STAC62 – 2015 Assignment 1 – Solutions

① (i) $\phi \in \mathcal{F}$ and $\Omega = \phi^c$ so $\Omega \in \mathcal{F}$

(ii) put $A_{n+1} = A_{n+2} = \cdots = \phi$ so $\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{\infty} A_i$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ since $A_i \in \mathcal{F}\ \forall i$. Also

(iii) put $A_{n+1} = A_{n+2} = \cdots = \Omega$ so $\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{F}$ since $A_i^c \in \mathcal{F}\ \forall i$ which implies $\bigcup_{i=1}^{\infty} A_i^c \in \mathcal{F}$ which implies $\left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{F}$.

② Let $\mathcal{F} = \bigcap_{i \in I} \mathcal{F}_i$. Then $\phi \in \mathcal{F}_i\ \forall i$ and so $\phi \in \mathcal{F}$. If $A_1, A_2, \ldots \in \mathcal{F}$ then $A_1, A_2, \ldots \in \mathcal{F}_i\ \forall i$ and thus $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}_i\ \forall i$ which implies $\bigcup_{j=1}^{\infty} A_j \in \mathcal{F}$. Finally if $A \in \mathcal{F}$ then $A \in \mathcal{F}_i\ \forall i$ which implies $A^c \in \mathcal{F}_i\ \forall i$ which implies $A^c \in \mathcal{F}$. Therefore $\mathcal{F}$ is a σ-field.

③ (i) Put $A_i = \phi$ for $i = 1, 2, \ldots$ Then $\phi = \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ and $A_i \cap A_j = \phi$ for $i \neq j$. Therefore $P(\phi) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) = \infty P(\phi)$ which implies $P(\phi) = 0$ since $P(\phi) \le 1$. (ii) Put $A_{n+1} = A_{n+2} = \cdots = \phi$ so $P\left(\bigcup_{i=1}^{n} A_i\right) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$ (since mut. disj.) $= \sum_{i=1}^{n} P(A_i)$ (since $P(A_i) = 0$ for all $i > n$ by (i)).

(iii) $\Omega = A \cup A^c$ and $A \cap A^c = \phi$. Therefore by (ii) $1 = P(\Omega) = P(A) + P(A^c)$ and $P(A^c) = 1 - P(A)$.

(iv) If $A \subseteq B$ then $B = A \cup (A^c \cap B)$ and $A \cap (A^c \cap B) = \phi$. Therefore, $P(B) = P(A) + P(A^c \cap B) \ge P(A)$ since $P(A^c \cap B) \ge 0$. (v) For $n = 2$, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$. Assume true for all $k \le n$. Then $P\left(\bigcup_{i=1}^{n+1} A_i\right) = P\left(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right) = P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right)$ (by $n=2$ case) Now apply induction to $P\left(\bigcup_{i=1}^{n} A_i\right)$ and $P\left(\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right)$,

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \wedge A_j) + \cdots + (-1)^{n+1} P(A_1 \wedge \cdots \wedge A_n)$$

$$P(\bigcup_{i=1}^{n} A_i \wedge A_{n+1}) = \sum_{i=1}^{n} P(A_i \wedge A_{n+1}) - \sum_{i<j} P(A_i \wedge A_j \wedge A_{n+1}) + \cdots +$$
$$(-1)^{n+1} P(A_1 \wedge \cdots \wedge A_n \wedge A_{n+1})$$

Combining we obtain

$$P(\bigcup_{i=1}^{n+1} A_i) = \sum_{i=1}^{n+1} P(A_i) - \sum_{i<j} P(A_i \wedge A_j) + \cdots +$$
$$(-1)^{n+1} \sum_{i_1 < \cdots < i_n} P(A_{i_1} \wedge \cdots A_{i_n}) + (-1)^{n+2} P(A_1 \wedge \cdots \wedge A_{n+1})$$

which gives the result.

④ (i) $P(\Omega \mid B) = P(\Omega \wedge B) / P(B) = P(B)/P(B) = 1$

(ii) If $A_1, A_2, \ldots \in \mathcal{F}$ are mut. disj. then $A_1 \wedge B, A_2 \wedge B, \ldots \in \mathcal{F}$ are also mut. disj. Therefore

$$P(\bigcup_{i=1}^{\infty} A_i \mid B) = P(\bigcup_{i=1}^{\infty} A_i \wedge B)/P(B)$$
$$= \sum_{i=1}^{n} P(A_i \wedge B)/P(B) = \sum_{i=1}^{\infty} P(A_i \mid B).$$

Therefore $P(\cdot \mid B)$ is a prob. measure on $\mathcal{F}$.

