STAC58 : 2017 Assignment (3) - Solutions

(1.) $m_{(\bar{x}, \underset{\sim}{r})}(t, \underset{\sim}{u}) = E_\mu\left(\exp\{t\bar{x} + \underset{\sim}{u}'\underset{\sim}{r}\}\right)$

$$= \int_{\mathbb{R}^n} \exp\{t\bar{x} + \underset{\sim}{u}'\underset{\sim}{r}\} (2\pi\sigma_0^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i - \mu_0)^2\right\} dx_1 \cdots dx_n$$

$$= \int_{\mathbb{R}^n} \exp\left\{\sum_{i=1}^n u_i x_i + t\bar{x} - \sum_{j=1}^n u_j \bar{x}\right\} dx_1 \cdots dx_n$$

$$= \int_{\mathbb{R}^n} \prod_{i=1}^n \exp\left\{\left(u_i + \frac{t}{n} - \frac{1}{n}\sum_{j=1}^n u_j\right) x_i\right\} (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_0^2}(x_i - \mu)^2\right\} dx_i$$

$$= \prod_{i=1}^n E_\mu\left(\exp\{(u_i + \tfrac{t}{n} - \bar{u}) x_i\}\right)$$

$$= \prod_{i=1}^n m_{(\mu, \sigma_0^2)}\left(u_i + \tfrac{t}{n} - \bar{u}\right)$$

where $m_{(\mu, \sigma_0^2)}(t) = \exp\{t\mu - \sigma_0^2 \frac{t^2}{2}\}$

is the mgf of a $N(\mu, \sigma_0^2)$ r.v.

$$= \prod_{i=1}^n \exp\left\{(u_i - \bar{u} + \tfrac{t}{n})\mu + \frac{\sigma_0^2}{2}(u_i - \bar{u} + \tfrac{t}{n})^2\right\}$$

$$= \exp\left\{t\mu_0 + \frac{\sigma_0^2}{2}\left(\sum_{i=1}^n (u_i - \bar{u})^2 + \frac{t^2}{n}\right)\right\}$$

since $\sum_{i=1}^n (u_i - \bar{u}) = 0$

$$= \exp\left\{\mu t + \sigma_0^2 \frac{t^2}{2}\right\} \exp\left\{\frac{\sigma_0^2}{2}\sum_{i=1}^n (u_i - \bar{u})^2\right\}$$

$= (\text{mgf of } \bar{x}) \times (\text{mgf of } \underset{\sim}{r})$

and so $\bar{x}$ and $\underset{\sim}{r}$ are statistically independent with $\bar{x} \sim N(\mu, \sigma_0^2/n)$ and $\underset{\sim}{r} \sim N_n(\underset{\sim}{0}, \sigma_0^2(I - \underset{\sim}{1}(\underset{\sim}{1}'\underset{\sim}{1})^{-1}\underset{\sim}{1}'))$

since a $N_n(\underset{\sim}{\mu}, \Sigma)$ dist'n has mgf

$$m_{(\mu,\Sigma)}(\underset{\sim}{t}) = \exp\{\underset{\sim}{t}'\underset{\sim}{\mu} + \tfrac{1}{2}\underset{\sim}{t}'\Sigma\underset{\sim}{t}\}$$

and $\underset{\sim}{t}'(I - \underset{\sim}{1}(\underset{\sim}{1}'\underset{\sim}{1})^{-1}\underset{\sim}{1}')t = \sum_{i=1}^{n}(t_i - \bar{t})^2$.

2. (a) $R(a, d_1) = L(a, d_1(1))\frac{1}{3} + L(a, d_1(2))\frac{1}{3} + L(a, d_1(3))\frac{1}{3}$
$= 0 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = \frac{1}{3}$

$R(b, d_1) = L(b, d_1(1))\frac{1}{3} + L(b, d_1(2))\frac{2}{3} + L(b, d_1(3))\,0$
$= 0\,\frac{1}{3} + 0 \cdot \frac{2}{3} + 0 \cdot 0 = 0$

$R(c, d_1) = L(c, d_1(1)) \cdot 0 + L(c, d_1(2))\frac{1}{3} + L(c, d_1(3))\frac{2}{3}$
$= 1 \cdot 0 + 1 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} = \frac{1}{3}$

(b) $R(a, d_2) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{1}{3}$
$R(b, d_2) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} + 1 \cdot 0 = \frac{2}{3}$
$R(c, d_2) = 1 \cdot 0 + 1 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} = 1$

(c) Since $R(\theta, d_1) \leq R(\theta, d_2)$ $\forall \theta$, and $R(b, d_1) < R(b, d_2)$, d_1 is preferred to d_2.

