STAD57: Time Series Analysis
Problem Set 3 Solutions

1. Exercise 3.1 from the textbook.

SOL:

We have $X_t = W_t + \theta W_{t-1}$, so that

$$\begin{aligned}\gamma_X(h) &= Cov\left(X_{t+h}, X_t\right) = Cov\left(W_{t+h} + \theta W_{t+h-1}, W_t + \theta W_{t-1}\right) = \\ &= Cov\left(W_{t+h}, W_t + \theta W_{t-1}\right) + \theta Cov\left(W_{t+h-1}, W_t + \theta W_{t-1}\right) = \\ &= Cov\left(W_{t+h}, W_t\right) + \theta Cov\left(W_{t+h}, W_{t-1}\right) + \theta Cov\left(W_{t+h-1}, W_t\right) + \theta^2 Cov\left(W_{t+h-1}, W_{t-1}\right) = \\ &= \begin{cases} \sigma_W^2(1+\theta^2), & h=0 \\ \sigma_W^2\theta, & h=1 \\ 0, & h\ge 2 \end{cases} \Rightarrow \rho_X(h) = \begin{cases} 1, & h=0 \\ \theta/(1+\theta^2), & h=1 \\ 0, & h\ge 2 \end{cases}\end{aligned}$$

To find the maximum/minimum of $\rho_X(1)$ we differentiate it w.r.t. to θ and set to 0:

$$\frac{d}{d\theta}\rho_X(h) = \frac{d}{d\theta}\left(\frac{\theta}{1+\theta^2}\right) = \frac{(1+\theta^2) - \theta(2\theta)}{(1+\theta^2)^2} = \frac{1-\theta^2}{(1+\theta^2)^2} = 0 \Rightarrow \theta = \pm 1$$

Substituting $\theta = \pm 1$ into $\rho_X(1)$ we get that the maximum/minimum values are:

$$\rho_X(1) = \frac{\theta}{1+\theta^2} = \frac{\pm 1}{1+(\pm 1)^2} = \pm\frac{1}{2}$$

(Note: to be technically correct we also have to check that the 2nd derivatives are negative/positive at the maximum/minimum, but you can check this is the case).

2. Exercise 3.2 from textbook.

SOL:

We have $\begin{cases} X_1 = W_1 \\ X_t = \varphi X_{t-1} + W_t, t \ge 2 \end{cases} \Rightarrow X_t = \varphi X_{t-1} + W_t = \varphi\left(\varphi X_{t-2} + W_{t-1}\right) + W_t =$

$$= \varphi^2\left(\varphi X_{t-3} + W_{t-2}\right) + \varphi W_{t-1} + W_t = \cdots = \varphi^{t-1}\underbrace{X_1}_{=W_1} + \varphi^{t-2}W_2 + \cdots + \varphi W_{t-1} + W_t \Rightarrow$$

$$\Rightarrow X_t = \varphi^{t-1}W_1 + \varphi^{t-2}W_2 + \cdots + \varphi W_{t-1} + W_t = \sum_{j=0}^{t-1}\varphi^j W_{t-j}$$

a) $\mathbb{E}\left[X_t\right] = \mathbb{E}\left[\sum_{j=0}^{t-1}\varphi^j W_{t-j}\right] = \sum_{j=0}^{t-1}\varphi^j \underbrace{\cancel{\mathbb{E}\left[W_{t-j}\right]}}_{=0} = 0$

$$\mathbb{V}\left[X_t\right] = \mathbb{V}\left[\sum_{j=0}^{t-1}\varphi^j W_{t-j}\right] = \sum_{j=0}^{t-1}\varphi^{2j}\underbrace{\mathbb{V}\left[W_{t-j}\right]}_{=\sigma_W^2} = \sigma_W^2\sum_{j=0}^{t-1}\varphi^{2j} = \sigma_W^2\frac{1-\varphi^{2t}}{1-\varphi^2}.$$

Since the variance depends on t, the series is not stationary.

