Introduction to Control

Jaehyun Lim[1]

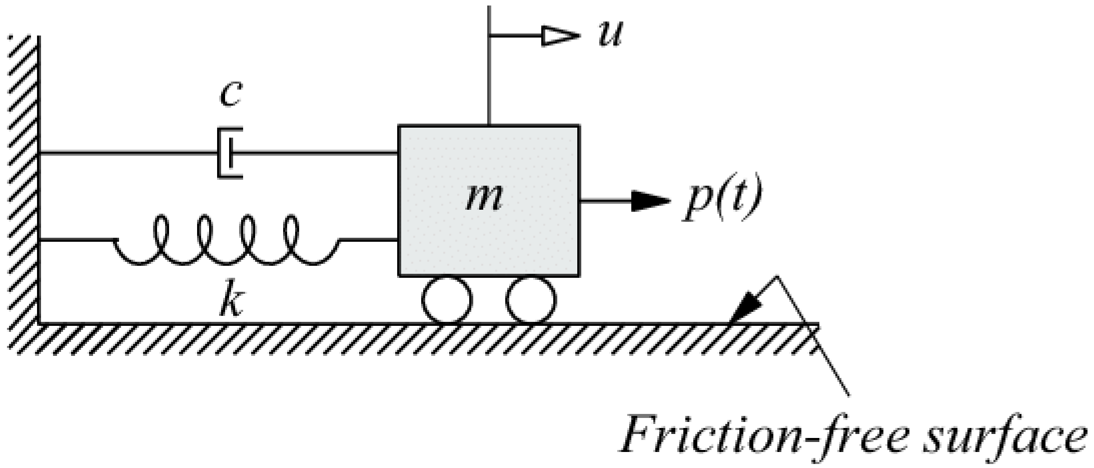
[1]Machine Learning and Control Systems (MLCS) Laboratory

[1]School of Mechanical Engineering, Yonsei University

Linear systems

Machine Learning and
Control Systems Lab.

Mass-spring-damper



- Equation of motion: $m\ddot{p} + c\dot{p} + kp = F \ (= u)$
- How to minimize the error to a reference point $e = p_{\text{ref}} - p$?:
  Set $u = k_p e + k_i \int_0^t e dt + k_d \dot{e}$ (PID control)

State-space representation

- State-space representation:

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
$$

  If A, B, C and D are constant, the system is called linear-time invariant (LTI) system.

- Convert mass-spring-damper system into state-space representation:

$$
m\ddot{p} + c\dot{p} + kp = F
$$

$$
\Downarrow
$$

$$
\underbrace{\begin{bmatrix} \dot{p} \\ \ddot{p} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p \\ \dot{p} \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} F \end{bmatrix}}_{u}
$$

Feedback control with state-space representation

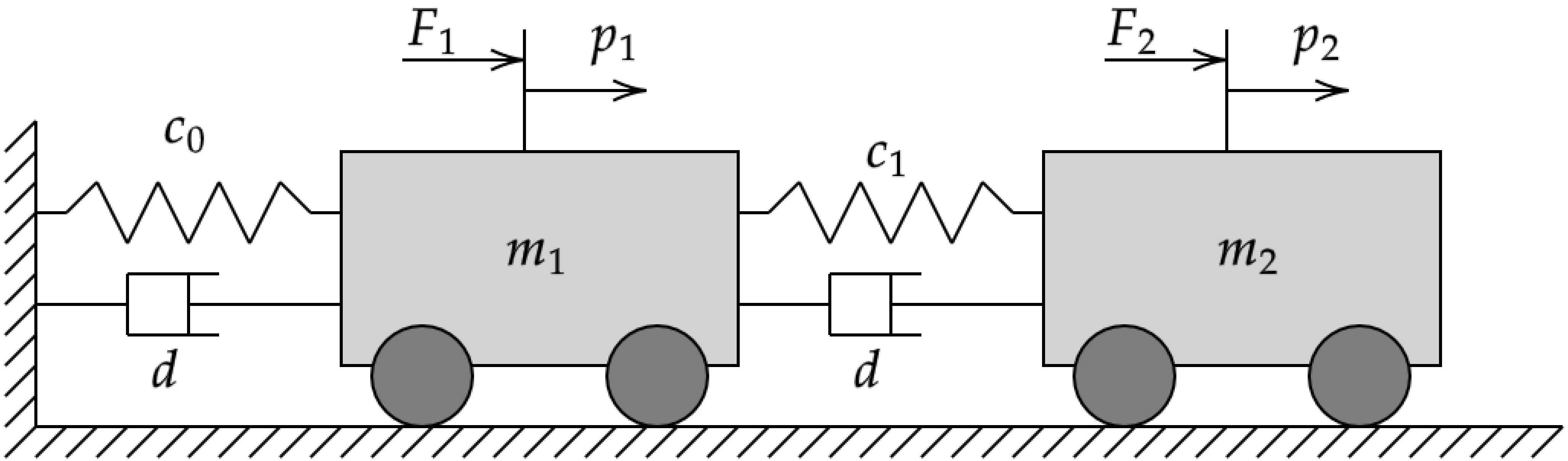
- If we set $u = K(x_{\text{ref}} - x)$,

