STAC62 : 2015 Assignment 4 - Solutions

1. (a) $\mu(n) = E(S_n) = E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i) = 0$,

$cov(S_m, S_n) = E((S_m - E(S_m))(S_n - E(S_n)))$
$= E(S_m S_n) = \sum_{i=1}^{m}\sum_{j=1}^{n} E(X_i X_j) = \sum_{i=1}^{\min(m,n)} E(X_i^2)$
since $E(X_i X_j) = E(X_i)E(X_j) = 0$ when $i \neq j$
Therefore, $\sigma(m,n) = \min(m,n)E(X_i^2) = \min(m,n)$.

(b) Consider $(S_1, S_2, \ldots, S_n)$
$= (X_1, X_1+X_2, \ldots, X_1+\cdots+X_n) = A\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ where

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & & & \\ 1 & 1 & \cdots & 1 \end{pmatrix} \text{ and } \underset{\sim}{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \sim N_n(\underset{\sim}{0}, I).$$

Then by result proved in class $(S_1, \ldots, S_n) \sim N_n(A\underset{\sim}{0}, AA')$ and this establishes that $\{S_n : n = 0, 1, \ldots\}$ is a Gaussian process.

2. We have $q_k = P_k(S_n = N$ for some n where $S_1, \ldots, S_{n-1} \neq 0 \mid S_0 = k) = q_{k-1}q + q_{k+1}p$ ⊛
with boundary conditions $q_0 = 0$ $q_N = 1$.
Now note $q_k \equiv 1$ and $q_k = (\frac{q}{p})^k$ both satisfy ⊛ and so a general solution is of the form $A + B(\frac{q}{p})^k$ and bdry conditions imply $A + B = 0$, $A + B(\frac{q}{p})^N = 1$.
Then provided $p \neq q$ these imply
$B = -(1-(q/p)^N)^{-1}$ and $A = (1-(q/p)^N)^{-1}$ so
$q_k = (1-(q/p)^N)^{-1}(1-(q/p)^k) \to k/N$ as $p \to \frac{1}{2}$.

(3) The question is equivalent to asking how many paths there are from 0 to $\alpha-\beta$ in $\alpha+\beta$ steps that never revisit the x-axis. From the result proved in class we know that there are $\frac{\alpha-\beta}{\alpha+\beta} N_{\alpha+\beta}(0,\alpha-\beta)$ such paths. Since there are in total $N_{\alpha+\beta}(0,\alpha-\beta)$ equally likely paths this implies that the relevant probability is $\frac{\alpha-\beta}{\alpha+\beta} N_{\alpha+\beta}(0,\alpha-\beta) / N_{\alpha+\beta}(0,\alpha-\beta) = \frac{\alpha-\beta}{\alpha+\beta}$.

4. 3.11.18

(a) $e(p) = E_p(f(x)) = \sum_{i_1=0}^{1} \cdots \sum_{i_n=0}^{1} f(i_1,\dots,i_n) \prod_{j=1}^{n} p^{i_j}(1-p)^{1-i_j}$

For $n=1$, $e(p_1) = f(0)(1-p_1) + f(1)(p_1) = f(0) + (f(1)-f(0))\,p_1$

$\le f(0) + (f(1)-f(0))\,p_2$ when $p_2 \ge p_1$ since $f(1) \ge f(0)$.

Suppose the result holds for n. Then for $n+1$

$E_p(f(\underset{\sim}{x}) \mid X_1 = x_1,\dots,X_n = x_n) = (1-p_1) f(x_1,\dots,x_n,0) +$

$p_1 f(x_1,\dots,x_n,1) \le (1-p_2) f(x_1,\dots,x_n,0) + p_2 f(x_1,\dots,x_n,1)$

(since $f(x_1,\dots,x_n,\cdot)$ is increasing. Then $e(p_1) =$

$E_{p_1}(f) = E_{p_1}(E(f(\underset{\sim}{x}) \mid X_1 = x_1,\dots,X_n = x_n))$

$= (1-p_1) E_{p_1}(f(X_1,\dots,X_n,0)) + p_1 E_{p_1}(f(X_1,\dots,X_n,1))$

$\le (1-p_1) E_{p_2}(f(X_1,\dots,X_n,0)) + p_1 E_{p_2}(f(X_1,\dots,X_n,1))$

by induction since $f(\cdot,0)$ and $f(\cdot,1)$ are coordinatewise increasing

$= E_{p_2}(f(X_1,\dots,X_n,0)) + p_1(E_{p_2}(f(X_1,\dots,X_n,1) - f(X_1,\dots,X_n,0)))$

