Income Taxation

1 Labor income taxation

1.1 Non-discriminatory taxation

In this section, taxation is assumed to be non-scheduled, meaning that the government applies a uniform tax on labor income without differentiating between individuals according to their wage level. Each agent faces the same proportional tax rate, regardless of their productivity or earnings.

1.1.1 The Representative Household

The household's utility maximization porblem holds as a starting point for the primal problem:

$$\int_0^{\infty} e^{-(\rho-\lambda)t} U\left[c(t), l(t)\right] dt$$

subject to the dynamic budget constraint:

$$\begin{cases} \dot{a} = \bar{r}(t)a(t) + \bar{w}(t)l(t) - c(t) \\ a(0) = a_0 \end{cases}$$

We know that the conditions then:

$$\begin{cases} u_c = \tilde{q} \\ u_l = -\bar{w}\tilde{q} \\ \dot{\tilde{q}} = (\rho - r)\tilde{q} \end{cases}$$

If we integrate the budget constraint, we get the intertemporal constraint:

$$\underbrace{\int_0^{\infty} \bar{w}(t)l(t)e^{-\int_0^t \bar{r}(s)ds} dt + a_0}_{\text{Present value of lifetime wealth after tax}} = \underbrace{\int_0^{\infty} c(t)e^{-\int_0^t \bar{r}(s)ds} dt}_{\text{Present value of lifetime consumption}}$$

And from the last first order condition we have:

$$\tilde{q}(t) = q(0)e^{\int_0^t \rho - r(s)ds} = \tilde{q}(0)e^{\rho t - \int_0^t r(s)ds} \implies e^{-\int_0^t r(s)ds} = \frac{\tilde{q}(t)}{q(0)}e^{-\rho t}$$

Then if we replaced everything in the intertemporl budget constraint and add a fictional lump sum tax T we ended up with:

$$u_c(0)a_0 - \int_0^\infty e^{-\rho t}\left(u_l l + u_c c + \tilde{T}\right) dt = 0$$

Where $\tilde{T} \equiv q(t)T = u_c(c(t), l(t))T$

1.1.2 The Primal Approach

Once the restrictions are incorporated, the maximization problem can be stated as:

$$\max \int_0^\infty e^{-\rho t} u(c, l)$$

$$\text{s.t.} \quad \begin{cases} u_c(c(0), l(0))a_0 - \int_0^\infty e^{-\rho t}\left[u_l(c(t), l(t))l(t) + u_c(c(t), l(t))c(t) + \tilde{T}\right] dt = 0 \\ \dot{k} = f(k, l) - c - g \end{cases}$$

This is an **isoperimetric problem**, which can be rewritten using a Lagrange multiplier λ for the implementability constraint:

$$\max \int_0^\infty e^{-\rho t}\left\{u(c, l) - \lambda\left[u_l(c(t), l(t))l(t) + u_c(c(t), l(t))c(t) + \tilde{T}\right]\right\} dt$$

$$\text{s.t.} \quad \dot{k} = f(k, l) - c - g$$

The **current-value Hamiltonian** is

$$\mathscr{H}_c = u(c, l) - \lambda\left\{u_l(c(t), l(t))l(t) + u_c(c(t), l(t))c(t) + \tilde{T}(t)\right\} + \phi\left(f(k, l) - c - g\right)$$

Where ϕ is expressed in current value.

Differentiating with respect to the lump-sum term $\tilde{T}$ gives

$$\frac{\partial \mathscr{H}_c}{\partial \tilde{T}} = -\lambda$$

Since introducing a lump-sum tax would improve welfare

$$\frac{\partial \mathscr{H}_c}{\partial \tilde{T}} > 0 \implies \lambda > 0$$

Given that in the second-best (distortionary) setting, the lump-sum tax is not used, and the multiplier λ associated with the implementability constraint is negative whenever the incentive (implementability) constraint is binding.

If initial assets are sufficient (or lump-sum taxation is feasible) so that the government could finance transfers without distorting other choices, the first-best allocation can be implemented without intervention. In this case, the implementability constraint is not binding, and the associated multiplier satisfies $\lambda = 0$

FOC

$\frac{\partial \mathscr{H}_c}{\partial c} = u_c(c,l) - \lambda\left(u_{lc}l(t) + u_{cc}c(t) + u_c\right) - \phi = 0$

$\frac{\partial \mathscr{H}_c}{\partial l} = u_l(c,l) - \lambda\left(u_{ll}l(t) + u_{cl}c(t) + u_l\right) + f_k\phi = 0$

$\frac{\partial \mathscr{H}_c}{\partial k} = \phi f_k = -\dot{\phi} + \rho\phi$

$$\frac{\dot{\phi}}{\phi} = \rho - f_k$$

Working the FOC

From the first one

$1 - \lambda\left(H_c + 1\right) - \frac{\phi}{u_c} = 0$

From the second one

$1 - \lambda\left(H\_l + 1\right) + f\_l\frac{\phi}{u_l} = 0$

Where H_i denotes the elasticity of the marginal utility with respect to i.

> **Note**: This follows Atkinson and Stiglitz (1972), although they define the elasticity with a negative sign; here it is defined with a positive sign for convenience.

Using the household FOC:

$1 - \lambda\left(H\_c + 1\right) = \frac{\phi}{\tilde{q}}$

Given that H_c is constant (in steady state c and l are constant, so it must be constant) then $\frac{\phi}{\tilde{q}}$ is also constant the ratio of the co-states is constant over time. Then it must be that:

$$\log\frac{\phi}{\tilde{q}} = \log\phi - \log\tilde{q} \implies \dot{\phi} - \dot{\tilde{q}} = \rho - f_k - (\rho - \bar{r}) = \bar{r} - f_k = (1-\theta)f_k - f_k$$

Thus, again, $\theta = 0$, which corresponds either to **steady state** or to a **CRRA utility** $u = \frac{c^{-1\sigma}}{1-\sigma}$.

$1 - \lambda\left(H\_l + 1\right) = \frac{f_l}{\bar{w}}\frac{\phi}{\tilde{q}}$

Taking the ratio:

$$\frac{1 - \lambda\left(H_l + 1\right)}{1 - \lambda\left(H_c + 1\right)} = \frac{w}{\bar{w}} = \frac{1}{1-\tau}$$

Substracting from both sides -1

$\frac{1-\lambda(H_l+1)}{1-\lambda(H_c+1)} - 1 = \frac{1}{1-\tau} - 1 = \frac{\tau}{1-\tau}$

$\frac{1-\lambda(H_l+1)-1+\lambda(H_c+1)}{1-\lambda(H_c+1)} = \frac{\tau}{1-\tau}$

$$\frac{\lambda\left(H_c-H_l\right)}{1-\lambda\left(H_c+1\right)}=\frac{\tau}{1-\tau} \rightarrow \text{ Atkinson and Stiglitz (1972) condition}$$

1.2 An Exploration in the Theory of Optimum Income Taxation: A problem of mechanism design

1.2.1 Optimal Non-linear tax problem

Here, tax rates vary across individuals according to their labor income, allowing the government to design an **optimal wage schedule**. The simplicity of the non-scheduled system avoids issues of self-selection and information asymmetry but limits redistributive possibilities. (Signaling game)

While similar distinctions could, in principle, be applied to consumption taxation, designing progressive schedules for consumption is far more complex due to what is known as the "parking lot problem": under a progressive consumption tax, agents would have incentives to exchange purchase orders or reallocate spending among themselves to exploit lower tax brackets. Conversely, a regressive schedule would encourage joint purchases to minimize the total tax burden. Hence, consumption taxation is typically implemented uniformly, whereas labor income taxation admits both non-scheduled and scheduled forms.

One would suppose that in any economic system where equality is valued, progressive income taxation would be an important instrument of policy. Even in a highly socialist economy, where all who work are employed by the state, the shadow price of highly skilled labor should surely be considerably greater than the disposable income actually available to the laborer. (Mirrlees, 1971)

In **designing progressive labor income taxation**, the government seeks a reliable signal of an individual's productivity. While observable characteristics such as IQ, education, age, location, or even ethnicity might correlate with earnings, they are imperfect proxies. The most direct and informative indicator of an individual's income-earning potential is their actual income (Mirrlees, 1971).

