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Topic: Seminar. Using of the Pontryagin maximum principle. Differential game with discounting and infinite time horizon. Logarithmic utility function.

Time to perform: 14 April, 2020

Questions, answers: by email e.v.gromova @ spbu. ru

A model with logarithmic utility function Consider the model of joint exploitation of a non-renewable resource, e.g., an oil reservoir. The dynamics of the total resource stock is described by the DE

$$\dot{x} = -\sum_{i=1}^{3} u_i,$$

where $u(t) \in [0, u_{max}]$ for all t and $u_{max} \geq 0$. **Note the minus sign in front of u_i in the DE above.**

The payoff of the i-th player is defined as

$$K_i(x_0, u) = \int_0^\infty e^{-\rho t} \ln(u_i(t))dt, \quad i = 1, 2, 3.$$

Find the optimal controls guaranteeing maximization of the total payoff

$$\sum_{i=1}^{3} K_i(x_0, u) \to \max .$$

Problem:

1. Find $u_1^*(t)$, $u_2^*(t)$, $u_3^*(t)$ by the Pontryagin maximum principle.
2. Plot the graphs of optimal controls.
3. Find $x^*(t)$ — cooperative (optimal) trajectory. Plot the graph of optimal trajectory.
4. Calculate value of the total maximal payoff $\sum_{i=1}^{3} K_i(x_0, u^*)$.

Solution: Write down the Hamiltonian:

$$H(t, x, \psi, u) = e^{-\rho t} \ln(u_1 u_2 u_3)dt - \psi(u_1 + u_2 + u_3).$$

The optimal controls are

$$u_i^*(t) = \frac{e^{-\rho t}}{\psi} \tag{1}$$

Now we write the differential equation for ψ. Since the Hamiltonian does not depend on the state x we have $\dot{\psi} = 0$ and so, $\psi(t)$ is constant for all t. Let's set $\psi(t) = \psi^*$, where ψ^* is some yet unknown value.

Now we need to employ a boundary condition. If we take the simpler one, i.e., $\lim_{t\to\infty}\psi(t) = 0$, we get $\psi^* = 0$ and so, all optimal controls turn out to be infinity. Instead of this, we choose $u_i^* = u_{\max}$. With such controls, the solution $x(t)$ takes the following form:

$$x(t) = x_0 - 3u_{max}t.$$

Our resource cannot be negative. Therefore, at time $\bar{t} = x_0/3u_{max}$ we have to stop extraction because there is nothing left.

Let's compute the optimal total payoff:

$$\sum_{i=1}^{3} K_i(x_0, u) = \int_0^{\bar{t}} e^{-\rho t}\ln(3u_{max})dt + \int_{\bar{t}}^{\infty} e^{-\rho t}\ln(0)dt.$$

Since $\ln(0) = -\infty$, the total payoff will be $-\infty$. Quite a bad performance. Let's try another solution.

Instead of the "simple" boundary condition we take a more general one: $\lim_{t\to\infty} x(t)\psi(t) = 0$. Since our $\psi(t)$ is constant and non-zero, we have

$$\lim_{t\to\infty} x(t) = 0.$$

In this case ψ^* is not zero any longer, so we can use the equation (1). Substitute it in the DE for the state and compute $x(t)$:

$$x(t, \psi^*) = x_0 - \frac{3}{\rho\psi^*}(1 - e^{-\rho t})$$

As t goes to infinity, we get

$$\lim_{t\to\infty} x(t, \psi^*) = x_0 - \frac{3}{\rho\psi^*} = 0.$$

Now we can compute ψ^*:

$$\psi^* = \frac{3}{\rho x_0}.$$

Substitute ψ^* in (1) to get an expression for the optimal control:

$$u^*(t) = \frac{e^{-\rho t}\rho x_0}{3}$$

Finally, we compute the optimal value of the total payoff:

$$\sum_{i=1}^{3} K_i(x_0, u) = x_0 + 3\int_0^{\infty} e^{-\rho t}\ln\left[\frac{e^{-\rho t}\rho x_0}{3}\right] dt = \frac{\ln(\rho) + \ln(x_0) - 1 - \ln(3)}{\rho}$$

The total payoff obviously grow with x_0 – the more resources we have, the more money we get. Let's see how the value of ρ influences the payoff. We can differentiate the above expression w.r.t. ρ and set the derivative to 0 to obtain the optimal value of ρ. This last part is left as an exercise.