UNIVERSITY OF TORONTO SCARBOROUGH
Department of Statistics

Time Series Analysis

STAD57H3F – November 5, 2014

Midterm Exam

Duration – 110 minutes

Examination aids allowed: Scientific Calculator

Last Name: ____

First Name: ____

Student #: ____

Instructions:

1. There are 4 questions on 10 pages in total (including this cover sheet) for this exam.
2. Write your student number at the top of each page.
3. Answer all questions directly on the examination paper.
4. Show your intermediate work, and write clearly and legibly.
5. Read the questions carefully and answer the question that is being asked.

1.	2.	3.	4.	Total

1. (25 marks)

Consider the Gaussian MA(1) process $X_t = W_t + \theta W_{t-1}$, where $\{W_t\} \sim^{iid} \mathrm{N}(0, \sigma_W^2)$.

a. Find the coefficients $\varphi_{1,1}, (\varphi_{2,1}, \varphi_{2,2})$ of the 1-step-ahed predictors $X_2^1 = \varphi_{1,1} X_1$ and $X_3^2 = \varphi_{2,1} X_2 + \varphi_{2,2} X_1$ in terms of the parameters (θ, σ_W^2).

SOL:

For the MA(1) model we have $\gamma(h) = \begin{cases} \sigma_W^2(1+\theta^2), & h=0 \\ \sigma_W^2\theta, & h=1 \\ 0, & h\ge 2 \end{cases} \Rightarrow \rho(h) = \begin{cases} 1, & h=0 \\ \theta/(1+\theta^2), & h=1 \\ 0, & h\ge 2 \end{cases}$

The 1-step ahead predictor X_2^1 has coefficient $\varphi_{1,1} = \rho(1) = \dfrac{\theta}{1+\theta^2}$

The 2-step ahead predictor X_3^2 has coefficients given by the system $\mathbf{\Gamma}_2 \boldsymbol{\varphi}_2 = \boldsymbol{\gamma}_2 \Rightarrow$

$$\Rightarrow \begin{bmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} \varphi_{2,1} \\ \varphi_{2,2} \end{bmatrix} = \begin{bmatrix} \gamma(1) \\ \gamma(2) \end{bmatrix} \Rightarrow \begin{bmatrix} \varphi_{2,1} \\ \varphi_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & \theta/(1+\theta^2) \\ \theta/(1+\theta^2) & 1 \end{bmatrix}^{-1} \begin{bmatrix} \theta/(1+\theta^2) \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \varphi_{2,1} \\ \varphi_{2,2} \end{bmatrix} = \frac{1}{1 - \dfrac{\theta^2}{(1+\theta^2)^2}} \begin{bmatrix} 1 & -\theta/(1+\theta^2) \\ -\theta/(1+\theta^2) & 1 \end{bmatrix} \begin{bmatrix} \theta/(1+\theta^2) \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix} \varphi_{2,1} \\ \varphi_{2,2} \end{bmatrix} = \frac{(1+\theta^2)^2}{1+\theta^2+\theta^4} \begin{bmatrix} \theta/(1+\theta^2) \\ -\theta^2/(1+\theta^2)^2 \end{bmatrix} = \frac{1}{1+\theta^2+\theta^4} \begin{bmatrix} \theta+\theta^3 \\ -\theta^2 \end{bmatrix}$$

b. Write down the likelihood function of the parameters (θ, σ_W^2) for the first three observations of the process (x_1, x_2, x_3).

