CSC 225 - Summer 2019

Hashing I

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The Pigeonhole Principle (1)

Exercise: Prove that in any group of 8 people, there must be at least two people whose birthdays will fall on the same day of the week in 2019.

The Pigeonhole Principle (2)

Exercise: Prove that in any group of 8 people, there must be at least two people whose birthdays will fall on the same day of the week in 2019.

Proof: Everyone in the group will have a birthday in 2019, and everyone's birthday must fall on one of the seven days of the week. Since there are only 7 days of the week, at least two birthdays must fall on the same day of the week.

The Pigeonhole Principle (3)

Exercise: Prove that in any group of 8 people, there must be at least two people whose birthdays will fall on the same day of the week in 2019.

Proof: Everyone in the group will have a birthday in 2019[a], and everyone's birthday must fall on one of the seven days of the week. Since there are only 7 days of the week, at least two birthdays must fall on the same day of the week.

[a] Arguments about February 29th birthdays notwithstanding

The Pigeonhole Principle (4)

Exercise: Prove that in any group of 8 people, there must be at least two people whose birthdays will fall on the same day of the week in 2019.

- The proof is almost trivial, but it relies on an intuitive assumption: If any collection of n items is assigned to a set of m possible outcomes and $n > m$, at least two items must share an outcome.
- This is called the **Pigeonhole Principle**.

The Pigeonhole Principle (5)

Theorem: (The Pigeonhole Principle)
Suppose a group of n pigeons occupy a collection of m pigeonholes. If $m < n$, then at least one pigeonhole must contain multiple pigeons.

The Pigeonhole Principle (6)

Exercise: Consider a group of 1000 people with English names (both first and last). Prove that at least two people must have the same first and last initials.

The Pigeonhole Principle (7)

Exercise: Consider a group of 1000 people with English names (both first and last). Prove that at least two people must have the same first and last initials.

Since everyone has an English name, there are 26 possible choices for each of their initials. The total number of possible first/last initial pairs is

$$26 \cdot 26 = 676$$

and since $676 < 1000$, by the Pigeonhole Principle, two people in the group must have the same first and last initials.

The Pigeonhole Principle (8)

Exercise: Consider an array A of 114 distinct values in the range $\{0, 1, 2, \ldots, 225\}$. Prove that A must contain a pair of values x and y such that $x + y = 225$.

The Pigeonhole Principle (9)

Exercise: Consider an array A of 114 distinct values in the range $\{0, 1, 2, \ldots, 225\}$. Prove that A must contain a pair of values x and y such that $x + y = 225$.

There are 113 possible pairs that add to 225:

$$(0, 225), \quad (1, 224), \quad \ldots, \quad (111, 114), \quad (112, 113).$$

Each of the 114 elements of A falls into one of the pairs. Since $114 > 113$, by the Pigeonhole Principle there must be two elements $A[i]$ and $A[j]$ which fall into the same pair, so $A[i] + A[j] = 225$.

Tables (1)

Exercise: Describe a dictionary implementation with fast searching and insertion, but where all of the elements are integers in the range $0 - 10$ and no duplicates are allowed.

Tables (2)

Index	**Value**
0	`false`
1	`true`
2	`false`
3	`true`
4	`false`
5	`false`
6	`false`
7	`false`
8	`true`
9	`true`
10	`true`

Using a simple array to create a table of `boolean` values with indices $1, 2, \ldots, 10$ would be a compact and efficient solution in this case. The table above represents a dictionary containing the elements $1, 3, 8, 9, 10$.

Tables (3)

Index	Value
0	`false`
1	`true`
2	`false`
3	`true`
4	`false`
5	`false`
6	`false`
7	`false`
8	`true`
9	`true`
10	`true`

This solution only applies when the elements are integers and the range is small. However, the advantages are easy to see: searching and inserting both require a simple array access.

Tables (4)

Exercise: Describe a dictionary implementation which can store n distinct keys in the range $\{0, 1, 2, 3, \ldots, 2n\}$ with $\Theta(1)$ FIND and INSERT operations.

Tables (5)

Index	**Value**
0	
1	
2	
3	
$\vdots$	$\vdots$
2n-1	
2n	

Since table lookups are $\Theta(1)$ and $2n \in \Theta(n)$, a table of size $2n$ provides the desired running times and requires $\Theta(n)$ space.

Tables (6)

Index	Value
0	
1	
2	
3	
⋮	⋮
2n-1	
2n	

In general, if a problem requires a dictionary with integer keys in a relatively constrained range, a simple table like the one above is often a good choice.

Tables (7)

Exercise: Describe a dictionary implementation which can store n distinct keys in the range $\{0, 1, 2, 3, \ldots, 2^n\}$ with $\Theta(1)$ $\textsc{Find}$ and $\textsc{Insert}$ operations.

Tables (8)

Index	Value
0	
1	
2	
3	
$\vdots$	$\vdots$
$2^n - 1$	
2^n	

In this case, a table would provide the desired running times for FIND and INSERT, but would require $\Theta(2^n)$ space and initialization time.

Tables (9)

Index	**Value**
0	
1	
2	
3	
$\vdots$	$\vdots$
$2^n - 1$	
2^n	

The vast majority of the space would be wasted, since the table would only contain n elements.

Compressed Tables (1)

Index	Value
0	
1	
2	
3	
⋮	⋮
2n-1	
2n	

Idea: Instead of mapping an element *i* to slot *i* of the array, use a different indexing scheme which wastes less space.

Compressed Tables (2)

Index	Value
0	
1	
2	
3	
⋮	⋮
2n-1	
2n	

Since most of the table is empty anyway, it should be possible to compress all of the data into a small number of indices.

Compressed Tables (3)

Index	Value
0	10
1	
2	92
3	
4	
5	
6	6
7	17
8	
9	29

For example, the elements $6, 10, 17, 29, 92$, which are in the range $0 - 99$, can be inserted into a table of size 10 by using the last digit as the index.

Compressed Tables (4)

Index	Value
0	10
1	
2	92
3	
4	
5	
6	6
7	17
8	
9	29

In this example, the index of a particular element k is computed with the formula

$$g(k) = k \bmod 10$$

Compressed Tables (5)

Index	Value
0	10
1	
2	92
3	
4	
5	
6	6
7	17
8	
9	29

$$g(k) = k \bmod 10$$

- **Task**: Search for the element 15
- Since $g(k) = 5$, index 5 is inspected. Since it is empty, the value 15 cannot be in the data structure.

Compressed Tables (6)

Index	Value
0	10
1	
2	92
3	
4	
5	
6	6
7	17
8	
9	29

$$g(k) = k \bmod 10$$

- **Task**: Search for the element 16
- Since $g(k) = 6$, index 6 is inspected. Index 6 is not empty, but it does not contain 16, so the element is cannot be in the data structure.

Compressed Tables (7)

Index	**Value**
0	10
1	
2	92
3	
4	
5	
6	6
7	17
8	
9	29

$$g(k) = k \bmod 10$$

- **Task**: Insert the value 16.
- Since $g(k) = 6$, the key 16 should be inserted at index 6.
- But index 6 is full, so insertion is impossible without discarding an existing element.

Compressed Tables (8)

Index	Value
0	10
1	
2	92
3	
4	
5	
6	6
7	17
8	
9	29

- **General Issue**: What if every key in the dictionary has the same last digit?
- In practice, it is very common for a data structure to contain a large collection of very similar data.

Compressed Tables (9)

Index	Value
0	10
1	
2	92
3	
4	
5	
6	6
7	17
8	
9	29

The compression approach is useful for reducing table size, but not for spreading similar data out evenly among indices.

Hash Tables (1)

Index	Value
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Idea: What if the indexing scheme incorporates both digits?

More generally, what if the indexing scheme is designed to incorporate all aspects of each element into its assigned index?

Hash Tables (2)

Index	Value
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Exercise: Insert the elements $1, 11, 21, 31$ into the table using the **hash function**

$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

to determine the index of each key.

Hash Tables (3)

Index	Value
0	
1	1
2	11
3	
4	
5	
6	
7	
8	
9	

$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(1) = (1 + 0) \bmod 10 = 1$
- $h(11) = (11 + 1) \bmod 10 = 2$

Hash Tables (4)

Index	Value
0	
1	1
2	11
3	21
4	31
5	
6	
7	
8	
9	

$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(21) = (21 + 2) \bmod 10 = 3$
- $h(31) = (31 + 3) \bmod 10 = 4$

Hash Tables (5)

Index	**Value**
0	
1	1
2	11
3	21
4	31
5	
6	
7	
8	
9	

$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

Using the new function $h(k)$, a set of keys with the same last digit are mapped to different indices of the table.

Hash Tables (6)

Index	Value
0	
1	1
2	11
3	21
4	31
5	
6	
7	
8	
9	

$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

However, there are still multiple keys which receive the same index (such as $2, 11$ and 20).

Hash Tables (7)

A **hash table** is a data structure which implements the dictionary ADT using a table of size M and a hash function $h(k)$ which maps each key k (which may be be of any type) to values in the range $0, 1, \ldots, M - 1$.

The set of all possible input keys k is called the **universe** of keys and is denoted by U. The actual set of keys inserted into the table is not necessarily equal to U.

Hash tables are only effective if the structure and size of the input data is reasonably well understood. In particular, the universe U is normally much larger than the table size, so it is necessary to assume that only a small fraction of possible keys will actually be inserted into the table.

Hash Tables (8)

Index	Value
0	
1	1
2	11
3	21
4	31
5	
6	
7	
8	
9	

$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- **Task**: Insert the value 22 into the hash table above.
- $h(22) = (22 + 2) \bmod 10 = 4$.
- The insertion results in a **collision**.

Collisions

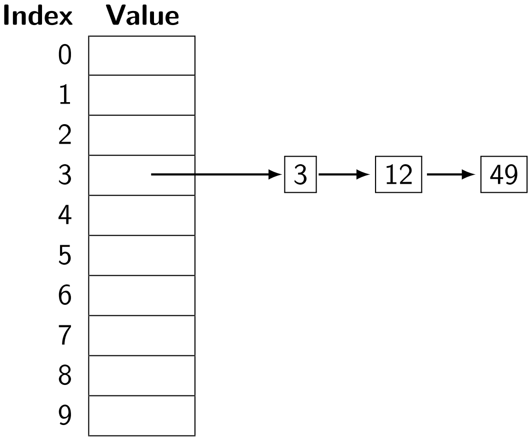
A **collision** in a hashing scheme is a pair of keys $k_1, k_2 \in U$ such that

$$h(k_1) = h(k_2)$$

In cases where the table size M is smaller than the number of possible keys in U, collisions are unavoidable due to the Pigeonhole Principle (since the exact input data is not known in advance).

Since collisions are unavoidable, hash tables must include a **collision resolution** scheme to accommodate keys with equal hash values.

Chaining (1)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

Idea: Instead of each index containing a single element, store a linked list of elements in each position of the table (to allow multiple elements to share each index). This is called **chaining**.

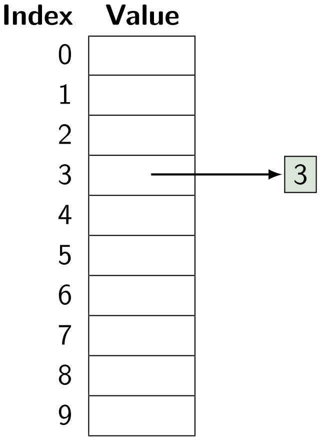
Chaining (2)

Index	**Value**
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

Example: Insert the sequence $3, 14, 15, 9, 26, 5, 35$ into the hash table above using chaining.

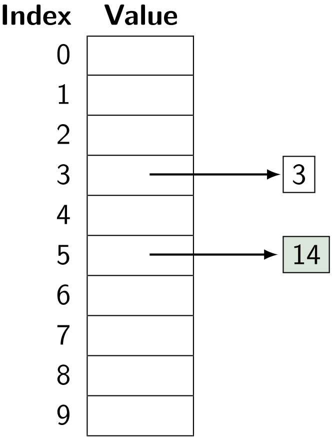
Chaining (3)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(3) = 3$

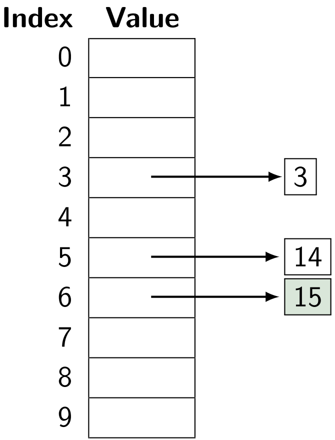
Chaining (4)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(14) = 5$

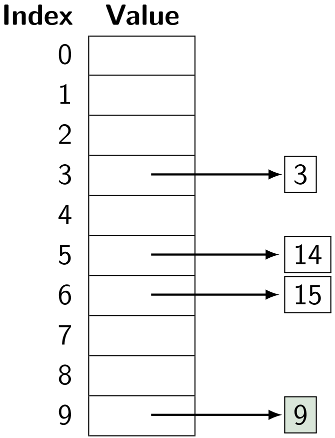
Chaining (5)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(15) = 6$

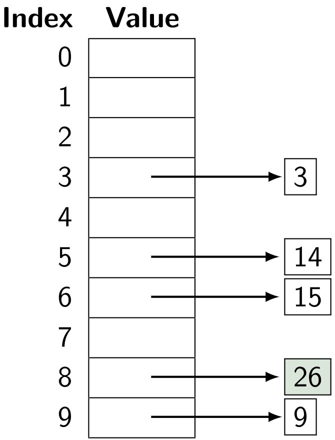
Chaining (6)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(9) = 9$

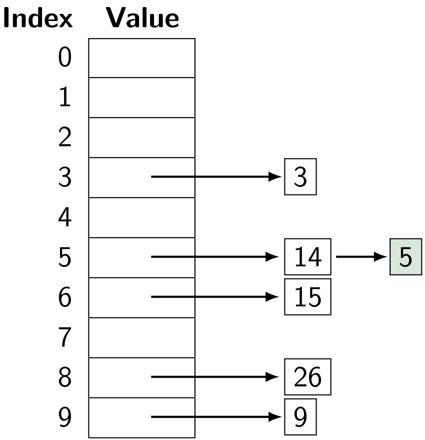
Chaining (7)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(26) = 8$

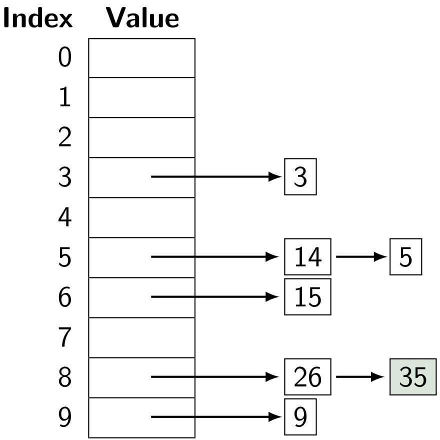
Chaining (8)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(5) = 5$

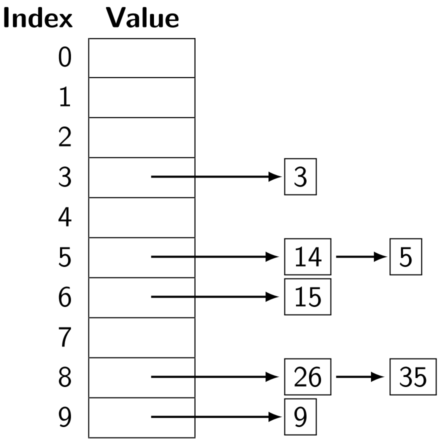
Chaining (9)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(35) = 8$

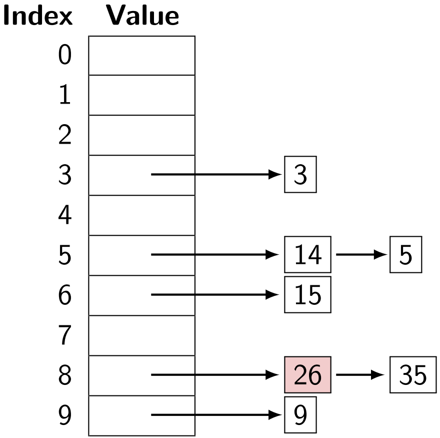
Chaining (10)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- **Task**: $\textsc{Find}(35)$
- First, evaluate the hash function: $h(35) = 8$.

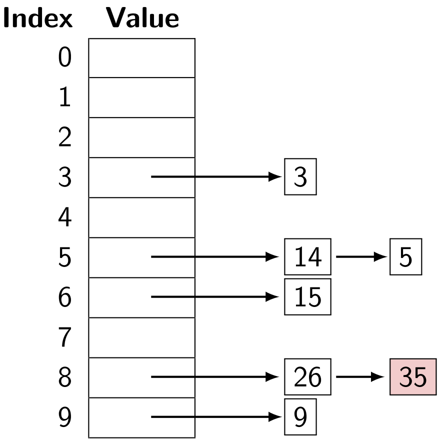
Chaining (11)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- **Task**: $\textsc{Find}(35)$
- Search through the list at index 8.

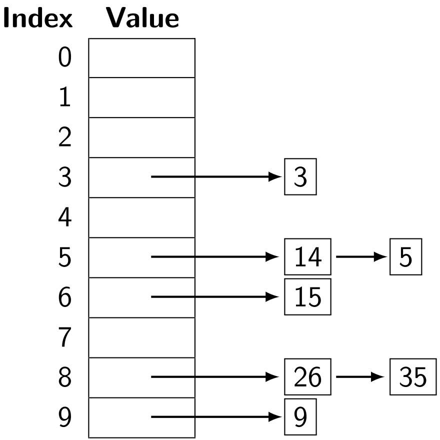
Chaining (12)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- **Task**: $\textsc{Find}(35)$
- If the key is found, return it (and any other associated data).

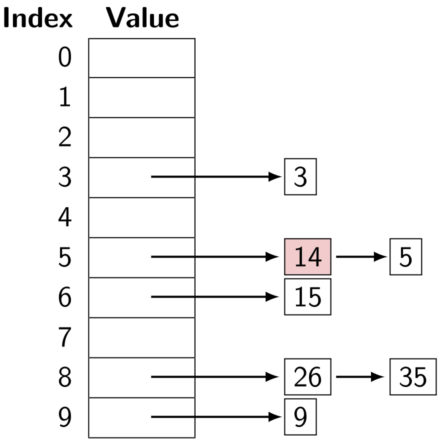
Chaining (13)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- **Task**: $\textsc{Find}(23)$
- The hash code for 23 is $h(23) = 5$.

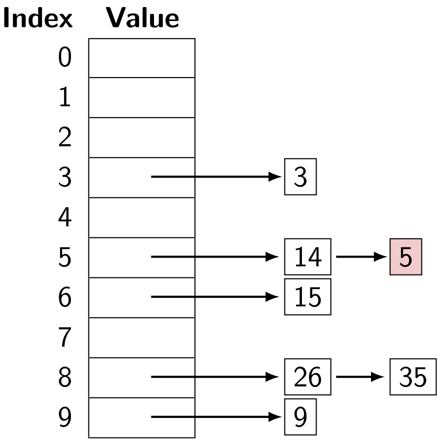
Chaining (14)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- The entire contents of the list at index 5 must be inspected during an unsuccessful query.

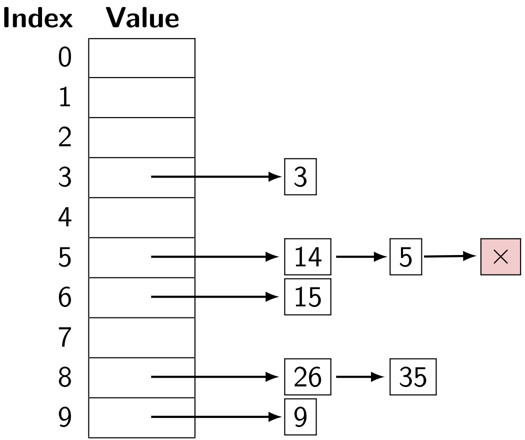
Chaining (15)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- The entire contents of the list at index 5 must be inspected during an unsuccessful query.

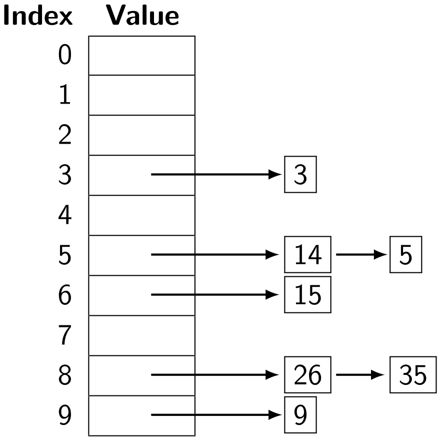
Chaining (16)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- If the key is not found among the items in the list, the key is not in the table.

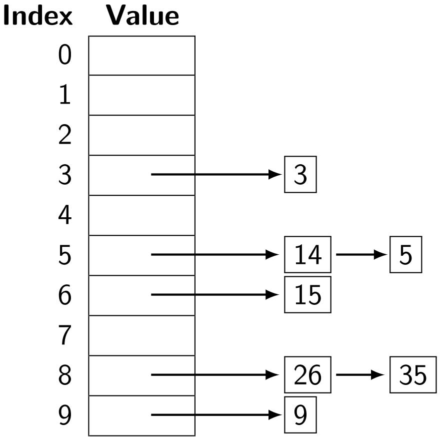
Chaining (17)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

Hash tables can also be used with non-numerical data, by devising a hash function to map the element type (e.g. strings) to integer indices.

Chaining (18)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

Question: What is the worst-case running time of FIND in a hash table with chaining?

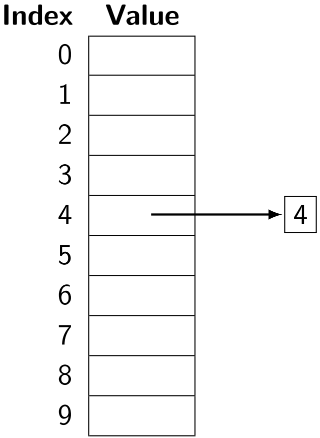
Clustering (1)

Index	Value
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- **Exercise**: Insert the values $4, 13, 22, 31$ into the hash table above, using chaining for collision resolution.

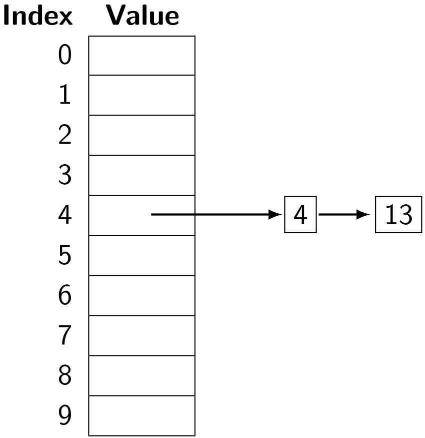
Clustering (2)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(4) = 4$

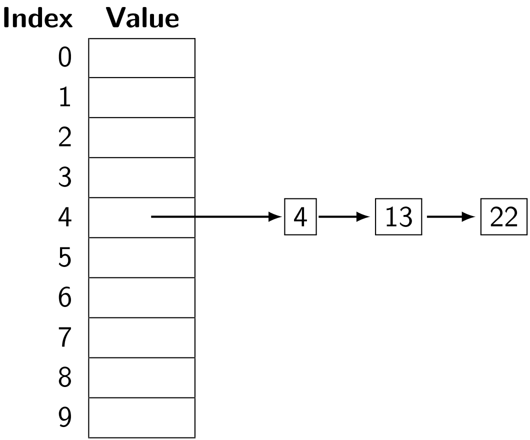
Clustering (3)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(13) = 4$

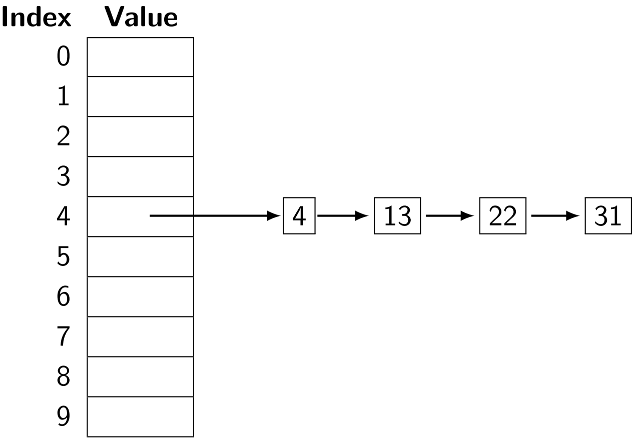
Clustering (4)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(22) = 4$

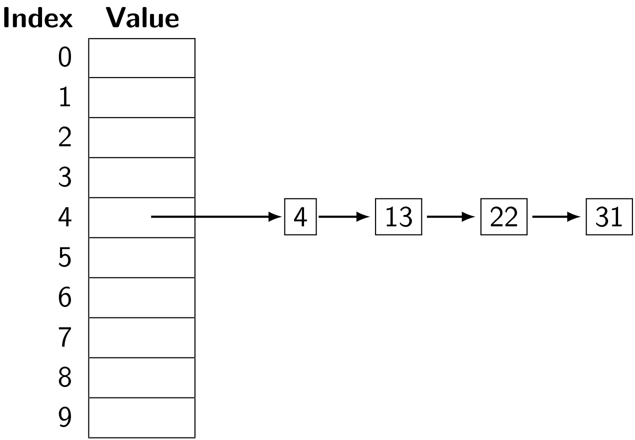
Clustering (5)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- $h(31) = 4$

