UNIVERSITY OF TORONTO SCARBOROUGH
Department of Statistics

Time Series Analysis

STAD57H3F – November 11, 2015

Midterm Exam

Duration – 110 minutes

Examination aids allowed: Scientific Calculator

Last Name: Solution

First Name: ______________________

Student #: ______________________

Instructions:

1. There are 4 questions on 10 pages in total (including this cover sheet) for this exam.
2. Write your student number at the top of each page.
3. Answer all questions directly on the examination paper.
4. Show your intermediate work, and write clearly and legibly.
5. Read the questions carefully and answer the question that is being asked.

1.	2.	3.	4.	Total

1. (30 marks)

Consider the stationary AR(1) series $X_t = \varphi X_{t-1} + W_t$, where $\{W_t\} \sim \mathrm{WN}(0, \sigma_W^2)$.

a. Find the 1- & 2-step-ahead best linear predictors X_{n+1}^n & X_{n+2}^n and express them as linear combinations of the first n random variables $\{X_1, \ldots, X_n\}$.

b. Find the Mean Square Prediction Error (MSPE) of X_{n+1}^n & X_{n+2}^n, i.e. find

$P_{n+1}^n = \mathbb{E}\left[\left(X_{n+1} - X_{n+1}^n\right)^2\right]$ & $P_{n+2}^n = \mathbb{E}\left[\left(X_{n+2} - X_{n+2}^n\right)^2\right]$, expressed in terms of φ, σ_W^2.

c. Find the covariance between the 1- & 2-step-ahead prediction errors, i.e. find $\mathrm{Cov}\left[\left(X_{n+1} - X_{n+1}^n\right), \left(X_{n+2} - X_{n+2}^n\right)\right]$ expressed in terms of φ, σ_W^2.

a. For the AR(1) model the BLP's are following the form of the model, i.e. $X_{n+1}^n = \varphi X_n$ &

$X_{n+2}^{n+1} = \varphi X_{n+1}^n = \varphi^2 X_n$

b. $P_{n+1}^n = \mathbb{E}\left[(X_{n+1} - X_{n+1}^n)^2\right] = \mathbb{E}\left[(\underbrace{\varphi X_n + W_{n+1}}_{= X_{n+1}} - \varphi X_n)^2\right] =$

$= \mathbb{E}[W_{n+1}^2] = \sigma_W^2$

$P_{n+2}^n = \mathbb{E}\left[(X_{n+2} - X_{n+2}^n)^2\right] = \mathbb{E}\left[(\varphi X_{n+1} + W_{n+2} - \varphi^2 X_n)^2\right] =$

$= \mathbb{E}\left[(\varphi(\varphi X_n + W_{n+1}) + W_{n+2} - \varphi^2 X_n)^2\right] =$

$= \mathbb{E}\left[(\cancel{\varphi^2 X_n} + \varphi W_{n+1} + W_{n+2} - \cancel{\varphi^2 X_n})^2\right] =$

$= \mathbb{E}\left[(\varphi W_{n+1} + W_{n+2})^2\right] = \mathrm{Var}\left[\varphi W_{n+1} + W_{n+2}\right] =$

(lag uncorrelated W's) $= \varphi^2 \mathrm{Var}[W_{n+1}] + \mathrm{Var}[W_{n+2}] = \sigma_W^2 \cdot (1 + \varphi^2)$

c. $$\mathrm{Cov}\left[\underbrace{(X_{n+1}-X_{n+1}^{n})}_{\downarrow},\ \underbrace{(X_{n+2}-X_{n+2}^{n})}_{\downarrow}\right] =$$

$$= \mathrm{Cov}\left[\quad W_{n+1}\ ,\ \varphi W_{n+1}+W_{n+2}\right] =$$

$$= \varphi\,\mathrm{Cov}\underbrace{\left(W_{n+1},W_{n+1}\right)}_{=Var[W_{n+1}]=\sigma_w^2} + \mathrm{Cov}\left(W_{\cancel{n+1}},W_{n+2}\right)^{=0}$$

$$= \varphi\,\sigma_W^{2}$$

2. (20 marks)
Let $\{X_t\}$ be a white noise sequence. Is the squared series $\{X_t^2\}$ also a white noise? Prove or disprove; a simple yes/no answer will get 0 marks.