$\le E_{p_2}(f(X_1,\dots,X_n,0)) + p_2(E_{p_2}(f(X_1,\dots,X_n,1) - f(X_1,\dots,X_n,0)))$

since $f(X_1,\dots,X_n,1) - f(X_1,\dots,X_n,0) \ge 0$

$= E_{p_2}(f)$

(b) We want to show $E(f(x)g(\underset{\sim}{x})) \ge E(f(\underset{\sim}{x}))E(g(\underset{\sim}{x}))$

When $n=1$, $E(f(x)g(x)) = (1-p) f(0) g(0) + p f(1) g(1)$

and $E(f(x))E(g(x)) = [(1-p) f(0) + p f(1)][(1-p) g(0) + p g(1)]$

$= (1-p)^2 f(0) g(0) + p(1-p)(f(0)g(1) + f(1)g(0)) + p^2 f(1) g(1)$

Then $E(f(x)g(x)) - E(f(x))E(g(x))$

$= (1-p)(1-(1-p)) f(0)g(0) - p(1-p)(f(0)g(1) + f(1)g(0)) + p(1-p) f(1)g(1)$

$= p(1-p)[f(0)g(0) - f(0)g(1) - f(1)g(0) + f(1)g(1)]$

$= p(1-p)(f(1) - f(0))(g(1) - g(0)) \ge 0.$

Now assume true for n and consider $n+1$ case. Then

$$E(f(\underset{\sim}{x})g(\underset{\sim}{x})) = E\left(E(f(\underset{\sim}{x})g(\underset{\sim}{x}) \mid x_{n+1})\right)$$
$$\geq E\left(E(f(\underset{\sim}{x}) \mid x_{n+1})E(g(\underset{\sim}{x}) \mid x_{n+1})\right)$$ by induction as in (a).

Now $E(f(\underset{\sim}{x}) \mid x_{n+1})(i) = E(f(x_1, \ldots, x_n, i))$ and since $f(x_1, \ldots, x_n, 1) \geq f(x_1, \ldots, x_n, 0)$ then $E(f(\underset{\sim}{x}) \mid x_{n+1})(i)$ is increasing in i. Therefore

$$E(E(f(\underset{\sim}{x}) \mid x_{n+1})E(g(\underset{\sim}{x}) \mid x_{n+1})) \geq E(E(f(\underset{\sim}{x}) \mid x_{n+1}))E(E(g(\underset{\sim}{x}) \mid x_{n+1}))$$
$$= E(f(\underset{\sim}{x}))E(g(\underset{\sim}{x}))$$ and we are done.

5 5.1.2 $T(s) = \sum_{n=0}^{\infty} t(n) s^n = \sum_{n=0}^{\infty} P(T>n) s^n$

$= \sum_{n=0}^{\infty} (1-P(T\le n)) s^n = \sum_{n=0}^{\infty} s^n - \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} P(X=i)\right) s^n$

$\overset{|s|<1}{=} \frac{1}{1-s} - \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} P(X=i) s^i\right) s^{n-i}$

$= \frac{1}{1-s} - \left(\sum_{n=0}^{\infty} s^n\right)\left(\sum_{n=0}^{\infty} P(X=n) s^n\right) = \frac{1}{1-s} - \frac{1}{1-s} G(s)$

$= \frac{1-G(s)}{1-s}$, so $G(s) = 1 - (1-s)T(s)$.

Now $E(X) = \lim_{s\uparrow 1} G'(s) = \lim_{s\uparrow 1} \left(T(s) - (1-s)T'(s)\right)$

$= \lim_{s\uparrow 1} T(s) \overset{\text{def}}{=} T(1)$ and

$E(X(X-1)) = \lim_{s\uparrow 1} G''(s) = \lim_{s\uparrow 1} \left(T'(s) + T'(s) - (1-s)T''(s)\right)$

$= 2T'(1)$.

Therefore, $Var(X) = E(X(X-1)) - E(X)(E(X)-1)$

$= 2T'(1) - T(1)(T(1)-1) = 2T'(1) + T(1) - T^2(1)$.

