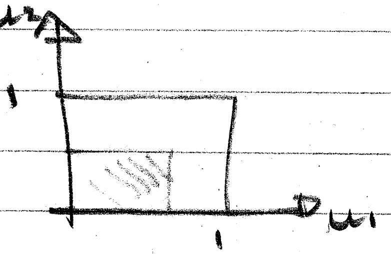
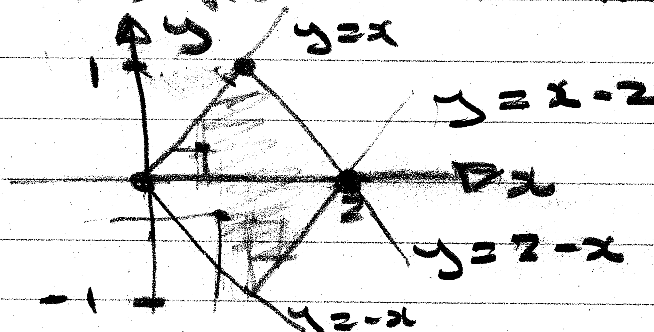
STA 262 : 2015 Assignment 2 – Solutions

1 (i)
$$F_{(U_1,U_2)}(u_1,u_2) = \begin{cases} 0 & \text{if } u_1<0 \text{ or } u_2<0 \\ \min(u_1,1)\min(u_2,1) & \text{if } u_1 \geq 0,\ u_2 \geq 0 \end{cases}$$



(ii)
$$f_{(U_1,U_2)}(u_1,u_2) = \frac{\partial^2 F_{(U_1,U_2)}(u_1,u_2)}{\partial u_1 \partial u_2}$$
$$= \begin{cases} 1 & \text{if } 0 \leq u_1 \leq 1,\ 0 \leq u_2 \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

