Pollution Control 2

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Question

In this project we consider the good old pollution control problem in a different setting. We assume that there is a production unit whose (harmful) byproduct is discharged into a lake at the rate $u \geq 0$, $[u]$ = ton/day. The lake can accommodate a fixed **fraction** of this pollutant: δ, $[\delta] = 1/\text{day}$. From now on we will denote the stock of pollution by x, $[x]$ = ton.

For simplicity we assume that the amount of production is related to the amount of discharged pollutant. However, this dependence is not linear: $p = f(u)$. The function p satisfies a number of natural conditions:

1. $p(0) = 0$: no polluton, no production;
2. $p'(u) > 0$ for $u > 0$: the more we pollute, the more we product;
3. $p''(u) < 0$ for $u > 0$: in fancy words it is called the *law of decreasing marginal return*. That is to say, we cannot increase our production infinitely.

Find a function that satisfies these rules. You might try several functions. One possible approach consists in assuming that $u \in [0, u_{max}]$ (which is perfectly natural: there must be a limit to any pollution). In this case, your function $p(u)$ has to satisfy the conditions above only for u that belong to this interval. We can also assume that the production is measured in dollars (or any other currency at your choice).

We assume that the company wants to maximize its **total** profit over the fixed time interval of 5 years. The company does not bear any costs related to the pollutant discharge. However, **at the end of the period**, the company has to pay a fine that is proportional to the current amount of the pollutant, say D dollars per ton.

1. Describe the problem in mathematical terms. Write down the differential equation for the dynamics of the stock of pollutant and the functional to be maximized.
2. Choose 2 different models for the function $p(x)$. You may try any function that satisfies the conditions 1.–3. above. Note that some functions can be more convenient to deal with when you solve the optimal control problem. If you see that the equations turn out to be too complex, try another model.
3. Solve the optimal control problem for two different functions $p(x)$. Analyze the obtained results for some meaningful values of parameters. How does the choice of the productivity function $p(x)$ influence the result?
4. Systematize your results and write a report.

Solution

Question 1: Describe the problem in mathematical terms. Write down the differential equation for the dynamics of the pollutant and the functional to be maximized.

From the Question description, it's not hard to find that as the time change, the ratio u is directly proportional to the total pollutant x. Besides, consider the absorption of pollutants by the lake itself. Therefore we can write down the relationship in differential equation form.

$$\dot{x} = u - \delta, x(0) = x_0 \text{ and } x(T) = x_T \text{ only for fixed endpoint} \tag{1}$$

There are many function $p(u)$ satisfy the following natural conditions:

1. $p(0) = 0$;
2. $p'(u) > 0$ for $u \in (0, u_{max}]$;
3. $p''(u) < 0$ for $u \in (0, u_{max}]$.

$p(u)$ is the amount of production, assume the profit is a dollars/unit. Therefore, the instantaneous payoff function is defined as:

$$f_0(u) = ap(u) \tag{2}$$

The company want to maximize it's **total** profit over the fixed time interval of 5 years and the company does not bear any costs related to the pollutant discharge. However, at **the end of the period**, the company has to pay a fine that is proportional to the current amount of the pollutant, say D dollars per ton. Therefore we have a terminal cost.

$$F_0(x(T)) = Dx(T) \tag{3}$$

If **endpoint** is **fixed**, then Equation (3) is a constant. It doesn't influence out optimization problem. And a is a constant, hence maximize Function (2) is equivalent to maximize Function $p(u)$.

The total payoff function to be maximized is:

$$J(u, T) = \int_0^T p(u)dt \to \max_u \tag{4}$$

To simplisfy the calculation, in the following I use $D(0, 1]$ instead of $\frac{D}{a}$. [1]

If **endpoint** is **free**, then the total payoff function to be maximized is:

$$J(u, T) = \int_0^T p(u)dt - Dx(T) \to \max_u \tag{5}$$

[1] In deed, the fundamental operation of the optimized equation and constant does not affect the value of the solution. The original expression $\frac{D}{a} \leq 1$, which means the penalty in the terminal time doesn't large than profit per/ton.

Question 2: Choose 2 different models for the function $p(u)$ that satisfies the conditions 1.-3. above.

Assume, $u_{max} = b$, the following two models satisfy conditions 1-3.

$$\begin{aligned} &\textbf{Model 1: } p(u) = -\frac{1}{2}u^2 + bu \\ &\textbf{Model 2: } p(u) = \ln(u+1) \end{aligned} \tag{6}$$

Question 3: Solve the optimal control problem for two different functions $p(u)$. Analyze the obtained results for some meaningful values of parameters. How does the choice of the productivity function $p(u)$ influence the results.

Don't consider leap years, in this case, $T = 5 \times 365 = 1825$.

