CSC 226: Summer 2019: Lab 3

May 29, 2019

1 Walk, Trail and Path

Let $G = (V,E)$ be an undirected graph with vertex set V and edge set E. Let x,y be two (not necessarily distinct) vertices of G.

Walk: An $x - y$ walk of G is an alternating sequence of vertices and edges starting from x and ending at y. The sequence may look like this:

$$x = x_0, e_1, x_1, e_2, \ldots, e_n, x_n = y$$

The *length* of a walk is the number of edges in the walk. In the example above the length of the x–y walk is n. There might be repeated vertices and/or repeated edges in a walk.

Closed and Open Walks: An x–y walk is *closed* if $x = y$, otherwise it is *open*.

Trail and Circuit: A *trail* is an $x - y$ walk where no edge is repeated. A closed trail is called a *circuit*.

Path and Cycle: A *trail* is an x–y walk where no edge is repeated. A closed trail is called a *circuit*.

Exercise

Based on the definitions above, answer the following questions.

1. In each of the following pairs, which one is a subset of the other? For example in the pair "path, circuit", is a path always a circuit? or is a circuit always a path? or neither is true?
   - path, circuit Ans. Path ⊂ trail and circuit ⊂ trail. But a path might or might not have the same endpoints, whereas a circuit starts and ends at the same vertex. So, path is not always a circuit. A circuit may have repeated vertices, but a path does not. So, a circuit is not always a path.
   - cycle, trail Ans. Cycle ⊂ path ⊂ trail.

- trail, open walk Ans. Trail $\subset$ walk, but a trail may be closed. So trail is not always an open walk. An open walk may have repeated edges, so it is not always a trail.

2. Draw the graph with the following edges and let's call it T_t. Try to draw it without crossing edges.

   $(a,b),(b,c),(c,a),(d,e),(e,f),(f,d),(a,d),(b,e),(c,f),(a,e),(b,f),(c,d)$

   Ans. See Figure 1.

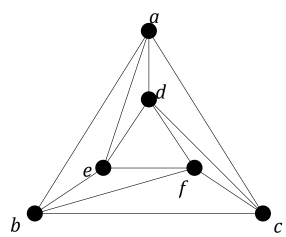

Figure 1: The graph T_t.

3. How many a,c paths are there in graph T_t in Exercise 2? How many of those paths have length 4?

   Ans. The 26 a–c paths in T_t are shown below. Among them, 10 has length 4.

   - $a \to c$
   - $a \to b \to c$
   - $a \to d \to c$
   - $a \to b \to f \to c$
   - $a \to d \to f \to c$
   - $a \to e \to b \to c$
   - $a \to e \to d \to c$
   - $a \to e \to f \to c$
   - $a \to b \to e \to d \to c$
   - $a \to b \to e \to f \to c$
   - $a \to b \to f \to d \to c$
   - $a \to d \to e \to b \to c$
   - $a \to d \to e \to f \to c$
   - $a \to d \to f \to b \to c$

- $a \to e \to b \to f \to c$
- $a \to e \to d \to f \to c$
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- $a \to e \to b \to f \to d \to c$
- $a \to e \to d \to f \to b \to c$

4. Let G be an undirected graph and let x,y (x 6= y) be two distinct vetices of G. If there is an $x - y$ trail in G, prove that there is an $x - y$ path in G.

   Ans. See Page 517 of [**?**].

2 Subgraphs and Isomorphism

Let $G = (V,E)$ be an undirected graph with vertex set V and edge set E and let $G_1 = (V_1,E_1)$ be another undirected graph with vertex set V_1 and edge set E_1.

Subgraph: G_1 is a *subgraph* of G if $V_1 \subset V$, where V_1 6= $\emptyset$, and $E_1 \subset E$ such that each edge in E_1 is incident to vertices in V_1.

G_1 is a *spanning subgraph* of G if $V_1 = V$ and $E_1 \subset E$.

G_1 is an *induced subgraph* of G if all the edges incident to vertices in $V_1 \subset V$ are in $E_1 \subset E$.

