Capital Taxation - Primal Problem

Note

These notes combine material directly taken from Lucas (1990) and the lectures' explanations and commentary. When content is reproduced verbatim or closely follows the original phrasing, the source is indicated.

The main difference with lectures is that here the Hamiltonian is written in current-value form.

1 Suply Side: An Analytical Review

1.1 A theorical framework

Following Lucas (1990), the analysis begins with **Ramsey's (1927)** normative question:

> *What configuration of tax rates maximizes household utility, given a level of government spending and market-determined quantities and prices?*

A second, positive and quantitative question follows:

> *How large are the welfare and behavioral effects of such tax choices?*

To address these, we consider a representative household that maximizes the discounted sum of utilities from consumption and leisure over an infinite horizon. The household both supplies labor and owns the productive capital of the economy. Nevertheless, the same results would hold if ownership and labor supply decisions were separated. This follows from the **Fisher Separation Theorem**, which states that under perfect capital markets, individual consumption and production decisions can be separated.

So instead of doing Lucas approach, we can follow the separated approach and arrive to the same conclusions.

1.1.1 Representative Household

The household's optimization problem is:

$$\int_0^\infty e^{-(\rho-n)t} U\left[c(t), l(t)\right] dt$$

subject to the dynamic budget constraint:

$$\dot{a} + na(t) = r(t)a(t) + w(t)l(t) - c(t)$$

where: $a(t)$ is the household's financial wealth, - $r(t)$ is the return on capital, $w(t)$ is the wage rate, n is the population growth rate, and ρ is the subjective discount rate.

1.1.2 Firms

Firms maximize profits according to:

$$\pi(t) = F\left[k(t), l(t)h(t)\right] - w(t)l(t) - r(t)k(t)$$

where $h(t)$ denotes the average skill level of labor, which grows exogenously at a Harrod-neutral technological rate.

In the extended version of the model, individuals may allocate time to skill accumulation:

$$\frac{d}{dt}h(t) = h(t)G\left[v(t)\right]$$

where $v(t)$ is the share of time devoted to learning.

(Assuming competitive market then $\pi(t) = 0$ for all periods, then we get the same restriction)

(If we had $\pi(t) > 0$ then we would have to suppose that the firms have DRS, and that would change the standards results. If we IRS, then we would have that one firm has market power and then no-competition, profit would be negative if they act competitive)

1.1.3 Government

Introducing the government, suppose there is a labor income tax $\tau(t)$ and a capital income tax $\theta(t)$.

The household's budget constraint becomes:

$$\dot{a} + na(t) = (1 - \theta(t))r(t)a(t) + (1 - \tau(t))w(t)l(t) - c(t)$$

Rearranging terms:

$\dot{a} + na(t) = r(t)a(t) + w(t)l(t) - c(t) - \theta(t)r(t)a(t) - \tau(t)w(t)l(t)$

$$\dot{a} + na(t) = r(t)a(t) + w(t)l(t) - c(t) - G(t)$$

where $G(t) = \theta(t)r(t)a(t) + \tau(t)w(t)l(t)$ represents the tax revenues collected by the government. This is the restriction presented in the paper Lucas 1990.

If the government can issue debt $b(t)$, not presented in Lucas paper, the aggregate resource constraint becomes:

$$\dot{k} + \dot{b} + nk(t) + nb(t) = r(t)k(t) + r(t)b(t) + w(t)l(t) - c(t) - \theta(t)r(t)a(t) - \tau(t)w(t)l(t)$$

If the government can issue debt $b(t)$, the aggregate resource constraint becomes:

$$\dot{k} + \dot{b} + n[k(t) + b(t)] = r(t)[k(t) + b(t)] + w(t)l(t) - c(t) - \theta(t)r(t)a(t) - \tau(t)w(t)l(t)$$

Simplifying:

$\dot{k} + nk(t) = r(t)k(t) + w(t)l(t) - c(t) + r(t)b(t) - \dot{b} - nb(t) - \theta(t)r(t)a(t) - \tau(t)w(t)l(t)$

$$\dot{k} + nk(t) = r(t)k(t) + w(t)l(t) - c(t) - g(t)$$

Where $g(t)$ is government spending $(\dot{b} + nb(t) + \theta(t)r(t)a(t) + \tau(t)w(t)l(t) - r(t)b(t))$.

The government's budget constraint can thus be written as:

$\dot{b} = g(t) + r(t)b(t) - nb(t) - \theta(t)r(t)a(t) - \tau(t)w(t)l(t)$

or equivalently:

$\dot{b} = g(t) + r(t)b(t) - nb(t) - (r(t) - \bar{r}(t))\, a(t) - (w(t) - \bar{w}(t))\, l(t)$

The difference between public debt and household asset is the negative value of physical capital, therefore

$$\dot{b} = g(t) + r(t)(b(t) - a(t)) - nb(t) + \bar{r}(t)a(t) - (w(t) - \bar{w}(t))\, l(t)$$

We can also rewrite it as:

$$\dot{b} = g(t) - (f(k,l) - \bar{r}(t)a(t) - \bar{w}(t)l(t))$$

The difference $f(k,l) - \bar{r}(t)a(t) - \bar{w}(t)l(t)$ is the **net resources available to the government through taxes**.

