Linear Model Predictive Control

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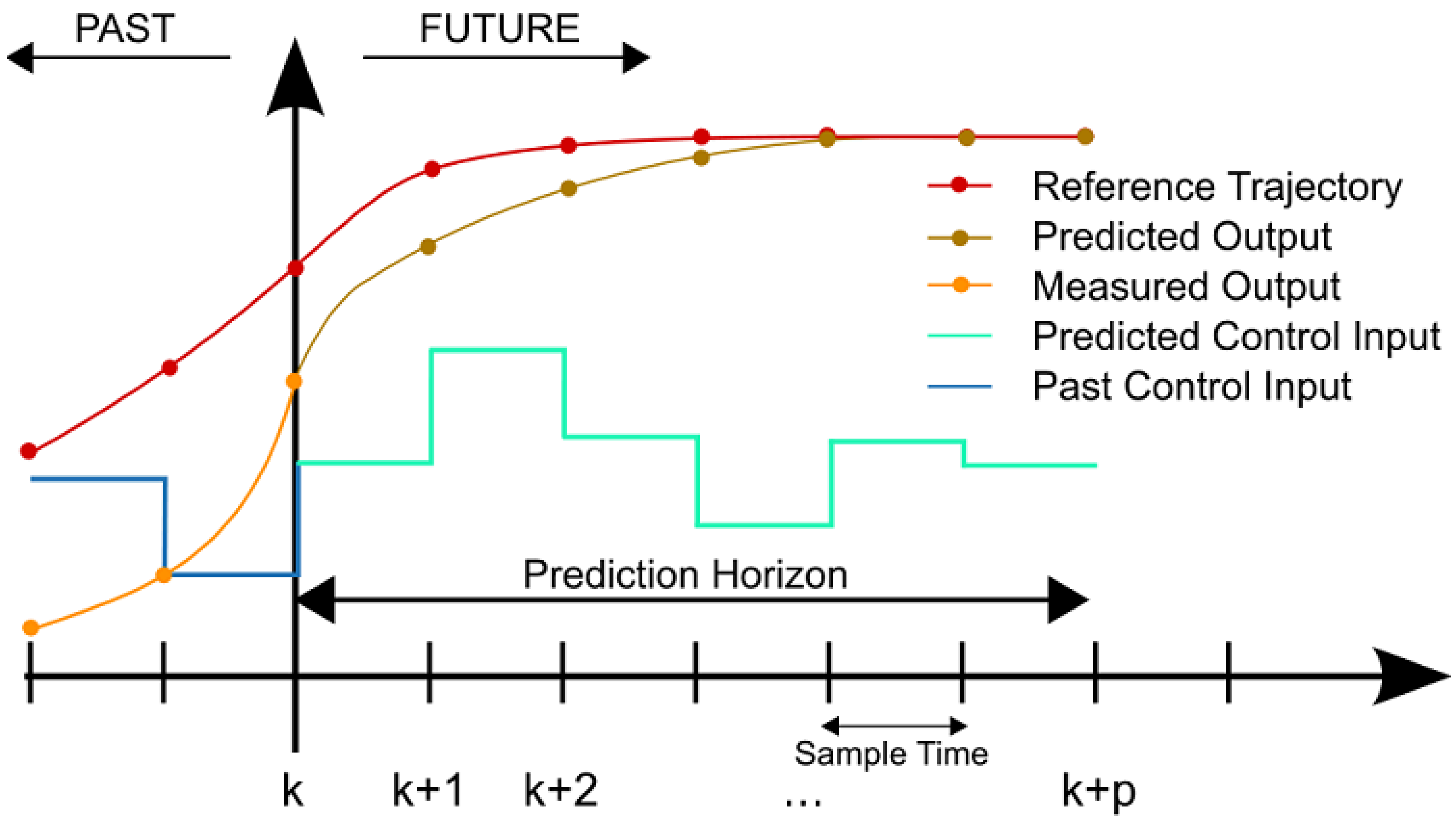
Model predictive control

- Use a dynamical model of the process to predict its future evolution and choose the (sub)optimal control input.