b) $Cor(X_t, X_{t-h}) = Cor\left(\sum_{j=0}^{t-1} \varphi^j W_{t-j}, \sum_{k=0}^{t-1} \varphi^k W_{t-h-k}\right) = \sum_{j=0}^{t-1} \sum_{k=0}^{t-h-1} \varphi^j \varphi^k \underbrace{Cov\left(W_{t-j}, W_{t-h-k}\right)}_{=\begin{cases}\sigma_W^2, & j=h+k \\ 0, & j\neq h+k\end{cases}} =$

$$= \sigma_W^2 \sum_{k=0}^{t-h-1} \varphi^{k+h} \varphi^k = \varphi^h \left(\sigma_W^2 \sum_{k=0}^{t-h-1} \varphi^{2k}\right) = \varphi^h \left(\sigma_W^2 \frac{1-\varphi^{2(t-h)}}{1-\varphi^2}\right) = \varphi^h \mathbb{V}[X_{t-h}] \Rightarrow$$

$$\Rightarrow Cor(X_t, X_{t-h}) = \frac{Cov(X_t, X_{t-h})}{\sqrt{\mathbb{V}[X_t]\mathbb{V}[X_{t-h}]}} = \frac{\varphi^h \mathbb{V}[X_{t-h}]}{\sqrt{\mathbb{V}[X_t]\mathbb{V}[X_{t-h}]}} = \varphi^h \sqrt{\frac{\mathbb{V}[X_{t-h}]}{\mathbb{V}[X_t]}}$$

c) Simply take limits of the above expressions as $t \to \infty$. We have:

$$\lim_{t\to\infty} \mathbb{V}[X_t] = \lim_{t\to\infty}\left(\sigma_W^2 \frac{1-\varphi^{2t}}{1-\varphi^2}\right) = \sigma_W^2 \frac{1-\lim_{t\to\infty}\left(\varphi^{2t}\right)}{1-\varphi^2} = \sigma_W^2 \frac{1}{1-\varphi^2} \text{ (since } |\varphi|<1)$$

$$\lim_{t\to\infty} Cor(X_t, X_{t-h}) = \lim_{t\to\infty}\left(\sigma_W^2 \varphi^h \frac{1-\varphi^{2(t-h)}}{1-\varphi^2}\right) = \sigma_W^2 \varphi^h \frac{1-\lim_{t\to\infty}\left(\varphi^{2(t-h)}\right)}{1-\varphi^2} = \sigma_W^2 \frac{\varphi^h}{1-\varphi^2}.$$

d) To generate n values from the Gaussian AR(1) series, you can iteratively generate $N \gg n$ values starting from $X_1 = W_1 = \sigma_W Z_1 \sim N(0, \sigma_W^2)$ and using $X_t = \varphi X_{t-1} + W_t = \varphi X_{t-1} + \sigma_W Z_t, \ \forall t \geq 2$, where $\sigma_W Z_t = W_t \sim N(0, \sigma_W^2)$, but only keep the last n values.

e) If you start with $X_t = \frac{W_1}{\sqrt{1+\varphi^2}} \sim N\left(0, \frac{\sigma_W^2}{1-\varphi^2}\right)$, the resulting series will be stationary. We have

$X_t = \varphi^{t-1} W_1 + \varphi^{t-2} W_2 + \cdots + \varphi W_{t-1} + W_t = \varphi^{t-1} W_1 + \sum_{j=0}^{t-2} \varphi^j W_{t-j}$. The mean will trivially be zero again, but the variance and covariances will be given by:

$$\mathbb{V}[X_t] = \mathbb{V}\left[\varphi^{t-1} X_1 + \sum_{j=0}^{t-2} \varphi^j W_{t-j}\right] = \varphi^{t-1}\mathbb{V}[X_1] + \sum_{j=0}^{t-2} \varphi^j \mathbb{V}\left[W_{t-j}\right] =$$

$$= \sigma_W^2 \frac{\varphi^{t-1}}{1-\varphi^2} + \sigma_W^2 \sum_{j=0}^{t-2} \varphi^j = \sigma_W^2 \left(\frac{\varphi^{t-1}}{1-\varphi^2} + \frac{1-\varphi^{t-1}}{1-\varphi^2}\right) = \frac{\sigma_W^2}{1-\varphi^2} \text{ (indep. of } t)$$

