1.

A	B	C	D	E	F	G
0	2	9	3	+∞	+∞	+∞
0	2	7	3	6	5	11
0	2	7	3	6	5	9
0	2	7	3	6	5	9
0	2	7	3	6	5	9
0	2	7	3	6	5	9

2.

$Let\ p\ be\ the\ shortest\ path\ from\ u\ to\ v\ on\ a\ DAG\ G$

$Then\ no\ other\ path\ from\ u\ to\ v\ has\ a\ total\ weight\ less\ than\ that\ of\ p$

$Let\ |p|\ denote\ the\ total\ weight\ of\ path\ p$

$We\ have: \forall\ path\ q\ from\ u\ to\ v\ on\ G, |q| \geq |p|$

$Consider\ G', which\ is\ a\ copy\ of\ G\ but\ with\ all\ edge\ weights\ negated$

$(every\ weight\ is\ multiplied\ by\ a\ factor\ of\ -1)$

$Because\ G'\ has\ the\ same\ set\ of\ vertices\ and\ edges, every\ path\ on\ G\ is\ also$

$valid\ on\ G', but\ with\ opposite\ total\ weights.$

$We\ have: \forall\ path\ q\ from\ u\ to\ v\ on\ G', |q| \leq |p|$

$\Rightarrow The\ shortest\ path\ from\ u\ to\ v\ on\ DAG\ G\ is\ the\ longest\ path\ from\ u\ to\ v$

$on\ G', a\ negated\ copy\ of\ G$

Now we just need a good way of finding the shortest path on a DAG.

Input: A DAG G, starting vertex u, destination v

Output: The longest path from u to v on G, represented by a stack, with u on top

Algorithm LongestPathOnDAG(G, u, v):

```
    // Initialization
    T ← a queue of topological ordering for G
    // Let G' be the copy of G with all weights negated
    G' ← G
    for each edge e in G' do:
        weight(e) ← - weight(e)
    end
    // Let Dis be an array, where Dis[i] is the known least cost from u to i so far
    for all Dis[i] do:
        Dis[i] ← ∞
    end
    Dis[u] ← 0
    // Let Pre be an array, where Pre[i] represents the previous vertex for i
    for all Pre[i] do:
        Pre[i] ← unknown
    end
    // Let Path be a stack for the longest path
    Path ← empty stack
```

```
// Find shortest path on G'
for each vertex i in T do:
    for each reachable neighbor j from i do:
        if Dis[i] + weight(i, j) < Dis[j] do:
            Dis[j] ← Dis[i] + weight(i, j)
            Pre[j] ← i
        end
    end
end
// Build stack
current ← v
while current != unknown do:
    Path.push(current)
    current ← Pre[current]
end
return Path
```

Run time analyze:

Topological sort: $O(|V| + |E|)$

Change edge weights: $O(|E|)$

Initialize Dis and Pre: $O(|V|)$

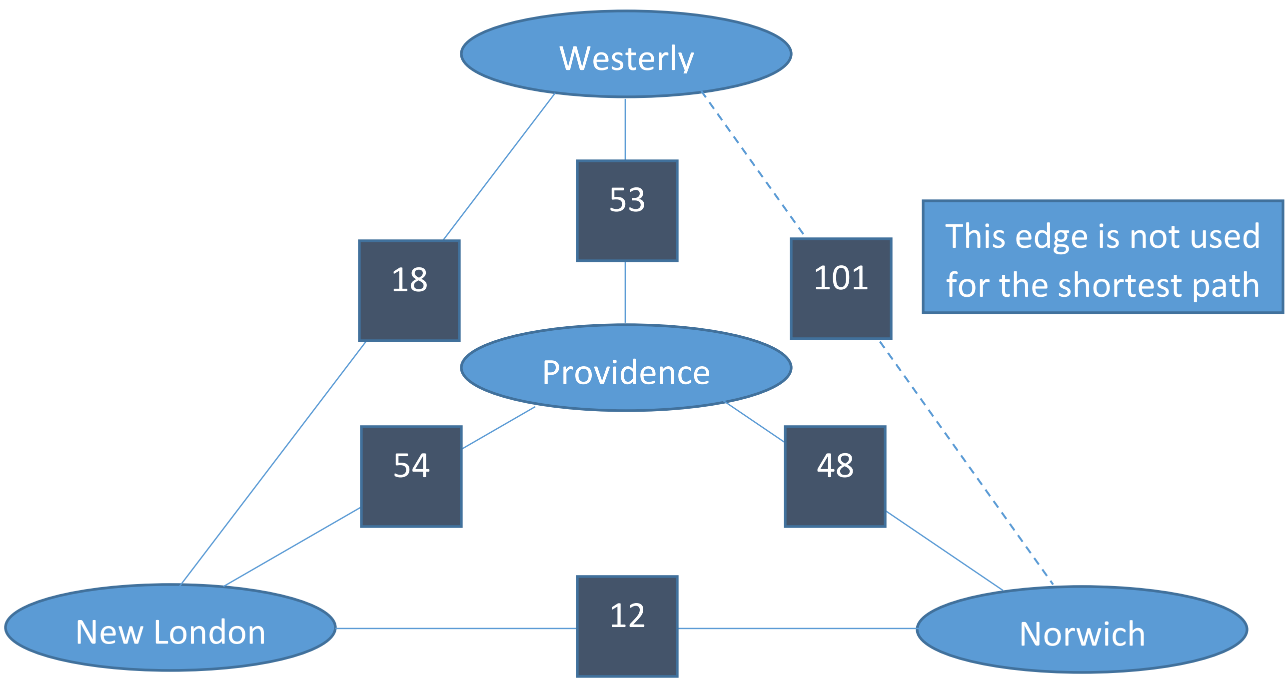
Finding shortest path on G': Simply a BFS with topological order, $O(|V| + |E|)$

Build stack: $O(|V|)$

Total run time: $O(|V| + |E| + |E| + |V| + |V| + |E| + |V|) \in O(|V| + |E|)$

3.

	Providence	Westerly	New London	Norwich
Providence	0	53	54	48
Westerly	53	0	18	30
New London	54	18	0	12
Norwich	48	30	12	0



4.

a.

$if\ M^2(i,j) = 1$

$\Rightarrow \exists\ k, M(i,k) \cdot M(k,j) = 1$

$\Rightarrow M(i,k) = 1 \wedge M(k,j) = 1$

$\Rightarrow (i,k),(k,j) \in G$

$\therefore M^2(i,j) = 1 \Rightarrow \exists\ k, (i,k),(k,j) \in G$

There is at least one intermediate vertex k, such that i-k-j is a path on G.

$if\ M^2(i,j) = 0$

$\Rightarrow \forall\ k, M(i,k) \cdot M(k,j) = 0$

$\Rightarrow M(i,k) = 0 \vee M(k,j) = 0$

$\Rightarrow (i,k) \notin G \vee (k,j) \notin G$

$\therefore M^2(i,j) = 0 \Rightarrow \forall\ k, (i,k) \notin G \vee (k,j) \notin G$

$\Leftrightarrow \nexists\ k, (i,k),(k,j) \in G$

There is no intermediate vertex k, such that i-k-j is a path on G.

b.

$if\ M^4(i,j) = 1$

$\Rightarrow \exists\ k, M^2(i,k) \cdot M^2(k,j) = 1$

$\Rightarrow M^2(i,k) = 1 \wedge M^2(k,j) = 1$

$\Rightarrow \exists\ l, (i,l),(l,k) \in G \wedge \exists\ m, (k,m),(m,j) \in G$

$\Rightarrow (i,l),(l,k),(k,m),(m,j) \in G$

$\therefore M^4(i,j) = 1 \Rightarrow \exists\ \{k,l,m\}, (i,l),(l,k),(k,m),(m,j) \in G$

There is at least one set of vertices {k, l, m}, such that i-l-k-m-j is a path on G.

$similarly$

$M^4(i,j) = 0 \Rightarrow \nexists \{k,l,m\}, (i,l), (l,k), (k,m), (m,j) \in G$

There is no such set of vertices {k, l, m}, such that i-l-k-m-j is a path on G.

