UNIVERSITY OF TORONTO SCARBOROUGH
Department of Computer & Mathematical Sciences

December 2015 Final Examination

STAD57 Time Series Analysis
Instructor: Sotirios Damouras

Duration: 3 hours
Examination aids allowed: Scientific calculator & one double-sided, standard-sized (8½×11) aid sheet

Last Name: S015 ______________________
First Name: ______________________
Student #: ______________________

Instructions:
1. There are 8 questions on 12 pages in total (including this cover sheet) for this exam.
2. Write your student number at the top of each page.
3. Answer all questions directly on the examination paper. Use the backs of the pages or the last page if more space is needed, and provide clear pointers to your work.
4. Show your intermediate work, and write clearly and legibly.
5. Read the questions carefully and answer the question that is being asked.

1.	2.	3.	4.	5.	6.	7.	8.	Total

1. (15 marks)

Consider the *integrated* AR(1) series $(1-\varphi B)(1-B)X_t = W_t \Leftrightarrow \nabla X_t = \varphi \nabla X_{t-1} + W_t$, where $\{W_t\} \sim \text{WN}(0, \sigma_W^2)$.

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a. Find the 1- & 2-step-ahead best linear predictors X_{n+1}^n & X_{n+2}^n and express them as linear combinations of the first n random variables $\{X_1, \dots, X_n\}$.
(Hint: $X_{n+m} = X_n + \sum_{j=1}^m \nabla X_{n+j}$)

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b. Find the Mean Square Prediction Error (MSPE) of X_{n+1}^n & X_{n+2}^n, i.e. find $P_{n+1}^n = \mathbb{E}\left[\left(X_{n+1} - X_{n+1}^n\right)^2\right]$ & $P_{n+2}^n = \mathbb{E}\left[\left(X_{n+2} - X_{n+2}^n\right)^2\right]$, expressed in terms of φ, σ_W^2.

$\nabla X_{n+1} = \phi \nabla X_n + W_{n+1} \Rightarrow X_{n+1} = X_n + \varphi(X_n - X_{n-1}) + W_{n+1}$

$\nabla X_{n+2} = \varphi \nabla X_{n+1} + W_{n+2} \Rightarrow X_{n+2} = X_{n+1} + \varphi(X_{n+1} - X_n) + W_{n+2} = (1+\varphi)[X_n + \varphi(X_n - X_{n-1})$

$-\varphi X_n + W_{n+2}$

$= (1+\varphi+\varphi^2)X_n +$

$+ \varphi(1+\varphi)X_{n-1}$

$+ (1+\varphi)W_{n+1}$

$+ W_{n+2}$

a. $\nabla X_t \sim AR(1) \Rightarrow \nabla X_{n+1}^n = \varphi \nabla X_n = \varphi(X_n - X_{n-1}) \Rightarrow$

$\nabla X_{n+2}^n = \varphi \nabla X_{n+1}^n = \varphi^2 \nabla X_n$

$X_{n+1}^n = X_n + \nabla X_{n+1}^n = X_n + \varphi X_n - \varphi X_{n-1}$

$X_{n+2}^n = X_n + \nabla X_{n+1}^n + \nabla X_{n+2}^n = X_n + \varphi(1+\varphi)\nabla X_n$

$= X_n + \varphi X_n + \varphi^2 X_n + \varphi X_{n-1} + \varphi^2 X_{n-1}$

$\mathbb{E}[(X_{n+1} - X_{n+1}^n)^2] = \mathbb{E}[W_{n+1}^2] = \sigma_w^2$

$\mathbb{E}[(X_{n+2} - X_{n+2}^n)^2] = \mathbb{E}[[(1+\varphi)W_{n+1} + W_{n+2}]^2] =$

$= (1+\varphi)^2 \sigma_w^2 + \sigma_w^2 = \sigma_w^2[1 + (1+\varphi)^2]$

2. (12 marks)

Consider a time series $\{X_t\}$ whose autocovariance matrix has the following form:

$$\left[\{\mathrm{Cov}(X_s, X_t)\}_{s,t\geq 1}\right] = \begin{bmatrix} \sigma^2 & 0 & \rho\sigma^2 & 0 & 0 & \cdots \\ 0 & \sigma^2 & 0 & \rho\sigma^2 & 0 & \cdots \\ \rho\sigma^2 & 0 & \sigma^2 & 0 & \rho\sigma^2 & \ddots \\ 0 & \rho\sigma^2 & 0 & \sigma^2 & 0 & \ddots \\ 0 & 0 & \rho\sigma^2 & 0 & \sigma^2 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \ \sigma^2 > 0, |\rho| < \tfrac{1}{2}.$$

Find the simplest possible SARIMA model specification that can accurately represent this autocovariance structure.