(Hint: Use the 1-step-ahead predictors you derived from the previous part)

SOL:

Use the law of multiplication to write $f(x_1, x_2, x_3) = f(x_1) f(x_2 | x_1) f(x_3 | x_1, x_2)$

Since the process is Gaussian all densities are Normal, so we just have to find their mean and variance. For the marginal density $f(x_1)$, the mean and variance are $\mathbb{E}[X_1] = 0$ and $\mathbb{V}[X_1] = \gamma(0) = \sigma_W^2(1+\theta^2)$. For the conditional densities, the means and variances are given by the 1-step-ahed predictors X_{n+1}^n and their MSE P_{n+1}^n, for $n = 1, 2$. We already know the 1-step-ahead coefficients, and their MSE is given by $P_{n+1}^n = (1-\varphi_{n,n}^2) P_n^{n-1}$, which can also be calculated based the coefficients. In particular, we have:

$$\mathbb{E}[X_2 | X_1] = X_2^1 = \varphi_{1,1} X_1,\ \ \mathbb{V}[X_2 | X_1] = P_1^0(1-\varphi_{1,1}^2) = \gamma(0)(1-\varphi_{1,1}^2) = \sigma_W^2(1+\theta^2)(1-\varphi_{1,1}^2)$$

$$\mathbb{E}[X_3 | X_2] = X_3^2 = \varphi_{2,1} X_2 + \varphi_{2,2} X_1,\ \ \mathbb{V}[X_3 | X_2] = P_2^1(1-\varphi_{2,2}^2) = \sigma_W^2(1+\theta^2)(1-\varphi_{2,2}^2)(1-\varphi_{1,1}^2)$$

Substituting everything into the density we get:

$$\begin{aligned} f(x_1, x_2, x_3) &= f(x_1) f(x_2 | x_1) f(x_3 | x_1, x_2) = \\ &= \frac{1}{\sqrt{2\pi P_1^0}} \exp\left\{ -\frac{\left(x_1 - x_1^0\right)^2}{2P_1^0} \right\} \times \frac{1}{\sqrt{2\pi P_2^1}} \exp\left\{ -\frac{\left(x_2 - x_2^1\right)^2}{2P_2^1} \right\} \times \\ &\quad \times \frac{1}{\sqrt{2\pi P_3^2}} \exp\left\{ -\frac{\left(x_3 - x_3^2\right)^2}{2P_3^2} \right\} \end{aligned}$$

$$\text{where } \begin{cases} x_1^0 = 0,\ x_2^1 = \varphi_{1,1} x_1,\ x_3^2 = \varphi_{2,1} x_2 + \varphi_{2,2} x_1 \\ P_1^0 = \sigma_W^2(1+\theta^2),\ P_2^1 = \sigma_W^2(1+\theta^2)(1-\varphi_{1,1}^2),\ P_3^2 = \sigma_W^2(1+\theta^2)(1-\varphi_{1,1}^2)(1-\varphi_{2,2}^2) \\ \varphi_{1,1} = \dfrac{\theta}{1+\theta^2},\ \varphi_{2,1} = \dfrac{\theta+\theta^3}{1+\theta^2+\theta^4},\ \varphi_{2,2} = -\dfrac{\theta^2}{1+\theta^2+\theta^4} \end{cases}$$

2. (25 marks)

Consider two time series following: $\begin{cases} \text{AR}(1): & X_t = \varphi X_{t-1} + W_t \\ \text{MA}(2): & Y_t = W_t + \theta W_{t-2} \end{cases}$, where $\{W_t\} \sim \text{WN}(0, \sigma_W^2)$ is the same white noise process used for both X_t & Y_t.

a. Find the autocovariance functions $\gamma_X(h)$ and $\gamma_Y(h)$ in terms of the parameters $(\varphi, \theta, \sigma_W^2)$.

SOL:

For the AR(1) model, we can equivalently write the series in its causal form as $X_t = \sum_{j=0}^{\infty} \varphi^j W_{t-j}$, from which we can easily get $\gamma_X(h) = \sigma_W^2 \dfrac{\varphi^h}{1-\varphi^2}, \forall h \geq 0$.

