Solutions

STAC62F: 2015 Final Exam Time: 3 hours

Aids: This is an open book exam but no computers are allowed.

Name:

Student Number:

1. Suppose we have a probability model for a response $\omega = (\omega_1, \omega_2) \in \Omega = \{(0,0), (0,1), (1,0)\}$ and $p(0,0) = 1/2, p(0,1) = 1/3, p(1,0) = 1/6$.

(a) (5 marks) Determine the marginal distribution of $X(\omega_1, \omega_2) = \omega_1 + \omega_2$.

ω_1	ω_2	$p(\omega_1,\omega_2)$	$\omega_1+\omega_2$
0	0	1/2	0
0	1	1/3	1
1	0	1/6	1

X	p_X
0	1/2
1	1/2

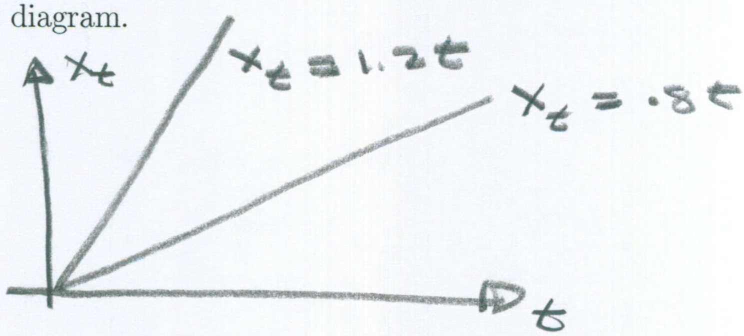
(b) (5 marks) Determine the conditional expectation of ω_1 given $X = x$ for each possible value of x.

$E(\omega_1 | X)(0) = 0 \cdot 1 = 0$

$E(\omega_1 | X)(1) = 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} = \frac{1}{3}$

2. Suppose that for $t \geq 0$ we have that $X_t = \omega t$ for $t \geq 0$ where $\omega \sim N(0, 1)$.

(a) (5 marks) Draw two possible sample paths for this stochastic process. Clearly label your diagram.



(b) (5 marks) Compute $P(\sup_{t \geq 0} X_t > 5)$.

$$P(\sup_{t \geqslant 0} X_t > 5) = P(\omega > 0) = \frac{1}{2}$$

since $\sup_{t \geqslant 0} \omega t > 5$ iff $\omega > 0$

(c) (5 marks) Explain what justifies the above definition as a valid definition of a stochastic process.

For any choice $n \in \mathbb{N}$ and $t_1, \ldots, t_n$ the joint distribution of $(X_{t_1}, \ldots, X_{t_n})'$

$$= \omega \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} \sim N_n\left(\underset{\sim}{0}, \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix}(t_1 \cdots t_n)\right)$$

and these distributions marginalize appropriately so KCT applies.

3. Suppose that Ω is a set and $\mathcal{F}$ is a σ-field on Ω and $A_1, A_2, \ldots \in \mathcal{F}$.

(a) (5 marks) Prove that $\cap_{n=1}^{\infty} \cup_{i=n}^{\infty} A_i \in \mathcal{F}$.

Since $\mathcal{F}$ is closed under countable unions we have $\bigcup_{i=n}^{\infty} A_i \in \mathcal{F}$ $\forall n$. But if $B_i \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} B_i^c \in \mathcal{F}$ and so $\left(\bigcup_{i=1}^{\infty} B_i^c\right)^c = \bigcap_{i=1}^{\infty} B_i \in \mathcal{F}$ and $\mathcal{F}$ is closed under countable intersections. Therefore $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i \in \mathcal{F}$.

(b) (5 marks) Prove that the set in (a) equals the set of all $\omega \in \Omega$ that are in infinitely many of the A_i.

Suppose $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$. Then $\omega \in \bigcup_{i=n}^{\infty} A_i$ for every n. But if ω was an element of only finitely many A_i then $\exists$ a N s.t. $\omega \notin A_i$ $\forall i \geq N$. But this implies $\omega \notin \bigcup_{i=N}^{\infty} A_i$ which is a contradiction. Therefore ω is in infinitely many of the A_i.