$$Cor(X_t, X_{t-h}) = Cor\left(\varphi^{t-1} X_1 + \sum_{j=0}^{t-2} \varphi^j W_{t-j}, \varphi^{t-h-1} X_1 + \sum_{k=0}^{t-h-2} \varphi^k W_{t-h-k}\right) =$$

$$= \varphi^{t-1}\varphi^{t-h-1} \underbrace{Cov(X_1, X_1)}_{=\mathbb{V}[X_1]} + \sum_{j=0}^{t-2} \sum_{k=0}^{t-h-2} \varphi^j \varphi^k \underbrace{Cov\left(W_{t-j}, W_{t-h-k}\right)}_{=\begin{cases}\sigma_W^2, & j=h+k \\ 0, & j\neq h+k\end{cases}} \Rightarrow$$

$$Cor(X_t, X_{t-h}) = \sigma_W^2 \varphi^h \frac{\varphi^{2(t-h-1)}}{1-\varphi^2} + \sigma_W^2 \sum_{k=0}^{t-h-2} \varphi^{k+h}\varphi^k = \sigma_W^2 \varphi^h \left(\frac{\varphi^{2(t-h-1)}}{1-\varphi^2} + \sum_{k=0}^{t-h-2} \varphi^{2k}\right) =$$

$$= \sigma_W^2 \varphi^h \left(\frac{\varphi^{2(t-h-1)}}{1-\varphi^2} + \frac{1-\varphi^{2(t-h-1)}}{1-\varphi^2}\right) = \sigma_W^2 \frac{\varphi^h}{1-\varphi^2} \text{ (indep. of } t)$$

$$\Rightarrow Cor(X_t, X_{t-h}) = \frac{Cov(X_t, X_{t-h})}{\sqrt{\mathbb{V}[X_t]\mathbb{V}[X_{t-h}]}} = \frac{\sigma_W^2 \frac{\varphi^h}{1-\varphi^2}}{\sigma_W^2 \frac{1}{1-\varphi^2}} = \varphi^h, \ \forall h \geq 0$$

3. Exercise 3.4 from the textbook.

SOL:

a) $X_t = .8X_{t-1} - .15X_{t-2} + W_t - .3W_{t-1} \Leftrightarrow X_t - .8X_{t-1} + .15X_{t-2} = W_t - .3W_{t-1} \Leftrightarrow$

$\Leftrightarrow (1-.8B+.15B^2)X_t = (1-.3B)W_t \Leftrightarrow (1-.5B)\cancel{(1-.3B)}X_t = \cancel{(1-.3B)}W_t \Leftrightarrow$

$\Leftrightarrow (1-.5B)X_t = W_t \Leftrightarrow X_t = .5X_{t-1} + W_t$

which is an AR(1) model with $\varphi_1 = .5$. Since $|\varphi_1| < 1$, the model is causal (→stationary) and invertible (b/c all pure AR models are invertible).

b) $X_t = X_{t-1} - .5X_{t-2} + W_t - W_{t-1} \Leftrightarrow X_t - X_{t-1} + .5X_{t-2} = W_t - W_{t-1} \Leftrightarrow \underbrace{(1-B+.5B^2)}_{\varphi(B)}X_t = \underbrace{(1-B)}_{\theta(B)}W_t$

The roots of the AR polynomial are given by:

$$\varphi(z) = 0 \Rightarrow .5z^2 - z + 1 = 0 \Rightarrow z = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(.5)(1)}}{2(.5)}$$

$$\Rightarrow z = \frac{1 \pm \sqrt{1-2}}{1} = 1 \pm \sqrt{-1} = 1 \pm i$$

These are conjugate complex roots with norm $|1 \pm i| = \sqrt{1^2+1^2} = \sqrt{2} > 1$, so the model is causal. However, the root of the MA polynomial $\theta(z) = 1 - z = 0$ is trivially $z = 1$, which lies *on* the unit circle → the model is *not* invertible.

