VII Gaussian Processes

- probably the most commonly used in applications e.g. time series, finance, physics, ...

Def $\{X_t : t \in T\}$ is a Gaussian process with mean function $\mu : T \to \mathbb{R}$ and autocovariance function $\sigma : T \times T \to \mathbb{R}$ if for any distinct $t_1, \dots, t_k \in T$, $k (\in \mathbb{N}) \leq \#(T)$

$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_k} \end{pmatrix} \sim N_k\left(\begin{pmatrix} \mu(t_1) \\ \vdots \\ \mu(t_k) \end{pmatrix}, \begin{pmatrix} \sigma(t_1,t_1) & \cdots & \sigma(t_1,t_k) \\ \vdots & & \vdots \\ \sigma(t_1,t_k) & \cdots & \sigma(t_k,t_k) \end{pmatrix} \right)$$

- mean and autocovariance function are all we need to characterize a Gaussian process
- recall we can also think of a Gaussian process as a probability measure P on $\left(\times_{t \in T} \mathbb{R}^1, \times_{t \in \mathbb{R}} \mathcal{B}^1\right)$ that is such that $P_{(X_{t_1}, \dots, t_k)}$ has the above multivariate normal distribution.

- $f \in \times_{t \in T} \mathbb{R}^1$ is a function $f : T \to \mathbb{R}$

eg $\underset{\sim}{X} \in \mathbb{R}^n$ and $\underset{\sim}{X} \sim N_n(\underset{\sim}{\mu}, \Sigma)$

- then take $T = \{1, \dots, n\}$, $\mu(i) = \mu_i$, $\sigma(i,j) = \sigma_{ij}$

Prop A matrix $\Sigma \in \mathbb{R}^{k \times k}$ is a variance matrix iff Σ is positive semidefinite.

Proof: $\Rightarrow$] $\exists$ r. vector $\underset{\sim}{Y}$ s.t. $\Sigma = Var(\underset{\sim}{Y}) = E\left((\underset{\sim}{Y} - E(\underset{\sim}{Y}))(\underset{\sim}{Y} - E(\underset{\sim}{Y}))'\right)$

$= (cov(Y_i, Y_j))$ and so the matrix is symmetric since $cov(X,Y) = E((X-E(X))(Y-E(Y)))$
$= E((Y-E(Y))(X-E(X))) = cov(Y,X)$.

Also $\underset{\sim}{a}' \Sigma \underset{\sim}{a} = \underset{\sim}{a}' E((\underset{\sim}{Y}-E(\underset{\sim}{Y}))(\underset{\sim}{Y}-E(\underset{\sim}{Y}))')\underset{\sim}{a}$
$= E(\underset{\sim}{a}'(\underset{\sim}{Y}-E(\underset{\sim}{Y}))(\underset{\sim}{Y}-E(\underset{\sim}{Y}))'\underset{\sim}{a})$
$= E((\underset{\sim}{a}'\underset{\sim}{Y}-E(\underset{\sim}{a}'\underset{\sim}{Y}))^2) = Var(\underset{\sim}{a}'\underset{\sim}{Y}) \geq 0$
since $E(\underset{\sim}{a}'\underset{\sim}{Y}) = \underset{\sim}{a}'E(\underset{\sim}{Y})$. Therefore Σ is psd.

4] By the spectral decomposition $\Sigma = Q \Lambda Q'$ where $Q \in \mathbb{R}^{k \times k}$ is orthogonal and $\Lambda = diag(\lambda_1, \ldots, \lambda_k)$ where $\lambda_i \geq 0$. Put $\Sigma^{1/2} = Q \Lambda^{1/2} Q'$.

Suppose $\underset{\sim}{Z} \sim N_k(\underset{\sim}{0}, I)$ and put $\underset{\sim}{Y} = \Sigma^{1/2}\underset{\sim}{Z}$ then $Var(\underset{\sim}{Y}) = \Sigma^{1/2} Var(\underset{\sim}{Z})(\Sigma^{1/2})' = \Sigma^{1/2} I \Sigma^{1/2}$
$= \Sigma$. Therefore Σ is a variance matrix.

