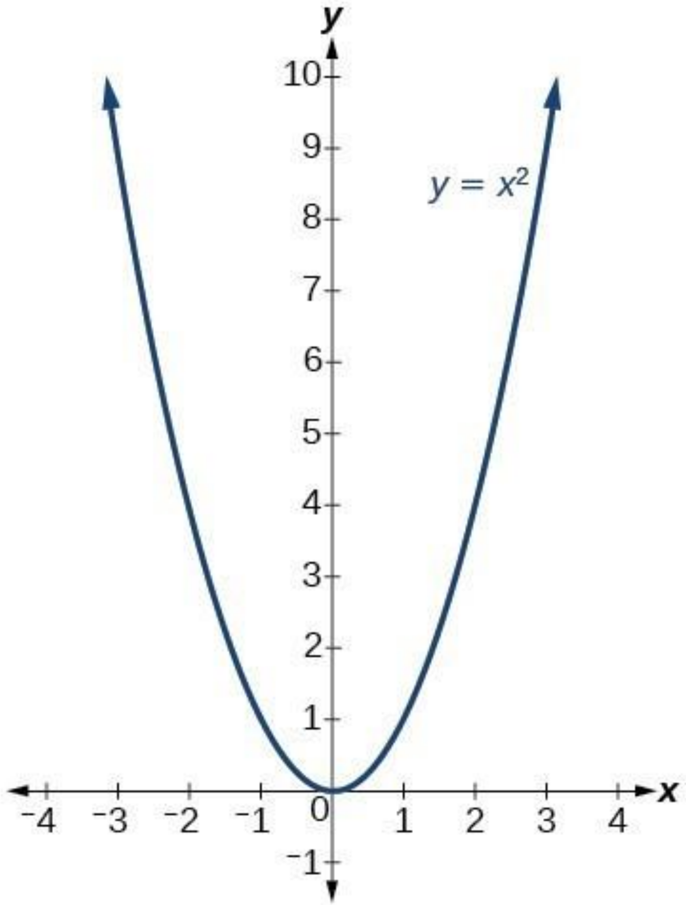
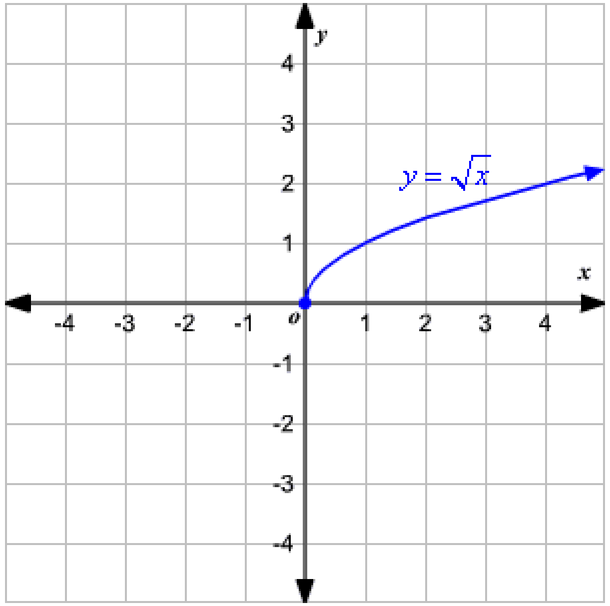


Roots have a similar effect

$$\sqrt{x} = x^{1/2}$$



Logarithms

- The power you need to raise a base to in order to get a certain number
- The log of a negative number is undefined

If:
$b^y = x$

Then:
$\log_b x = y$

Where:
b = “Base” of the logarithm

What does $\log_a x = y$ tell us about the effect of x on y?