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= Ax + BK(x_{\text{ref}} - x) \\ &= (A - BK)x + BKx_{\text{ref}}\end{aligned}$$

- If we set $\dot{p}_{\text{ref}} = 0$, it is equivalent as PD control:

$$K = \begin{bmatrix} k_p & k_d \end{bmatrix}$$

Multi-input multi-output (MIMO) systems



$$\underbrace{\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{c_0+c_1}{m_1} & \frac{c_1}{m_1} & -\frac{2d}{m_1} & \frac{d}{m_1} \\ \frac{c_1}{m_2} & -\frac{c_1}{m_2} & \frac{d}{m_2} & -\frac{2d}{m_2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p_1 \\ p_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}}_{u}$$

Multi-input multi-output (MIMO) systems

- How to design K for MIMO system?
  - Find K which makes real parts of $\mathrm{eig}(A - BK)$ to be negative (for stability).
  - Eigenstructure assignment (if the system is completely controllable)

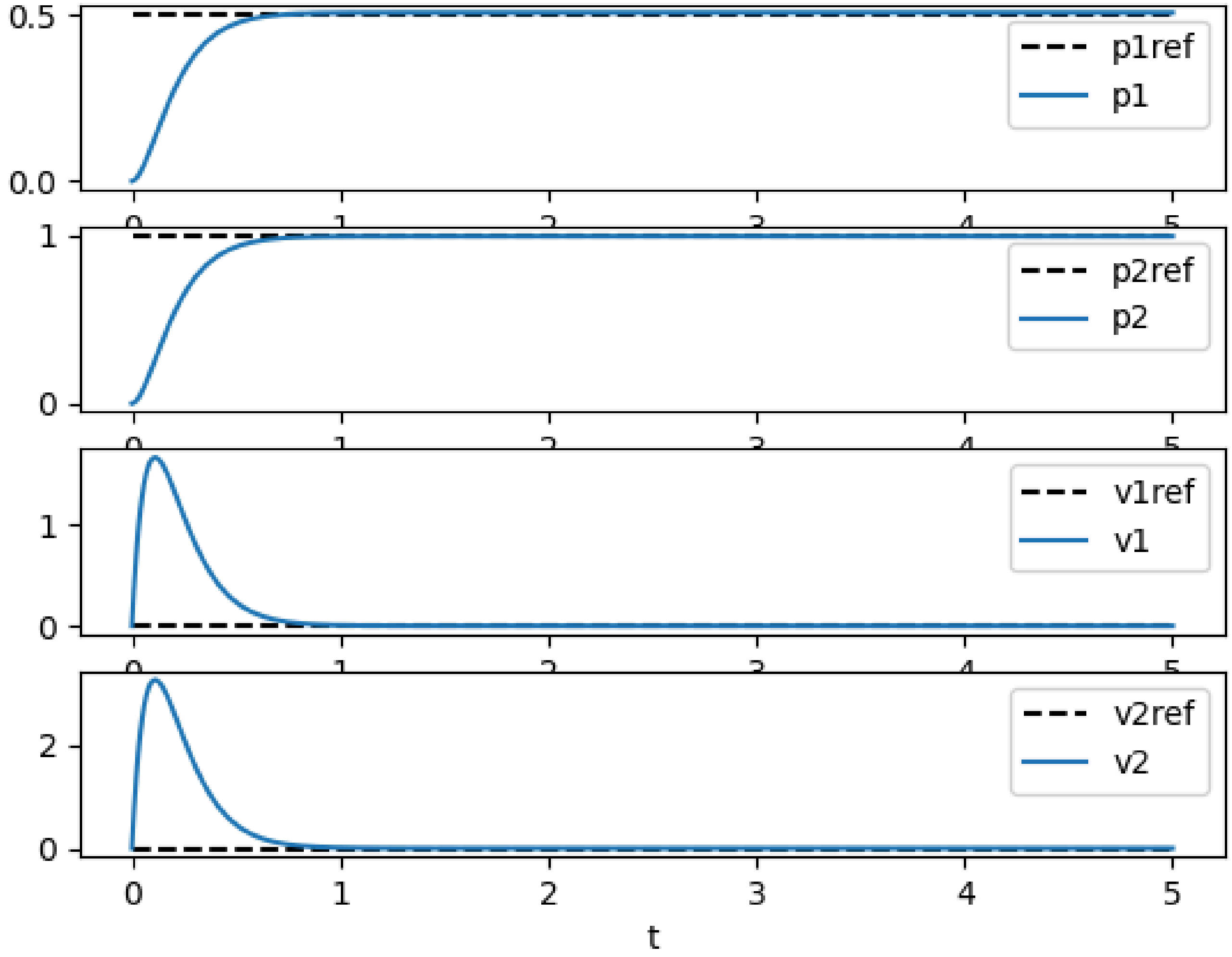
$$(A - BK)v_i = \lambda_i v_i \Rightarrow \begin{bmatrix} \lambda_i I - A & B \end{bmatrix} \begin{bmatrix} v_i \\ K v_i \end{bmatrix} = 0$$