- so provided the autocovariance function σ produces variance matrices then μ and σ define a Gaussian process
- checking that σ gives a valid autocovariance fn (Bochner's theorem)

$\sigma(s,s) = v^2$ small $\ell \Rightarrow$ small correlations
large $|t-s| \Rightarrow$ " "

eg $T = \mathbb{R}$

- $\sigma(s,t) = v^2 \exp\{-(t-s)^2/2\ell^2\}$ for constants v^2, ℓ^2 is a valid autocovariance fn

- note - we can also define complex-valued random variables

  - let X, Y be (real) r.v.'s and put $Z = X + iY$ where $i = \sqrt{-1}$

  - if $E|X| < \infty$, $E|Y| < \infty$ then define $E(Z) = E(X) + iE(Y)$

  - now $\bar{Z} = X - iY$ and $E(\bar{Z}) = E(X) - iE(Y)$ $= \overline{E(Z)}$

  - $|Z|^2 = Z\bar{Z} = X^2 + Y^2$ so $E|Z|^2 = EX^2 + EY^2$

- now recall $e^{ix} = \cos x + i\sin x$ for $x \in \mathbb{R}$ so $\overline{e^{ix}} = \cos x - i\sin x$ and $|e^{ix}|^2 = \cos^2 x + \sin^2 x = 1$

- fact - $\exp\{-t^2/2\} = E(\exp\{itZ\})$ when $Z \sim N(0,1)$

Def A process $\{X_t : t \in T\}$ where $T \subseteq \mathbb{R}$ is stationary if $E X_t^2 < \infty$ $\forall t$ and $\mu(t) = E X_t$ is constant and $\sigma(s,t) = \sigma(s+h, t+h)$ $\forall s,t \in T$ and h s.t. $s+h, t+h \in T$

- when $T = \mathbb{Z}$ (time series) and $\{X_t : t \in \mathbb{Z}\}$ stationary then $\sigma(s,t) = \sigma(s-t, 0) = \sigma(0, t-s)$ then define $\gamma(h) = \sigma(h,0)$ and note $\gamma(-h) = \sigma(-h,0) = \sigma(0,h) = \sigma(h,0) = \gamma(h)$ an even function $(\gamma(0) = Var(X_0) = Var(X_t))$

ARMA(p,q) process (autoregressive moving average)

Def Suppose $\{X_t : t \in \mathbb{Z}\}$ is a mean 0 stationary process then the process is an ARMA(p,q) process if $\exists$ constants $\phi_1, \ldots, \phi_p, \theta_1, \ldots \theta_q$ st

$$X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + Z_t + \sum_{j=1}^{q} \theta_j Z_{t-j}$$

where $\{Z_t : t \in \mathbb{Z}\}$ is white noise (stationary, mean 0, $cov(Z_s, Z_t) = \begin{cases} \sigma^2 & s=t \\ 0 & \text{otherwise} \end{cases}$)

- note $\{Z_t : t \in \mathbb{Z}\}$ with Z_t iid $N(0, \sigma^2)$ is a particular example of white noise.

eg AR(1) = ARM(1,0)

- $X_t = \phi X_{t-1} + Z_t = \phi^2 X_{t-2} + Z_t + \phi Z_{t-1}$

$$= \cdots = \phi^n X_{t-n} + \sum_{i=0}^{n-1} \phi^i Z_{t-i}$$

- then $\left\| X_t - \sum_{i=0}^{n-1} \phi^i Z_{t-i} \right\|_2^2 = |\phi|^{2n} \| X_{t-n} \|_2^2$

stationary

$$= |\phi|^{2n} E X_0^2 \to 0 \quad \text{iff } |\phi| < 1$$

- so $\sum_{i=0}^{n-1} \phi^i Z_{t-i} \xrightarrow{2} X_t$ also

$$\sum_{i=0}^{n-1} \phi^i Z_{t-i} \xrightarrow{P} X_t$$

$\sum_{i=0}^{n} \phi^i Z_{t-i}$ converges to something with finite expectation

- in fact $X_t = \sum_{i=0}^{\infty} \phi^i Z_{t-i}$ a.s.

