CSC 225 - Summer 2019

Algorithm Analysis III

Bill Bird

Department of Computer Science
University of Victoria

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Comparing Algorithms (1)

We have seen how to find an expression for the worst case running time of an algorithm by counting primitive operations. Suppose we design two algorithms with running times

$$f_1(n) = 3n\log_2 n + \frac{1}{10^7}n^3 + 5n$$
$$f_2(n) = \frac{1}{2}n^2 - 5n + 100$$

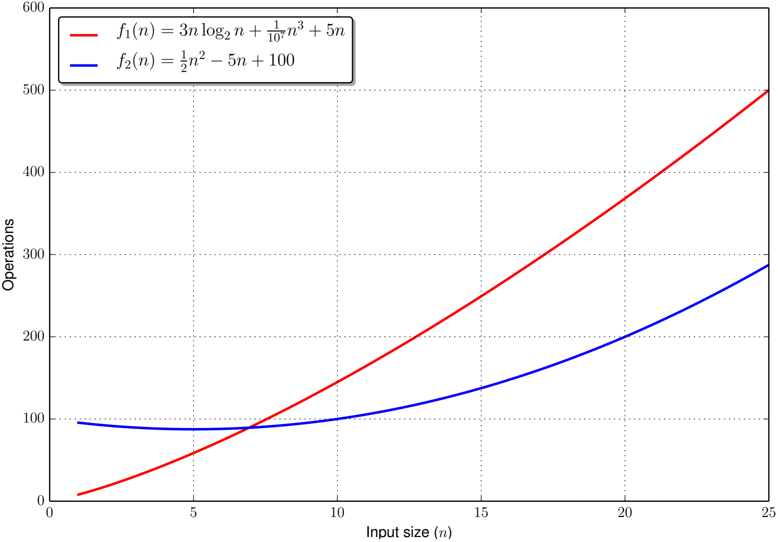
We want to compare these two algorithms and decide which one is faster. Essentially, we want to say

$$f_1(n) \leq f_2(n)$$

or

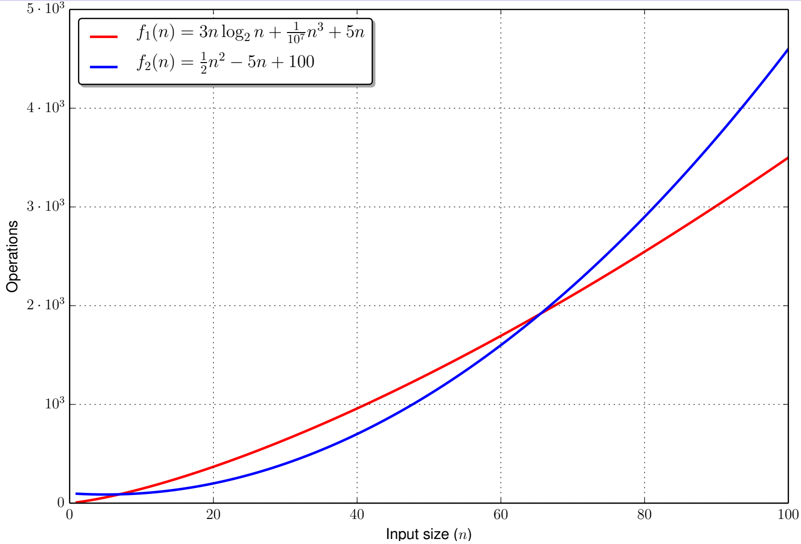
$$f_2(n) \leq f_1(n)$$

Comparing Algorithms (2)



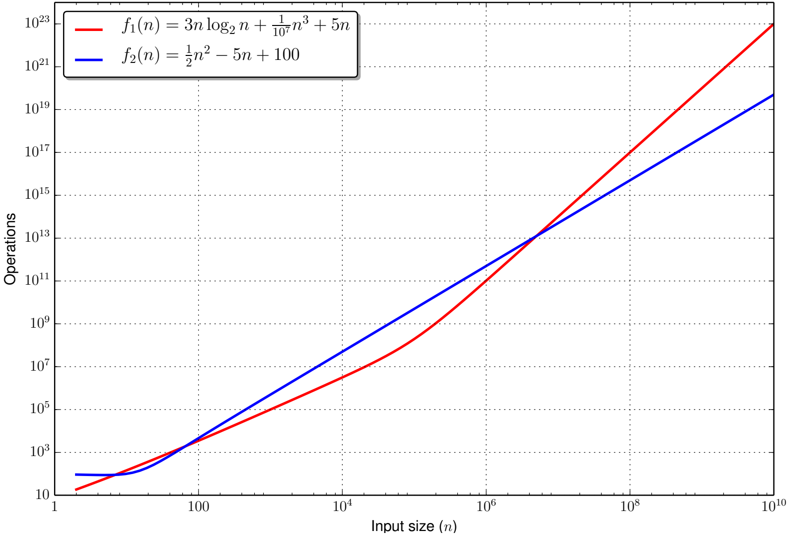
However, the $\leq$ operator only applies if one function is *always* less than the other.

Comparing Algorithms (3)



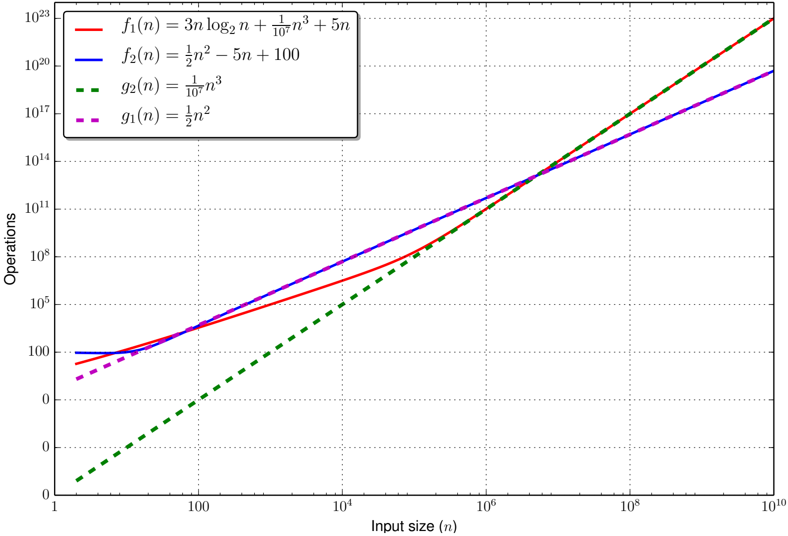
And, as the input size *n* gets larger, which algorithm is faster may change.

Comparing Algorithms (4)



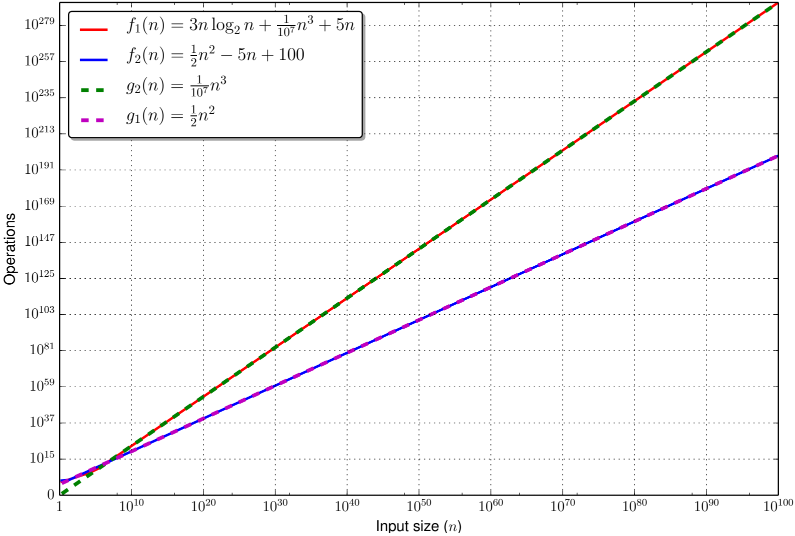
And, as the input size *n* gets larger, which algorithm is faster may change.

