Generating Functions

Def For sequence $\{a_n : n \in \mathbb{N}\}$ define the generating function $G_a(s) = \sum_{i=0}^{\infty} a_i s^i$ at each s for which the series converges absolutely

- notes ① - if $\sum_{i=0}^{\infty} a_i |s^i|$ converges then $\sum_{i=0}^{\infty} a_i t^i$ converges for all $|t| \le |s|$

② - if $G_a(s) = \sum_{i=0}^{\infty} a_i s^i$ is defined for all $|s| < h$ where $h > 0$ then G_a has all its derivatives at 0 and

$$\left.\frac{d^k G_a(s)}{ds^k}\right|_{s=0} = a_k k!$$

value - sometimes a nice simple expression for $G_a(s)$

and in this sense G_a "generates" the sequence, and thus G_a is unique.

- hence not all sequences have "generating functions" eg $a_n = n^n$

Def The convolution of sequences $\{a_n\}$, $\{b_n\}$ is the sequence $\{c_n\}$ where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$

Lemma ① If $\{a_n\}$, $\{b_n\}$ have gen. fns G_a, G_b defined for $|s| < h_a$, $|s| < h_b$ respectively then the convolution has gen. fn given by $G_c(s) = G_a(s) G_b(s)$ defined for $|s| < \min\{h_a, h_b\}$

Proof: Let $|s| < \min\{h_a, h_b\}$. Then

$$\sum_{i=0}^{n} |c_i|\,|s|^i = \sum_{i=0}^{n} \Big|\sum_{j=0}^{i} a_j b_{i-j}\Big|\,|s|^i$$

$$\le \sum_{i=0}^{n} \Big[\sum_{j=0}^{i} |a_j|\,|b_{i-j}| \Big] |s|^i \le \sum_{i=0}^{n} |a_i|\,|s|^i \sum_{j=0}^{n} |b_j|\,|s|^j .$$

__Lemma 2 (Abel's Theorem)__ If $a_i \geq 0\ \forall i$ and $G_a(s)$ is finite for $|s| < 1$ then $\lim_{s \uparrow 1} G_a(s) = \sum_{i=0}^{\infty} a_i$

could be infinite

Proof: <complex analysis text>

(a) Probability Generating Functions

Def If X is a r.v. s.t. $P_X(\mathbb{N}_0) = 1$ then the probability generating function of X is
$$G_X(s) = \sum_{i=0}^{\infty} P(X=i)s^i$$

- note ① - $\sum_{i=0}^{\infty} P(X=i)s^i$ is abs. conv. for $|s| \leq 1$

- note ② - $P_X \Leftrightarrow G_X$

- note ③ $G_X(s) = E[s^X]$.

Prop ① If X, Y are stat. ind with pgfs G_X, G_Y then $W = X+Y$ has pgf $G_W(s) = G_X(s)G_Y(s)$.

Proof: $P(W=n) = \sum_{i=0}^{n} P(X=i, Y=n-i) = \sum_{i=0}^{n} P(X=i)P(Y=n-i)$

and now apply Lemma ①.

Prop ② If X has pgf G_X then

(i) $E[X] = \lim_{s \uparrow 1} G'(s)$

(ii) $E[X(X-1)\cdots(X-k+1)] = \lim_{s\uparrow 1} G^{(k)}(s)$

Proof: For $0 < s < 1$ $\quad G^{(k)}(s) = \sum_{i=k}^{\infty} P(X=i)\, i(i-1)\cdots(i-k+1)s^{i-k}$

and by Abels Theorem $\lim_{s\uparrow 1} G^{(k)}(s) = \sum_{i=k}^{\infty} i(i-1)\cdots(i-k+1)P(X=i)$

$= E[X(X-1)\cdots(X-k+1)]$.