Model predictive control

- MPC example: autonomous racing

Vázquez, J. L., Brühlmeier, M., Liniger, A., Rupenyan, A., & Lygeros, J. (2020, October). Optimization-based hierarchical motion planning for autonomous racing. In 2020 IEEE/RSJ international conference on intelligent robots and systems (IROS) (pp. 2397-2403). IEEE.

Model predictive control

- Model predictive control (MPC) problem:

$$\begin{aligned} \min_{x_0,\ldots,x_N,u_0,\ldots,u_{N-1}} \quad & \sum_{k=0}^{N-1} \ell_k(x_k, u_k) + \ell_N(x_N) \quad && \text{cost} \\ \text{s. t.}\ x_0 &= x_{\text{init}} && \text{initial state constraint} \\ x_{k+1} &= f_k(x_k, u_k), \quad k = 0, \ldots, N-1 && \text{equation of motion constraint} \\ h_k(x_k, u_k) &\le 0, \quad k = 0, \ldots, N-1 \\ h_N(x_N) &\le 0 \end{aligned}$$

Linear MPC with quadratic cost

- Problem formulation

$$
\begin{aligned}
\min_{x_0,\ldots,x_N,u_0,\ldots,u_{N-1}} \ & \frac{1}{2}\sum_{k=0}^{N-1}[x_k^{\mathrm{T}} Q_k x_k + u_k^{\mathrm{T}} R_k u_k] + \frac{1}{2} x_N^{\mathrm{T}} Q_N x_N \\
\text{s. t. } x_0 &= x_{\mathrm{init}} \\
x_{k+1} &= A_k x_k + B_k u_k + d_k, \quad k = 0,\ldots,N-1 \\
x_{\min} &\le x_k \le x_{\max}, \quad k = 1,\ldots,N \\
u_{\min} &\le u_k \le u_{\max}, \quad k = 0,\ldots,N-1 \\
C_k x_k + D_k u_k &\le h_k, \quad k = 0,\ldots,N-1 \\
C_N x_N &\le h_N
\end{aligned}
$$

- Linear time-invariant (LTI) system: $A_0 = A_1 = \cdots = A_{N-1}$, $B_0 = B_1 = \cdots = B_{N-1}$, $d_0 = d_1 = \cdots = d_{N-1}$

Convert LTI-MPC into general QP formulation

- Cost function and optimization variables:

$$\frac{1}{2}\sum_{k=0}^{N-1}[x_k^{\mathrm{T}}Qx_k + u_k^{\mathrm{T}}Ru_k] + \frac{1}{2}x_N^{\mathrm{T}}Q_Nx_N \;\Rightarrow\; \frac{1}{2}z^{\mathrm{T}}\bar{P}z$$

where

$$z = \begin{bmatrix} x_0 \\ \vdots \\ x_N \\ u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}, \bar{P} = \begin{bmatrix} \bar{Q} & \\ & \bar{R} \end{bmatrix}, \bar{Q} = \begin{bmatrix} Q & & & \\ & \ddots & & \\ & & Q & \\ & & & Q_N \end{bmatrix}, \bar{R} = \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix}$$

Convert LTI-MPC into general QP formulation

- Equation of motion into equality constraints:

$$
\begin{aligned}
x_0 &= x_{\text{init}} \\
x_1 &= Ax_0 + Bu_0 + d \\
&\vdots \\
x_N &= Ax_{N-1} + Bu_{N-1} + d
\end{aligned}
$$

$$\Downarrow$$

$$
\underbrace{\begin{bmatrix} -I & & & & & & \\ A & -I & & & B & & \\ & \ddots & \ddots & & & \ddots & \\ & & A & -I & & & B \end{bmatrix}}_{\bar{A}} z = \underbrace{\begin{bmatrix} -x_{\text{init}} \\ -d \\ \vdots \\ -d \end{bmatrix}}_{\bar{b}}
$$

