STAC62 : 2016 Assignment 1 - Solutions

1. (a) Let $A_1 = \phi$, $A_2 = \Omega$, $A_3 = \{1,2\}$, $A_4 = \{3,4,5\}$.
So $A_1 \in \mathcal{F}$ and (i) of the Definition is satisfied.
Also $A_1^c = A_2$, $A_2^c = A_1$, $A_3^c = A_4$ and $A_4^c = A_3$
and so (iii) of the Definition is satisfied.
Since there are only finitely many elements in $\mathcal{F}$
and infinite union is really only a finite union
so if $\mathcal{F}$ is closed under pairwise unions it is
closed under infinite unions. We have $A_1 \cup A_i = A_i$,
for all i, $A_i \cup A_2 = A_2$ for all i, $A_3 \cup A_4 = A_2$
and so $\mathcal{F}$ is closed under unions and satisfies (ii)
of the Definition. Therefore $\mathcal{F}$ is a σ-field.

(b) Clearly $\phi, \Omega \in \mathcal{F}(\mathcal{A})$, $\{1\}^c = \{2,3,4,5\} \in \mathcal{F}(\mathcal{A})$,
$\{2,3,4\}^c = \{1,5\} \in \mathcal{F}(\mathcal{A})$, $\{5\}^c = \{1,2,3,4\} \in \mathcal{F}(\mathcal{A})$
Arguing as in (a) then shows that
$\{\phi, \Omega, \{1\}, \{2,3,4,5\}, \{2,3,4\}, \{1,5\}, \{5\},$
$\{1,2,3,4\}\}$ is a σ-field and so this must
be $\mathcal{F}(\mathcal{A})$

(c) If P determines a probability measure on
$\mathcal{F}(\mathcal{A})$, then we must have $P(\{2,3,4,5\}) = 1 - P(\{1\})$
$= 1 - 1/4 = 3/4$, $P(\{1,5\}) = 1 - P(\{2,3,4\}) = 1 - 1/2 = 1/2$,
$P(\{1,2,3,4\}) = 1 - P(\{5\}) = 1 - 1/4 = 3/4$, $P(\phi) = 0$
and $P(\Omega) = 1$. Suppose then that $A_1, A_2 \in \mathcal{F}(\mathcal{A})$
are disjoint. Then $P(A_1 \cup A_2) = P(A_1) + P(A_2)$
for every possible choice of A_1 and A_2 and
so P is additive with $P(\Omega) = 1$. Therefore
P is a probability measure on $\mathcal{F}(\mathcal{A})$ and it
is unique.

2 (b) Note that $(a-\frac{1}{n}, a]$ is a Borel set for $n=1,2,\ldots$. Also $\bigcap_{n=1}^{\infty} A_n$ is a Borel set since it is formed from countable intersections of Borel sets. Now $a \in A_n$ $\forall n$ and so $a \in \bigcap_{n=1}^{\infty} A_n$ which gives $\{a\} \subseteq \bigcap_{n=1}^{\infty} A_n$. Now if $\varepsilon > 0$ then $\exists N$ s.t. $\forall n \geq N$ $a - 1/n > a - \varepsilon$. Therefore, for every $n \geq N$ we have that $a-\varepsilon \notin A_n$ and so $a - \varepsilon \notin \bigcap_{n=1}^{\infty} A_n$. This says that if $x \in \bigcap_{n=1}^{\infty} A_n$ then $x \geq a$ and since any $x > a$ cannot be in any A_n we have $\bigcap_{n=1}^{\infty} A_n \subseteq \{a\}$. Therefore $\{a\} = \bigcap_{n=1}^{\infty} A_n$ and $\{a\}$ is a Borel set.

(a) We have $(-\infty, a] = \bigcup_{n=1}^{\infty} (a-n, a]$ since $\forall x \in (-\infty, a]$ $\exists n$ s.t. $a-n < x$ and so $x \in (a-n, a]$ while if $x \in (a-n, a]$ then $x \leq a$ so $x \in (-\infty, a]$. Since $(a-n, a]$ is a Borel set for each n we have that $\bigcup_{n=1}^{\infty} (a-n, a]$ is a Borel set and the result is established.

3 Since $\underline{\lim}_{n\to\infty} A_n = \{\omega : \omega \text{ is in all but finitely many of the } A_i\}$
$\subseteq \{\omega : \omega \text{ is all infinitely many of the } A_i\} = \overline{\lim}_{n\to\infty} A_n$
So we always have $\underline{\lim}_{n\to\infty} A_n \subseteq \overline{\lim}_{n\to\infty} A_n$.

(a) Suppose $\omega \in \overline{\lim}_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$. Since $A_1 \subseteq A_2 \subseteq \cdots$ we have $\bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=n}^{\infty} A_i$ $\forall n$ and clearly $\bigcup_{i=n}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} A_i$ and so $\bigcup_{i=n}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i$ $\forall n$. Therefore $\overline{\lim}_{n\to\infty} A_n = \bigcup_{i=1}^{\infty} A_i$. If $\omega \in \bigcup_{i=1}^{\infty} A_i$ then $\exists$ a first n s.t. $\omega \in A_n$ which implies $\omega \in A_i$ for all $i \geq n$.

This implies that $\omega \in \underline{\lim}_{n\to\infty} A_n$ and so $\underline{\lim}_{n\to\infty} A_n \subseteq \overline{\lim}_{n\to\infty} A_n$ which implies the result.