eg ① Poisson(λ)

$$p_X(i) = \frac{\lambda^i e^{-\lambda}}{i!} \qquad i = 0, 1, \ldots$$

$$G_X(s) = \sum_{i=0}^{\infty} \frac{(\lambda s)^i e^{-\lambda}}{i!} = e^{-\lambda} e^{\lambda s} = \exp\{\lambda(s-1)\}$$

$$\frac{d\, G_X(s)}{ds} = \lambda \exp\{\lambda(s-1)\} \,, \quad \frac{d^2 G_X(s)}{ds^2} = \lambda^2 \exp\{\lambda(s-1)\}$$

$$\mathbb{E}[X] = \lambda \qquad , \qquad \mathbb{E}[X(X-1)] = \lambda^2$$

$$\begin{aligned} \mathrm{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \mathbb{E}[X(X-1)] - \mathbb{E}[X](\mathbb{E}[X]-1) \\ &= \lambda^2 - \lambda(\lambda-1) = \lambda . \end{aligned}$$

- $X_i \sim$ Poisson (λ_i) and $X_1, \ldots, X_n$ mut. stat. ind then $Y = X_1 + \cdots + X_n$ has pgf given by

$$G_Y(s) = \prod_{i=1}^{n} G_{X_i}(s) = \exp\{\sum_{i=1}^{n} \lambda_i (s-1)\}$$

$\therefore$ by uniqueness $Y \sim$ Poisson ($\lambda_1 + \cdots + \lambda_n$)

eg ② <u>Compound distributions</u> (insurance $N = \#$ of claims, X_i = size of ith claim)

- $N, X_1, X_2, \ldots$ mut. stat. ind. with $X_1, X_2, \ldots$ identically distributed with pgf G_X and N with pgf G_N

- put $Y = X_1 + \cdots + X_N$

$$G_Y(s) = \mathbb{E}[s^Y] = \mathbb{E}[s^{X_1 + \cdots + X_N}]$$
$$\overset{TTE}{=} \mathbb{E}[\mathbb{E}[s^{X_1 + \cdots + X_N} | N]]$$
$$= \mathbb{E}[G_X^N(s)]$$
$$= G_N(G_X(s))$$

- $X \sim 1 + \text{Bernoulli}(p)$, $G_X(s) = qs + ps^2$

$N \sim \text{Poisson}(\lambda)$, $G_N(s) = \exp\{\lambda(s-1)\}$

$$G_Y(s) = \exp\{\lambda(qs + ps^2 - 1)\}$$

$$E(Y) = \lim_{s \uparrow 1} G_Y'(s) = \lim_{s \uparrow 1} \lambda(q + 2ps)\exp\{\lambda(qs + ps^2 - 1)\} = \lambda(q + 2p)$$

eg ③ Recurrent events

- concerned with the occurrence of some recurrent event H e.g. the failure of a light-bulb

- $X_1, X_2, \ldots$ mut. stat. ind. and $X_2, X_3, \ldots$ (interoccurrence times) iid

- $T_m = X_1 + \ldots + X_m$ = time of m-th occurrence

- T is a random walk

- suppose $P(X_1 \in \mathbb{N}) = P(X_i \in \mathbb{N}) = 1 \quad \forall i \geq 2$

- let H_n = H occurs at time $n = 0, 1, \ldots$ and

$$u_n = P(H_n) \overset{TTP}{=} \sum_{j=1}^{n} P(H_n \mid X_1 = j)\, P(X_1 = j)$$

- now $P(H_n \mid X_1 = j) = \sum_{i=1}^{n-j+1} P(T_i = n \mid X_1 = j)$

since $H_n = \bigcup_{i=1}^{n} \{T_i = n\}$ mut. disj.

$$= \sum_{i=1}^{n-j+1} P\left(\overset{j + X_2 + \ldots + X_i = n}{1 + X_2 + \ldots + X_i = n - j + 1} \right)$$

$$= \sum_{i=1}^{n-j+1} P(T_i = n-j+1 \mid X_1 = 1)$$

$$= P(H_{n-j+1} \mid H_1)$$

- for $m \geq 2$

$$P(H_m \mid H_1) \overset{TTP}{=} \sum_{j=1}^{m-1} P(H_m \mid H_1, X_2 = j)\, P(X_2 = j \mid H_1)$$

$$= \sum_{j=1}^{m-1} P(H_m \mid H_1, X_2 = j)\, P(X_2 = j)$$

since X_1, X_2 are stat. ind.

