STA C58 Assignment 2 - Solutions

1. (a) $L_{\Psi}(1\,|\,1) = \frac{1}{2}$, $\quad L(2\,|\,1) = \frac{1}{3}$

$L_{\Psi}(1\,|\,2) = \frac{1}{3}$, $\quad L(2\,|\,2) = \frac{4}{6}$

$L_{\Psi}(1\,|\,3) = \frac{1}{2}$, $\quad L(2\,|\,3) = \frac{1}{6}$

and any positive multiple of these give all possible profile likelihoods.

(b) $C_{1/2}(1) = \{\psi : L_{\Psi}(\psi\,|\,1) \ge \frac{1}{2}\sup_{\psi} L(\psi\,|\,1)\}$

$= \{\psi : L_{\Psi}(\psi\,|\,1) \ge 1/4\} = \{1, 2\}$

$C_{1/2}(2) = \{\psi : L_{\Psi}(\psi\,|\,2) \ge \frac{2}{6}\} = \{2\}$

$C_{1/2}(3) = \{\psi : L_{\Psi}(\psi\,|\,3) \ge \frac{1}{4}\} = \{1\}$

2. (a) $L(\mu, \sigma^2\,|\,x) = (\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{n}{2\sigma^2}(\bar{x}-\mu)^2 - \frac{(n-1)s_x^2}{2\sigma^2}\}$

and $L(\psi, \sigma^2\,|\,x) = (\sigma^2)^{-\frac{n}{2}} \exp\{-\frac{n}{2\sigma^2}(\bar{x}-\frac{\sigma}{\psi})^2 - \frac{(n-1)s_x^2}{2\sigma^2}\}$

and note that $\psi \in \mathbb{R}\setminus 0$, $\sigma^2 > 0$ are unconstrained so $L_{\Psi}(\psi\,|\,x) = \sup_{\sigma^2>0} L(\psi, \sigma^2\,|\,x)$ and we need to maximize $L(\psi, \sigma^2\,|\,x)$ as a function of σ^2.

(b) Taking the log of the likelihood we obtain

$l(\psi, \sigma^2\,|\,x) = -\frac{n}{2}\log\sigma^2 - \frac{n}{2\sigma^2}(\bar{x}-\frac{\sigma}{\psi})^2 - \frac{(n-1)s_x^2}{2\sigma^2}$ and

$$\frac{\partial l(\psi, \sigma^2 | x)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{n}{2\sigma^4}\left(\bar{x} - \frac{(\sigma^2)^{1/2}}{\psi}\right)^2$$

$$+ \frac{n}{2\sigma^2} 2\left(\bar{x} - \frac{(\sigma^2)^{1/2}}{\psi}\right) \frac{1}{2}\frac{(\sigma^2)^{-1/2}}{\psi} + \frac{(n-1)s_x^2}{2\sigma^4}$$

Then setting this equal to 0 and multiplying through by $-2\sigma^5/n$ we get

$$\sigma^3 - \sigma(\bar{x}^2 - 2\bar{x}\sigma/\psi + \sigma^2/\psi^2) - \sigma^3(\bar{x} - \sigma/\psi)\frac{\sigma^{-1}}{\psi} - \frac{n-1}{n} s_x^2 \sigma$$

$$= \sigma^3 - \bar{x}^2\sigma + \frac{2\bar{x}}{\psi}\sigma^2 - \cancel{\frac{\sigma^3}{\psi^2}} - \frac{\bar{x}}{\psi}\sigma^2 + \cancel{\frac{\sigma^3}{\psi^2}} - \frac{n-1}{n} s_x \sigma$$

$$= \sigma^3 + \frac{\bar{x}}{\psi}\sigma^2 - \left(\bar{x}^2 + \frac{n-1}{n} s_x\right)\sigma = 0$$

or $$\sigma^2 + \frac{\bar{x}}{\psi}\sigma - \left(\bar{x}^2 + \frac{n-1}{n} s_x\right) = 0$$

so $$\sigma = \frac{-\frac{\bar{x}}{\psi} \pm \sqrt{\frac{\bar{x}^2}{\psi^2} + 4\left(\bar{x}^2 + \frac{n-1}{n} s_x^2\right)}}{2}$$

$$= \frac{1}{2}\left(\frac{-\bar{x}}{\psi} + \sqrt{\left(\frac{\bar{x}}{\psi}\right)^2 + \frac{4}{n}\sum_{i=1}^{n} x_i^2}\right)$$ since $\bar{x}^2 + \frac{n-1}{n}s_x^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

and since σ is not negative the root must be

$$\sigma(\psi) = \frac{1}{2}\left(-\frac{\bar{x}}{\psi} + \sqrt{\frac{\bar{x}^2}{\psi^2} + \frac{4}{n}\sum_{i=1}^{n} x_i^2}\right)$$