6 5.7.3

Note $K_X(t) = \ln m_X(t) = \ln(1-(1-m_X(t)))$

$$= -\sum_{k=1}^{\infty} \frac{(1-m_X(t))^k}{k} \text{ for all } t \text{ s.t. } m_X(t) \le 1$$

which is an open interval about 0

(a) Write $m_X(t) = \sum_{k=0}^{\infty} \frac{E(X^k)}{k!} t^k$

$$= 1 + E(X)t + \frac{E(X^2)}{2} t^2 + \dots$$

So $k_X(t) = -(1-m_X(t)) - \frac{(1-m_X(t))^2}{2} - \frac{(1-m_X(t))^3}{3} - \dots$

$$= \left(E(X)t + \frac{E(X^2)}{2} t^2 + \frac{E(X^3)}{3!} t^3 + \dots\right)$$

$$- \frac{\left((E(X))^2 t^2 + E(X)E(X^2)t^3 + \dots\right)}{2} + \frac{\left((E(X))^3 t^3 + \dots\right)}{3!}$$

collecting powers of t $=$ $E(X)t + \left(E(X^2) - (E(X))^2\right)\frac{t^2}{2} +$

$$\left(E(X^3) - 3!\frac{E(X)E(X^2)}{2} + 2(E(X))^3\right)\frac{t^3}{3!} + \dots$$

Therefore $K_1(X) = E(X)$

$K_2(X) = E(X^2) - (E(X))^2 = \mathrm{Var}(X)$

$K_2(X) = E(X^3) - 3E(X)E(X^2) + 2(E(X))^3$

(b) $m_{X+Y}(t) = E(e^{tX}e^{tY}) = E(e^{tX})E(e^{tY})$ since e^{tX}, e^{tY} are independent and so $m_{X+Y}(t) = m_X(t)\,m_Y(t)$ giving $K_{X+Y}(t) = K_X(t) + K_Y(t)$.

7. 5.8.6 We proved in class that $m_{\underset{\sim}{x}}(t) = \exp\{\underset{\sim}{t}'\underset{\sim}{\mu} + \frac{1}{2}\underset{\sim}{t}'\Sigma\underset{\sim}{t}\}$ when $\underset{\sim}{x} \sim N_n(\underset{\sim}{\mu}, \Sigma)$ and so $c_{\underset{\sim}{x}}(\underset{\sim}{t}) = \exp\{i\underset{\sim}{t}'\underset{\sim}{\mu} - \frac{1}{2}\underset{\sim}{t}'\Sigma\underset{\sim}{t}\}$.

Now $a_1X_1 + \cdots + a_nX_n \sim N(\mu_{a_1,\ldots,a_n}, \sigma^2_{a_1,\ldots,a_n})$ for every $a_1,\ldots,a_n$. Thus

$$c_{a_1X_1+\cdots+a_nX_n}(t) = \exp\{i\mu_{a_1,\ldots,a_n}t - \frac{1}{2}\sigma^2_{a_1,\ldots,a_n}t^2\}$$

where $\mu_{a_1,\ldots,a_n} = E(a_1X_1 + \cdots + a_nX_n) = \underset{\sim}{a}'E(\underset{\sim}{x})$ and

$\sigma^2_{a_1,\ldots,a_n} = Var(a_1X_1 + \cdots + a_nX_n) = Var(\underset{\sim}{a}'\underset{\sim}{x})$

$= \underset{\sim}{a}'Var(\underset{\sim}{x})\underset{\sim}{a}$. Now $c_{a_1X_1+\cdots+a_nX_n}(t)$

$= E(\exp(it(a_1X_1 + \cdots + a_nX_n))) = E(\exp\{it\underset{\sim}{a}'\underset{\sim}{x}\})$

$= c_{\underset{\sim}{x}}(t\underset{\sim}{a}) = \exp\{i(t\underset{\sim}{a})'E(\underset{\sim}{x}) - \frac{1}{2}(t\underset{\sim}{a}')Var(\underset{\sim}{x})(t\underset{\sim}{a})\}$

and since $t\underset{\sim}{a}$ can take any value in $\mathbb{R}^k$

this proves (by the Uniqueness Theorem) that

$\underset{\sim}{x} \sim N_n(E(\underset{\sim}{x}), Var(\underset{\sim}{x}))$

<u>Note</u> - this result says that if every linear combination of $(X_1,\ldots,X_n)$ is normally dist'd then $(X_1,\ldots,X_n)$ is multivariate normal -