4. Exercise 3.5 from the textbook.

SOL:

For the AR(2) model $\overbrace{(1-\varphi_1 B - \varphi_2 B^2)}^{=\varphi(B)}X_t = W_t$ to be stationary, we need the roots of the characteristic equation $1 - \varphi_1 z - \varphi_2 z^2 = 0$ to be outside the unit disk. Let z_1, z_2 be the possibly complex roots; we can equivalently write the equation as $1 - \varphi_1 z - \varphi_2 z^2 = \left(1 - z_1^{-1}z\right)\left(1 - z_2^{-1}z\right) =$ $= 1 - (z_1^{-1} + z_2^{-1})z + (z_1^{-1}z_2^{-1})z^2$, from which we get that $\varphi_1 = z_1^{-1} + z_2^{-1}$ & $\varphi_2 = -z_1^{-1}z_2^{-1}$. We want $|z_1|, |z_2| > 1 \Rightarrow |z_1^{-1}|, |z_2^{-1}| < 1$, which implies:

$$\varphi_1 = z_1^{-1} + z_2^{-1} \Rightarrow |\varphi_1| = |z_1^{-1} + z_2^{-1}| \le |z_1^{-1}| + |z_2^{-1}| < 1 + 1 = 2$$

$$\varphi_2 = -z_1^{-1}z_2^{-1} \Rightarrow |\varphi_2| = |-z_1^{-1}z_2^{-1}| \le |z_1^{-1}||z_2^{-1}| < 1$$

Moreover, the roots are given by $z = \dfrac{\varphi_1 \pm \sqrt{\varphi_1^2 + 4\varphi_2}}{-2\varphi_2}$, and they are real provided that

$\varphi_1^2 + 4\varphi_2 \ge 0 \Rightarrow \varphi_2 \ge -\dfrac{\varphi_1^2}{4}$. Assuming we have two real roots, let $z_1 \le z_2$ without loss of generality. For both roots to be outside the unit circle, we have 3 cases $\begin{cases} a: \quad 1 < z_1 \le z_2 \\ b: \quad z_1 \le z_2 < -1 \\ c: z_1 < -1 \,\&\, 1 < z_2 \end{cases}$. Let

$\varphi_2 < 0$, so that $z_1 = \dfrac{\varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_2}}{-2\varphi_2} \le \dfrac{\varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_2}}{-2\varphi_2} = z_2$; the 3 case become:

$$a.\ 1 < z_1 \Rightarrow 1 < \frac{\varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_2}}{-2\varphi_2} \Rightarrow \sqrt{\varphi_1^2 + 4\varphi_2} < \varphi_1 + 2\varphi_2 \Rightarrow \varphi_1^2 + 4\varphi_2 < \left(\varphi_1 + 2\varphi_2\right)^2 \Rightarrow$$

$$\Rightarrow \cancel{\varphi_1^2} + \cancel{4}\varphi_2 < \cancel{\varphi_1^2} + \cancel{4}\varphi_1\varphi_2 + \cancel{4}\varphi_2^2 \overset{(\text{div } \varphi_2)}{\Rightarrow} \boxed{1 > \varphi_1 + \varphi_2}$$

$$b.\ z_2 < -1 \Rightarrow \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2}}{-2\varphi_2} < -1 \Rightarrow \sqrt{\varphi_1^2 + 4\varphi_2} < 2\varphi_2 - \varphi_1 \Rightarrow \varphi_1^2 + 4\varphi_2 < \left(2\varphi_2 - \varphi_1\right)^2 \Rightarrow$$

$$\Rightarrow \cancel{\varphi_1^2} + \cancel{4}\varphi_2 < \cancel{4}\varphi_2^2 - \cancel{4}\varphi_1\varphi_2 + \cancel{\varphi_1^2} \overset{(\text{div } \varphi_2)}{\Rightarrow} \boxed{1 > \varphi_2 - \varphi_1}$$

$$c.\ z_2 > 1 \Rightarrow \frac{\varphi_1 + \sqrt{\varphi_1^2 + 4\varphi_2}}{-2\varphi_2} > 1 \Rightarrow \sqrt{\varphi_1^2 + 4\varphi_2} > 2\varphi_2 + \varphi_1 \Rightarrow \varphi_1^2 + 4\varphi_2 > \left(2\varphi_2 + \varphi_1\right)^2 \Rightarrow$$