$\{X_t\} \sim WN(\mu, \sigma_w^2) \Rightarrow \mathbb{E}[X_t] = \mu$, $Var[X_t] = \sigma_w^2$,

and $Cov(X_t, X_s) = \mathbb{E}[X_t X_s] - \mathbb{E}[X_t]\mathbb{E}[X_s] = \mathbb{E}[X_t X_s] - \mu^2 = 0, \forall s \neq t$

For simplicity, assume $\mathbb{E}[X_t] = \mu = 0 \Rightarrow Var[X_t] =$
$= \mathbb{E}[X_t^2] = \sigma_w^2$ (similar arguments work for $\mu \neq 0$)

For $\{X_t^2\}$ to be WN, we want:

1. $\mathbb{E}[X_t^2]$ constant
2. $Var[X_t^2]$ constant
3. $Cov(X_t^2, X_s^2) = 0$, $\forall s \neq t$

The 1st requirement holds, since $\mathbb{E}[X_t^2] = \sigma_w^2$.

But the other two do **NOT** necessarily hold.

In particular, $Cov(X_t^2, X_s^2) = \mathbb{E}[X_t^2 X_s^2] - \mathbb{E}[X_t^2]\mathbb{E}[X_s^2] =$
$= \mathbb{E}[X_t^2 X_s^2] - \sigma_w^4 \neq 0$, because $\mathbb{E}[X_t^2 X_s^2]$ is not necessarily equal to $\mathbb{E}[X_t^2] \cdot \mathbb{E}[X_s^2] = \sigma_w^4$ just because $Cov(X_t, X_s) = \mathbb{E}[X_t X_s] = 0$

As a concrete counter example, consider the following time series:

$X_1 = Z$ & $X_2 = \frac{Z^2-1}{\sqrt{2}}$, where $Z \sim N(0,1)$

We have $\mathbb{E}[X_1] = \mathbb{E}[Z] = 0$, $Var[X_1] = \mathbb{E}[Z^2] = 1$

$\mathbb{E}[X_2] = \mathbb{E}\left[\frac{Z^2-1}{\sqrt{2}}\right] = \frac{\mathbb{E}[Z^2]-1}{\sqrt{2}} = 0$,

$Var(X_2] = \mathbb{E}\left[\left(\frac{Z^2-1}{\sqrt{2}}\right)^2\right] = \frac{\mathbb{E}[Z^4]-1}{2} = \frac{3-1}{2} = 1$

$Cov(X_1, X_2) = \mathbb{E}[X_1 X_2] = \mathbb{E}\left[Z \cdot \frac{Z^2-1}{\sqrt{2}}\right] =$

$= \mathbb{E}[Z^3/\sqrt{2}] - \mathbb{E}[Z/\sqrt{2}] = 0$,

which shows $\{X_1, X_2\}$ is WN.

But: $Var[X_1^2] = \mathbb{E}[X_1^4] - \mathbb{E}[X_1^2]^2 = \mathbb{E}[Z^4] - \mathbb{E}[Z^2]^2 = 3 - 1^2 = 2$

$Var[X_2^2] = \mathbb{E}[X_2^4] - \underbrace{\mathbb{E}[X_2^2]}_{=Var[X_2]=1}{}^2 = \mathbb{E}\left[\left(\frac{Z^2-1}{\sqrt{2}}\right)^4\right] - 1^2 =$

$= \frac{1}{4}\mathbb{E}[Z^8 - 4Z^6 + 6Z^4 - 4Z^2 + 1] = \frac{1}{4}[\overset{\mathbb{E}[Z^8]}{105} - 4 \cdot \overset{\mathbb{E}[Z^6]}{15} + 6 \cdot \overset{\mathbb{E}[Z^4]}{3} - 4 \cdot 1 + 1]$

$= \ldots = 15 \neq 2 = Var[X_1^2]$

& $Cov(X_1^2, X_2^2) = \mathbb{E}[X_1^2 \cdot X_2^2] - \overset{=1}{\mathbb{E}[X_1^2]}\,\overset{=1}{\mathbb{E}[X_2^2]} =$

$= \mathbb{E}\left[Z^2 \cdot \left(\frac{Z^2-1}{\sqrt{2}}\right)^2\right] - 1 = \frac{1}{2}\mathbb{E}[Z^2 \cdot (Z^4 - 2Z^2 + 1)] - 1 =$

$= \frac{1}{2}\mathbb{E}[Z^6 - 2Z^4 + Z^2] - 1 = \frac{1}{2}(15 - 2 \cdot 3 + 1) - 1 = 5 - 1 = \underline{4 \neq 0}$

3. (35 marks)

Consider the AR(2) model $X_t = \varphi_1 X_{t-1} + \varphi_2 X_{t-2} + W_t$, where $W_t \sim \mathrm{WN}(0, \sigma_W^2)$.

a. Find the theoretical values of $\gamma(0)$ & $\gamma(1)$ in terms of $\varphi_1, \varphi_2, \sigma_W^2$.

b. Write the 1-step-ahead predictor X_{n+1}^n as a function of $X_1, \ldots, X_n$.