1.2 Consumers Maximizing Utility

For simplicity, we suppose that there's no possibility for skill improvement ($\dot{h} = 0$), and that the rate of growth of population is equal to zero ($n = 0$). [This differs from Lucas' framework, which incorporates endogenous human capital accumulation and population growth, his focus is on modeling a growing economy. Here, we focus exclusively on the role of taxation.]

The problem becomes:

$$\max_{c(t),l(t),a(t)} \int_0^\infty e^{-\rho t} U\left[c(t), l(t)\right] dt$$

$$\text{s.t.} \quad \begin{cases} \dot{a} = \bar{r}(t)a(t) + \bar{w}(t)l(t) - c(t) \\ a(0) = a_0 \end{cases}$$

where $\bar{r}(t)$ and $\bar{w}(t)$ denote after-tax returns on capital and labor.

1.2.1 The Hamiltonian

The current-value Hamiltonian is:

$$\mathscr{H}_c(t) = U\left[c(t), l(t)\right] + \tilde{q}(t)\left[\bar{r}(t)a(t) + \bar{w}(t)l(t) - c(t)\right]$$

FOC

$\frac{\partial \mathscr{H}_c}{\partial c(t)} = \frac{\partial U(c(t),l(t))}{\partial c(t)} - \tilde{q}(t) = 0$

$\frac{\partial \mathscr{H}_c}{\partial l(t)} = \frac{\partial U(c(t),l(t))}{\partial l(t)} + \bar{w}\tilde{q}(t) = 0$

$\frac{\partial \mathscr{H}_c}{\partial a(t)} = \bar{r}\tilde{q}(t) = -\dot{\tilde{q}}(t) + \rho\tilde{q}(t)$

The objetive of the model is to embed government public expenditure in the resource constraint of the government.

Working the first two conditions we have:

$$\frac{\partial U(c(t), l(t))/\partial l(t)}{\partial U(c(t), l(t))/\partial c(t)} = \bar{w}$$

The ratio of marginal utility of labor and marginal utility of consumption tells how much consumption the household would be willing to give up to avoid working a bit more.

Meanwhile from the last condition.

$$\frac{\dot{\tilde{q}}(t)}{\tilde{q}(t)} = \rho - \bar{r}$$

The rate of change of the **shadow value of wealth** (in current value) over time is given by the difference between ρ and $\bar{r}$.

When the subjective discount rate exceeds the after-tax return on capital, $\rho > \bar{r}$, the **shadow value of wealth**—that is, the marginal utility of an additional unit of savings—increases over time. In this case, the representative household is relatively **impatient**: it places a higher value on future utility than the market return would justify. Put differently, the household is willing to postpone consumption, preferring to transfer resources to future periods or future generations.

Conversely, when $\rho < \bar{r}$, the shadow value of wealth **declines** over time. The return on saving is high relative to the household's rate of time preference, which implies that the household is **impatient in the opposite direction**—it values present consumption more highly and discounts the future heavily. In this scenario, current consumption is preferred to saving for the benefit of future generations, reflecting a lower concern for intertemporal welfare.

Taking logs and differentiating with respect to time the first equilibrium condition:

$\log\left(\frac{\partial U(c(t),l(t))}{\partial c(t)}\right) = \log(\tilde{q}(t))$

$$\frac{\left[\partial^2 U(c(t), l(t))/\partial c(t)^2\right]\dot{c} + \left[\partial^2 U(c(t), l(t))/\partial c(t)\partial l(t)\right]\dot{l}}{\partial U(c(t), l(t))/\partial c(t)} = \frac{\dot{\tilde{q}}(t)}{\tilde{q}(t)}$$

Using the last first order condition:
$\frac{\partial^2 U(c(t),l(t))}{\partial c(t)^2}\dot{c} + \frac{\partial^2 U(c(t),l(t))}{\partial c(t)\partial l(t)}\dot{l} = [\rho - \bar{r}]\frac{\partial U(c(t),l(t))}{\partial c(t)}$

$$U_{cc}\dot{c} + U_{cl}\dot{l} = (\rho - \bar{r})U_c$$

We can then write it as:

$$\frac{U_{cc}\dot{c} + U_{cl}\dot{l}}{U_c} \equiv H_c = (\rho - \bar{r})$$

Also:
$\log\left(-\frac{\partial U(c(t),l(t))}{\partial l(t)}\right) = \log(w(t)) + \log(\tilde{q}(t))$

$$\frac{\left[\partial^2 U(c(t),l(t))/\partial l(t)^2\right]\dot{l} + \left[\partial^2 U(c(t),l(t))/\partial c(t)\partial l(t)\right]\dot{c}}{\partial U(c(t),l(t))/\partial l(t)} = \frac{\dot{w}(t)}{w(t)} + \frac{\dot{\tilde{q}}(t)}{\tilde{q}(t)}$$

