STAC62F: 2016 Assignment 1

1. Suppose $\Omega = \{1, 2, 3, 4, 5\}$

(a) Consider $\mathcal{F} = \{\phi, \Omega, \{1,2\}, \{3,4,5\}\}$. Establish whether or not $\mathcal{F}$ is a σ-field.

(b) Suppose $\mathcal{A} = \{\{1\}, \{2,3,4\}, \{5\}\}$. Determine $\mathcal{F}(\mathcal{A})$.

(c) Suppose $P(\{1\}) = 1/4$, $P(\{2,3,4\}) = 1/2$ and $P(\{5\}) = 1/4$. Does this determine a probability measure on $\mathcal{F}(\mathcal{A})$ and if so what is it?

2. Consider the Borel sets $\mathcal{B}^1$ on $\mathbb{R}^1$

(a) Prove that any set of the form $(-\infty, a]$, for $a \in \mathbb{R}^1$, is a Borel set.

(b) Prove that the set $\{a\}$ is a Borel set for any $a \in \mathbb{R}^1$

3. (a) Suppose that $A_1 \subseteq A_2 \subseteq \cdots \subseteq \Omega$. Then prove that $\overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n = \bigcup_{i=1}^{\infty} A_i$

(b) Suppose that $\Omega \supseteq A_1 \supseteq A_2 \supseteq \cdots$. Then prove that $\overline{\lim_{n\to\infty}} A_n = \underline{\lim_{n\to\infty}} A_n = \bigcap_{i=1}^{\infty} A_i$.

4. Suppose $(\Omega, \mathcal{F}, P)$ is a probability model and $A_1, A_2, \ldots, A_n \in \mathcal{F}$. Then prove

$$P(A_1 \cap \cdots \cap A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cup A_j) + \cdots + (-1)^{n+1} P(A_1 \cup \cdots \cup A_n)$$

(Hint: do it for $n=2$ and then use induction).

(5) Suppose $\Omega = \{0,1\}^{\mathbb{N}}$, $\mathcal{F} = \bigtimes_{i=1}^{\infty} \{\phi, \{0\}, \{1\}, \{0,1\}\}$ and P is the probability measure on $\mathcal{F}$ formed via independence with P_i on $\{\phi, \{0\}, \{1\}, \{0,1\}\}$ given by $P_i(\{0\}) = 1/2$.

(a) Let A = after 10 tosses there are more heads than tails. Determine $P(A)$.

(b) What is the probability that 30-th toss yields a head?

(c) What is the probability that only finitely many heads are obtained?

(d) Let $T(\omega) = n$ if $\omega_n = 1$, $\omega_{n-1} = 1$ and $T(\omega) = \infty$ otherwise. (T is a stopping time and this case we stop at time n if we get a head at times $n-1$ and n). What is the probability that the process is stopped?

(e) Define stopping time $T(\omega) = n$ if the number of heads before time n is $\geq j$. Determine the probability distribution of T and note you have to consider $T(\omega) = \infty$.