Comparing Algorithms (5)

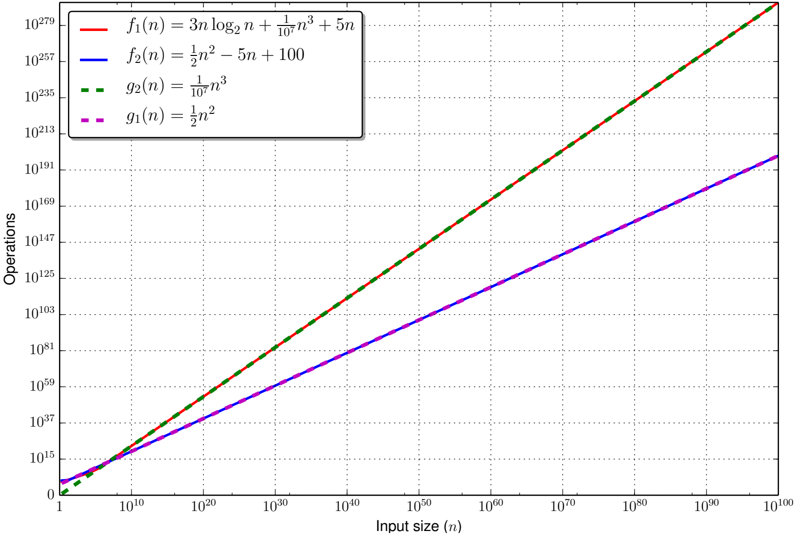


It turns out that most terms in the functions f_1 and f_2 are irrelevant at large enough scales.

Comparing Algorithms (6)

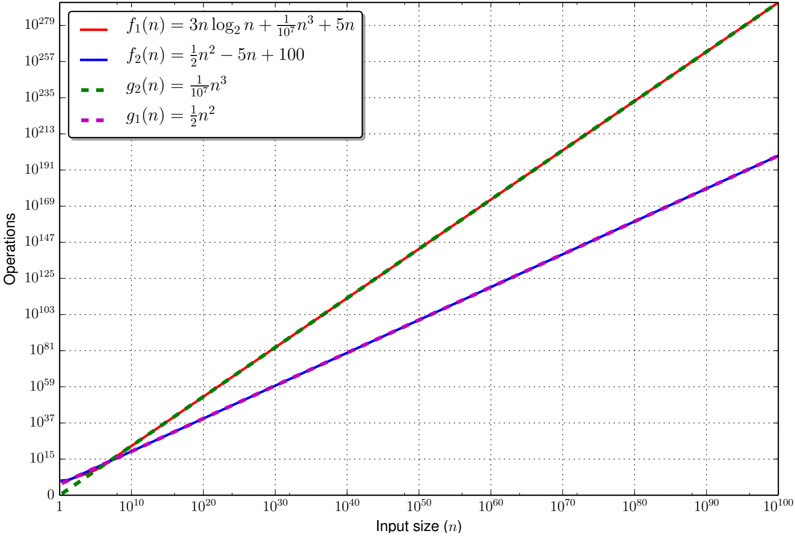


Eventually, $f_1(n)$ becomes indistinguishable from $\frac{1}{10^7}n^3$, and $f_2(n)$ becomes indistinguishable from $\frac{1}{2}n^2$.



(They are so indistinguishable that their lines on the plot above appear identical)

Comparing Algorithms (8)



We say that $f_1(n)$ **asymptotically approaches** $\frac{1}{10^7}n^3$.

Asymptotic Analysis (1)

A function like

$$f(n) = 5n^2 + 100n + 10000 \log n$$

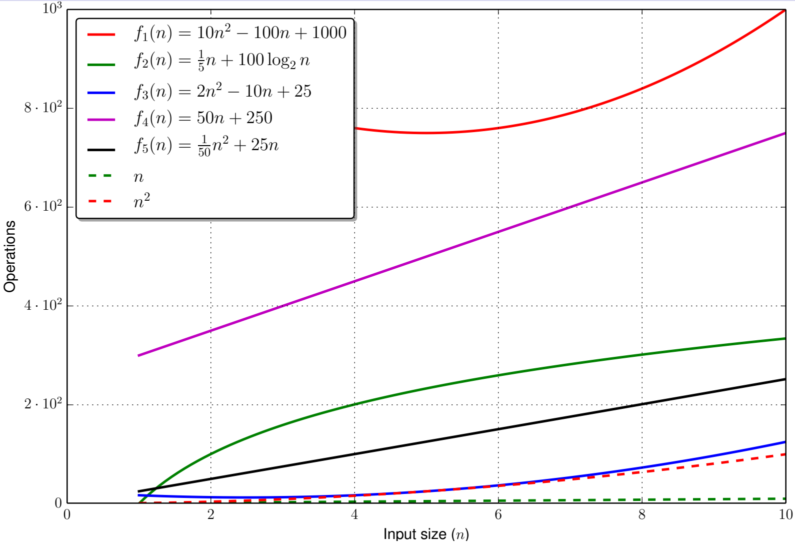
will eventually be dominated by its largest term (in this case $5n^2$). As a result, $f(n)$ is asymptotic to $5n^2$.

Formally, a function $f(n)$ asymptotically approaches a function $g(n)$ if and only if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.$$

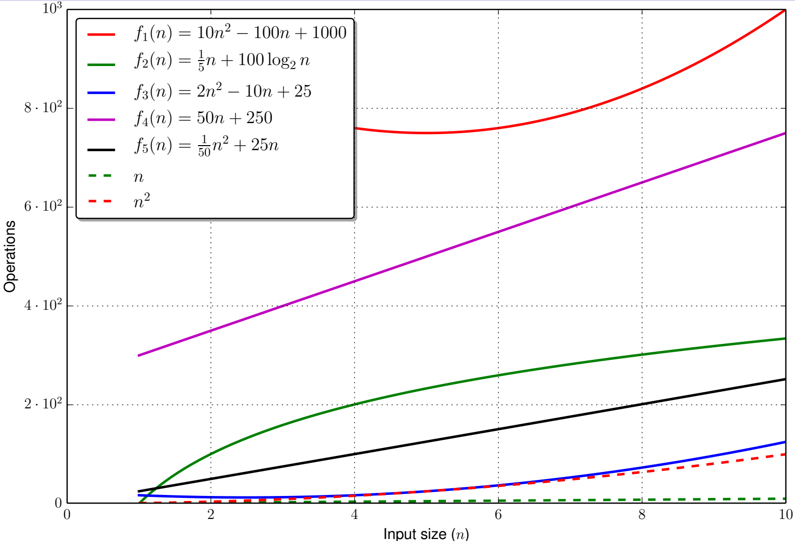
When $f(n)$ asymptotically approaches $g(n)$, we can write $f(n) \sim g(n)$, although we don't use this notation much in CSC 225.

