STAC62 : 2016 Assignment 8

1. Suppose we have probability model $(\Omega, \mathcal{F}, P)$

(a) Establish the following for events in $\mathcal{F}$

(i) $I_{A^c} = 1 - I_A$, (ii) $I_{\bigcap_{i=1}^n A_i} = \prod_{i=1}^n I_{A_i}$,

(iii) $I_{\bigcup_{i=1}^n A_i} = 1 - \prod_{i=1}^n (1 - I_{A_i})$

(b) Use (iii) to prove the inclusion-exclusion relation for $P(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots$

2. Suppose we have a probability model $(\{1,2,3\}, 2^{\{1,2,3\}}, P)$ and $Y_n : \{1,2,3\}$ is given by $Y_n(\omega) = 1$ when $\omega = 1$, $Y_n(\omega) = 1/n$ when $\omega = 2$ or 3 for $n = 1, 2, \ldots$

(a) Establish that each Y_n is a random variable and determine P_{Y_n}.

(b) Calculate $E(Y_n)$.

(c) Prove that $Y_n \xrightarrow{a.s.} Y$ where $Y(\omega) = 1$ when $\omega = 1$ and $Y(\omega) = 0$ when $\omega = 2$ or 3.

(d) Why does $E(Y_n) \to E(Y)$?

3. Suppose that $X \sim N(0,1)$ and let $Y = |X|$.

(a) Calculate $E(|X|)$

(b) Find the density of Y and use this to compute $E(Y)$.

4. Suppose that $\underset{\sim}{X} \in \mathbb{R}^k$ is a random vector and $\underset{\sim}{Y} = \underset{\sim}{a} + B\underset{\sim}{X}$ where $\underset{\sim}{a} \in \mathbb{R}^l$, $B \in \mathbb{R}^{l \times k}$ are fixed. Then prove $E(\underset{\sim}{Y}) = \underset{\sim}{a} + BE(\underset{\sim}{X})$ whenever $E(|X_i|) < \infty$ for $i = 1, \ldots, k$.

5. Suppose $X \in \mathbb{R}^{k \times l}$ is a random matrix; namely the (i,j)-th element x_{ij} is a random variable. Define $E(X)$ to be the matrix with (i,j)-th element $E(x_{ij})$ provided $E(|x_{ij}|) < \infty$ $\forall\, i,j$. If $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times k}$ and $C \in \mathbb{R}^{l \times n}$ are fixed and $Y = A + BXC$ then prove $E(Y) = A + B\,E(X)\,C$.

6. Prove the Corollary to Jensen's Inequality (in the notes).

7. Using the definition of conditional expectation show that (i) $E(1|X) = 1$ and (ii) $E(aY_1 + bY_2 | X) = aE(Y_1|X) + bE(Y_2|X)$ (assume all conditional expectations exist).