Pollution Control 2

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Question

In this project we consider the good old pollution control problem in a different setting. We assume that there is a production unit whose (harmful) byproduct is discharged into a lake at the rate $u \geq 0$, $[u]$ = ton/day. The lake can accommodate a fixed **fraction** of this pollutant: δ, $[\delta] = 1/\text{day}$. From now on we will denote the stock of pollution by x, $[x]$ = ton.

For simplicity we assume that the amount of production is related to the amount of discharged pollutant. However, this dependence is not linear: $p = f(u)$. The function p satisfies a number of natural conditions:

1. $p(0) = 0$: no polluton, no production;
2. $p'(u) > 0$ for $u > 0$: the more we pollute, the more we product;
3. $p''(u) < 0$ for $u > 0$: in fancy words it is called the *law of decreasing marginal return*. That is to say, we cannot increase our production infinitely.

Find a function that satisfies these rules. You might try several functions. One possible approach consists in assuming that $u \in [0, u_{max}]$ (which is perfectly natural: there must be a limit to any pollution). In this case, your function $p(u)$ has to satisfy the conditions above only for u that belong to this interval. We can also assume that the production is measured in dollars (or any other currency at your choice).

We assume that the company wants to maximize its **total** profit over the fixed time interval of 5 years. The company does not bear any costs related to the pollutant discharge. However, **at the end of the period**, the company has to pay a fine that is proportional to the current amount of the pollutant, say D dollars per ton.

1. Describe the problem in mathematical terms. Write down the differential equation for the dynamics of the stock of pollutant and the functional to be maximized.
2. Choose 2 different models for the function $p(x)$. You may try any function that satisfies the conditions 1.–3. above. Note that some functions can be more convenient to deal with when you solve the optimal control problem. If you see that the equations turn out to be too complex, try another model.
3. Solve the optimal control problem for two different functions $p(x)$. Analyze the obtained results for some meaningful values of parameters. How does the choice of the productivity function $p(x)$ influence the result?
4. Systematize your results and write a report.

Solution

Question 1: Describe the problem in mathematical terms. Write down the differential equation for the dynamics of the pollutant and the functional to be maximized.

From the Question description, it's not hard to find that as the time change, the total amount of pollution increases u per unit time. Besides, consider the absorption of pollutants by the lake itself. Therefore we can write down the relationship in differential equation form.

$$\dot{x} = u - \delta x, x(0) = x_0 \geq 0 \tag{1}$$

There are many function $p(u)$ satisfy the following natural conditions:

1. $p(0) = 0$;
2. $p'(u) > 0$ for $u \in (0, u_{max}]$;
3. $p''(u) < 0$ for $u \in (0, u_{max}]$.

Before, $p(u)$ is the amount of production, now we assume $p(u)$ is the profit dollars/unit. Indeed do that change doesn't influence our optimal solution. Therefore, the instantaneous payoff function is defined as:

$$f_0(u) = p(u) \tag{2}$$

The company want to maximize it's **total** profit over the fixed time interval of 5 years and the company does not bear any costs related to the pollutant discharge. However, at **the end of the period**, the company has to pay a fine that is proportional to the current amount of the pollutant, say D dollars per ton. Therefore we have a terminal cost.

$$F_0(x(T)) = Dx(T) \tag{3}$$

Assume **endpoint** is **free**, then the total payoff function to be maximized is:

$$J(u, T) = \int_0^T p(u)dt - Dx(T) \to \max_u \tag{4}$$

Question 2: Choose 2 different models for the function $p(u)$ that satisfies the conditions 1.-3. above.

We need to assure the first order derivative large than 0, assume $u_{max} = b$. the following two models satisfy conditions 1-3[1].