$$P(H_m \mid H_1, X_2 = j) = \sum_{i=2}^{m-j+2} P(T_i = m \mid H_1, X_2 = j)$$

$$= \sum_{i=2}^{m-j+2} P(1 + j + X_3 + \cdots + X_i = m)$$

$1 + X_3 + \cdots + X_i \sim 1 + X_2 + \cdots + X_{i-1}$

$$= \sum_{i=2}^{m-j+2} P(T_{i-1} = m - j \mid H_1)$$

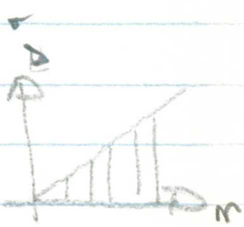
$$\overset{\text{change index}}{=} \sum_{i=1}^{m-j+1} P(T_i = m - j \mid H_1)$$

$$= P(H_{m-j} \mid H_1)$$

- now put

$$G_H(x) - 1 = \sum_{m=2}^{\infty} x^{m-1} P(H_m \mid H_1)$$

$$= \sum_{m=2}^{\infty} x^{m-1} \sum_{j=1}^{m-1} P(H_{m-j} \mid H_1)\, P(X_2 = j)$$



$$= \sum_{m=1}^{\infty} x^{m} \sum_{j=1}^{m} P(H_{m-j+1} \mid H_1)\, P(X_2 = j)$$

$$\overset{\text{change order}}{=} \sum_{j=1}^{\infty} \sum_{m=j}^{\infty} x^{m} P(H_{m-j+1} \mid H_1)\, P(X_2 = j)$$

$x^m = x^{j + m - j}$
$n = m - j + 1$

$$= \sum_{j=1}^{\infty} x^{j} \left(\sum_{n=1}^{\infty} x^{n-1} P(H_m \mid H_1) \right) P(X_2 = j)$$

$$= G_{X_2}(x)\, G_H(x)$$

and so $G_H(x) = \dfrac{1}{1 - G_{X_2}(x)}$

– then $U(x) = \sum_{n=1}^{\infty} x^n u_n = \sum_{n=1}^{\infty} x^n \sum_{j=1}^{n} P(H_{n-j+1} | H_1) P(X_1 = j)$

$$= \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} x^n P(H_{n-j+1} | H_1) P(X_1 = j) \qquad m = n-j+1$$

$$= \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} x^{m+j-1} P(H_m | H_1) P(X_1 = j)$$

$$= \sum_{j=1}^{\infty} x^j P(X_1 = j) G_H(x) = G_{X_1}(x) G_H(x)$$

$\therefore U(x) = \dfrac{G_{X_1}(x)}{1 - G_{X_2}(x)}$ and we can calculate the u_n by differentiating this n times

– <u>note①</u> – if $E X_2 = \mu$ then $G_{X_1}(x) = x \dfrac{1 - G_{X_2}(x)}{\mu(1-x)}$ is a valid pgf (Ex)

– then $U(x) = \dfrac{x}{\mu(1-x)} = \sum_{n=1}^{\infty} \dfrac{1}{\mu} x^n \quad |x|<1$

and $u_n = \dfrac{1}{\mu}$ is constant (not a pgf)

<u>eg</u> – $X_1, X_2, \ldots$ iid Geometric (p)

– $G_{X_1}(x) = \sum_{i=1}^{\infty} x^i p(1-p)^{i-1} = \dfrac{xp}{1-x(1-p)}$

– $\mu = \lim_{x \uparrow 1} G'_{X_1}(x) = \dfrac{1}{p}$

– $U(x) = \dfrac{xp}{1-x} = \dfrac{1}{\mu} \dfrac{x}{1-x}$

eg 4 Simple random walk

- let $p_0(n) = P(S_n = 0 \mid S_0 = 0) = \begin{cases} \binom{n}{\frac{n}{2}} p^{\frac{n}{2}}(1-p)^{\frac{n}{2}} & n \text{ even} \\ 0 & \text{otherwise} \end{cases}$

$$P_0(x) = \sum_{n=0}^{\infty} p_0(n) x^n$$

$$= \sum_{n=0}^{\infty} \binom{2n}{n} (p(1-p))^n x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(2n-1)(2n-3)\cdots 3 \cdot 1}{n!} (2p(1-p))^n x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n+1)}{n!} (-4p(1-p))^n x^{2n}$$