- $\sigma(s,t) = \operatorname{cov}(X_s, X_t)$

$$= E\left(\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \phi^{i+j} Z_{s-i} Z_{t-j} \right)$$

$t \geq s$

DCT

$$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \phi^{i+j} E(Z_{s-i} Z_{t-j}) = \begin{cases} \sigma^2, & j = t-s+i \\ 0, & \text{otherwise} \end{cases}$$

$$= \sigma^2 \sum_{i=0}^{\infty} \phi^{i+t-s+i} = \sigma^2 \phi^{t-s} \sum_{i=0}^{\infty} \phi^{2i}$$

$$= \sigma^2 \phi^{|t-s|} / (1-\phi^2)$$

- suppose $\{Z_t : t \in \mathbb{Z}\}$ is $N(0, \sigma^2)$

- $c_{X_t}(u) = E(e^{iuX_t}) = E(\exp\{iu \sum_{j=0}^{\infty} \phi^j Z_{t-j}\})$

$= E\left(\prod_{j=0}^{\infty} \exp\{iu\phi^j Z_{t-j}\}\right) = E\left(\lim_{n\to\infty} \prod_{j=0}^{n} \exp\{\ \}\right)$

$\overset{DCT}{=} \prod_{j=0}^{\infty} \exp\{-\frac{\sigma^2 u^2 \phi^{2j}}{2}\} = \exp\{-\frac{\sigma^2 u^2}{2} \sum_{j=0}^{\infty} \phi^{2j}\}$

$= \exp\{-\frac{\sigma^2}{1-\phi^2} \frac{u^2}{2}\}$

$\therefore X_t \sim N(0, \frac{\sigma^2}{1-\phi^2})$

- similarly $c_{(X_{t_1}, \ldots, X_{t_k})}(u_1, \ldots, u_k)$

$= \exp\{-\frac{1}{2} \frac{\sigma^2}{1-\phi^2} \underline{u}' (\phi^{|t_i - t_j|}) \underline{u}$

$\therefore (X_{t_1}, \ldots, X_{t_k}) \sim N_k(\underline{0}, \frac{\sigma^2}{1-\phi^2}(\phi^{|t_i-t_j|})$

- so $\{X_t : t \in \mathbb{Z}\}$ is a Gaussian process

- <u>note</u> - if $\{X_t : t \in \mathbb{Z}\}$ is a stationary Gaussian process, then it is strictly stationary, namely $(X_{t_1}, \ldots, X_{t_k}) \sim (X_{t_1+h}, \ldots, X_{t_k+h})$ $\forall\, t_1, \ldots, t_k, h \in \mathbb{Z}$

Brownian Motion (the Wiener Process)

Def $\{W_t : t \geq 0\}$ is a standard <u>Wiener process</u> if

(i) $P(W_0 = 0) = 1$

(ii) the process has <u>independent increments</u>
($t_1 < t_2 < \cdots < t_n$, then $W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ are mutually stat. indep.)

(iii) $W_t - W_s \sim N(0, t-s)$ for $0 \leq s \leq t$.

- σW_t a general Wiener process

Prop $\{\sigma W_t : t \geq 0\}$ is a Gaussian process.