For the MA(2) model we have:

$$\gamma_Y(0) = \text{Var}[Y_t] = \text{Var}[W_t + \theta W_{t-2}] = \text{Var}[W_t] + \theta^2 \text{Var}[W_{t-2}] = \sigma_W^2 (1 + \theta^2)$$

$$\begin{aligned} \gamma_Y(1) &= \text{Cov}[Y_{t+1}, Y_t] = \text{Cov}[W_{t+1} + \theta W_{t-1}, W_t + \theta W_{t-2}] = \\ &= \text{Cov}[W_{t+1}, W_t] + \theta \text{Cov}[W_{t+1}, W_{t-2}] + \theta \text{Cov}[W_{t-1}, W_t] + \theta^2 \text{Cov}[W_{t-1}, W_{t-2}] = 0 \end{aligned}$$

$$\begin{aligned} \gamma_Y(2) &= \text{Cov}[Y_{t+2}, Y_t] = \text{Cov}[W_{t+2} + \theta W_t, W_t + \theta W_{t-2}] = \\ &= \text{Cov}[W_{t+2}, W_t] + \theta \text{Cov}[W_{t+2}, W_{t-2}] + \theta \text{Cov}[W_t, W_t] + \theta^2 \text{Cov}[W_t, W_{t-2}] = \sigma_W^2 \theta \end{aligned}$$

$$\gamma_Y(h) = 0, \ \forall h \geq 3$$

b. Find the cross-covariance function $\gamma_{XY}(h)$ in terms of the parameters $(\varphi,\theta,\sigma_W^2)$.

SOL:

$$\gamma_{XY}(0)=\mathrm{Cov}[X_t,Y_t]=\mathrm{Cov}[\sum_{j=0}^{\infty}\varphi^j W_{t-j},W_t+\theta W_{t-2}]=$$

$$=\sum_{j=0}^{\infty}\varphi^j\left\{\mathrm{Cov}[W_{t-j},W_t]+\theta\,\mathrm{Cov}[W_{t-j},W_{t-2}]\right\}=$$

$$=\mathrm{Cov}[W_t,W_t]+\varphi^2\theta\mathrm{Cov}[W_{t-2},W_{t-2}]+0=\sigma_W^2\left(1+\varphi^2\theta\right)$$

$$\gamma_{XY}(-1)=\mathrm{Cov}[X_{t-1},Y_t]=\sum_{j=0}^{\infty}\varphi^j\left\{\mathrm{Cov}[W_{t-1-j},W_t]+\theta\,\mathrm{Cov}[W_{t-1-j},W_{t-2}]\right\}=$$

$$=\varphi\theta\mathrm{Cov}[W_{t-2},W_{t-2}]+0=\varphi\theta\sigma_W^2$$

$$\gamma_{XY}(-2)=\mathrm{Cov}[X_{t-2},Y_t]=\sum_{j=0}^{\infty}\varphi^j\left\{\mathrm{Cov}[W_{t-2-j},W_t]+\theta\,\mathrm{Cov}[W_{t-2-j},W_{t-2}]\right\}=$$

$$=\theta\mathrm{Cov}[W_{t-2},W_{t-2}]=\theta\sigma_W^2$$

$$\gamma_{XY}(h)=0,\ \ \forall h\le -3$$

For the following, assume $h\ge 1$

$$\gamma_{XY}(h)=\mathrm{Cov}[X_{t+h},Y_t]=\sum_{j=0}^{\infty}\varphi^j\left\{\mathrm{Cov}[W_{t+h-j},W_t]+\theta\,\mathrm{Cov}[W_{t-h-j},W_{t-2}]\right\}=$$

$$=\varphi^h\mathrm{Cov}[W_t,W_t]+\varphi^{h+2}\theta\mathrm{Cov}[W_{t-2},W_{t-2}]+0=\sigma_W^2\varphi^h(1+\theta\varphi^2)$$

3. (35 marks)

a. Consider the ARMA(2,1) model $X_t = -.7X_{t-1} + .2X_{t-2} + W_t + .5W_{t-1}$. Calculate its ACF for the first 5 lags, i.e. find the values of $\rho(h)$ for $h = 1,2,3,4,5$.

