University of Toronto Scarborough

Department of Computer & Mathematical Sciences

STAD57 Time Series Analysis

December 2016 Final Examination

Instructor: Sotirios Damouras
Duration: 3hours

Examination aids allowed: Non-programmable scientific calculator, open book/notes

Last Name: Sol Key
First Name: ______
Student #: ______

Instructions:

- Read the questions carefully and answer only what is being asked.
- Answer all questions directly on the examination paper; use the last pages if you need more space, and provide clear pointers to your work.
- Show your intermediate work, and write clearly and legibly.

Question:	1	2	3	4	5	6	7	Total
Points:	20	15	20	20	15	10	10	110
Score:								

1. Consider the time series $X_t = 5 + W_t + .5W_{t-1} - .25W_{t-2}$, where $W_t \sim \mathrm{WN}(0, 1)$.

   (a) (4 points) The series follows an ARMA(p, q) model. Find the order of the model (i.e. p, q) and determine whether it is stationary and/or invertible.

   (b) (4 points) Find the ACF of the series.

   (c) (4 points) Find the PACF of the series for lags $h = 1, 2, 3$.

   Let X_{n+m}^n be the m-step-ahead Best Linear Predictor (BLP) based on n series values, and let P_{n+m}^n be its associated Mean Square Prediction Error (MSPE).

   (d) (4 points) Find X_{n+m}^n and P_{n+m}^n for any $m \geq 3$.

   (e) (4 points) Find X_{2+1}^2 (as a function of X_1, X_2) and P_{2+1}^2.

2

(a) ARMA(0, 2) = MA(2) ⇒ causal/stationary

2

roots of MA poly: $\underbrace{1}_{\gamma} + \underbrace{\tfrac{1}{2}}_{\beta}z - \underbrace{\tfrac{1}{4}}_{a}z^2 = 0 \Rightarrow \Delta = \beta^2 - 4a\gamma = \frac{1}{4} - 4\left(-\frac{1}{4}\right) \cdot 1$

$$\mathbb{R} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-1/2 \pm \sqrt{5/4}}{-2/4 \; 1/2} = -\left(1 \pm 2 \cdot \sqrt{5/4}\right) = 1 + \frac{1}{4} = 5/4$$

which is > 0 in abs. val.

(b) 1 $\gamma(0) = (1 + \theta_1^2 + \theta_2^2)\cancel{\sigma^2}\,1 \cdot 1 + \frac{1}{4} + \frac{1}{16} = \frac{16 + 4 + 1}{16} = \frac{21}{16}$

1 $\gamma(1) = (1\theta_1 + \theta_1\theta_2) = \frac{1}{2} \cdot \left(1 - \frac{1}{4}\right) = \frac{3}{8} = \frac{6}{16} \Rightarrow \rho(1) = \frac{6}{21}$ 1

1 $\gamma(2) = \theta_2 = -\frac{1}{4} = -\frac{4}{16} \Rightarrow \rho(2) = -\frac{4}{21}$

$\gamma(h) = 0 \quad , h > 0$

(c) $\varphi_{11} = \rho(1) = \frac{6}{21} = .2857143$ 1

$$\varphi_{2,2} = \frac{\gamma(2) - \sum \varphi_{11} \cdot \gamma(1)}{\gamma(0) - \sum \varphi_{11}\gamma(1)} = \frac{-4/21 - \frac{6}{21} \cdot \frac{6}{16}}{\frac{21}{16} - \frac{6}{21} \cdot \frac{6}{16}} = \frac{-4 \cdot 21 - 6^2}{21^2 - 6^2} = -0.29629$$