Convert LTI-MPC into general QP formulation

- Inequality constraints:

$$\underbrace{\begin{bmatrix} x_{\min} \\ \vdots \\ x_{\min} \\ u_{\min} \\ \vdots \\ u_{\min} \end{bmatrix}}_{z_{\min}} \leq z \leq \underbrace{\begin{bmatrix} x_{\max} \\ \vdots \\ x_{\max} \\ u_{\max} \\ \vdots \\ u_{\max} \end{bmatrix}}_{z_{\max}}, \quad \underbrace{\begin{bmatrix} C & & & D & & \\ & \ddots & & & \ddots & \\ & & C & & & D \\ & & & C_N & & \end{bmatrix}}_{\bar{G}} z \leq \underbrace{\begin{bmatrix} h \\ \vdots \\ h \\ h_N \end{bmatrix}}_{\bar{h}}$$

Convert LTI-MPC into general QP formulation

- Finally,

$$
\begin{aligned}
\min_{x_0,\ldots,x_N,u_0,\ldots,u_{N-1}} \ & \frac{1}{2}\sum_{k=0}^{N-1}[x_k^{\mathrm{T}} Q_k x_k + u_k^{\mathrm{T}} R_k u_k] + \frac{1}{2} x_N^{\mathrm{T}} Q_N x_N \\
\text{s. t. } x_0 &= x_{\text{init}} \\
x_{k+1} &= A x_k + B u_k + d, \quad k = 0,\ldots,N-1 \\
x_{\min} &\le x_k \le x_{\max}, \quad k = 1,\ldots,N \\
u_{\min} &\le u_k \le u_{\max}, \quad k = 0,\ldots,N-1 \\
C x_k + D u_k &\le h, \quad k = 0,\ldots,N-1 \\
C_N x_N &\le h_N
\end{aligned}
$$

$$\Downarrow$$

$$\min_z \ \frac{1}{2} z^{\mathrm{T}} \bar{P} z \quad \text{s. t. } \bar{A} z = \bar{b}; \bar{G} z \le \bar{h}; z_{\min} \le z \le z_{\max}$$

Condensed form

- Equation of motion:

$$
\begin{aligned}
x_1 =& Ax_0 + Bu_0 + d \\
x_2 =& Ax_1 + Bu_1 + d = A(Ax_0 + Bu_0 + d) + Bu_1 + d \\
&\vdots \\
x_N =& A^N x_0 + A^{N-1}Bu_0 + \cdots + Bu_{N-1} + (A^{N-1} + \cdots + A^0)d
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \underbrace{\begin{bmatrix} B & & & \\ AB & B & & \\ \vdots & & \ddots & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix}}_{\bar{S}} \underbrace{\begin{bmatrix} u_0 \\ \vdots \\ u_{N-1} \end{bmatrix}}_{z} + \underbrace{\begin{bmatrix} A \\ \vdots \\ A^N \end{bmatrix}}_{\bar{T}} x_0 + \underbrace{\begin{bmatrix} \sum_{k=0}^{0} A^k \\ \vdots \\ \sum_{k=0}^{N-1} A^k \end{bmatrix}}_{\bar{t}} d
$$

Condensed form

- Cost function:

$$
\begin{aligned}
&\frac{1}{2}\sum_{k=0}^{N-1}[x_k^\mathrm{T} Q_k x_k + u_k^\mathrm{T} R_k u_k] + \frac{1}{2} x_N^\mathrm{T} Q_N x_N \\
&\quad = \frac{1}{2}(\bar{S}z + \bar{T}x_0 + \bar{t})^\mathrm{T} \bar{Q}(\bar{S}z + \bar{T}x_0 + \bar{t}) + \frac{1}{2} z^\mathrm{T} \bar{R} z + \frac{1}{2} x_0^\mathrm{T} Q x_0 \\
&\quad = \frac{1}{2} z^\mathrm{T} \underbrace{(\bar{S}^\mathrm{T} \bar{Q} \bar{S} + \bar{R})}_{\bar{P}} z + \underbrace{(\bar{T}x_0 + \bar{t})^\mathrm{T} \bar{Q} \bar{S}}_{\bar{q}^\mathrm{T}} z + \frac{1}{2}(x_0^\mathrm{T} Q x_0 + (\bar{T}x_0 + \bar{t})^\mathrm{T} \bar{Q}(\bar{T}x_0 + \bar{t})) \\
&\quad = \frac{1}{2} z^\mathrm{T} \bar{P} z + \bar{q}^\mathrm{T} z + \underbrace{\frac{1}{2}(x_0^\mathrm{T} Q x_0 + (\bar{T}x_0 + \bar{t})^\mathrm{T} \bar{Q}(\bar{T}x_0 + \bar{t}))}_{\text{constant (can be ignored)}}
\end{aligned}
$$

Condensed form

- Control input bounds and linear inequality constraints:

$$u_{\min} \leq u_k \leq u_{\max} \Rightarrow \underbrace{\begin{bmatrix} u_{\min} \\ \vdots \\ u_{\min} \end{bmatrix}}_{z_{\min}} \leq z \leq \underbrace{\begin{bmatrix} u_{\max} \\ \vdots \\ u_{\max} \end{bmatrix}}_{z_{\max}}$$

$$Cx_k + Du_k \leq h \Rightarrow \begin{bmatrix} & & D & \\ C & & & \ddots \\ & \ddots & & & D \\ & & C_N & \end{bmatrix} \left(\begin{bmatrix} \bar{S} \\ I \end{bmatrix} z + \begin{bmatrix} \bar{T}x_0 + \bar{t} \\ 0 \end{bmatrix} \right) \leq \begin{bmatrix} h - Cx_0 \\ \vdots \\ h \\ h_N \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} & & D & \\ C & & & \ddots \\ & \ddots & & & D \\ & & C_N & \end{bmatrix} \begin{bmatrix} \bar{S} \\ I \end{bmatrix}, \; \bar{d} = \begin{bmatrix} h - Cx_0 \\ \vdots \\ h \\ h_N \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 \\ C & & \\ & \ddots & \\ & & C_N \end{bmatrix} (\bar{T}x_0 + \bar{t})$$

Condensed form

- State bounds:

$$\begin{bmatrix} x_{\min} \\ \vdots \\ x_{\min} \end{bmatrix} \leq \underbrace{\bar{S}z + \bar{T}x_0 + \bar{t}}_{[x_1^{\mathrm{T}} \cdots x_N^{\mathrm{T}}]^{\mathrm{T}}} \leq \begin{bmatrix} x_{\max} \\ \vdots \\ x_{\max} \end{bmatrix} \Rightarrow \begin{bmatrix} -\bar{S} \\ \bar{S} \end{bmatrix} z \leq \begin{bmatrix} x_{\min} \\ \vdots \\ x_{\max} \\ \vdots \end{bmatrix} + \begin{bmatrix} \bar{T}x_0 + \bar{t} \\ -\bar{T}x_0 - \bar{t} \end{bmatrix}$$

- Merged inequality constraints:

$$\bar{G} = \begin{bmatrix} -\bar{S} \\ \bar{S} \\ \bar{C} \end{bmatrix}, \bar{h} = \begin{bmatrix} x_{\min} \\ \vdots \\ x_{\max} \\ \vdots \\ 0 \\ \vdots \end{bmatrix} + \begin{bmatrix} \bar{T}x_0 + \bar{t} \\ -\bar{T}x_0 - \bar{t} \\ \bar{d} \end{bmatrix},$$