$$\Rightarrow \cancel{\varphi_1^2} + \cancel{4}\varphi_2 > \cancel{4}\varphi_2^2 + \cancel{4}\varphi_1\varphi_2 + \cancel{\varphi_1^2} \overset{(\text{div } \varphi_2)}{\Rightarrow} \boxed{1 < \varphi_2 - \varphi_1},\ AND$$

$$z_1 < -1 \Rightarrow \frac{\varphi_1 - \sqrt{\varphi_1^2 + 4\varphi_2}}{-2\varphi_2} < -1 \Rightarrow \sqrt{\varphi_1^2 + 4\varphi_2} > 2\varphi_2 + \varphi_1 \Rightarrow \varphi_1^2 + 4\varphi_2 > \left(2\varphi_2 + \varphi_1\right)^2 \Rightarrow$$

$$\Rightarrow \cancel{\varphi_1^2} + \cancel{4}\varphi_2 > \cancel{4}\varphi_2^2 + \cancel{4}\varphi_1\varphi_2 + \cancel{\varphi_1^2} \overset{(\text{div } \varphi_2)}{\Rightarrow} \boxed{1 < \varphi_2 + \varphi_1}$$

Note that for case c. we need $\left(1 < \varphi_2 + \varphi_1\right) \& \left(1 < \varphi_2 - \varphi_1\right) \Rightarrow \varphi_2 > 1$, which the assumption $\varphi_2 < 0$, so we reject it. Repeating the above analysis for $\varphi_2 > 0$, we get similar results. So, overall, we have the conditions $\left(|\varphi_2| < 1\right)$ & $\left(\varphi_2 + \varphi_1 < 1\right)$ & $\left(\varphi_2 - \varphi_1 < 1\right)$.

5. Exercise 3.6 from the textbook.

SOL:

We have $X_t = -.9X_{t-2} + W_t \Leftrightarrow X_t + .9X_{t-2} = W_t \Leftrightarrow \underbrace{(1 + 0B + .9B^2)}_{\varphi(B)} X_t = W_t$. The roots of the AR polynomial are: $\varphi(z) = 1 + .9z^2 = 0 \Rightarrow z^2 = -\dfrac{1}{.9} \Rightarrow z = \pm\dfrac{i}{\sqrt{.9}}$. The ACF is given by

$\rho(h) = -.9\rho(h-2), \forall h \geq 2$ with initial conditions $\rho(0) = 1$ $\rho(1) = -.9\rho(-1) = -.9\rho(1) \Rightarrow \rho(1) = 0$

$$\Rightarrow \begin{cases} \rho(2) = -.9\rho(0) = -.9 \\ \rho(3) = -.9\rho(1) = 0 \\ \rho(4) = -.9\rho(2) = (-.9)^2 \Rightarrow \rho(h) = \begin{cases} (-.9)^{h/2}, & h = 0, 2, 4, \ldots \\ 0, & h = 1, 3, 5, \ldots \end{cases} \\ \rho(5) = -.9\rho(3) = 0 \\ \vdots \end{cases}$$

6. Exercise 3.7 from the textbook.

SOL:

a) $X_t + 1.6X_{t-1} + .64X_{t-2} = W_t \Rightarrow X_t = -1.6X_{t-1} - .64X_{t-2} + W_t \Rightarrow \rho(h) = -1.6\rho(h-1) - .64\rho(h-2) \Rightarrow$

$\rho(0) = 1$

$$\rho(1) = -1.6\rho(0) - .64\rho(-1) = -1.6 - .64\rho(1) \Rightarrow \rho(1) = -\frac{1.6}{1.64} = -0.9756098$$

$$\rho(2) = -1.6\rho(1) - .64\rho(0) = 1.6\frac{1.6}{1.64} - .64 = 0.9209756$$

$$\rho(3) = -1.6\rho(2) - .64\rho(1) = \cdots = -0.8491707$$

$\vdots$

```
From R, we get:
> ARMAacf(ar=c(-1.6,-.64), ma=0, 10)
         0          1          2          3          4 
 1.0000000 -0.9756098  0.9209756 -0.8491707  0.7692488 
         5          6          7          8          9 
-0.6873288  0.6074068 -0.5319605  0.4623964 -0.3993796 
        10 
 0.3430736 
```

b) $X_t - .4X_{t-1} - .45X_{t-2} = W_t \Rightarrow X_t = .4X_{t-1} + .45X_{t-2} + W_t \Rightarrow \rho(h) = .4\rho(h-1) + .45\rho(h-2) \Rightarrow$

