University of Toronto Scarborough

Department of Computer & Mathematical Sciences

STAD57H3 Time Series Analysis

Midterm Examination
October 25, 2019

Duration: 110 minutes

Examination aids allowed: Non-programmable Scientific Calculator, formula sheet (provided by instructor)

Last Name: ______________________________

First Name: ______________________________

Student #: ______________________________

Instructions:

- Read the questions carefully and answer only what is being asked.
- Answer all questions directly on the examination paper; use the last pages if you need more space, and provide clear pointers to your work.
- Show your intermediate work, and write clearly and legibly.

Question:	1	2	3	4	Total
Points:	23	10	10	12	55
Score:					

1. Let $Y_t = \sum_{s=1}^{t} W_s$, where $W_t \sim WN(0, \sigma_w^2)$, i.e. Y_t is a random walk process.

   (a) [5 points] Show that the m-step-ahead BLP (best linear predictor) for Y_{n+m} based on $Y_1, \ldots, Y_n$ is given by $Y_{n+m}^n = Y_n$. This implies that the best prediction for any future value is just the current value (which should not be surprising since the random walk is a Markov process).
   Hint: You can show that the predictor satisfies the normal equations, or that it is equal to the minimum MSE predictor.

   (b) [3 points] Find the MSPE (mean square prediction error) P_{n+m}^n of the m-step-ahead BLP.

   For the remaining questions, consider the random walk process $Y_t = \sum_{s=1}^{t} X_s$, where X_t follows the AR(1) process $X_t = \phi X_{t-1} + W_t$ and $W_t \sim WN(0, \sigma_w^2)$.

   (c) [5 points] Show that the 1-step-ahead BLP is $Y_{n+1}^n = (1+\phi)Y_n - \phi Y_{n-1}$.
   Hint: $X_n = (Y_n - Y_{n-1})$

   (d) [2 points] Show that the MSPE is $P_{n+1}^n = \sigma_w^2$.

   (e) [5 points] Find the 2-step-ahead BLP Y_{n+2}^n.

   (f) [3 points] Find its MSPE P_{n+2}^n.

Solution:

(a) Using normal equations:

$$\mathbb{E}[(Y_{n+m} - Y_{n+m}^n)Y_i] = \mathbb{E}[((\cancel{Y_n} + \sum_{j=1}^{m} W_{n+j}) - \cancel{Y_n})Y_i]$$
$$= \sum_{j=1}^{m} \overbrace{\mathbb{E}[W_{n+j}Y_i]}^{=0} = 0 \quad \forall i \leq n$$

Using conditional expectation:

$$\mathbb{E}[Y_{n+m}|Y_n, \ldots, Y_1] = \mathbb{E}[Y_n + \sum_{j=1}^{m} W_{n+j}|Y_n, \ldots, Y_1]$$
$$= Y_n + \sum_{j=1}^{m} \overbrace{\mathbb{E}[W_{n+j}|Y_n, \ldots, Y_1]}^{=0} = Y_n$$

Both proofs rely on the fact that $W_t \perp Y_s, \ \forall t > s$.

(b)

$$\mathbb{E}[(Y_{n+m} - Y^n_{n+m})^2] = \mathbb{E}[((\cancel{Y_n} + \sum_{j=1}^{m} W_{n+j}) - \cancel{Y_n})^2]$$
$$= \sum_{j=1}^{m} \overbrace{\mathbb{E}[W^2_{n+j}]}^{=\sigma_w^2} = m\sigma_w^2$$

(c) Note that $Y_n = Y_{n-1} + X_n \Rightarrow X_n = (Y_n - Y_{n-1})$. Using conditional expectation:

$$\begin{aligned}
\mathbb{E}[Y_{n+1}|Y_n, \ldots, Y_1] &= \mathbb{E}[Y_n + X_{n+1}|Y_n, \ldots, Y_1] \\
&= Y_n + \mathbb{E}[(\phi X_n + W_{n+1})|Y_n, \ldots, Y_1] \\
&= Y_n + \phi\,\mathbb{E}[Y_n - Y_{n-1}|Y_n, \ldots, Y_1] + \mathbb{E}[W_{n+1}|Y_n, \ldots, Y_1] \\
&= Y_n + \phi(Y_n - Y_{n-1}) = (1+\phi)Y_n - \phi Y_{n-1} = Y^n_{n+1}
\end{aligned}$$

(d)

$$\begin{aligned}
\mathbb{E}[(Y_{n+1} - Y^n_{n+1})^2] &= \mathbb{E}[((Y_n + X_{n+1}) - ((1+\phi)Y_n - \phi Y_{n-1}))^2] \\
&= \mathbb{E}[(\cancel{(1+\phi)Y_n - \phi Y_{n-1}} + W_{n+1}) - \cancel{((1+\phi)Y_n - \phi Y_{n-1})})^2] \\
&= \mathbb{E}[(W_{n+1})^2] = \sigma_w^2
\end{aligned}$$