[1]For model 2, $p'(u) > 0$ for $u > 0$ the maximal value occur at the boundary point b. But for model 1, $p'(u) > 0$ for $u \in (0, b)$ and $p'(b) = 0, p''(u)|_{u=b} < 0$, the maximal value occur at $u = b$.

$$\begin{aligned} &\textbf{Model 1: } p(u) = -\frac{1}{2}u^2 + bu \\ &\textbf{Model 2: } p(u) = \ln(u+1) \end{aligned} \tag{5}$$

Question 3: Solve the optimal control problem for two different functions $p(u)$. Analyze the obtained results for some meaningful values of parameters. How does the choice of the productivity function $p(u)$ influence the results.

Model 1: Our optimization problem is:

$$J(u,T) = \int_0^T \left(-\frac{1}{2}u^2 + bu\right)dt - Dx(T) \to \max_u \; s.t.(1) \tag{6}$$

a) Write down the Hamiltonian Function:

$$H(x,u,\psi) = -\frac{1}{2}u^2 + bu + \psi(u - \delta x) \tag{7}$$

b) It's first order partial derivatives w.r.t u is:

$$\frac{\partial}{\partial u}H(x,u,\psi) = -u + b + \psi \tag{8}$$

According to the first order extreme condition:

$$u^*(t) = b + \psi(t) \tag{9}$$

And $\frac{\partial^2}{\partial u^2}H(x,u,\psi)\Big|_{u=u^*} < 0$, we can conclude that the Hamiltonian H is concave w.r.t u.

c) We substituete (9) into (7) to get the maximal Hamiltonian Function:

$$\mathcal{H}(x,\psi) = H(x,u^*,\psi) = \frac{1}{2}(b+\psi)^2 - \delta\psi x \tag{10}$$

d) The canonical form is written as:

$$\begin{cases} \dot{x} = \dfrac{\partial}{\partial \psi}H(x,u,\psi)\Big|_{u=u^*} = b + \psi(t) - \delta x \\ \dot{\psi} = -\dfrac{\partial}{\partial x}H(x,u,\psi)\Big|_{u=u^*} = \delta\psi(t) \end{cases} \tag{11}$$

e) From D.E.S (11), it's not hard to find:

$$\psi(t) = \psi_0 e^{\delta t} \tag{12}$$

f) According to D.E.S (11) and Equation (12), we can find the optimal trajectory:

$$x^*(t) = x_0 e^{-\delta t} + \frac{b}{\delta}(1 - e^{-\delta t}) + \frac{\psi_0}{2\delta}(e^{\delta t} - e^{-\delta t}) \tag{13}$$

g) There is boundary condition on ψ:

$$\psi(T) = \psi_0 e^{\delta T} = -\frac{d}{dx} Dx(t)\bigg|_{t=T} = -D \tag{14}$$

Hence:

$$\psi_0 = -De^{-\delta T} \tag{15}$$

Then,

$$\psi(t) = -De^{\delta(t-T)} \tag{16}$$

h) Substitute Equation (16) into Equation (9), we can find the optimal control:

$$u^*(t) = b - De^{\delta(t-T)}, \text{ where } b \geq D \tag{17}$$

We need to assure $u^*(t) \geq 0$ for $t \in [0, T]$, so condition $b > D$ is necessary.

i) Substitute Equation (15) into Equation (13), we get:

$$x^*(t) = x_0 e^{-\delta t} + \frac{b}{\delta}(1 - e^{-\delta t}) - D\frac{e^{-\delta T}}{2\delta}(e^{\delta t} - e^{-\delta t}) \tag{18}$$

j) Find the first order derivative w.r.t t:

$$\frac{dx^*(t)}{dt} = -\mathrm{e}^{-\delta\,(T+t)} \left(\frac{D}{2} - b\,\mathrm{e}^{T\,\delta} + \frac{D\,\mathrm{e}^{2\,\delta\,t}}{2} + \delta\,x_0\,\mathrm{e}^{T\,\delta}\right) \tag{19}$$