Asymptotic Analysis (2)



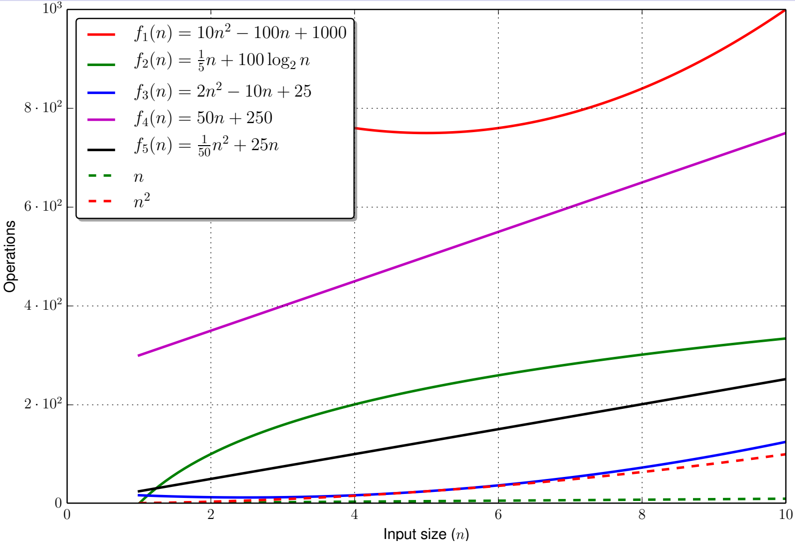
Consider the set of functions above.

Asymptotic Analysis (3)



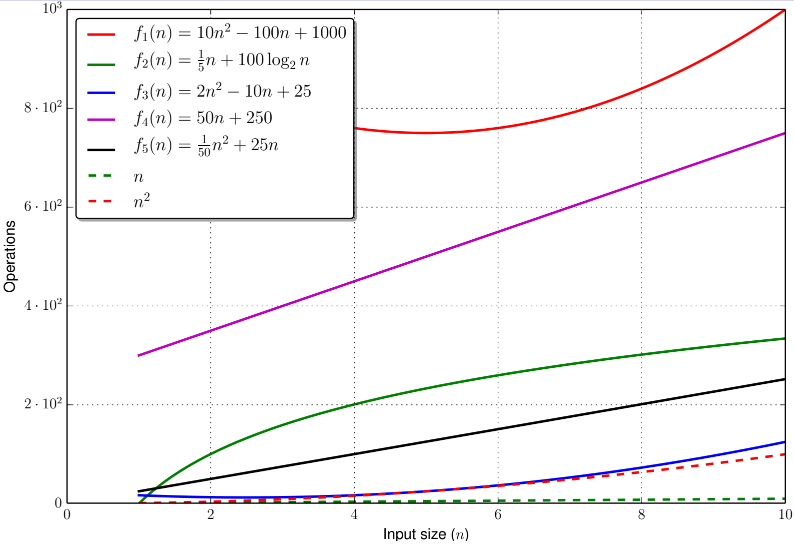
$$f_1(n) = 10n^2 - 100n + 1000 \sim 10n^2$$

Asymptotic Analysis (4)



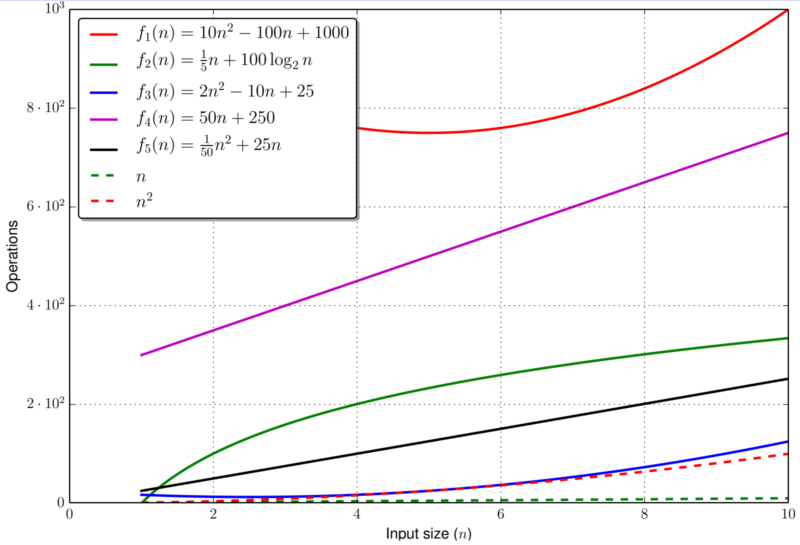
$$f_2(n) = \frac{1}{5}n + 100\log_2 n \sim \frac{1}{5}n$$

Asymptotic Analysis (5)



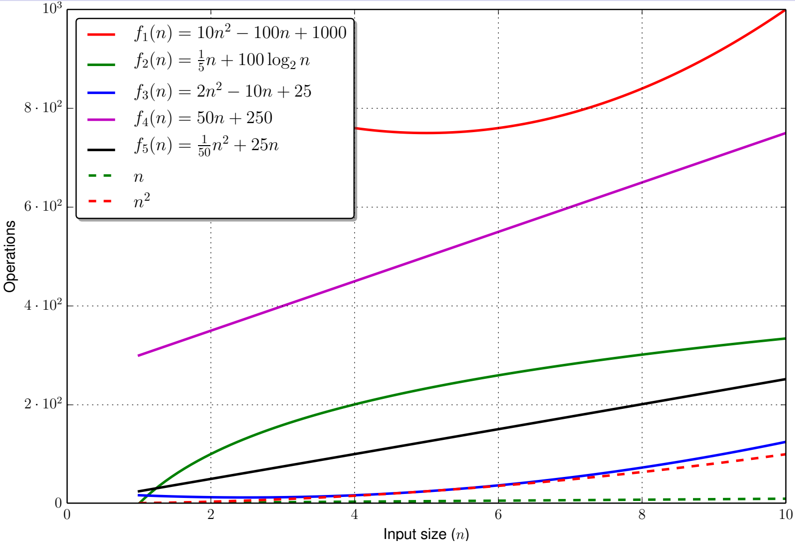
$$f_3(n) = 2n^2 - 10n + 25 \sim 2n^2$$

Asymptotic Analysis (6)



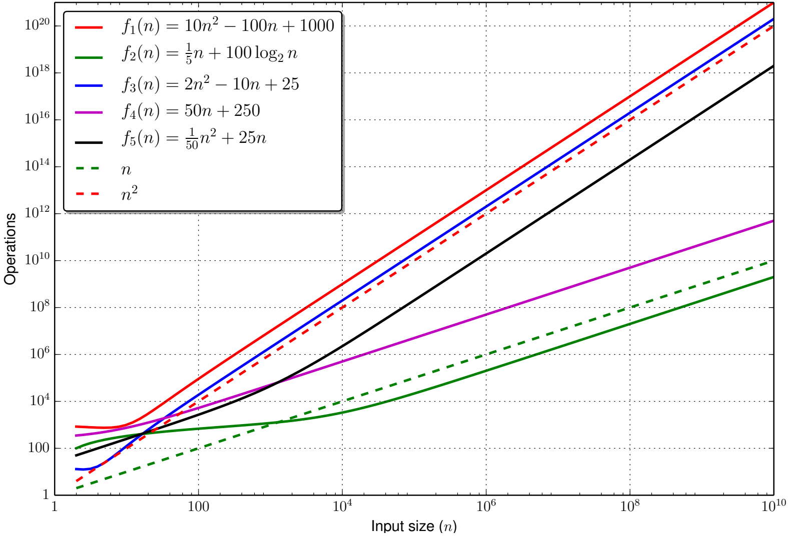
$$f_4(n) = 50n + 250 \sim 50n$$

Asymptotic Analysis (7)