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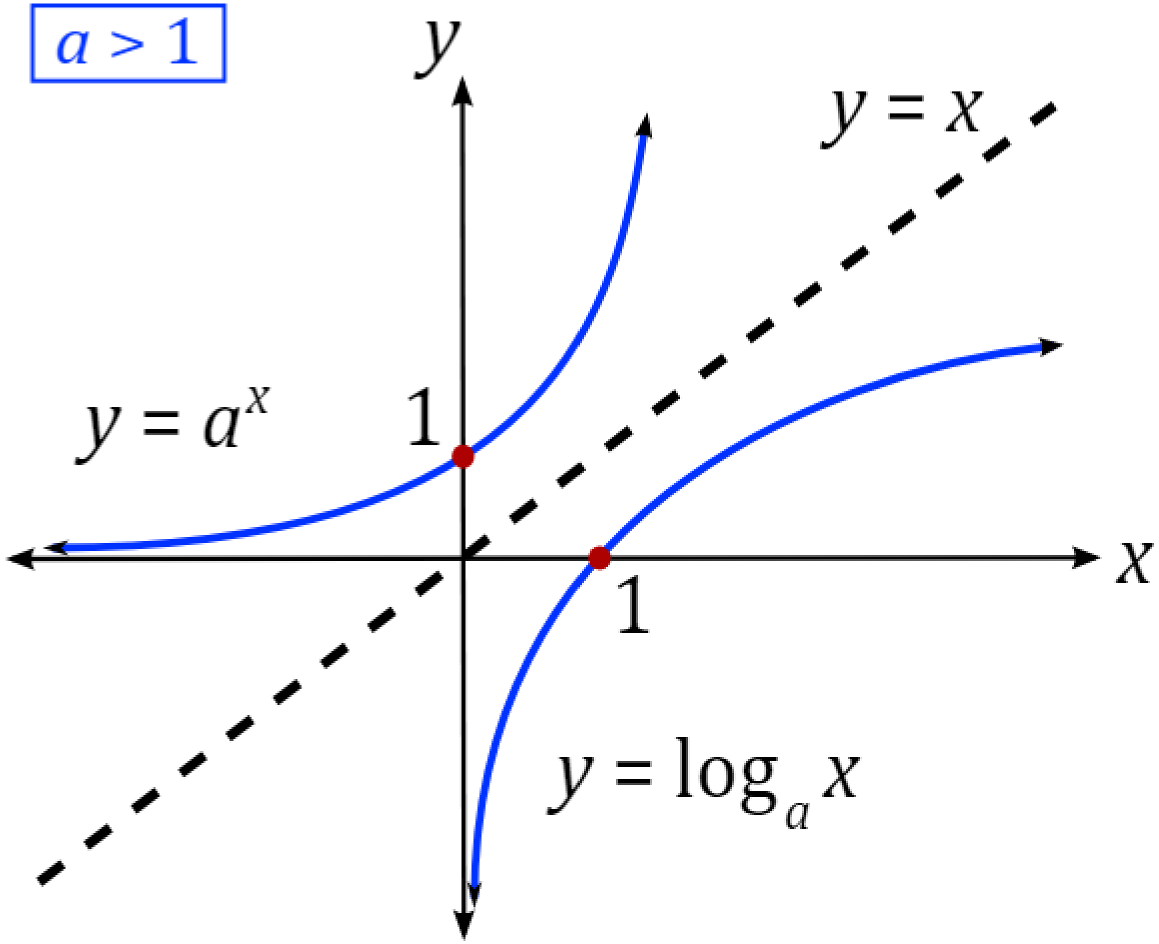
- Consider:

$$a^y = x$$

What happens to x as y gets larger?