1

$$\varphi_{2,1} = \varphi_{1,1} - \varphi_{22} \cdot \varphi_{11} = \varphi_{11} \cdot (1 - \varphi_{22})$$
$$= \frac{6}{21} \cdot (1 - \quad) \quad = .3703704$$ 1

$$\varphi_{33} = \frac{0 - \varphi_{21}\,\gamma(2) - \varphi_{22}\,\gamma(1)}{\gamma(0) - \varphi_{21}\,\gamma(1) - \varphi_{22}\,\gamma(2)} = \cdots = .1852632$$ 1

d) $X^n_{n+m} = 5$ 2, $\quad P^n_{n+m} = \gamma(0) = \frac{21}{16}$ 2

e) $X^2_{2+1} = 5 + \varphi_{21}(X_2 - 5) + \varphi_{22}(X_1 - 5)$ 2

$P^2_{2+1} = \gamma(0) \cdot (1 - \varphi_{11})^2 \cdot (1 - \varphi_{22})^2$ 2

2. Consider the zero-mean, stationary *bivariate* series $\begin{bmatrix} X_t \\ Y_t \end{bmatrix}$ with individual auto-covariance functions $\gamma_X(h)$,$\gamma_Y(h)$, $\forall h \geq 0$, and cross-covariance function $\gamma_{X,Y}(h)$, $\forall h \in \mathbb{Z}$.

(a) (7 points) Find the auto-covariance function of $V_t = X_t - Y_{t-\ell}$, for some $\ell \in \mathbb{Z}$, expressed in terms of $\gamma_X(h)$,$\gamma_Y(h)$, $\gamma_{X,Y}(h)$.

(b) (8 points) Find the cross-covariance function $\gamma_{Z,W}(h)$ of the bivariate linear transformation $\begin{bmatrix} Z_t \\ W_t \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \begin{bmatrix} X_t \\ Y_t \end{bmatrix}$, expressed in terms of $\gamma_X(h)$,$\gamma_Y(h)$, $\gamma_{X,Y}(h)$ and the constants a, b, c.

(a) $\gamma_V(h) = Cov(V_{t+h}, V_t) = Cov(X_{t+h} - Y_{t+h-\ell}, X_t - Y_{t-\ell}) =$ 2

$= Cov(X_{t+h}, X_t) - Cov(X_{t+h}, Y_{t-\ell}) -$ 2

$- Cov(X_t, Y_{t+h-\ell}) + Cov(Y_{t+h-\ell}, Y_{t-\ell}) =$

$= \gamma_X(h) - \gamma_{XY}(h+\ell) - \gamma_{XY}(\ell-h) + \gamma_Y(h)$, $\forall h \in \mathbb{Z}$ 3

(b) $\gamma_{ZW}(h) = Cov(Z_{t+h}, W_t) =$ 2 $\begin{cases} Z_t = aX_t + bY_t \\ W_t = cX_t \end{cases}$

$= Cov(aX_{t+h} + bY_{t+h}, cX_t) =$ 3

$= ac\,Cov(X_{t+h}, X_t) + bc\,Cov(X_t, Y_{t+h}) =$

$= c \cdot (a \cdot \gamma_X(h) + b\,\gamma_{XY}(-h))$ 3

(contd...)

3. The time series $\{X_t\}$ follows a zero-mean SARIMA$(0,1,0) \times (1,0,0)_{[3]}$ model with a single parameter Φ and an i.i.d. Normal$(0,\sigma^2)$ white noise sequence $\{W_t\}$.

(a) (4 points) Write down the linear equation describing the evolution of X_t based on its past $(X_{t-1}, X_{t-2}, \ldots)$ and the white noise $(W_t, W_{t-1}, \ldots)$.

(b) (4 points) Can you find a *causal* representation for X_t? (If yes, provide the representation; if no, explain why not.)

(c) (4 points) Find the ACF of $\nabla X_t = X_t - X_{t-1}$.

(d) (8 points) Write the conditional likelihood, given $X_0 = 0$, of the first 4 observations of the series $(x_1, \ldots, x_4)$, expressed as a function of the parameters Φ, σ^2 and the values $x_1, \ldots, x_4$.