Using the last first order condition:
$\frac{\partial^2 U(c(t),l(t))}{\partial c(t)\partial l(t)}\dot{c} + \frac{\partial^2 U(c(t),l(t))}{\partial l(t)^2}\dot{l} = [\gamma_w + \rho - \bar{r}]\frac{\partial U(c(t),l(t))}{\partial l(t)}$

$$U_{lc}\dot{c} + U_{ll}\dot{l} = (\gamma_w + \rho - \bar{r})U_l$$

We can then write it as:

$$\frac{U_{cc}\dot{c} + U_{cl}\dot{l}}{U_h} \equiv H_l = (\gamma_w + \rho - \bar{r})$$

So the left-hand side is the rate of change of marginal disutility of labor, and the right-hand side is the "force" driving the labor supply adjustment due to intertemporal considerations and wage growth.

> **Note** If we suppose that labor is exogenous or that utility function is additive separable between consumption and labor, we have the following condition:
>
> $$\frac{U_{cc}}{U_c}\dot{c} \equiv H_c = (\rho - \bar{r}) \rightarrow \text{ Euler's Equation}$$

Transversality Condition

$$\lim_{t\to\infty} e^{-\rho t}\tilde{q}(t)a(t) = 0$$

This means that the present value of wealth, measured in current-value shadow price terms and discounted by $e^{-\rho t}$, must approach zero as $t \to \infty$. Since $\tilde{q}(t)$ reflects the marginal utility value of wealth, the condition does not imply that the asset stock $a(t)$ itself goes to zero.

We work the budget constraint.

$$\dot{a} - \bar{r}(t)a(t) = \bar{w}(t)l(t) - c(t)$$

Solving with an Integrating Factor

$$a(t)e^{-\int_0^t \bar{r}(s)ds} \implies \frac{d}{dt}a(t)e^{-\int_0^t \bar{r}(s)ds} = e^{-\int_0^t \bar{r}(s)ds}\left[\dot{a}(t) - \bar{r}a(t)\right]$$

Then we can write
$\dot{a}e^{-\int_0^t \bar{r}(s)ds} - \bar{r}(t)a(t)e^{-\int_0^t \bar{r}(s)ds} = \left[\bar{w}(t)l(t) - c(t)\right]e^{-\int_0^t \bar{r}(s)ds}$

$$\frac{d}{dt}a(t)e^{-\int_0^t \bar{r}(s)ds} = \left[\bar{w}(t)l(t) - c(t)\right]e^{-\int_0^t \bar{r}(s)ds}$$

Then the increase (decrease) of the discounted assets, is given by the discounted net income after taxes.

Integrating both sides:

$$\int_0^\infty \frac{d}{dt}a(t)e^{-\int_0^t \bar{r}(s)ds}dt = \int_0^\infty \left[\bar{w}(t)l(t) - c(t)\right]e^{-\int_0^t \bar{r}(s)ds}dt$$

Then we have:

$$\lim_{t\to\infty} a(t)e^{-\int_0^t \bar{r}(s)ds} = \int_0^\infty \left[\bar{w}(t)l(t) - c(t)\right]e^{-\int_0^t \bar{r}(s)ds}dt + a_0$$

If $\bar{r}(t) = \rho$ then we have the transversality condition.

1. If the LHS is $\lim_{t\to\infty} a(t)e^{-\int_0^t \bar{r}(s)ds} \to \infty$, the household would be accumulating assets indefinitely. Such a path cannot be optimal, because at any time the household could increase its utility by consuming part of these assets instead of leaving them unspent. Hence, infinite discounted wealth violates optimality.

2. If the LHS is $\lim_{t\to\infty} a(t)e^{-\int_0^t \bar{r}(s)ds} \to -\infty$, the household would be financing consumption through perpetual borrowing—essentially playing a Ponzi game. In practice, this is impossible since lenders would eventually stop providing credit. Theoretically, such a path would also be ruled out because it makes lifetime utility undefined or unbounded, violating the feasibility and solvency conditions of the optimization problem.

