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Topic: Seminar. Using of the Pontryagin maximum principle. Linear- state differential game. Solution

Let us consider the following differential game of 3 persons [2]. Dynamic equation:

$$\begin{aligned} &\dot{z} = u + v + w, && (1)\\ &z = (x, y)^T; \quad z(0) = z_0 = (x_0, y_0)^T; \\ &u = \{u_1; u_2\}, \quad v = \{v_1; v_2\}, \quad w = \{w_1; w_2\}; \\ &|u| \le 1; \quad |v| \le 1; \quad |w| \le 1. && (2) \end{aligned}$$

Instantaneous payoff at τ, $\tau \in [0; T]$:

$$h_i(z(\tau)) = a_i \cdot x(\tau) + b_i \cdot y(\tau) + c_i, \quad a_i, b_i, c_i \ge 0; \tag{3}$$
$$a_i^2 + b_i^2 + c_i^2 \ne 0, \quad i = 1, 2, 3.$$

Integral payoff of the player i:

$$K_i(z_0, u, v, w) = \int_0^T h_i(z)dt, \quad i = 1, 2, 3. \tag{4}$$

The problem has to be solved in the class of open-loop strategies [1].

Let us consider the cooperative form of the game. It means, that players join together, i.e. use controls $u^*(t)$, $v^*(t)$, $w^*(t)$ to maximize the total payoff:

$$\sum_{i=1}^{3} K_i(z_0, u, v, w) = \sum_{i=1}^{3} \int_0^T h_i(z)dt, \qquad i = 1, 2, 3. \tag{5}$$

Denote by $u^*(t)$, $v^*(t)$, $w^*(t)$, $t \in [0; T]$ optimal controls, $z^*(t)$ — cooperative (optimal) trajectory for cooperative game.

Problem:

1. Find $u^*(t)$, $v^*(t)$, $w^*(t)$ by the Pontryagin maximum principle.
2. Find $z^*(t)$ — cooperative (optimal) trajectory.
3. Calculate value of the total maximal payoff $\sum_{i=1}^{3} K_i(z_0, u^*, v^*, w^*)$.

Solution:

Since the function $h_i(z(\tau))$ is continious then we get

$$\sum_{i=1}^{3} K_i(z_0, u, v, w) = \int_0^T \sum_{i=1}^{3} h_i(z)dt, \qquad i = 1, 2, 3. \tag{6}$$

Let us denote

$$a_{123} = a_1 + a_2 + a_3; \qquad b_{123} = b_1 + b_2 + b_3; \qquad c_{123} = c_1 + c_2 + c_3;$$
$$a_{ij} = a_i + a_j; \qquad b_{ij} = b_i + b_j; \qquad c_{ij} = c_i + c_j.$$

We get

$$\sum_{i=1}^{3} K_i(z_0, u, v, w) = \max_{u,v,w} \int_0^T (a_{123} \cdot x(t) + b_{123} \cdot y(t) + c_{123}) dt, \tag{7}$$

The integral payoff here is $\sum_{i=1}^{3} K_i(z_0, u, v, w$. Then we actually solve optimal control problem with controls $u(t), v(t), w(t)$ (they are 2-dimensional variables!).

Since state variable z is 2-dimensional, you shouldl get the Hamiltonian

$$H = \psi_1(u_1 + v_1 + w_1) + \psi_2(u_2 + v_2 + w_2) + (a_{123} \cdot x(.) + b_{123} \cdot y(.) + c_{123}). \tag{8}$$

From PMP (Pintryagin maximum principle) we get the following euations for adjoint variables ψ_1, ψ_2:

$$\begin{aligned} \frac{\partial \psi_1}{\partial \tau} &= -\frac{\partial H}{\partial x} = -a_{123}; \\ \frac{\partial \psi_2}{\partial \tau} &= -\frac{\partial H}{\partial y} = -b_{123}, \end{aligned} \tag{9}$$

with condition $\psi_1(T) = 0$; $\psi_2(T) = 0$.

Maximization of Hamiltonian H is given with respect to controls which included in H in linear form. From this we conclude that maximum is attended in boundes. Moreover, take into account (2) ($u_1^2 + u_2^2 \leq 1$; the same for (v_1, v_2), (w_1, w_2)) we get

$$u_2 = \sqrt{1 - u_1^2}; \quad v_2 = \sqrt{1 - v_1^2}; \quad w_2 = \sqrt{1 - w_1^2}. \tag{10}$$

If we substitute (10) at Hamiltonian H after calculations we get

$$u_1^* = \sqrt{\frac{\psi_1^2}{\psi_1^2 + \psi_2^2}}; \qquad u_2^* = \sqrt{1 - u_1^{*2}} = \sqrt{\frac{\psi_2^2}{\psi_1^2 + \psi_2^2}}. \tag{11}$$

$$v_1^* = w_1^* = \sqrt{\frac{\psi_1^2}{\psi_1^2 + \psi_2^2}}; \qquad v_2^* = w_2^* = \sqrt{1 - v_1^{*2}} = \sqrt{\frac{\psi_2^2}{\psi_1^2 + \psi_2^2}}. \tag{12}$$

From (9) we get $\psi_1(\tau)$, $\psi_2(\tau)$:

$$\psi_1(t) = -a_{123}(t - T); \quad \psi_2(t) = -b_{123}(t - T). \tag{13}$$

Finally from (11) and (13) we have:

$$u_1^* = v_1^* = w_1^* = \frac{a_{123}}{\sqrt{a_{123}^2 + b_{123}^2}}; \qquad u_2^* = v_2^* = w_2^* = \frac{b_{123}}{\sqrt{a_{123}^2 + b_{123}^2}}. \tag{14}$$

Note, that we take square root with "plus" because we have the maximization problem.

We obtained constant controls! (They do not depend on t)

Then from (1) we have

$$x(t) = (u_1 + v_1 + w_1) \cdot t + x_0; \qquad y(t) = (u_2 + v_2 + w_2) \cdot t + y_0,$$

and the coopertive (optimal) trajectory is as follows:

$$x^*(t) = \frac{3a_{123}}{\sqrt{a_{123}^2 + b_{123}^2}} \cdot t + x_0; \qquad y^*(t) = \frac{3b_{123}}{\sqrt{a_{123}^2 + b_{123}^2}} \cdot t + y_0. \tag{15}$$

Now it is possible to calculate (7) by substituting optimal controls and trajectory to subintegral function:

$$V_T(z_0, N) = \int_0^T ((3\sqrt{a_{123}^2 + b_{123}^2}\ t + a_{123} \cdot x_0 + b_{123} \cdot y_0 + c_{123})dt =$$

$$= 3\sqrt{a_{123}^2 + b_{123}^2}\frac{T^2}{2} + \sum_{i=1}^{3} h_i(z_0)T. \tag{16}$$

References:

1. Pontryagin, Lev Semenovich. Mathematical theory of optimal processes. Routledge, 2018.

2. Petrosyan, L. A., and N. N. Danilov. "Cooperative differential games and their applications." Izd. Tomskogo University, Tomsk (1982).