Under the condition below:

$$\begin{aligned} b &\geq D \\ D - 2\,b\,\mathrm{e}^{T\,\delta} + 2\,\delta\,x_0\,\mathrm{e}^{T\,\delta} &< 0 \\ \frac{\sqrt{2\,b\,\mathrm{e}^{T\,\delta} - D - 2\,\delta\,x_0\,\mathrm{e}^{T\,\delta}}}{\sqrt{D}} &> 1 \\ D + 2\,\delta\,x_0\,\mathrm{e}^{T\,\delta} &\neq 2\,b\,\mathrm{e}^{T\,\delta} \\ \text{all parameters are large than } 0 \end{aligned} \tag{20}$$

We can find the stationary point:

$$t^* = \frac{\ln\left(\frac{\sqrt{2\,b\,\mathrm{e}^{T\,\delta} - D - 2\,\delta\,x_0\,\mathrm{e}^{T\,\delta}}}{\sqrt{D}}\right)}{\delta} \tag{21}$$

Obviously, $\frac{dx^*(t)}{dt} < 0$ for $t \in [0, t^*)$ and $\frac{dx^*(t)}{dt} \geq 0$ for $t \in [t^*, T]$.
So, $x^*(t)$ reachs maximal value at $t = t^*$.

k) Check the second sufficient condition for the existence of extreme values. Find the second order derivative w.r.t t:

$$\frac{d^2x^*(t)}{dt^2} = \delta^2\, x_0\, \mathrm{e}^{-\delta\, t} - b\, \delta\, \mathrm{e}^{-\delta\, t} - \frac{D\, \mathrm{e}^{-T\, \delta}\, \left(\delta^2\, \mathrm{e}^{\delta\, t} - \delta^2\, \mathrm{e}^{-\delta\, t}\right)}{2\, \delta} \tag{22}$$

Substitute t^* into Equation (22):

$$\left.\frac{d^2x^*(t)}{dt^2}\right|_{t=t^*} = -\sqrt{D}\, \delta\, \mathrm{e}^{-T\, \delta}\, \sqrt{2\, b\, \mathrm{e}^{T\, \delta} - D - 2\, \delta\, x_0\, \mathrm{e}^{T\, \delta}} < 0 \tag{23}$$

l) Assure $x^*(t) \geq 0,$ for $t \in [0, T]$.

Since the function $x^*(t)$ is concave for $t \in [0, T]$. To assure $x^*(t) \geq 0$ for all $t \in [0, T]$. We only need to make sure the boundary points large than 0, w.r.t $x_0 \geq 0$ and $x^*(T) \geq 0$.

Substitute $t = T$ into Equation (18), we can get the terminal value of x^* w.r.t $x^*(T)$:

$$x^*(T) = \frac{2\, b - D + D\, \mathrm{e}^{-2\, T\, \delta} - 2\, b\, \mathrm{e}^{-T\, \delta} + 2\, \delta\, x_0\, \mathrm{e}^{-T\, \delta}}{2\, \delta} \tag{24}$$

Obviously, $\delta > 0$, so we need add the condition:

$$2\, b - D + D\, \mathrm{e}^{-2\, T\, \delta} - 2\, b\, \mathrm{e}^{-T\, \delta} + 2\, \delta\, x_0\, \mathrm{e}^{-T\, \delta} \geq 0 \tag{25}$$

m) If we need to assure that the terminal value of pollutions is less than the initial value, we can abtain the conditon by the following Expression:

$$x^*(T) < x_0 \tag{26}$$

From the Expression (26), we can get :

$$\frac{2\, b - D + D\, \mathrm{e}^{-2\, T\, \delta} - 2\, b\, \mathrm{e}^{-T\, \delta} + 2\, \delta\, x_0\, \mathrm{e}^{-T\, \delta}}{2\, \delta} < x_0 \tag{27}$$

Summarize of Model 1:

After the above steps, we got some Conditions (20), (25) and (27), combine them, I get:

$$\begin{aligned} b &\geq D \\ b\, \mathrm{e}^{T\, \delta} -\ \delta\, x_0\, \mathrm{e}^{T\, \delta} &> D \\ D + 2\, \delta\, x_0\, \mathrm{e}^{T\, \delta} &\neq 2\, b\, \mathrm{e}^{T\, \delta} \\ 2b(1 - e^{-T\delta}) - D(1 - e^{-2T\delta}) + 2\delta x_0 e^{-T\delta} &\geq 0 \\ 2b(1 - e^{-T\delta}) - D(1 - e^{-2T\delta}) + 2\delta x_0 (e^{-T\delta} - 1) &< 0 \\ \text{all parameters are large than } 0 & \end{aligned} \tag{28}$$

The following is also those conditions but genereted by MATLAB. The condition above is equivalent to the condition below.

$$\left(\begin{array}{c} D \leq b \\ 0 \leq 2\,b - D + D\,\mathrm{e}^{-2\,T\,\delta} - 2\,b\,\mathrm{e}^{-T\,\delta} + 2\,\delta\,x_0\,\mathrm{e}^{-T\,\delta} \\ 2\,b - D + D\,\mathrm{e}^{-2\,T\,\delta} - 2\,b\,\mathrm{e}^{-T\,\delta} + 2\,\delta\,x_0\,\mathrm{e}^{-T\,\delta} \leq 2\,\delta\,x_0 \\ 0 < D \\ 0 < T \\ 1 < \frac{\sqrt{2\,b\,\mathrm{e}^{T\,\delta} - D - 2\,\delta\,x_0\,\mathrm{e}^{T\,\delta}}}{\sqrt{D}} \\ 0 < b \\ 0 < \delta \\ 0 < t \\ 0 < x_0 \\ D - 2\,b\,\mathrm{e}^{T\,\delta} + 2\,\delta\,x_0\,\mathrm{e}^{T\,\delta} < 0 \\ D + 2\,\delta\,x_0\,\mathrm{e}^{T\,\delta} \neq 2\,b\,\mathrm{e}^{T\,\delta} \end{array}\right) \tag{29}$$

Now, I choose some parameters which satisfy the conditions (28). When T is too large, it is difficult to select other parameters that meet the above conditions. So, I choose a small one that $T = 5$. Corresponding, we need to change unit of our time from day to year. $x_0 = 900, b = 1000, D = 500, \delta = 1$.

The Figure of optimal control and total pollution as shown in Figure 1 and Figure 2:

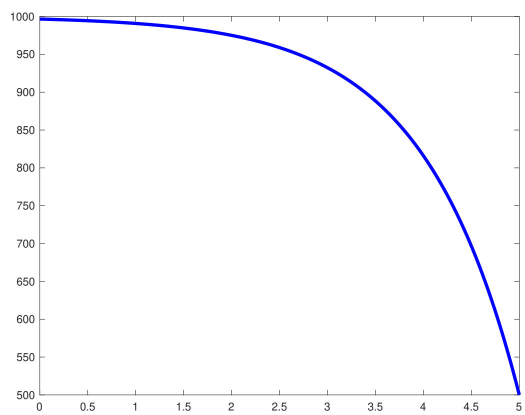

Figure 1: Optimal Control - Model 1

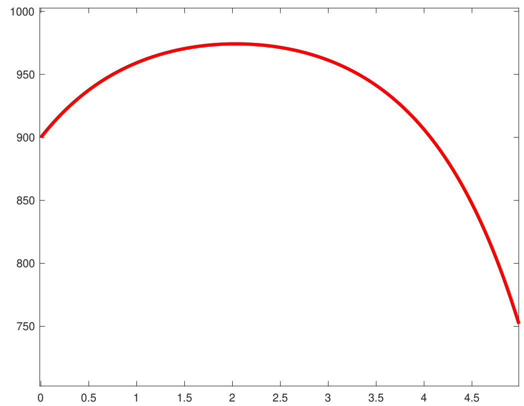

Figure 2: Total Pollution - Model 1

Model 2: Our optimization problem is:

$$J(u,T) = \int_0^T \ln(u+1)dt - Dx(T) \to \max_u \; s.t.(1) \tag{30}$$

a) Write down the Hamiltonian Function:

$$H(x,u,\psi) = \ln(u+1) + \psi(u - \delta x) \tag{31}$$

b) It's first order partial derivatives w.r.t u is:

$$\frac{\partial}{\partial u} H(x,u,\psi) = \frac{1}{u+1} + \psi \tag{32}$$

According to the first order extreme condition:

$$u^*(t) = -\frac{1}{\psi} - 1 \tag{33}$$

And $\frac{\partial^2}{\partial u^2} H(x,u,\psi)\big|_{u=u^*} < 0$, we can conclude that the Hamiltonian H is concave w.r.t u.

c) We substituete (33) into (31) to get the maximal Hamiltonian Function:

$$\mathcal{H}(x,\psi) = H(x,u^*,\psi) = -\ln(-\psi) - (1+\psi) - \psi\delta x \tag{34}$$

d) The canonical form is written as:

$$\begin{cases} \dot{x} = \dfrac{\partial}{\partial \psi} H(x,u,\psi)\Big|_{u=u^*} = -\dfrac{1}{\psi} - 1 - \delta x \\ \dot{\psi} = -\dfrac{\partial}{\partial x} H(x,u,\psi)\Big|_{u=u^*} = \delta\psi(t) \end{cases} \tag{35}$$

e) From D.E.S (35), it's not hard to find:

$$\psi(t) = \psi_0 e^{\delta t} \tag{36}$$

f) According to D.E.S (35) and Equation (36), we can find the optimal trajectory:

$$x^*(t) = \mathrm{e}^{-\delta\, t} \left(x_0 + \frac{1}{\delta} \right) - \mathrm{e}^{-\delta\, t} \left(\frac{t}{\psi_0} + \frac{\mathrm{e}^{\delta\, t}}{\delta} \right) \tag{37}$$

g) There is boundary condition on ψ:

$$\psi(T) = \psi_0 e^{\delta T} = -\frac{d}{dx} Dx(t) \bigg|_{t=T} = -D \tag{38}$$

Hence:

$$\psi_0 = -De^{-\delta T} \tag{39}$$

Then,

$$\psi(t) = -De^{\delta(t-T)} \tag{40}$$

h) Substitute Equation (40) into Equation (33), we can find the optimal control:

$$u^*(t) = \frac{e^{\delta(T-t)}}{D} - 1, \text{ where } \frac{e^{\delta T}}{D} - 1 \leq b, \text{ and } 0 \leq D \leq 1 \tag{41}$$

We need to assure $u^*(t) \in [0, b]$ for $t \in [0, T]$, so condition $\frac{e^{\delta T}}{D} - 1 \leq b$ and $0 \leq D \leq 1$ is necessary,

i) Substitute Equation (39) into Equation (37), we get:

$$x^*(t) = \mathrm{e}^{-\delta\, t} \left(x_0 + \frac{1}{\delta} \right) - \mathrm{e}^{-\delta\, t} \left(-\frac{te^{\delta T}}{D} + \frac{\mathrm{e}^{\delta\, t}}{\delta} \right) \tag{42}$$

j) Find the first order derivative w.r.t t:

$$\frac{dx^*(t)}{dt} = -\frac{\mathrm{e}^{-\delta\, t} \left(D - \mathrm{e}^{T\,\delta} + D\,\delta\, x_0 + \delta\, t\, \mathrm{e}^{T\,\delta} \right)}{D} \tag{43}$$

Under the condition below:

$$\begin{aligned} \frac{e^{\delta T}}{D} - 1 &\leq b \\ D &\leq 1 \\ D + D\,\delta\, x_0 &< \mathrm{e}^{T\,\delta} \\ \text{all parameters are large than } 0 \end{aligned} \tag{44}$$

We can find the stationary point:

$$t^* = -\frac{\mathrm{e}^{-T\,\delta}\,\left(D - \mathrm{e}^{T\,\delta} + D\,\delta\,x_0\right)}{\delta} \tag{45}$$

Obviously, $\frac{dx^*(t)}{dt} < 0$ for $t \in [0, t^*)$ and $\frac{dx^*(t)}{dt} \geq 0$ for $t \in [t^*, T]$.

