STAC58 : 2017 Assignment 1 - Solutions

1 (a) $f_X(0) = \frac{3}{10}$, $f_X(1) = \frac{3}{10}$, $f_X(-1) = \frac{4}{10}$

(b) $\mu_X = (-1)\frac{4}{10} + (0)\frac{3}{10} + (1)\frac{3}{10} = -\frac{1}{10}$

$$\sigma_X^2 = E(X^2) - \mu_X^2 = (-1)^2\frac{4}{10} + (0)^2\frac{3}{10} + (1)^2\frac{3}{10} - \left(-\frac{1}{10}\right)^2$$

$$= \frac{4}{10} + \frac{3}{10} - \frac{1}{100} = \frac{69}{100}$$

(c) $$f_{Y|X}(0|-1) = \frac{\#(\{2,4,6,8,10\} \cap \{3,6,7,9\})}{10} \Big/ f_X(-1)$$

$$= \frac{\#(\{6\})}{10} \Big/ f_X(-1) = \frac{1}{4}$$

$$f_{Y|X}(0|0) = \frac{\#(\{2,4,6,8,10\} \cap \{1,5,10\})}{10} \Big/ f_X(0)$$

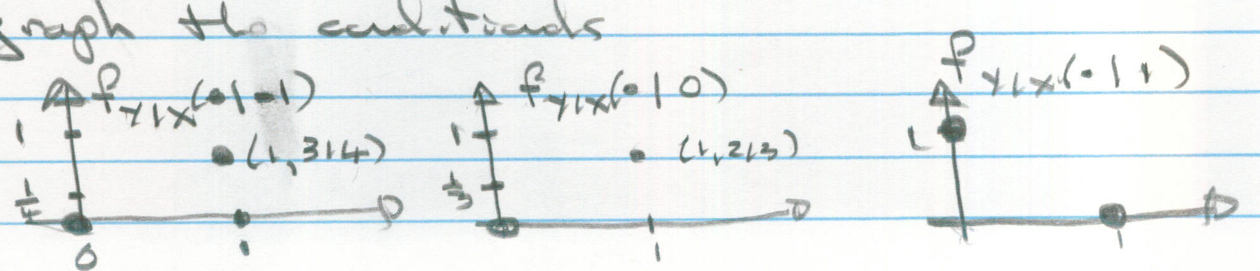
$$= \frac{1}{10} \Big/ \frac{3}{10} = \frac{1}{3}$$

$$f_{Y|X}(0|1) = \frac{\#(\{2,4,6,8,10\} \cap \{2,4,8\})}{10} \Big/ f_X(1)$$

$$= \frac{3}{10} \Big/ \frac{3}{10} = 1$$

and note $f_{Y|X}(1|x) = 1 - f_{Y|X}(0|x)$.

Since $f_{Y|X}(\cdot|x)$ changes as x changes Y and X are related. To see the relationship graph the conditionals



2d) $\mu_X \overset{\text{def}}{=} \sum_{\omega \in \Omega} X(\omega) / \#(\Omega)$

$$= \sum_{x \in \mathbb{R}} \sum_{\omega : X(\omega) = x} \frac{X(\omega)}{\#(\Omega)}$$

$$= \sum_{x \in \mathbb{R}} x \sum_{\omega : X(\omega) = x} \frac{1}{\#(\Omega)} = \sum_{x \in \mathbb{R}} x \frac{\#\{\omega : X(\omega) = x\}}{\#(\Omega)}$$

$$= \sum_x x f_X(x)$$

3.) Suppose $f_{(X,Y)}(x,y) = f_X(x) f_Y(y)$ $\forall x, y$.
Then $f_{Y|X}(y|x) = \frac{\#(\{\omega : X(\omega) = x, Y(\omega) = y\})}{\#(\{\omega : X(\omega) = x\})}$