This series has autocovariance at lag 2 only

Can describe it as seasonal MA(1) w/ $s=2$ ⇒

$\Rightarrow \quad X_t = W_t + \Theta W_{t-2} \Rightarrow$

$\Rightarrow \quad \mathrm{Cov}(X_t, X_{t+1}) = \mathrm{Cov}(\underbrace{W_t + \Theta W_{t-2}, W_{t-1} + \Theta W_{t-3}}_{\text{no overlap}})$

$\Rightarrow \quad \mathrm{Cov}(X_t, X_{t+2}) = \mathrm{Cov}(W_t + \Theta W_{t-2}, W_{t+2} + \Theta W_t) =$

$= \Theta\, \mathrm{Cov}(W_t) = \Theta \sigma_w^2$

Note: $\mathrm{Cov}(X_t, X_{t+h}) = 0, \ \forall h \geq 3$

3. (15 marks)

Consider the following VAR(1) model: $\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} \varphi_{11} & 0 \\ 0 & \varphi_{22} \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} W_{1,t} \\ W_{2,t} \end{bmatrix}$, where

$\mathbf{W}_t = \begin{bmatrix} W_{1,t} \\ W_{2,t} \end{bmatrix} \sim \mathrm{WN}\left(\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{\Sigma} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)$.

5

a. Show that each coordinate $X_{1,t}, X_{2,t}$ individually follows a (univariate) AR(1) process.
(Hint: use the causal representation of the VAR(1) model)

10

b. Find the cross-covariance function $\gamma_{1,2}(h) = \mathrm{Cov}(X_{1,t+h}, X_{2,t})$ in terms of the parameters $(\varphi_{11}, \varphi_{22}, \rho)$.

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a $\underline{X}_t = \Phi \underline{X}_{t-1} + \underline{W}_t = \Phi^2 \underline{X}_{t-2} + \Phi \underline{W}_{t-1} + W_t = \cdots = \sum_{j=0}^{\infty} \Phi^j \cdot \underline{W}_{t-j}$

where $\Phi^j = \begin{bmatrix} \varphi_{11} & 0 \\ 0 & \varphi_{22} \end{bmatrix}^j = \begin{bmatrix} \varphi_{11}^j & 0 \\ 0 & \varphi_{22}^j \end{bmatrix}$

$\underline{X}_t = \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \sum \Phi^j \cdots = \begin{bmatrix} \sum \varphi_{11}^j W_{1,t-j} \\ \sum \varphi_{22}^j W_{2,t-j} \end{bmatrix} \Rightarrow X_{1,t} = \sum \varphi_{11}^j W_{1,t-j} \rightarrow$ AR(1)

1

b. $\mathrm{Cov}(X_{1,t+h}, X_{2,t}) = \mathrm{Cov}\left(\sum_{j=0}^{\infty} \varphi_{11}^j W_{1,t+h-j}, \sum_{k=0}^{\infty} \varphi_{22}^k \cdot W_{2,t-k} \right) =$

cross term $\neq 0$ only if $t+h-j = t-k \Rightarrow \begin{cases} j = k+h, & h \geq 0 \\ k = j-h, & h < 0 \end{cases}$

$= \begin{cases} \sum_{k\geq 0} \varphi_{11}^{k+h} \cdot \varphi_{22}^{k} \cdot \rho \rightarrow \rho\varphi_{11}^h \sum_{k\geq 0} (\varphi_{11}\varphi_{22})^k, & h \geq 0 \quad 6 \\ \sum_{j\geq 0} \varphi_{11}^j \cdot \varphi_{22}^{j-h} \cdot \rho, & h < 0 \quad 3 \end{cases}$

$\boxed{= \dfrac{\rho\varphi_{11}^h}{1-\varphi_{11}\varphi_{22}}}$

$\rightarrow \dfrac{\rho\varphi_{22}^{-h}}{1-\varphi_{11}\varphi_{22}}$

4. (15 marks)

Consider the random variables $X \sim \mathrm{N}(0,1)$ and $Y = X^2$.

