STAC62F: 2016 Assignment (3) - Solutions

(1.) (a)(i) $I_{A^c}(\omega) = \begin{cases} 1 & \omega \in A^c \\ 0 & \omega \in A \end{cases}$ iff $I_A(\omega) = \begin{cases} 0 & \omega \in A^c \\ 1 & \omega \in A \end{cases}$

iff $I_{A^c}(\omega) = 1 - I_A(\omega)$ and so $I_{A^c} = 1 - I_A$.

(ii) $I_{\bigcap_{i=1}^{n} A_i}(\omega) = \begin{cases} 1 & \omega \in \bigcap_{i=1}^{n} A_i \\ 0 & \omega \notin \bigcap_{i=1}^{n} A_i \end{cases}$ iff $\begin{matrix} \omega \in A_i \ \forall i \\ \exists i \text{ s.t. } \omega \notin A_i \end{matrix}$

iff $\prod_{i=1}^{n} I_{A_i}(\omega) = \begin{cases} 1 & \omega \in A_i \ \forall i \\ 0 & \exists i \text{ s.t. } \omega \notin A_i \end{cases}$

(iii) $I_{\bigcup_{i=1}^{n} A_i} \overset{(i)}{=} 1 - I_{(\bigcup_{i=1}^{n} A_i)^c} = 1 - I_{\bigcap_{i=1}^{n} A_i^c} \overset{(ii)}{=} 1 - \prod_{i=1}^{n} I_{A_i^c}$

$\overset{(i)}{=} 1 - \prod_{i=1}^{n} (1 - I_{A_i})$

(iv) $E(I_{\bigcup_{i=1}^{n} A_i}) = E(1 - \prod_{i=1}^{n} (1 - I_{A_i}))$

$= 1 - E\left(1 - \sum_{i=1}^{n} I_{A_i} + \sum_{i<j} I_{A_i \cap A_j} - \cdots + (-1) I_{\bigcap_{i=1}^{n} A_i}\right)$

since $\prod_{i=1}^{n} (1 - I_{A_i}) = 1 - \sum_{i=1}^{n} I_{A_i} + \sum_{i<j} I_{A_i} I_{A_j}$

$- \sum_{i<j<k} I_{A_i} I_{A_j} I_{A_k} + \cdots + (-1)^n I_{A_1} \cdots I_{A_n}$

$= 1 - \sum_{i=1}^{n} I_{A_i} + \sum_{i<j} I_{A_i \cap A_j} + \cdots + (-1)^n I_{A_1 \cap \cdots \cap A_n}$

$= \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \cdots + (-1)^{n+1} P(A_1 \cap \cdots \cap A_n)$

2. (a) Since $\mathcal{F} = 2^{\{1,2,3\}}$ (the power set) we have immediately that $Y_n^{-1}B \in \mathcal{F}$ for every Borel set $B \in \mathbb{R}^1$ (actually $Y_n^{-1}B \in \mathcal{F}$ for any $B \subseteq \mathbb{R}$). Therefore, Y_n is a r.v. We have that for a Borel set B

$$P_{Y_n}(B) = \begin{cases} 1 & \text{if } \{1/n, 1\} \subseteq B \\ P(\{1\}) & \text{if } 1 \in B,\ \frac{1}{n} \notin B \\ P(\{2,3\}) & \text{if } 1 \notin B,\ \frac{1}{n} \in B \\ 0 & \text{if } 1 \notin B,\ \frac{1}{n} \notin B. \end{cases}$$

(b) $E(Y_n) = 1 \cdot P(Y_n = 1) + \frac{1}{n} P(Y_n = \frac{1}{n})$

$= P(\{1\}) + \frac{1}{n} P(\{2,3\})$

(c) We have that $Y_n(1) = 1\ \forall n$ and $Y_n(2) = Y_n(3) = \frac{1}{n} \to 0$ as $n \to \infty$. Therefore $\{\omega : \lim_{n\to\infty} Y_n(\omega) = Y(\omega)\} = \{1,2,3\}$ and $P(\{1,2,3\}) = 1$. This proves $Y_n \xrightarrow{a.s.} Y$.

(d) $E(Y_n) = P(\{1\}) + \frac{1}{n} P(\{2,3\}) \to P(\{1\})$ and $E(Y) = 1 \cdot P(Y=1) + 0 \cdot P(Y=0) = P(Y=1) = P(\{1\})$. Alternatively you could note that $|Y_n(\omega)| \leq 2\ \forall n$ and thus $Y_n \xrightarrow{a.s.} Y$ by the Dominated Convergence Theorem.

3 (a) $E(|X|) = \int_{-\infty}^{\infty} |x| \, (2\pi)^{-\frac{1}{2}} \exp\{-\frac{1}{2}x^2\} \, dx$

$= 2\int_0^{\infty} x \, (2\pi)^{-\frac{1}{2}} \exp\{-\frac{1}{2}x^2\} \, dx$ by symmetry

$= 2\left(-(2\pi)^{-\frac{1}{2}} \exp\{-\frac{1}{2}x^2\} \Big|_0^{\infty}\right)$

$= 2\left(-0 + (2\pi)^{-\frac{1}{2}} \exp\{-\frac{1}{2}0\}\right) = \sqrt{\frac{2}{\pi}}$

(b) $F_Y(y) \overset{y \geq 0}{=} P(Y \leq y) = P(|X| \leq y)$

$= P(-y \leq X \leq y) = \Phi(y) - \Phi(-y)$

$= 2\Phi(y) - 1$. So $f_Y(y) = \frac{dF_Y(y)}{dy}$

$= 2\phi(y)$. Then $E(Y) = \int_0^{\infty} y \, 2\phi(y) \, dy$

$= 2\int_0^{\infty} y\,\phi(y)\,dy = \sqrt{\frac{2}{\pi}}$ as above.