SOL:

We have that $\psi_1 = \varphi_1 + \theta_1 = -.7 + .5 = -.2$. The initial conditions are

$$\gamma(h) - \sum_{j=1}^{2} \varphi_j \gamma(h-j) = \sigma_W^2 \sum_{j=h}^{1} \theta_j \psi_{j-h},\ 0 \le h < 2 = \max(2,1+1) \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{ll} (h=0) & \gamma(0) - \varphi_1\gamma(1) - \varphi_2\gamma(2) = \sigma_W^2\left(\theta_0\psi_0 + \theta_1\psi_1\right) \\ (h=1) & \gamma(1) - \varphi_1\gamma(0) - \varphi_2\gamma(1) = \sigma_W^2\left(\theta_1\psi_0\right) \end{array} \right\} \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{c} \gamma(0) + .7\gamma(1) - .2\gamma(2) = \sigma_W^2\left(1 + .5(-.2)\right) \\ \gamma(1) + .7\gamma(0) - .2\gamma(1) = \sigma_W^2\left(.5\right) \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \gamma(0) + .7\gamma(1) - .2\gamma(2) = \sigma_W^2\left(.9\right) \\ .7\gamma(0) + .8\gamma(1) = \sigma_W^2\left(.5\right) \end{array} \right\}$$

Using the fact that for $(h=2)$ $\gamma(2) = \varphi_1\gamma(1) + \varphi_2\gamma(0) = -.7\gamma(1) + .2\gamma(0)$, we get

$$\gamma(0) + .7\gamma(1) - .2\gamma(2) = \gamma(0) + .7\gamma(1) - .2\left(-.7\gamma(1) + .2\gamma(0)\right) = .96\gamma(0) + .84\gamma(1)$$

$$\Rightarrow \left\{ \begin{array}{c} .96\gamma(0) + .84\gamma(1) = \sigma_W^2\left(.9\right) \\ .7\gamma(0) + .8\gamma(1) = \sigma_W^2\left(.5\right) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \gamma(0) = 10\sigma_W^2/6 \\ \gamma(1) = -5\sigma_W^2/6 \end{array} \right\}$$

For $h \ge 2$, $\gamma(h) = \varphi_1\gamma(h-1) + \varphi_2\gamma(h-1) = -.7\gamma(h-1) + .2\gamma(h-2) \Rightarrow$

$$\Rightarrow \left\{ \begin{array}{c} \gamma(2) = -.7\gamma(1) + .2\gamma(0) = 5.5\sigma_W^2/6 \\ \gamma(3) = -.7\gamma(2) + .2\gamma(1) = -4.85\sigma_W^2/6 \\ \gamma(4) = -.7\gamma(3) + .2\gamma(2) = 4.495\sigma_W^2/6 \\ \gamma(5) = -.7\gamma(4) + .2\gamma(3) = -4.1165\sigma_W^2/6 \end{array} \right\} \Rightarrow$$

$$\Rightarrow \left\{ \begin{array}{c} \rho(1) = \gamma(1)/\gamma(0) = -5/10 = -.5 \\ \rho(2) = \gamma(2)/\gamma(0) = 5.5/10 = .55 \\ \rho(3) = \gamma(3)/\gamma(0) = -4.85/10 = -.485 \\ \rho(4) = \gamma(4)/\gamma(0) = 4.495/10 = .4495 \\ \rho(5) = \gamma(5)/\gamma(0) = -4.1165/10 = -.41165 \end{array} \right.$$

b. Consider two jointly stationary, zero-mean series $\{X_t, Y_t\}$ with individual autocovariance functions $\gamma_X(h), \gamma_Y(h)$ and cross-covariance function $\gamma_{XY}(h)$. Assume you want to forecast X_3 based on X_1, X_2, Y_1, Y_2 using the best linear predictor (BLP) $\hat{X}_3 = \alpha_1 X_1 + \alpha_2 X_2 + \beta_1 Y_1 + \beta_2 Y_2$. Give the equations that the BLP coefficients $\alpha_1, \alpha_2, \beta_1, \beta_2$ must satisfy.
(Hint: these equations have terms involving auto/cross-covariances evaluated at different lags.)