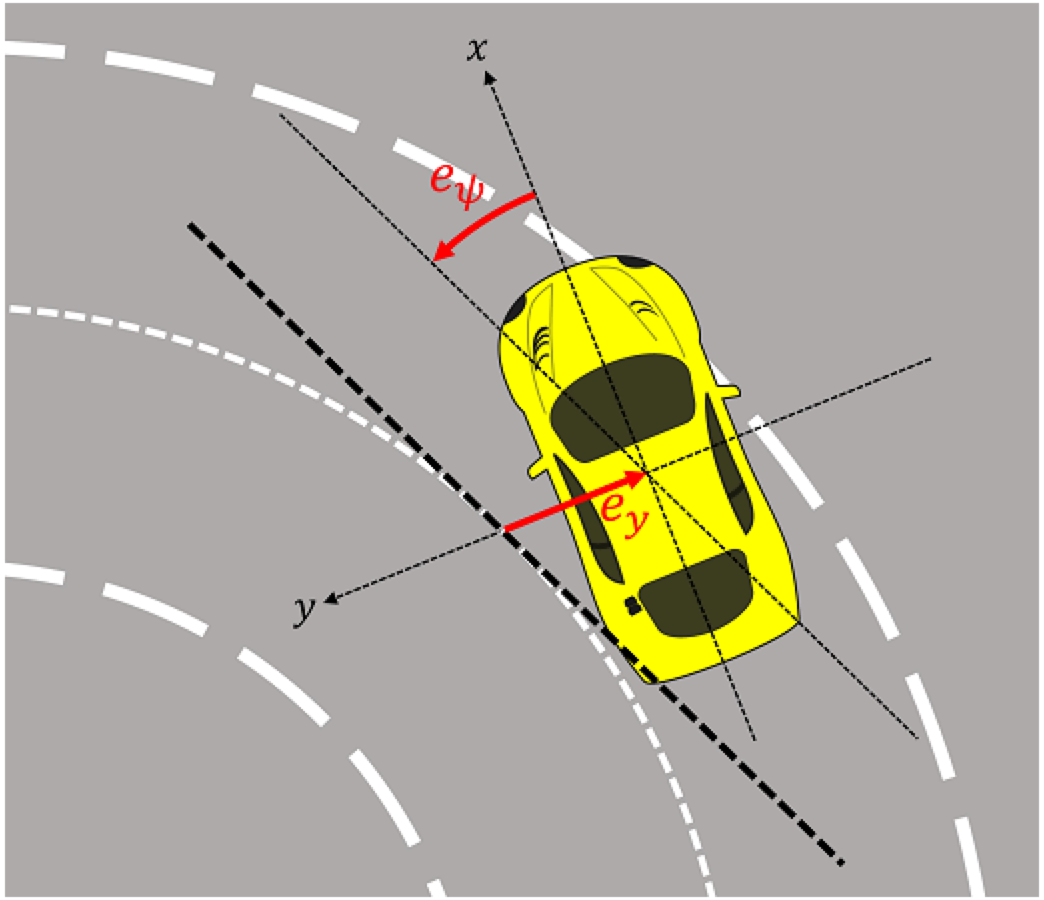
Condensed form

- Finally,

$$
\begin{aligned}
\min_{x_0,\ldots,x_N,u_0,\ldots,u_{N-1}} \frac{1}{2}\sum_{k=0}^{N-1}[x_k^{\mathrm{T}}Q_k x_k + u_k^{\mathrm{T}}R_k u_k] + \frac{1}{2}x_N^{\mathrm{T}}Q_N x_N& \\
\mathrm{s.\,t.}\ x_0 &= x_{\mathrm{init}} \\
x_{k+1} &= A_k x_k + B_k u_k + d_k, \quad k = 0,\ldots,N-1 \\
x_{\min} &\le x_k \le x_{\max}, \quad k = 1,\ldots,N \\
u_{\min} &\le u_k \le u_{\max}, \quad k = 0,\ldots,N-1 \\
C_k x_k + D_k u_k &\le h_k, \quad k = 0,\ldots,N-1 \\
C_N x_N &\le h_N \\
&\Downarrow
\end{aligned}
$$

$$
\min_z \frac{1}{2}z^{\mathrm{T}}\bar{P}z + \bar{q}^{\mathrm{T}}z \quad \mathrm{s.\,t.}\ \bar{G}z \le \bar{h};\, z_{\min} \le z \le z_{\max}
$$

Example: Autonomous vehicle



- Vehicle's pose is defined as lateral position error e_y and yaw angle error e_ψ to the center of the road.