Now suppose ω is in infinitely many of the A_i. Then $\omega \in \bigcup_{i=n}^{\infty} A_i$ $\forall n$ as there are only finitely A_i not part of this union. Therefore $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$.

4. Suppose that $X = (X_1, X_2)' \sim N_2(\mu, \Sigma)$ where $\mu = (1, -1)'$ and

$$\Sigma = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

(a) (5 marks) Determine $E(X_2 \mid X_1)(2)$.

From class $X_2 | X_1 = x_1 \sim N\left(\mu_2 + \sigma_{21} \frac{(x_1 - \mu_1)}{\sigma_{11}}, \sigma_{22} - \frac{\sigma_{12}^2}{\sigma_{11}}\right)$

$x_1 = 2$

$= N\left(-1 - \left(\frac{2-1}{2}\right), 2 - \frac{1}{2}\right) = N\left(-\frac{3}{2}, \frac{3}{2}\right)$

$\therefore E(X_2 | X_1)(2) = -\frac{3}{2}$.

(b) (5 marks) Let $\Sigma^{1/2}$ denote the symmetric square root of Σ and put $Z = \Sigma^{-1/2}(X - \mu)$. Are the coordinates Z_1, Z_2 statistically independent? Justify your conclusion.

We have that $Z = \Sigma^{-\frac{1}{2}}(X - \mu) \sim N_2(\underset{\sim}{0}, I)$.

So $\text{cov}(Z_1, Z_2) = 0$ and since the joint distribution is normal this implies that Z_1 and Z_2 are independent.

5. (a) (5 marks) Suppose that X and Y are statistically independent random variables and $h, k : (R^1, B^1) \to (R^1, B^1)$. Prove that $h(X)$ and $k(Y)$ are statistically independent.

$P((h(X), k(Y)) \in B_1 \times B_2)$

$= P((X, Y) \in (h, k)^{-1}(B_1 \times B_2))$

$= P((X, Y) \in h^{-1}B_1 \times k^{-1}B_2)$

$= P(X \in h^{-1}B_1) P(Y \in k^{-1}B_2)$ by independence of X and Y

$= P(h(X) \in B_1) P(k(Y) \in B_2)$

and so $h(X)$ and $k(Y)$ are independent.

(b) (5 marks) For general random variables X and Y (so they are not necessarily independent) prove that $E(h(X)g(Y) \,|\, X) = h(X)E(g(Y) \,|\, X)$ a.s.

We have that $E(g(Y)|X)$ satisfies $E(k(X)g(Y)) = E(k(X)E(g(Y)|X))$ for every k. Therefore $E(k(X)h(X)g(Y))$
$= E(k(X)E(h(X)g(Y)|X)) = E(k(X)h(X)E(g(Y)|X))$ for every k and so $E(h(X)g(Y)|X) = h(X)E(g(Y)|X)$ a.s. by the Proposition stated in the notes.

6. Suppose that $X \sim \text{binomial}(n, \theta)$.

(a) (5 marks) Determine the probability generating function of X.

$$r_X(t) = E(t^X) = \sum_{x=0}^{n} \binom{n}{x} (\theta t)^x (1-\theta)^{n-x}$$

$$= (\theta t + 1 - \theta)^n \quad \forall\, t \in \mathbb{R}.$$

(b) (5 marks) Suppose that $Y \sim \text{binomial}(m, \theta)$ statistically independent of X. Using probability generating functions determine the distribution of $X+Y$ and fully justify your reasoning.

$$r_{X+Y}(t) = E(t^{X+Y}) = E(t^X t^Y)$$

independence

$$= E(t^X) E(t^Y) = r_X(t) r_Y(t)$$

$= (\theta t + 1 - \theta)^{m+n}$ and therefore, by the uniqueness of the pgf $X + Y \sim \text{binomial}(m+n, \theta)$.

7. (a) (5 marks) Suppose that $X_n \xrightarrow{a.s.} Z$ and $Y_n \xrightarrow{a.s.} Z$ as $n \to \infty$. Prove that $Y_n - X_n \xrightarrow{P} 0$ as $n \to \infty$.