(iii) This transformation maps the unit square to

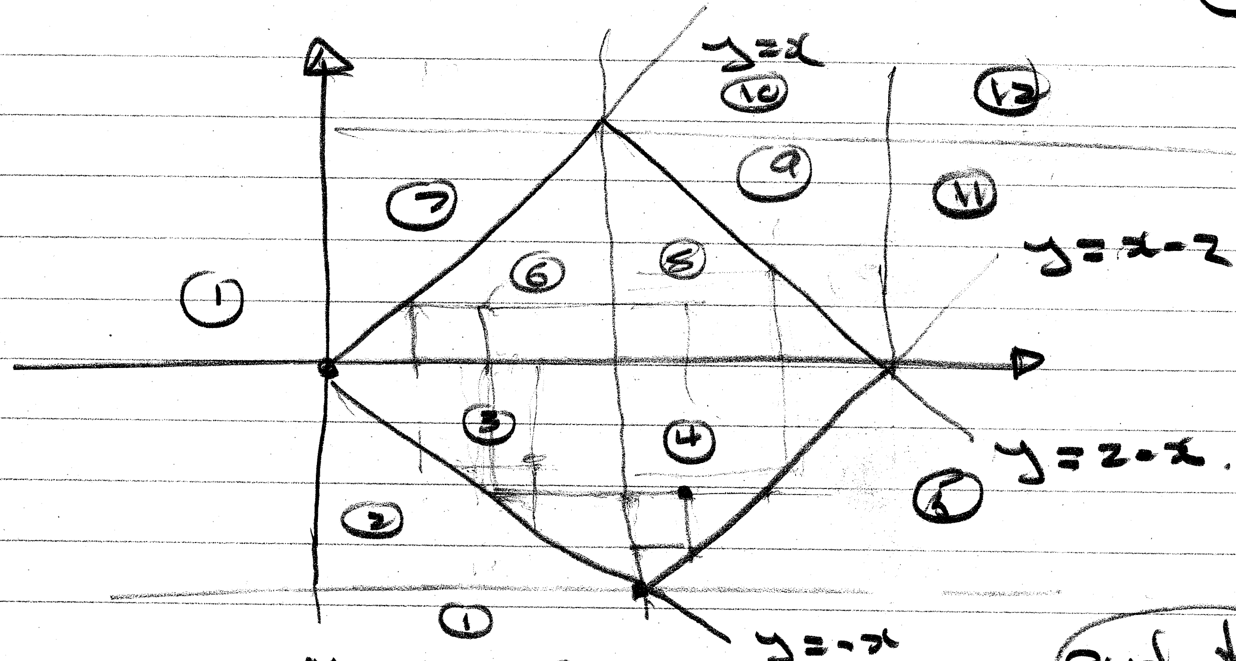


The inverse of the transformation is given by $u_1 = (x+y)/2$, $u_2 = (x-y)/2$. So

$$J_T(u_1,u_2) = \left|\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right|^{-1} = \frac{1}{2}$$

and $J_T((x+y)/2, (x-y)/2) = \frac{1}{2}$. Therefore

$$f_{(X,Y)}(x,y) = f_{U_1}\left(\frac{x+y}{2}\right) f_{U_2}\left(\frac{x-y}{2}\right)\frac{1}{2}$$
$$= \begin{cases} 1/2 & \text{if } 0 \leq \frac{x+y}{2} \leq 1,\ 0 \leq \frac{x-y}{2} \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$F_{(X,Y)}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{(X,Y)}(u,v)\,du\,dv$$ evaluate areas and divide by 2

$= 0$ $\qquad x \le 0$ or $y \le -1$ 1

$= 0$ $\qquad 0 \le x \le 1,\ y \le -x$ 2

$= \frac{1}{4}(x+y)^2$ $\qquad 0 \le x \le 1,\ -x \le y \le 0$ 3

$= \frac{1}{4}(1+y)^2 + \frac{1}{2}(x-1)(y-x+2) + \frac{1}{4}(x-1)^2$ $\qquad 1 \le x \le 2,\ x-2 \le y \le 0$ 4

$= \frac{1}{2}(y+1)^2$ $\qquad 2+y \le x,\ -1 \le y \le 0$ 5

$= \frac{1}{4}x^2 + \frac{1}{4}y^2 + \frac{1}{2}(x-y)y$ $\qquad 0 \le x \le 1,\ 0 \le y \le x$ 6

$= \frac{1}{2}x^2$ $\qquad 0 \le x \le 1,\ x \le y$ 7

$= \frac{1}{4} + \left(\frac{1}{4} + \frac{1}{4}y^2 + \frac{1}{2}(1-y)y\right) + \frac{1}{2}(x-1)y + \left(\frac{1}{4} + \frac{1}{2}(x-1)(2-x) + \frac{1}{4}(x-1)^2\right.$ 8

$1 \le x \le 2,\ 0 \le y \le 2-x$

$= \frac{1}{4} + \left(\frac{1}{4} + \frac{1}{4}y^2 + \frac{1}{2}(1-y)y\right)$ $\qquad 1 \le x \le 2,\ 2-x \le y \le 1$ 9

$+ \left(\frac{1}{4} + \frac{1}{2}(x-1)(2-x) + \frac{1}{4}(x-1)^2\right) + \frac{1}{2}(x-1)(2-x)$

$$= \frac{1}{2} + 2\left(\frac{1}{4} - \frac{1}{4}(2-x)^2\right) \qquad 1 \le x \le 2,\ y \ge 1 \qquad (10)$$

$$= \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2}(1-y)^2\right) \qquad 2 \le x,\ 0 \le y \le 1 \qquad (11)$$

$$= 1 \qquad 2 \le x,\ y \ge 1 \qquad (12)$$

(2) (i) $\omega \in T^{-1}A^c$ iff $T(\omega) \in A^c$ iff $T(\omega) \notin A$ iff $\omega \notin T^{-1}A$ iff $\omega \in (T^{-1}A)^c$ qed

(ii) $\omega \in T^{-1}\bigcup_{i\in I} A_i$ iff $T(\omega) \in \bigcup_{i\in I} A_i$

iff $\exists\, i$ s.t. $T(\omega) \in A_i$ iff $\exists\, i \in I$ s.t. $\omega \in T^{-1}A_i$

iff $\omega \in \bigcup_{i\in I} T^{-1}A_i$ qed.