3. Then it must be that the rate of growth of assets is less than the rate of growth of the discounting factor such that $\lim_{t\to\infty} a(t)e^{-\int_0^t \bar{r}(s)ds} = 0$

Then we have:

$$\underbrace{\int_0^\infty \bar{w}(t)l(t)e^{-\int_0^t \bar{r}(s)ds}dt + a_0}_{\text{Present value of lifetime wealth after tax}} = \underbrace{\int_0^\infty c(t)e^{-\int_0^t \bar{r}(s)ds}dt}_{\text{Present value of lifetime consumption}}$$

Consequently, the household chooses initial consumption c_0 as high as possible while still satisfying the intertemporal budget.

From the first order conditions, and assuming the implicit function theorem, we know that the households' maximized utility makes consumption and labor supply choices a function of after-tax remuneration and of the co-state variable.

$$\begin{cases} u_c(c,l) = \tilde{q} \\ u_l(c,l) = -\bar{w}\tilde{q} \end{cases} \implies \begin{cases} c = c(\bar{w},\tilde{q}) \\ l = l(\bar{w},\tilde{q}) \end{cases}$$

Then we have the **Frisch compensated demand**, how an individual adjusts consumption or labor in response to a change in the wage or interest rate, **holding the marginal utility of wealth constant** ($\tilde{q}$).

Then, $\tilde{q}_0$ must be chosen as low as possible while still satisfying the budget constraint. A low $\tilde{q}_0$ corresponds to a household that cannot increase its lifetime utility by reallocating wealth between consumption and leisure. If $\tilde{q}_0$ were higher than necessary, the household would have leftover "marginal value" of wealth, indicating that utility could still be increased by adjusting consumption or leisure over time.

The marginal conditions together with the equations of motion and the constraints for system of Euler equations that can be solved for the full dynamics of this model economy given the initial stock of physical capital and debt (In Lucas paper also include the initial stock of human capital). (Lucas, 1990)

By setting the tax rates τ and θ equal to zero, these same equalities also serve to characterize the first-best allocation. (Lucas, 1990)

1.3 Efficient taxes

1.3.1 Primal Problem

We think of the government as directly choosing a feasible resource allocation, subject to constraints that express the assumption that it is possible to find prices such that price-taking household will be willing to consume their part of this allocation. We can then work backward from such an ***implementable*** *allocation to the set of taxes that will implement it.*

In an implementable allocation, the household budget constraint must be satisfied, and so must the marginal conditions. (Lucas, 1990)

Rewriting the marginal conditions:

$U_c = \tilde{q}(t)$

$-U_l = \bar{w}\tilde{q}(t)$

$\frac{\dot{\tilde{q}}(t)}{\tilde{q}(t)} = \rho - \bar{r}$

And the intertemporal constraint is:

$a_0 + \int_0^\infty e^{-\int_0^t r(t)ds} \left[\bar{w}(t)l(t) - c(t)\right] dt = 0$

A fictional lump-sum tax T is introduced solely to provide an interpretation of the associated multiplier. Since lump-sum taxes are assumed to be available (second-best setting), we set $T = 0$ in the analysis.

$$a_0 + \int_0^\infty e^{-\int_0^t r(s)ds} \left[\bar{w}(t)l(t) - c(t) - T\right] dt = 0$$

Consider the first-order differential condition:

$$\frac{\dot{\tilde{q}}(t)}{\tilde{q}(t)} = \rho - \bar{r}(t)$$

Integrating booth sides from 0 to t:

$$\int_0^t \frac{\dot{\tilde{q}}(s)}{\tilde{q}(s)} ds = \int_0^t \rho - \bar{r}(s) ds$$

which gives

$$\log \tilde{q}(t) - \log \tilde{q}(0) = \int_0^t \rho - \bar{r}(s) ds$$

Exponentiation yields

$$\tilde{q}(t) = \tilde{q}(0) e^{\int_0^t \rho - \bar{r}(s) ds} = \tilde{q}(0) e^{\rho t - \int_0^t \bar{r}(s) ds}$$

Then we have:

$$e^{-\int_0^t \bar{r}(s)ds} = \frac{\tilde{q}(t)}{\tilde{q}(0)} e^{-\rho t} = \frac{u_c(c(t), l(t))}{u_c(c(0), l(0))} e^{-\rho t}$$

Having worked the conditions we can add them to our intertemporal constraint

$$a_0 + \int_0^\infty \frac{u_c(c(t), l(t))}{u_c(c(0), l(0))} e^{-\rho t} \left[\bar{w}(t)l(t) - c(t) - T\right] dt = 0$$

Multiplying both sides by $u_c(c(0), l(0))$

$$u_c(c(0), l(0)) a_0 + \int_0^\infty u_c(c(t), l(t)) e^{-\rho t} \left[\bar{w}(t)l(t) - c(t) - T\right] dt = 0$$

Distributing $u_c(c(t), l(t))$

$u_c(c(0), l(0)) a_0 + \int_0^\infty e^{-\rho t} \left[u_c(c(t), l(t))\bar{w}(t)l(t) - u_c(c(t), l(t))c(t) - u_c(c(t), l(t))T\right] dt = 0$