$\rho(0) = 1$

$$\rho(1) = .4\rho(0) + .45\rho(-1) = .4 + .45\rho(1) \Rightarrow \rho(1) = \frac{.4}{.55} = 0.7272727$$

$$\rho(2) = .4\rho(1) + .45\rho(0) = .4\frac{.4}{.55} - .45 = 0.7409091$$

$$\rho(3) = .4\rho(2) + .45\rho(1) = \cdots = 0.6236364$$

$\vdots$

```
> ARMAacf(ar=c(.4,.45), ma=0, 10)
        0         1         2         3         4         5 
1.0000000 0.7272727 0.7409091 0.6236364 0.5828636 0.5137818 
        6         7         8         9        10 
0.4678014 0.4183224 0.3778396 0.3393809 0.3057802 
```

c) $X_t - 1.2X_{t-1} + .85X_{t-2} = W_t \Rightarrow X_t = 1.2X_{t-1} - .85X_{t-2} + W_t \Rightarrow \rho(h) = 1.2\rho(h-1) - .85\rho(h-2) \Rightarrow$

$$\rho(0) = 1$$

$$\rho(1) = 1.2\rho(0) - .85\rho(-1) = 1.2 - .85\rho(1) \Rightarrow \rho(1) = \frac{1.2}{1.85} = 0.64864865$$

$$\rho(2) = 1.2\rho(1) - .85\rho(0) = 1.2\frac{1.2}{1.85} - .85 = -0.07162162$$

$$\rho(3) = 1.2\rho(2) - .85\rho(1) = \cdots = -0.63729730$$

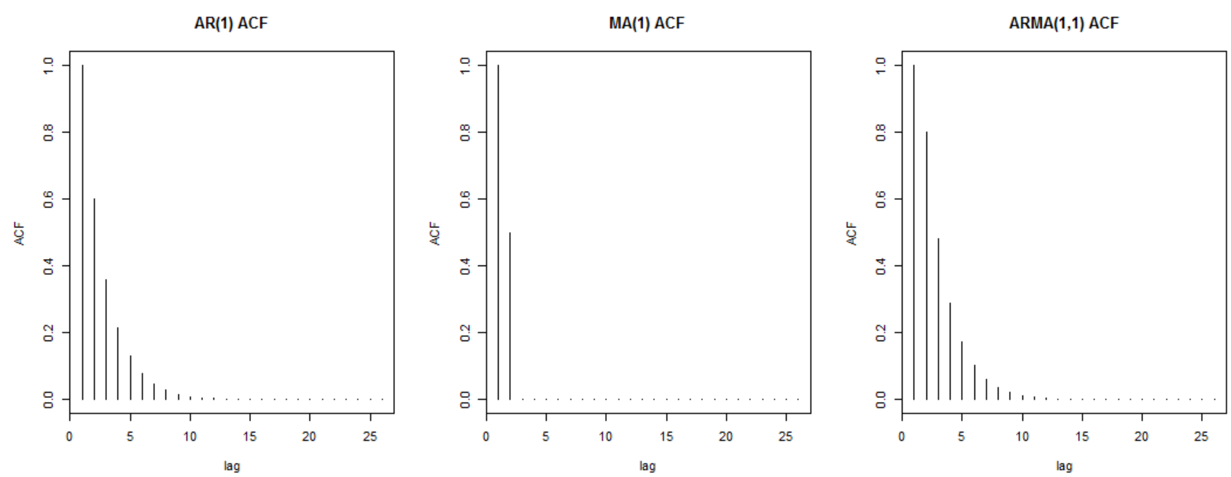
$$\vdots$$

```
> ARMAacf(ar=c(1.2,-.85), ma=0, 10)
          0           1           2           3           4
 1.00000000  0.64864865 -0.07162162 -0.63729730 -0.70387838
          5           6           7           8           9
-0.30295135  0.23475500  0.53921465  0.44751583  0.07868654
         10
-0.28596460
```