6 **a.** Find the Minimum Mean Square Error (MMSE) predictor of Y based on X, and its corresponding mean square prediction error.

9 **b.** Find the Best Linear Predictor (BLP) of Y based on X, and its corresponding mean square prediction error.

(Hint: If $X \sim \mathrm{N}(0,1)$, then $\mathbb{E}[X^2]=1$, $\mathbb{E}[X^3]=0$ and $\mathbb{E}[X^4]=3$)

a. The MMSE is $\mathbb{E}[Y|X] = \mathbb{E}[X^2|X] = X^2$. 4

It's prediction error is 0, since $E[(Y-E[Y|X])^2]=0$ 2

b. The BLP of Y based on X is $\overset{g(x)=}{\alpha+\beta X}$, where α, β are such that $\begin{cases} \mathbb{E}[(Y-(\alpha+\beta X))\cdot 1] = 0 \\ \mathbb{E}[(Y-(\alpha+\beta X))\cdot X] = 0 \end{cases}$ 3

$\Rightarrow \begin{cases} \mathbb{E}[X^2-\alpha-\beta X]=0 \Rightarrow \overset{1}{\mathbb{E}[X^2]} - \alpha - \beta \overset{0}{\cancel{\mathbb{E}[X]}} = 0 \Rightarrow \boxed{\alpha=1} \\ \mathbb{E}[(X^2-\alpha-\beta X)X]=0 \Rightarrow \overset{0}{\cancel{\mathbb{E}[X^3]}} - \alpha \overset{0}{\cancel{\mathbb{E}[X]}} - \beta \overset{1}{\mathbb{E}[X^2]} = 0 \Rightarrow \boxed{\beta=0} \end{cases}$ 3

$\Rightarrow$ The BLP is constant, equal to 1

It's prediction error is $\mathbb{E}[(X^2-\alpha-\beta X)^2] =$

$= \mathbb{E}[(X^2-1)^2] = \mathbb{E}[X^4+1-2X^2] =$ 3

$= 3+1-2\cdot 1 = 2.$

5. (10 marks)

Consider the MA(1) process $X_t = V_t + \theta V_{t-1}$ which is defined in terms of the zero-mean stationary process $\{V_t\}$ whose autocovariance function is $\gamma_V(h)$. Show that $\{X_t\}$ stationary and find its autocovariance function in terms of θ and γ_V.

We have $\mathbb{E}[V_t]=0$, $Cov(V_{t+h}, V_t) = \gamma_V(h)$

2

- $\mathbb{E}[X_t] = \mathbb{E}[V_t] + \theta\,\mathbb{E}[V_{t-1}] = 0$

1

- $Cov(X_{t+h}, X_t) = Cov(V_{t+h} + \theta V_{t+h-1}, V_t + \theta V_{t-1}) =$

1 $= Cov(V_{t+h}, V_t) + \theta \cdot Cov(V_{t+h}, V_{t-1}) + \theta\, Cov(V_{t+h-1}, V_t) + \theta^2 Cov(V_{t+h-1}, V_{t-1}) =$

2 $= \gamma_V(h) + \theta\,\gamma_V(h+1) + \theta \cdot \gamma_V(h-1) + \theta^2 \cdot \gamma(h)$

1 $= (1+\theta^2)\cdot\gamma_V(h) + \theta\gamma_V(h+1) + \theta\gamma_V(h-1) =$

indep. of t ⇒ stationary

3 $\circ$ $Var[X_t] = Var(V_t + \theta V_{t-1}) =$

$= Var[V_t] + \theta^2 Var[V_{t-1}] + 2\theta\, Cov(V_t, V_{t-1})$

6. (13 marks)

Consider two independent AR(1) series $\begin{cases} X_t = aX_{t-1} + U_t \\ Y_t = bY_{t-1} + V_t \end{cases}$, where $\{U_t\} \sim \mathrm{WN}(0, \sigma_U^2)$ is independent of $\{V_t\} \sim \mathrm{WN}(0, \sigma_V^2)$. Does their sum $Z_t = X_t + Y_t$ necessarily follow an AR(1) series? Prove or disprove.