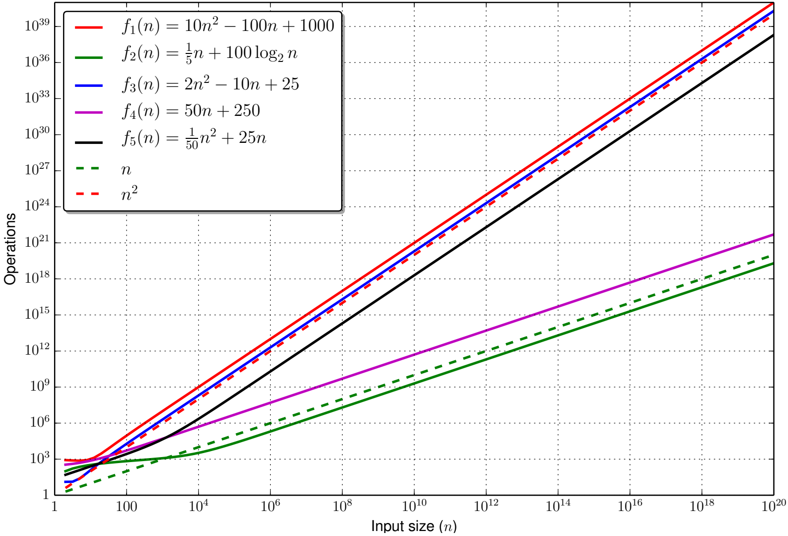
$$f_5(n) = \frac{1}{50}n^2 + 25n \sim \frac{1}{50}n^2$$

Asymptotic Analysis (8)



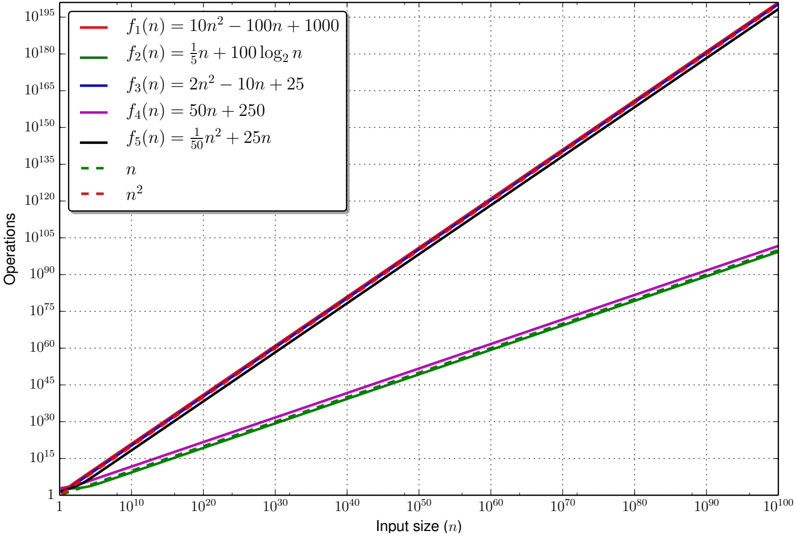
As n becomes large, the functions asymptotically approaching $c \cdot n^2$ form one group, and the functions asymptotic to $c \cdot n$ form another.

Asymptotic Analysis (9)



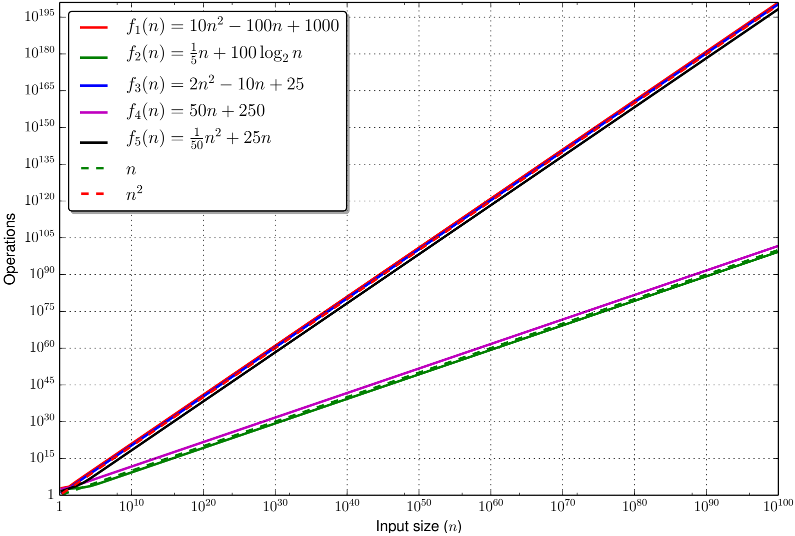
Eventually, all of the n^2 functions are virtually indistinguishable from each other.

Asymptotic Analysis (10)



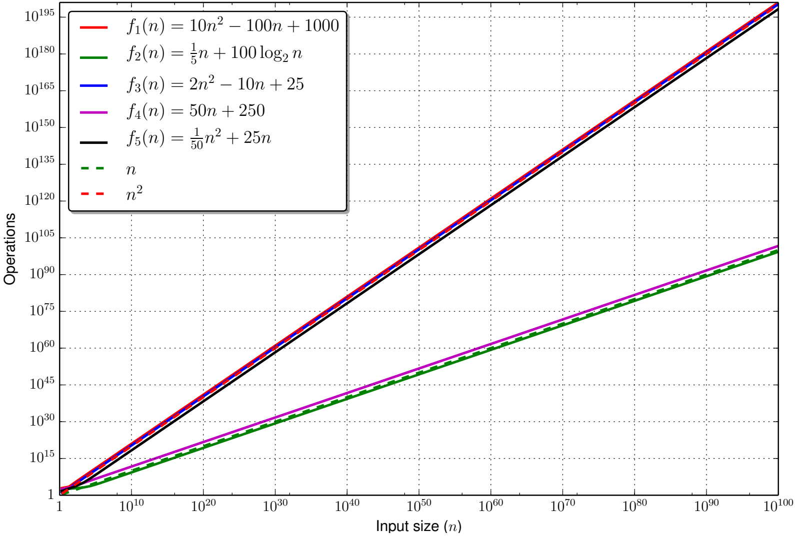
The function $f_1(n)$ is always going to be about five times as large as $f_3(n)$.

Asymptotic Analysis (11)



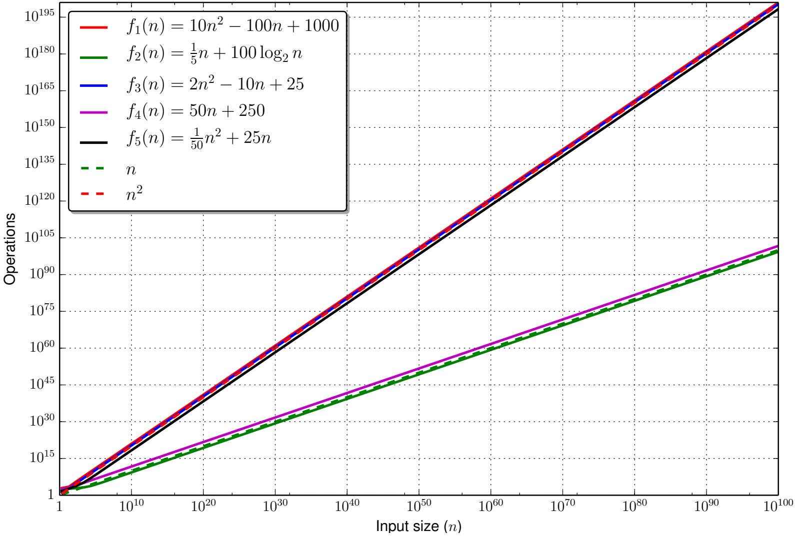
But, as such a massive scale, a constant multiple is not significant.

