STAC58: 2017 Assignment 4 - Solutions

1. (8.2.4)
(a) Let C be the 0.975-confidence interval for μ. Then, $P_\mu(C) = 0.975$. The size of the test is the rejecting probability of H_0. Hence, the size is $\alpha = P_0(0 \notin C) = 1 - P_0(C) = 1 - 0.975 = 0.025$.
(b) The confidence interval C is $[\bar{x} - z_{0.9875}/\sqrt{20}, \bar{x} + z_{0.9875}/\sqrt{20}]$. Since $\bar{x} \sim N(\theta, 1/20)$ if θ is true, the power function is given by

$$\begin{aligned}\beta(\theta) &= P_\theta(0 \notin C) = P_\theta(\bar{x} < -z_{0.9875}/\sqrt{20} \text{ or } \bar{x} > z_{0.9875}/\sqrt{20}) \\ &= \Phi(-(z_{0.9875} + \theta)/\sqrt{20}) + 1 - \Phi((z_{0.9875} - \theta)/\sqrt{20}).\end{aligned}$$

2. (8.2.16) Without loss of generality, assume $\mu_0 = 0$. Then for $H_0 : \sigma^2 = \sigma_0^2$ versus $H_a : \sigma^2 = \sigma_1^2$, the UMP size α test rejects H_0 whenever

$$\frac{L\left(\sigma_1^2 \,|\, x_1, \ldots, x_n\right)}{L\left(\sigma_0^2 \,|\, x_1, \ldots, x_n\right)} = \frac{\sigma_1^{-2n} \exp\left\{-\frac{1}{2\sigma_1^2}\sum_{i=1}^n x_i^2\right\}}{\sigma_0^{-2n} \exp\left\{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n x_i^2\right\}} > c_0$$

or, equivalently, whenever $n\left(\sigma_0^2 - \sigma_1^2\right) + \frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum_{i=1}^n x_i^2 > \ln c_0$ or, using $\sigma_0^2 < \sigma_1^2$, whenever

$$\frac{1}{\sigma_0^2}\sum_{i=1}^n x_i^2 > \frac{2}{\sigma_0^2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)^{-1}\left(\ln c_0 - n\left(\sigma_0^2 - \sigma_1^2\right)\right).$$

Under H_0 we have that $\frac{1}{\sigma_0^2}\sum_{i=1}^n x_i^2 \sim \chi^2(n)$, so the test is to reject whenever $\frac{1}{\sigma_0^2}\sum_{i=1}^n x_i^2 > x_{1-\alpha}$, where $x_{1-\alpha}$ is the $(1-\alpha)$th quantile of the $\chi^2(n)$ distribution. Since the test does not involve σ_1^2, it is UMP size α for $H_0 : \sigma^2 = \sigma_0^2$ versus $H_a : \sigma^2 > \sigma_0^2$. The power function of this test is given by $P_{\sigma^2}\left(\frac{1}{\sigma_0^2}\sum_{i=1}^n x_i^2 \geq x_{1-\alpha}\right) = P_{\sigma^2}\left(\frac{1}{\sigma^2}\sum_{i=1}^n x_i^2 \geq \frac{\sigma_0^2}{\sigma^2}x_{1-\alpha}\right) = P\left(Z \geq \frac{\sigma_0^2}{\sigma^2}x_{1-\alpha}\right)$ where $Z = \left(\sum_{i=1}^n x_i^2\right)/\sigma^2 \sim \chi^2(n)$, so the power function is increasing in σ^2. This implies that the above test is of size α for $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_a : \sigma^2 > \sigma_0^2$. Now suppose φ is also size α for $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_a : \sigma^2 > \sigma_0^2$. Then φ is also size α for $H_0 : \sigma^2 = \sigma_0^2$ versus $H_a : \sigma^2 > \sigma_0^2$ and so must have its power function uniformly less than or equal to the power function for the above test when $\sigma^2 > \sigma_0^2$. This implies that the above test is UMP size α for $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_a : \sigma^2 > \sigma_0^2$.

3. (8.3.4) From Example 7.1.1 we have that the posterior distribution of θ is Beta$(n\bar{x} + \alpha, n(1 - \bar{x}) + \beta)$. The Bayes rule is given by the posterior mean and this is evaluated in Example 7.2.2 to be $(n\bar{x} + \alpha)/(n + \alpha + \beta)$.

4. (8.3.9) Suppose $T(s) \in \{\theta_1, \theta_2\}$ for each s. The Bayes rule will minimize

$$\begin{aligned}&E_\Pi\left(P_\theta\left(T(s) \neq_\theta\right)\right) = E_\Pi\left(E_\theta\left(1 - I_{\{\theta\}}\left(T(s)\right)\right)\right) \\ &= 1 - E_\Pi\left(E_\theta\left(I_{\{\theta\}}\left(T(s)\right)\right)\right) = 1 - E_M\left(E_{\Pi(\cdot\,|\,s)}\left(I_{\{\theta\}}\left(T(s)\right)\right)\right).\end{aligned}$$

Therefore, the Bayes rule at s is given by $T(s)$ which maximizes

$$E_{\Pi(\cdot\,|\,s)}\left(I_{\{\theta\}}\left(T(s)\right)\right) = \Pi\left(\{\theta_1\}\,|\,s\right) I_{\{\theta_1\}}\left(T(s)\right) + \Pi\left(\{\theta_2\}\,|\,s\right) I_{\{\theta_2\}}\left(T(s)\right)$$

and this is clearly given by

$$T(s) = \begin{cases} \theta_1 & \Pi\left(\{\theta_1\}\,|\,s\right) > \Pi\left(\{\theta_2\}\,|\,s\right) \\ \theta_2 & \Pi\left(\{\theta_2\}\,|\,s\right) > \Pi\left(\{\theta_1\}\,|\,s\right) \end{cases}$$

and when $\Pi\left(\{\theta_1\}\,|\,s\right) = \Pi\left(\{\theta_2\}\,|\,s\right)$ we can take $T(s)$ to be either θ_1 or θ_2. So the Bayes rule is given by the posterior mode.

5. (7.2.22) The posterior distribution of μ given σ^2 is the $N(\mu_x, \left(n + 1/\tau_0^2\right)^{-1}\sigma^2)$ distribution where μ_x is given by (7.1.7). The posterior distribution of σ^2 is the Gamma$(\alpha_0 + n/2, \beta_x)$ distribution, where β_x is given by (7.1.8). Therefore, the integral (7.2.2) is given by

$$\psi_0^{-2} \int_0^\infty \frac{1}{\sqrt{2\pi}} \left(n + \frac{1}{\tau_0^2}\right)^{1/2} \exp\left(-\frac{\lambda}{2}\left(n + \frac{1}{\tau_0^2}\right)\left(\psi_0^{-1}\lambda^{-\frac{1}{2}} - \mu_x\right)^2\right) \times$$
$$\frac{\left(\beta_x\right)^{\alpha_0 + n/2}}{\Gamma\left(\alpha_0 + n/2\right)} \lambda^{\alpha_0 + n/2 - 1} \exp\left(-\beta_x \lambda\right)\, d\lambda.$$