Maclaurin series $= (1-4p(1-p)x^2)^{-\frac{1}{2}} \qquad |x|<1$

- put $f_0(n) = P(S_1 \cdots S_{n-1} \neq 0, S_n = 0 \mid S_0 = 0)$, $f_0(0) = 0$
= prob hit 0 for 1st time after 0 is n

$$F_0(x) = \sum_{n=0}^{\infty} f_0(n) x^n = E[x^{T_0}]$$

where T_0 = hitting time of 0 after $t=0$

- note - could happen that $\sum_{n=1}^{\infty} f_0(n) < 1$ and then T_0 is defective (because $P(T_0 = \infty) > 0$)

- for $n>0$, $P(S_n = 0 \mid S_0 = 0)$

$$\overset{TTP}{=} \sum_{k=1}^{n} P(S_n = 0 \mid S_0 = 0, S_1 \cdots S_{k-1} \neq 0, S_k = 0) f_0(k)$$

$$\overset{MP}{=} \sum_{k=1}^{n} P(S_n = 0 \mid S_k = 0) f_0(k)$$

$$\overset{\text{temporally homogeneous}}{=} \sum_{k=1}^{n} P(S_{n-k} = 0 \mid S_0 = 0) f_0(k) = \sum_{k=1}^{n} p_0(n-k) f_0(k)$$

$$= \sum_{k=0}^{n} p_k(n-k) f_0(k) \qquad f_0(0)=0$$

$$\therefore \quad P_0(x) = 1 + P_0(x) F_0(x)$$

$$\therefore \quad F_0(x) = \frac{P_0(x)-1}{P_0(x)} = 1 - (1-4pqx^2)^{\frac{1}{2}}$$

- then $\lim_{x \uparrow 1} F_0(x) = 1 - (1-4pq)^{\frac{1}{2}} \overset{\text{Ex}}{=} 1 - |p-q|$

$\therefore$ SRW returns to 0 (after starting at 0) with prob. 1 iff $p=q=\frac{1}{2}$.

$$P_0(x) = \sum_{n=0}^{\infty} P(S_n=0 \mid S_0=0)\, x^n$$

$$= 1 + \sum_{n=1}^{\infty} P(S_n=0 \mid S_0=0)\, x^n$$

$$= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n} p_k(n-k) f_0(k) \right) x^n$$

$$= 1 + \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} p_k(n-k) f_0(k) \right) x^n$$

$$= 1 + P_0(x) F_0(x)$$

eg ⑤ Branching processes

- X_{ij} iid where $P_{X_{ij}}(\mathbb{N}_0) = 1$ with pgf G_X
- put $Z_0 = 1$ and for $n > 0$

$$Z_n = \sum_{j=1}^{Z_{n-1}} X_{nj} = \text{size of pop. at time } n.$$

- let G_n be the pgf of Z_n
- $G_{n+m}(x) = \mathbb{E}[x^{Z_{n+m}}]$

$$= \mathbb{E}[x^{Y_1 + \cdots + Y_{Z_n}}]$$

where $Y_1, Y_2, \ldots$ are the mth gen of ind branching processes (Y_i arising from each member of nth gen) independent of Z_n

$$= G_n(G_m(x))$$

- thus $G_2(x) = G_X(G_X(x))$, $G_3(x) = G_X(G_X(G_X(x)))$ etc.
- $\mathbb{E}[Z_n] = \lim_{x \uparrow 1} \frac{d}{dx} G_n(x) \overset{\text{Chain rule}}{=} \lim_{x \uparrow 1} \prod_{i=1}^{n} G_X'(G_{i-1}(x))$

where $G_0(x) \overset{\text{formal}}{=} x$

$$= (G_X'(1))^n = \mu^n \text{ where } \mu = \mathbb{E}[X_{ij}]$$

- want to calculate

$$\eta = P(\text{ultimate extinction}) = P\left(\bigcup_{n=1}^{\infty} \{Z_n = 0\}\right)$$

and note $\{Z_n = 0\} \subseteq \{Z_{n+1} = 0\}$ so $\{Z_n = 0\} \uparrow \bigcup_{j=1}^{\infty} \{Z_j = 0\}$