(a) $(1-\Phi B^3)\cdot\nabla X_t = W_t \Rightarrow$ (2)

$\Rightarrow (1-\Phi B^3)(1-B)X_t = (1-B-\Phi B^3+\Phi B^4)X_t = W_t$

$\Rightarrow X_t - X_{t-1} - \Phi X_{t-3} + \Phi X_{t-4} = W_t$

$\Rightarrow X_t = X_{t-1} + \Phi X_{t-3} - \Phi X_{t-4} + W_t$

(b) No, not stationary.

(c) $\rho(h) = \begin{cases} \Phi^{h/3}/1-\phi^2, & h = n\cdot 3, \ n \in \mathbb{N} \\ 0, & o/w \end{cases}$

(d) The marginal distribution of ∇X_t is $N(0, \frac{\sigma^2}{1-\Phi^2})$

The conditional distr. of $\nabla X_1 | X_0$ is $N(0, \frac{\sigma^2}{1-\Phi^2})$

— " — $X_2 | X_1$ is $N(X_1, \frac{\sigma^2}{1-\Phi^2})$

— " — $X_3 | X_2$ is $N(X_2, \frac{\sigma^2}{1-\Phi^2})$

(contd...)

— " — $X_4 | X_3, X_1$ is $N(X_3 + \Phi X_1, \sigma^2$

$$f(x_1, x_2, x_3, x_4 \mid X_0 = 0) = \left(\frac{1}{\sqrt{2\pi\, \sigma^2/1-\varphi^2}}\right)^3 \cdot \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) \cdot$$

$$\cdot \exp\left\{-\frac{1}{2} \cdot \left(\frac{x_1^{\,2}}{\sigma^2/1-\varphi^2} + \frac{(x_2-x_1)^2}{\sigma^2/1-\varphi^2} + \frac{(x_3-x_2)^2}{\sigma^2/1-\Phi^2}\right) + \frac{(x_4-x_3-\Phi x_1)^2}{\sigma^2}\right.$$

4. Consider two jointly stationary time series $\{X_t, Y_t\}$, with individual auto-covariance functions $\gamma_X(h), \gamma_Y(h)$, $\forall h \geq 0$ and cross-covariance function $\gamma_{X,Y}(h)$, $\forall h \in \mathbb{Z}$.

   (a) (7 points) Find the Best Linear Predictor (BLP) of Y_t given X_t, and its Mean Square Prediction Error (MSPE), expressed in terms of $\gamma_X(h), \gamma_Y(h), \gamma_{X,Y}(h)$.

   (b) (13 points) Find the BLP of Y_t given X_t, X_{t-1}, Y_{t-1}, and its MSPE, expressed in terms of $\gamma_X(h), \gamma_Y(h), \gamma_{X,Y}(h)$.
   (Note: you don't need to solve the system of equations defining the BLP coefficients.)

Let $\hat{Y}_t = aX_t$

(a) Want $E((Y_t - \hat{Y}_t)X_t) = 0 \Rightarrow$

$\Rightarrow E[(Y_t - aX_t)X_t] = 0 \Rightarrow Cov(Y_t, X_t) - a \cdot Cov(X_t X_t) = 0$

$\Rightarrow \hat{a} = \dfrac{\gamma_{XY}(0)}{\gamma_X(0)}$

MSPE $E[(\hat{Y}_t - Y_t)^2] = E(\hat{a}X_t - Y_t)^2] =$

$= E[a^2 X_t^2 + Y_t^2 - 2\hat{a} X_t Y_t) \Rightarrow$

$= \gamma_Y(0) + a^2 \gamma_X(0) - 2a\,\gamma_{XY}(0) = \gamma_Y(0) - a\,\gamma_{XY}(0) =$

$= \gamma_Y(0) - \dfrac{(\gamma_{XY}(0))^2}{\gamma_X(0)}$

 (contd...)