$$p_i = \mathrm{Null}(\begin{bmatrix} \lambda_i I - A & B \end{bmatrix})$$

$$\begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix} = \begin{bmatrix} V \\ Q \end{bmatrix}$$

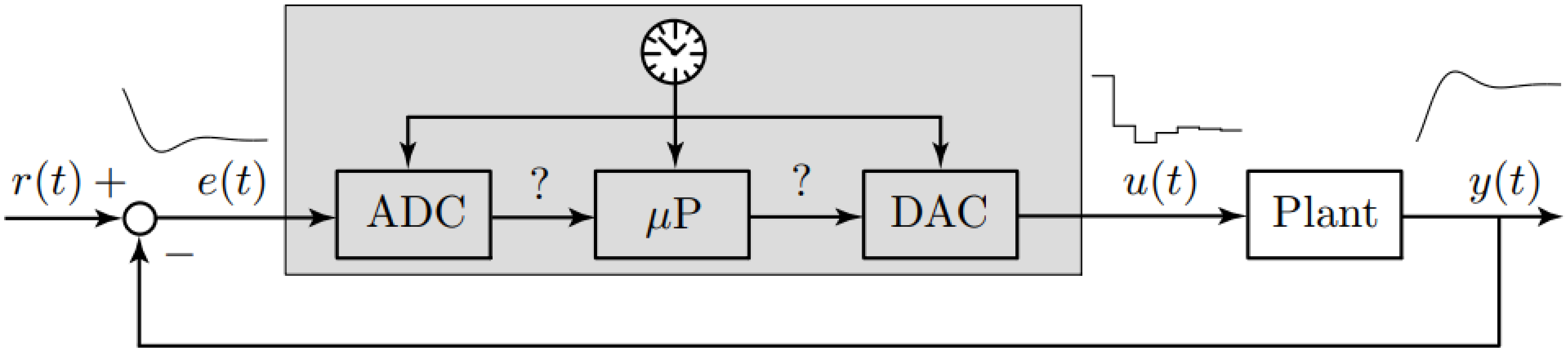
$$K = QV^{-1}$$

Simulation



Discrete-time linear systems

- Most control systems today use digital computers (usually microprocessors) to implement the controllers, which receives measurement only at specific intervals and holds its control input for the specified sample period.