⑤ $0 = P(\phi) = P(\phi \wedge \phi) = P(\phi)P(\phi) = P(\phi \wedge B)$
$= P(\phi)P(B) = P(\phi \wedge B^c) = P(\phi)P(B^c) = P(\phi \wedge \Omega)$
$= P(\phi)P(\Omega)$ and so ϕ is ind. of each element of $\{\phi, B, B^c, \Omega\}$. Also $P(A) = P(A \wedge \Omega) = P(A)P(\Omega)$
$P(A \wedge B) = P(A)P(B)$, $P(A \wedge B^c) = P(A) - P(A \wedge B)$
(since $A = A \wedge B \cup A \wedge B^c$) $= P(A) - P(A)P(B) = P(A)(1 - P(B))$

$= P(A)P(B^c)$ and so A is ind. of each element of $\{\phi, B, B^c, \Omega\}$. Also $P(A^c \cap B)$
$= P(B) - P(A \cap B) = P(B)P(A^c)$ and
$P(A^c \cap B^c) = P(A^c) - P(A^c \cap B) = P(A^c)P(B^c)$
and we are done.

6. First since $\mathcal{F} = 2^\Omega$ we must have $X^{-1}B \in \mathcal{F}$ for every $B \in \mathcal{B}'$ and so X is a r.v. But we need to evaluate $X^{-1}B$ to calculate P_X. We have the following:

(i) if $0, 1 \notin B$ then $X^{-1}B = \phi$ and $P_X(B) = 0$.
(ii) if $0 \in B$ but $1 \notin B$ then $X^{-1}B = \{1, 2\}$ and
$P_X(B) = 1/2 + 1/4 = 3/4$
(iii) if $0 \notin B$ but $1 \in B$ then $X^{-1}B = \{3\}$ so $P_X(B) = 1/4$
(iv) if $0, 1 \in B$ then $X^{-1}B = \Omega$ and $X^{-1}B = \Omega$ so $P_X(B) = 1$

This determines $P_X(B)$ at all $B \in \mathcal{B}'$.

7. 1.4.6 We have $P(G|T) = P(G \cap T)/P(T)$ while $P(T|G) = P(G \cap T)/P(G)$. Therefore $P(G|T) = P(T|G)$ iff $P(T) = P(G)$ so the probability of the defendant being guilty given the testimony being true is equal to the probability of the testimony being true given the defendant is guilty iff the prob. of the testimony being true equals the probability of the defendant being guilty.

8. 1.5.8 The sample space is $\Omega = \{H,T\} \times \{H,T\} \times \{H,T\}$ and the event C = "at least two are alike"
$= \{(H,H,H), (T,T,T), (H,H,T), (H,T,H), (T,H,H), (H,T,T), (T,H,T), (T,T,H)\} = \underline{\Omega}$
Then $P(\text{"all alike"} \mid C)$
$= P(\{(H,H,H),(T,T,T)\}) / P(C)$
$= (2/8)/P(\underline{\Omega}) = 1/4 \neq 1/2.$

9. 1.8.14 $P(A_j \mid B) = P(A_j \cap B)/P(B)$

$$= \frac{P(A_j \cap B)}{P(A_j)} P(A_j) \frac{1}{\sum_{i=1}^{n} P(A_i \cap B)}$$

since $B = \bigcup_{i=1}^{n} A_i \cap B$ and $(A_i \cap B) \cap (A_j \cap B) = \emptyset$ when $i \neq j$

$$= \frac{P(B \mid A_j) P(A_j)}{\sum_{i=1}^{n} P(B \mid A_i) P(A_i)}$$

10. 1.8.29 We have $P(A \mid B) = P(A \cap B)/P(B) > P(A)$ and so $P(A \cap B)/P(A) = P(B \mid A) > P(B)$.
Also $P(A \mid B^c) = P(A \cap B^c)/P(B^c)$
$= (P(A) - P(A \cap B))/P(B^c) < (P(A) - P(A)P(B))/P(B^c)$
$= P(A)$ and so B^c repels A.

Finally suppose $P(B \mid A) > P(B)$ and $P(C \mid B) > P(C)$ then $P(C \mid A) = P(A \cap C)/P(A)$ and nothing forces this to be greater than $P(C)$. For example, consider tossing a symmetrical die A = "even", $B = \{2,4\}$, $C = \{1,2,3,4,5\}$ then

$P(B|A) = 2/3 > P(B) = 1/3$ and $P(C|B) = 1$
$> P(C) = 5/6$ but $P(C|A) = 2/3 < P(C) = 5/6$