SOL:
The BLP $\hat{X}_3 = \alpha_1 X_1 + \alpha_2 X_2 + \beta_1 Y_1 + \beta_2 Y_2$ minimizes the MSE $\mathbb{E}\left[(X_3 - \hat{X}_3)^2\right]$, which is implies that the coefficients satisfy the normal equations: $\mathbb{E}\left[(X_3 - \hat{X}_3)U\right] = 0 \Rightarrow$

$$\Rightarrow \mathbb{E}[X_3 U] = \mathbb{E}\left[\hat{X}_3 U\right] = \mathbb{E}[(\alpha_1 X_1 + \alpha_2 X_2 + \beta_1 Y_1 + \beta_2 Y_2)U] \Rightarrow$$

$$\Rightarrow \alpha_1 \mathbb{E}[X_1 U] + \alpha_2 \mathbb{E}[X_2 U] + \beta_1 \mathbb{E}[Y_1 U] + \beta_2 \mathbb{E}[Y_2 U] = \mathbb{E}[X_3 U], \ \forall U = X_1, X_2, Y_1, Y_2$$

Using the fact that the series are zero-mean, i.e. $\mathbb{E}[X_i Y_j] = \mathrm{Cov}[X_i, Y_j] = \gamma_{XY}(i-j)$, we get

$$\left\{\begin{array}{c} (U = X_1)\ \ \alpha_1 \mathbb{E}[X_1 X_1] + \alpha_2 \mathbb{E}[X_2 X_1] + \beta_1 \mathbb{E}[Y_1 X_1] + \beta_2 \mathbb{E}[Y_2 X_1] = \mathbb{E}[X_3 X_1] \\ (U = X_2)\ \ \alpha_1 \mathbb{E}[X_1 X_2] + \alpha_2 \mathbb{E}[X_2 X_2] + \beta_1 \mathbb{E}[Y_1 X_2] + \beta_2 \mathbb{E}[Y_2 X_2] = \mathbb{E}[X_3 X_2] \\ (U = Y_1)\ \ \alpha_1 \mathbb{E}[X_1 Y_1] + \alpha_2 \mathbb{E}[X_2 Y_1] + \beta_1 \mathbb{E}[Y_1 Y_1] + \beta_2 \mathbb{E}[Y_2 Y_1] = \mathbb{E}[X_3 Y_1] \\ (U = Y_2)\ \ \alpha_1 \mathbb{E}[X_1 Y_2] + \alpha_2 \mathbb{E}[X_2 Y_2] + \beta_1 \mathbb{E}[Y_1 Y_2] + \beta_2 \mathbb{E}[Y_2 Y_2] = \mathbb{E}[X_3 Y_2] \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{c} \alpha_1 \gamma_X(0) + \alpha_2 \gamma_X(1) + \beta_1 \gamma_{XY}(0) + \beta_2 \gamma_{XY}(-1) = \gamma_X(2) \\ \alpha_1 \gamma_X(1) + \alpha_2 \gamma_X(0) + \beta_1 \gamma_{XY}(1) + \beta_2 \gamma_{XY}(0) = \gamma_X(1) \\ \alpha_1 \gamma_{XY}(0) + \alpha_2 \gamma_{XY}(1) + \beta_1 \gamma_Y(0) + \beta_2 \gamma_Y(1) = \gamma_{XY}(2) \\ \alpha_1 \gamma_{XY}(-1) + \alpha_2 \gamma_{XY}(0) + \beta_1 \gamma_Y(1) + \beta_2 \gamma_Y(0) = \gamma_{XY}(1) \end{array}\right\}$$

4. (15 marks)

A time series dataset has sample moments $\hat{\gamma}(0)=2.35, \hat{\gamma}(1)=1.69, \hat{\gamma}(2)=1.17,$ and $\hat{\gamma}(3)=0.48$. Use Yule-Walker estimation to fit an AR(3) model and report the estimated coefficients $\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3$ & $\hat{\sigma}_w^2$.