3. The likelihood function is
$L(\sigma^2 | x) = (\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu_0)^2\}$ so by the Factorization Theorem, $\sum_{i=1}^n (x_i - \mu_0)^2$ is a sufficient statistic. (The log-likelihood function is $l(\sigma^2 | x) = -\frac{n}{2}\log \sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu_0)^2$ so $\frac{\partial l(\sigma^2|x)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n (x_i - \mu_0)^2$. Setting the derivative equal to 0 gives the solution $\sigma^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \mu_0)^2$. So we can compute $\sum_{i=1}^n (x_i - \mu_0)^2$ and this proves this is a MSS.) Now $E_{\sigma^2}(\sum_{i=1}^n (x_i - \mu_0)^2)$
$= \sum_{i=1}^n E_{\sigma^2}((x_i - \mu_0)^2) = \sum_{i=1}^n Var_{\sigma^2}(x_i) = \sum_{i=1}^n \sigma^2 = n\sigma^2$.
Therefore, $\frac{1}{n}\sum_{i=1}^n (x_i - \mu_0)^2$ is an unbiased estimator of σ^2. Since $\sum_{i=1}^n (x_i - \mu_0)^2$ is a sufficient statistic the Rao-Blackwell theorem implies that any unbiased estimator can be (possibly) improved wrt variance by Rao-Blackwellization. Since $\sum_{i=1}^n (x_i - \mu_0)^2$ is complete there is only one unbiased estimator that is a function of this statistic so $\frac{1}{n}\sum_{i=1}^n (x_i - \mu_0)^2$ is UMVU.

Note - the material in ( ) is not necessary for a solution
- also a solution can be based on the Lehman-Scheffé theorem.

4. $L(\theta, aa_1 + (1-a)a_2) = |\theta - aa_1 - (1-a)a_2|$
$= |a(\theta - a_1) + (1-a)(\theta - a_2)| \le |a(\theta - a_1)| + |(1-a)(\theta - a_2)|$
(by the triangle inequality) $= a|\theta - a_1| + (1-a)|\theta - a_2|$
when $a \in [0,1]$ $= aL(\theta, a_1) + (1-a)L(\theta, a_2)$

So for any estimator $d(x)$ of θ then when using the loss L and T a sufficient statistic

$$R(\theta, d) = E_\theta(L(\theta, d(x)))$$

$$= E_\theta(E(L(\theta, d(x))|T)(T(x)))$$

$$\overset{\text{Jensen}}{\geq} E_\theta(L(\theta, E(d(x)|T)(T(x)))) = R(\theta, d_T)$$

since L is a convex fn by above.

and note that $d_T(x) = E(d(x)|T)(T(x))$ does not depend on θ and has risk no greater than risk of d.

5. We have that $(\bar{x}, \|\underset{\sim}{x} - \bar{x}\underset{\sim}{1}\|^2)$ is a complete MSS for this problem and $E_{(\mu,\sigma^2)}(\bar{x}) = \mu$ while (as shown in class)

$E_{(\mu,\sigma^2)}(\|\underset{\sim}{x} - \bar{x}\underset{\sim}{1}\|) = \sigma\, \Gamma(n/2)\, 2^{1/2} / \Gamma(\frac{n-1}{2})$.

Therefore $d(x) = \bar{x} + \dfrac{\Gamma(\frac{n-1}{2})}{2^{1/2}\Gamma(\frac{n}{2})} z_{0.25} \|\underset{\sim}{x} - \bar{x}\underset{\sim}{1}\|$

is an unbiased estimator of $\mu + \sigma z_{0.25}$. By Lehman-Scheffé it is UMVU.

6. The UMVU estimator of $\mu^2+\sigma_0^2$ (in 8.1.8) is $d(x) = \bar{x}^2 + \frac{n-1}{n}\sigma_0^2$. By the information inequality the estimator that achieves the lower bound when μ is true is of the form $(\mu^2+\sigma_0^2) + (2\mu) I^{-1}(\mu)$ where $I(\mu)$ is the information. Since the log-likelihood equals $l(\mu \mid x) = -\frac{n}{2\sigma_0^2}(\bar{x}-\mu)^2$ the score function equals $S(\mu \mid x) = \frac{n}{\sigma_0^2}(\bar{x}-\mu)$ so $I(\mu) = Var_\mu(S(\mu \mid x))$ $= \frac{n^2}{\sigma_0^2} Var_\mu(\bar{x}) = \frac{n^2}{\sigma_0^4}\frac{\sigma_0^2}{n} = \frac{n}{\sigma_0^2}$.

Therefore, the estimator achieving the lower bound when μ is true is $(\mu^2+\sigma_0^2) + 2\mu\left(\frac{\sigma_0^2}{n}\right)S(\mu \mid x)$ $= (\mu^2+\sigma_0^2) + 2\frac{\mu\sigma_0^2}{n}\frac{n}{\sigma_0^2}(\bar{x}-\mu) = \sigma_0^2 - \mu^2 + 2\mu\bar{x}$ which depends on μ so there is no unbiased estimator that achieves the lower bound and, in particular, the UMVU estimator doesn't.

7. Let $\{f_\theta : \theta \in \Theta_0\}$ be a model and $\{f_\theta : \theta \in \Theta\}$, with $\Theta_0 \subseteq \Theta$ the larger model and T a complete statistic for $\{f_\theta : \theta \in \Theta_0\}$. Now suppose that the function h is such that $E_\theta(h(T)) = 0$ $\forall\, \theta \in \Theta$. Then $E_\theta(h(T)) = 0$ $\forall\, \theta \in \Theta_0$ which implies $P_\theta(\{x : h(T(x)) \neq 0\}) = 0$ $\forall\, \theta \in \Theta_0$. But by assumption $P_\theta(\{x : h(T(x)) \neq 0\})$ $= 0$ $\forall\, \theta \in \Theta$ and this implies that T is complete for $\{f_\theta : \theta \in \Theta\}$.