So, $x^*(t)$ reachs maximal value at $t = t^*$.

k) Check the second sufficient condition for the existence of extreme values. Find the second order derivative w.r.t t:

$$\frac{d^2x^*(t)}{dt^2} = \frac{\delta\,\mathrm{e}^{-\delta\,t}\,\left(D - 2\,\mathrm{e}^{T\,\delta} + D\,\delta\,x_0 + \delta\,t\,\mathrm{e}^{T\,\delta}\right)}{D} \tag{46}$$

Substitute t^* into Equation (46):

$$\left.\frac{d^2x^*(t)}{dt^2}\right|_{t=t^*} = -\frac{\delta\,\mathrm{e}^{T\,\delta}\,\mathrm{e}^{D\,\mathrm{e}^{-T\,\delta} + D\,\delta\,x_0\,\mathrm{e}^{-T\,\delta} - 1}}{D} < 0 \tag{47}$$

So, $x^*(t)$ reachs maximal value at $t = t^*$.

l) Assure $x^*(t) \geq 0$, for $t \in [0, T]$.

Since the function $x^*(t)$ is concave for $t \in [0, T]$. To assure $x^*(t) \geq 0$ for all $t \in [0, T]$. We only need to make sure the boundary points large than 0, w.r.t $x_0 \geq 0$ and $x^*(T) \geq 0$.

Substitute $t = T$ into Equation (42), we can get the terminal value of x^* w.r.t $x^*(T)$:

$$x^*(T) = \frac{T\,\delta - D + D\,\mathrm{e}^{-T\,\delta} + D\,\delta\,x_0\,\mathrm{e}^{-T\,\delta}}{D\,\delta} \tag{48}$$

Obviously, $D\delta > 0$, so we need add the condition:

$$T\,\delta - D + D\,\mathrm{e}^{-T\,\delta} + D\,\delta\,x_0\,\mathrm{e}^{-T\,\delta} \geq 0 \tag{49}$$

m) If we need to assure that the terminal value of pollutions is less than the initial value, we can abtain the conditon by the following Expression:

$$x^*(T) < x_0 \tag{50}$$

From the Expression (50), we can get :

$$\frac{T\,\delta - D + D\,\mathrm{e}^{-T\,\delta} + D\,\delta\,x_0\,\mathrm{e}^{-T\,\delta}}{D\,\delta} < x_0 \tag{51}$$

Summarize of Model 2:

After the above steps, we got some Conditions (44), (49) and (51), combine them, I get:

$$\begin{aligned} \frac{e^{\delta T}}{D} - 1 &\le b \\ D &\le 1 \\ D + D\,\delta\,x_0 &< \mathrm{e}^{T\,\delta} \\ T\,\delta - D + D\,\mathrm{e}^{-T\,\delta} + D\,\delta\,x_0\,\mathrm{e}^{-T\,\delta} &\ge 0 \\ T\delta + D(e^{-T\delta} - 1)(1 + \delta x_0) &< 0 \\ \text{all parameters are large than } 0 & \end{aligned} \tag{52}$$

The following is also those conditions but genereted by MATLAB. The condition above is equivalent to the condition below.