(b). similarly, we want $\hat{Y}_t = a_1 X_t + a_2 X_{t-1} + a_3 Y_{t-1}$

$$E\left[(Y_t - \hat{Y}_t) \begin{bmatrix} X_t \\ X_{t-1} \\ Y_{t-1} \end{bmatrix} \right] = 0 \Rightarrow$$

$$\Rightarrow \underbrace{\begin{bmatrix} \gamma_{XX}(0) & \gamma_X(1) & \gamma_{XY}(1) \\ \gamma_X(1) & \gamma_X(0) & \gamma_{XY}(0) \\ \gamma_{XY}(1) & \gamma_{XY}(0) & \gamma_Y(0) \end{bmatrix}}_{\underline{\Gamma}} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{\underline{a}} = \underbrace{\begin{bmatrix} \gamma_{XY}(0) \\ \gamma_{XY}(-1) \\ \gamma_Y(1) \end{bmatrix}}_{\underline{\gamma}}$$

$$\Rightarrow \underline{\alpha} = \underline{\Gamma}^{-1} \underline{\gamma}$$

$$P = \gamma_Y(0) - \underline{\gamma}\, \underline{\Gamma}^{-1} \underline{\gamma}$$

5. Consider a zero-mean (i.e. no drift) random walk $X_t = X_{t-1} + W_t$, $t \geq 1$, where $X_0 = 0$ and $W_t \sim$WN(0,σ^2).

(a) (5 points) Assume you try to estimate the auto-covariance at lag 0 (i.e. the variance) based on n observations, *as if* the process was stationary. More specifically, you use

$$\hat{\gamma}(0) = \frac{1}{n}\sum_{t=1}^{n} X_t^2$$

Find the expected value of $\hat{\gamma}(0)$, as a function of n, σ.

(b) (5 points) Assume you try to estimate the auto-covariance at lag 1 based on n observations, *as if* the process was stationary. More specifically, you use

$$\hat{\gamma}(1) = \frac{1}{n}\sum_{t=2}^{n} X_t X_{t-1}.$$

Find the expected value of $\hat{\gamma}(1)$, as a function of n, σ.

(c) (5 points) Assume you are fitting an AR(1) model to n observations of X_t using Yule-Walker estimation. Furthermore, assume you replace all sample moment estimators with their expected values. Find the resulting estimate $\hat{\phi}_1$ of the AR(1) coefficient, as a function of n, σ.

(a) $X_t = \sum_{s=1}^{t} W_s \Rightarrow E[X_t^2] = E\left[\left(\sum_{s=1}^{t} W_s\right)^2\right] =$ by uncorrelated W_t

$= \sum_{s=1}^{t} E[W_s^2] = t \cdot \sigma_w^2$

$\Rightarrow E(\hat{\gamma}(0)) = E\left(\frac{1}{n}\sum_{t=1}^{n} X_t^2\right) = \frac{1}{n}\cdot\sum_{t=1}^{n} E(X_t^2) =$

$= \frac{1}{n}\cdot\sum_{t=1}^{n} t\sigma_w^2 = \frac{1}{\cancel{n}}\cdot\frac{\cancel{n}\cdot(n-1)}{2}\cdot\sigma_w^2 = \frac{n-1}{2}\sigma_w^2$

(b). $E(X_t X_{t-1}) = E\left[\left(\sum_{s=1}^{t} W_s\right)\left(\sum_{r=1}^{t-1} W_r\right)\right] =$ by uncorrelated W_t $\sum_{r=1}^{t-1} E[W_r^2] = (t-1)\sigma_w^2$

$\Rightarrow E[\hat{\gamma}(1)] = E\left[\frac{1}{n}\sum_{t=2}^{n} X_t X_{t-1}\right] = \frac{1}{n}\sum_{t=2}^{n} E[X_t X_{t-1}] =$

$= \frac{1}{n}\cdot\sum_{t=2}^{n}(t-1)\sigma_w^2 = \frac{\sigma_w^2}{n}\sum_{t=1}^{n-1} t = \frac{(n-1)\cdot(n-2)}{2n}\sigma_w^2$ -

(contd...)

(c) $\hat{\varphi}_1 = \frac{\hat{\gamma}(1)}{\hat{\gamma}(0)} \overset{\text{replace}}{\approx} \frac{\cancel{\frac{n-1}{2}} \cancel{\sigma_w^2}}{\frac{\cancel{(n-1)}(n-2)}{\cancel{2}n} \cancel{\sigma_w^2}} = \frac{n}{n-2}$

6. Consider the ARCH(1) process $X_t = \sigma_t \cdot \varepsilon_t$, where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$ and $\varepsilon_t \sim^{iid} N(0,1)$.

