CSC 226: Summer 2019: Lab 1

May 15, 2019

1 Asymptotic Notation

Let f and g be two functions that take integers as input and outputs real numbers.
Big-Oh: $f(n)$ is $O(g(n))$ if and only if there exists a *real* constant $c > 0$ and an integer $n_0 > 0$ such that $f(n) \leq c.g(n)$ $\forall n \geq n_0$.
Big-Omega: $f(n)$ is $\Omega(g(n))$ if and only if there exists a *real* constant $c > 0$ and an integer $n_0 > 0$ such that $f(n) \geq c.g(n)$ $\forall n \geq n_0$.
Big-Theta: $f(n)$ is $\Theta(g(n))$ if and only if $f(n)$ is $O(g(n))$ and $f(n)$ is $\Omega(g(n))$. Based on the definitions above, prove the followings.

1. $5n^2 + 6n + 12$ is $O(n^3)$. Ans: Here, $f(n) = 5n^2 + 6n + 12$ and $g(n) = n^3$. $5n^2 + 6n + 12 \leq 5n^3 + 6n^3 + 12n^3 = 23n^3$. For $c = 23$ and $n_0 = 1$, $f(n) \leq c.g(n)$, and therefore, $f(n)$ is $O(n^3)$.

2. $5n^2 + 6n + 12$ is $\Omega(n^2)$. Ans: Here, $f(n) = 5n^2 + 6n + 12$ and $g(n) = n^2$. $5n^2 + 6n + 12 \geq 5n^2$. For $c = 5$ and $n_0 = 1$, $f(n) \geq c.g(n)$, and therefore, $f(n)$ is $\Omega(n^2)$.

3. $5n^2 + 6n + 12$ is $\Theta(n^2)$. Ans: Here, $f(n) = 5n^2 + 6n + 12$ and $g(n) = n^2$. We have to prove that $f(n)$ is $O(n^2)$ and $f(n)$ is $\Omega(n^2)$. We have proved that $f(n)$ is $\Omega(n^2)$. We now prove that $f(n)$ is $O(n^2)$. $5n^2 + 6n + 12 \leq 5n^2 + 6n^2 + 12n^2 = 23n^2$. For $c = 23$ and $n_0 = 1$, $f(n) \leq c.g(n)$, and therefore, $f(n)$ is $O(n^2)$. By definition of Big-Theta, $f(n)$ is $\Theta(n^2)$.

2 Rules of Big-Oh

Prove the following theorems using the definition of Big-Oh from above.

1. **R1 (Scaling):** If $f(n)$ is $O(g(n))$ then $af(n)$ is $O(g(n))$, $a > 0$. Ans: If $f(n)$ is $O(g(n))$, then there exists $c > 0$ and $n_0 > 0$ such that $f(n) \leq c.g(n)$ for all $n \geq n_0$. Then $af(n) \leq ac.g(n) = c^0.g(n)$, where $c^0 = ac$. For $c^0 > 0$, since $a > 0$ and $c > 0$, and $n_0 > 0$, $af(n) \leq c^0 g(n)$. Therefore, $af(n)$ is $O(g(n))$.
2. **R4 (Transitivity):** If $d(n)$ is $O(f(n))$ and $f(n)$ is $O(g(n))$, then $d(n)$ is $O(g(n))$. Ans: If $d(n)$ is $O(f(n))$, then there exists $c > 0$ and $n_0 > 0$ such that $d(n) \leq c.f(n)$ for all $n \geq n_0$. Since $f(n)$ is $O(g(n))$, then there exists $a > 0$ and $n_1 > 0$ such that

$f(n) \leq a.g(n)$ for all $n \geq n_1$. Then $d(n) \leq ac.g(n) = c^0.g(n)$, where $c^0 = ac$, for $c^0 > 0$ and $\max\{n_0,n_1\} > 0$. Therefore, $d(n)$ is $O(g(n))$.

3. **R7:** $\log(n^x)$ is $O(\log n)$ for any fixed $x > 0$. Ans: Since $\log(n^x) = x\log n$, for $c = x > 0$ and $n_0 = 1$, $\log(n^x) \leq c\log n$. Therefore, $\log(n^x)$ is $O(\log n)$.

4. **R6:** n^x is $O(a^n)$ for any fixed $x > 0$ and $a > 1$. Ans: We have to show that a^n grows faster than n^x.

   The Limit Rule [3]: Suppose $\lim_{n\to\infty} \frac{f(n)}{g(n)} = L$ exists. Then,
   $$f(n) = \begin{cases} O(g(n)) & \text{if } L = 0 \\ \Theta(g(n)) & \text{if } 0 < L < \infty \\ \Omega(g(n)) & \text{if } L = \infty \end{cases}$$
   In our case, $f(n) = n^x$ and $g(n) = a^n$. To find L, we use L'Hopital rule.

   L'Hopital Rule [3]: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$.