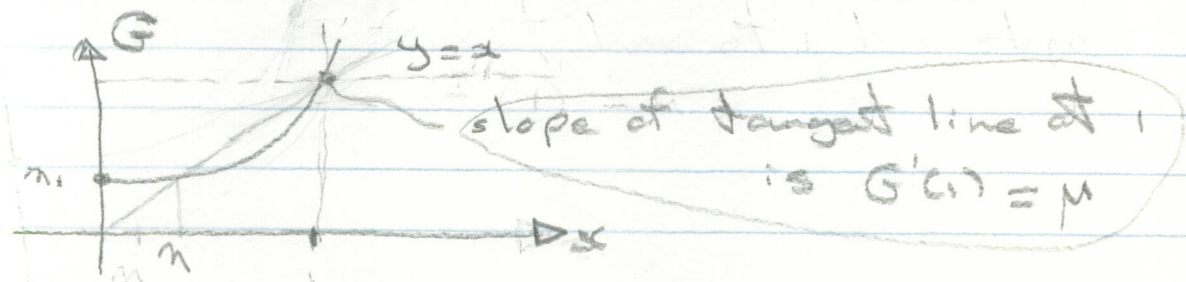
- put $\eta_n = P(Z_n = 0) = G_n(0) \uparrow \eta$ as $n \to \infty$

- further $\eta_n = G_n(0) = G_X(G_{n-1}(0)) = G_X(\eta_{n-1})$

$$\therefore \quad \eta = G_X(\eta) \quad \text{by cty of } G_X$$

- note that if $e \in [0,1]$ sats $e = G_X(e)$ then $\eta_1 = G_X(0) \le G_X(e) = e$ and $\eta_2 = G_X(\eta_1) \le G(e) \le e$ etc and thus η is the smallest nonnegative fixed point of G_X (also $G(1) = 1$)

- now $G_X'(x) = E\left[\frac{d}{dx} x^X\right] = E\left[X x^{X-1}\right] \ge 0$ and G_X is nondecreasing, further $G_X''(x) = E\left[X(X-1) x^{X-2}\right] \ge 0$ and G_X is convex



- then if $\mu > 1$, then $G_X(s) = s$ has 1 root in $[0,1)$ and $\eta < 1$; if $\mu < 1$ then $G_X(s) = s$ has no roots in $[0,1)$ and $\eta = 1$; if $\mu = 1$ then $G_X(s) \equiv s$ which implies $P(X=1) = 1$ and $\eta = 0$ or $G_X(s) \not\equiv s$ and $\eta = 1$.

(b) Characteristic and Moment Generating Functions

Def For random vector $X \in \mathbb{R}^k$

(i) the characteristic function is defined by
$c_{\underset{\sim}{x}}(\underset{\sim}{t}) = \mathbb{E}[\exp\{i\underset{\sim}{t}'\underset{\sim}{x}\}] \quad \forall \underset{\sim}{t} \in \mathbb{R}^k \qquad i = \sqrt{-1}$

(ii) the moment generating function is defined by
$m_{\underset{\sim}{x}}(\underset{\sim}{t}) = \mathbb{E}[\exp\{\underset{\sim}{t}'\underset{\sim}{x}\}]$ provided $\exists\, h > 0$ s.t. this expectation is finite $\forall \underset{\sim}{t} \in B_h(\underset{\sim}{0})$.

-note- $c_{\underset{\sim}{x}}(\underset{\sim}{t}) = \mathbb{E}[\cos(\underset{\sim}{t}'\underset{\sim}{x}) + i\sin(\underset{\sim}{t}'\underset{\sim}{x})]$

$$= \mathbb{E}[\cos(\underset{\sim}{t}'\underset{\sim}{x})] + i\,\mathbb{E}[\sin(\underset{\sim}{t}'\underset{\sim}{x})] \in \mathbb{C}$$

always exists since $|\cos(x)| \le 1$, $|\sin(x)| \le 1$

Prop ① (Uniqueness) If $m_{\underset{\sim}{x}}$ exists it is unique and $c_{\underset{\sim}{x}}$ is unique.