Consider $X_n - Y_n$ and let $A_X = \{\omega : X_n(\omega) \to Z(\omega)\}$, $A_Y = \{\omega : Y_n(\omega) \to Z(\omega)\}$. Then $P(A_X) = P(A_Y) = 1$ and $P(A_X \cap A_Y) = -P(A_X \cup A_Y) + P(A_X) + P(A_Y) = -1+1+1 = 1$. So, since $X_n(\omega) - Y_n(\omega) \to Z(\omega) - Z(\omega) = 0$ for every $\omega \in A_X \cap A_Y$ we have $X_n - Y_n \to 0$ a.s. From a result in class this implies $X_n - Y_n \xrightarrow{P} 0$.

(b) (5 marks) Suppose that $X_1, X_2, \ldots$ is an i.i.d. sequence with $E(X_1) = \mu, Var(X_1) = \sigma^2$. Prove that $\frac{1}{n} S_n \xrightarrow{2} \mu$.

We have $E\left(\left(\frac{1}{n} S_n - \mu\right)^2\right) = E\left(\left(\frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)\right)^2\right)$

$= \frac{1}{n^2} \sum_{i=1}^{n} E\left((X_i - \mu)^2\right) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} E\left((X_i - \mu)(X_j - \mu)\right)$

$= \frac{\sigma^2}{n} + \frac{1}{n^2} 0 = \frac{\sigma^2}{n} \to 0$ as $n \to \infty$

Therefore $\frac{1}{n} S_n \xrightarrow{2} \mu$.

(c) (5 marks) Suppose that $X_1, X_2, \ldots$ is an i.i.d. sequence with X_1 having characteristic function $c(t) = \exp\{-|t|\}$ for $t \in R^1$ (note X_1 is distributed Cauchy). Determine the characteristic function of $\frac{1}{n}S_n$ and from this determine its limiting distribution. What does this tell you about $\frac{1}{n}S_n$ as an estimator?

$$c_{\frac{1}{n}S_n}(t) = E\left(\exp\{i \tfrac{1}{n} S_n t\}\right)$$

$$= E\left(\prod_{j=1}^{n} \exp\{\tfrac{i}{n} x_j t\}\right) \overset{\text{iid}}{=} \prod_{j=1}^{n} E(\exp\{i x_j \tfrac{t}{n}\})$$

$$= \prod_{j=1}^{n} \exp\{-\tfrac{|t|}{n}\} = \exp\{-|t|\}$$

Therefore $\frac{1}{n}S_n \sim$ Cauchy $\forall n$ and so $\frac{1}{n}S_n \xrightarrow{D}$ Cauchy. This implies that the distribution of $\frac{1}{n}S_n$ does not become more concentrated as n increases and, in particular, $\frac{1}{n}S_n$ is not consistent for μ.

8. (a) (5 marks) Suppose that $\{X_t : t \in \mathbb{Z}\}$ is a stochastic process satisfying $X_t = Z_t + \theta Z_{t-1}$ where $\{Z_t : t \in \mathbb{Z}\}$ is white noise. Determine the mean and autocovariance function of $\{X_t : t \in \mathbb{Z}\}$. Is $\{X_t : t \in \mathbb{Z}\}$ a stationary process? Justify your answer.

$\mu(t) = E(X_t) = E(Z_t) + \theta E(Z_{t-1}) = 0 + \theta \cdot 0 = 0$

$\sigma(s,t) = \mathrm{cov}(X_s, X_t) = E(X_s X_t) = E((Z_s + \theta Z_{s-1})(Z_t + \theta Z_{t-1}))$

$= E(Z_s Z_t + \theta Z_s Z_{t-1} + \theta Z_{s-1} Z_t + \theta^2 Z_{s-1} Z_{t-1})$

and using $E(Z_s Z_t) = \sigma^2 \delta_{st}$

$= \sigma^2 \delta_{st} + \theta \sigma^2 \delta_{s\,t-1} + \theta \sigma^2 \delta_{s-1\,t} + \theta^2 \sigma^2 \delta_{st}$

$= \begin{cases} \sigma^2(1+\theta^2) & s = t \text{ or } |s-t| = 0 \\ \theta\sigma^2 & s = t-1, t+1 \text{ or } |s-t| = 1 \\ 0 & \text{otherwise or } |s-t| > 1 \end{cases}$

Since $\mu(t)$ is constant and since $\sigma(s,t) = \sigma(|s-t|)$ the process is stationary.