   Now, $f^0 = xn^{x-1}$, $f^{00} = x(x-1)n^{x-2}$. In this way, $f^k = xx - 1(x-2)...2.1 = x!$ and $f^{k+1} = 0$. $g^0 = a^n \ln a$, $g^{00} = a^n(\ln a)^2$, ..., $g^{k+1} = a^n(\ln a)^{k+1}$.

   Therefore, $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f^{k+1}(n)}{g^{k+1}(n)} = \lim_{n\to\infty} \frac{0}{a^n(\ln a)^{k+1}} = 0$ and $f(n)$ is $O(g(n))$.

3 Permutations and Combinations

3.1 Poker Hands

If you have played poker, you probably know some or all the hands below [1]. You can choose 5 cards from 52 in $\binom{52}{5}$ ways. But how many of them would be a *Royal Flush* or a *Four-of-a-Kind*? Let's try to calculate the numbers for all the following hands.

1. **Royal Flush:** All five cards are of the same suit and are of the sequence 10 J Q K A. Ans:There are 4 suits and each suit can have exactly 1 royal flush suit. So, the number of possible royal flush is 4.

2. **Straight Flush:** All five cards are of the same suit and are sequential in rank. Ans: In one suit, there are 10 sequences possible, $A-4$, $1-5$, ..., $10-A$, since A can be assumed to be both higher than K and lower 1.
   Since, there are four suits, there are 40 straight flushes.
3. **Four-of-a-Kind:** Four cards are all of the same rank. Ans:To be of the same rank, the 4 cards have to come from 4 suits. There are 13 such sets of four

and we can choose 1 in $\binom{13}{1}$ ways. The fifth card in the hand can be any of the remaining 48 cards. So, number of possible hands is 13 × 48 = 624.

4. **Flush:** All five cards are of the same suit but not all sequential in rank. Ans: We can choose a suit from 4 in $\binom{4}{1}$= 4 ways, 5 cards can be chosen from 13 in $\binom{13}{5}$ ways. Therefore, the number is 4 $\binom{13}{5}$.

5. **Straight:** All five cards are sequential in rank but are not all of the same suit. Ans: There are 10 possible sequences as in Straight Flush. For each of the 5 cards, we can choose one from four suit. Then we have to deduct the 40 Straight Flushes. So, the total number is $\binom{10}{1}4^5 - 40 = 10200.$

6. **Three-of-a-Kind:** Three cards are all of the same rank and the other two are each of different ranks from the first three and each other. Ans: Choosing a rank has $\binom{13}{1}$= 13 ways. For a chosen rank, we can choose 3 cards out of 4 in $\binom{4}{3}$ = 4 ways. Then we choose the forth card from the remaining 48, and then one from remaining 44. Since the order of the last two cards do not matter, the number is 13 × 4 × 48 × 44/2

7. **One Pair:** Only two cards of the five are of the same rank with the other three cards all having different ranks from each other and from that of the pair. Ans:Choosing a pair can be done in $\binom{13}{1}\binom{4}{2}$ ways. The remaining three cards can be chosen in (48 × 44 × 40)/3! ways. So the total number is 13 $\times \binom{4}{2} \times (48 \times 44 \times 40)/6.$

8. **Two Pair:** Two pairs of two cards of the same rank (the ranks of each pair are different in rank, obviously, to avoid a Four-of-a-Kind). Ans:Choosing two pairs can be done in $\binom{13}{2}$ ways and choosing two cards for each rank can be done in $\binom{4}{2}$ ways. The remaining card can be chosen in 44 ways.
So the total number is $\binom{13}{2} \times \binom{4}{2} \times 44.$

9. **Full House:** A hand consisting of one pair and a three-of-a-kind of a different rank than the pair. Ans: We choose a rank for the pair in 13 ways and a rank for the three-of-a-kind in 12 ways. We did not divide by 2! to remove permutation, because pair of 2s and three 4s is different than a pair of 4s and three 2s. So we do not need to remove permutations. For the pair, we can choose two cards in $\binom{4}{2}$ ways and for the three-of-a-kind we can choose in $\binom{4}{3}$ ways. The total number is 13$\times 12 \times \binom{4}{2} \times \binom{4}{2}$.

3.2 Some other problems

1. Six friends want to play enough games of chess and every one wants toplay everyone else. How many games will they have to play? Ans: Two players can be chosen in $\binom{6}{2}$ ways.
2. There are five flavors of ice cream: banana, chocolate, lemon, strawberryand vanilla. We can have three scoops. How many variations will there be? [2] Ans: Check out the link.

References

[1] Jeff Duda, *Probabilities of Poker Hands with Variations.* http://www.meteor.iastate.edu/~jdduda/portfolio/492.pdf

[2] https://www.mathsisfun.com/combinatorics/combinationspermutations.html

[3] https://www.cs.auckland.ac.nz/courses/compsci220s1c/lectures/2016S1C/CS220Lecture04.pdf