Proof: accept.

-note- there are inversion results that show how to compute densities, prob. fns and dist'n fns from mgfs and cfs

Def For $i_1, \ldots, i_k \in \mathbb{N}_0$, $\mathbb{E}[X_1^{i_1} \cdots X_k^{i_k}]$ is the $(i_1, \ldots, i_k)$-th moment of $\underset{\sim}{x}$ provided that $\mathbb{E}|X_1^{i_1} \cdots X_k^{i_k}| < \infty$.

Prop ② If $i_1 \le j_1, \dots, i_k \le j_k$ and $E[|X_1^{j_1}\cdots X_k^{j_k}|] < \infty$ then $E[|X_1^{i_1}\cdots X_k^{i_k}|] < \infty$

higher order moments existing implies existence of lower order moment

Proof: $(k=1)$ $E[|X_1|^{i_1}] = E[I_{[0,1]}(|X_1|)\,|X_1|^{i_1}]$

$+ E[I_{(1,\infty)}(|X_1|)\,|X_1|^{i_1}] \le P(|X_1| \le 1) + E|X_1|^{j_1}$

Prop ③ If $m_X(t)$ exists $\forall t \in B_r(0)$ with $r>0$ then all moments exist and

$$E[X_1^{i_1}\cdots X_k^{i_k}] = \frac{\partial^{i_1+\dots+i_k} m_X(t)}{\partial t_1^{i_1}\cdots\partial t_k^{i_k}}\bigg|_{t=0}.$$

Proof: First let $k=1$ then for $t \in B_r(0)$ $m_X(t)$ exists. Then $m_X(t) = E[e^{tX}] = E[e^{tX_+ - tX_-}]$

$$= \underbrace{E[I_{\{X\ge 0\}}e^{tX_+}]}_{m_{X_+}(t) - P(X<0)} + \underbrace{E[I_{\{X<0\}}e^{-tX_-}]}_{m_{X_-}(-t) - P(X\ge 0)} < \infty$$

and $m_{X_+}(t)$, $m_{X_-}(-t)$ are defined and similarly $m_{X_+}(-t)$, $m_{X_-}(t)$ are defined. This implies $m_{|X|}(t)$ is defined.

Now put $Y_n = \sum_{k=0}^{n} \frac{t^k X^k}{k!}$. Then

$Y_n \to Y = \sum_{k=0}^{\infty} \frac{t^k X^k}{k!} = e^{tX}$ and $|Y_n| \le \sum_{k=0}^{n} \frac{|t|^k|X|^k}{k!} \le e^{|t||X|}$.

Then $\forall k$ $E[|X|^k] \le \frac{k!}{t^k} E(e^{|t||X|}) < \infty$ and all moments exist.

By DCT

$E[Y_n] \to \sum_{k=0}^{\infty} \frac{t^k}{k!} EX^k = m_X(t)$.

~~(the rest of the proof)~~

Prop If $m_{\underset{\sim}{x}}(\underset{\sim}{t}) < \infty \ \forall \ \underset{\sim}{t} \in B_h(\underset{\sim}{0})$ for some $h > 0$ then $c_{\underset{\sim}{x}}(\underset{\sim}{t}) = m_{\underset{\sim}{x}}(i\underset{\sim}{t})$

Proof: Accept.

- **note** - some r.v.'s do not have all moments and thus cannot have a mgf

eg $X \sim$ Cauchy, $f_X(x) = \frac{1}{\pi(1+x^2)} \quad x \in (-\infty, \infty)$

$$\mathbb{E}[X^+] = \frac{1}{\pi} \int_0^\infty \frac{x}{1+x^2} \, dx = \frac{1}{2\pi} \ln(1+x^2) \Big|_0^\infty = \infty$$

$\therefore \mathbb{E}[X^+] = \mathbb{E}[X^-] = \infty$ and $\mathbb{E}[X]$ is not defined (so no moments are defined).