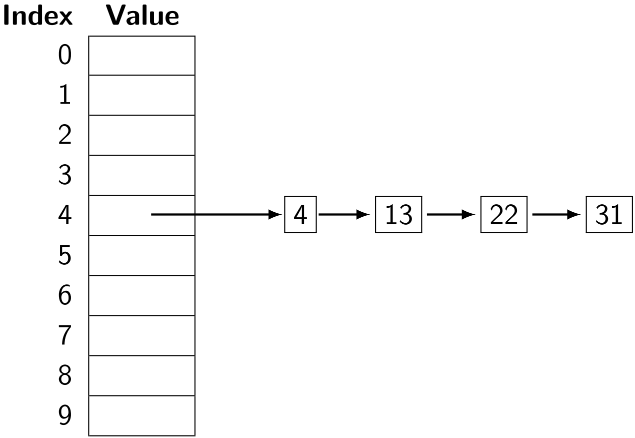
Clustering (6)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- The input data in this example exhibits **clustering** with this hash function.

Clustering (7)



$$h(k) = (k + \lfloor k/10 \rfloor) \bmod 10$$

- Since the data may be clustered, the worst case running time of FIND is $\Theta(n)$.

Clustering (8)

There is no general way to prevent clustering unless all of the input data is known in advance. Collisions are always inevitable, so the goal of a hash function designer must be to minimize the likelihood that similar keys are hashed to similar values.

Ideally, the result of a hash function on a given input sequence should be **indistinguishable from random numbers**. In practice, this is an extremely difficult standard to meet. Hash functions whose output appears to be random are useful for some fields, especially cryptography (CSC 429).

We will focus on choosing hash functions which distribute inputs among indices with equal probability, and which do not have any obvious clustering behavior.

String Hashing (1)

Index	Value
0	
1	
2	
3	
4	
5	
6	

- **Exercise**: Design a hashing scheme to insert the strings below into the hash table above (with size 7).

`ocean   boat   tide   sand   canoe`

String Hashing (2)

Index	**Value**
0	
1	
2	
3	
4	
5	
6	

- Since the output of a hash function must be an integer index, the first challenge is finding a way to convert compound data types (like strings or structures) into a single hash code.
- The characters of a string are represented by numbers, so we can define a function which uses the numerical values of each character to determine the hash code of the string.

String Hashing (3)

Index	Value
0	
1	
2	
3	
4	
5	
6	

- **First try**: For a string $s = c_1 c_2 c_3 \ldots c_k$, define

$$h(s) = (c_1 + c_2 + \ldots c_k) \bmod 7$$

- We will see that this is a **bad idea**.

String Hashing (4)

Index	**Value**
0	
1	
2	
3	
4	
5	
6	

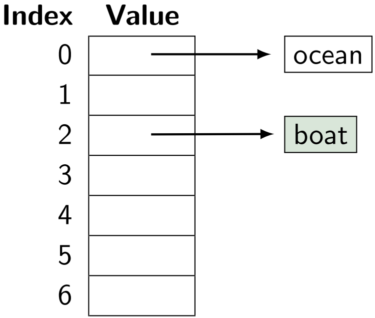
The numerical values of characters can vary (and is basically irrelevant to the theoretical discussion of hashing). However, most machines use the ASCII character set, in which the lowercase letters are numbered starting at 97, so this example will use ASCII values.

String Hashing (5)

Index	**Value**
0	→ ocean
1	
2	
3	
4	
5	
6	

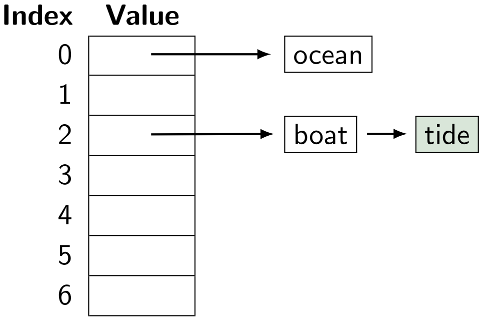
$$\begin{aligned} h(\text{`ocean'}) &= (\text{`o'} + \text{`c'} + \text{`e'} + \text{`a'} + \text{`n'}) \bmod 7 \\ &= (111 + 99 + 101 + 97 + 110) \bmod 7 \\ &= 518 \bmod 7 \\ &= 0 \end{aligned}$$

String Hashing (6)