Example: Autonomous vehicle

- Objective: minimizes the lateral position error e_y, yaw angle error e_ψ, and its time derivatives.

$$\min_{x_0,\ldots,x_N,u_0,\ldots,u_{N-1}} \frac{1}{2}\sum_{k=0}^{N-1}[x_k^{\mathrm{T}}Q_k x_k + u_k^{\mathrm{T}}R_k u_k] + \frac{1}{2}x_N^{\mathrm{T}}Q x_N$$
$$\begin{aligned} \text{s. t. } x_0 &= x_{\text{init}} \\ x_{k+1} &= A_k x_k + B_k u_k + d_k, \quad k = 0,\ldots,N-1 \\ x_{\min} &\le x_k \le x_{\max}, \quad k = 1,\ldots,N \\ u_{\min} &\le u_k \le u_{\max}, \quad k = 0,\ldots,N-1 \end{aligned}$$

Example: Autonomous vehicle

- Equation of motion:

$$\underbrace{\begin{bmatrix} \dot{e}_y \\ \ddot{e}_y \\ \dot{e}_\psi \\ \ddot{e}_\psi \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-2(C_f+C_r)}{mv_x} & \frac{2(C_f+C_r)}{m} & \frac{-2(C_f l_f - 2C_r l_r)}{mv_x} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-2(C_f l_f - C_r l_r)}{I_z v_x} & \frac{2(C_f l_f - C_r l_r)}{I_z} & \frac{-2(C_f l_f^2 + 2C_r l_r^2)}{I_z v_x} \end{bmatrix}}_{A_c} \underbrace{\begin{bmatrix} e_y \\ \dot{e}_y \\ e_\psi \\ \dot{e}_\psi \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ \frac{2C_f}{m} \\ 0 \\ \frac{2C_f l_f}{I_z} \end{bmatrix}}_{B_{u,c}} \underbrace{\delta}_{u} + \underbrace{\begin{bmatrix} 0 \\ \frac{-2(C_f l_f - C_r l_r)}{mv_x} - v_x \\ 0 \\ \frac{-2(C_f l_f^2 + C_r l_r^2)}{I_z v_x} \end{bmatrix}}_{B_{w,c}} \underbrace{\dot{\psi}_{\text{ref}}}_{w}$$

- m, I_z: the vehicle total mass and the yaw moment of inertia.
- C_f, C_r: the cornering stiffness of the fornt and rear tires.
- l_f, l_r: distances from the center of mass to the fornt and rear tires.
- v_x, δ: longitudinal speed of the vehicle and the steering angle.
- $\dot{\psi}_{\text{ref}}$: reference yaw rate, $\dot{\psi}_{\text{ref}} = v_x / r$ where r is the radius of curvature of the road.

Example: Autonomous vehicle

- Convert continuous-time EOM into discrete-time EOM:

$$\dot{x} = A_c x + B_{u,c} u + B_{w,c} w \;\Rightarrow\; x_{k+1} = A_d x_k + B_{u,d} u_k + \underbrace{B_{w,d} w_k}_{d_k}$$

$$A_d = e^{A_c T_s}$$

$$\begin{bmatrix} B_{u,d} & B_{w,d} \end{bmatrix} = \int_{\tau=0}^{T_s} e^{AT_s} d\tau \begin{bmatrix} B_u & B_w \end{bmatrix} = A^{-1}(A_d - I) \begin{bmatrix} B_u & B_w \end{bmatrix}$$

- In general, the above system is LTV since $d_k = B_{w,d}\dot{\psi}_{\text{ref},k}$ is time-varying. Therefore, in this example, we will only consider roads with a constant radius of curvature so that the system to be LTI.

Example: Autonomous vehicle

- Simulation result