STAC62F : 2016 Assignment 4 - Solutions

1. Letting $q_k = P(S_n = N$ for some n and $S_1, \ldots, S_{n-1} \neq 0 \mid S_0 = k)$ we have
$q_k = p q_{k+1} + q q_{k-1}$ which has solutions $q_k \equiv 1$ and $q_k = (q/p)^k$ subject to the boundary conditions $q_0 = 0$, $q_N = 1$. The general solution is of the form $A + B\left(\frac{q}{p}\right)^k$ for some $A, B \in \mathbb{R}$. Then from the boundary conditions A and B must satisfy $0 = A + B$ and $1 = A + B(q/p)^N$. So $1 = B[(q/p)^N - 1]$ and $B = [(q/p)^N - 1]^{-1}$ and $A = [1 - (q/p)^N]^{-1}$ which requires $q \neq p$. Therefore,

$$q_k = \frac{1 - (q/p)^k}{1 - (q/p)^N} \quad \text{when } q \neq p.$$

When $q = p$, $q_k = \lim_{x \to 1} \dfrac{1 - x^k}{1 - x^N} \overset{\text{L'Hospital}}{=} \lim_{x \to 1} \dfrac{-k x^{k-1}}{-N x^{N-1}}$

$$= \frac{k}{N}$$

2. (a) Suppose $X \sim$ Poisson(λ) then $m_X(t)$
$= E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \lambda^x e^{-\lambda}/x!$
$= e^{-\lambda} \sum_{x=0}^{\infty} (\lambda e^t)^x / x! = \exp\{\lambda(e^t - 1)\}$ and this holds for all $t \in \mathbb{R}$.

(b) $E(X) = \dfrac{d\, m_X(t)}{dt}\Big|_{t=0} = \lambda e^t \exp\{\lambda(e^t - 1)\}\Big|_{t=0}$
$= \lambda e^0 \exp\{\lambda(e^0 - 1)\} = \lambda.$

$E(X^2) = \dfrac{d^2 m_X(t)}{dt^2}\Big|_{t=0} = \left(\lambda e^t \exp\{\lambda(e^t - 1)\} + \lambda^2 e^{2t} \exp\{\lambda(e^t - 1)\}\right)\Big|_{t=0}$
$= \lambda + \lambda^2$

Therefore $Var(X) = E(X^2) - (E(X))^2 = \lambda$.

(c) $m_Y(t) = E(e^{tY}) = E(e^{t(X_1+\dots+X_n)})$
$= E\left(\prod_{i=1}^n e^{tX_i}\right) \overset{indep.}{=} \prod_{i=1}^n E(e^{tX_i})$
$= \prod_{i=1}^n m_{X_i}(t) \overset{(a)}{=} \prod_{i=1}^n \exp\{\lambda(e^t-1)\}$
$= \exp\{n\lambda(e^t-1)\}$

Therefore by the Uniqueness Thm.
$Y \sim$ Poisson $(n\lambda)$

(d) $r_X(t) = E(t^X) = E(\exp\{X \log t\})$
$= m_X(\log t)$ and note that
if $m_X(t)$ exists for all $|t| \le h$ where $h > 0$
then $m_X(t)$ exists for all $t \in [-\infty, h]$.
Therefore $r_X(t) = m_X(\log t)$ $\forall t \le 1$ since $\exp\{h\} > 1$.
This gives an inversion theorem since when
we know m_X we can compute $P(X = k)$
$= (d^k m_X(\log t) / dt^k)|_{t=0}$.

(3) We have that $E(X^k) = E((\mu + X - \mu)^k)$
$= \sum_{i=0}^{k} \binom{k}{i} \mu^i E((X-\mu)^{k-i}) = \sum_{i=0}^{k} \binom{k}{i} \mu^i \sigma^{k-i} E\left(\left(\frac{X-\mu}{\sigma}\right)^{k-i}\right)$

and $(X-\mu)/\sigma \sim N(0,1)$. For $Z \sim N(0,1)$

$E(Z^k) = \int_{-\infty}^{\infty} z^k \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$ put $u = z^{k-1}$, $dv = z e^{-z^2/2}$

integration by parts $= \frac{-z^{k-1} e^{-z^2/2}}{\sqrt{2\pi}}\Big|_{-\infty}^{\infty} + (k-1)\int_{-\infty}^{\infty} z^{k-2} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$

$= (k-1) E(Z^{k-2})$. Therefore $E(Z^3) = 2E(Z) = 0$

$E(Z^4) = 3E(Z^2) = 3 \cdot 1 = 3$, $E(Z^5) = 4E(Z^3) = 0$, $E(Z^6) = 5E(Z^4) = 5 \cdot 3 = 15$.

Therefore $\bar{Y}$ has mean

$$E(Y) = \sum_{i=0}^{3} \binom{3}{i} \mu^i \sigma^{3-i} E(Z^{3-i})$$

$$= \binom{3}{0}\mu^0\sigma^3 \cdot 0 + \binom{3}{1}\mu^1\sigma^2 \cdot 1 + \binom{3}{2}\mu^2\sigma \cdot 0 + \binom{3}{3}\mu^3\sigma^0 \cdot 1 = 3\mu\sigma^2 + \mu^3$$

and $Var(\bar{Y}) = \frac{1}{n} Var(Y) = \frac{1}{n}\left(E(Y^2) - (E(Y))^2\right)$

$$E(Y^2) = E(X^6) = \sum_{i=0}^{6} \binom{6}{i} \mu^i \sigma^{6-i} E(Z^{6-i})$$

$$= \sigma^6 \cdot 15 + \binom{6}{2}\mu^2\sigma^4 \cdot 3 + \binom{6}{4}\mu^4\sigma^2 \cdot 1 + \binom{6}{6}\mu^6$$

$$= 15\sigma^6 + 45\mu^2\sigma^4 + 15\mu^4\sigma^2 + \mu^6$$

Therefore by the CLT

$$\frac{\bar{Y} - (3\mu\sigma^2 + \mu^3)}{\sqrt{\frac{15\sigma^6 + 45\mu^2\sigma^4 + 15\mu^4\sigma^2 + \mu^6}{n}}} \xrightarrow{D} N(0,1)$$

as $n \to \infty$. This implies $P(\bar{Y} \le k)$

$$\approx \Phi\left(\frac{k - (3\mu\sigma^2 + \mu^3)}{\sqrt{\frac{15\sigma^6 + 45\mu^2\sigma^4 + 15\mu^4\sigma^2 + \mu^6}{n}}}\right)$$