Asymptotic Analysis (12)



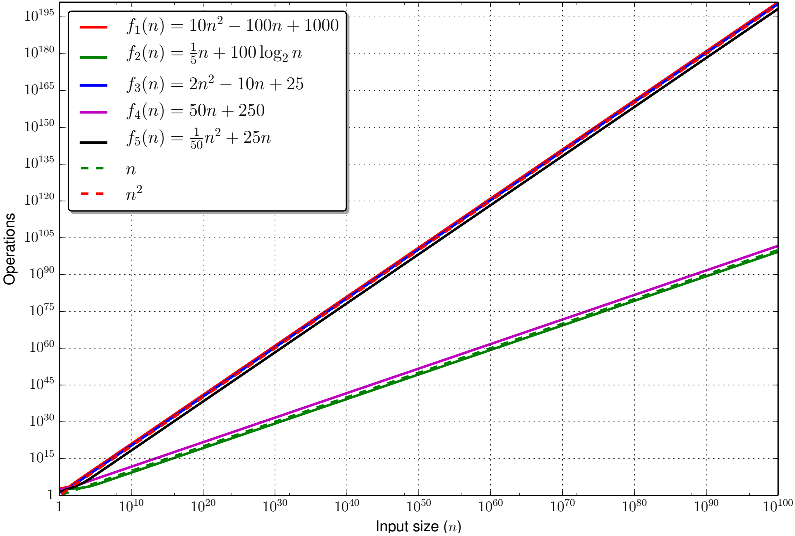
We can informally classify f_1, f_3 and f_5 as being 'basically equal to n^2' asymptotically.

Asymptotic Analysis (13)



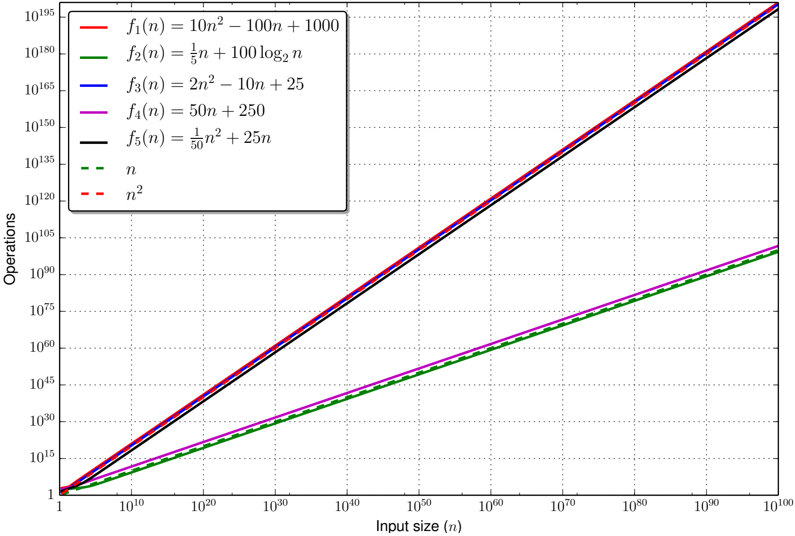
Formally, we say that $f_1(n) \in \Theta(n^2)$ (pronounced 'f_1 is big theta of n squared').

Asymptotic Analysis (14)



Similarly, $f_3(n) \in \Theta(n^2)$ and $f_5(n) \in \Theta(n^2)$.

Asymptotic Analysis (15)



We can also say that $f_2(n) \in \Theta(n)$ and $f_4(n) \in \Theta(n)$.

Big-Theta (1)

Big-Theta functions exactly like a generalized asymptotic version of the equals sign.

Reflexive:

- $a = a$
- Similarly, $f_1(n) \in \Theta(f_1(n))$

Symmetric:

- If $a = b$, then $b = a$.
- Similarly, since $f_1(n) \in \Theta(n^2)$, $n^2 \in \Theta(f_1(n))$

Transitive:

- If $a = b$, and $b = c$, then $a = c$
- Similarly, since $f_3(n) \in \Theta(n^2)$ and $n^2 \in \Theta(f_1(n))$, $f_3(n) \in \Theta(f_1(n))$

We will see the formal definition of Big-Theta (and the terms 'reflexive', 'transitive' and 'symmetric') later.

Big-Theta (2)

The following theorems give sufficient conditions for Big Theta.

Theorem: If

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$$

for some constant $c > 0$, then

$$f(n) \in \Theta(g(n)) \text{ and } g(n) \in \Theta(f(n))$$

Theorem: If $f(n) \sim cg(n)$ for some constant $c > 0$, then

$$f(n) \in \Theta(g(n)).$$

ContainsDuplicate Revisited (1)

Going back to the three running times of the algorithms for CONTAINSDUPLICATE:

Nested Loops (Original):

$$F_1(n) = 4n^2 + n + 2$$

Nested Loops (Improved):

$$F_2(n) = n^2 + 2$$

Sort and Scan:

$$F_3(n) = 10n \log_2 n + 2n$$

We can use the theorems on the previous slide to show

$$F_1(n) \in \Theta(n^2)$$
$$F_2(n) \in \Theta(n^2)$$
$$F_3(n) \in \Theta(n \log_2 n)$$

ContainsDuplicate Revisited (2)

$$F_1(n) \in \Theta(n^2)$$
$$F_2(n) \in \Theta(n^2)$$
$$F_3(n) \in \Theta(n \log_2 n)$$

Therefore, we can see that $F_1(n)$ and $F_2(n)$ are asymptotically equal (up to a constant factor). To compare the two nested loop algorithms to the Sort and Scan algorithm, we need the asymptotic analogue of the "$\leq$" operator.

We will also define asymptotic analogues for the $<, \leq, \geq$ and $>$ operators.

Asymptotic Operators

Analogy	Asymptotic Operator	Name
"$f(n) < g(n)$"	$f(n) \in o(g(n))$	Little-O
"$f(n) \leq g(n)$"	$f(n) \in O(g(n))$	Big-O
"$f(n) = g(n)$"	$f(n) \in \Theta(g(n))$	Big-Theta
"$f(n) \geq g(n)$"	$f(n) \in \Omega(g(n))$	Big-Omega
"$f(n) > g(n)$"	$f(n) \in \omega(g(n))$	Little-Omega

Big-O (1)

Definition:
Let $f(n)$ and $g(n)$ be functions defined for all $n \geq 0$. If there exist constants $c > 0, n_0 \geq 0$ such that

$$f(n) \leq cg(n)$$

for all $n > n_0$, then we write

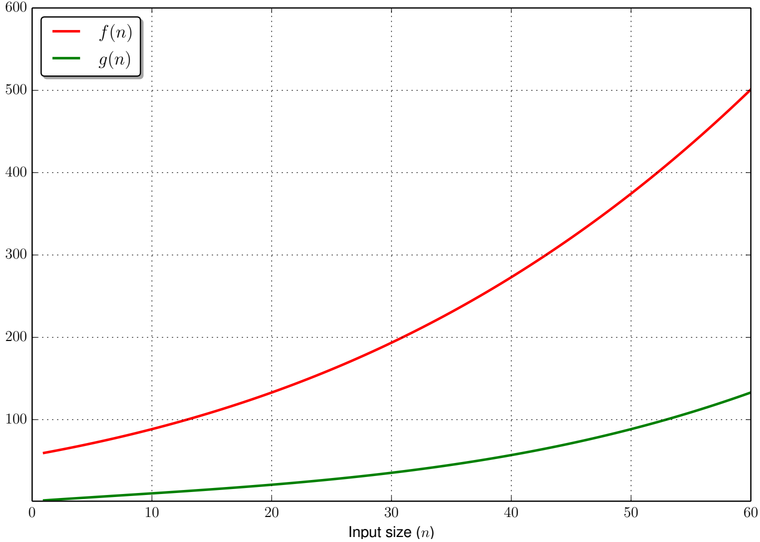
$$f(n) \in O(g(n))$$

(pronounced 'f is Big-O of g')