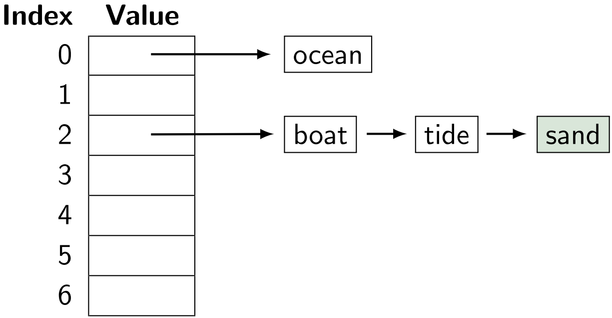
$$
\begin{aligned}
h(\text{‘boat’}) &= (\text{‘b’} + \text{‘o’} + \text{‘a’} + \text{‘t’}) \bmod 7 \\
&= (98 + 111 + 97 + 116) \bmod 7 \\
&= 422 \bmod 7 \\
&= 2
\end{aligned}
$$

String Hashing (7)



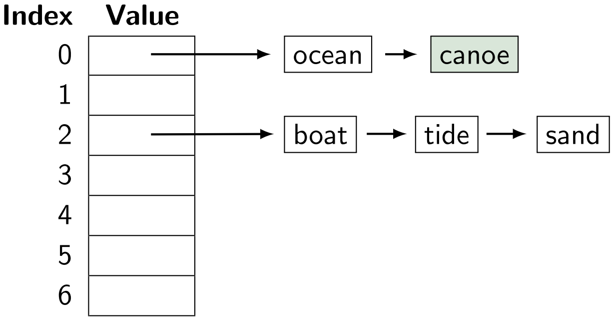
$$
\begin{aligned}
h(\text{'tide'}) &= (\text{'t'} + \text{'i'} + \text{'d'} + \text{'e'}) \bmod 7 \\
&= (116 + 105 + 100 + 101) \bmod 7 \\
&= 422 \bmod 7 \\
&= 2
\end{aligned}
$$

String Hashing (8)



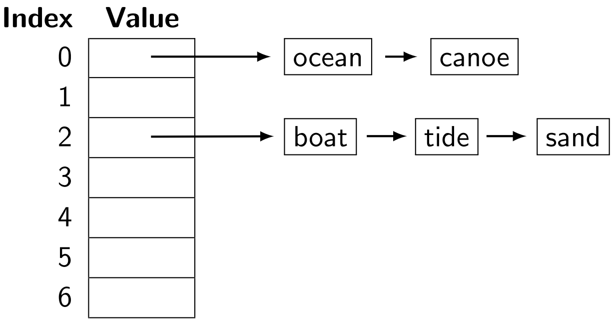
$$
\begin{aligned}
h(\text{‘sand’}) &= (\text{‘s’} + \text{‘a’} + \text{‘n’} + \text{‘d’}) \bmod 7 \\
&= (115 + 97 + 110 + 100) \bmod 7 \\
&= 422 \bmod 7 \\
&= 2
\end{aligned}
$$

String Hashing (9)



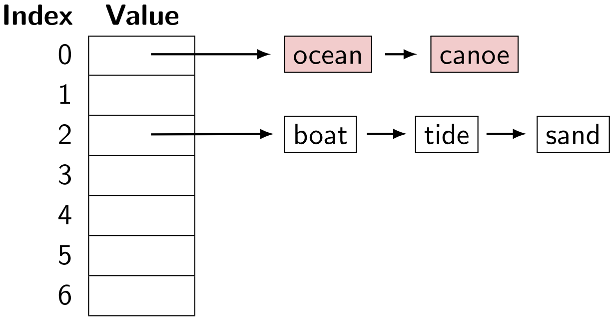
$$
\begin{aligned}
h(\text{`canoe'}) &= (\text{`c'} + \text{`a'} + \text{`n'} + \text{`o'} + \text{`e'}) \bmod 7 \\
&= (99 + 97 + 110 + 111 + 101) \bmod 7 \\
&= 518 \bmod 7 \\
&= 0
\end{aligned}
$$

String Hashing (10)



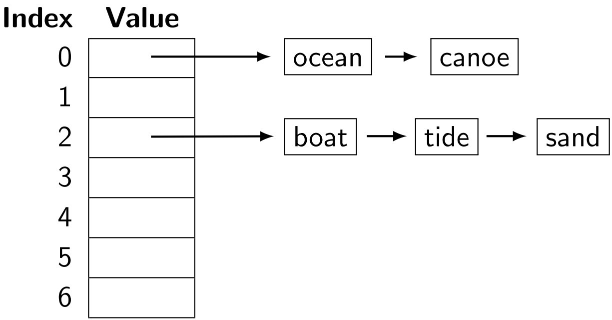
- For any hash function $h(k)$, there will exist input sequences which cause extreme clustering.
- The clustering in index 2 has no obvious cause.
- The two items in index 0 are anagrams, which is evidence of a serious deficiency of the hash function.

String Hashing (11)



$$h(\text{‘ocean'}) = (111 + 99 + 101 + 97 + 110) \bmod 7$$
$$h(\text{‘canoe'}) = (99 + 97 + 110 + 111 + 101) \bmod 7$$

String Hashing (12)



- Any two words with the same letters will be hashed to the same value, since the hash function does not incorporate any position information into the hash value.