(iii) $\omega \in T^{-1}\bigcap_{i\in I} A_i$ iff $T(\omega) \in \bigcap_{i\in I} A_i$

iff $T(\omega) \in A_i\ \forall i \in I$ iff $\omega \in T^{-1}A_i\ \forall i \in I$

iff $\omega \in \bigcap_{i\in I} T^{-1}A_i$

(3) There is one joint distribution of (X,Y) and two marginal distributions, one for X and one for Y. The marginal distributions of a $N_2(\mu, \Sigma)$ were determined in class. When $(X,Y) \sim N_2(\underset{\sim}{0}, \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix})$ then $X \sim N(0,1)$ and $Y \sim N(0,1)$. Therefore, since these distributions are consistent by KCT this is a valid definition of a stochastic process.

(4) The density of $\underset{\sim}{x}$ is $(2\pi)^{-k/2}(\det\Sigma)^{-\frac{1}{2}}\exp\{-\frac{1}{2}(\underset{\sim}{x}-\underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})\}$ and the transformation is $T(\underset{\sim}{x}) = B^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})$ with inverse $T^{-1}(\underset{\sim}{z}) = \underset{\sim}{\mu} + B\underset{\sim}{z}$. Therefore $J_T(\underset{\sim}{z}) = |\det B^{-1}|^{-1} = |\det B| = (\det\Sigma)^{1/2}$. The density of $\underset{\sim}{z}$ is thus given by

$$(2\pi)^{-k/2}(\det \Sigma)^{-1/2}\exp\{-\tfrac{1}{2}(T^{-1}(\underset{\sim}{z})-\underset{\sim}{\mu})'\Sigma^{-1}(T^{-1}(\underset{\sim}{z})-\underset{\sim}{\mu})\} \times (\det \Sigma)^{1/2}$$

$$= (2\pi)^{-k/2}\exp\{-\tfrac{1}{2}\underset{\sim}{z}'B'\Sigma^{-1}B\underset{\sim}{z}\}$$

$$= (2\pi)^{-k/2}\exp\{-\tfrac{1}{2}\underset{\sim}{z}'\underset{\sim}{z}\}$$

since $B'\Sigma^{-1}B = B'(BB')^{-1}B$
$= (B')(B')^{-1}B^{-1}B = \underline{I}$

Since this is the density of the $N_k(\underset{\sim}{0}, I)$ distribution we have $\underset{\sim}{z} \sim N_k(\underset{\sim}{0}, I)$.

5. $p_{\underset{\sim}{y}}(x_1, \ldots, x_{l-1}, y)$ where $0 \le x_i$, $0 \le y$ and $x_1 + \cdots + x_{l-1} + y = n$

$$= \sum_{\substack{x_l \ge 0, \ldots, x_k \ge 0 \\ x_l + \cdots + x_k = y}} \binom{n}{x_1 \cdots x_k} p_1^{x_1} \cdots p_{l-1}^{x_{l-1}} p_l^{x_l} \cdots p_k^{x_k}$$

$$= \frac{n!}{x_1! \cdots x_{l-1}!\, y!} p_1^{x_1} \cdots p_{l-1}^{x_{l-1}} \sum_{\substack{x_l \ge 0, \ldots, x_k \ge 0 \\ x_l + \cdots + x_k = y}} \binom{y}{x_l \cdots x_k} p_l^{x_l} \cdots p_k^{x_k}$$

$$= \binom{n}{x_1 \cdots x_{l-1}\, y} p_1^{x_1} \cdots p_{l-1}^{x_{l-1}} (p_l + \cdots + p_k)^y$$

since $\ast = (p_l + \cdots + p_k)^y$ by the multinomial theorem

$$(a_1 + \cdots + a_k)^n = \sum_{\substack{x_1 \ge 0, \ldots, x_k \ge 0 \\ x_1 + \cdots + x_k = n}} \binom{n}{x_1 \cdots x_k} a_1^{x_1} \cdots a_k^{x_k}$$