Prop ④ If $\underset{\sim}{x}, \underset{\sim}{y} \in \mathbb{R}^k$ are stat ind then
$m_{\underset{\sim}{x}+\underset{\sim}{y}}(\underset{\sim}{t}) = m_{\underset{\sim}{x}}(\underset{\sim}{t})\, m_{\underset{\sim}{y}}(\underset{\sim}{t})$ (provided $m_{\underset{\sim}{x}}, m_{\underset{\sim}{y}}$ def'd)
and $c_{\underset{\sim}{x}+\underset{\sim}{y}}(\underset{\sim}{t}) = c_{\underset{\sim}{x}}(\underset{\sim}{t})\, c_{\underset{\sim}{y}}(\underset{\sim}{t})$.

Proof: $m_{\underset{\sim}{x}+\underset{\sim}{y}}(\underset{\sim}{t}) = \mathbb{E}[e^{\underset{\sim}{t}'(\underset{\sim}{x}+\underset{\sim}{y})}] = \mathbb{E}[e^{\underset{\sim}{t}'\underset{\sim}{x}} e^{\underset{\sim}{t}'\underset{\sim}{y}}]$

$= \mathbb{E}[e^{\underset{\sim}{t}'\underset{\sim}{x}}]\, \mathbb{E}[e^{\underset{\sim}{t}'\underset{\sim}{y}}] = m_{\underset{\sim}{x}}(\underset{\sim}{t})\, m_{\underset{\sim}{y}}(\underset{\sim}{t})$.

eg multivariate normal

- $X \sim N(0,1)$
- $m_x(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$

$$= \int_{-\infty}^{\infty} e^{-t^2/2} \frac{1}{\sqrt{2\pi}} \exp\{-\tfrac{1}{2}(x-t)^2\} dx$$

$u = x - t$, $du = dx$

$$= e^{t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = e^{t^2/2}$$

$c_x(t) = m_x(it) = e^{-t^2/2}$

- $\underset{\sim}{X} \sim N_k(\underset{\sim}{0}, I)$

$m_{\underset{\sim}{x}}(\underset{\sim}{t}) = \mathbb{E}[e^{\underset{\sim}{t}'\underset{\sim}{x}}] = \mathbb{E}[\prod_{i=1}^{k} e^{t_i X_i}]$

$\overset{\text{stat. ind}}{=} \prod_{i=1}^{k} e^{-t_i^2/2} = e^{\frac{1}{2}\underset{\sim}{t}'\underset{\sim}{t}}$

$c_{\underset{\sim}{x}}(\underset{\sim}{t}) = e^{-\frac{1}{2}\underset{\sim}{t}'\underset{\sim}{t}}$

- $\mathbb{E}[X] = 0$, $Var[X] = \mathbb{E}[X^2] - \mathbb{E}^2[X] = 1$

$\mathbb{E}[\underset{\sim}{X}] = \underset{\sim}{0}$, $Var[\underset{\sim}{X}] = (cov[X_i, X_j]) = I$

- let $\mu \in \mathbb{R}^k$, $C \in \mathbb{R}^{k}$ nonsingular s.t.
$CC' = \Sigma$

$\underset{\sim}{Y} = \mu + C\underset{\sim}{X}$

- $\mathbb{E}[\underset{\sim}{Y}] = \underset{\sim}{\mu}$, $\mathrm{Var}[\underset{\sim}{Y}] = C\,\mathrm{Var}[\underset{\sim}{X}]\,C' = CC' = \Sigma$
- now $P(\underset{\sim}{Y} \in B) = P(\underset{\sim}{\mu} + C\underset{\sim}{X} \in B)$

$$= P(\underset{\sim}{X} \in C^{-1}(B - \underset{\sim}{\mu}))$$

$$= \int_{C^{-1}(B-\underset{\sim}{\mu})} f_{\underset{\sim}{X}}(\underset{\sim}{x})\, d\underset{\sim}{x}$$

$$\underset{\sim}{y} = \underset{\sim}{\mu} + C\underset{\sim}{x} \quad d\underset{\sim}{y} = J(\underset{\sim}{y} \to \underset{\sim}{x})\, d\underset{\sim}{x}$$

