Optimal Control of Linear Systems

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Optimal control problems

- Predictive control allows the controller to make better decisions at the current time to account for future possibilities
- For example, controlling only to reduce current costs without considering future trajectories can lead to failure.

Optimal control problems

- Continuous-time OCP:

$$V(x(0),0) = \min_u \left\{ \int_0^T \ell_t(x(t),u(t))dt + \ell_T(x(T)) \right\}$$
$$\text{subject to } \dot{x}(t) = f(x(t),u(t)) \quad \forall t \in [0,T]$$
$$h(x(t),u(t)) \in \mathcal{Y} \quad \forall t \in [0,T]$$

- Discrete-time OCP:

$$V(x[0],0) = \min_u \left\{ \sum_{k=0}^{N} \ell_k(x[k],u[k]) + \ell_N(x[N]) \right\}$$
$$\text{subject to } x[k+1] = f(x[k],u[k]) \quad k = 0,\ldots,N-1$$
$$h(x[k],u[k]) \in \mathcal{Y} \quad k = 0,\ldots,N-1$$

Infinite-horizon linear quadratic OCP

- Contiunous-time:

$$V(x(0)) = \min_u \int_0^\infty \left[x(t)^\mathrm{T} Q x(t) + u(t)^\mathrm{T} R u(t)\right] dt$$
$$\text{subject to } \dot{x}(t) = Ax(t) + Bu(t) \quad \forall t \in [0, T]$$

- Discrete-time:

$$V(x[0]) = \min_u \sum_{k=0}^{\infty} \left[x[k]^\mathrm{T} Q x[k] + u[k]^\mathrm{T} R u[k]\right]$$
$$\text{subject to } x[k+1] = Ax[k] + Bu[k] \quad k = 0, \ldots, N-1$$

Continuous-time Linear Quadratic OCP

Continuous-time Lyapunov equation

- Consider a linear system $\dot{x} = Ax$ and quadratic cost $\ell(x) = x^{\mathrm{T}}Qx$:

$$V(x) = x^{\mathrm{T}}Px = \int_0^{\infty} x(t)^{\mathrm{T}}Qx(t)dt.$$

  The time derivative of V is:

$$\dot{V} = \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} = x^{\mathrm{T}}A^{\mathrm{T}}Px + x^{\mathrm{T}}PAx.$$

  Since $\dot{V} = \lim_{\delta \to 0^+} \int_\delta^\infty x(t)^{\mathrm{T}}Qx(t)dt - \int_0^\infty x(t)^{\mathrm{T}}Qx(t)dt = -x^{\mathrm{T}}Qx$,

$$x^{\mathrm{T}}A^{\mathrm{T}}Px + x^{\mathrm{T}}PAx = -x^{\mathrm{T}}Qx$$

$$A^{\mathrm{T}}P + PA = -Q$$

- Given any $Q \succ 0$, there exists a unique $P \succ 0$ satisfying $A^{\mathrm{T}}P + PA + Q = 0$ if and only if the linear system $\dot{x} = Ax$ is globally asymptotically stable.

Continuous-time Lyapunov equation

- Using The mixed Kronecker matrix-vector product $\mathrm{vec}(ABC) = (C^{\mathrm{T}} \otimes A)\mathrm{vec}(B)$,

$$\begin{aligned} -\mathrm{vec}(Q) &= \mathrm{vec}(A^{\mathrm{T}}P + PA) \\ &= \mathrm{vec}(A^{\mathrm{T}}P + PA) \\ &= \mathrm{vec}(A^{\mathrm{T}}PI + IPA) \\ &= (I \otimes A^{\mathrm{T}} + A^{\mathrm{T}} \otimes I)\mathrm{vec}(P) \end{aligned}$$

- Analytic solution of continuous-time Lyapunov equation P can be found as:

$$\begin{aligned} (I \otimes A^{\mathrm{T}} + A^{\mathrm{T}} \otimes I)\mathrm{vec}(P) &= -\mathrm{vec}(Q) \\ \mathrm{vec}(P) &= -(I \otimes A^{\mathrm{T}} + A^{\mathrm{T}} \otimes I)^{-1}\mathrm{vec}(Q) \end{aligned}$$

Continuous-time linear quadratic regulator (LQR)

- Consider a linear system $\dot{x} = Ax + Bu$ and quadratic cost $\ell(x,u) = x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru$:

$$V(x) = x^{\mathrm{T}}Px = \min_{u} \int_{0}^{\infty} x(t)^{\mathrm{T}}Qx(t) + u(t)^{\mathrm{T}}Ru(t)dt.$$

Since $\dot{V} + x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru = 0$,

$$\begin{aligned}
0 &= \dot{V} + x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru \\
&= \dot{x}^{\mathrm{T}}Px + x^{\mathrm{T}}P\dot{x} + x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru \\
&= (Ax + Bu)^{\mathrm{T}}Px + x^{\mathrm{T}}P(Ax + Bu) + x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru \\
&= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA + Q)x + u^{\mathrm{T}}B^{\mathrm{T}}Px + x^{T}PBu + u^{\mathrm{T}}Ru
\end{aligned}$$