Then the profile likelihood fn is

$$L_\Psi(\psi | x) = (\sigma^2(\psi))^{-\frac{n}{2}} \exp\left\{-\frac{n}{2\hat{\sigma}^2(\psi)}\left(\bar{x} - \frac{\sigma(\psi)}{\psi}\right)^2 - \frac{(n-1)s_x^2}{\sigma^2(\psi)}\right\}.$$

3 (6.2.20) The likelihood function for μ is given by

$$L(\mu|x) = c\exp\{-\tfrac{n}{2}(\bar{x}-\mu)^2\}$$

and $\psi = \Phi(\mu) = \Phi(1-\mu)$. Now $d\Phi(\mu)/d\mu = -\varphi(1-\mu) = -(2\pi)^{-1/2}e^{-\frac{1}{2}(1-\mu)^2} < 0$ and so Φ is a strictly decreasing function of μ which implies it is 1-1. Now $\mu = \Phi^{-1}(\psi) = 1-\Phi^{-1}(\psi)$ and so the likelihood fn is

$$L(\psi|x) = c\exp\{-\tfrac{n}{2}(\bar{x}-1+\Phi^{-1}(\psi))^2\}.$$

The MLE of μ is $\bar{x}$ and so the MLE of ψ is $\Phi(1-\bar{x})$.

4. (6.3.25) We have that $\bar{x} \sim N(\mu, \sigma_0^2/n)$ so

$$P_\mu\left(\mu \le \bar{x} + k\frac{\sigma_0}{\sqrt{n}}\right) = P_\mu\left(-k \le \frac{\bar{x}-\mu}{\sigma_0/\sqrt{n}}\right)$$

$$= 1 - P_\mu\left(\frac{\bar{x}-\mu}{\sigma_0/\sqrt{n}} \le -k\right) = 1-\Phi(-k) = \gamma$$

and so $k = -\Phi^{-1}(1-\gamma) = -z_{1-\gamma} = z_\gamma$

a γ-confidence interval for μ.

5. 6.3.26 We have that

$$\max_{\mu \le \mu_0} P_\mu(\bar{X} > \bar{x}) = \max_{\mu \le \mu_0} P_\mu\left(\frac{\bar{X}-\mu}{\sigma_0/\sqrt{n}} > \frac{\bar{x}-\mu}{\sigma_0/\sqrt{n}}\right)$$

$$= \max_{\mu \le \mu_0}\left(1 - \Phi\left(\frac{\bar{x}-\mu}{\sigma_0/\sqrt{n}}\right)\right)$$

$$= 1 - \min_{\mu \le \mu_0} \Phi\left(\frac{\bar{x}-\mu}{\sigma_0/\sqrt{n}}\right) = 1 - \Phi\left(\frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}}\right)$$

since $\Phi\left(\frac{\bar{x}-\mu}{\sigma_0/\sqrt{n}}\right)$ decreases as μ increases.

6. 6.3.27 The p-value

$1 - \Phi\left(\frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}}\right) \le \alpha$ iff $\frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}} \ge \Phi^{-1}(1-\alpha) = z_{1-\alpha}$

The power function is then given by

$$\beta(\mu) = P_\mu\left(\frac{\bar{x}-\mu_0}{\sigma_0/\sqrt{n}} \ge z_{1-\alpha}\right)$$

$$= P_\mu\left(\frac{\bar{x}-\mu}{\sigma_0/\sqrt{n}} \ge \frac{\mu_0-\mu}{\sigma_0/\sqrt{n}} + z_{1-\alpha}\right)$$

$$= 1 - \Phi\left(\frac{\mu_0-\mu}{\sigma_0/\sqrt{n}} + z_{1-\alpha}\right)$$