1.(a).

$$\nabla^2 X_t = (1-B)^2 X_t = (1-2B+B^2)X_t =$$

$$= X_t - 2X_{t-1} + X_{t-2} =$$

$$= (t^2+Z_t) - 2((t-1)^2 + Z_{t-1}) + ((t-2)^2 + Z_{t-2}) =$$

$$= t^2 + Z_t - 2(t^2 - 2t - 1 + Z_{t-1}) + (t^2 - 4t + 2^2 + Z_{t-2}) =$$

$$= \cancel{t^2} + Z_t - \cancel{2t^2} + \cancel{4t} - 2 + 2Z_{t-1} + \cancel{t^2} - \cancel{4t} + 4 + Z_{t-2} =$$

$$= Z_t - 2Z_{t-1} + Z_{t-2} + 2 =$$

$$= \cancel{Z_{t-1}} + W_t - \cancel{2}Z_{t-1} + Z_{t-2} + 2 =$$

$$= W_t - (\cancel{Z_{t-2}} + W_{t-1}) + \cancel{Z_{t-2}} + 2 =$$

$= W_t + W_{t-1} + 2$, which is a MA(2) process $\Rightarrow$ stationary

1.(b) $\nabla_2 Y_t = (1-B^2)Y_t = Y_t - Y_{t-2} =$ $\left(\forall\ t \geq 2,\ t \in \mathbb{N}\ \ (-1)^t = (-1)^{t-2}\right)$

$$= \cancel{(-1)^t} + Z_t - \cancel{(-1)^{t-2}} - Z_{t-2} =$$

$$= Z_{t-1} + W_t - Z_{t-2} = \cancel{Z_{t-2}} + W_{t-1} + W_t - \cancel{Z_{t-2}}$$

$= W_t + W_{t-1}$, which is also a MA(2) process $\Rightarrow$

$\Rightarrow$ stationary

2. (a) For the model to be stationary, the roots of the AR characteristic polynomial must lie outside the unit disk: $\varphi(z) = \underbrace{1}_{c} - \underbrace{\varphi_1}_{b} z + \underbrace{\frac{1}{4}}_{a} z^2 = 0 \Rightarrow$

$\Rightarrow$ roots: $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{\varphi_1 \pm \sqrt{\varphi_1^2 - 1}}{1/2} = 2 \cdot (\varphi_1 \pm \sqrt{\varphi_1^2 - 1})$

For positive φ_1 (w/ real roots, ie $\varphi_1^2 > 1$), the smallest root is $2 \cdot (\varphi_1 - \sqrt{\varphi_1^2 - 1})$, which must be $> 1 \Rightarrow$

$\Rightarrow 2(\varphi_1 - \sqrt{\varphi_1^2 - 1}) > 1 \Rightarrow \varphi_1 - \sqrt{\varphi_1^2 - 1} > 1/2 \Rightarrow$

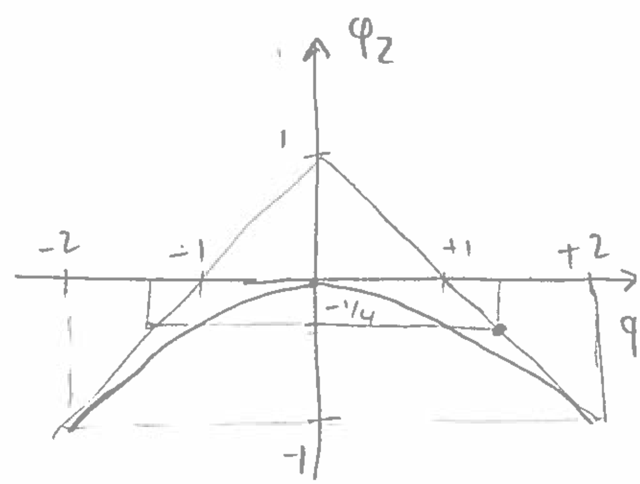
$\Rightarrow \varphi_1 - 1/2 > \sqrt{\varphi_1^2 - 1} \Rightarrow (\varphi_1 - 1/2)^2 > \varphi_1^2 - 1 \Rightarrow$

$\Rightarrow \cancel{\varphi_1^2} + \frac{1}{4} - \varphi_1 > \cancel{\varphi_1^2} - 1 \Rightarrow \varphi_1 < 1 + \frac{1}{4} \Rightarrow \boxed{\varphi_1 < \frac{5}{4}}$

Similarly for negative φ_1 (w/ real roots), the biggest root is $2(\varphi_1 + \sqrt{\varphi_1^2 - 1})$ which must be $< -1 \Rightarrow$

$\Rightarrow 2(\varphi_1 + \sqrt{\varphi_1^2 - 1}) < -1 \Rightarrow \ldots \Rightarrow \boxed{\varphi_1 > -5/4}$

As long as $-\frac{5}{4} < \varphi_1 < 5/4$, the model will be stationary, (even for complex roots), as shown from the stationarity region for the general AR(2) model



2.(b) For the AR(2) model; we have the recurrence equation: $\rho(h) = \varphi_1 \rho(h-1) + \varphi_2 \rho(h-2)$, $\forall\ h \geq 2$,

with initial conditions: $\begin{cases} \rho(0) = 1 \\ \rho(1) = \varphi_1 \rho(0) + \varphi_2 \rho(1) \end{cases}$ $\Rightarrow$

$$\Rightarrow \begin{cases} \rho(0) = 1 \\ \rho(1) = 1 \cdot 1 - \frac{1}{4}\rho(1) \Rightarrow \rho(1) = 4/5 \\ \rho(h) = \rho(h-1) - \frac{1}{4}\rho(h-2), \ \forall\ h \geq 2 \end{cases}$$

