# STAC62F: 2016 Assignment (2) - Solutions

(1) (a) We have that $\mathcal{F}(\Omega) = \{\phi, \Omega, \{1,2\}, \{3,4,5\}, \{3,4\}, \{1,2,5\}, \{5\}, \{1,2,3,4\}\}$. Now observe $X^{-1}\{3\} = \{4,5\}$. Since $\{3\}$ is a Borel subset of $\mathbb{R}^1$ and $\{4,5\} \notin \mathcal{F}(\Omega)$ then $X$ is not a random variable.

(b) We have
$$Y^{-1}B = \begin{cases} \phi & \text{if } 0,1,2 \notin B \\ \{1,2\} & \text{if } 0 \in B,\ 1,2 \notin B \\ \{1,2,3,4\} & \text{if } 0,1 \in B,\ 2 \notin B \\ \Omega & \text{if } 0,1,2 \in B \\ \{3,4\} & \text{if } 1 \in B\ \ 0,2 \notin B \\ \{3,4,5\} & \text{if } 1,2 \in B\ \ 0 \notin B \\ \{5\} & \text{if } 2 \in B,\ 0,1 \in B \end{cases}$$
for Borel set $B$. Therefore, since $Y^{-1}B \in \mathcal{F}(\Omega)$ for every Borel set $B$ then $Y$ is a r.v. Then
$$P_Y(B) = \begin{cases} 0 & \text{if } 0,1,2 \notin B \\ \frac{1}{4} & 0 \in B,\ 1,2 \notin B \\ \frac{1}{2} & 0,1 \in B,\ 2 \notin B \\ 1 & 0,1,2 \in B \\ \frac{1}{4} & 1 \in B,\ 0,2 \notin B \\ \frac{3}{4} & 1,2 \in B,\ 0 \notin B \\ \frac{1}{2} & 2 \in B,\ 0,1 \notin B \end{cases}.$$

(2.) (i) $P_{\underset{\sim}{X}}(\mathbb{R}^k) = P(\underset{\sim}{X}^{-1}\mathbb{R}^k) = P(\Omega) = 1$

(ii) Suppose $B_1, B_2, \ldots \in \mathcal{B}^k$ are mutually disjoint Borel sets. Then if $i \neq j$, $\underset{\sim}{X}^{-1}(B_i) \cap \underset{\sim}{X}^{-1}(B_j)$
$= \{\omega : X(\omega) \in B_i\} \cap \{\omega : X(\omega) \in B_j\}$
$= \{\omega : X(\omega) \in B_i \text{ and } X(\omega) \in B_j\} = \phi$
since $B_i \cap B_j = \phi$.

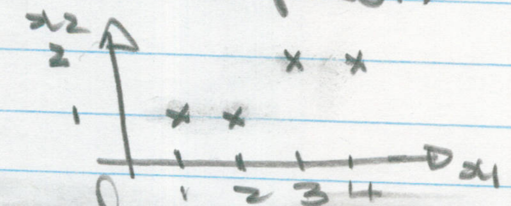
Therefore, $\underset{\sim}{X}^{-1}B_1, \underset{\sim}{X}^{-1}B_2, \ldots$ are mutually disjoint elements of $\mathcal{F}$ and $\underset{\sim}{X}^{-1}\bigcup_{i=1}^{\infty} B_i$
$= \{\omega : \underset{\sim}{X}(\omega)\in B_i \text{ some } i\}$
$= \bigcup_{i=1}^{\infty} \{\omega : \underset{\sim}{X}(\omega)\in B_i\} = \bigcup_{i=1}^{\infty} \underset{\sim}{X}^{-1}B_i$. This gives

$$P_{\underset{\sim}{X}}\left(\bigcup_{i=1}^{\infty} B_i\right) = P\left(\underset{\sim}{X}^{-1}\bigcup_{i=1}^{\infty} B_i\right) = P\left(\bigcup_{i=1}^{\infty} \underset{\sim}{X}^{-1}B_i\right)$$

$= \sum_{i=1}^{\infty} P(\underset{\sim}{X}^{-1}B_i)$ since the $\underset{\sim}{X}^{-1}B_i$ are disjoint

$= \sum_{i=1}^{\infty} P_{\underset{\sim}{X}}(B_i)$ which completes the proof.