String Hashing (13)

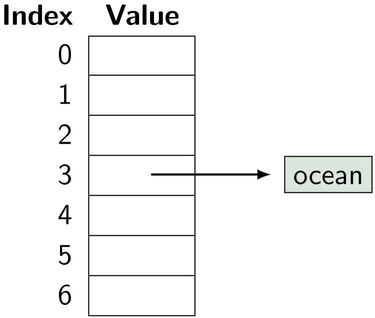
Index	**Value**
0	
1	
2	
3	
4	
5	
6	

- **Second try**: For a string $s = c_1 c_2 c_3 \dots c_k$, define

$$h(s) = (c_1^1 + c_2^2 + c_3^3 + \dots c_k^k) \bmod 7$$

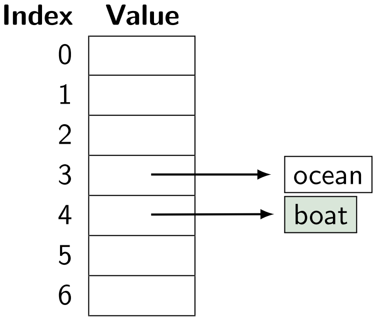
- This hash function weights each letter differently.

String Hashing (14)



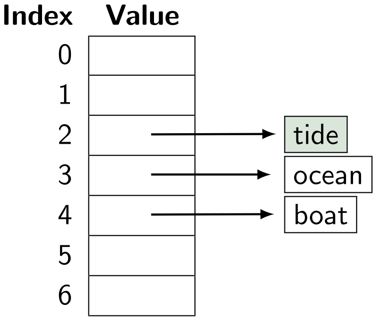
$$
\begin{aligned}
h(\text{'ocean'}) &= ((\text{'o'})^1 + (\text{'c'})^2 + (\text{'e'})^3 + (\text{'a'})^4 + (\text{'n'})^5) \bmod 7 \\
&= (111 + 9801 + 1030301 + 88529281 + 16105100000) \bmod 7 \\
&= 16194669494 \bmod 7 \\
&= 3
\end{aligned}
$$

String Hashing (15)



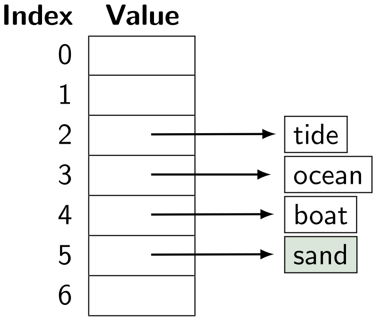
$$
\begin{aligned}
h(\text{`boat'}) &= ((\text{`b'})^1 + (\text{`o'})^2 + (\text{`a'})^3 + (\text{`t'})^4) \bmod 7 \\
&= (98 + 12321 + 912673 + 181063936) \bmod 7 \\
&= 181989028 \bmod 7 \\
&= 4
\end{aligned}
$$

String Hashing (16)



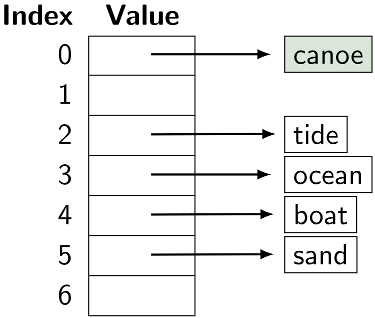
$$
\begin{aligned}
h(\text{'tide'}) &= ((\text{'t'})^1 + (\text{'i'})^2 + (\text{'d'})^3 + (\text{'e'})^4) \bmod 7 \\
&= (116 + 11025 + 1000000 + 104060401) \bmod 7 \\
&= 105071542 \bmod 7 \\
&= 2
\end{aligned}
$$

String Hashing (17)



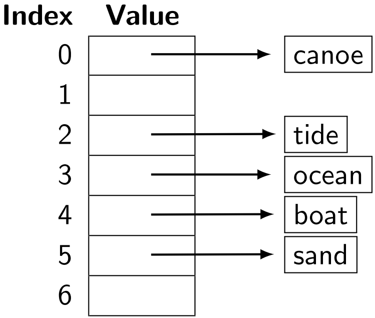
$$
\begin{aligned}
h(\text{'sand'}) &= ((\text{'s'})^1 + (\text{'a'})^2 + (\text{'n'})^3 + (\text{'d'})^4) \bmod 7 \\
&= (115 + 9409 + 1331000 + 100000000) \bmod 7 \\
&= 101340524 \bmod 7 \\
&= 5
\end{aligned}
$$

String Hashing (18)



$$
\begin{aligned}
h(\text{`canoe'}) &= ((\text{`c'})^1 + (\text{`a'})^2 + (\text{`n'})^3 + (\text{`o'})^4 + (\text{`e'})^5) \bmod 7 \\
&= (99 + 9409 + 1331000 + 151807041 + 10510100501) \bmod 7 \\
&= 10663248050 \bmod 7 \\
&= 0
\end{aligned}
$$

String Hashing (19)



- The second hash function produces an optimal distribution.
- With a well-chosen hash function, hash tables have $\Theta(1)$ expected running times for both Insert and Find.
- Experimental analysis is the often best way to find the best hash function for a particular task.

Load Factor

Some degree of clustering is inevitable with any hashing scheme, especially when the table size is not much larger than the number of entries.

The **load factor** of a hash table of size M containing n keys is

$$\alpha = \frac{n}{M}.$$

Rule of thumb: Choose the table size to keep $\alpha \leq 0.6$.