Finally using the marginal conditions

$$u_c(c(0),l(0))a_0 - \int_0^\infty e^{-\rho t}\left[u_l(c(t),l(t))l(t) + u_c(c(t),l(t))c(t) - u_c(c(t),l(t))T\right]dt = 0$$

For better visualization, we drop the explicit dependence on time t and let $\tilde{T} \equiv u_c(c(t),l(t))T$:

$$u_c(0)a_0 - \int_0^\infty e^{-\rho t}\left(u_l l + u_c c + \tilde{T}\right)dt = 0$$

As we can now see, the budget constraint is solely expressed in terms of quantities and their associated shadow prices, rather than nominal monetary values. The integrand represents the present value of marginal contributions of consumption, labor, and transfers, weighted by their respective marginal utilities.

No-confiscation constraint: The government cannot confiscate household wealth in the initial period. To enforce this, we ignore the first period (or the period where the confiscation constraint would bind) in the implementability condition. This ensures that households remain solvent initially, while the dynamic budget constraint and optimality conditions determine feasible paths of consumption, labor, and transfers over time.

Once the restrictions are incorporated, the maximization problem can be stated as:

$$\max \int_0^\infty e^{-\rho t}u(c,l)$$

$$\text{s.t.} \begin{cases} u_c(c(0),l(0))a_0 - \int_0^\infty e^{-\rho t}\left[u_l(c(t),l(t))l(t) + u_c(c(t),l(t))c(t) + \tilde{T}\right]dt = 0 \\ \dot{k} = f(k,l) - c - g \end{cases}$$

This is an **isoperimetric problem**, which can be rewritten using a Lagrange multiplier λ for the implementability constraint:

$$\max \int_0^\infty e^{-\rho t}\left\{u(c(t),l(t)) - \lambda\left[u_l(c(t),l(t))l(t) + u_c(c(t),l(t))c(t) + \tilde{T}\right]\right\}dt$$

$$\text{s.t. } \dot{k} = f(k,l) - c - g$$

The **current-value Hamiltonian** is

$$\mathscr{H}_c = u(c(t),l(t)) - \lambda\left\{u_l(c(t),l(t))l(t) + u_c(c(t),l(t))c(t) + \tilde{T}(t)\right\} + \phi\left(f(k,l) - c - g\right)$$

Where ϕ is expressed in current value.

Differentiating with respect to the lump-sum term $\tilde{T}$ gives

$$\frac{\partial \mathscr{H}_c}{\partial \tilde{T}} = -\lambda$$

Since introducing a lump-sum tax would improve welfare

$$\frac{\partial \mathscr{H}_c}{\partial \tilde{T}} > 0 \implies \lambda < 0$$

Given that in the second-best (distortionary) setting, the lump-sum tax is not used, and the multiplier λ associated with the implementability constraint is negative whenever the incentive (implementability) constraint is binding.

If initial assets are sufficient (or lump-sum taxation is feasible) so that the government could finance transfers without distorting other choices, the first-best allocation can be implemented without intervention. In this case, the implementability constraint is not binding, and the associated multiplier satisfies $\lambda = 0$

FOC

$\frac{\partial \mathscr{H}_c}{\partial c} = u_c - \lambda \left(u_{lc} l(t) + u_{cc} c(t) + u_c \right) - \phi = 0$

$\frac{\partial \mathscr{H}_c}{\partial l} = u_l - \lambda \left(u_{ll} l(t) + u_{cl} c(t) + u_l \right) + f_l \phi = 0$

$\frac{\partial \mathscr{H}_c}{\partial k} = \phi f_k = -\dot{\phi} + \rho \phi$

$$\frac{\dot{\phi}}{\phi} = \rho - f_k$$

In steady state c, l is constant consumption optimum, then we can prove that $\tilde{q}$ is also constant

$$0 = \underbrace{U_{cc}\dot{c} + U_{cl}\dot{l}}_{\text{first Household FOC}} = \dot{\tilde{q}} = \underbrace{(\rho - \bar{r})\tilde{q}}_{\text{third Household FOC}}$$

Since $\dot{c} = \dot{l} = 0$ in the steady state, it follows that

$$\rho = \bar{r} = (1-\theta) f_k$$

> **Theorem:** In the steady state, if consumption c, labor l, and the costate ϕ are constant, then the capital tax distortion is zero:
>
> $$\frac{\dot{\phi}}{\phi} = \rho - f_k = -\theta f_k$$
>
> Since ϕ is constant ($\dot{\phi} = 0$), it follows that
>
> $$\theta = 0$$
>
> **Note**
>
> When the utility function is $u(c) = \ln c$, it turns out that $\theta \neq 0$. In this case, the costate ϕ is **not constant**—it may grow or "explode" over time.