Proof: Let $t_1 < t_2 < \cdots < t_n$, then put $0 = t_0$

$$(W_{t_1}, W_{t_2}, \ldots, W_{t_n}) = \left((W_{t_1} - W_{t_0}), \sum_{i=1}^{2}(W_{t_i} - W_{t_{i-1}}), \ldots, \sum_{i=1}^{n}(W_{t_i} - W_{t_{i-1}})\right) = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots \\ & & & \\ 1 & \cdots & & 1 \end{pmatrix}}_{A} \begin{pmatrix} W_{t_1} - W_0 \\ W_{t_2} - W_{t_1} \\ \vdots \\ W_{t_n} - W_{t_{n-1}} \end{pmatrix}$$

$$\sim N_n\left(\underset{\sim}{0}, \sigma^2 A \begin{pmatrix} t_1 & & 0 \\ & t_2 - t_1 & \\ 0 & & \ddots \\ & & & t_n - t_{n-1} \end{pmatrix} A'\right)$$

$$= N_n\left(\underset{\sim}{0}, \sigma^2 \begin{pmatrix} t_1 & t_1 & \cdots & t_1 \\ t_1 & t_2 & \cdots & t_2 \\ \vdots & & & \\ t_1 & t_2 & & t_n \end{pmatrix}\right) = N_n(\underset{\sim}{0}, \sigma^2(\min(t_i, t_j)))$$

Corollary $\mathrm{cov}(\sigma W_t, \sigma W_s) = \sigma^2 \min(t, s)$

- **note** – the existence of $\{\omega W_t : t \geq 0\}$ is guaranteed by KCT
- in addition the following result can be proved.

Prop $\exists$ process $\{\omega W_t : t \geq 0\}$ satisfying (i)–(iii) and (iv) $P(\omega W_t$ is continuous in $t) = 1$

Prop For process $\{W_t : t \geq 0\}$ sat. (i)–(iv) $P(\omega W_t$ is nowhere differentiable in $t) = 1$

- how does BM arise generally?
- suppose $Z_1, Z_2, \ldots$ iid with mean 0 and variance 1
- put $S_0 = 0$, $S_n = \sum_{i=1}^{n} Z_i$ (a random walk)
  $\frac{1}{\sqrt{n}} S_n \xrightarrow{d} N(0,1)$,
- Donsker's Theorem $\left\{ \frac{1}{\sqrt{n}} S_{\lfloor nt \rfloor} : t \in [0,1] \right\}$

space shrunk by factor $\frac{1}{\sqrt{n}}$ and time speed by factor n

$\xrightarrow{d} \{ W_t : t \in [0,1] \}$ standard Wiener process

$\frac{\lfloor nt \rfloor}{n} \to t$

- **note** – for $[0,T]$ put $\Delta T = T/n$
  - consider $\{ \sqrt{\Delta T}\, S_{\lfloor t/\Delta T \rfloor} : t \in [0,T] \}$

$$= \left\{ \sqrt{T} \left(\frac{1}{\sqrt{n}} S_{\lfloor n \frac{t}{T} \rfloor} \right) : \frac{t}{T} \in [0,1] \right\} \xrightarrow{d} \left\{ \sqrt{T}\, W_{\frac{t}{T}} : \frac{t}{T} \in [0,1] \right\}$$

$$\overset{d}{=} \{ W_t : t \in [0,T] \}$$

Diffusin processes

- suppose $X_1, X_2, \ldots$ i.i.d $P(X_i = k) = p$, $P(X_i = -k) = q$.
- $E(X_i) = (p-q)k$, $Var(X_i) = 4pqk^2$

$$C_{X_i}(s) = pe^{iks} + qe^{-iks}$$

- put $p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta t}\right)$ $q = \frac{1}{2}\left(1 - \frac{\mu}{\sigma}\sqrt{\Delta t}\right)$

$$k = \sigma\sqrt{\Delta t}$$

$$\therefore (p-q)k = \mu \Delta t$$

$$2\sqrt{pq} = \left(1 - \frac{\mu^2}{\sigma^2}\Delta t\right)^{1/2}, \quad \frac{k}{2\sqrt{pq}\,k} = \frac{1}{\left(1-\frac{\mu^2}{\sigma^2}\Delta t\right)^{1/2}}$$

- define $Z_{t,\Delta t} = \sum_{i=1}^{\lfloor t/\Delta t \rfloor} X_i$
- then $E(Z_{t,\Delta t}) = \lfloor t/\Delta t \rfloor \Delta t\, \mu \sim \mu t$

$$Var(Z_{t,\Delta t}) = \lfloor t/\Delta t \rfloor \Delta t\, \sigma^2 \left(1 - \frac{\mu^2}{\sigma^2}\Delta t\right) \sim t\sigma^2$$