Model 1-1(no terminal cost): Our optimization problem is:

$$J(u,T) = \int_0^T \left(-\frac{1}{2}u^2 + bu\right)dt \to \max_u \ s.t.(1) \tag{7}$$

a) Write down the Hamiltonian Function:

$$H(x,u,\psi) = -\frac{1}{2}u^2 + bu + \psi(u - \delta) \tag{8}$$

b) It's first order partial derivatives w.r.t u is:

$$\frac{\partial}{\partial u}H(x,u,\psi) = -u + b + \psi \tag{9}$$

According to the first order extremality condition:

$$u^*(t) = b + \psi(t) \tag{10}$$

And $\frac{\partial^2}{\partial u^2}H(x,u,\psi)\Big|_{u=u^*} < 0$, we can conclude that the Hamiltonian H is concave w.r.t u.

c) We substituete (10) into (8) to get the maximal Hamiltonian Function:

$$\mathcal{H}(x,\psi) = H(x,u^*,\psi) = \frac{1}{2}(b+\psi)^2 - \delta\psi \tag{11}$$

d) The canonical form is writen as:

$$\begin{cases} \dot{x} = \dfrac{\partial}{\partial \psi}H(x,u,\psi)\Big|_{u=u^*} = b + \psi(t) - \delta \\ \dot{\psi} = -\dfrac{\partial}{\partial x}H(x,u,\psi)\Big|_{u=u^*} = 0 \end{cases} \tag{12}$$

e) From D.E.S (12), it's not hard to find that $\psi(t) \equiv \psi_0$, ψ_0 is a constant.

f) According to D.E.S (12) and $\psi(t) \equiv \psi_0$, we can find the optimal trajectory:

$$x^*(t) = x_0 + (b + \psi_0 - \delta)t \tag{13}$$

g) From Equation (13) and initial condition $x(T) = x_T$, we can find:

$$\psi(t) \equiv \psi_0 = \frac{x_T - x_0}{T} - b + \delta \tag{14}$$

h) Substitute ψ_0 into Equation (10), we can find the optimal control:

$$u^*(t) \equiv \frac{x_T - x_0}{T} + \delta, \text{ where } 0 \leq \frac{x_T - x_0}{T} + \delta \leq b \tag{15}$$

i) Substitute ψ_0 into Equation (13), we can find the optimal trajectory:

$$x^*(t) = x_0 + \frac{x_T - x_0}{T}t \tag{16}$$

j) During the optimal control (15), the total profit is:

$$V = aJ(u^*, T) = \frac{a(x_T - x_0)^2}{2T} + a\delta(x_T - x_0) + \frac{a\delta^2 T}{2} + ab(x_T - x_0) + ab\delta T \tag{17}$$

Model 1-2(terminal cost): Our optimization problem is:

$$J(u, T) = \int_0^T \left(-\frac{1}{2}u^2 + bu\right)dt - Dx(T) \rightarrow \max_u \ s.t.(1) \tag{18}$$

Same procedure as **Model 1-1** item a) to f). The difference between them is there are no boundary conditions on ψ in Model 1-1. But for Model 1-2 is not.

$$\psi(T) = \psi_0 = -\frac{d}{dx}Dx(t)\bigg|_{t=T} = -D \tag{19}$$

Hence, $\psi(t) \equiv -D$.

- Substitute $\psi(t) \equiv -D$ into Equation (10), we can find the optimal control:

$$u^*(t) = b - D, \text{ where } b \geq D \tag{20}$$

  To maximize the total profit, we should set the maximum control to be greater than D.

- Substitute $\psi(t) \equiv -D$ into Equation (13), we get:

$$x^*(t) = x_0 + (b - D - \delta)t \tag{21}$$

  Define $t^* = \frac{-x_0}{b-D-\delta}$.

Since, $x(t) \geq 0$, there are two cases for optimal trajectory:

Case 1: $b - D \geq \delta$

$$x^*(t) = x_0 + (b - D - \delta)t \tag{22}$$

Case 2: $b - D < \delta$

$$\begin{cases} x^*(t) = x_0 + (b - D - \delta)t & 0 \leq t \leq t^* \\ x^(t) = 0 & t > t^* \end{cases} \tag{23}$$

- Finally, the optimal profit is:
  - Fot $T \leq t^*$:

$$\begin{aligned} V = aJ(u^*, T) &= a \int_0^T \big(-\frac{1}{2}u^2 + bu \big)\Big|_{u=u^*} dt - aDx(T) \\ &= \frac{aT}{2}(b^2 - D^2) - aD\big[x_0 + (b - D - \delta)T\big] \\ &= \frac{aT}{2}(b^2 + D^2) - aD\big[x_0 + (b - \delta)T\big] \end{aligned} \tag{24}$$