If u is optimal control input given P, $\frac{\partial}{\partial u}(\dot{V} + x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru) = 2Ru + 2B^{\mathrm{T}}Px = 0$,

$$u = -R^{-1}B^{\mathrm{T}}Px$$

- Therefore, an optimal control law is $u = -Kx$ where $K = R^{-1}B^{\mathrm{T}}P$

Continuous algebraic Ricatti equation

- Using $u = -R^{-1}B^{\mathrm{T}}Px$, $\dot{x} = (A - BR^{-1}B^{\mathrm{T}}P)x$. then

$$
\begin{aligned}
0 &= \dot{V} + x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru \\
&= x^{\mathrm{T}}(A - BR^{-1}B^{\mathrm{T}}P)^{\mathrm{T}}Px + x^{\mathrm{T}}P(A - BR^{-1}B^{\mathrm{T}}P)x + x^{\mathrm{T}}Qx + x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px \\
&= x^{\mathrm{T}}A^{\mathrm{T}}Px + x^{\mathrm{T}}PAx + x^{\mathrm{T}}Qx - x^{\mathrm{T}}PBR^{-1}B^{\mathrm{T}}Px \\
&= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q)x
\end{aligned}
$$

$A^{\mathrm{T}}P + PA - PBR^{-1}B^{\mathrm{T}}P + Q$ is known as the continuous time algebraic Riccati equation (CARE), and P can be found by solving it.

Continuous algebraic Ricatti equation

- Iterative solution:

$$
\begin{aligned}
&K \leftarrow 0\\
&\text{repeat until } P \text{ converges}\\
&\quad P \leftarrow \operatorname{vec}_{n\times n}^{-1}\left(-\left(I \otimes (A-BK)^{\mathrm{T}} + (A-BK)^{\mathrm{T}} \otimes I\right)^{-1} \operatorname{vec}(Q + K^{\mathrm{T}} R K)\right)\\
&\quad K \leftarrow R^{-1} B^{\mathrm{T}} P\\
&\text{end repeat}
\end{aligned}
$$

Discrete-time Linear Quadratic OCP

Discrete-time LQR

- Consider a linear system $x[k+1] = Ax[k] + Bu[k]$ and quadratic cost $\ell(x[k], u[k]) = x[k]^{\mathrm{T}} Q x[k] + u[k]^{\mathrm{T}} R u[k]$:

$$V(x[0]) = x[0]^{\mathrm{T}} P x[0] = \min_u \sum_{k=0}^{\infty} x[k]^{\mathrm{T}} Q x[k] + u[k]^{\mathrm{T}} R u[k].$$

Since $V(x[k]) = V(x[k+1]) + x[k]^{\mathrm{T}} Q x[k] + u[k]^{\mathrm{T}} R u[k]$,

$$\begin{aligned} x^{\mathrm{T}} P x &= (Ax + Bu)^{\mathrm{T}} P (Ax + Bu) + x^{\mathrm{T}} Q x + u^{\mathrm{T}} R u \\ &= x^{\mathrm{T}} (A^{\mathrm{T}} P A + Q) x + x^{\mathrm{T}} A^{\mathrm{T}} P B u + u^{\mathrm{T}} B^{\mathrm{T}} P A x + u^{\mathrm{T}} (R + B^{\mathrm{T}} P B) u \end{aligned}$$

If u is optimal control input given P, $(R + B^{\mathrm{T}} P B) u + B^{\mathrm{T}} P A x = 0$,

$$u = -(R + B^{\mathrm{T}} P B)^{-1} B^{\mathrm{T}} P A x$$

Discrete algebraic Ricatti equation

- Using $u = -(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PAx$,

$$\begin{aligned} x^{\mathrm{T}}Px &= x^{\mathrm{T}}(A^{\mathrm{T}}PA + Q)x + x^{\mathrm{T}}A^{\mathrm{T}}PBu + u^{\mathrm{T}}B^{\mathrm{T}}PAx + u^{\mathrm{T}}(R + B^{\mathrm{T}}PB)u \\ &= x^{\mathrm{T}}(A^{\mathrm{T}}PA + Q - A^{\mathrm{T}}PB(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA)x \end{aligned}$$

- Discrete algebraic RIcatti equation

$$P = A^{\mathrm{T}}PA + Q - A^{\mathrm{T}}PB(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA$$

- Iterative solution:

$$\begin{aligned} &P \leftarrow Q \\ &\text{repeat until } P \text{ converges} \\ &\quad P \leftarrow A^{\mathrm{T}}PA + Q - A^{\mathrm{T}}PB(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PA \\ &\text{end repeat} \end{aligned}$$

Simulation