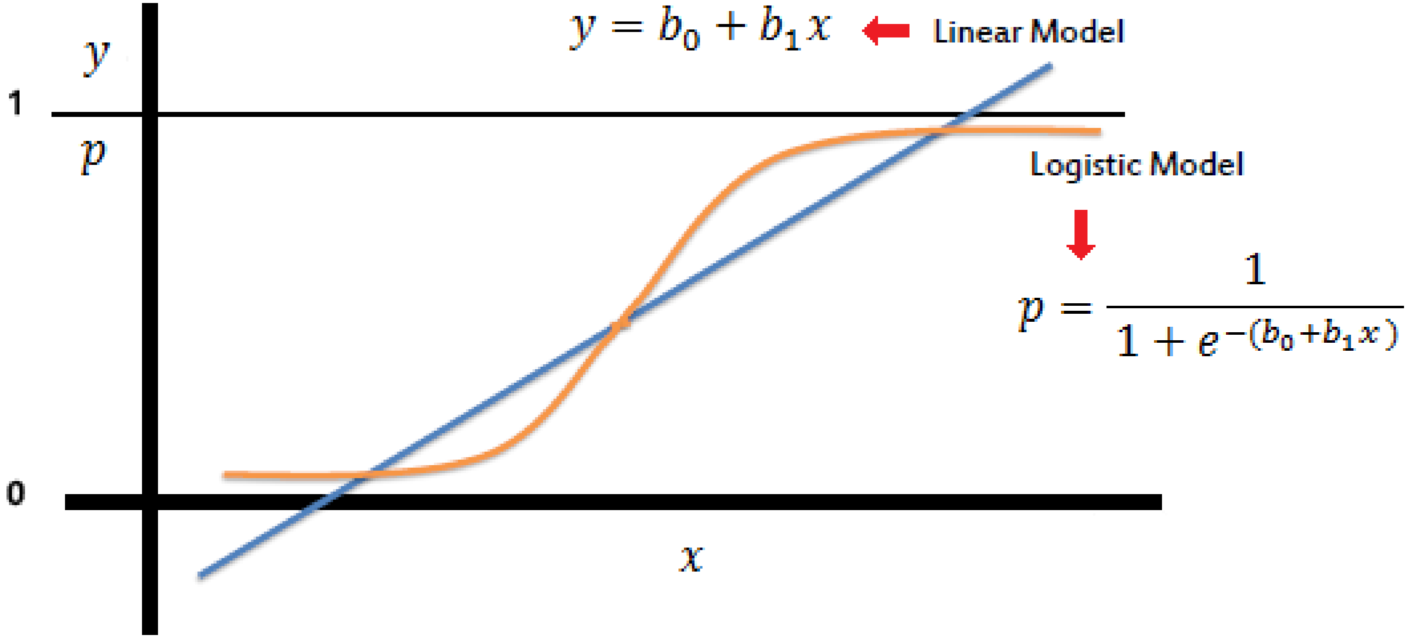
Therefore, what happens to y as x approaches ∞?

What does $\log_a x = y$ tell us about the effect of x on y?



Linear vs. Logistic model

Under what circumstances would we want to use a logistic model rather than a linear one?



Working With Polynomials

Why you need to know this

- You won't be doing much, if any, manual algebra in grad school
- It helps to be familiar with the basics in order to follow the logic behind some statistical procedures or models.

Arithmetic with exponents

Rule or special case	Formula	Example
Product	$x^a x^b = x^{a+b}$	$2^2 2^3 = 2^5 = 32$
Quotient	$\frac{x^a}{x^b} = x^{a-b}$	$\frac{2^3}{2^2} = 2^1 = 2$
Power of power	$(x^a)^b = x^{ab}$	$(2^3)^2 = 2^6 = 64$
Power of a product	$(xy)^a = x^a y^a$	$36 = 6^2 = (2 \cdot 3)^2 = 2^2 \cdot 3^2$ $= 4 \cdot 9 = 36$
Power of one	$x^1 = x$	$2^1 = 2$
Power of zero	$x^0 = 1$	$2^0 = 1$
Power of negative one	$x^{-1} = \frac{1}{x}$	$2^{-1} = \frac{1}{2}$
Change sign of exponents	$x^{-a} = \frac{1}{x^a}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
Fractional exponents	$x^{m/n} = \sqrt[n]{x^m}$ $= (\sqrt[n]{x})^m$	$4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$

Arithmetic with logarithms

1. $\log_b(xy) = \log_b x + \log_b y$	$b^{x+y} = b^x b^y$
2. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$	$b^{x-y} = \frac{b^x}{b^y}$
3. $\log_b(x^y) = y \cdot \log_b x$	$(b^x)^y = b^{xy}$
4. $\log_b 1 = 0$	$b^0 = 1$
5. $\log_b(b) = 1$	$b^1 = b$
6. $\log_b(b^n) = n$	$b^n = b^n$

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5. $\log_b(b) = 1$	$b^1 = b$
6. $\log_b(b^n) = n$	$b^n = b^n$

Notice how logarithms can turn complicated expressions into simpler ones.

- Often used as a convenience for algebraic manipulation
- Often used to make exponential functions linear. Convenient for visualization as well as algebra.

Factoring

- Factoring is rewriting an expression as a multiplication of simpler expressions
- Generally something you develop an intuition for

$$x^2 + 6x + 8 = (x + 2)(x + 4)$$

$$x^2 + 2x - 8 = (x - 2)(x + 4)$$