Author: Ekaterina Gromova

Date: April 14, 2020

Topic: Seminar. Using of the Pontryagin maximum principle. Differential game with discounting and infinite time horizon. Isoelastic utility function.

Time to perform: 21 April, 2020

Questions, answers: by email e.v.gromova @ spbu. ru

A model with isoelastic utility function Consider the model of joint exploitation of a non-renewable resource, e.g., an oil reservoir. The dynamics of the total resource stock is described by the DE

$$\dot{x} = -\sum_{i=1}^{3} u_i.$$

Again, note the minus sign in front of u_i. The payoff of the i-th player is defined as

$$K_i(x_0, u) = \int_0^\infty e^{-\rho t} \frac{u_i^{1-\eta}}{1-\eta} dt, \quad \eta \neq 1, \quad i = 1, 2, 3.$$

Find the optimal controls guaranteeing maximization of the total payoff

$$\sum_{i=1}^{3} K_i(x_0, u) \to \max .$$

Problem:

1. Find $u_1^*(t)$, $u_2^*(t)$, $u_3^*(t)$ by the Pontryagin maximum principle.
2. Plot the graphs of optimal controls.
3. Find $x^*(t)$ — cooperative (optimal) trajectory. Plot the graph of optimal trajectory.
4. Calculate value of the total maximal payoff $\sum_{i=1}^{3} K_i(x_0, u^*)$.

Solution: We will follow the same program as in the previous problem. We start by writing down the Hamiltonian:

$$H(t, x, \psi, u) = e^{-\rho t} \sum_{i=1}^{3} \frac{u_i^{1-\eta}}{1-\eta} dt - \psi(u_1 + u_2 + u_3).$$

The optimal controls are obtained as solutions of the following first order optimality condition:

$$e^{-\rho t} u_i^{-\eta} - \psi = 0,$$

whence we get

$$u_i^*(t) = \left[e^{\rho t}\psi\right]^{-\frac{1}{\eta}} \tag{1}$$

The differential equation for ψ is again $\dot{\psi} = 0$ and so, we can set $\psi(t) = \psi^*$, where ψ^* is some yet unknown value.

Now we use the boundary condition. In the previous case we learned that setting $\psi^* = 0$ is not a good idea. So, we skip this step and go over to the second one (however, you are advised to check $\psi^* = 0$).

In this case ψ^* is not zero any longer, so we can use the equation (1). Substitute it in the DE for the state and compute $x(t)$:

$$x(t, \psi^*) = x_0 + \frac{3\,\eta\,\left(\mathrm{e}^{-\frac{\rho\,t}{\eta}} - 1\right)}{(\psi^*)^{1/\eta}\,\rho}$$

As t goes to infinity, we get

$$\lim_{t\to\infty} x(t, \psi^*) = x_0 - \frac{3\,\eta}{(\psi^*)^{1/\eta}\,\rho} = 0.$$

Now we can compute ψ^*:

$$\psi^* = \left[\frac{3\,\eta}{x_0\rho}\right]^{\eta}.$$

Substitute ψ^* in (1) to get an expression for the optimal control:

$$u^*(t) = e^{-\frac{\rho t}{\eta}}\,\frac{x_0\rho}{3\,\eta}$$

We can easily see that letting η go to 1, we recover the $\ln$-case.

The rest, i.e. the computation of the optimal total payoff is done in the same way as in the previous case.