A Cycle Constraint

O'Gorman et al. 2014 used $\sum_{\substack{a,b,c\\a<b}} \delta_{\text{trans}}^{(a,b,c)} R'_{a,b,c}$ as the cycle constraint of the Hamiltonian, and showed the sufficient lower bound $\delta_{\text{lower}}^{(\text{full})} < \delta_{\text{trans}}^{(a,b,c)} = \delta^{(\text{full})} < \frac{\delta_{n',n}^{(\text{full})}}{N-2}$, where $\delta_{\text{trans}}^{(a,b,c)} \in \mathbb{R}$, $R'_{a,b,c} = r'_{a,c} + r'_{a,b}r'_{b,c} - r'_{a,b}r'_{a,c} - r'_{b,c}r'_{a,c}$, and $r'_{n',n} = r_{n',n}$ if $n' < n$, $1 - r_{n,n'}$ otherwise. We replaced $\sum_{\substack{a,b,c\\a<b}} \delta_{\text{trans}}^{(a,b,c)} R'_{a,b,c}$ with $\sum_{a<b<c} \delta_{a,b,c}^{(\text{full})} R_{a,b,c}$ for simplicity, and set $\delta_{\text{trans}}^{(a,b,c)} = \frac{\delta_{a,b,c}^{(\text{full})}}{3}$ for the following reasons:

$$
\begin{aligned}
\sum_{\substack{a,b,c\\a<b}} R'_{a,b,c} = \sum_{\substack{a,b,c\\a<b}} &(r'_{a,c} + r'_{a,b}r'_{b,c} - r'_{a,b}r'_{a,c} - r'_{b,c}r'_{a,c}) \\
= &\sum_{a<b<c} (r_{a,c} + r_{a,b}r_{b,c} - r_{a,b}r_{a,c} - r_{b,c}r_{a,c}) \\
&+ \sum_{a<c<b} (r_{a,c} + r_{a,b}(1 - r_{c,b}) - r_{a,b}r_{a,c} - (1 - r_{c,b})r_{a,c}) \\
&+ \sum_{c<a<b} ((1 - r_{c,a}) + r_{a,b}(1 - r_{c,b}) - r_{a,b}(1 - r_{c,a}) - (1 - r_{c,b})(1 - r_{c,a})) \\
= &\sum_{a<b<c} (r_{a,c} + r_{a,b}r_{b,c} - r_{a,b}r_{a,c} - r_{b,c}r_{a,c}) \\
&+ \sum_{a<c<b} (r_{a,b} + r_{a,c}r_{c,b} - r_{a,c}r_{a,b} - r_{c,b}r_{a,b}) \\
&+ \sum_{c<a<b} (r_{c,b} + r_{c,a}r_{a,b} - r_{c,a}r_{c,b} - r_{a,b}r_{c,b}) \\
=&3 \sum_{a<b<c} (r_{a,c} + r_{a,b}r_{b,c} - r_{a,b}r_{a,c} - r_{b,c}r_{a,c}) \\
=&3 \sum_{a<b<c} R_{a,b,c}.
\end{aligned}
$$

B Sufficient Lower Bound

It is clear that

$$
\delta_{\text{lower}}^{(Z)} = \max_n \max\{\max_{\lambda,\lambda'}(-s_{1,\lambda,n}^{(Z)} - t_{\lambda,\lambda',n}^{(Z)}), \max_{\lambda,\lambda'}(-s_{2,\lambda',n}^{(Z)} - t_{\lambda,\lambda',n}^{(Z)})\} < \xi_{o,\lambda,\lambda',n}^{(Z)}
$$

so that $\sum_\lambda p_{o,\lambda,n} \leq 1$ for all n, o. In addition, it does not lose its generality by assuming that $s_{1,\lambda,n}^{(Z)} + t_{\lambda,\lambda',n}^{(Z)} \leq s_{1,0,n}^{(Z)} + t_{0,\lambda',n}^{(Z)} = 0, s_{2,\lambda',n}^{(Z)} + t_{\lambda,\lambda',n}^{(Z)} \leq s_{2,0,n}^{(Z)} + t_{\lambda,0,n}^{(Z)} = 0$ for all λ, λ', n because $U_{o,0,n} = \phi$ is always most advantageous in the term of the DAG constraint.

Under these conditions and $\Lambda_{1,n}, \Lambda_{2,n} > 1$, the sufficient lower bound $\delta_{\text{lower}}^{(Z)}$ satisfies

$$
\begin{aligned}
\delta_{\text{lower}}^{(\text{full})} &= \max\{0, \max_{n' \neq n} \max_{\boldsymbol{d}_{\cdot,n} \setminus d_{n',n}} (-H_{\text{score}}^{n(\text{full})}(\boldsymbol{d}_{\cdot,n} \mid d_{n',n} = 1) + H_{\text{score}}^{n(\text{full})}(\boldsymbol{d}_{\cdot,n} \mid d_{n',n} = 0))\} \\
&\leq \max_n \max_{\substack{o',o,\lambda\\o' \neq o}} \max_{\boldsymbol{p}_{o',\cdot,n}} (-H_{\text{score}}^{n(Z)}(\boldsymbol{p}_{1,\cdot,n}, \boldsymbol{p}_{2,\cdot,n} \mid p_{o,\lambda,n} = 1) + H_{\text{score}}^{n(Z)}(\boldsymbol{p}_{1,\cdot,n}, \boldsymbol{p}_{2,\cdot,n} \mid \boldsymbol{p}_{o,\cdot,n} = \boldsymbol{0})) \\
&= \max_n \max\{\max_{\lambda,\lambda'}(-s_{1,\lambda,n}^{(Z)} - t_{\lambda,\lambda',n}^{(Z)}), \max_{\lambda,\lambda'}(-s_{2,\lambda',n}^{(Z)} - t_{\lambda,\lambda',n}^{(Z)})\} \\
&= \delta_{\text{lower}}^{(Z)}.
\end{aligned}
$$

It is obvious that $\delta_{\text{lower}}^{(Z)}$ is also the sufficient bound when $\Lambda_{1,n} = 1$ or $\Lambda_{2,n} = 1$.

Correction and Apology. Equation (24) in the main text is a sufficient lower bound. However, the above result is better as a sufficient lower bound.