SOL:

Use Yule-Walker estimation, and save time by starting with the solution of the

2×2system $\hat{\boldsymbol{\varphi}}_2 = \hat{\boldsymbol{\Gamma}}_2^{-1}\hat{\boldsymbol{\gamma}}_2 \Rightarrow \begin{bmatrix} \hat{\varphi}_{2,1} \\ \hat{\varphi}_{2,2} \end{bmatrix} = \begin{bmatrix} \hat{\gamma}(0) & \hat{\gamma}(1) \\ \hat{\gamma}(1) & \hat{\gamma}(0) \end{bmatrix}^{-1} \begin{bmatrix} \hat{\gamma}(1) \\ \hat{\gamma}(2) \end{bmatrix} \Rightarrow$

$$\begin{bmatrix} \hat{\varphi}_{2,1} \\ \hat{\varphi}_{2,2} \end{bmatrix} = \begin{bmatrix} 2.35 & 1.69 \\ 1.69 & 2.35 \end{bmatrix}^{-1} \begin{bmatrix} 1.69 \\ 1.17 \end{bmatrix} = \frac{1}{2.35^2 - 1.69^2} \begin{bmatrix} 2.35 & -1.69 \\ -1.69 & 2.35 \end{bmatrix} \begin{bmatrix} 1.69 \\ 1.17 \end{bmatrix} = \begin{bmatrix} 0.7479 \\ -0.04 \end{bmatrix}$$

For $\hat{\boldsymbol{\varphi}}_3$ we have from the Durbin-Levinson algorithm:

$$\hat{\varphi}_{3,3} = \frac{\hat{\gamma}(3) - \sum_{k=1}^{2} \hat{\varphi}_{2,k}\hat{\gamma}(3-k)}{\hat{\gamma}(0) - \sum_{k=1}^{2} \hat{\varphi}_{2,k}\hat{\gamma}(k)} = \frac{\hat{\gamma}(3) - \hat{\varphi}_{2,1}\hat{\gamma}(2) - \hat{\varphi}_{2,2}\hat{\gamma}(1)}{\hat{\gamma}(0) - \hat{\varphi}_{2,1}\hat{\gamma}(1) - \hat{\varphi}_{2,2}\hat{\gamma}(2)} =$$

$$= \frac{.48 - (0.7479)(1.17) - (-0.04)(1.69)}{2.35 - (0.7479)(1.69) - (-0.04)(1.17)} = \cdots = -0.2891$$

$$\hat{\varphi}_{3,2} = \hat{\varphi}_{2,2} - \hat{\varphi}_{3,3}\hat{\varphi}_{2,1} = (-0.04) - (-0.2891)(0.7479) = 0.1762$$

$$\hat{\varphi}_{3,1} = \hat{\varphi}_{2,1} - \hat{\varphi}_{3,3}\hat{\varphi}_{2,2} = (0.7479) - (-0.2891)(-0.04) = 0.7363$$

Finally, the estimated variance is

$$\hat{\sigma}_w^2 = \hat{\gamma}(0) - \hat{\varphi}_{3,1}\hat{\gamma}(1) - \hat{\varphi}_{3,2}\hat{\gamma}(2) - \hat{\varphi}_{3,3}\hat{\gamma}(3) =$$

$$= 2.35 - (0.7363)(1.69) - (0.1762)(0.7479) - (-0.2891)(0.48) = 1.038157$$

Extra Space -- Use if needed and indicate clearly which questions you are answering