STAC62F: 2016 Assignment 3

1. Suppose $\underset{\sim}{x}$ is a random vector in $\mathbb{R}^k$ with mean μ. Show that $E(\underset{\sim}{a} + B\underset{\sim}{x}) = \underset{\sim}{a} + BE(\underset{\sim}{x})$ for any fixed $B \in \mathbb{R}^{\ell \times k}$, $\underset{\sim}{a} \in \mathbb{R}^{\ell}$.

2. Suppose $X \in \mathbb{R}^{m \times n}$ is random with $E(X) = M \in \mathbb{R}^{m \times n}$. For $A \in \mathbb{R}^{\ell \times m}$, $B \in \mathbb{R}^{n \times k}$ and $C \in \mathbb{R}^{\ell \times k}$ fixed, prove that $E(AXB + C) = AMB + C$.

3. Suppose that $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$ is a sample (i.i.d.) from a distribution on $\mathbb{R}^k$ with mean μ and variance Σ.
   (a) If $\bar{\underset{\sim}{x}} = \frac{1}{n} \sum_{i=1}^{n} \underset{\sim}{x}_i$, then prove that $E(\bar{\underset{\sim}{x}}) = \underset{\sim}{\mu}$.
   (b) Prove that $Var(\bar{\underset{\sim}{x}}) = \frac{1}{n} \Sigma$.
   (c) Put $X = (\underset{\sim}{x}_1\, \underset{\sim}{x}_2 \cdots \underset{\sim}{x}_n)' \in \mathbb{R}^{n \times k}$ and $S = \frac{1}{n-1}(X - \underset{\sim}{1}_n \bar{\underset{\sim}{x}}')'(X - \underset{\sim}{1}_n \bar{\underset{\sim}{x}}')$. Prove that $E(S) = \Sigma$ (Show first $X'X = \sum_{i=1}^{n} \underset{\sim}{x}_i \underset{\sim}{x}_i'$ and $X' \underset{\sim}{1}_n \bar{\underset{\sim}{x}}' = n \bar{\underset{\sim}{x}} \bar{\underset{\sim}{x}}'$).

4. Suppose $X \sim N(0,1)$. Determine $E(X^2)$ two ways. First compute $\int_{-\infty}^{\infty} x^2 f_X(x)\, dx$ directly and second determine the density of $Y = X^2$ and calculate $\int_0^{\infty} y f_Y(y)\, dy$.

5. Suppose that $\{X_t : t \in \mathbb{R}\}$ is such that $\mu(t) = E(X_t)$, $\sigma(t,s) = v^2 \exp\{-\frac{(t-s)^2}{2\ell^2}\}$ and for any $t_1, \ldots, t_k \in \mathbb{R}$ and $k \in \mathbb{R}$

$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_k} \end{pmatrix} \sim N_k\left(\begin{pmatrix} \mu(t_1) \\ \vdots \\ \mu(t_k) \end{pmatrix}, \Sigma(t_1, \ldots, t_k) \right)$$

where $\Sigma(t_1, \ldots, t_n) \equiv (\sigma(t_i, t_j))$. Using the fact that $\exp\{-t^2/2\} = \int_{-\infty}^{\infty} \exp\{ixt\}\varphi(x)dx$ where φ is $N(0,1)$ density and $i = \sqrt{-1}$ prove that this defines a stochastic process (called a Gaussian process with mean fn μ and covariance fn σ.)

Hint 1. Recall properties of complex numbers $z = x+iy$ for $x, y \in \mathbb{R}$, namely $\bar{z} = x - iy$, $|z|^2 = x^2+y^2 = z\bar{z}$ and $\exp\{ix\} = \cos x + i \sin x$.

Hint 2. Prove $\underline{a}'\Sigma(t_1, \ldots, t_n)\underline{a} > 0$ for any $\underline{a} \in \mathbb{R}^n \setminus \{0\}$ and conclude that $\Sigma(t_1, \ldots, t_n)$ is a variance matrix using the spectral decomposition of a positive definite matrix. Finally apply KCT.

6. 3.2.1

7. 3.5.4

8. 3.6.5

9. 3.7.4

10. 4.5.4