ASSIGNMENT 2: THEORY
DUE FRIDAY, SEPTEMBER 20, AT 12:30 PM

On the top left of every assignment please write two things only: your V number, and the assignment number. Please, **do not write your name on your assignment**. Whether or not it is explicitly stated, you always must justify your answers. Steps in your work should be explained clearly enough to be understandable to your classmates. If you think you have found a typo or error in the assignment, please tell me right away.

If part of a problem is solved exactly in the textbook, this part probably will not be graded and is included on the homework only to help you learn the material (e.g. for exams). If you copy the answer from the textbook, this defeats the purpose.

Problem 1. Suppose the polynomial $p(x)$ interpolates $f(x) = \sin(\pi x)$ at $x_0 = -1/2$, $x_1 = 0$, $x_2 = 1/2$. (Unless otherwise stated, we're always talking about "the interpolating polynomial" – the one with the minimum possible degree.)

(a) Write this $p(x)$ explicitly in the Lagrange form. Do not simplify.
(b) Simplify to find $p(x)$ in the monomial basis.
(c) By definition, p is the unique polynomial of degree ≤ 2 that interpolates f at the specified x_i. What is the degree of p in this example? What geometric property of the interpolation points does this reflect?
(d) What is the exact interpolation error at $x = 1/4$?
(e) Using the interpolation error theorem from class, find an upper bound on $|f(1/4) - p(1/4)|$. How does it compare to the actual error from part (d)?
(f) Using the interpolation error theorem from class, find an upper bound on $\max_{x \in [-1/2, 1/2]} |f(x) - p(x)|$.

Problem 2. Let us *define* the Chebyshev polynomials for all $x \in \mathbb{R}$ by the recurrence relation

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

As will be shown in class, the $T_n(x)$ defined as above also satisfy the equation $T_n(x) = \cos\left(n \cos^{-1} x\right)$ for all $x \in [-1, 1]$.

(a) Using the recurrence relation, find simplified expressions for T_2, T_3, and T_4.
(b) Using the cosine formula for T_n, find all of its roots on the interval $[-1, 1]$.
(c) How do you know that T_n has no other roots outside of $[-1, 1]$? (In this region, the above cosine formula has no relevance; T_n is defined by the recurrence relation.)
(d) We will show in class that the coefficient on the x^n term in $T_n(x)$ is 2^{n-1}. For the polynomial $(x - x_1)(x - x_2) \cdots (x - x_n)$ that has the same n roots as $T_n(x)$, give the value of

$$\max_{x \in [-1,1]} |(x - x_1)(x - x_2) \cdots (x - x_n)|.$$