Assume you observe *n* values from this model, but you mistakenly fit an AR(1) model $X_t = \varphi X_{t-1} + W_t$ using Yule-Walker estimation.

c. Express the Yule-Walker estimates $\hat{\varphi}, \hat{\sigma}_W^2$ in terms of the sample moments $\hat{\gamma}(0), \hat{\gamma}(1)$.

d. Assume the sample moments are equal to the theoretical moments of the true AR(2) model from part **a.** Write the 1-step-ahead predictor $X_{n+1}'^n$ you would get from the incorrect AR(1) model as a function of $X_1, \ldots, X_n$ and the estimated coefficients. Show that the difference in prediction from using the incorrect model is

$$X_{n+1}^n - X_{n+1}'^n = -\frac{\varphi_1 \varphi_2}{1 - \varphi_2} X_n + \varphi_2 X_{n-1}.$$

e. Find the 1-step-ahead mean square prediction error (MSPE) from using the incorrect model in **d.**, i.e. $P_{n+1}'^n = \mathbb{E}\left[\left(X_{n+1} - X_{n+1}'^n\right)^2\right]$, in terms of $\varphi_1, \varphi_2, \sigma_W^2$. Compare that to the MSPE $P_{n+1}^n = \mathbb{E}\left[\left(X_{n+1} - X_{n+1}^n\right)^2\right]$ of the correct model.

a. For the AR(2) model we have: $\gamma(0) = \varphi_1 \gamma(1) + \varphi_2 \gamma(2) + \sigma_w^2 \Rightarrow$

$\Rightarrow \gamma(0) \cdot [1 - \varphi_1 \rho(1) - \varphi_2 \rho(2)] = \sigma_w^2 \Rightarrow \boxed{\gamma(0) = \dfrac{\sigma_w^2}{1 - \varphi_1 \rho(1) - \varphi_2 \rho(2)}}$

where $\rho(0) = 1$, $\rho(1) = \varphi_1 \rho(0) + \varphi_2 \rho(-1) = \varphi_1 + \varphi_2 \rho(1) \Rightarrow$

$\Rightarrow \rho(1)(1 - \varphi_2) = \varphi_1 \Rightarrow \boxed{\rho(1) = \dfrac{\varphi_1}{1 - \varphi_2}}$ & $\rho(2) = \varphi_1 \rho(1) + \varphi_2 \underbrace{\rho(0)}_{=1} \Rightarrow$

$\Rightarrow \boxed{\rho(2) = \dfrac{\varphi_1^2}{1 - \varphi_2} + \varphi_2} \Rightarrow \gamma(0) = \dfrac{\sigma_w^2}{1 - \dfrac{\varphi_1^2}{1 - \varphi_2} - \dfrac{\varphi_2 \varphi_1^2}{1 - \varphi_2} - \varphi_2^2} \Rightarrow$

$\Rightarrow \boxed{\gamma(0) = \dfrac{\sigma_w^2}{1 - \dfrac{\varphi_1^2 \cdot (1 + \varphi_2)}{1 - \varphi_2} - \varphi_2^2}}$, & $\boxed{\gamma(1) = \rho(1) \cdot \gamma(0)}$

b. $X_{n+1}^n = \varphi_1 X_n + \varphi_2 X_{n-1}$ for the AR(2) model

c. $\hat{\varphi} = \dfrac{\hat{\gamma}(1)}{\hat{\gamma}(0)} = \hat{\rho}(1)$ & $\hat{\sigma}_w^2 = \hat{\gamma}(0)[1-\hat{\varphi}^2] = \hat{\gamma}(0)\cdot[1-\hat{\rho}^2(1)]$

d. $X'^{n}_{n+1} = \hat{\varphi} X_n$, where $\hat{\varphi} = \dfrac{\gamma(1)}{\gamma(0)} = \rho(1) = \dfrac{\varphi_1}{1-\varphi_2} \Rightarrow$

