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Topic: Seminar. Using of the Pontryagin maximum principle. Linear- quadratic differential game. Linear- quadratic differential game with terminal payoff.

A Cooperative Differential Game of Pollution Control

Model

Consider a game-theoretic model of pollution control. There are 3 players (companies, countries) that participate in the game, $N = \{1, 2, 3\}$. Each player has an industrial production site. It is assumed that the production is proportional to the pollutions u_i. Thus, the strategy of a player is to choose the amount of pollutions emited to the atmosphere, $u_i \in [0; b_i]$. In this example the solution will be considered in the class of open-loop strategies $u_i(t)$.

The dynamics of the total amount of pollution $x(t)$ is described by

$$\dot{x} = u_1 + u_2 + u_3 - \delta x, \quad x(t_0) = x_0,$$

where δ is the absorption coefficient corresponding to the natural purification of the atmosphere.

In the following we assume that the absorption coefficient δ is equal to zero:

$$\dot{x} = u_1 + u_2 + u_3, \quad x(t_0) = x_0. \tag{1}$$

The instantaneous payoff of i-th player is defined as:

$$R(u_i(t)) = b_i u_i(t) - \frac{1}{2}u_i^2(t), \qquad i \in N.$$

Each player has to bear expences due to the pollution removal. Thus the instantaneous payoff (utility) of the i-th player is equal to $R(u_i(t)) - d_i x(t), d_i > 0$.

Thus the integral payoff of the i-th player is defined as

$$\int\limits_{t_0}^{T} \left(\left(b_i - \frac{1}{2}u_i \right) u_i - d_i x \right) dt, \quad i = 1, 2, 3. \tag{2}$$

Let us consider the same game with additional cost (which is proportional to the amount of pollution) at the terminal time T.

Thus the payoff of the i-th player is defined as

$$K_i(x_0, T - t_0, u) = \int\limits_{t_0}^{T} \left(\left(b_i - \frac{1}{2}u_i \right) u_i - d_i x \right) dt - Dx_i(T), \quad i = 1, 2, 3. \tag{3}$$

Then the optimization problem is as follows:

$$\sum_{i=1}^{3} K_i(t_0, x_0, T, u) = \sum_{i=1}^{3} \int\limits_{t_0}^{T} \left(\left(b_i - \tfrac{1}{2}u_i \right) u_i - d_i x \right) dt - \sum_{i=1}^{3} D_i x(T) \to \max_{u_1, u_2, u_3}, \tag{4}$$

$s.t. x(t)$ satisfies (1).

Problem:

1. Find $u_1^*(t)$, $u_2^*(t)$, $u_3^*(t)$ by the Pontryagin maximum principle.
2. Plot the graphs of optimal controls (for several sets of parameters b_i, d_i, D_i). How the solution will be changed depending on parameters?
3. Find $x^*(t)$ — cooperative (optimal) trajectory. Plot the graph of optimal trajectory (for several sets of parameters b_i, d_i, D_i). How the solution will be changed depending on parameters?
4. Calculate value of the total maximal payoff $\sum_{i=1}^{3} K_i(x_0, u^*)$.

Hints:

1. The algorithm of the PMP using is changed for modified model. We consider problem with integral and terminal payoff $H(x(T))$. So we have the following bound condition on adjoint variable $\psi(t)$:

$$\psi(T) = \frac{d}{dt} H(x(t))|_{t=T}. \tag{5}$$

Solution First write down the Hamiltonian function:

$$H(x,\psi,u) = u_1\left(b_1 - \frac{u_1}{2}\right) - d\,x + u_2\left(b_2 - \frac{u_2}{2}\right) + u_3\left(b_3 - \frac{u_3}{2}\right) + \psi\,\left(u_1 + u_2 + u_3\right),$$

where ψ is the adjoint variable and $d = d_1 + d_2 + d_3$.

Taking the first derivative with respect to u_i we get the expressions for the optimal controls:

$$u_i^* = b_i + \psi, \quad i = 1, 2, 3.$$

One can check that the respective second order derivatives are negative, hence the computed controls do indeed maximize the Hamiltonian.

The canonical system is written as

$$\begin{cases} \dot{x} = u_1 + u_2 + u_3, \\ \dot{\psi} = d. \end{cases}$$

Now substitute the optimal controls u_i^* to get the final form

$$\begin{cases} \dot{x} = b + 3\psi, \\ \dot{\psi} = d, \end{cases} \tag{6}$$

where $b = b_1 + b_2 + b_3$. So, we see that the canonical system does not depend on x which makes it easy to solve. Recall that the initial condition is $x(0) = x_0$. We need another boundary condition, which is obtained from the rule (5):

$$\psi(T) = \frac{d}{dx} Dx(t)\bigg|_{t=T} = -D,$$

where $D = D_1 + D_2 + D_3$. Now we can compute

$$\psi(t) = \psi(0) + dt,$$

which yields $\psi_0 + dT = \psi(T) = -D$ and $\psi_0 = -D - d \cdot T$. Finally, we get

$$\psi(t) = -D - d \cdot T + dt = -D - d(T - t).$$

Substitute this solution to the first differential equation in (6) to obtain the expression for $x(t)$:

$$x(t) = x(0) + \frac{3d\,t^2}{2} + (b - 3D - 3T\,d)\ t.$$

The optimal controls are

$$u_i^*(t) = b_i - D - d(T - t).$$

Using all these data we can compute the optimal value of the payoff function to be

$$K_i(0, x_0, T, u^*) = \frac{3\,D^2\,T}{2} + \frac{3\,D\,T^2\,d}{2} - b\,D\,T - x_0\,D + \frac{T^3\,d^2}{2} - \frac{b\,T^2\,d}{2} + \frac{T}{2}(b_1^2 + b_2^2 + b_3^2) - x_0\,T\,d$$

References:

1. Lecture course on Control theory, Dr. Dmitry Gromov
2. Pontryagin, Lev Semenovich. Mathematical theory of optimal processes. Routledge, 2018.
3. Gromova, Ekaterina. "The Shapley value as a sustainable cooperative solution in differential games of three players." Recent Advances in Game Theory and Applications. Birkhauser, Cham, 2016. 67–89.

Good luck!