> The constancy of costates is **not necessary** for optimality. Whatmatters is that the **transversality conditions** are satisfied, ensuring the solution is feasible and welfare-maximizing.
>
> Thus, constant consumption and labor in steady state do **not always imply constant costates**; the specific utility functional form can affect the dynamics of the multipliers.

Working the FOC

From the first one

$1-\lambda\left(H_c+1\right)-\frac{\phi}{u_c}=0$

From the second one

$1-\lambda\left(H_l+1\right)+f_l\frac{\phi}{u_l}=0$

Where H_i denotes the elasticity of the marginal utility with respect to i.

> **Note**: This follows Atkinson and Stiglitz (1972), although they define the elasticity with a negative sign; here it is defined with a positive sign for convenience.

Using the household FOC:

$1-\lambda\left(H_c+1\right)=\frac{\phi}{\tilde{q}}$

Given that H_c is constant (in steady state c and l are constant, so it must be constant) then $\frac{\phi}{\tilde{q}}$ is also constant the ratio of the co-states is constant over time. Then it must be that:

$$\log\frac{\phi}{\tilde{q}}=\log\phi-\log\tilde{q}\implies\dot{\phi}-\dot{\tilde{q}}=\rho-f_k-(\rho-\bar{r})=\bar{r}-f_k=(1-\theta)f_k-f_k$$

Thus $\theta=0$, which corresponds either to **steady state** or to a **CRRA utility** $u=\frac{c^{-1\sigma}}{1-\sigma}$.

The full solution to the Ramsey problem, then, must involve heavy initial capital taxation followed by lower and ultimately zero taxation (...) Capital income taxation will initially be high, imitating capital levy on the initial stock. If the system converges to a balanced path, capital taxation will converge to zero. (Lucas, 1990)

$1-\lambda\left(H_l+1\right)=\frac{f_l}{\bar{w}}\frac{\phi}{\tilde{q}}$

Taking the ratio:

$$\frac{1-\lambda\left(H_c+1\right)}{1-\lambda\left(H_l+1\right)}=\frac{\bar{w}}{w}=1-\tau$$

If the numerator is smaller than the denominator, then $\tau>0$ (and conversely). So:

$$1-\lambda\left(H_c+1\right)<1-\lambda\left(H_l+1\right)\implies\tau>0$$

Then continue:

$1-\lambda\left(H_c+1\right)<1-\lambda\left(H_l+1\right)\implies\tau>0$

Given that $\lambda < 0$ then we can write.

$$(H_c + 1) < (H_l + 1) \implies \tau > 0$$

$$H_c < H_l \implies \tau > 0$$

So we can write it as:

$$\frac{u_{cc}c + u_{cl}l}{u_c} < \frac{u_{lc}c + u_{ll}l}{u_l} \implies \tau > 0$$

This inequality states that the marginal disutility of labor increases more rapidly than the marginal utility of consumption as the household scales up both c and l proportionally. Equivalently, if the household increases consumption and leisure together, the additional "pain" from working rises faster than the additional "benefit" from consuming.

In other words, labor is relatively more concave in the household's utility function than consumption.

- If the utility function is **additively separable**, i.e., $u(c,l) = u(c) + v(l)$ then the cross-derivative $u_{cl} = 0$ and the inequality holds, so $\tau > 0$ by construction.
- If both consumption and leisure are normal goods, this condition is also satisfied: as income rises, both c and l increase, implying that the marginal utility of consumption declines more slowly than the marginal disutility of labor. *See **appendix** for a formal derivation in the general case.*

Appendix: Comparative static

Here we analyze how **equilibrium values** of variables change in response to changes in parameters holding the system in **steady state**.

For the maximizing utility problem of household we can summarize the results in:

$$\begin{cases} u_c \equiv \tilde{q} \\ -u_l \equiv \bar{w}\tilde{q} \end{cases}$$

Taking total differential:

$$u_{cc}dc + u_{cl}dl = d\tilde{q}$$

This shows how a small change in the marginal utility of consumption can be archived by adjusting consumption dc and/or labor dl.

$$u_{lc}dc + u_{ll}dl = -\tilde{q}d\bar{w} - \bar{w}d\tilde{q}$$

Then the change in the shadow value of labor income, must be compensated by changes in the consumption and labor affecting the marginal disutility of labor.

Putting the results in matrix form

$$\begin{bmatrix} u_{cc} & u_{cl} \\ u_{lc} & u_{ll} \end{bmatrix} \begin{bmatrix} dc \\ dl \end{bmatrix} = \begin{bmatrix} d\tilde{q} \\ -\tilde{q}d\bar{w} - \bar{w}d\tilde{q} \end{bmatrix}$$

Let H denote the determinant of the Hessian of $u(c,l)$ given that we assume concavity of the utility function then $H > 0$.

