STAD57: Time Series Analysis
Problem Set 5

1. Exercise 3.18 from the textbook

a. $\varphi_1^{ols} = 0.4285906$ & $\varphi_2^{ols} = 0.4417874$

$\varphi_1^{yw} = 0.4339481$ & $\varphi_2^{yw} = 0.4375768$

b. $s.e.(\varphi_1^{ols}) = 0.03979433$ & $\text{s.e.}(\varphi_2^{ols}) = 0.03976163$

$s.e.(\varphi_1^{yw}) = 0.04001303$ & $\text{s.e.}(\varphi_2^{yw}) = 0.04001303$

The standard errors for Yule-Walker estimation are higher than those of regression (ordinary least squares). This is typical since for regression we treat the lagged TS values as fixed (i.e. non-random), which reduces the uncertainty in estimation.

2. Exercise 3.20 from the textbook.

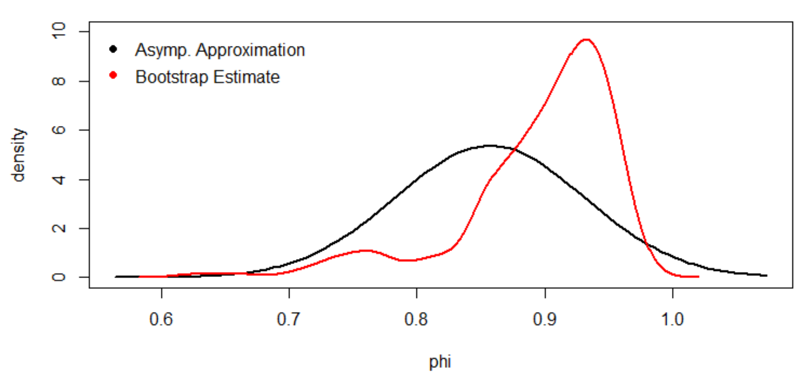
The ACF & PACF do not appear significantly different from 0 at all lags h≥1, which means that the simulated series resembles a White Noise. This is correct since the ARMA(1,1) model is redundant: $X_t = .9X_{t-1} + W_t - .9W_{t-1} \Leftrightarrow \cancel{(1-.9B)}X_t = \cancel{(1-.9B)}W_t \Leftrightarrow X_t = W_t$

3. Exercise 3.21 from the textbook.

	ϕ	θ	σ^2
True	.9	.5	1
Sim 1 Estimates	0.8935255	0.6206927	1.04339
Sim 2 Estimates	0.9305838	0.5301091	1.041926
Sim 3 Estimates	0.874694	0.513024	0.9972095

4. Exercise 3.22 from the textbook.

The asymptotic approximate distribution of $\hat{\varphi}$ and its Bootstrap estimate are shown below:



The asymptotic approximation is not so good when the number of data n is small, and the value of the parameter φ is close to the boundary of the stationary region (in this case, close to 1).

5. Exercise 3.24 from the textbook.

a. We have $\mathbb{E}[X_t] = \mathbb{E}[\alpha + \varphi X_{t-1} + W_t + \theta W_{t-1}] \Rightarrow \mu = \alpha + \varphi \mathbb{E}[X_{t-1}] + \mathbb{E}[W_t] + \theta \mathbb{E}[W_{t-1}] \Rightarrow$

$\Rightarrow \mu = \alpha + \varphi\mu \Rightarrow \mu = \dfrac{\alpha}{1-\varphi}$. We know that for the ARMA(1,1) model the autocovariance function is $\gamma(0) = \sigma_W^2 \dfrac{1+2\varphi\theta+\theta^2}{1-\varphi^2}, \quad \gamma(1) = \sigma_W^2 \dfrac{(1+\varphi\theta)(\varphi+\theta)}{1-\varphi^2}, \quad \gamma(h) = \varphi^{h-1}\gamma(1) \ \ \forall h \geq 2$. The ACF is thus $\rho(0) = 1, \quad \rho(1) = \dfrac{(1+\varphi\theta)(\varphi+\theta)}{1+2\varphi\theta+\theta^2}, \quad \rho(h) = \varphi^{h-1}\rho(1) \ \ \forall h \geq 2$. The series is weakly strationary, but not necessarily strictly stationary. It would be strictly stationary if the series were also Gaussian. (Note: the time series' mean does not affect its autocovariance or ACF)

b. From theorem A.5 in the Appendix (basically, a version of the Central Limit Theorem for stationary sequences of variables) we have that $\bar{X}_n \to AN\left(\mu, \frac{1}{n}V\right)$, where $\mu = \dfrac{\alpha}{1-\varphi}$, and

$$V = \sum_{h=-\infty}^{\infty} \gamma(h) = \gamma(0) + 2\sum_{h=1}^{\infty} \gamma(h) = \gamma(0) + 2\gamma(1)\sum_{h=1}^{\infty} \varphi^{h-1} \Rightarrow$$

$$\Rightarrow V = \gamma(0) + 2\gamma(1)\sum_{h=0}^{\infty} \varphi^{h} = \gamma(0) + \frac{2\gamma(1)}{1-\varphi} = \sigma_W^2 \frac{1+2\varphi\theta+\theta^2 + 2\dfrac{(1+\varphi\theta)(\varphi+\theta)}{1-\varphi}}{1-\varphi^2}$$

6. Consider a zero-mean, stationary *Gaussian* ARMA(p,q) model with autocovariance function $\gamma(h), \forall h \geq 0$.

a. Write down the likelihood $L = f(x_1, x_2)$ of the first two observations from the model using the fact that $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix}\right)$.