4. $E(\underset{\sim}{Y}) = \begin{pmatrix} E(Y_1) \\ \vdots \\ E(Y_2) \end{pmatrix}$ and $Y_i = a_i + \sum_{j=1}^{k} b_{ij} X_j$

so $E(Y_i) = E\left(a_i + \sum_{j=1}^{k} b_{ij} X_j\right) = a_i + \sum_{j=1}^{k} b_{ij} E(X_j)$

Therefore $E(\underset{\sim}{Y}) = \underset{\sim}{a} + B\,E(\underset{\sim}{X})$.

5. $E(Y)$ is the matrix with (i,j)-th element equal to $E(Y_{ij})$. Now the (i,j)-th element of $A+BXC$ is given by $a_{ij} + \underset{\sim}{b}'_{(i)} X \underset{\sim}{c}_j$ where $\underset{\sim}{b}_{(i)}$ is the i-th row of B and $\underset{\sim}{c}_j$ is the j-th row of C. Therefore $E(Y_{ij}) = a_{ij} + E(\underset{\sim}{b}'_{(i)} X \underset{\sim}{c}_j) = a_{ij} + \underset{\sim}{b}'_{(i)} E(X\underset{\sim}{c}_j)$ by ④ and $E(X\underset{\sim}{c}_j) = E\begin{pmatrix} \underset{\sim}{x}'_{(1)} \underset{\sim}{c}_j \\ \vdots \\ \underset{\sim}{x}'_{(k)} \underset{\sim}{c}_j \end{pmatrix}$ where $\underset{\sim}{x}_{(i)}$ is the i-th row of X. Then $E(\underset{\sim}{x}'_{(i)} \underset{\sim}{c}_j) = E\left(\sum_{r=1}^{l} x_{ir} c_{rj}\right)$

$$= \sum_{r=1}^{l} E(x_{ir}) c_{rj} \text{. Therefore } \underset{\sim}{b}'_{(i)} E(X\underset{\sim}{c}_j)$$

$$= \sum_{s=1}^{k} b_{ir} \sum_{r=1}^{l} E(x_{sr}) c_{rj}$$

$$= \sum_{s=1}^{k} \sum_{r=1}^{l} b_{ir} E(x_{sr}) c_{rj}$$

Therefore $E(Y) = A + B\,E(X)\,C$.

6. In the proof of Jensen we obtained $c_{k+1} h(E(\underset{\sim}{x})) \le c_{k+1} E(h(\underset{\sim}{x}))$ where $c_{k+1} \ge 0$. If $c_{k+1} > 0$ then equality in Jensen occurs iff $c_{k+1} h(E(\underset{\sim}{x})) = c_{k+1} E(h(\underset{\sim}{x}))$ which occurs iff $c_{k+1}(h(\underset{\sim}{x}) - h(E(\underset{\sim}{x}))) + \sum_{i=1}^{k} c_i (x_i - E(x_i))$ (which is nonnegative) has mean 0. This occurs iff $c_{k+1}(h(\underset{\sim}{x}) - h(E(\underset{\sim}{x}))) + \sum_{i=1}^{k} c_i(x_i - E(x_i)) = 0$ with probability 1. But then $h(\underset{\sim}{x}) = a + \underset{\sim}{c}'\underset{\sim}{x}$ where $a = h(E(\underset{\sim}{x})) + \sum_{i=1}^{k} (c_i/c_{k+1}) E(x_i)$ and $\underset{\sim}{c} = (c_1/c_{k+1}, \ldots, c_k/c_{k+1})$. If $c_{k+1} = 0$ the result follows by induction. If $h(\underset{\sim}{x}) = a + \underset{\sim}{c}'\underset{\sim}{x}$ for some $a, \underset{\sim}{c}$ then the result is immediate.

7. (i) We have $\mathbb{E}(1\, h(\underset{\sim}{x})) = \mathbb{E}(h(\underset{\sim}{x}))$ for every h and so by definition of conditional expectation $\mathbb{E}(1 \mid \underset{\sim}{x}) = 1$

(ii) We have $\mathbb{E}((aY_1 + bY_2)\, h(\underset{\sim}{x}))$

$= a\,\mathbb{E}(Y_1\, h(\underset{\sim}{x})) + b\,\mathbb{E}(Y_2\, h(\underset{\sim}{x}))$ by linearity of $\mathbb{E}$

$= a\,\mathbb{E}(\mathbb{E}(Y_1 \mid \underset{\sim}{x})\, h(\underset{\sim}{x})) + b\,\mathbb{E}(\mathbb{E}(Y_2 \mid \underset{\sim}{x})\, h(\underset{\sim}{x}))$ by def'n of conditional expectation

$= \mathbb{E}((a\,\mathbb{E}(Y_1 \mid \underset{\sim}{x}) + b\,\mathbb{E}(Y_2 \mid \underset{\sim}{x}))\, h(\underset{\sim}{x}))$ by linearity of $\mathbb{E}$

and since this holds for every h we have, by the definition of conditional expectation, that $\mathbb{E}(aY_1 + bY_2 \mid \underset{\sim}{x}) = a\,\mathbb{E}(Y_1 \mid \underset{\sim}{x}) + b\,\mathbb{E}(Y_2 \mid \underset{\sim}{x})$.