1. [4 marks]
   a) 9! = 362880
   b) 5! * 4! = 120 * 24 = 2880
   c) 5! * 5! = 120 * 120 = 14400
   d) 5! * 4! * 2 = 5760
2. [4 marks]
   a) $\binom{13}{2}\binom{39}{3} = 78 * 9139 = 712842$
   b) $\binom{13}{2}\binom{39}{3} + \binom{13}{1}\binom{39}{4} + \binom{13}{0}\binom{39}{5} = 78 * 9139 + 13 * 82251 + 1 * 575757 = 2357862$
   c) $\binom{13}{3}\binom{13}{2} = 286 * 78 = 22308$
   d) $\binom{12}{1}\binom{12}{2}\binom{2}{2} = 12 * 66 * 1 = 792$
3. [3 + 3 marks]

   a) Let n be a positive integer with n>2, then

$$\binom{n}{2} + \binom{n-1}{2} = \frac{n!}{2!\,(n-2)!} + \frac{(n-1)!}{2!\,(n-3)!}$$

$$= \frac{n!}{2(n-2)!} + \frac{(n-2)(n-1)!}{(n-2)2(n-3)!}$$

$$= \frac{n! + (n-2)(n-1)!}{2(n-2)!}$$

$$= \frac{\big(n + (n-2)\big)(n-1)!}{2(n-2)!}$$

$$= \frac{2(n-1)(n-1)!}{2(n-2)!}$$

$$= \frac{(n-1)(n-1)(n-2)!}{(n-2)!}$$

$$= (n-1)^2$$

which is a perfect square for all $n > 2$.

b) From the binomial theorem, we know that for any real numbers a and b, and any positive integer n,

$$(a+b)^n = \binom{n}{0}a^0b^n + \binom{n}{1}a^1b^{n-1} + \binom{n}{2}a^2b^{n-2} + \cdots + \binom{n}{n}a^nb^0$$

So, if we let $a = -x$ and $b = 1 + x$ in the binomial theorem, then

$$(-x + (1+x))^n = (1)^n = 1$$

$$= \binom{n}{0}(-x)^0(1+x)^n + \binom{n}{1}(-x)^1(1+x)^{n-1} + \binom{n}{2}(-x)^2(1+x)^{n-2} + \cdots + \binom{n}{n}(-x)^n(1+x)^0$$

$$= (1+x)^n - \binom{n}{1}x(1+x)^{n-1} + \binom{n}{2}x^2(1+x)^{n-2} - \cdots + (-1)^n\binom{n}{n}x^n$$

4. [2 + 3 marks]

a) Here we are counting combinations with repetition where n = 4 and r = 32, thus

$$\binom{4+32-1}{32} = \binom{35}{32} = \frac{35!}{32!\,3!} = 6545$$

b) For this one we start with $x1$ = 1, $x2$ = 2, $x3$ = 3, $x4$ = 4, leaving r = 32-1-2-3-4 = 22, thus

$$\binom{4+22-1}{22} = \binom{25}{22} = 2300$$

5. [2 + 2 + 2 marks]

Let $(A, \mathfrak{R}_1)$ and $(B, \mathfrak{R}_2)$ be two posets. By definition of partial orders we know the following to be true,

a) For all $a \in A$ and $b \in B$, $(a, a) \in \mathfrak{R}_1$ and $(b, b) \in \mathfrak{R}_2$, respectively.

b) For all $a, x \in A$, if $(a, x) \in \mathfrak{R}_1$ and $(x, a) \in \mathfrak{R}_1$ then $a = x$.
For all $b, y \in B$, if $(b, y) \in \mathfrak{R}_2$ and $(y, b) \in \mathfrak{R}_2$ then $b = y$.

c) For all $a, b, c \in A$, if $(a, b) \in \mathfrak{R}_1$ and $(b, c) \in \mathfrak{R}_1$ then $(a, c) \in \mathfrak{R}_1$.
For all $x, y, z \in B$, if $(x, y) \in \mathfrak{R}_2$ and $(y, z) \in \mathfrak{R}_2$ then $(x, z) \in \mathfrak{R}_2$.

Now, let $\mathfrak{R}$ be a relation on $A \times B$ defined by

$$\mathfrak{R} = \{((a, b), (x, y)) | (a, x) \in \mathfrak{R}_1, (b, y) \in \mathfrak{R}_2\}$$

To show that $\mathfrak{R}$ is a partial order on $A \times B$ we must show that it is a) reflexive, b) antisymmetric, and c) transitive.

a) Let $(a, b) \in A \times B$. Since $(a, a) \in \mathfrak{R}_1$ and $(b, b) \in \mathfrak{R}_2$ from above, then by definition of $\mathfrak{R}$, $((a, b), (a, b)) \in \mathfrak{R}$. Thus, $\mathfrak{R}$ is reflexive.

b) Let $(a, b), (x, y) \in A \times B$ and suppose that $((a, b), (x, y))$ and $((x, y), (a, b))$ are in $\mathfrak{R}$. By definition of $\mathfrak{R}$, it is true that (a, x) and (x, a) are in $\mathfrak{R}_1$ and (b, y) and (y, b) are in $\mathfrak{R}_2$. Because $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are both antisymmetric, we may conclude that $a = x$ and $b = y$, respectively. Therefore, $(a, b) = (x, y)$ and thus $\mathfrak{R}$ is antisymmetric.

c) Let $(a, b), (x, y), (c, d) \in A \times B$ and suppose that $((a, b), (x, y))$ and $((x, y), (c, d))$ are in $\mathfrak{R}$. By definition of $\mathfrak{R}$, it is true that (a, x) and (x, c) are in $\mathfrak{R}_1$ and (b, y) and (y, d) are in $\mathfrak{R}_2$. Because $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are both transitive, we may conclude that $(a, c) \in \mathfrak{R}_1$ and $(b, d) \in \mathfrak{R}_2$, respectively. Therefore, by the definition of $\mathfrak{R}$, this implies that $((a, b), (c, d)) \in \mathfrak{R}$ and thus $\mathfrak{R}$ is transitive.

By the above properties, $\mathfrak{R}$ is a partial order on $A \times B$.