(3.) Let $B\in\mathcal{B}^2$. Then




$\underset{\sim}{X}^{-1}B$ is a subset of $\Omega$ and thus an element of $\mathcal{F} = 2^{\Omega}$. Therefore $\underset{\sim}{X}$ is a random vector.

(b) $$P_{\underset{\sim}{X}}(B) = \frac{\#(\{(1,1),(2,1),(3,2),(4,2)\}\cap B)}{4}$$

(c) We have $Y(1)=2$, $Y(2)=3$, $Y(3)=5$, $Y(4)=6$. For any $B\in\mathcal{B}^1$ then $Y^{-1}B\in 2^{\Omega}$ so $Y$ is a r.v. Also $P_Y(B) = P(Y^{-1}B) = \#(\{2,3,5,6\}\cap B)/4$.

(4) (i) When $0\le a\le b$, $\Delta_{a,b}F = F(b)-F(a) = 1-e^{-b}-1+e^{-a}$
$= e^{-a}-e^{-b} \geq 0$ since $e^{-x}$ is decreasing.
If $a\le 0$, $b\le 0$, then $\Delta_{a,b}F = 0-0 = 0$. If $a\le 0$, $b>0$, then $\Delta_{a,b}F = 1-e^{-b} \geq 0$.
So condition (i) is satisfied.

(ii) $F(x) = 0$ when $x < 0$ so $\lim_{x \to -\infty} F(x) = 0$. Also $\lim_{x \to \infty} F(x) = \lim_{x \to \infty}(1 - e^{-x}) = 1 - \lim_{x \to \infty} e^{-x} = 1$

Therefore property (ii) is satisfied.

(iii) F is clearly continuous at $x < 0$ and at $x > 0$, and $\lim_{x \downarrow 0} F(x) = 1 - \lim_{x \downarrow 0} e^{-x} = 1 - 1 = 0$

and so F is continuous everywhere and thus right continuous.

(5.) When $\underset{\sim}{X} \sim N_k(\underset{\sim}{0}, I)$ then $F_{\underset{\sim}{X}}(x_1, \ldots, x_k)$

$$= \prod_{i=1}^{k} \left( \int_{-\infty}^{x_i} (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{z^2}{2}\}\, dz \right).$$ Then

$$F_{\underset{\sim}{Y}}(x_1, \ldots, x_{k-1}) = F_{\underset{\sim}{X}}(x_1, \ldots, x_{k-1}, \infty)$$

$$= \prod_{i=1}^{k-1} \left( \int_{-\infty}^{x_i} (2\pi)^{-\frac{1}{2}} e^{-z^2/2}\, dz \right) \underbrace{\int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{z^2}{2}\}\, dz}_{=1}$$

$$= \prod_{i=1}^{k} \left( \int_{-\infty}^{x_i} (2\pi)^{-\frac{1}{2}} e^{-z^2/2}\, dz \right)$$

and so $\underset{\sim}{Y} \sim N_{k-1}(\underset{\sim}{0}, I)$.

This shows that if we have a collection of r.v.'s $\{X_t : t \in T\}$ s.t. for any $t_1, \ldots, t_k$ we have $(X_{t_1}, \ldots, X_{t_k}) \sim N_k(\underset{\sim}{0}, I)$ then, by the Kolmogorov Consistency Theorem, this defines a unique stochastic process with time domain T, as all the distribution fns are defined consistently.

6. (a) $1 = \int_{[0,1]^2} c y e^{-xy}\,dx\,dy = c\int_0^1 \left(\int_0^1 y e^{-xy}\,dx\right) dy$

$= c\int_0^1 \left(-e^{-xy}\Big|_0^1\right) dy = c\int_0^1 (1-e^{-y})\,dy$

$= c\left(1 + e^{-y}\Big|_0^1\right) = c e^{-1}$ so $c = e$.

