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**Topic:** Seminar. Using of the Pontryagin maximum principle. Differential game with negative externality.

**Time to perform:** 28 April, 2020

**Questions, answers:** by email e.v.gromova @ spbu. ru

### Model

We consider a simple model of nonrenewable resource extraction process modeled as a game. There are three players (countries). Each player $i$ has strategic variable $u_i(t)$ that corresponds to the rate of emission at time $t$. We assume $u_i(t) \in [0; b_i], i = \{1, 2, 3\}$. The game starts at time instant $t_0$ and ends at time $T$.

The state equations have the form:

$$\dot{x}(t) = -\sum_{i=1}^{3} u_i(t), \quad x(t_0) = x_0. \tag{1}$$

The integral payoff of player $i$ is

$$K_i(x_0, t_0, T, u(t)) = \int_{t_0}^{T} \left(b_i\sqrt{u_i(t)} - d_i x(t)\right) dt, \quad d_i > 0.$$

We consider the cooperative form of the game. This means that all players join to maximize their total payoff

$$V(N, x_0, T - t_0) = \sum_{i=1}^{3} K_i(x_0, T - t_0; u_1, ..., u_n). \tag{2}$$

The controls $\{u_1^*, ..., u_n^*\}$ which maximize (2) are said to be optimal controls and the corresponding trajectory $x^*(t)$, $t \in [t_0, T]$ is said to be the cooperative trajectory.

### Problem:

1. Find $u_1^*(t)$, $u_2^*(t)$, $u_3^*(t)$ by the Pontryagin maximum principle.
2. Plot the graphs of optimal controls.
3. Find $x^*(t)$ — cooperative (optimal) trajectory. Plot the graph of optimal trajectory.
4. Calculate value of the total maximal payoff $\sum_{i=1}^{3} K_i(x_0, u^*)$.

### Hints:

1. This is a classical problem for using Pontryagin's maximum principle

2. The Hamiltonian for this problem takes the form

$$H = -\psi(t)\sum_{i=1}^{3} u_i(t) + \sum_{i=1}^{3} b_i\sqrt{u_i(t)} - \sum_{i=1}^{3} d_i x(t).$$

We define $d = \sum_{i=1}^{3} d_i$. The adjoint equations and the related transversality conditions are:

$$\frac{d\psi(t)}{dt} = d, \quad \psi(T) = 0.$$

3. From the necessary first order conditions we get:

$$u_i^* = \frac{b_i^2}{4\psi(t)^2}.$$

4. The only problem here to be sure that $u_i^*(t)$ belongs to $[0; b_i]$.

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**References:**


1. Lecture course on Control theory, Dr. Dmitry Gromov
2. Pontryagin, Lev Semenovich. Mathematical theory of optimal processes. Routledge, 2018.


Good luck!