7. Exercise 3.8 from the textbook.

SOL:

The calculations of the general ARMA(1,1) ACF are verified in the example from Lecture 7, p. 14-15. The resulting ACF's for the various models using $\varphi = .6$ & $\theta = .9$ are:



Code:

```
# Q3.8

phi=.6; theta=.9
AR_1.acf=ARMAacf(ar=phi, lag=25)
MA_1.acf=ARMAacf(ma=theta, lag=25)
ARMA_1.1.acf=ARMAacf(ar=phi, ma=theta, lag=25)

par(mfrow=c(1,3))
```

```
plot(AR_1.acf, type='h', main="AR(1) ACF", xla='lag', ylab='ACF' )
plot(MA_1.acf, type='h', main="MA(1) ACF", xla='lag', ylab='ACF' )
plot(ARMA_1.1.acf, type='h', main="ARMA(1,1) ACF", xla='lag',
ylab='ACF' )
```

8. Determine whether the following AR models are stationary, and calculate the first 4 coefficients of their causal representation.

a. $X_t = -.8X_{t-1} + .4X_{t-2} + W_t$

b. $X_t = -.5X_{t-1} + .4X_{t-2} + W_t$

SOL:

a.

The AR model is stationary if its characteristic polynomial has all roots outside the unit circle.

We have $X_t + .8X_{t-1} - .4X_{t-2} = W_t \Rightarrow \varphi(B)X_t = W_t$ where $\varphi(z) = 1 + .8z - .4z^2 \Rightarrow$

$$z = \frac{-.8 \pm \sqrt{(.8)^2 - 4\times(-.4)\times 1}}{2\times(-.4)} = \frac{.8 \pm \sqrt{.64+1.6}}{.8} = 1 \pm \sqrt{\frac{2.24}{.8}} = 1 \pm \sqrt{3.5} = \begin{cases} 2.870829 \\ -0.8708287 \end{cases}$$

Since only one root is outside the unit circle, the model is not stationary, and it does not have a causal representation (i.e. you can try to invert the AR polynomial operator, but the resulting ψ weights will be exploding to ∞. The first 4 terms are $\psi_1 = -0.8,\ \psi_2 = 1.04,\ \psi_3 = -1.152,$ $\psi_4 = 1.3376,\ \psi_5 = -1.53088$.

b.

We have $X_t + .5X_{t-1} - .4X_{t-2} = W_t \Rightarrow \varphi(B)X_t = W_t$ where $\varphi(z) = 1 + .5z - .4z^2 \Rightarrow$

$$z = \frac{-.5 \pm \sqrt{(.5)^2 - 4\times(-.4)\times 1}}{2\times(-.4)} = \frac{.5 \pm \sqrt{.25+1.6}}{.8} = \frac{.5 \pm \sqrt{1.85}}{.8} = \begin{cases} 2.325184 \\ -1.075184 \end{cases}$$

Since *both* roots are outside the unit disk (i.e. $|z| > 1$), the model is stationary and its causal representation is given by:

$$\psi(B)\varphi(B) = 1 \Leftrightarrow (1 + \psi_1 B + \psi_2 B^2 + \cdots)(1 + .5B - .4B^2) = 1$$

$$\Rightarrow 1 + (\psi_1 + .5)B + (\psi_2 + .5\psi_1 - .4)B^2 + \cdots = 1$$

$$\Rightarrow \begin{cases} \psi_1 + .5 = 0 \Rightarrow \psi_1 = -.5 \\ \psi_2 + .5\psi_1 - .4 = 0 \Rightarrow \psi_2 = -.5\psi_1 + .4 = .5^2 + .4 = .65 \\ \psi_3 + .5\psi_2 - .4\psi_1 = 0 \Rightarrow \psi_3 = -.5\psi_2 + .4\psi_1 = -.5\times.65 - .4\times.5 = -.525 \\ \psi_4 + .5\psi_3 - .4\psi_2 = 0 \Rightarrow \psi_4 = -.5\psi_3 + .4\psi_2 = +.5\times.525 + .4\times.65 = .5225 \end{cases}$$