- Because of these characteristics, control gain designed on the continuous-time model can make the system to be unstable.

Discrete-time linear systems

- With sampling time T_s, discrete-time linear system with $x[k] := x(kT_s)$ is

$$\begin{aligned} x[k+1] &= A_d x[k] + B_d u[k] \\ y[k] &= C_d x[k] + D_d u[k] \end{aligned}$$

- Stability condition

$$\mathrm{Re}(\mathrm{eig}(A_d - B_d K)) < 1$$

Discrete-time linear systems

- Convert continuous-time linear systems into discrete-time linear systems:

$$
\begin{aligned}
A_d &= e^{AT_s} \\
B_d &= \int_{\tau=0}^{T_s} e^{A\tau} d\tau B = A^{-1}(A_d - I)B \\
C_d &= C \\
D_d &= D
\end{aligned}
$$

- or $\begin{bmatrix} A_d & B_d \\ 0 & I \end{bmatrix} = e^{\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} T_s}$

Nonlinear systems

Nonlinear systems

- Nonlinear continuous-time systems:

$$\dot{x} = f(x, u)$$

- Nonlinear discrete-time systems:

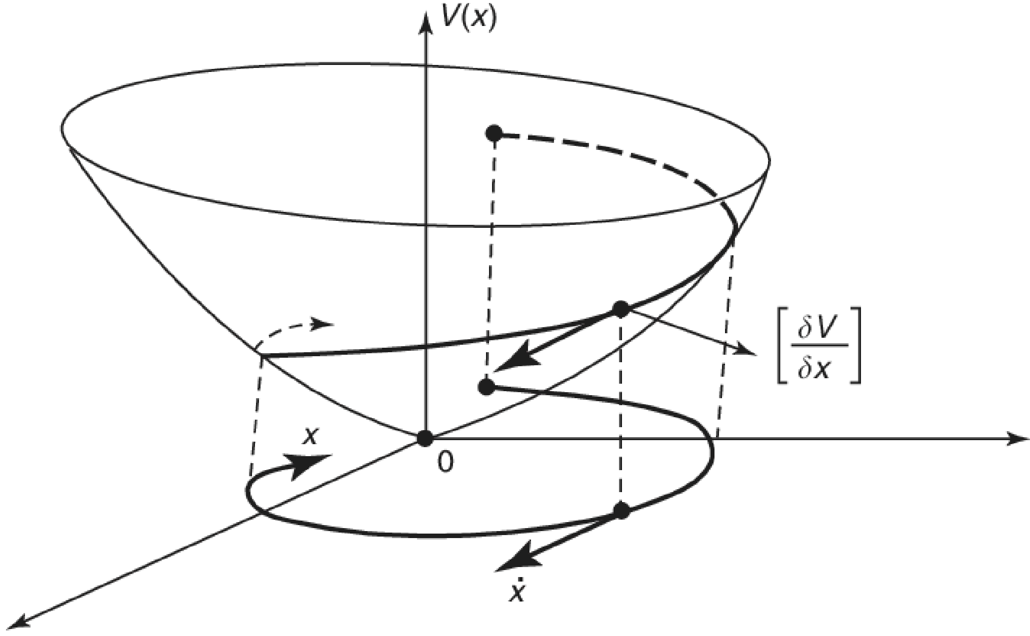
$$x[k+1] = f(x[k], u[k])$$

where f is arbitrary nonlinear function $f : \mathbb{R}^{\dim(x)} \times \mathbb{R}^{\dim(u)} \to \mathbb{R}^{\dim(x)}$

Lyapunov function

- Consider a function $V : \mathbb{R}^n \to \mathbb{R}$ such that
  - $V(x) = 0$ if and only if $x = 0$
  - $V(x) > 0$ if and only if $x \neq 0$
  - $\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} \leq 0$ for all values of $x \neq 0$