(e) Similarly,

$$\begin{aligned}
\mathbb{E}[Y_{n+2}|Y_n, \ldots, Y_1] &= \mathbb{E}[Y_n + X_{n+2} + X_{n+1}|Y_n, \ldots, Y_1] \\
&= Y_n + \mathbb{E}[(\phi X_{n+1} + W_{n+2}) + X_{n+1}|Y_n, \ldots, Y_1] \\
&= Y_n + \mathbb{E}[(1+\phi)X_{n+1} + W_{n+2})|Y_n, \ldots, Y_1] \\
&= Y_n + (1+\phi)\,\mathbb{E}[X_{n+1}|Y_n, \ldots, Y_1] + \cancelto{0}{\mathbb{E}[W_{n+2}|Y_n, \ldots, Y_1]} \\
&= Y_n + (1+\phi)\,\mathbb{E}[\phi X_n + W_{n+1}|Y_n, \ldots, Y_1] \\
&= Y_n + (1+\phi)\phi\,\mathbb{E}[(Y_n - Y_{n-1})|Y_n, \ldots, Y_1] + \phi\,\cancelto{0}{\mathbb{E}[W_{n+1}|Y_n, \ldots, Y_1]} \\
&= Y_n + (1+\phi)\phi(Y_n - Y_{n-1}) = Y^n_{n+2}
\end{aligned}$$

(f) [3 points] Similarly,

$$\begin{aligned}
\mathbb{E}[(Y_{n+2} - Y^n_{n+2})^2] &= \mathbb{E}[(Y_{n+2} - (Y_n + (1+\phi)\phi(Y_n - Y_{n-1})))^2] \\
&= \vdots \\
&= \mathbb{E}[(W_{t+2} + \phi W_{t+1})^2] = \mathbb{E}[(W_{n+2})^2] + \phi^2\,\mathbb{E}[(W_{n+1})^2] = (1+\phi^2)\sigma_w^2
\end{aligned}$$

2. Consider the AR(2) model $X_t = -.3X_{t-1} + .4X_{t-2} + W_t$.

   (a) [3 points] Determine whether the series is stationary.

   (b) [7 points] Find a linear filter $\tau(B)$ such that the filtered series $Y_t = \tau(B)X_t = \sum_{i=0}^{\infty} \tau_i X_{t-i}$ follows the AR(1) process $Y_t = .5Y_{t-1} + W_t$.
   Hint: Factorize the AR operator $\phi(B) = (1 + c_1 B)(1 + c_2 B)$.

Solution:

(a) The roots of the characteristic polynomial $\phi(z) = 1 + .3z - .4z^2$ are given by

$$\begin{aligned}
r_{1/2} &= \frac{-.3 \pm \sqrt{(.3)^2 - 4(-.4)1}}{2(-.4)} = \frac{.3 \pm \sqrt{.09 + 1.6}}{.8} = \frac{.3 \pm \sqrt{1.69}}{.8} \\
&= \frac{.3 \pm 1.3}{.8} = \begin{cases} (.3 - 1.3)/.8 = -1/.8 = -5/4 \\ (.3 + 1.3)/.8 = 1.6/.8 = 2 \end{cases}
\end{aligned}$$

Since both roots are outside the unit disk ($r_1 = -5/4 < 1,\ r_2 = 2 > 1$), the model is stationary.

(b) The AR polynomial can be factorized based on the roots as follows:

$$\begin{aligned}
\phi(z) &= 1 + .3z - .4z^2 = (1 - z/r_1)(1 - z/r_1) \\
&= (1 + z4/5)(1 - z/2) = (1 + .8z)(1 - .5z)
\end{aligned}$$

We have

$$\begin{aligned}
X_t &= -.3X_{t-1} + .4X_{t-2} + W_t \\
\Rightarrow (1 + .3B - .4B^2)X_t &= W_t \\
\Rightarrow (1 - .5B)\overbrace{(1 + .8B)X_t}^{=Y_t} &= W_t \\
\Rightarrow (1 - .5B)Y_t &= W_t \\
\Rightarrow Y_t &= .5Y_{t-1} + W_t \\
\Rightarrow \tau(B) &= (1 + .8B)
\end{aligned}$$

More generally, for any initial model $\phi(B)X_t$ and target model $\xi(B)Y_t = W_t$, you can find the desired transformation as follows:

$$\begin{aligned}
\phi(B)X_t &= W_t \quad \text{(multiply l.h.s. by } \xi(B)\xi^{-1}(B) = 1) \\
\Rightarrow \xi(B)\overbrace{\xi^{-1}(B)\phi(B)X_t}^{=Y_t} &= W_t \\
\Rightarrow \tau(B) &= \xi^{-1}(B)\phi(B)
\end{aligned}$$