Big-O (2)

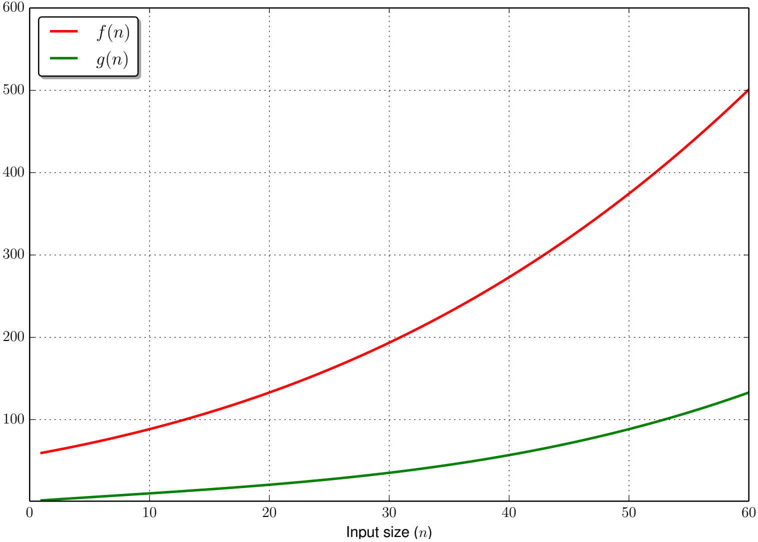
Interpretation:
$f(n) \in O(g(n))$ if there is some multiple of $g(n)$ (the constant c) such that $f(n)$ is **always** less than or equal $cg(n)$ after a certain point n_0.

Big-O Illustration (1)



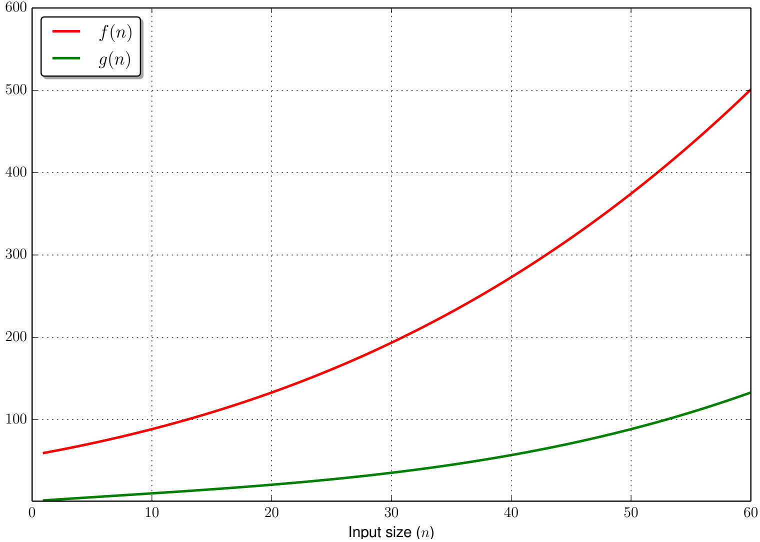
Big-O is a generalization of the regular $\leq$ operator, and behaves the same way.

Big-O Illustration (2)



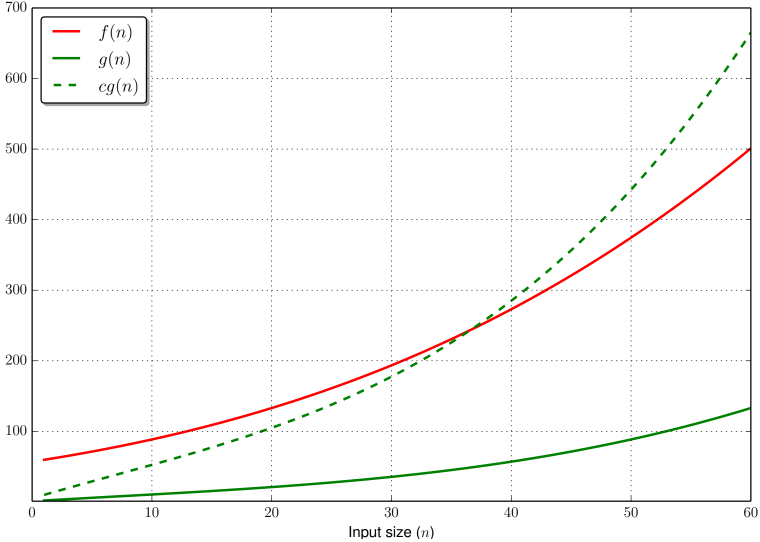
If $g(n) \leq f(n)$ for all n, then $g(n) \in O(f(n))$
(We can choose $c = 1$ and $n_0 = 0$).

Big-O Illustration (3)



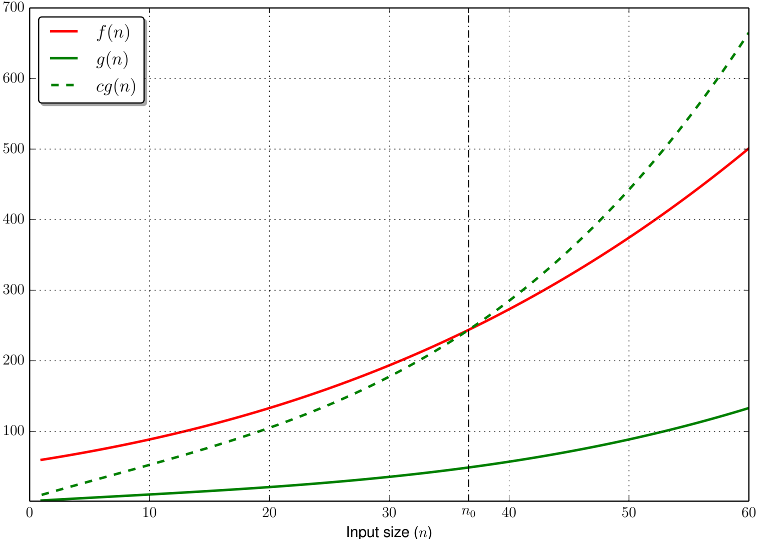
Suppose we want to show that $f(n) \in O(g(n))$.

Big-O Illustration (4)



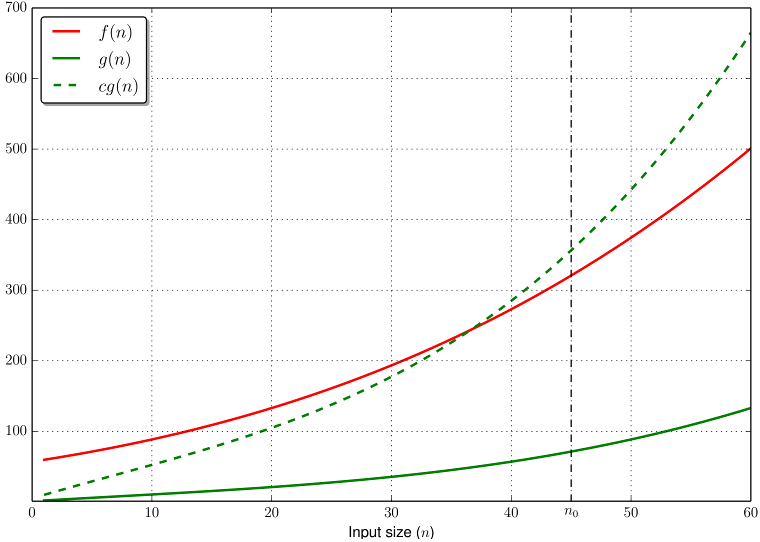
Find a constant c such that $cg(n)$ becomes greater than $f(n)$.

