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Using Cauchy–Schwarz inequality we get for all $x \in \Re^n$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} |x_i| \cdot 1 \leq \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} 1^2 \right)^{1/2} = \sqrt{n}\|x\|_2$$

Such a bound does exist. Recall Hölder's inequality

$$\sum_{i=1}^{n} |a_i||b_i| \leq \left(\sum_{i=1}^{n} |a_i|^r \right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} |b_i|^{\frac{r}{r-1}} \right)^{1-\frac{1}{r}}$$

Apply it to the case $|a_i| = |x_i|^p$, $|b_i| = 1$ and $r = q/p > 1$

$$\sum_{i=1}^{n} |x_i|^p = \sum_{i=1}^{n} |x_i|^p \cdot 1 \leq \left(\sum_{i=1}^{n} (|x_i|^p)^{\frac{q}{p}} \right)^{\frac{p}{q}} \left(\sum_{i=1}^{n} 1^{\frac{q}{q-p}} \right)^{1-\frac{p}{q}} = \left(\sum_{i=1}^{n} |x_i|^q \right)^{\frac{p}{q}} n^{1-\frac{p}{q}}$$

Then

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \leq \left(\left(\sum_{i=1}^{n} |x_i|^q \right)^{\frac{p}{q}} n^{1-\frac{p}{q}} \right)^{1/p} = \left(\sum_{i=1}^{n} |x_i|^q \right)^{\frac{1}{q}} n^{\frac{1}{p}-\frac{1}{q}} = n^{1/p-1/q}\|x\|_q$$

In fact $C = n^{1/p-1/q}$ is the best possible constant.