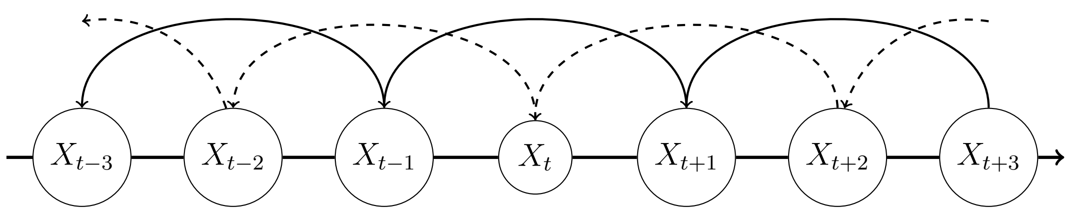
3. Consider the AR(2) model $X_t = \phi_2 X_{t-2} + W_t$ (i.e. $\phi_1 = 0$).
   (a) [5 points] Find the causal representation of X_t.
   (b) [5 points] Find the ACF $\rho(h)$ of X_t.

Solution:

(a) By back-substitution:

$$\begin{aligned}
X_t &= \phi_2 X_{t-2} + W_t \\
&= \phi_2(\phi_2 X_{t-4} + W_{t-2}) + W_t \\
&= \phi_2^2(\phi_2 X_{t-6} + W_{t-4}) + \phi_2 W_{t-2} + W_t \\
&= \phi_2^3(\phi_2 X_{t-8} + W_{t-6}) + \phi_2^2 W_{t-4} + \phi_2 W_{t-2} + W_t \\
&= \vdots \\
&= \sum_{j=0}^{\infty} \phi_2^{2j} W_{t-2j}
\end{aligned}$$

(b) Note that, by back-substitution, only values that are an even number of lags apart are related (odd lags have no correlation).



It is straightforward to show that:

$$\gamma_(h) = \begin{cases} \phi_2^{h/2} \sigma_w^2/(1-\phi_2^2), & h \in \{0, 2, 4, \ldots\} \\ 0, & h \in \{1, 3, 5, \ldots\} \end{cases}$$

$$\Rightarrow \rho(h) = \begin{cases} \phi_2^{h/2}, & h \in \{0, 2, 4, \ldots\} \\ 0, & h \in \{1, 3, 5, \ldots\} \end{cases}$$

4. Consider the AR(1) process $X_t = \frac{1}{2}X_{t-1} + W_t$ and the MA(1) process $Y_t = W_t + \frac{1}{2}W_{t-1}$, defined on the same white noise $W_t \sim WN(0, \sigma_w^2)$. Define their average $Z_t = (X_t + Y_t)/2$.

(a) [6 points] Find the ψ-weights of the causal representation $Z_t = \sum_{i=0}^{\infty} \psi_i W_{t-i}$.

(b) [6 points] Show that the ARMA(1,2) model $Z_t = \frac{1}{2}Z_{t-1} + W_t - \frac{1}{8}W_{t-2}$ has the same ψ-weights you got in (a).
(In other words, the sum of an AR(1) and MA(1) model is not an ARMA(1,1), but rather an ARMA(1,2) model).

Solution:

(a) The MA model is already expressed in terms of causal weights. For the AR(1) model, the causal weights are $\psi_j = \phi^j = 1/2^j$, $\forall j \geq 0$. Thus:

$$\begin{aligned} X_t &= W_t + \tfrac{1}{2}W_{t-1} + \tfrac{1}{2^2}W_{t-2} + \ldots \\ Y_t &= W_t + \tfrac{1}{2}W_{t-1} \\ \Rightarrow (X_t + Y_t)/2 &= W_t + \tfrac{1}{2}W_{t-1} + \tfrac{1}{2^3}W_{t-2} + \tfrac{1}{2^4}W_{t-3} + \ldots \\ Z_t &= W_t + \tfrac{1}{2}W_{t-1} + \sum_{t=2}^{\infty} \tfrac{1}{2^{j+1}}W_{t-j} \end{aligned}$$

(b) For $\phi_1 = \frac{1}{2}$ and $\theta_1 = 0, \theta_2 = \frac{1}{8}$, we have:

$$\begin{aligned} \psi_0 &= 1 \\ \psi_1 - \phi_1\psi_0 = \theta_1 \Rightarrow \psi_1 - \tfrac{1}{2} = 0 \Rightarrow \psi_1 &= \tfrac{1}{2} \\ \psi_2 - \phi_1\psi_1 = \theta_2 \Rightarrow \psi_2 - \tfrac{1}{4} = \tfrac{1}{8} \Rightarrow \psi_2 &= \tfrac{1}{8} = \tfrac{1}{2^3} \\ \psi_j - \phi_1\psi_{j-1} = 0 \Rightarrow \psi_j = \psi_{j-1}/2 \Rightarrow \psi_j &= \tfrac{1}{2^{j+1}}, \quad \forall j \geq 3 \end{aligned}$$