(Hint: Compare the causal representation of the sum to that of an AR(1) process)

The causal representation of each series is

$$\begin{cases} X_t = \sum_{j\geq 0} a^j U_{t-j} \\ Y_t = \sum_{j\geq 0} b^j V_{t-j} \end{cases} \quad 3$$

Their sum is $X_t + Y_t = \sum_{j \geq 0} a^j U_{t-j} + b^j V_{t-j}$ 1

Unless $a = b$, there is no way to represent the sum as an AR(1) causal process, which has

$$\sum \varphi^j \cdot W_{t-j} \quad 9$$

7. (15 marks)

Find the general form of the *truncated* m-step-ahead predictor $\tilde{X}^n_{n+m}$ and its mean square error P^n_{n+m} for the SMA(1)$_2$ model $X_t = W_t + \Theta W_{t-2}$, $W_t \sim \mathrm{WN}(0, \sigma^2_W)$.

(Hint: Use the recursive formula $\tilde{X}^n_{n+m} = -\sum_{j=1}^{m-1} \pi_j \tilde{X}^n_{n+m-j} - \sum_{j=m}^{n+m-1} \pi_j X^n_{n+m-j}$, where $\{\pi_j\}$ are the invertible weights)

The invertible weights for an SMA(1)$_2$ model are such that $\pi(B) \cdot \Theta(B) = 1 \Rightarrow$

2

$\Rightarrow (\pi_0 + \pi_1 B + \pi_2 B^2 + \dots)(1 + \Theta B^2) = 1 \Rightarrow$

$$\Rightarrow \begin{cases} \pi_0 = 1 \\ \pi_1 = 0 \\ \pi_2 = -\Theta \\ \pi_3 = 0 \\ \pi_4 = \Theta^2 \\ \vdots \end{cases} \rightarrow \pi_j = \begin{cases} (-\Theta)^{j/2}, & j = 0, 2, \dots \\ 0, & j \text{ odd} \end{cases}$$

4

$$\tilde{X}^n_{n+m} = -\sum_{\substack{j=1 \\ j\text{ even}}}^{m-1} (-\Theta)^j - \sum_{\substack{j=m \\ j\text{ even}}}^{n+m-1} (-\Theta)^j \Rightarrow$$

$\tilde{X}^n_{n+1} = -(-\Theta)X_{n-1} - (-\Theta)^2 X_{n-3} - \dots$

$\tilde{X}_{n+2} = -(-\Theta)X_n - (-\Theta)^2 X_{n-2} - \dots$

$\tilde{X}^n_{n+m} = 0, \quad \forall\, m \geq 3$

4

$$P^n_{n+m} = \sigma^2_W \sum_{j=0}^{m-1} \psi_j^2 = \begin{cases} \sigma_w^2, & m = 1 \\ \sigma_w^2, & m = 2 \\ \sigma_w^2 (1 + \Theta^2), & m \geq 3 \end{cases}$$

5

$\psi_0 = 1$

$\psi_1 = 0$

$\psi_2 = \Theta$

$\psi_k = 0, \forall\, k \geq 3$

8. (15 marks)

Consider the AR(1) process $X_t = \varphi X_{t-1} + Y_t$ with errors $\{Y_t\}$ following the ARCH(1) process $\left\{\begin{matrix} Y_t = \sigma_t \cdot Z_t, \; Z_t \sim^{iid} \mathrm{N}(0,1) \\ \sigma_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 \end{matrix}\right\}$. Find the auto-covariance function of $\{X_t\}$ in terms of the parameters $(\varphi, \alpha_0, \alpha_1)$.

We know that the ARCH(1) process is a WN:

with mean 0 & variance $\frac{\alpha_0}{1-\alpha_1}$ 5

$\Rightarrow$ the ACVF is $\gamma_X(h) = \varphi^h \cdot \mathbb{E}[Y_t] =$ 5

$= \frac{\varphi^h}{1-\varphi^2} \cdot \frac{\alpha_0}{1-\alpha_1}, \quad \forall\, h \geq 0$

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Extra Space -- Use if needed and indicate clearly which questions you are answering

Student #:____

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---- End of Exam ---- **(Total Marks = 110)**