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Topic: Seminar. Using of the Pontryagin maximum principle. Differential game with discounting and infinite time horizon. Isoelastic utility function.

Time to perform: 21 April, 2020

Questions, answers: by email e.v.gromova @ spbu. ru

A model with isoelastic utility function Consider the model of joint exploitation of a non-renewable resource, e.g., an oil reservoir. The dynamics of the total resource stock is described by the DE

$$\dot{x} = \sum_{i=1}^{3} u_i.$$

The payoff of the i-th player is defined as

$$K_i(x_0, u) = \int_0^\infty e^{-\rho t} \frac{u_i^{1-\eta}}{1-\eta} dt, \quad \eta \neq 1, \quad i = 1, 2, 3.$$

Find the optimal controls guaranteeing maximization of the total payoff

$$\sum_{i=1}^{3} K_i(x_0, u) \to \max.$$

Problem:

1. Find $u_1^*(t)$, $u_2^*(t)$, $u_3^*(t)$ by the Pontryagin maximum principle.
2. Plot the graphs of optimal controls.
3. Find $x^*(t)$ — cooperative (optimal) trajectory. Plot the graph of optimal trajectory.
4. Calculate value of the total maximal payoff $\sum_{i=1}^{3} K_i(x_0, u^*)$.

Hints:

1. Note that this is an optimal control problem with discounting and infinite time horizon. This means that the expression under the integral depends on the time t. See the notes on Control theory by Prof. D. Gromov on how to deal with such problems (it will be given today).
2. If you let $\eta \to 1$ you should recover the solution $(x^*(t),\ u^*(t)$ from the previous class (07/04/2020) with logarithmic utility function.

References:

1. Lecture course on Control theory, Dr. Dmitry Gromov
2. Pontryagin, Lev Semenovich. Mathematical theory of optimal processes. Routledge, 2018.

Good luck!