Note: if $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the d-dimensional multivariate Normal density is given by

$$f(\mathbf{x}) = (2\pi)^{-d/2} |\boldsymbol{\Sigma}|^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

b. Write down the likelihood of the first two observations from the model using the fact that $f(x_1, x_2) = f(x_1) \times f(x_2 \mid x_1)$, for $X_1 \sim N(0, \gamma(0))$ and $X_2 \mid X_1 \sim N(X_2^1, P_2^1)$, where X_2^1 is the 1-step-ahead BLP of X_2 given X_1 & P_2^1 is its corresponding MSE. Show that this is the same as the expression in **a.**

(This problem justifies the likelihood breakdown using 1-step-ahead forecasts that we used in class for deriving the ML estimation)

a.

$$L = f(x_1, x_2) = (2\pi)^{-2/2} \left|\mathbf{\Sigma}\right|^{-1/2} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})'\mathbf{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\} =$$

$$= \frac{1}{2\pi}\sqrt{\begin{vmatrix}\gamma(0) & \gamma(1)\\ \gamma(1) & \gamma(0)\end{vmatrix}} \exp\left\{-\frac{1}{2}\begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}\gamma(0) & \gamma(1)\\ \gamma(1) & \gamma(0)\end{bmatrix}^{-1}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right\} =$$

$$= \frac{1}{2\pi\sqrt{\gamma^2(0)-\gamma^2(1)}} \exp\left\{-\frac{1}{2\left(\gamma^2(0)-\gamma^2(1)\right)}\begin{bmatrix}x_1\gamma(0)-x_2\gamma(1) & -x_1\gamma(1)+x_2\gamma(0)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right\} =$$

$$= \frac{1}{2\pi\sqrt{\gamma^2(0)-\gamma^2(1)}} \exp\left\{-\frac{x_1^2\gamma(0)-2x_1x_2\gamma(1)+x_2^2\gamma(0)}{2\left(\gamma^2(0)-\gamma^2(1)\right)}\right\}$$

b. The 1-step-ahead BLP is given by $X_2^1 = \varphi_{1,1}X_1 = \dfrac{\gamma(1)}{\gamma(0)}X_1$ (by solving the prediction eqn $\gamma(1) = \varphi_{1,1}\gamma(0)$), with MSE given by $P_2^1 = P_1^0 - \varphi_{1,1}^2\gamma(0) = \gamma(0) - \dfrac{\gamma^2(1)}{\gamma^2(0)}\gamma(0) = \gamma(0) - \dfrac{\gamma^2(1)}{\gamma(0)}$. The likelihood is:

$$L = f(x_1, x_2) = f(x_1)\times f(x_2 \mid x_1) =$$

$$= \left(\frac{1}{\sqrt{2\pi\gamma(0)}}\exp\left\{-\frac{x_1^2}{2}\right\}\right)\times\left(\frac{1}{\sqrt{2\pi\left(\gamma(0)-\gamma^2(1)/\gamma(0)\right)}}\exp\left\{-\frac{\left(x_2-x_1\gamma(1)/\gamma(0)\right)^2}{2\left(\gamma(0)-\gamma^2(1)/\gamma(0)\right)}\right\}\right) =$$

$$= \frac{1}{2\pi\sqrt{\gamma(0)\left(\gamma(0)-\gamma^2(1)/\gamma(0)\right)}}\exp\left\{-\frac{x_1^2}{2\gamma(0)} - \frac{x_2^2-2x_1x_2\,\gamma(1)/\gamma(0)+x_1^2\,\gamma^2(1)/\gamma^2(0)}{2\left(\gamma(0)-\gamma^2(1)/\gamma(0)\right)}\right\} =$$

$$= \frac{1}{2\pi\sqrt{\gamma^2(0)-\gamma^2(1)}}\exp\left\{-\frac{x_1^2\left(1-\cancel{\gamma^2(1)/\gamma^2(0)}\right)+x_2^2-2x_1x_2\,\gamma(1)/\gamma(0)+\cancel{x_1^2\,\gamma(1)/\gamma(0)}}{2\left(\gamma(0)-\gamma^2(1)/\gamma(0)\right)}\right\} =$$

$$= \frac{1}{2\pi\sqrt{\gamma^2(0)-\gamma^2(1)}}\exp\left\{-\frac{x_1^2-2x_1x_2\,\gamma(1)/\gamma(0)+x_2^2}{2\left(\gamma(0)-\gamma^2(1)/\gamma(0)\right)}\right\} =$$

$$= \frac{1}{2\pi\sqrt{\gamma^2(0)-\gamma^2(1)}}\exp\left\{-\frac{x_1^2\gamma(0)-2x_1x_2\gamma(1)+x_2^2\gamma(0)}{2\left(\gamma^2(0)-\gamma^2(1)\right)}\right\}$$

which is exactly the same as in part **a.**