  - For $T \geq t^*$, there are no terminal costs:

$$V = \frac{aT}{2}(b^2 - D^2) \tag{25}$$

Model 2-1(no terminal cost): Our optimization problem is:

$$J(u, T) = \int_0^T \ln(u + 1)dt \to \max_u \ s.t.(1) \tag{26}$$

a) Write down the Hamiltonian Function:

$$H(x, u, \psi) = \ln(u + 1) + \psi(u - \delta) \tag{27}$$

b) It's first order partial derivatives w.r.t u is:

$$\frac{\partial}{\partial u} H(x, u, \psi) = \frac{1}{u + 1} + \psi \tag{28}$$

According to the first order extremality condition:

$$u^*(t) = -\frac{1 + \psi}{\psi} \tag{29}$$

And $\frac{\partial^2}{\partial u^2} H(x, u, \psi)\big|_{u=u^*} < 0$, we can conclude that the Hamiltonian H is concave w.r.t u.

c) We substituete (29) into (27) to get the maximal Hamiltonian Function:

$$\mathcal{H}(x, \psi) = H(x, u^*, \psi) = -\ln(-\psi) - (1 + \psi) - \psi\delta \tag{30}$$

d) The canonical form is writen as:

$$\begin{cases} \dot{x} = \dfrac{\partial}{\partial \psi} H(x,u,\psi)\Big|_{u=u^*} = -\dfrac{1+\psi}{\psi} - \delta \\ \dot{\psi} = -\dfrac{\partial}{\partial x} H(x,u,\psi)\Big|_{u=u^*} = 0 \end{cases} \tag{31}$$

e) From D.E.S (31), it's not hard to find that $\psi(t) \equiv \psi_0$, ψ_0 is a constant.

f) According to D.E.S (31) and $\psi(t) \equiv \psi_0$, we can find the optimal trajectory:

$$x^*(t) = x_0 - \left(\frac{1+\psi_0}{\psi_0} + \delta\right)t \tag{32}$$

g) From Equation (32) and initial condition $x(T) = x_T$, we can find:

$$\psi(t) \equiv \psi_0 = \frac{T}{x_0 - x_T - T(1+\delta)} \tag{33}$$

h) Substitute ψ_0 into Equation (29), we can find the optimal control:

$$u^*(t) \equiv \frac{x_T - x_0}{T} + \delta, \text{ where } 0 \leq \frac{x_T - x_0}{T} + \delta \leq b \tag{34}$$

i) Substitute ψ_0 into Equation (32), we can find the optimal trajectory:

$$x^*(t) = x_0 + \frac{x_T - x_0}{T} t \tag{35}$$

j) During the optimal control (34), the total profit is:

$$V = aJ(u^*,T) = -aT\ln(-\psi_0) = aT\Big[\ln\big(x_T - x_0 + T(1+\delta)\big) - \ln T\Big] \tag{36}$$

Model 2-2(terminal cost): Our optimization problem is:

$$J(u,T) = \int_0^T \ln(u+1)dt - Dx(T) \to \max_u \ s.t.(1) \tag{37}$$

Same procedure as **Model 2-1** item a) to f). The difference between them is there are no boundary conditions on ψ in Model 2-1. But for Model 2-2 is not.

$$\psi(T) = \psi_0 = -\frac{d}{dx} Dx(t)\Big|_{t=T} = -D \tag{38}$$

Hence, $\psi(t) \equiv -D$.

- Substitute $\psi(t) \equiv -D$ into Equation (29), we can find the optimal control:

$$u^*(t) = \frac{1-D}{D}, \text{ where } b \geq \frac{1-D}{D} \tag{39}$$

- Substitute $\psi(t) \equiv -D$ into Equation (32), we get:

$$x^*(t) = x_0 + \left(\frac{1-D}{D} - \delta\right)t \tag{40}$$

Define $t^* = \frac{-x_0 D}{1-D-D\delta}$.
There are also two cases for x:
Case 1: $u^* \geq \delta$

$$x^*(t) = x_0 + \left(\frac{1-D}{D} - \delta\right)t \tag{41}$$

Case 2: $u^* < \delta$

$$\begin{cases} x^*(t) = x_0 + \left(\frac{1-D}{D} - \delta\right)t & 0 \leq t \leq t^* \\ x^*(t) = 0 & t > t^* \end{cases} \tag{42}$$

- Finally, the optimal profit is:
  - For $T \leq t^*$

$$V = aJ(u^*, T) = -aT \ln D - a\big[Dx_0 + (1 - D - \delta D)T\big] \tag{43}$$

  - For $T > t^*$, there are no terminal costs:

$$V = -aT \ln D \tag{44}$$