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Topic: Seminar. Using of the Pontryagin maximum principle. Linear- state differential game

Let us consider the following differential game of 3 persons [2]. Dynamic equation:

$$\begin{aligned} \dot{z} &= u + v + w, & (1)\\ z &= (x, y)^T; \quad z(0) = z_0 = (x_0, y_0)^T; \\ u &= \{u_1; u_2\}, \quad v = \{v_1; v_2\}, \quad w = \{w_1; w_2\}; \\ |u| &\leq 1; \quad |v| \leq 1; \quad |w| \leq 1. & (2) \end{aligned}$$

Instantaneous payoff at τ, $\tau \in [0; T]$:

$$h_i(z(\tau)) = a_i \cdot x(\tau) + b_i \cdot y(\tau) + c_i, \quad a_i, b_i, c_i \geq 0; \quad (3)$$
$$a_i^2 + b_i^2 + c_i^2 \neq 0, \quad i = 1, 2, 3.$$

Integral payoff of the player i:

$$K_i(z_0, u, v, w) = \int_0^T h_i(z)dt, \quad i = 1, 2, 3. \quad (4)$$

The problem has to be solved in the class of open-loop strategies [1].

Let us consider the cooperative form of the game. It means, that players join together, i.e. use controls $u^*(t)$, $v^*(t)$, $w^*(t)$ to maximize the total payoff:

$$\sum_{i=1}^{3} K_i(z_0, u, v, w) = \sum_{i=1}^{3} \int_0^T h_i(z)dt, \qquad i = 1, 2, 3. \quad (5)$$

Denote by $u^*(t)$, $v^*(t)$, $w^*(t)$, $t \in [0; T]$ optimal controls, $z^*(t)$ — cooperative (optimal) trajectory for cooperative game.

Problem:

1. Find $u^*(t)$, $v^*(t)$, $w^*(t)$ by the Pontryagin maximum principle.
2. Find $z^*(t)$ — cooperative (optimal) trajectory.
3. Calculate value of the total maximal payoff $\sum_{i=1}^{3} K_i(z_0, u^*, v^*, w^*)$.

Time to perform: 24 March, 2020

Questions, answers: by email e.v.gromova @ spbu. ru

Hints:

1. Since the function $h_i(z(\tau))$ is continious then we get

$$\sum_{i=1}^{3} K_i(z_0, u, v, w) = \int_0^T \sum_{i=1}^{3} h_i(z)dt, \qquad i = 1, 2, 3. \quad (6)$$

2. The integral payoff here is $\sum_{i=1}^{3} K_i(z_0, u, v, w$. Then we actually solve optimal control problem with controls $u(t), v(t), w(t)$ (they are 2-dimensional variables!).

3. State variable z is 2-dimensional, you will get in Hamiltonian $H = \ldots + \psi_1(u_1 + v_1 + w_1) + \psi_2(u_2 + v_2 + w_2)$.

4. Maximization of Hamiltonian H is given with respect to controls which included in H in linear form. From this we conclude that maximum is attended in boundes. Moreover, take into account (2), we get

$$u_2 = \sqrt{1 - u_1^2}; \quad v_2 = \sqrt{1 - v_1^2}; \quad w_2 = \sqrt{1 - w_1^2}. \tag{7}$$

5. If you substitute (7) at Hamiltonian H after calculations you should get

$$u_1^* = \sqrt{\frac{\psi_1^2}{\psi_1^2 + \psi_2^2}}; \qquad u_2^* = \sqrt{1 - u_1^{*2}} = \sqrt{\frac{\psi_2^2}{\psi_1^2 + \psi_2^2}}. \tag{8}$$

$$v_1^* = w_1^* = \sqrt{\frac{\psi_1^2}{\psi_1^2 + \psi_2^2}}; \qquad v_2^* = w_2^* = \sqrt{1 - v_1^{*2}} = \sqrt{\frac{\psi_2^2}{\psi_1^2 + \psi_2^2}}. \tag{9}$$

(Continue: find $\psi_i(t)$)

6. Optimal trajectory is obtained from (1) if you substitute in r.h.s. optimal controls (final solution).

7. Calculate value of the total maximal payoff $\sum_{i=1}^{3} K_i(z_0, u^*, v^*, w^*)$: put in subintegral function $\sum_{i=1}^{3} h_i(z)dt$ optimal trajectory and find an integral.

References:

1. Pontryagin, Lev Semenovich. Mathematical theory of optimal processes. Routledge, 2018.

2. Petrosyan, L. A., and N. N. Danilov. "Cooperative differential games and their applications." Izd. Tomskogo University, Tomsk (1982).

Good luck!