- $Y_{t,\Delta t} = (Z_{t,\Delta t} - \mu t)/\sqrt{t}\,\sigma$

N sum

$$= \sqrt{\frac{\lfloor t/\Delta t \rfloor}{t/\Delta t}\left(1 - \frac{\mu^2}{\sigma^2}\Delta t\right)^{1/2}} \left(\frac{Z_{t,\Delta t} - \lfloor t/\Delta t \rfloor \Delta t\, \mu}{\sqrt{\lfloor t/\Delta t \rfloor \Delta t\left(1 - \frac{\mu^2}{\sigma^2}\Delta t\right)^{1/2}}} \right)$$

1 as $\Delta t \to 0$

$$+ \frac{\lfloor t/\Delta t \rfloor \Delta t\, \mu - t\mu}{\sqrt{t}\,\sigma}$$

$$\frac{t}{\Delta t} - 1 \le \lfloor t/\Delta t \rfloor \le \frac{t}{\Delta t} + 1$$

$$1 - \frac{\Delta t}{t} \le \frac{\lfloor t/\Delta t \rfloor}{t/\Delta t} \le 1 + \frac{\Delta t}{t}$$

$$= \sqrt{t}\left(\frac{\lfloor t/\Delta t \rfloor \Delta t}{t} - 1\right)\frac{\mu}{\sigma}$$

$\to 0$ as $\Delta t \to 0$.

- and $N_{sum} \to N(0,1)$ as $\Delta t \to 0$ and note this not by the standard CLT since the $x_i - \Delta t \mu$ have mean 0 and are independent but $Var(x_i) = 4pqh^2 = \Delta t\left(1 - \frac{\mu^2}{\sigma^2}\Delta t\right)\sigma^2$ changes with Δt ; i.e. $x_i - \Delta t \mu$ $i = 1, \ldots$ are not iid.

- but the CLT can be proved under more general conditions

- now for $\delta > 0$ $E(|x_i - \Delta t \mu|^{2+\delta})$

$$= \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\Delta t\right)|\Delta t\,\sigma - \Delta t \mu|^{2+\delta} + \frac{1}{2}\left(1 - \frac{\mu}{\sigma}\Delta t\right)|-\Delta t\,\sigma - \Delta t\mu|^{2+\delta}$$

$$\Delta t^{2+\delta}\left[\frac{1}{2}\left(1 + \frac{\mu}{\sigma}\Delta t\right)|\sigma - \mu|^{2+\delta} + \frac{1}{2}\left(1 - \frac{\mu}{\sigma}\Delta t\right)|\sigma + \mu|^{2+\delta}\right]$$

- so $$\frac{\sum_{i=1}^{\lfloor t/\Delta t \rfloor} E|x_i - \Delta t \mu|^{2+\delta}}{\left(\sum_{i=1}^{\lfloor t/\Delta t \rfloor} Var(x_i)\right)^{1+\frac{\delta}{2}}}$$ Lyapunov

$$= \frac{\lfloor t/\Delta t \rfloor (\Delta t)^{2+\delta}[\quad]}{(\lfloor t/\Delta t \rfloor \Delta t)^{1+\delta}[\quad]^{1+\frac{\delta}{2}}} = O\left(\frac{\Delta t}{\lfloor t/\Delta t \rfloor^{\delta}}\right) \to 0 \text{ as } \Delta t \to 0$$

- with an appropriate generalization of Donsker's Theorem we have that

$$\left\{\frac{1}{\sqrt{n}}\sum_{i=1}^{\lfloor nt \rfloor} x_i : t \in [0,T]\right\}$$

$$\xrightarrow{D} \left\{\mu t + \sqrt{t}\,\sigma W_t : t \in [0,T]\right\}$$ Brownian motion with drift