$The\ pattern\ is\ obvious$

$Here\ is\ a\ simple\ proof\ by\ induction$

$Hypothesis$:

$\exists\ k, \forall\ 1 \le m \le k, M^m(i,j) = 1 \Leftrightarrow There\ is\ a\ path\ of\ length\ m\ from\ i\ to\ j$

$Base\ cases$:

$Already\ discussed\ for\ k = 1, k = 2, k = 4$

$Induction\ Steps$:

$M^{k+1} = M^k M$

$M^k M(i,j) = 1 \Leftrightarrow \exists\ l, M^k(i,l) = 1 \wedge M(l,j) = 1$

$\therefore There\ is\ a\ path\ of\ length\ k\ from\ i\ to\ l\ and\ a\ path\ of\ length\ 1\ from\ l\ to\ j$

$We\ take\ the\ path\ from\ i\ to\ l\ then\ go\ from\ l\ to\ n\ via\ the\ second\ path$

$This\ gives\ a\ path\ from\ i\ to\ j\ of\ length\ k + 1$

$By\ P.M.I$:

$\forall\ 1 \le k \le n, M^k(i,j) = 1 \Leftrightarrow There\ is\ a\ path\ of\ length\ n\ from\ i\ to\ j$

$\blacksquare$

$M^k(i, j)$ tells the existence of a path of length k from i to j on the graph G.

5.

This is very similar to the situation in question 4.

$If\ d = \infty, then\ \forall\ k, M(i,k) + M(k,j) = \infty$

$\Rightarrow M(i,k) = \infty \vee M(k,j) = \infty$

$We\ know\ M(i,i) = 0, so\ if\ M^2(i,j) = \infty, then\ M(i,j) = \infty$

$\Rightarrow there\ is\ no\ path\ of\ length \leq 2\ (number\ of\ edges)\ from\ i\ to\ j$

$If\ d\ is\ finite, then\ \exists\ k, M(i,k) + M(k,j) = d < \infty$

$\Rightarrow M(i,k) < \infty \wedge M(k,j) < \infty$

$If\ i \neq k, j \neq k, this\ means\ that\ i - k - j\ is\ a\ path\ of\ length\ 2$

$If\ i = k, or\ j = k, this\ means\ that\ i - j\ is\ an\ edge\ in\ G$

$\Rightarrow There\ is\ a\ path\ of\ length \leq 2\ from\ i\ to\ j$

$Total\ weight\ of\ path\ i - k - j = d\ (i\ may\ equal\ j\ and\ possibly\ just\ i - j)$

$and\ d\ is\ the\ minimum\ of\ all\ possible\ k's\ that\ satisfy\ M(i,k), M(k,j) < \infty$

$Therefore\ d\ is\ the\ weight\ of\ the\ lightest\ path\ of\ length \leq 2\ from\ i\ to\ j$

$Here\ is\ a\ simple\ proof\ by\ induction$

$Hypothesis$:

$\exists\ k, \forall\ 1 \leq m \leq k, M^m(i,j) = d$

$\Leftrightarrow The\ lightest\ path\ of\ length \leq m\ from\ i\ to\ j\ has\ weight\ d$

$Base\ cases$:

$Already\ discussed\ for\ k = 1, k = 2$

$Induction\ Steps$:

$M^{k+1} = M^k M$

$M^k M(i,j) = d \Leftrightarrow \nexists\, l, M^k(i,l) + M(l,j) < d$

$\therefore There\ is\ no\ vertex\ l, where\ the\ lightest\ path\ from\ i\ to\ l\ with\ length$

$\leq k\ has\ weight\ w_{il}, such\ that\ w_{il} + weight\ of\ (l,j) < d$

$\therefore M^{k+1}(i,j)\ is\ the\ least\ cost\ from\ i\ to\ j\ via\ some\ path\ of\ length \leq k+1$

$By\ P.M.I$:

$\forall\ 1 \leq k \leq n, M^k(i,j) = d$

$\Leftrightarrow The\ lightest\ path\ of\ length \leq k\ from\ i\ to\ j\ has\ weight\ d$

■

$M^k(i, j)$ is the least cost from i to j via some path of length (number of edges) ≤ k on G.