(b) Suppose $A_1 \supseteq A_2 \supseteq \ldots$. Then $\bigcap_{i=n}^{\infty} A_i \subseteq A_i$ for $i=1,\ldots,n-1$ so $\bigcap_{i=n}^{\infty} A_i \subseteq \bigcap_{i=1}^{\infty} A_i$ and obviously $\bigcap_{i=1}^{\infty} A_i \subseteq \bigcap_{i=n}^{\infty} A_i$. Therefore $\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=n}^{\infty} A_i$ for every n. This implies $\underline{\lim}_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty} A_i = \bigcap_{i=1}^{\infty} A_i$. Now suppose $\omega \in \overline{\lim}_{n\to\infty} A_n$. Then $\omega \in \bigcup_{i=n}^{\infty} A_i \;\forall n$ but $\bigcup_{i=n}^{\infty} A_i = A_n$ since $A_{n+i} \subseteq A_n \;\forall i \geq 1$. Thus $\omega \in \bigcap_{n=1}^{\infty} A_n$ which proves $\omega \in \underline{\lim}\, A_n$. Therefore, we have proved that $\underline{\lim}\, A_n = \overline{\lim}\, A_n = \bigcap_{n=1}^{\infty} A_n$.

(4) When $n=2$ we have $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$ and so $P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$. Now assume the result holds for $A_1, \ldots, A_{n-1}$ with $n-1 \geq 2$. Then

$$P(A_1 \cap \cdots \cap A_n) = P((A_1 \cap \cdots \cap A_{n-1}) \cap A_n)$$

$$\overset{\text{for } n=2}{=} P(A_1 \cap \cdots \cap A_{n-1}) + P(A_n) - P((A_1 \cap \cdots A_{n-1}) \cup A_n)$$

and since $(A_1 \cap \cdots \cap A_{n-1}) \cup A_n = (A_1 \cup A_n) \cap \cdots \cap (A_{n-1} \cup A_n)$

$$= P(A_1 \cap \cdots A_{n-1}) + P(A_n) - P((A_1 \cup A_n) \cap \cdots \cap (A_{n-1} \cup A_n))$$

and so we can apply the induction hypothesis to the outer two terms to obtain

$$= \sum_{i=1}^{n-1} P(A_i) - \sum_{\substack{i<j \\ \leq n-1}} P(A_i \cup A_j) + \cdots + (-1)^n P(A_1 \cup \cdots \cup A_{n-1})$$

$$+ P(A_n) - \Big[\sum_{i=1}^{n-1} P(A_i \cup A_n) - \sum_{i<j\leq n-1} P((A_i \cup A_n) \cup (A_j \cup A_n))$$

$$+ \cdots + (-1)^n P((A_1 \cup A_n) \cup \cdots \cup (A_{n-1} \cup A_n)) \Big]$$

and since $(A_i \cup A_n) \cup (A_j \cup A_n) = A_i \cup A_j \cup A_n$ etc

$$= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cup A_j) + \cdots + (-1)^{n+1} P(A_1 \cup \cdots \cup A_n).$$

as required.

(5) Note that this is the model of independent tossing of a fair coin

(a) $P($"after 10 tosses there are more heads than tails"$) = P($"6 or more heads in 10 tosses"$)$

$$= \sum_{k=6}^{\infty} \binom{10}{6} \left(\tfrac{1}{2}\right)^{10}$$

since the number of heads in 10 tosses is distributed binomial$(10, \tfrac{1}{2})$

(b) $P($"30-th toss is a head"$) = \tfrac{1}{2}$ since the 30-th toss is independent of all others and is distributed Bernoulli$(1/2)$.

(c) $P($"finitely many heads are obtained"$)$

$$= \sum_{i=1}^{\infty} P(\text{"exactly } i \text{ heads are obtained"})$$

$P($"exactly i heads are obtained"$) =$

$$= \sum_{n=i}^{\infty} P(\text{"}i\text{-th and last head is obtained on } n\text{th toss"})$$

and $P(\text{" "}) \le P(\text{"}n+1, n+2, \ldots \text{ tosses are tails"})$

since $\subseteq$

$$\le \lim_{m \to \infty} P(\text{"toss } n+1, \ldots, n+m \text{ are tails"})$$

$$= \lim_{m \to \infty} \left(\tfrac{1}{2}\right)^m = 0$$ and so $P($"finitely many heads obtained"$) = 0$.

(d) $P(\text{"process stops"}) = P(T<\infty) = 1 - P(T=\infty)$

$= 1 - P(\text{" there are never two heads in a row"})$

and $P(\text{"}\dagger\text{"}) = \lim_{n\to\infty} P((\omega_1, \dots, \omega_n) \in \{(0,1,0,\dots), (1,0,1,0,\dots)\})$

$= \lim_{n\to\infty} 2\left(\frac{1}{2}\right)^n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-1} = 0.$

Therefore, $P(T<\infty) = 1.$

(e) $P(T=n) = P(\text{"exactly } j-1 \text{ heads occur in } (\omega_1, \dots, \omega_{n-1}) \text{ and } \omega_n = 1\text{"})$

and since each sequence of n has prob $\left(\frac{1}{2}\right)^n$

$$= \begin{cases} 0 & n<j \\ \left(\frac{1}{2}\right)^n \times \#\{(\omega_1, \dots, \omega_{n-1}) : \text{exactly } j-1 \text{ of the } \omega_i = 1\} \end{cases}$$

$$= \begin{cases} 0 & n<j \\ \binom{n-1}{j-1}\left(\frac{1}{2}\right)^n & n \geq j \end{cases}$$

Now $P(T=\infty) = P(\text{"}\#\text{ of heads} \leq j-1\text{" in infinitely many tosses"})$

$\leq P(\text{"only finitely many heads are obtained"})$

$= 0$ by (c).