$$\Rightarrow X'^{n}_{n+1} = \frac{\varphi_1}{1-\varphi_2} X_n \Rightarrow X^n_{n+1} - X'^{n}_{n+1} = \varphi_1 X_n + \varphi_2 X_{n-1} - \frac{\varphi_1}{1-\varphi_2} X_n =$$

$$= \varphi_1\cdot\left(1-\frac{1}{1-\varphi_2}\right)X_n + \varphi_2 X_{n-1} = -\frac{\varphi_1\varphi_2}{1-\varphi_2} X_n + \varphi_2 X_{n-1}$$

e. For the correct model we have $P^n_{n+1} = \sigma_w^2$.

For the incorrect model we have:

$$P'^{n}_{n+1} = \mathbb{E}\left[(X_{n+1} - X'^{n}_{n+1})^2\right] = \mathbb{E}\left[\left(\varphi_1 X_n + \varphi_2 X_{n-1} + W_{n+1} - \frac{\varphi_1}{1-\varphi_2} X_n\right)^2\right] =$$

$$= \mathbb{E}\left[\left(-\frac{\varphi_1\varphi_2}{1-\varphi_2} X_n + \varphi_2 X_{n-1} + W_{n+1}\right)^2\right] = \left(\text{by uncorrelated } W_{n+1} \text{ with } X_n / X_{n-1}\right)$$

$$= \left(\frac{\varphi_1\varphi_2}{1-\varphi_2}\right)^2 \underbrace{\mathbb{E}[X_n^2]}_{=\gamma(0)} + \varphi_2^2 \underbrace{\mathbb{E}[X_{n-1}^2]}_{=\gamma(0)} - 2\frac{\varphi_1\varphi_2^2}{1-\varphi_2} \underbrace{\mathbb{E}[X_n X_{n-1}]}_{=\gamma(1)=\rho(1)\gamma(0)} + \underbrace{\mathbb{E}[W_{n+1}^2]}_{=\sigma_w^2} =$$

$$= \gamma(0)\cdot\left\{\left(\frac{\varphi_1\varphi_2}{1-\varphi_2}\right)^2 + \varphi_2^2 - \cancel{2}\cdot\frac{\varphi_1\varphi_2^2}{1-\varphi_2}\cdot\frac{\varphi_1}{1-\varphi_2}\right\} + \sigma_w^2 =$$

$$= \sigma_w^2 + \gamma(0)\,\varphi_2^2\left[1-\Big(\underbrace{\frac{\varphi_1}{1-\varphi_2}}_{=\rho(1)}\Big)^2\right] = \sigma_w^2 + \gamma(0)\cdot\varphi_2^2\left[1-\rho^2(1)\right] \geq \sigma_w^2 = P^n_{n+1}$$

4. (15 marks)
A time series dataset has sample moments $\hat{\gamma}(0)=1.37, \hat{\gamma}(1)=.656$ based on a sample of size n=400. Use Yule-Walker estimation to fit the AR(1) model $X_t=\varphi X_{t-1}+W_t$ and create an approximate 95% confidence interval for φ.

$$\hat{\varphi}=\hat{\rho}(1)=\frac{\hat{\gamma}(1)}{\hat{\gamma}(0)}=\frac{.656}{1.37}=.4788321$$

$$\hat{\sigma}_W^2=\hat{\gamma}(0)\cdot[1-\hat{\varphi}^2]=1.37\cdot[1-.4788321^2]=1.055886$$

Asymptotically, $\hat{\varphi}\sim N\left(\varphi, \frac{\sigma_w^2}{n}\cdot\Gamma_1^{-1}\right)$,

where $\Gamma_1=[\gamma(0)]=1.37 \Rightarrow \hat{\varphi}\sim N\left(\varphi, \frac{1.055886}{400}\cdot\frac{1}{1.37}\right)$

$\Rightarrow \hat{\varphi}\sim N(\varphi, .0019268) \Rightarrow$

$\Rightarrow$ 95% confidence interval for φ is:

$$\hat{\varphi}\pm 1.96\times \text{s.e.}(\hat{\varphi}) = .4788321 \pm 1.96\times\sqrt{.0019268}$$

$$=[.3927973, .564867]$$

Extra Space -- Use if needed and indicate clearly which questions you are answering