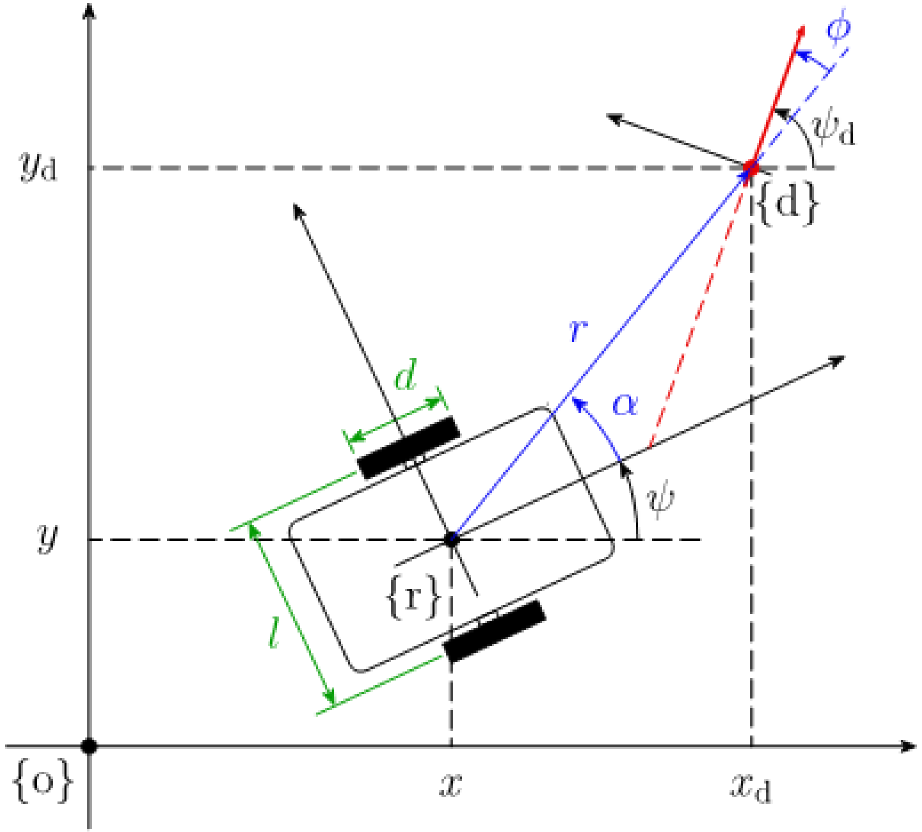
  Then $V(x)$ is called a Lyapunov function and the system is stable in the sense of Lyapunov (if $\dot{V}(x) < 0$, the system is asymtotically stable).



Differential-drive mobile robot

- Kinematic equation of motion (considers only geometric properties):

$$\dot{r} = -v\cos\alpha$$
$$\dot{\alpha} = v\sin\alpha/r - \omega$$
$$\dot{\phi} = -v\sin\alpha/r$$



Lyapunov function of differential drive robot

- If we define Lyapunov function of the differential drive robot as follows:

$$V = \frac{1}{2}(\lambda_r r^2 + \lambda_\alpha \alpha^2 + \lambda_\phi \phi^2)$$

then $V(x) \begin{cases} = 0 & \text{if } x = 0 \\ > 0 & \text{if } x \neq 0 \end{cases}$

-The time derivative of V is:

$$\begin{aligned} \dot{V} &= \lambda_r r\dot{r} + \lambda_\alpha \alpha\dot{\alpha} + \lambda_\phi \phi\dot{\phi} \\ &= -\lambda_r r v \cos\alpha + \lambda_\alpha \alpha(v \sin\alpha / r - \omega) - \lambda_\phi \phi v \sin\alpha / r \\ &= -\lambda_r r v \cos\alpha + (\lambda_\alpha \alpha - \lambda_\phi \phi) v \sin\alpha / r - \lambda_\alpha \alpha\omega \end{aligned}$$

Control design

- Control design by making $\dot{V} \leq 0$ for all values of $x \neq 0$:
   i. Set $v(r, \alpha) = k_v r \cos\alpha$:

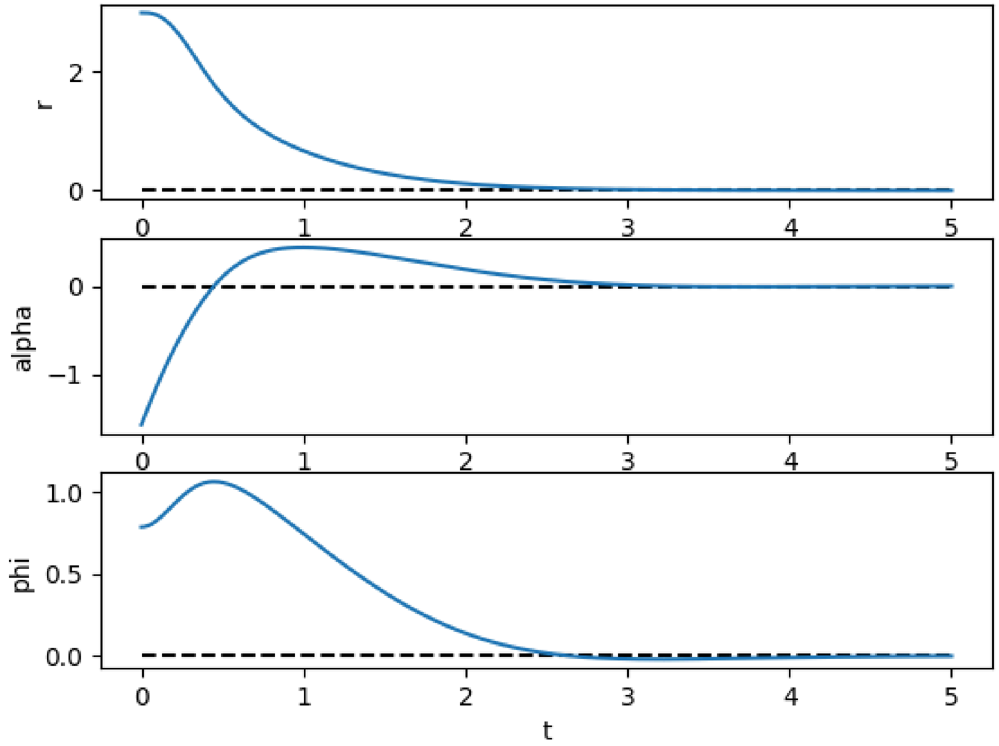
$$\begin{aligned}\dot{V} &= -\lambda_r r v \cos\alpha + (\lambda_\alpha \alpha - \lambda_\phi \phi) v \sin\alpha / r - \lambda_\alpha \alpha \omega \\ &= -\lambda_r k_v r^2 \cos^2\alpha + (\lambda_\alpha \alpha - \lambda_\phi \phi) k_v \cos\alpha \sin\alpha - \lambda_\alpha \alpha \omega\end{aligned}$$