> **Note**: For static utility maximization, concavity is sufficient but not necessary; quasi-concavity of $u(c,l)$ is enough to guarantee a global maximum. In intertemporal optimization, **concavity of the instantaneous utility** and the quasi-concavity of the overall objective ensures that the Hamiltonian maximization yields well-defined FOCs and stable solutions.

Then, using Cramer's rule, the solutions are:

$$dc = \frac{1}{H}\begin{vmatrix} d\tilde{q} & u_{cl} \\ -\tilde{q}d\bar{w} - \bar{w}d\tilde{q} & u_{ll} \end{vmatrix} = \frac{(u_{ll} + u_{cl}\bar{w})d\tilde{q} + u_{cl}\tilde{q}d\bar{w}}{H}$$

$$dl = \frac{1}{H}\begin{vmatrix} u_{cc} & d\tilde{q} \\ u_{lc} & -\tilde{q}d\bar{w} - \bar{w}d\tilde{q} \end{vmatrix} = -\frac{(u_{lc} + u_{cc}\bar{w})d\tilde{q} + u_{cc}\tilde{q}d\bar{w}}{H}$$

Let o denote leisure in the economy (o comes from ocio, Spanish for leisure). Given that total time is normalized: $T = l + o$, a change in labor implies an opposite change in leisure $dl = -do$

Using this, we can restate the comparative statics in terms of leisure instead of labor:

$$do = -\frac{1}{H}\begin{vmatrix} u_{cc} & d\tilde{q} \\ u_{lc} & -\tilde{q}d\bar{w} - \bar{w}d\tilde{q} \end{vmatrix} = \frac{(u_{cc}\bar{w} + u_{lc})d\tilde{q} + u_{cc}\tilde{q}d\bar{w}}{H}$$

Comparative static with respect to the wage $\bar{w}$

This measures how consumption c and leisure o respond to changes in the price of leisure.

Assuming $d\tilde{q} = 0$ (holding the shadow value constant):

$$do = \frac{u_{cc}\tilde{q}d\bar{w}}{H} \implies \left.\frac{do}{d\bar{w}}\right|_{d\tilde{q}=0} = \frac{\partial o}{\partial \bar{w}} = \frac{u_{cc}\tilde{q}}{H} < 0$$

This indicates that leisure is an ordinary good: an increase in the wage (price of leisure) reduces leisure.

$$dc = \frac{u_{cl}\tilde{q}d\bar{w}}{H} \implies \left.\frac{dc}{d\bar{w}}\right|_{d\tilde{q}=0} = \frac{\partial c}{\partial \bar{w}} = \frac{u_{cl}\tilde{q}}{H}$$

Where the sign of $dc/d\bar{w}$ depends on the cross-derivative u_{cl}, if $u_{cl} > 0$, leisure and consumption are substitutes; if $u_{cl} < 0$, they are complements.

Comparative static with respect to the shadow value $\tilde{q}$

The shadow value $\tilde{q}$ represents the marginal utility of wealth, or equivalently, the intertemporal price of resources. If the shadow value increases, the marginal utility of wealth rises. Given the concavity of the utility function, this implies that effective wealth has decreased: the household is "poorer" in utility terms.

Assuming $d\bar{w} = 0$ (holding the wage constant):

$$dc = \frac{(u_{ll} + u_{cl}\bar{w})d\tilde{q}}{H} \implies \left.\frac{dc}{d\tilde{q}}\right|_{d\bar{w}=0} = \frac{\partial c}{\partial \tilde{q}} = \frac{u_{ll} + u_{cl}\bar{w}}{H}$$

$$do = \frac{(u_{cc}\bar{w} + u_{cl})d\tilde{q}}{H} \implies \left.\frac{do}{d\tilde{q}}\right|_{d\bar{w}=0} = \frac{\partial o}{\partial \tilde{q}} = \frac{u_{cc}\bar{w} + u_{cl}}{H}$$

If we assume that both goods are normal then it must be that:

$$u_{ll} + u_{cl}\bar{w} < 0, \qquad u_{cc}\bar{w} + u_{cl} < 0$$

Multiplying by l the first inequality and by c the second one:

$$u_{ll}l + u_{cl}l\bar{w} < 0, \qquad u_{cc}c\bar{w} + u_{cl}c < 0$$

Dividing by $\bar{w}\tilde{q} = \bar{w}u_c = -u_l > 0$

$$-\frac{u_{ll}l}{u_l} + \frac{u_{cl}l}{u_c} < 0, \qquad \frac{u_{cc}c}{u_c} - \frac{u_{cl}c}{u_l} < 0$$

Summing:

$$\frac{u_{cc}c}{u_c} - \frac{u_{cl}c}{u_l} - \frac{u_{ll}l}{u_l} + \frac{u_{cl}l}{u_c} < 0$$

Then it must be that:

$$\boxed{\frac{u_{cc}c + u_{cl}l}{u_c} < \frac{u_{ll}l + u_{cl}c}{u_l}}$$

Appendix: Ramsey Commodity Taxation - The collapse of the multiple household to the one household case

Property of homothetic preferences

When preferences are homothetic means that they are a monotonic transformation of homogeneous functions. Because we only care about ordinality of the utility function, the demands generated by $U(x,l)$ and $u(x,l)$, where $U(x,l) = g(u(x,l))$ and $g'(\cdot) > 0$, are the same for every (p,w).