Complement: A *complete graph* K_n of n vertices contains an edge between each pair of vertices. The complement graph of $\overline{G}$ of G is the spanning subgraph of K_n that contains all the edges that are not in G.

Isomorphism: Let $G_1 = (V_1,E_1)$ and $G_2 = (V_2,E_2)$ be two undirected graphs. G_1 and G_2 are *isomorphic* to each other if there exists a one-to-one and onto function $f : V_1 \to V_2$ such that for each $a,b \in V_1$, $(a,b) \in E_1$ if and only if $(f(a),f(b)) \in E_2$.

Exercise

Based on the definitions above, answer the following questions.

1. Draw the graph with the following edges: (a,b),(b,c),(c,a). How many subgraph does it have? You don't have to draw all the subgraphs, just calculate the number. How many of those subgraphs is a spanning subgraph? How many are induced subgraphs? Don't worry about isomorphism.

   Ans. The graph is a traingle with three corners a,b,c. There are 3 vertices and 3 edges. Each of these verices and edges might or might not be in a subgraph. So the number of possible subgraphs s 2^6. In a spanning subgraph we have all the three vertices, but we can choose any number of edges to appear in the subgraph. So the number of spanning subgraphs is 2^3 since we have 3 edges. When computing induced subgraphs, we can choose the vertices to be on the subgraph, but then we have to include all the edges that appear between those vertices. So the number of induced ubgraphs is 2^3 since we have 3 vertices.

2. Draw the complement of T_t from the previous exercise. How many edges are there in K_6?

   Ans. See Figure 2. It has only three edges (a,f),(b,d),(c,e). There are $\binom{6}{2}$= 15 edges in K_6. Since T_t had 12 edges, $\overline{T_t}$ must have 3 edges.

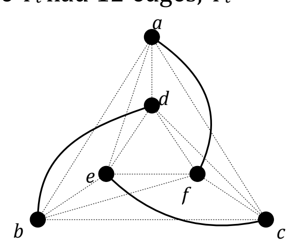
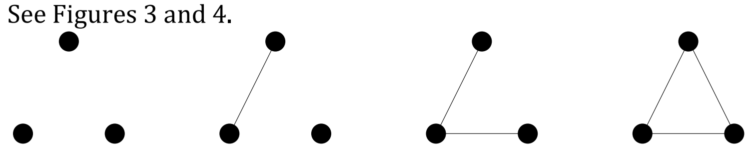


Figure 2: Complement graph $\overline{T_t}$ of the graph T_t.

3. Find all non-isomorphic (and loop-free) graphs of 3 vertices. Now find all non-isomorphic graphs of 4 vertices.

   Ans. See Figures 3 and 4.



(a) (b) (c) (d)

Figure 3: All loop-free non-isomorphic graphs on 3 vertices.

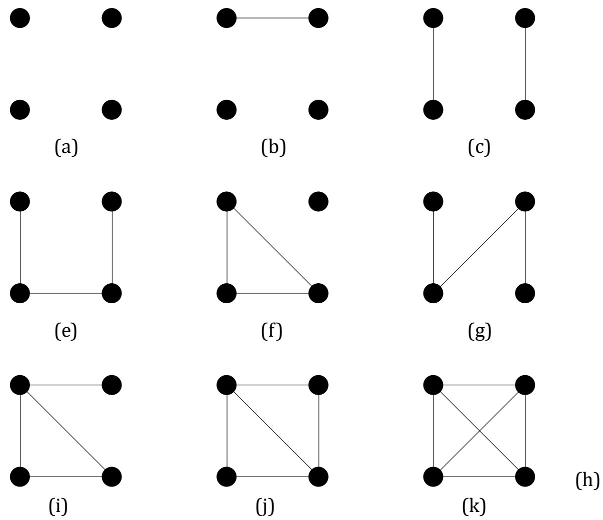


Figure 4: All loop-free non-isomorphic graphs on 4 vertices.

References

[1] Ralph P. Grimaldi. 2004. Discrete and Combinatorial Mathematics: An Applied Introduction (5th ed.). Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA.