GAUSSIAN PROCESS AMPLITUDE DEMODULATION BY MESSAGE-PASSING

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ABSTRACT

This supplementary document provides the computation rules of sum-product messages in the paper ”Gaussian Process Amplitude Demodulation by Message Passing”.

1 Exponential Node

The factor node function of the Exponential node is

$$f(y, x) = \delta(y - \exp(x)) \tag{1}$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}_{>0}$.

1.1 Forward messge

We want to find the message $\vec{\mu}_Y(y)$, given the incoming message $\vec{\mu}_X(x) = \mathcal{N}(x|m_x, \sigma_x^2)$. According to the sum-product rule, we have

$$\begin{aligned}
\vec{\mu}_Y(y) &= \int \delta(y - \exp(x))\, \vec{\mu}_X(x) dx && (2)\\
&= \int \frac{1}{u} \delta(y - u)\, \vec{\mu}_X(\log(u)) du && (3)\\
&= \int \delta(u - y)\, \text{Lognormal}(u|m_x, \sigma_x^2) du && (4)\\
&= \text{Lognormal}(y|m_x, \sigma_x^2), && (5)
\end{aligned}$$

and thus the forward message is a Log-Normal distribution.

1.2 Backward message

Now we would like to find the message $\overleftarrow{\mu}_X(x)$ given the message $\overleftarrow{\mu}_Y(y)$. According to the sum-product rule, we have

$$\begin{aligned}
\overleftarrow{\mu}_X(x) &= \int \delta(y - \exp(x)) \overleftarrow{\mu}_Y(y) dy && (6)\\
&= \overleftarrow{\mu}_Y(\exp(x)). && (7)
\end{aligned}$$

2 Multiplication Node

The factor node function of the Multiplication node is

$$f(z, x, y) = \delta(z - xy). \tag{8}$$

2.1 Forward message

We are interested in finding the message $\vec{\mu}_Z(z)$, given two Gaussian messages $\vec{\mu}(x) = \mathcal{N}(x|m_x, \sigma_x^2)$, $\vec{\mu}(y) = (y|m_y, \sigma_y^2)$. Since z is the product of two Gaussian random variables, its message will be the modified Bessel function of second kind [1] . If we further assume that x and y are independent, then the message $\vec{\mu}_Z(z)$ has the following formula

$$\begin{aligned}
\vec{\mu}_Z(z) = \exp\left(-\frac{1}{2}\left(\frac{m_x^2}{\sigma_x^2} + \frac{m_y^2}{\sigma_y^2}\right)\right) \times \sum_{n=0}^{\infty} \sum_{m=0}^{2n} \frac{z^{2n-m}|z|^{m-n}\sigma_x^{m-n-1}}{\pi\,(2n)!\,(\sigma_y)^{m-n+1}} \left(\frac{m_x}{\sigma_x^2}\right)^m && (9)\\
\times \binom{n}{r} \left(\frac{m_y}{\sigma_y^2}\right)^{2n-m} K_{m-n}\left(\frac{|z|}{\sigma_x\,\sigma_y}\right), && (10)
\end{aligned}$$

where $K_v(\cdot)$ denotes the modified Bessel function of the second kind and order v.

2.2 Backward message

The backward messages toward either x or y have similar functional forms, so we pick the edge x to compute the backward message. Assume we have the messages $\vec{\mu}_Y(y)$ and $\overleftarrow{\mu}_Z(z) = \delta(z - \hat{z})$ where $\hat{z}$ denotes an observation of z, then the formula of $\overleftarrow{\mu}_X(x)$ is

$$\begin{aligned}
\overleftarrow{\mu}_X(x) &= \int \vec{\mu}_Y(y)\, \overleftarrow{\mu}_Z(z)\, \delta(z - xy) dy dz \\
&= \int \vec{\mu}_Y(y)\, \delta(z - \hat{z})\, \delta(z - xy) dy dz \\
&= \int \vec{\mu}_Y(y)\, \delta(\hat{z} - xy) dy \\
&= \int \frac{1}{|x|}\, \vec{\mu}_Y(y)\; \delta\left(y - \frac{\hat{z}}{x}\right) dy \qquad (11) \\
&= \frac{1}{|x|}\, \vec{\mu}_Y \left(\frac{\hat{z}}{x}\right) \qquad (12)
\end{aligned}$$

3 References

[1] G. Cui, X. Yu, S. Iommelli, and L. Kong, "Exact Distribution for the Product of Two Correlated Gaussian Random Variables," *IEEE Signal Processing Letters*, vol. 23, no. 11, pp. 1662–1666, Nov. 2016, Conference Name: IEEE Signal Processing Letters.