3. (a) For AR(1) model: $X_{n+1}^{n} = \varphi X_n \quad , \forall n \geq 1$

(b) The 2-step-ahead BLP coefficient $\varphi_n^{(2)}$ must satisfy $\Gamma_n \cdot \varphi_n^{(2)} = \gamma_n^{(2)} \iff \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(n-1) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(n-2) \\ \vdots & \vdots & \ddots & \\ \gamma(n-1) & \gamma(n-2) & & \gamma(0) \end{bmatrix} \begin{bmatrix} \varphi_{n,1}^{(2)} \\ \vdots \\ \varphi_{n,n}^{(2)} \end{bmatrix} = \begin{bmatrix} \gamma(2) \\ \gamma(3) \\ \vdots \\ \gamma(n+1) \end{bmatrix}$

We have: $\varphi_n^{(2)} = \begin{bmatrix} \varphi^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} \quad -''- \quad \end{bmatrix} \begin{bmatrix} \varphi^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -''- \end{bmatrix} \Rightarrow$

$$\Rightarrow \varphi^2 \cdot \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \vdots \\ \gamma(n-1) \end{bmatrix} = \begin{bmatrix} \gamma(2) \\ \gamma(3) \\ \vdots \\ \gamma(n+1) \end{bmatrix}$$, which is satisfied b/c of the reccurence relation $\gamma(h) = \varphi\, \gamma(h-1) = \varphi^2 \gamma(h-2), \underline{\forall h \geq 2}$

(c) $$\mathbb{E}\left[\left(\underbrace{X_{n+1}}_{=\varphi X_n + W_{n+1}} - X_{n+1}^{n}\right)^2\right] = \mathbb{E}\left[\left(\cancel{\varphi X_n} + W_{n+1} - \cancel{\varphi X_n}\right)^2\right] = \mathbb{E}[W_{n+1}^2] = \sigma_W^2$$

$$\mathbb{E}\left[\left(X_{n+2} - X_{n+2}^{n}\right)^2\right] = \mathbb{E}\left[\left(\varphi \overbrace{X_{n+1}}^{=\varphi X_n + W_{n+1}} + W_{n+2} - \varphi^2 X_n\right)^2\right] =$$

$$= \mathbb{E}\left[\left(\cancel{\varphi^2 X_n} + \varphi W_{n+1} + W_{n+2} - \cancel{\varphi^2 X_n}\right)^2\right] =$$

$$= \mathbb{E}\left[\left(\varphi W_{n+1} + W_{n+2}\right)^2\right] = \varphi^2 \mathbb{E}[W_{n+1}^2] + \mathbb{E}[W_{n+2}^2] = (\varphi^2 + 1)\sigma_W^2$$

(d) $\text{Cov}[(X_{n+1} - X_{n+1}^{n}), (X_{n+2} - X_{n+2}^{n})] =$

$$= \text{Cov}[\varphi W_{n+1} + W_{n+2}, W_{n+1}] = \varphi \, \text{Cov}[W_{n+1}, W_{n+1}] = \varphi \sigma_w^2$$

$$\text{Corr}[\ \dots\ ] = \frac{\text{Cov}(\dots)}{\sqrt{\text{Var}(\cdot) \cdot \text{Var}(\cdot)}} = \frac{\varphi \cancel{\sigma_w^2}}{\sqrt{\cancel{\sigma_w^2} \cdot (1+\varphi^2) \cancel{\sigma_w^2}}} =$$

$$= \frac{\varphi}{\sqrt{1+\varphi^2}}$$

4.(a) For MA(1) model: $\gamma(h) = \begin{cases} (1+\theta^2)\sigma_w^2, & h=0 \\ \theta\sigma_w^2, & h=1 \\ 0, & h \geq 2 \end{cases}$

(b). Use Durbin-Levinson algorithm:

$$\varphi_{1,1} = \frac{\gamma(1)}{\gamma(0)} = \rho(1) = \frac{\theta}{1+\theta^2}$$

$$\varphi_{2,2} = \frac{\gamma(2) - \sum_{k=1}^{1} \varphi_{1,1}\gamma(1)}{\gamma(0) - \sum_{k=1} \varphi_{1,1}\gamma(1)} = \frac{0 - \frac{\theta}{1+\theta^2}\theta\cancel{\sigma_w^2}}{(1+\theta^2)\cancel{\sigma_w^2} - \frac{\theta}{(1+\theta^2)}\theta\cancel{\sigma_w^2}} = \frac{-\theta^2}{(1+\theta^2)^2 - \theta^2} = \frac{-\theta^2}{1+\theta^2+\theta^4}$$

$$\varphi_{2,1} = \varphi_{1,1}(1-\varphi_{2,2}) = \frac{\theta}{1+\theta^2}\cdot\left(1 - \frac{\theta^2}{1+\theta^2+\theta^4}\right)$$

$$\varphi_{3,3} = \frac{\gamma(3) - \sum_{k=1}^{2} \varphi_{2,k}\gamma(3-k)}{\gamma(0) - \sum_{k=1}^{n} \varphi_{2,k}\gamma(k)} = \frac{0 - \varphi_{2,2}\cdot\gamma(1)}{\gamma(0) - \varphi_{2,1}\gamma(1)} =$$

$$= \frac{0 + \frac{\theta^2}{1+\theta^2+\theta^4}\cdot\theta\cancel{\sigma_w^2}}{\frac{1}{(1+\theta^2)}\cancel{\sigma_w^2} - \{\varphi_{2,1}\}\,\theta\sigma_w^2}$$