$$= f_{(X,Y)}(x,y) / f_X(x) = f_Y(y)$$

which is constant in x and so X and Y are not related. Suppose now X and Y are not related. Then $f_{Y|X}(y|x) = g(y)$ for some function g.
Thus $f_{(X,Y)}(x,y) = \frac{\#\{\omega : X(\omega) = x, Y(\omega) = y\}}{\#(\Omega)}$

$$= \frac{\#\{\omega : X(\omega) = x, Y(\omega) = y\}}{\#\{\omega : X(\omega) = x\}} \frac{\#\{\omega : X(\omega) = x\}}{\#(\Omega)}$$

$$= f_{Y|X}(y|x) f_X(x) = f_X(x) g(y)$$

and $f_Y(y) = \frac{\#\{\omega : Y(\omega) = y\}}{\#(\Omega)} = \sum_x \frac{\#\{\omega : X(\omega) = x, Y(\omega) = y\}}{\#(\Omega)}$

$$= \sum_x f_{(X,Y)}(x,y) = g(y) \sum_x f_X(x) = g(y)$$

since $\sum_x f_X(x) = 1$. Therefore $f_{(X,Y)}(x,y) = f_X(x) f_Y(y)$.

(a) Generating the initial sample of 10 and computing the skewness statistic

```
> x=rnorm(10,5, 1.414214)
> xbar=mean(x)
> r=x-xbar
> ss=t(r)%*%r
> r=r/sqrt(ss)
> sum(r)
[1] -7.415943e-16
> t(r)%*%r
     [,1]
[1,]    1
>
> r3=r**3
> skew=sum(r3)/10
> skew
[1] 0.02933093
```

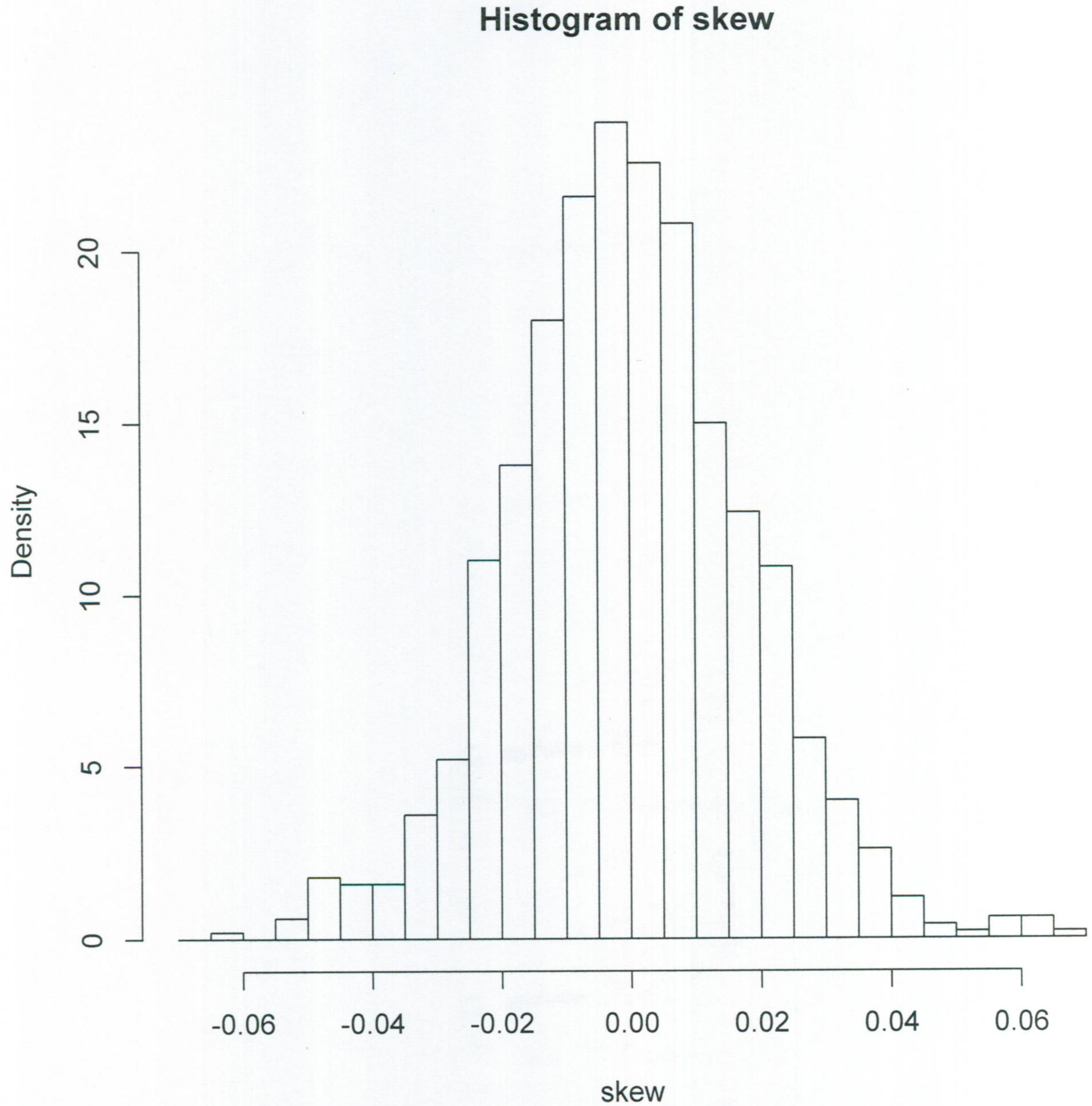
generate a sample of 10 from N(5,2) and compute skewness statistic