Therefore, the design of progressive taxation requires careful consideration of **incentive compatibility** to ensure that reported income accurately reflects productivity.

Simplifying assumptions

- Intertemporal problems are ignored.
- Difference in tastes, in family size and composition, and in voluntary transfers are ignored.
- **Individuals are supposed to determine the quantity and kind of labor they provide by rational calculation**.(Can this assumption be justified by saying that instead amount of labor what the individual is offering is amount of effort?)
- Migration is supposed to be impossible.
- The state is supposed to have perfect information about the individuals in the economy, their utilities and, consequently their actions.
- There is supposed to be one kind of labor.
- There is one consumer good.
- Welfare is separable in terms of the different individuals of the economy, and symmetric.
- The cost of administering the optimum tax schedule are assumed to be negligible.

Model We are in a world in which the first welfare theorem holds, competitive equilibrium is Pareto efficient. We suppose household's utility depends on consumption c and labor supply l.

$$\max_{c,l} U(c,l)$$

$$\text{s.t. } c = sl$$

where s denotes the individual's skill or productivity, i.e., their wage rate per unit of labor supplied. Hence, the labor choice depends on the individual's ability.

A more productive individual cannot earn less than a less productive one under this setup. Utility U is assumed to be strictly concave, continuously differentiable, strictly increasing in c, and strictly decreasing in l.

In this framework, an individual's type is fully characterized by their productivity s. Individuals with the same s will choose identical consumption and labor supply patterns, yielding the same labor-consumption combination in equilibrium.

If we introduce a tax schedule, the tax is a function of current labor income:

$$T(sl) = T(z)$$

so that the individual's optimization problem becomes:

$$\max_{c,l} U(c,l) \qquad \text{s.t. } c = sl - T(sl)$$

The Lagrangian is:

$$\mathscr{L} = U(c,l) + \lambda \left[sl - T(sl) - c \right]$$

FOC

$\frac{\partial \mathscr{L}}{\partial c} = u_c - \lambda = 0$

$\frac{\partial \mathscr{L}}{\partial l} = u_l + \lambda \left(s - T'(sl)s \right) = 0$

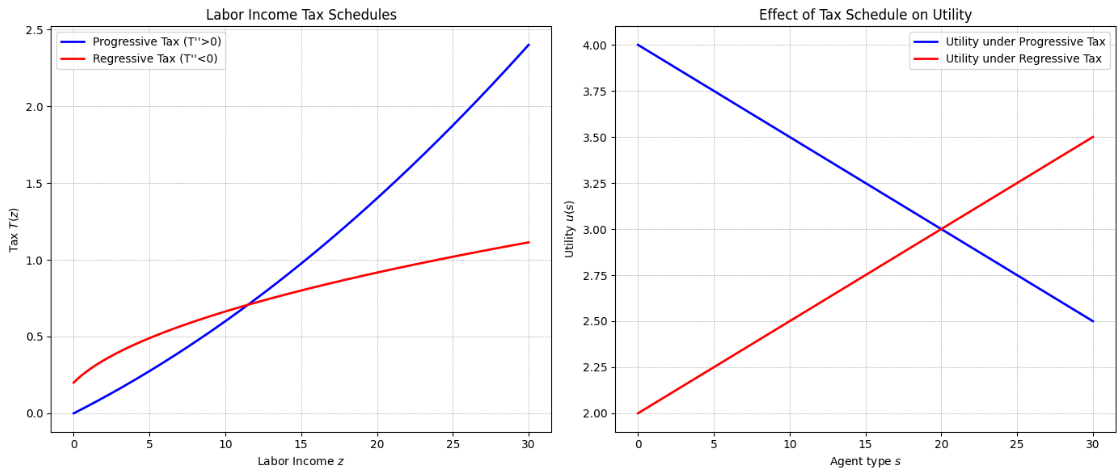
The equilibrium condition:

$$-\frac{u_l}{u_c} = s - T'(sl)s = (1 - T'(z))\, s \iff T'(z) = 1 - \frac{u_l}{u_c}\frac{1}{s} = u_l \left(\frac{1}{u_l} - \frac{1}{u_c}\frac{1}{s} \right)$$

Then the optimal units would be:

$$c^* = c(s), \qquad l^* = l(s), \qquad z^* = z(s) = sl(s)$$

A **progressive labor income tax schedule** characterizes $T(z)$ as an increasing and convex function of the labor income z. the slope of the fiscal rate (the **marginal income tax rate**) is itself increasing, so that $T'(z) > 0$ and $T''(z) > 0$. Conversely, when $T''(z) < 0$, the tax schedule is said to be **regressive**.



If the function $T(z)$ takes negative values for some z, the tax rate is negative, meaning the government subsidizes labor income for those values of z.

The indirect utility function then:

$$U\left(c(s), \frac{z(s)}{s}\right) \qquad \text{where} \qquad c(s) \equiv z(s) - T(z(s))$$

This formulation transforms the problem from one with identical utility functions but individual-specific constraints (dependent on s), to one with identical constraints across individuals but potentially different effective utilities. Given that preferences are identical across individuals, the functions $c(s)$ and $z(s)$ are the same for all agents.

- **Comparative statics with respect to** s

Differentiating U with respect to s gives:

$$dU = u_c \frac{dc}{ds} + u_l \frac{dz}{ds} - u_l \frac{z(s)}{s^2} ds \qquad \text{where} \qquad \frac{dc}{ds} = [1 - T'(z(s))] \frac{dz}{ds}$$

Given the Lagrange multiplier λ for the constraint $c(s) = z(s) - T(z(s))$, multiply the constraint by $-\lambda$ and sum:

$dU = u_c \frac{dc}{ds} + u_l \frac{dz}{ds} - u_l \frac{z(s)}{s^2} ds - \lambda \frac{dc}{ds} + \lambda [1 - T'(z(s))] \frac{dz}{ds}$

$$dU = (u_c - \lambda) \frac{dc}{ds} + \{u_l + \lambda [1 - T'(z(s))]\} \frac{dz}{ds} - u_l \frac{z(s)}{s^2} ds$$

Using the first-order conditions (FOC) for the household, we obtain the *envelope* result:

$$\frac{dU}{ds} = U'(s) = -u_l \frac{z(s)}{s^2}$$

This expression constitutes the **incentive-compatibility condition**. It ensures that, given the tax schedule $T(z)$, each individual of type s (with productivity s) prefers to choose the labor supply $z(s)$ designed for their own type rather than mimicking another type. In other words, it guarantees that **truthful self-selection** is optimal.

Because utility $U(s)$ must satisfy this differential condition, the planner **cannot freely equalize utility across individuals**. A more progressive tax schedule—characterized by a steeper $T'(z)$—reduces the net return to higher productivity, implying that $U(s)$ decreases with s. **This means that more productive agents obtain lower utility than less productive ones, even though they earn higher pre-tax income.**

If the tax schedule were excessively progressive, $U(s)$ would decline so sharply with s that high-productivity individuals would prefer to behave like low-productivity ones, supplying less labor and effectively “mimicking” them. In that case, the incentive-compatibility constraint would be violated.

Hence, the trade-off in optimal taxation arises: redistribution (through progressivity) must be balanced against incentive compatibility. Too much redistribution destroys incentives for high-productivity agents to exert effort or reveal their true type.

Taking c as a function of $u(s)$ and $z(s)$ then we have:

$$c(s) = X\left(u(s), z(s)\right)$$

Given that it has to fulfill the condition of:

$$U^*\left(c(s), \frac{z(s)}{s}\right)$$

$dU = u_c dc + u_l \frac{dz}{s} = 0$

Then we have:

$$\frac{dc}{dz} = -\frac{u_l}{u_c}\frac{1}{s}$$

$$\frac{dc}{dU} = \frac{1}{u_c}$$

Rather than assigning consumption and income schedules $(c(s), z(s))$ to each household type s, one can equivalently assign utility and income levels (U, z) to each skill type. Indeed, since the household's optimal utility level depends solely on its skill parameter s, we can represent preferences through the indirect utility function $U(s) = U^*\left(c(s), \frac{z(s)}{s}\right)$.

If the individual chooses income optimally given their skill, the problem can be written as

$$\max u\left(z, \frac{z}{s}\right)$$

(When there are no taxes (or when the tax schedule remains unchanged), income and consumption coincide, so that $c \equiv z$)

FOC

$u_c + u_l \frac{1}{s} = 0$

Evaluated at the optimum:

$u_c + u_l \frac{1}{s} \equiv 0$

So we can differentiate with respect to s to get the comparative static:

$u_{cc}\frac{dz}{ds} + u_{cl}\frac{1}{z}\frac{dz}{ds} + u_{lc}\frac{1}{s}\frac{dz}{ds} + u_{ll}\frac{1}{s}\frac{dz}{ds} - u_{cl}\frac{z}{s^2} - u_{ll}\frac{1}{s}\frac{z}{s^2} - u_l\frac{1}{s^2} = 0$

So we can write:

$\left(u_{cc} + u_{cl}\frac{1}{z} + u_{lc}\frac{1}{s} + u_{ll}\frac{1}{s}\right)\frac{dz}{ds} - u_{cl}\frac{z}{s^2} - u_{ll}\frac{1}{s}\frac{z}{s^2} - u_l\frac{1}{s^2} = 0$

$\left(u_{cc} + u_{cl}\frac{1}{z} + u_{lc}\frac{1}{s} + u_{ll}\frac{1}{s}\right)\frac{dz}{ds} = u_{cl}\frac{z}{s^2} + u_{ll}\frac{1}{s}\frac{z}{s^2} + u_l\frac{1}{s^2}$

$\left(u_{cc} + u_{cl}\frac{1}{z} + u_{lc}\frac{1}{s} + u_{ll}\frac{1}{s}\right)\frac{dz}{ds} = \frac{u_l}{s^2}\left[\frac{u_{cl}}{u_l} + \frac{u_{ll}}{u_l}\frac{z}{s} + 1\right]$

Using the first order condition

$$\underbrace{\text{S.O.C}}_{(-)}\frac{dz}{ds} = \underbrace{\frac{u_l}{s^2}}_{(-)}\underbrace{\left[-\frac{u_{cl}}{u_c}l + \frac{u_{ll}}{u_l}l + 1\right]}_{(+)\text{by assumption}}$$

Thus, implies that more skilled individuals choose higher income levels. In this framework, skill s raises the productivity of labor (since $l = z/s$), so supplying a given income requires less effort. The condition $\frac{dz}{ds} > 0$ therefore expresses that, as individuals become more productive, they optimally choose to earn (and consume) more.

1.3 Government welfare

Government welfare is supposed to be a concave function of individual utilities of all individuals having the level of skills s, $\gamma(s)$, for all individual levels of skills. Total social welfare $\mathscr{W}$ is the sum of all individuals' welfare contributions $W(u)$, so $W(u)$ depends on the s is how the social planner values each class.

$$\mathscr{W} = \int_{\min(s)}^{\max(s)} W\left(u\left(c(s), z(s)/s\right)\right)\gamma(s)ds$$

The income by person $z(s)$, deducting the consumption level $c(s)$, is the tax contribution of person s, $z(s) - c(s)$. The tax contributions over all groups is the total tax revenue

$$\mathscr{R} = \int_{\min(s)}^{\max(s)} T(z(s))\gamma(s)ds = \int_{\min(s)}^{\max(s)} \left(z(s) - c(s)\right)\gamma(s)ds = 0$$

The total tax revenue of the state is supposed to be null, the government is not collecting anything for itself: therefore, there will be groups/individuals that have a positive tax contribution, from which taxes are collected, while there will be groups with a negative tax contribution.

$$\max_{u(s),z(s)} \mathscr{W} = \int_{\underline{s}}^{\overline{s}} W\left(u\left(s\right)\right)\gamma(s)ds$$

$$\text{s.t.} \begin{cases} \int_{\underline{s}}^{\overline{s}} \left(z(s) - c(s)\right)\gamma(s)ds = 0 \\ \\ \frac{dU}{ds} = U'(s) = -u_l \frac{z(s)}{s^2} \\ \\ c(s) = X(u(s), z(s); s) \end{cases}$$

The second constraint ensures that the allocation is incentive-compatible: utility must be consistent with the optimal income choice at each skill level. Thus, the **utility level becomes a state variable**, increasing with s, **since more skilled individuals must achieve at least as high a utility as less skilled ones**.

This is an **isopremetric problem**, then we can write it like:

$$\max_{u(s),z(s)} \mathscr{W} = \int_{\underline{s}}^{\overline{s}} \left\{W\left(u\left(s\right)\right) + \lambda\left(z(s) - c(s)\right)\right\}\gamma(s)ds$$

$$\text{s.t.} \begin{cases} \dot{u} = -u_l \frac{z(s)}{s^2} \\ \\ c(s) = X(u(s), z(s)) \end{cases}$$

In such case, the Hamiltonian can be expressed as

$$\mathscr{H} = \left\{W\left(u(s)\right) + \lambda\left(z(s) - c(s)\right)\right\}\gamma(s) + \phi(s)\left[-u_l\left(X((u(s), z(s)), \frac{z(s)}{s}\right)\right]\frac{z(s)}{s^2}$$

FOC

$\frac{\partial \mathscr{H}}{\partial z(s)} = \lambda\left(1 - \frac{\partial X}{\partial z}\right)\gamma(s) - \phi(s)\left[u_{lc}\frac{\partial X}{\partial z} + u_{ll}\frac{1}{s}\right]\frac{z(s)}{s^2} - \phi(s)u_l\frac{1}{s^2} = 0$

$\frac{\partial \mathscr{H}}{\partial u(s)} = \left[W'\left(u(s)\right) - \lambda\frac{\partial X}{\partial U}\right]\gamma(s) - \phi(s)u_{lc}\frac{\partial X}{\partial U}\frac{z(s)}{s^2} = -\dot{\phi}$

Working the condition:

$\frac{\partial \mathscr{H}}{\partial z(s)} = \lambda\left(1 + \frac{u_l}{u_c}\frac{1}{s}\right)\gamma(s) - \phi(s)\left[-u_{lc}\left(\frac{u_l}{u_c}\frac{1}{s}\right) + u_{ll}\frac{1}{s}\right]\frac{z(s)}{s^2} - \phi(s)u_l\frac{1}{s^2} = 0$

Factoring out u_l:

$\frac{\partial \mathscr{H}}{\partial z(s)} = u_l\left\{\lambda\left(\frac{1}{u_l} + \frac{1}{u_c}\frac{1}{s}\right)\gamma(s) - \frac{\phi(s)}{s^2}\left[1 + \left(\frac{u_{ll}}{u_l} - \frac{u_{lc}}{u_c}\right)\frac{z(s)}{s}\right]\right\} = 0$

Then given that $l = \frac{z(s)}{s}$

$\frac{\partial \mathscr{H}}{\partial z(s)} = u_l\left\{\lambda\left(\frac{1}{u_l} + \frac{1}{u_c}\frac{1}{s}\right)\gamma(s) - \frac{\phi(s)}{s^2}\left[1 + \left(\frac{u_{ll}}{u_l} - \frac{u_{lc}}{u_c}\right)l\right]\right\} = 0$

The term $\frac{u_{ll}}{u_l} - \frac{u_{lc}}{u_c}$ captures the income effect on consumption. By assuming that consumption is a normal good, this bracketed expression is positive. Even if the good were inferior, as long as the income effect is less than one, the expression remains positive.

Economically, this ensures that individuals with higher skill demand higher income—a monotonicity restriction.

Moreover, given the assumption $\partial z/\partial s > 0$ the entire bracketed term is positive by construction.

The whole thing is going to depend on the co-state $\phi(s)$.

$$\underbrace{\lambda}_{(+)} \underbrace{\left(\frac{1}{u_l} + \frac{1}{u_c}\frac{1}{s}\right)}_{\text{FOC: } \frac{T'(z)}{u_l}} \gamma(s) - \frac{\phi(s)}{s^2} \underbrace{\left[1 + \left(\frac{u_{ll}}{u_l} - \frac{u_{lc}}{u_c}\right) l\right]}_{(+) \iff \frac{\partial z}{\partial s} > 0} = 0$$

Meanwhile the second condtion

$\frac{\partial \mathscr{H}}{\partial u(s)} = \left[W'\left(u(s)\right) - \lambda \frac{1}{u_c}\right] \gamma(s) - \phi(s) \frac{u_{lc}}{u_c} \frac{z(s)}{s^2} = -\dot{\phi}$

Im interested into know if weather a person with high ability chooses to earn more than a person with less productivity

Transversality Condition

Since $u(s)$ is a jump variable—its initial and terminal values are free—the corresponding co-state variable $\phi(s)$ must be zero at both boundaries. In other words, the transversality condition requires that the co-state variable $\phi(s)$ equals zero at both endpoints of the distribution, since the utility level $u(s)$ is free at $\underline{s}$ and $\overline{s}$. Formally,

$$\phi(\underline{s}) = \phi(\overline{s}) = 0$$

Intuitively, because the utility level $u(s)$ is not fixed at either endpoint, variations in $u(s)$ at the boundaries do not affect the value of the objective functional, which implies that the associated shadow price $\phi(s)$ must vanish there.

This boundary condition implies that, at the optimum, the marginal tax rate is zero at both ends of the skill distribution:

$$T'(z(\underline{s})) = T'(z(\overline{s})) = 0$$

In other words, there is no distortion at the bottom and at the top of the distribution of abilities. The highest-skilled and the lowest-skilled individuals face non-distorted (i.e., efficient) labor choices, while intermediate types may face positive marginal tax rates due to incentive-compatibility constraints.

At the lower bound, the least-skilled individual has no lower type to mimic, so distorting their labor supply provides no incentive benefit. At the upper bound, the most-skilled individual cannot imitate anyone above them, so distorting their choices also yields no informational gain. Consequently, both face non-distorted (efficient) labor-leisure allocations.

Solving the System Let us begin by assuming:

- Additive separability utility

$$u_{cl} = 0$$

Under this assumption, the first-order conditions reduce to:

$\frac{\partial \mathscr{H}}{\partial z(s)} = u_l \left\{ \lambda \left(\frac{1}{u_l} + \frac{1}{u_c}\frac{1}{s} \right) \gamma(s) - \frac{\phi(s)}{s^2} \left[1 + \frac{u_{ll}}{u_l} l \right] \right\} = 0$

$\frac{\partial \mathscr{H}}{\partial u(s)} = \left[W'\left(u(s)\right) - \lambda \frac{1}{u_c} \right] \gamma(s) = -\dot{\phi}$

Let us focus on the sign of $\dot{\phi}$:

$$-\left[W'\left(u(s)\right) - \lambda \frac{1}{u_c} \right]$$

We have assumed that $u(s)$ is increasing in s and that the marginal utility of consumption, $u_c(s)$, is decreasing in s, as implied by the concavity of the utility function. It follows that $\frac{1}{u_c(s)}$ is increasing in s, and consequently, $\frac{1}{u_c(s)}$ is increasing in s.

Now, if W is concave (for instance, in the utilitarian case $W(u) = u$, or more generally any concave welfare function such that $W'(u)$ is nonincreasing in u), then both terms in brackets are decreasing in s:

- $W'(u(s))$ decreases because $u(s)$ increases and W' is nonincreasing,
- $-\lambda/u_c(s)$ decreases because $\frac{1}{u_c(s)}$ increases.

Hence, the entire bracket is decreasing in s.

Integrating with respect to s from $\underline{s}$ to $\overline{s}$, we obtain:

$$\int_{\underline{s}}^{\overline{s}} \left[W'\left(u(s)\right) - \lambda \frac{1}{u_c} \right] \gamma(s) ds = -\int_{\underline{s}}^{\overline{s}} \dot{\phi} ds = \phi(\overline{s}) - \phi(\underline{s}) = 0$$

Equivalently, this can be written by splitting the integral at some intermediate point s:

$$\int_{\underline{s}}^{\overline{s}} \left[W'\left(u(s)\right) - \lambda \frac{1}{u_c} \right] \gamma(s) ds = \underbrace{\int_{\underline{s}}^{s} \left[W'\left(u(s)\right) - \lambda \frac{1}{u_c} \right] \gamma(s) ds}_{(+)} + \underbrace{\int_{s}^{\overline{s}} \left[W'\left(u(s)\right) - \lambda \frac{1}{u_c} \right] \gamma(s) ds}_{(-)} = 0$$

Since the bracket $[W'(u(s)) - \lambda/u_c(s)]$ is decreasing in s, for small s (the bottom part of the distribution), it is positive, meaning these individuals' marginal contribution to social welfare is above the average. For large s (top of the distribution), the bracket is negative, meaning these individuals' marginal contribution is below the average.

Regardless of the s:

$$\int_{\underline{s}}^{s} \left[W'\left(u(s)\right) - \lambda \frac{1}{u_c} \right] \gamma(s) ds = -\int_{s}^{\overline{s}} \left[W'\left(u(s)\right) - \lambda \frac{1}{u_c} \right] \gamma(s) ds$$

The LHS corresponds to the “lower-s” individuals, whose contribution to social welfare is positive. To redistribute welfare toward them, they should receive transfers from the tax system, i.e., $T(z(s)) < 0$. Conversely, the RHS corresponds to the “higher-s” individuals, who will need to make positive tax payments, i.e., $T(z(s)) > 0$.

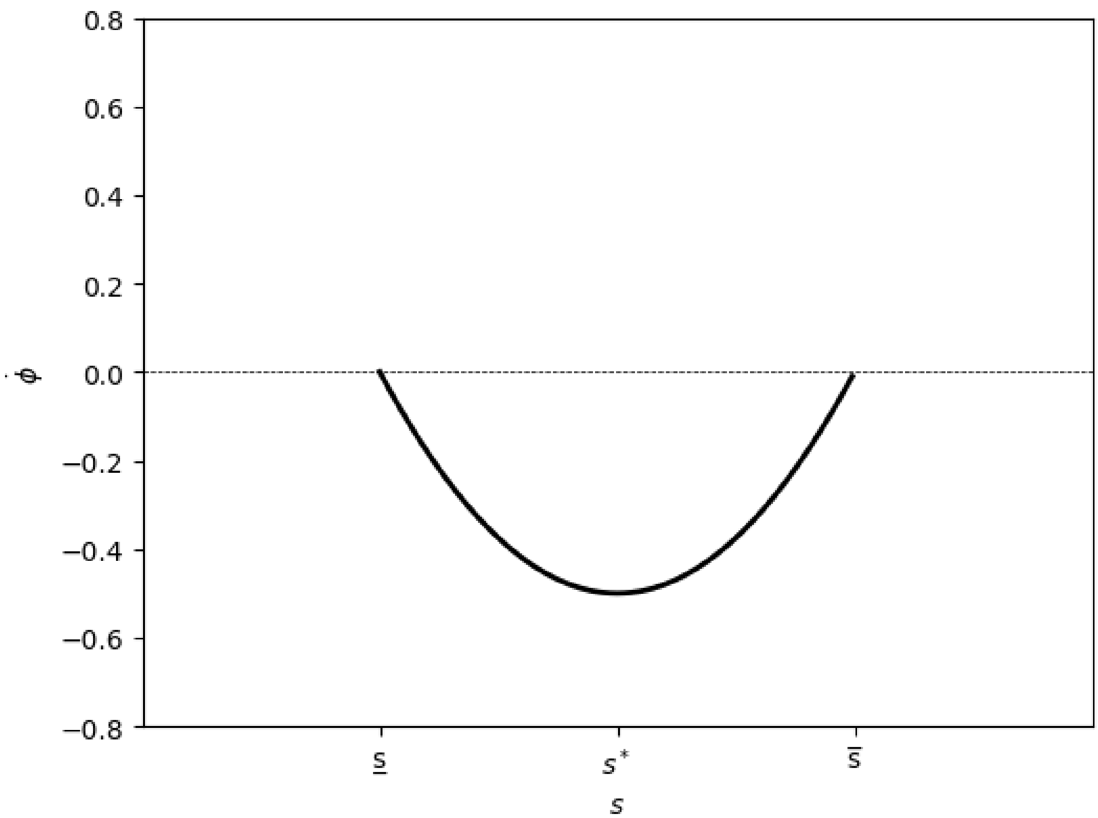
This means that there’s an intermediate consumer such that:

$$W'\left(u(s^*)\right) = \lambda \frac{1}{u_c}$$

$$\underbrace{\left[W'\left(u(s)\right) - \lambda \frac{1}{u_c}\right]}_{(+)\text{ for low } s (-) \text{ for high } s} \gamma(s) = -\dot{\phi}$$

Hence, the co-state $\phi(s)$ is decreasing for low-s individuals and increasing for high-s individuals. Overall, we have $\phi(s) < 0$ across the distribution.

Graphically, $\phi(s)$ has a U-shaped profile, reaching a minimum at the intermediate s where the bracket crosses zero.



Returning to the condition:

$$\underbrace{\lambda}_{(+)} \underbrace{\left(\frac{1}{u_l} + \frac{1}{u_c}\frac{1}{s}\right)}_{\text{FOC: } \frac{T'(z)}{u_l}} \gamma(s) - \underbrace{\frac{\phi(s)}{s^2}}_{(-)} \underbrace{\left[1 + \left(\frac{u_{ll}}{u_l} - \frac{u_{lc}}{u_c}\right) l\right]}_{(+) \Leftrightarrow \frac{\partial z}{\partial s} > 0} = 0 \implies T'(z) \leq 0$$

This shows that the marginal income tax is non-negative. That is, for an increase in $z(s)$, the household effectively reduces its net tax payment.

Economically, this implies that labor subsidies are not optimal under this setup: providing positive subsidies on labor would violate the first-order condition, because it would reverse the sign of $T'(z)$ and conflict with the concavity and monotonicity conditions that guarantee optimal redistribution.

- General utility function

We have the condition:

$\left[W'\left(u(s)\right) - \lambda \frac{1}{u_c}\right] \gamma(s) - \phi(s) \frac{u_{lc}}{u_c} \frac{z(s)}{s^2} = -\dot{\phi}$

$$\Delta - m\phi(s) = -\dot{\phi}$$

Where $\Delta \equiv \left[W'\left(u(s)\right) - \lambda \frac{1}{u_c}\right] \gamma(s)$ and $m \equiv \frac{u_{lc}}{u_c} \frac{z(s)}{s^2}$

Case 1: $m > 0$

Then i know that at $\underline{s}$

$$-\Delta = \dot{\phi}(\underline{s})$$

Given that we have demonstrated that $W'\left(u(s)\right) - \lambda \frac{1}{u_c}$ is positive for the "inferior" skill individuals, we have that $\dot{\phi}(\underline{s})$ is decreasing, until it reaches s^* individual.

The question is, can ϕ become positive before it hits $\underline{s}$?

Suppose, for contradiction, that ϕ becomes positive at some interior point s_0. At the crossing point s_0 where $\phi(s_0) = 0$ and ϕ would cross from negative to positive, we must have $\dot{\phi}(s_0) \geq 0$.

$$\dot{\phi}(s) = m\phi(s) - \Delta = (+)$$

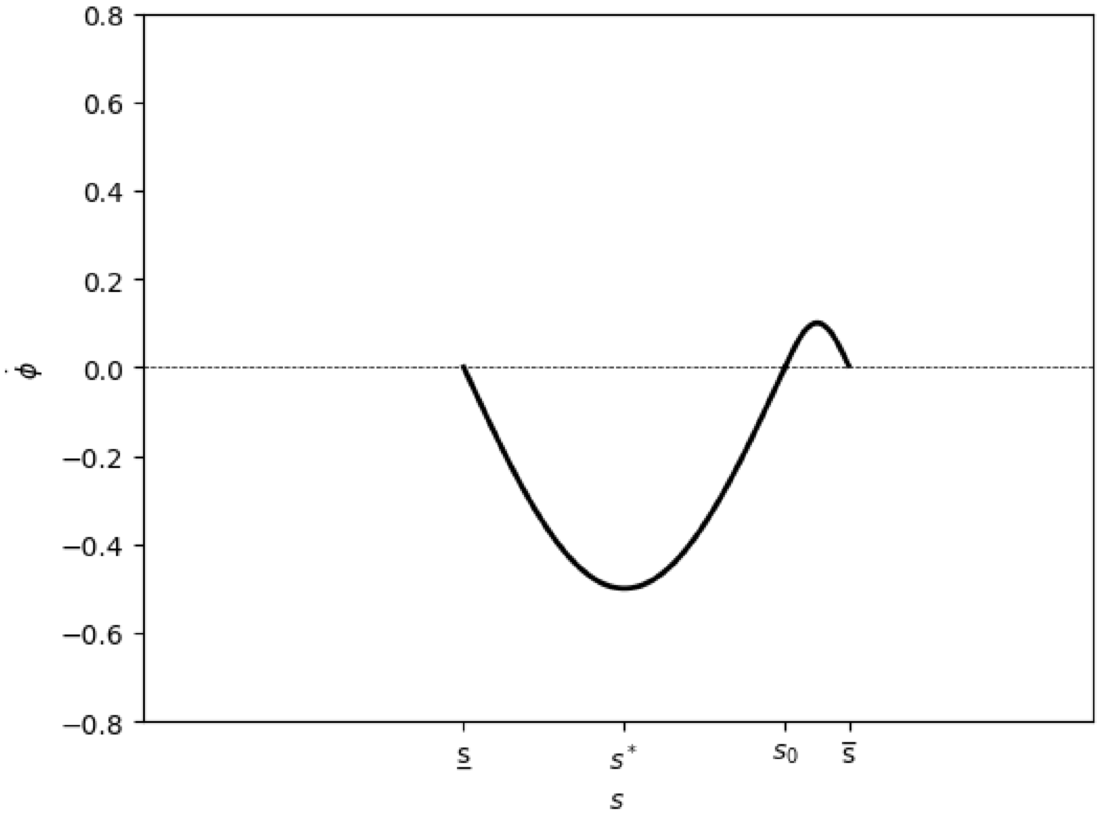
But once ϕ becomes strictly positive on a subinterval where $m > 0$, the homogeneous term $m\phi$ amplifies it and ϕ tends to grow (exponentially), making it impossible to return to zero later without violating boundary/terminal condition $\phi(\overline{s}) = 0$

Then if $m > 0$ i cannot have $\phi(s) > 0$. And we have that the U shape already described.

Case 2: $m < 0$

When $m < 0$ the homogeneous part of the ODE, $\dot{\phi} = m\phi$ is damping, i.e., the system resists large deviations and gradually pulls ϕ back down. Therefore, local positive excursion of ϕ cn in principle be created by the forcing term $-\Delta$ and later be driven back to zero by the negative m.

Given that $\Delta(s)$ is positive for all the individuals to the right of s^*, any positive excursion of the costate variable $\phi(s)$ could only occur among the high-skill individuals, i.e. to the right of s^*.



Then:

Theorem: Sufficient Condition. The marginal income tax is non-negative if $m \geq 0 \iff u_{cl} = -u_{co} \geq 0$.

Interpretation. The marginal utility of consumption decreases with leisure. Hence, leisure and consumption are gross substitutes (or, equivalently, not complements). **This condition is sufficient but not necessary for a non-negative marginal income tax**.

For low ability then $T(z) < 0$ (subsidied), for high ability $T(z) > 0$ (pay taxes)

Bunching case It may happen that leisure attains its maximum, $o = 1$, so labor is zero. In that case the household faces a corner solution: she would like to reduce labor supply further (i.e. increase leisure) but cannot.

Adding non-negative constraints

$$\mathscr{L} = U(c, l) + \lambda [sl - T(sl) - c]$$

FOC

$\frac{\partial \mathscr{L}}{\partial c} = u_c - \lambda = 0$

$\frac{\partial \mathscr{L}}{\partial l} = u_l + \lambda \left(s - T'(sl)s \right) \leq 0$

$$\boxed{u_l + u_c \left(s - T'(z)s \right) \leq 0}$$

or equivalently

$$T'(z) \leq 1 + \frac{1}{s} \frac{u_l}{u_c}$$

Thus the admissible slope $T'(z)$ for corner individuals is bounded above by a number that depends on preferences and s; the bound is typically smaller (more restrictive) for individuals at the corner.

Returning to the general optimality condition,

$$\underbrace{\lambda}_{(+)} \underbrace{\left(\frac{1}{u_l} + \frac{1}{u_c} \frac{1}{s} \right)}_{\text{FOC: } \geq \frac{T'(z)}{u_l}} \gamma(s) - \underbrace{\frac{\phi(s)}{s^2}}_{(-)} \underbrace{\left[1 + \left(\frac{u_{ll}}{u_l} - \frac{u_{lc}}{u_c} \right) l \right]}_{(+) \iff \frac{\partial z}{\partial s} > 0} = 0 \implies T'(z) \leq 0$$

under the same monotonicity assumptions. In the corner case the first-order condition for l is an inequality, so corner individuals admit slightly different bounds on $T'(z)$ compared with interior individuals.

Define s_{not} as the lowest skill for which the corner condition coincides with the interior condition (i.e. the individual for whom $l = 1$ is also the interior solution boundary). If such s_{not} exists, then all individuals with $s < s_{\text{not}}$ are in the corner and choose $z(s) = 0$ (voluntary non-employment). Because they all earn the same income $z = 0$, **individuals of different ability are bunched at zero income**: ability can no longer be inferred from observed income within the bunch. (Then, individuals with $z = 0$ **no longer signal their ability**)

At the threshold s_{not}, the marginal income tax satisfies the interior first-order equality

$$\lambda \frac{T'(z)}{u_l} \gamma(s) - \frac{\phi(s)}{s^2} \left[1 + \left(\frac{u_{ll}}{u_l} - \frac{u_{lc}}{u_c} \right) l \right] = 0$$

and because $\phi(s_{\text{not}}) < 0$, the implied marginal tax at the threshold is non-negative:

$$T'(z(s_{\text{not}})) \geq 0$$

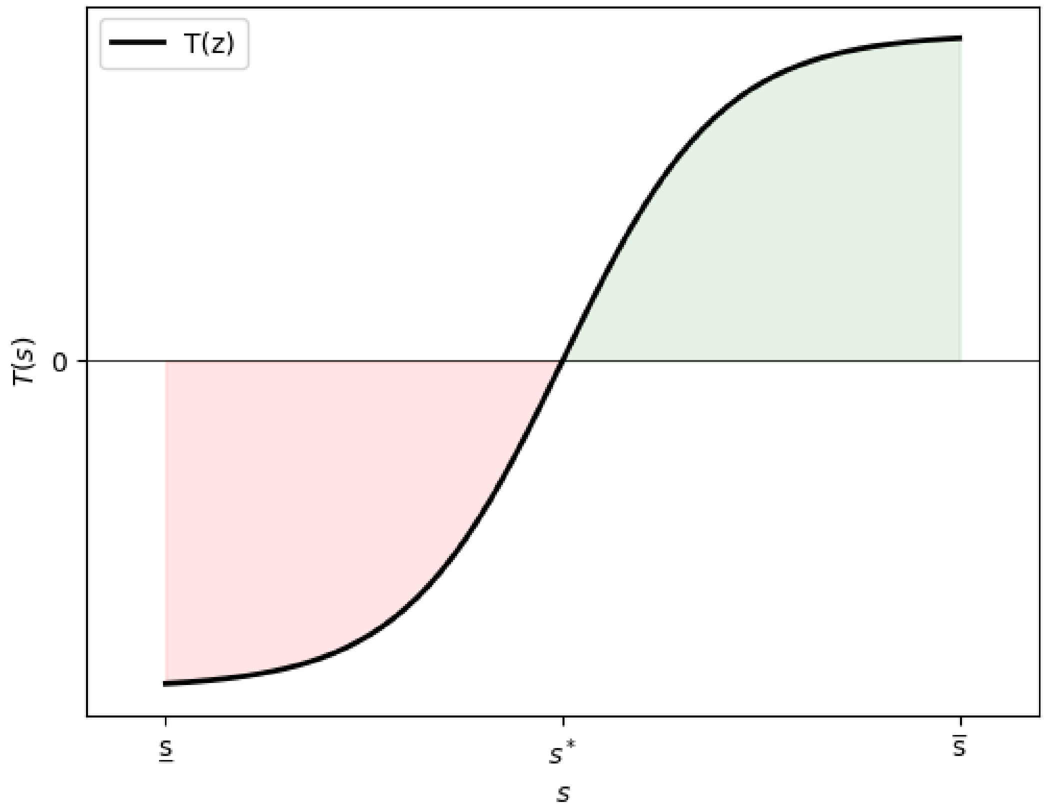
Consequently, the optimal marginal income tax at the bottom of the bunch (i.e. for $z = 0$) is non-negative, and typically strictly positive at the right end of the bunching interval. In words: the planner optimally sets a positive marginal tax rate at the boundary of the bunch, which sustains the bunching region to its left.

Bunching arises because, for sufficiently low skill s, the utility loss from working cannot be compensated by transfers under the tax schedule (or because the incentive constraints bind), so many low-ability types choose the same corner income $z = 0$. The planner places a non-negative (often positive) marginal tax at the boundary to balance redistribution and incentive costs: this is the marginal tax that makes the threshold individual indifferent between supplying (infinitesimally) more labor or staying at the corner.

Slope of $T'(z(s))$ From the first-order condition, the marginal income tax satisfies

$$T'(z(s)) < 1$$

This implies that tax rates above 100% are never optimal.



1.4 Signaling problem

To build intuition, consider a world with only two types, s_{high} and s_{low}. The utility function is:

$$u\left(c, \frac{z(s)}{s}\right)$$

1. Slope of Indifference Curves

Taking the total differential and setting $du = 0$ along a level curve:

$$du = u_c dc + \frac{u_l}{s} dz \implies \frac{dc}{dz} = -\frac{u_l}{u_c}\frac{1}{s} < 0$$

This shows that the indifference curves in the (z, c)-plane are upward-sloping in z.

2. Convexity of Indifference Curves

Compute the second derivative of c with respect to z:

$$\frac{d^2c}{dz^2} = -\frac{u_{lc}\frac{dc}{dz} + u_{ll}}{u_c}\frac{1}{s} + \frac{u_l}{u_c}\frac{u_{cc}\frac{dc}{dz} + u_{cl}}{u_c}\frac{1}{s} = -\frac{1}{s}\left(\frac{1}{u_c^3}\right)[u_{ll}u_c - 2u_{lc}u_c u_l + u_{cc}u_l] < 0$$

This can be rewritten in matrix form:

$$\frac{d^2c}{dz^2} = -\frac{1}{s}\left(\frac{1}{u_c^3}\right)\begin{bmatrix} u_{ll} & -u_{lc} \\ -u_{lc} & u_{cc} \end{bmatrix}\begin{bmatrix} u_c \\ u_l \end{bmatrix}$$

Since $u(c,l)$ is concave, its Hessian is negative semidefinite.This implies that the quadratic form

$$\begin{bmatrix} u_{ll} & -u_{lc} \\ -u_{lc} & u_{cc} \end{bmatrix}\begin{bmatrix} u_c \\ u_l \end{bmatrix}$$

is always negative:

$$\frac{d^2c}{dz^2} > 0$$

ensuring that the level curves are convex in (z,c).

3. Slope of Indifference Curves Across Different Abilities

Starting from the derivative of the slope with respect to z:

$$\frac{d}{dz}\left(\frac{dc}{dz}\right)_{c,z} = \frac{\partial d}{\partial dz}\frac{dc}{dz} = \frac{u_{ll}\frac{z(s)}{s^2}}{u_c}\frac{1}{s} - \frac{u_l}{u_c^2}u_{lc}\frac{z(s)}{s^2}\frac{1}{s} + \frac{u_l}{u_c}\frac{1}{s^2}$$

Factoring out $\frac{u_l}{u_c}\frac{1}{s^2}$:

$$\frac{\partial}{\partial z}\frac{dc}{dz} = \frac{u_l}{u_c}\frac{1}{s^2}\left[\frac{u_{ll}}{u_l}l - \frac{u_{lc}}{u_c}l + 1\right]$$

Noticing that the term in brackets is exactly $\frac{\partial z}{\partial s}$:

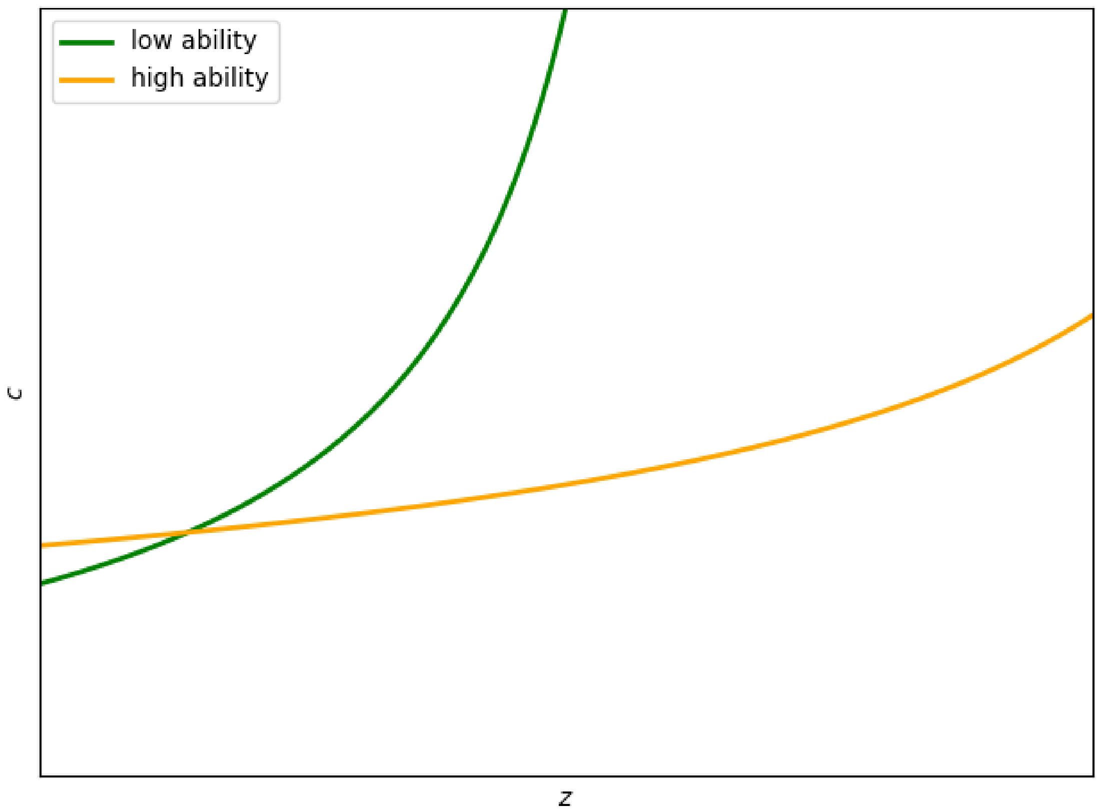
$$\frac{\partial}{\partial z}\frac{dc}{dz} = \frac{u_l}{u_c}\frac{1}{s^2}\frac{\partial z}{\partial s}$$

By assumption, $\frac{\partial z}{\partial s} > 0$, which implies:

$$\frac{\partial}{\partial z}\frac{dc}{dz} < 0$$

Intuitively, higher-ability individuals are more productive, so they need to give up less leisure to gain an additional unit of consumption. This makes their indifference curves flatter compared to those of lower-ability individuals at the same point.

This property implies the **single-crossing condition**: the indifference curves of low- and high-ability types intersect at most once. (Necessary for signaling model)



1.4.1 First-best solution

In the first-best allocation, there are no distortions, meaning that $\phi(s) = 0$ for all individuals. The first-order condition of the household problem then holds with equality:

$$\frac{u_l}{s} + u_c = 0$$

for every skill level s. Accordingly, the planner's first-order conditions are:

$\frac{\partial \mathscr{H}}{\partial z(s)} = u_l \lambda \left(\frac{1}{u_l} + \frac{1}{u_c} \frac{1}{s} \right) \gamma(s) = 0$

$\frac{\partial \mathscr{H}}{\partial u(s)} = \left[W'\left(u(s)\right) - \lambda \frac{1}{u_c} \right] \gamma(s) = 0$

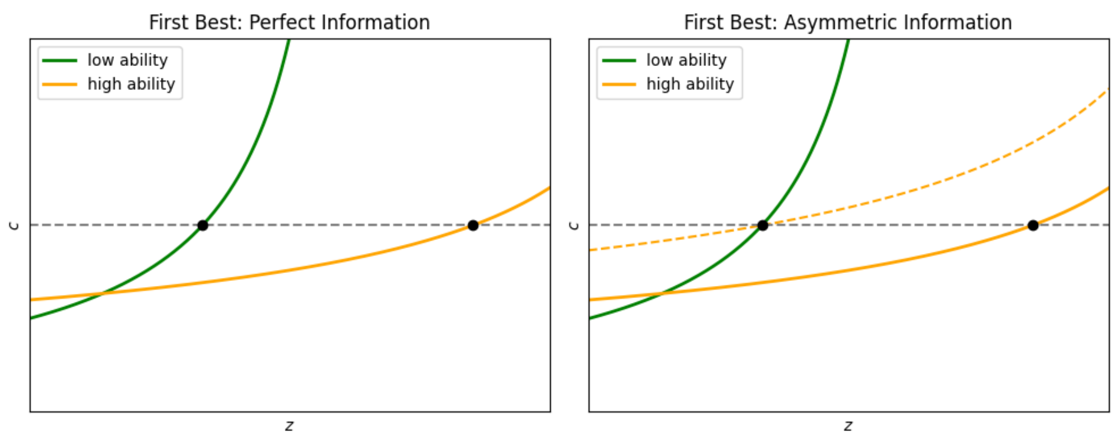
From the second condition it follows that:

$$\lambda = W'\left(u(s)\right) u_c$$

Hence, for the $W'(s)$ constant over s, i.e. the social welfare function is **linear** in individual utilities, u_c must be the same for all s, implying that the marginal utility of consumption is equalized across individuals. Similarly, the ratio between the marginal disutility of labor and the skill level must also be identical for all s in the first-best.

In this equilibrium, high-skilled individuals work more. As they supply more labor, their marginal utility of leisure decreases. Although they have higher productivity, they obtain the same consumption as others, which means they enjoy less utility from leisure and are therefore worse off.

The state cannot implement this allocation in practice because individuals have **private information about their skill level**. Without observing skills directly, the planner cannot enforce the first-best outcome.

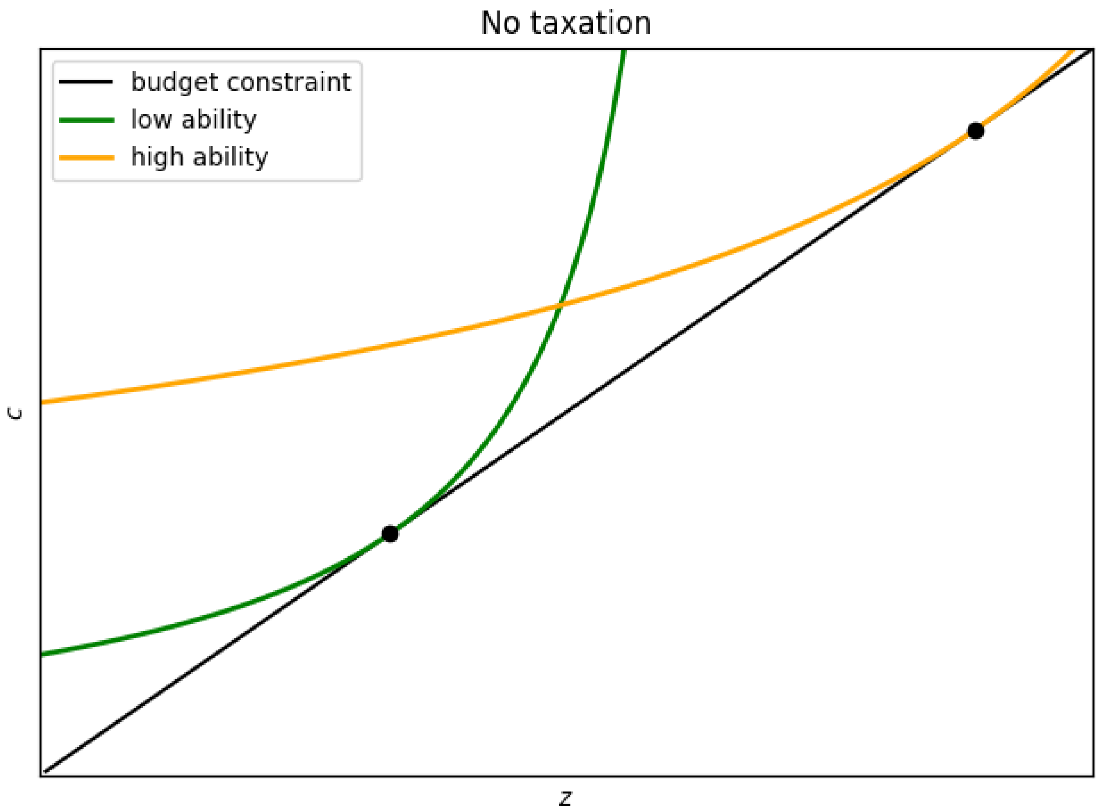


1.5 The optimal tax schedule

1.5.1 The linear tax schedule

In this framework, the tax rate corresponds to the slope of a ray originating from the origin in the income–consumption space. It represents the ratio between disposable income available for consumption and total income earned from labor. A slope of 1 defines a linear tax schedule with zero marginal tax rates.

If no taxes then:

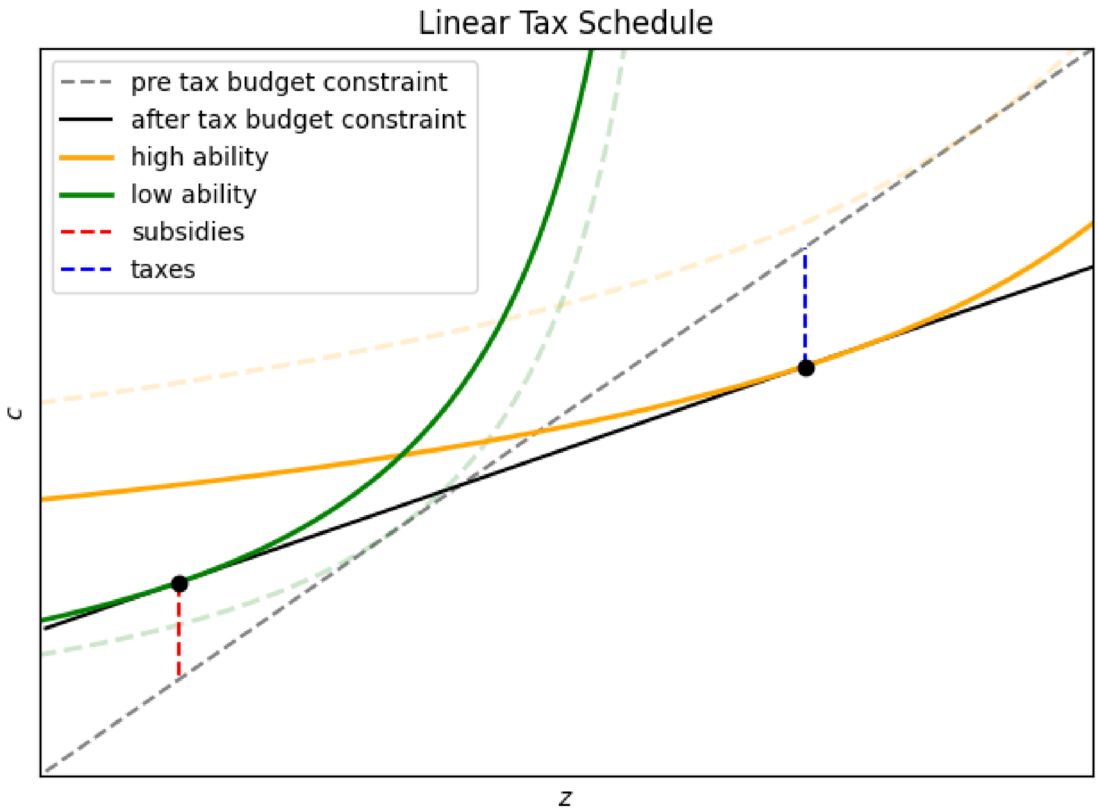


Under a linear schedule instead $T(z) = \tau z$, the budget constraint is given by:

$$c = z - \tau z + b$$

where b represents the redistribution of taxes to consumers in the form of a lump-sum transfer.

The slope of the linear tax schedule in the income–consumption space is given by $1 - T'(z)$. When the tax schedule is strictly positive, the slope is less than 1. At the same time, redistributive objectives targeting lower-income individuals raise the intercept of the line, so that it intersects the vertical axis at a positive level, reflecting transfers to individuals with zero income. The vertical distance between the line representing the linear tax schedule and the 45° line measures the tax payment at each income level on the horizontal axis.



From an economic perspective, redistributive objectives create a situation in which the linear tax schedule results in consumption lower than production for high-income individuals—subject to a positive tax rate—and consumption higher than production for low-income individuals, who benefit from a lump-sum transfer. In a linear income tax system, the sacrifice imposed on the top of the income distribution is proportional to the benefits received by the bottom. Consequently, the government must choose b and τ that the tax revenue collected from high-income individuals equals the subsidies provided to low-income individuals. This raises the question of whether a non-linear tax schedule could achieve more efficient or equitable outcomes.

1.5.2 The non-linear tax schedule

The only way to increase the tax rate on the highest-income individual without reducing their utility is to set the tax according to the **highest indifference curve** they can still reach. Let $\Omega(z)$ denote the pre-tax income function and $\Psi(z)$ the post-tax income function. To maximize the tax rate on the top earner, the vertical distance between $\Omega(z)$ and $\Psi(z)$ must be maximized, subject to keeping the individual on their attainable indifference curve.

$$\max \Omega(z) - \Psi(z) \implies \text{F.O.C: } \Omega'(z) = \Psi'(z)$$

The slope of the post-tax income function, $\Psi(z)$, corresponds to the **marginal rate of substitution** between income and consumption. Formally, it is given by

$$\frac{\partial x}{\partial z} = -\frac{u_l}{u_c}\frac{1}{s}$$

In contrast, the slope of the pre-tax income function, $\Omega(z)$, is equal to 1, as income and consumption coincide in the absence of taxation.

At the margin, the top earner should not face any **marginal income tax**,

$$T'(z) = 0,$$

which is known as the "no distortion at the top" result. This relies on the assumption that the utility of the top earner contributes positively to social welfare, so their utility should not be reduced. If this assumption does not hold, the framework changes.

Similarly, the same derivation applies if one wants to **minimize social welfare loss at the bottom**: the goal is to minimize the distance between pre-tax and post-tax income for the lowest-income individual, i.e.,

$$\min \Omega(z) - \Psi(z) \text{ for the bottom earner.}$$

Hence, **marginal tax rates at the top and bottom are not primarily about redistribution**, but about balancing the tax system. For example, if $z(s) = 0$ for an interval $s \in [s_1, s_2]$, it always holds that

$$0 < T'(z(s)) < 1.$$

Imposing a positive marginal tax at the top would prevent a **Pareto improvement**: setting $T'(z) = 0$ allows the highest-income individual to earn more, generating higher revenue that can be redistributed to low-income individuals. Similarly, taxing the lowest-income individual would reduce their utility unnecessarily; since they have no binding compatibility constraints, their marginal tax should avoid distortion.

In conclusion, the design of the tax system must carefully account for the incentives of the **top earner**, while ensuring that the bottom does not face unnecessary distortions.