$$= |\det C|\, d\underset{\sim}{x}$$

$$= \int_B |\det C|^{-1} f_{\underset{\sim}{X}}(C^{-1}(\underset{\sim}{y} - \underset{\sim}{\mu}))\, d\underset{\sim}{y}$$

$$\therefore f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = |\det C|^{-1} f_{\underset{\sim}{X}}(C^{-1}(\underset{\sim}{y} - \underset{\sim}{\mu}))$$

$$= |\det CC'|^{-\frac{1}{2}} (2\pi)^{-\frac{n}{2}} \exp\{-\tfrac{1}{2}(\underset{\sim}{y}-\underset{\sim}{\mu})'(CC')^{-1}(\underset{\sim}{y}-\underset{\sim}{\mu})\}$$

$$= (2\pi)^{-n/2} |\det \Sigma|^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}(\underset{\sim}{y}-\underset{\sim}{\mu})'\Sigma^{-1}(\underset{\sim}{y}-\underset{\sim}{\mu})\}$$

and we write $\underset{\sim}{Y} \sim N_n(\underset{\sim}{\mu}, \Sigma)$

- $m_{\underset{\sim}{Y}}(\underset{\sim}{t}) = \mathbb{E}[\exp\{\underset{\sim}{t}'(\underset{\sim}{\mu} + C\underset{\sim}{X})\}]$

$$= \mathbb{E}[\exp\{\underset{\sim}{t}'\underset{\sim}{\mu}\} \exp\{(C'\underset{\sim}{t})'\underset{\sim}{X}\}]$$

$$= \exp\{\underset{\sim}{t}'\underset{\sim}{\mu}\}\, m_{\underset{\sim}{X}}(C'\underset{\sim}{t})$$

$$= \exp\{\underset{\sim}{t}'\underset{\sim}{\mu} + \tfrac{1}{2}\underset{\sim}{t}'CC'\underset{\sim}{t}\} = \exp\{\underset{\sim}{t}'\underset{\sim}{\mu} + \tfrac{1}{2}\underset{\sim}{t}'\Sigma\underset{\sim}{t}\}.$$

- suppose $\underset{\sim}{X} \sim N_k(\underset{\sim}{\mu}, \Sigma)$ and $\underset{\sim}{y} = \underset{\sim}{g} + C\underset{\sim}{X}$ where $\underset{\sim}{g} \in \mathbb{R}^m$, $C \in \mathbb{R}^{m \times k}$ of rank m

$$\therefore m_{\underset{\sim}{y}}(\underset{\sim}{t}) = E[\exp\{\underset{\sim}{t}'(\underset{\sim}{g} + C\underset{\sim}{x})\}]$$

$$= \exp\{\underset{\sim}{t}'\underset{\sim}{g}\}\, m_{\underset{\sim}{x}}(C'\underset{\sim}{t})$$

$$= \exp\{\underset{\sim}{t}'(\underset{\sim}{g} + C\underset{\sim}{\mu}) + \tfrac{1}{2}\underset{\sim}{t}' C \Sigma C' \underset{\sim}{t}\}$$

and $\underset{\sim}{y} \sim N_m(\underset{\sim}{g} + C\underset{\sim}{\mu}, C\Sigma C')$

- if $\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n$ are mut. stat. ind. with $\underset{\sim}{x}_i \sim N_{k_i}(\underset{\sim}{\mu}_i, \Sigma_i)$ $\underset{\sim}{g}_1, \ldots, \underset{\sim}{g}_n \in \mathbb{R}^m$, $C_i \in \mathbb{R}^{m \times k_i}$ of rank m then $\underset{\sim}{y} = \sum_{i=1}^{n}(\underset{\sim}{g}_i + C_i\underset{\sim}{x}_i)$

$$\overset{Ex}{\sim} N_m\left(\sum_{i=1}^{n}(\underset{\sim}{g}_i + C_i\underset{\sim}{\mu}_i), \sum_{i=1}^{n} C_i \Sigma_i C_i'\right)$$