6) By # 5

$p(x_\ell, \ldots, x_k \mid \underset{\sim}{Y} = y)$

$$= \frac{P(X_\ell = x_\ell, \ldots, X_k = x_k, \underset{\sim}{Y} = y)}{P(\underset{\sim}{Y} = y)}$$

$$= \frac{P(X_1 = x_1, \ldots, X_{\ell-1} = x_{\ell-1}, X_\ell = x_\ell, \ldots, X_k = x_k)}{P(\underset{\sim}{Y} = y)}$$

$$= \frac{\binom{n}{x_1 \cdots x_k} p_1^{x_1} \cdots p_k^{x_k}}{\binom{n}{x_1 \cdots x_{\ell-1}\, y} p_1^{x_1} \cdots p_{\ell-1}^{x_{\ell-1}} (p_{\ell} + \cdots + p_k)^y}$$

$$= \binom{y}{x_\ell \cdots x_k} \left(\frac{p_\ell}{p_\ell + \cdots + p_k}\right)^{x_\ell} \cdots \left(\frac{p_k}{p_\ell + \cdots + p_k}\right)^{x_k}$$

and the result follows since $y = n - x_1 - \cdots - x_{\ell-1}$ and $p_\ell + \cdots + p_k = 1 - p_1 - \cdots - p_{\ell-1}$.

7. If $F(x,y) = \begin{cases} 1 - e^{-xy} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

and F is the cdf of (absolutely) continuous r.v.'s (X,Y) then the joint density is given by $f(x,y) = \partial^2 F(x,y)/\partial x \partial y = \begin{cases} (1-xy)e^{-xy} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

but this is not always nonnegative and so F can't be the cdf of an (absolutely) cont. pair (X,Y).

8 2.7.9 $P(X^+ \le x) = P(\max(0,X) \le x)$

$$= \begin{cases} 0 & x < 0 \\ F(x) & x \geq 0 \end{cases}$$

$$P(X^- \le x) = P(\min(0,X) \geq -x)$$

$$= 1 - P(\min(0,X) < -x)$$

$$= 1 - \begin{cases} 1 & x < 0 \\ F(-x-0) & x \geq 0 \end{cases}$$

where $F(x-0) = \lim_{\epsilon \downarrow 0} F(x-\epsilon)$

$$= \begin{cases} 0 & x < 0 \\ 1 - F(-x-0) & x \geq 0 \end{cases}$$

$$P(|X| \le x) = \begin{cases} 0 & x < 0 \\ P(-x \le X \le x) & x \geq 0 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ P(X \le x) - P(X < -x) & x \geq 0 \end{cases}$$

$$= \begin{cases} 0, & x < 0 \\ F(x) - F(-x-0), & x \geq 0 \end{cases}$$

9. 2.7.16

Consider the following probabilities where coin is picked by 1st person to pick.

i	j	P(" winning by going 1st)	P(winning by going 2nd")
1	2	$3/5$	$2/5$
1	3	$3/5 \cdot 3/5$	$1-(3/5 \cdot 3/5) > 3/5 \cdot 3/5$
2	1	$2/5$	$3/5 > 2/5$
2	3	$3/5 \cdot 3/5 + 2/5 \cdot 3/5 = 3/5$	$2/5$
3	1	$3/5 \cdot 2/5 + 2/5$	$1-(3/5 \cdot 2/5)$
3	2	$2/5$	$3/5 > 2/5$

So by going second you can always pick a coin that makes your probability of winning higher than your opponents.

10. 2.7.20

(a) $$P(U=V) = \int_{\{(x,y):x=y\}} f_{(U,V)}(u,v)\,du\,dv$$

= volume under the graph of $f_{(U,V)}$ over the line $\{(x,y): x=y\}$

$= 0$ since there is no volume.

(b) There is no contradiction because the joint distribution of (X,Y) is concentrated on the line $\{(x,y): x=y\}$ and so (X,Y) cannot have an absolutely continuous distribution in $\mathbb{R}^2$.