(b)

```
> # create a 1000X10 matrix where each row is a ssmple of 10 from a N(0,1)
> X=matrix(c(1:10000),nrow=1000,ncol=10)
> for (i in c(1:1000)) {X[i,]=rnorm(10,0,1)}
> # create a vector xbar of the 1000 sample means
> one=c(1:10)
> one=1+0*one
> one
 [1] 1 1 1 1 1 1 1 1 1 1
> xbar=X%*%one/10
> # create U a matrix of unit residuals from X
> Y=X-xbar%*%t(one)
> ss=diag(Y%*%t(Y))
> ss=sqrt(ss)
> ss=1/ss
> D=diag(ss)
> U=D%*%Y
> # cube the elements of U and calculate the individual skewness statistics
> U3=U**3
> skew=U3%*%one/10
> # plot a density histogram of the 1000 vales of skew
> max(skew)
[1] 0.06141137
> min(skew)
[1] -0.06906074
> boundaries <- seq(-.07,.07,by=.005)
>hist(skew,breaks=boundaries,freq=FALSE)
```

Plot the density histogram of skewness statistic (next page)

Note that the observed value of (from (a)) 0.02933093 is in the middle of this distribution so no indication model is wrong.



(c)

```
>x=rexp(10)
>xbar=mean(x)
> r=x-xbar
> ss=t(r)%*%r
> r=r/sqrt(ss)
> sum(r)
[1] 2.081668e-17
> t(r)%*%r
     [,1]
[1,]    1
> r3=r**3
> skew=sum(r3)/10
> skew
[1] 0.005190613
```

When generating from exp(1) distribution the observed value of skew is in the center even though the model is wrong!

0.06664907

When I generated from an exp(100) distribution, however, the observed value of skew is way out in the tails which indicates model is wrong

5. With the information generator discussed in class the unconditional prob. that III will be reported by the jailer is

$$1\cdot\frac{1}{3} + 0\cdot\frac{1}{3} + p\cdot\frac{1}{3} = (1+p)/3$$

and the unconditional prob. that I will live and III be reported is $p\frac{1}{3}$. Therefore the cond. prob. that I will live is $(p\cdot 1/3)\,/\,(1+p)\frac{1}{3} = p/(1+p)$.