(b) $P\left(\frac{1}{2} < X < 1, \frac{1}{2} < Y < 1\right)$

$= e\int_{\frac{1}{2}}^1 \left(\int_{\frac{1}{2}}^1 y e^{-xy}\,dx\right) dy = e\int_{\frac{1}{2}}^1 \left(-e^{-xy}\Big|_{\frac{1}{2}}^1\right) dy$

$= e\int_{\frac{1}{2}}^1 \left(e^{-\frac{y}{2}} - e^{-y}\right) dy = e\left(-2e^{-\frac{y}{2}}\Big|_{\frac{1}{2}}^1 + e^{-y}\Big|_{\frac{1}{2}}^1\right)$

$= e\left(2e^{-\frac{1}{4}} - 2e^{-\frac{1}{2}} + e^{-1} - e^{-\frac{1}{2}}\right)$

$= 1 + 2e^{3/4} - 3e^{\frac{1}{2}}$

(c) $f_X(x) = e\int_0^1 y e^{-xy}\,dy$ $\quad u = y \quad dv = e^{-xy}$, $du = 1 \quad v = -e^{-xy}/x$

$= e\left[\frac{-ye^{-xy}}{x}\Big|_{y=0}^{y=1} + \int_0^1 \frac{1}{x} e^{-xy}\,dy\right]$

$= e\left[\frac{-e^{-x}}{x} - \left(\frac{1}{x^2} e^{-xy}\right)\Big|_{y=0}^{y=1}\right]$

$= e\left[\frac{-e^{-x}}{x} - \frac{1}{x^2}\left(e^{-x} - 1\right)\right]$ for $0 \le x \le 1$.

$f_Y(y) = e\int_0^1 y e^{-xy}\,dx = e\left(-e^{-xy}\Big|_{x=0}^{x=1}\right)$

$= e\left[1 - e^{-y}\right]$ for $0 \le y \le 1$.

7. (a) $$p_{X_1}(x_1) = \sum_{x_2, x_3 \text{ s.t. } x_2+x_3 = n-x_1} \binom{n}{x_1\, x_2\, x_3} \theta_1^{x_1} \theta_2^{x_2} \theta_3^{x_3}$$

and using $\binom{n}{x_1\, x_2\, x_3} = \binom{n}{x_1}\binom{n-x_1}{x_2\, x_3}$

$$= \binom{n}{x_1} \theta_1^{x_1} \sum_{x_2+x_3=n-x_1} \binom{n-x_1}{x_2\, x_3} \theta_2^{x_2} (1-\theta_1-\theta_2)^{x_3}$$

$$= \binom{n}{x_1} \theta_1^{x_1} (1-\theta_1)^{n-x_1} \sum_{x_2=0}^{n-x_1} \binom{n-x_1}{x_2} \left(\frac{\theta_2}{1-\theta_1}\right)^{x_2} \left(1-\frac{\theta_2}{1-\theta_1}\right)^{n-x_1-x_2}$$

$= 1$ since sum of all binomial $(n-x_1, \theta_2/(1-\theta_1))$ probabilities.

Therefore $X_1 \sim$ binomial $(n, \theta_1)$

(b) $$P(X_2 = x_2 \mid X_1 = x_1) = \frac{P(X_1 = x_1, X_2 = x_2)}{P(X_1 = x_1)}$$

Now $P(X_1 = x_1, X_2 = x_2) = P(X_1 = x_1, X_2 = x_2, X_3 = n - x_1 - x_2)$

$$= \binom{n}{x_1\, x_2\, n-x_1-x_2} \theta_1^{x_1} \theta_2^{x_2} \theta_3^{n-x_1-x_2}$$

$$= \binom{n}{x_1}\binom{n-x_1}{x_2} \theta_1^{x_1} \theta_2^{x_2} (1-\theta_1-\theta_2)^{n-x_1-x_2}$$

$$= \binom{n}{x_1} \theta_1^{x_1} (1-\theta_1)^{n-x_1} \binom{n-x_1}{x_2} \left(\frac{\theta_2}{1-\theta_1}\right)^{x_2} \left(1-\frac{\theta_2}{1-\theta_1}\right)^{n-x_1-x_2}$$

Therefore $$p_{X_2|X_1}(x_2|x_1) = \binom{n-x_1}{x_2} \left(\frac{\theta_2}{1-\theta_1}\right)^{x_2} \left(1-\frac{\theta_2}{1-\theta_1}\right)^{n-x_1-x_2}$$

and $X_2 \mid X_1 = x_1 \sim$ binomial $\left(n-x_1, \frac{\theta_2}{1-\theta_1}\right)$.