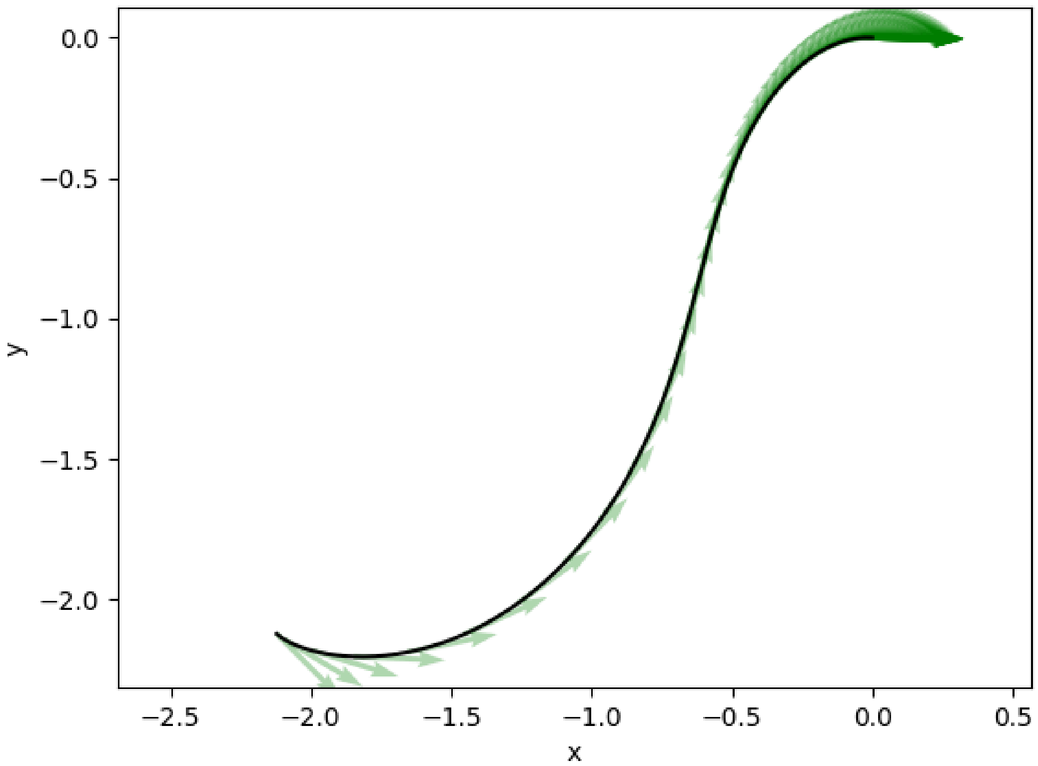
   ii. Set $\omega(\alpha, \phi) = (\lambda_\alpha \alpha - \lambda_\phi \phi) k_v \cos\alpha \sin\alpha / \lambda_\alpha \alpha + k_\omega \alpha$:

$$\begin{aligned}\dot{V} &= -\lambda_r k_v r^2 \cos^2\alpha + (\lambda_\alpha \alpha - \lambda_\phi \phi) k_v \cos\alpha \sin\alpha - \lambda_\alpha \alpha \omega \\ &= -\lambda_r k_v r^2 \cos^2\alpha - \lambda_\alpha k_\omega \alpha^2\end{aligned}$$

- Since $V > 0$ and $\dot{V} < 0$ for all values of $x \neq 0$, the system can be stabilized with above control law.

Simulation





Linearization of a nonlinear system

- Instead of designing complex nonlinear control law, a ststem can be handled by linearizing it.
- Linearize at an equilibrium point $\tilde{x}$ and $\tilde{u}$:

$$
\begin{aligned}
\dot{x} &= f(x, u) \\
&\approx \frac{\partial f}{\partial x}(x - \tilde{x}) + \frac{\partial f}{\partial u}(u - \tilde{u}) + f(\tilde{x}, \tilde{u}) \\
&= \underbrace{\frac{\partial f}{\partial x}}_{A} x + \underbrace{\frac{\partial f}{\partial u}}_{B} u + \underbrace{f(\tilde{x}, \tilde{u}) - \frac{\partial f}{\partial x}\tilde{x} - \frac{\partial f}{\partial u}\tilde{u}}_{d}
\end{aligned}
$$

- or linearize at an nominal trajectory $\{(\tilde{x}_1, \tilde{u}_1), \ldots, (\tilde{x}_n, \tilde{u}_n)\}$ (linear time-varying)

Example: linearization of the differential drive robot EoM

$$\begin{aligned}\dot{r} &= -v\cos\alpha \\ \dot{\alpha} &= v\sin\alpha/r - \omega \\ \dot{\phi} &= -v\sin\alpha/r\end{aligned}$$

$$\Downarrow$$

$$\underbrace{\begin{bmatrix}\dot{r}\\ \dot{\alpha}\\ \dot{\phi}\end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix}0 & -v\sin\alpha & 0\\ -v\sin\alpha/r^2 & v\cos\alpha/r & 0\\ v\sin\alpha/r^2 & v\cos\alpha/r & 0\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}r\\ \alpha\\ \phi\end{bmatrix}}_{x} + \underbrace{\begin{bmatrix}\cos\alpha & 0\\ \sin\alpha/r & -1\\ -\sin\alpha/r & 0\end{bmatrix}}_{B}\underbrace{\begin{bmatrix}v\\ \omega\end{bmatrix}}_{u}$$