(a) (5 points) Find $Cov(X_t, X_{t-1}^2)$.

(b) (5 points) Find $E(X_t^2 | X_{t-2})$ as a function of X_{t-2}.

uncorrelated

(a) $Cov(X_t, X_{t-1}^2) = Cov(\sigma_t \varepsilon_t, \sigma_{t-1}^2 \cdot \varepsilon_{t-1}^2) =$

$= E(\sigma_t \varepsilon_t \sigma_{t-1}^2 \varepsilon_{t-1}^2) =$

$= E(\varepsilon_t) \cdot E(\sigma_t \sigma_{t-1}^2) E(\varepsilon_{t-1}^2) = 0$

(b) For ARCH(1), we have $X_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \underbrace{\nu_t}_{WN}$.

$\Rightarrow X_t^2 = \alpha_0 + \alpha_1 (\alpha_0 + \alpha_1 X_{t-2}^2 + \nu_{t-1}) + \nu_t$

$\Rightarrow E[X_t^2 | X_{t-2}] = \alpha_0 + \alpha_1 . (\alpha_0 + \alpha_1 X_{t-2}^2) +$

$+ \underbrace{E(\alpha_1 \nu_{t-1} | X_{t-2}]}_{0} + \underbrace{E[\nu_t | X_{t-1}]}_{0}$

$= \alpha_0 + \alpha_1 \alpha_0 + \alpha_1^2 X_{t-1}^2$

Student #: ______________

7. (10 points) Consider the two time series $\begin{cases} X_t = .5X_{t-1} + W_t \\ Y_t = W_t + .5W_{t-1} \end{cases}$, defined based on the *common* white noise sequence $W_t \sim \text{WN}(0,1)$. Find the cross-covariance function $\gamma_{XY}(h), \forall h \in \mathbb{Z}$, and show that the series are jointly stationary.

3

$X_t \sim AR(1) \Rightarrow X_t = \sum_{j \geq 0} \left(\frac{1}{2}\right)^j W_{t-j}$

$\Rightarrow$ & stationary with $\gamma_X(h) = \frac{(1/2)^h}{1-\left(\frac{1}{2}\right)^2} = \frac{4}{3}(1/2)^h, \; h \geq 0$

3

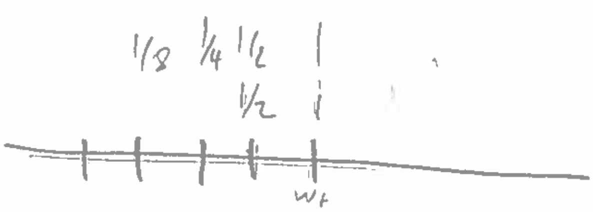
$Y_t \sim MA(1) \Rightarrow$ stationary with $\gamma_Y(h) = \begin{cases} 1+\frac{1}{4} = \frac{5}{4}, & h=0 \\ \frac{1}{2}, & h=1 \\ 0, & h \geq 2 \end{cases}$

4

$\gamma_{XY}(h) = Cov(X_{t+h}, Y_t) = Cov\left(\sum_{j \geq 0} \left(\frac{1}{2}\right)^j W_{t+h-j}, W_t + \frac{1}{2} W_{t-1}\right) =$

$= \sum_{j \geq 0} \left(\frac{1}{2}\right)^j \left[Cov(W_{t+h-j}, W_t) + \frac{1}{2} Cov(W_{t+h-j}, W_{t-1})\right] =$

$= \begin{cases} \left(\frac{1}{2}\right)^h \left(1+\left(\frac{1}{2}\right)^2\right) & , h \geq 0 \\ \left(\frac{1}{2}\right)^2 & , h=-1 \\ 0 & , h \leq -2 \end{cases}$



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