(b) (5 marks) If $\{Z_t : t \in \mathbb{Z}\}$ in (a) is an i.i.d. $N(0,1)$ process, then prove that $\{X_t : t \in \mathbb{Z}\}$ is a Gaussian process.

Consider the distribution of $(X_{t+1}, X_{t+2}, \ldots, X_{t+n})'$

$= \begin{pmatrix} \theta & 1 & 0 & \cdots & 0 & 0 \\ 0 & \theta & 1 & & & \vdots \\ \vdots & & & & & 0 \\ 0 & \cdots & & 0 & \theta & 1 \end{pmatrix} \begin{pmatrix} Z_t \\ Z_{t+1} \\ \vdots \\ Z_{t+n} \end{pmatrix} = A_{t,n} \underset{\sim}{Z}$ where

$\underset{\sim}{Z} \sim N_{n+1}(\underset{\sim}{0}, \sigma^2 I)$ and so $(X_{t+1}, \ldots, X_{t+n})$ $\sim N_n(\underset{\sim}{0}, \sigma^2 A_{t,n} A'_{t,n})$. Since the distribution of and $(X_{t_1}, \ldots, X_{t_n})$ is thus multivariate normal (using marginalization results from class) this implies $\{X_t : t \in \mathbb{Z}\}$ is a Gaussian process.

(c) (5 marks) Suppose that $\{W_{1t} : t \geq 0\}$ and $\{W_{2t} : t \geq 0\}$ are statistically independent standard Wiener processes. Explain what it means for these processes to be independent and prove that $\{W_{1t} + W_{2t} : t \geq 0\}$ is a Wiener process.

$\{W_{1t} ; t \geq 0\}$, $\{W_{2t} : t \geq 0\}$ are independent if for every $t_{11}, \ldots, t_{1k_1}, t_{21}, \ldots, t_{2k_2} \geq 0$ we have the random vectors $(W_{1t_{11}}, \ldots, W_{1t_{1k_1}})$ and $(W_{2t_{21}}, \ldots, W_{2t_{2k_2}})$ are independent. We have

(i) $P(W_{10} + W_{20} = 0) \geq P(W_{10} = 0) = 1$

(ii) For $t_1 < t_2 < \cdots < t_k$, $W_{1t_i} + W_{2t_i} - W_{1t_{i-1}} - W_{2t_{i-1}}$

$= (W_{1t_i} - W_{1t_{i-1}}) + (W_{2t_i} - W_{2t_{i-1}})$ are mut. ind. for $i = 1, \ldots, k$ since the processes are independent and each has independent increments.

(iii) $\sim N(0, (t_i - t_{i-1}) + (t_i - t_{i-1})) = N(0, 2(t_i - t_{i-1}))$

$\therefore \{W_{1t} + W_{2t} : t \geq 0\}$ is a Wiener process.

(d) (5 marks) Suppose that $\{W_t : t \geq 0\}$ is a standard Wiener process and $X_t = at + \sigma W_t$ (a Brownian motion with drift). Determine $\lim_{t\to\infty} P(|X_t| > t)$.

$$P(|X_t| > t) = P(X_t < -t) + P(X_t > t)$$

$$= P(at + \sigma W_t < -t) + P(at + \sigma W_t > t)$$

$$= P\left(W_t < -\frac{t + at}{\sigma}\right) + P\left(W_t > \frac{t - at}{\sigma}\right)$$

$$= P\left(W_t < -\frac{t(1+a)}{\sigma}\right) + P\left(W_t > \frac{t(1-a)}{\sigma}\right)$$

$$= \Phi\left(-\frac{t(1+a)}{\sigma}\right) + 1 - \Phi\left(\frac{t(1-a)}{\sigma}\right)$$

$\rightarrow \begin{cases} 0 + 1 - 1 = 0, & -1 < a < 1 \\ \frac{1}{2} + 1 - 1 = \frac{1}{2}, & a = -1 \\ 1 + 1 - 1 = 1, & a < -1 \end{cases}$

$0 + 1 - \frac{1}{2} = \frac{1}{2}, \ a = 1$

$0 + 1 + 0 = 1, \ a > 1$