Then we can demonstrate that by using $u(x,l)$ (being linearly homogeneous; and by the above explanation encompassing all the rest of the case of homothetic functions) we can write the demand as a separable function of p and I, such that:

$\frac{\partial u}{\partial x_i} = \lambda p_i$

Multiplying by x_i and summing over i (including labor):

$\frac{\partial u}{\partial \mathbf{x}}\mathbf{x} + \frac{\partial u}{\partial l} l = \lambda \left(\mathbf{p}\mathbf{x} - wl\right) = \lambda I$

But because preferences are homogeneous, we can use Euler's properties:

$$u(tx, tl) = t^r u(x,l) \iff \frac{\partial u}{\partial \mathbf{x}}\mathbf{x} + \frac{\partial u}{\partial l} l = rt^{r-1}u(x,l) = ru(x,l)$$

Replacing:

$$u(x,l) = \lambda I$$

Evaluted in the optimum this is an equivalence:

$$\mathscr{V}(p,I) \equiv \lambda^*(p,I)I$$

By the envelope theorem:

$$\frac{d\mathscr{V}(p,I)}{dI} = \lambda^*(p,I)$$

Then:

$$\mathscr{V}(p,I) = \frac{d\mathscr{V}(p,I)}{dI} I$$

Then we have an ordinary differential equation:

$$\frac{1}{I} = \frac{1}{\mathscr{V}(p,I)}\frac{d\mathscr{V}(p,I)}{dI} = \frac{d}{dI}\log \mathscr{V}(p,I)$$

Solving:

$$\log \mathscr{V}(p,I) = \log I + C \implies \mathscr{V}(p,I) = Ie^C$$

Where C is the constant of integration. Evaluating at $I = 1$:

$$e^C = \mathscr{V}(p, 1) = h(p)$$

So the solution of our ordinary differential equation is.

$$\mathscr{V}(p, I) = Ih(p)$$

Using Roy's identity:

$$x = -I\frac{h'(p)}{h(p)} \equiv Ig(p)$$

Then let's write:

$$\bar{x}_j = \sum_{h=1}^{H} I^h g(p) = \bar{I}g(p)$$

Then:

$$\frac{x_j^h}{\bar{x}_j} = \frac{I^h g(p)}{\bar{I}g(p)} = \frac{I^h}{\bar{I}} = \omega^h = \text{ Relative income share}$$

The Collapse of the Rule

When we have households with homothetic preferences that only difference their selves by the amount of income then the Ramsey rule collapses to the one person Ramsey Rule:

$$\frac{\Delta \mathbf{h}_j}{\mathbf{x}_j} = -\left[1 - \sum_{h=1}^{H}\left(\frac{\beta^h}{\lambda} + \sum_{i=1}^{n} t_i \frac{\partial x_i^h}{\partial I^h}\right)\frac{x_j^h}{\mathbf{x}_j}\right]$$

But the we have, check appendix, :

$$\frac{\Delta \mathbf{h}_j}{\mathbf{x}_j} = -\left[1 - \sum_{h=1}^{H}\left(\frac{\beta^h}{\lambda} + \sum_{i=1}^{n} t_i g_i(p)\right)\frac{I^h}{\bar{I}}\right]$$

Let's call the relative income share ω^h.

$$\frac{\Delta \mathbf{h}_j}{\mathbf{x}_j} = -\left[1 - \sum_{h=1}^{H}\left(\frac{\beta^h}{\lambda} + \sum_{i=1}^{n} t_i g_i(p)\right)\omega^h\right]$$

Then it doesn't depend on j. So what ultimately determines the differences between the many households case and the one-person case in commodity taxation is the distribution of consumption of that commodity.

This is the case of a linear income expansion path (Engel curve) passing from the origin, where the optimal of all the households' maximization problems will lie on the **same ray passing from the origin**. That is, many-household commodity taxation collapses onto single-household commodity taxation if and only if all households have homothetic preferences.

In this case, it is only a matter of overall equity and efficiency considerations on the extent of the distortion to be imposed on all commodities θ, but that distortion will be the same on every commodity and will be equal to θ for every good j.