Big-O Illustration (5)



Choose a constant n_0 such that $f(n) \leq cg(n)$ at all points greater than n_0

Big-O Illustration (6)



Note that n_0 does not have to be the smallest possible value (it is often easier to choose a more round number).

Big-O: Sufficient Condition

Theorem: If

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$$

for some constant $c \geq 0$, then

$$f(n) \in O(g(n))$$

Note that the theorem above is only a **sufficient** condition. If the limit converges, then $f(n) \in O(g(n))$. If the limit does not exist, $f(n)$ may still be in $O(g(n))$ (but we have to find a different way of proving it).

Big-O Examples (1)

Exercise: Prove that $6n^2 \in O(n^2)$.

Method 1:
Choose $c = 6$ and $n_0 = 0$.
Since $6n^2 \leq 6n^2$ for all $n \geq 0$, $6n^2 \in O(n^2)$

Method 2:
Since the limit

$$\lim_{n \to \infty} \frac{6n^2}{n^2} = 6$$

converges to a non-negative constant, $6n^2 \in O(n^2)$.

Big-O Examples (2)

Exercise: Prove that $n^2 \in O(6n^2)$.

Method 1:
Choose $c = 1$ and $n_0 = 0$.
Since $n^2 \leq 1 \cdot (6n^2)$ for all $n \geq 0$, $n^2 \in O(6n^2)$

Method 2:
Since the limit

$$\lim_{n\to\infty} \frac{n^2}{6n^2} = \frac{1}{6}$$

converges to a non-negative constant, $n^2 \in O(6n^2)$.

Big-O Examples (3)

Exercise: Prove that $n \in O(n^2)$.

Method 1:
Choose $c = 1$ and $n_0 = 1$.
Since $n \leq n^2$ for all $n \geq 1$, $n \in O(n^2)$

Method 2:
Since the limit

$$\lim_{n \to \infty} \frac{n}{n^2} = 0$$

converges to a non-negative constant, $n \in O(n^2)$.

Big-O Examples (4)

Exercise: Prove that $n \in O(n \log_2 n)$.

Method 1:
Choose $c = 1$ and $n_0 = 2$.
When $n \geq 2$, $\log_2 n \geq 1$. Therefore, $n \log_2 n \geq n$. Since $n \leq n \log_2 n$ for all $n \geq 2$, $n \in O(n \log_2 n)$

Method 2:
Since the limit

$$\lim_{n \to \infty} \frac{n}{n \log_2 n} = 0$$

converges to a non-negative constant, $n \in O(n \log_2 n)$.

Big-O Examples (5)

Exercise: Prove that $\log_2 n \in O(n)$.

Method 1:
Choose $c = 1$ and $n_0 = 1$.
Since $\log_2 n \leq n$ for all $n \geq 1$, $\log_2 n \in O(n)$

Method 2:
Since the limit

$$\lim_{n \to \infty} \frac{\log_2 n}{n} = 0$$

converges to a non-negative constant, $\log_2 n \in O(n)$.

Big-O Examples (6)

Exercise: Prove that $10^6 \in O(1)$.

Method 1:
Choose $c = 10^6$ and $n_0 = 1$.
Since $10^6 \leq 10^6 \cdot 1$ for all $n \geq 1$, $10^6 \in O(1)$

Method 2:
Since the limit

$$\lim_{n\to\infty} \frac{10^6}{1} = 10^6$$

converges to a non-negative constant, $10^6 \in O(1)$.

Exercise: Prove that $\sin(x) \in O(x)$.

Method 1:
$\sin(x) \leq 1$ for all $x \in \mathbb{R}$. Choose $c = 1$ and $n_0 = 1$.
Since $\sin(x) \leq 1 \cdot x$ for all $x \geq 1$, $\sin(x) \in O(x)$

Method 2:
Since the limit

$$\lim_{n \to \infty} \frac{\sin(x)}{x} = 0$$

converges to a non-negative constant, $\sin(x) \in O(x)$.

Big-O Examples (8)

Exercise: Prove that $\sin(x) \in O(1)$.

Method 1:
$\sin(x) \leq 1$ for all $x \in \mathbb{R}$. Choose $c = 1$ and $n_0 = 0$.
Since $\sin(x) \leq 1 \cdot 1$ for all $x \geq 0$, $\sin(x) \in O(1)$

Method 2:
The limit condition does not work in this case, since the limit

$$\lim_{n \to \infty} \frac{\sin(x)}{1}$$

does not converge.

Big-O Examples (9)

Exercise: Prove that $f(n) = 5n^3 + 100n^2 + n + 10 \in O(n^3)$.

Method 1:
Note that $n^k \leq n^{k+1}$ for $n \geq 1, k \geq 0$. For all $n \geq 1$,

$$\begin{aligned} f(n) &= 5n^3 + 100n^2 + n + 10 \\ &\leq 5n^3 + 100n^3 + n^3 + 10n^3 \text{ (when } n \geq 1) \\ &= 116n^3 \end{aligned}$$

Therefore, $f(n) \leq 116n^3$ for all $n \geq 1$, so $f(n) \in O(n^3)$
Method 2:
Since

$$\lim_{n \to \infty} \frac{5n^3 + 100n^2 + n + 10}{n^3} = 5$$

converges to a non-negative constant, $f(n) \in O(n^3)$.
(Several applications of L'Hôpital's rule are necessary; show your work).

Big-Omega: Definition 1

Definition:
Let $f(n)$ and $g(n)$ be functions defined for all $n \geq 0$. If there exist constants $c > 0, n_0 \geq 0$ such that

$$f(n) \geq cg(n)$$

for all $n > n_0$, then we write

$$f(n) \in \Omega(g(n))$$

(pronounced 'f is Big-Omega of g')

Big-Omega: Definition 2

Theorem: Let $f(n)$ and $g(n)$ be functions defined for all $n \geq 0$. Then $f(n) \in \Omega(g(n))$ if and only if

$$g(n) \in O(f(n))$$

Since Big-Omega behaves like an asymptotic $\geq$ operator, it can be treated as the reverse of Big-O (similar to how if $a \leq b$, then $b \geq a$).

Exercises

1. Prove that $5n^2 - 100n \in O(n^2)$
2. Prove that $n^2 \in \Omega(5n^2 - 100n)$
3. Prove that $\sum_{i=1}^{n} i \in O(n^2)$
4. Prove that $\sum_{i=1}^{n} i \in \Omega(n^2)$
5. Prove that $\sum_{i=1}^{n} i^2 \in O(n^3)$
6. Prove that $\sum_{i=1}^{n} i^2 \in \Omega(n^3)$

Big-Theta: Definition 1

Definition:
Let $f(n)$ and $g(n)$ be functions defined for all $n \geq 0$. Then $f(n) \in \Theta(g(n))$ ('f is Big-Theta of g') if and only if

$$\begin{aligned} &f(n) \in O(g(n)) \quad \text{and} \\ &f(n) \in \Omega(g(n)) \end{aligned}$$

Big-Theta: Definition 2

Definition:
Let $f(n)$ and $g(n)$ be functions defined for all $n \geq 0$. If there exist constants $c_1, c_2 > 0$ and $n_0 \geq 0$ such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for all $n > n_0$, then

$$f(n) \in \Theta(g(n))$$

Exercise: Prove that definition 1 implies definition 2.