$$\left(\begin{array}{c} D + D\,\delta\,x_0 < \mathrm{e}^{T\,\delta} \\ 0 \le T\,\delta - D + D\,\mathrm{e}^{-T\,\delta} + D\,\delta\,x_0\,\mathrm{e}^{-T\,\delta} \\ \frac{\mathrm{e}^{T\,\delta}}{D} - 1 \le b \\ 0 < D \\ D \le 1 \\ 0 < T \\ 0 < b \\ 0 < \delta \\ 0 < t \\ 0 < x_0 \\ \frac{T\,\delta - D + D\,\mathrm{e}^{-T\,\delta} + D\,\delta\,x_0\,\mathrm{e}^{-T\,\delta}}{D\,\delta} < x_0 \end{array}\right) \tag{53}$$

Now, I choose some parameters which satisfy the conditions (52). When T is too large, it is difficult to select other parameters that meet the above conditions. So, I choose a small one that $T = 5$. Corresponding, we need to change unit of our time from day to year. $x_0 = 100, b = 200, D = 0.9, \delta = 1$.

The Figure of optimal control and total pollution as shown in Figure 3 and Figure 4:

Summary:

From the first order derivative of $x^*(t)$ w.r.t t, Equation (19) and Equation (43), we can get some conditions which is necessary since we need to assure $x^*(t) \ge 0$ for all $t \in [0, T]$ and $x(T) \le x_0$. By comparing these conditions, it's easily to find that the conditions of logarithmic form utility function is simpler be find than quadratic form utility function. The reason is the different location of t marked in the following Equation. The previous one is exponential function, and another is linear function. So, the Expressions of Conditions of Model 2 is simpler than Model 1.

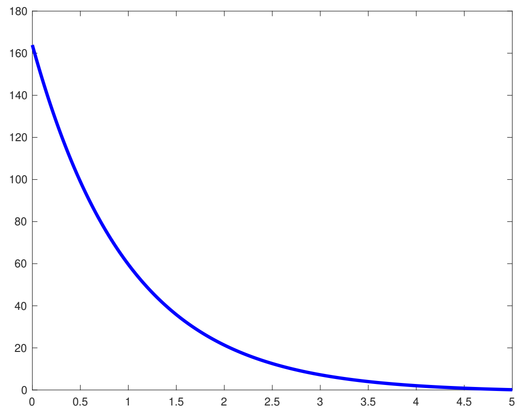

Figure 3: Optimal Control - Model 2

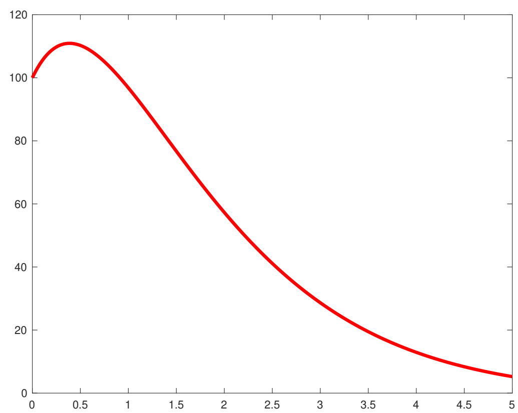

Figure 4: Total Pollution - Model 2

$$\textbf{Model 1}: \frac{dx^*(t)}{dt} = -\mathrm{e}^{-\delta\,(T+t)}\,\left(\frac{D}{2} - b\,\mathrm{e}^{T\,\delta} + \frac{D\,\mathrm{e}^{2\,\delta\,t}}{2} + \delta\,x_0\,\mathrm{e}^{T\,\delta}\right)$$

$$\textbf{Model 2}: \frac{dx^*(t)}{dt} = -\frac{\mathrm{e}^{-\delta\,t}\,\left(D - \mathrm{e}^{T\,\delta} + D\,\delta\,x_0 + \delta\,t\,\mathrm{e}^{T\,\delta}\right)}{D}$$

The utility function can influence the results. The optimal control is dependent to the utilify function. And the optimal trajectory is dependent to the optimal control. Then, for different utility functions, the company can use different controls to achieve the goal of maximizing profits.