More Exercises

1. Find a simple function $f(n)$ such that

$$\sum_{i=1}^{n} \log_2 i \in \Theta(f(n))$$

2. Prove that for any integer $k \geq 0$,

$$\sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$$

Little-O: Definition 1

Definition:
Let $f(n)$ and $g(n)$ be functions defined for all $n \geq 0$. If, for **all** constants $c > 0$, there exists a constant $n_0 > 0$ such that

$$f(n) \leq cg(n)$$

for all $n > n_0$, then $f(n) \in o(g(n))$ ('f is Little-O of g')

Little-O: Definition 2

Definition:
Let $f(n)$ and $g(n)$ be functions defined for all $n \geq 0$. Then $f(n) \in o(g(n))$ if and only if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Note that unlike the limit condition for Big-O, this limit condition is equivalent to the definition of Little-O.

Facts

You may use the relationships below without proof (unless you are specifically asked to prove one of them):

$$
\begin{aligned}
c &\in \Theta(1) \text{ for all constants } c > 0 \\
c &\in o(\log n) \text{ for all constants } c \geq 0 \\
\log_a n &\in \Theta(\log_b n) \text{ for all constant bases } a, b > 1 \\
\log n &\in o(n^k) \text{ for all } k > 0 \\
n &\in o(n \log n) \\
n \log n &\in o(n^k) \text{ for } k > 1 \\
n^k &\in o(n^{k+\varepsilon}) \text{ for } k > 0, \varepsilon > 0 \\
n^k &\in o(q^n) \text{ for } q > 1
\end{aligned}
$$

Little-Omega: Definition 1

Definition:
Let $f(n)$ and $g(n)$ be functions defined for all $n \geq 0$. If, for **all** constants $c > 0$, there exists a constant $n_0 > 0$ such that

$$f(n) \geq cg(n)$$

for all $n > n_0$, then $f(n) \in \omega(g(n))$ ('f is Little-Omega of g')

Little-Omega: Definition 2

Definition:
Let $f(n)$ and $g(n)$ be functions defined for all $n \geq 0$. Then $f(n) \in \omega(g(n))$ if and only if

$$\lim_{n \to \infty} \frac{g(n)}{f(n)} = 0$$

Asymptotic Operators

Analogy	Asymptotic Operator	Name
"$f(n) < g(n)$"	$f(n) \in o(g(n))$	Little-O
"$f(n) \leq g(n)$"	$f(n) \in O(g(n))$	Big-O
"$f(n) = g(n)$"	$f(n) \in \Theta(g(n))$	Big-Theta
"$f(n) \geq g(n)$"	$f(n) \in \Omega(g(n))$	Big-Omega
"$f(n) > g(n)$"	$f(n) \in \omega(g(n))$	Little-Omega

Relationships Between Inequalities

We are familiar with the relationships between regular inequalities:

$a = a$

$a \leq a$

$a \geq a$

If $a = b$, then $b = a$

If $a = b$ and $b = c$, then $a = c$

If $a \leq b$, then $b \geq a$

If $a < b$, then $b > a$

If $a \leq b$ and $b \leq c$, then $a \leq c$

If $a \leq b$ and $b < c$, then $a < c$

If $a = b$ and $b \leq c$, then $a \leq c$

If $a \leq b$ and $b \leq a$, then $a = b$

If $a \leq b$, then $a \not> b$

Asymptotic Relationships (1)

Similar relationships apply to the asymptotic operators:

$a = a$	$f(n) \in \Theta(f(n))$
$a \leq a$	$f(n) \in O(f(n))$
$a \geq a$	$f(n) \in \Omega(f(n))$
If $a = b$, then $b = a$	If $f(n) \in \Theta(g(n))$, then $g(n) \in \Theta(f(n))$
If $a = b$ and $b = c$, then $a = c$	If $f(n) \in \Theta(g(n))$ and $g(n) \in \Theta(h(n))$, then $f(n) \in \Theta(h(n))$
If $a \leq b$, then $b \geq a$	If $f(n) \in O(g(n))$, then $g(n) \in \Omega(f(n))$
If $a < b$, then $b > a$	If $f(n) \in o(g(n))$, then $g(n) \in \omega(f(n))$
If $a \leq b$ and $b \leq c$, then $a \leq c$	If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$, then $f(n) \in O(h(n))$
If $a \leq b$ and $b < c$, then $a < c$	If $f(n) \in O(g(n))$ and $g(n) \in o(h(n))$, then $f(n) \in o(h(n))$
If $a = b$ and $b \leq c$, then $a \leq c$	If $f(n) \in \Theta(g(n))$ and $g(n) \in O(h(n))$, then $f(n) \in O(h(n))$
If $a \leq b$ and $b \leq a$, then $a = b$	If $f(n) \in O(g(n))$ and $g(n) \in O(f(n))$, then $f(n) \in \Theta(g(n))$
If $a \leq b$, then $a \not> b$	If $f(n) \in O(g(n))$, then $f(n) \notin \omega(g(n))$

Asymptotic Relationships: Big-O, Big-Omega

If $f(n) \in O(g(n))$, then

- $g(n) \in \Omega(f(n))$
- $f(n) \notin \omega(g(n))$

If $f(n) \in \Omega(g(n))$, then

- $g(n) \in O(f(n))$
- $f(n) \notin o(g(n))$

Asymptotic Relationships: Big-Theta

If $f(n) \in \Theta(g(n))$, then

- $g(n) \in \Theta(f(n))$
- $f(n) \in O(g(n))$
- $f(n) \in \Omega(g(n))$
- $g(n) \in O(f(n))$
- $f(n) \notin \omega(g(n))$
- $f(n) \notin o(g(n))$
- $g(n) \notin o(f(n))$
- $g(n) \notin \omega(f(n))$

Asymptotic Relationships: Little-O, Little-Omega

If $f(n) \in o(g(n))$, then

- $f(n) \notin \Theta(g(n))$
- $f(n) \in O(g(n))$
- $f(n) \notin \Omega(g(n))$
- $g(n) \in \omega(f(n))$
- $g(n) \in \Omega(f(n))$
- $f(n) \notin \omega(g(n))$

If $f(n) \in \omega(g(n))$, then

- $f(n) \notin \Theta(g(n))$
- $f(n) \notin O(g(n))$
- $f(n) \in \Omega(g(n))$
- $g(n) \in o(f(n))$
- $g(n) \in O(f(n))$
- $f(n) \notin o(g(n))$

More Exercises

1. Prove that $5n \log_2 n \notin O(n)$
2. Prove that $\frac{5n}{\log_2 n} \notin \Theta(n)$
3. Prove that $\log_a n \in \Theta(\log_b n)$ for all $a, b > 1$
4. Determine and prove an asymptotic relationship between 2^n and $n!$
5. Using the definitions of Big-O and Little-Omega, prove that if $f(n) \in O(g(n))$, then $f(n) \notin \omega(g(n))$

Other Asymptotic Algebra

- For any constant $c > 0$, $cf(n) \in \Theta(f(n))$
- If $f(n) \in O(g(n))$, then $f(n) + g(n) \in \Theta(g(n))$
- If $f(n) \in O(g(n))$, then $f(n)g(n) \in O(g(n)^2)$