Exercise 1.5.1

Let $A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}$.

a)Prove that A is positive definite.

b)Calculate the Cholesky factor of A.

c)Find three other lower-triangular matrices G such that $A = GG^T$.

d)Let A be any positive-definite matrix of dimension $n - by - n$. How many lower-triangular matrices G such that $A + GG^T$ are there?

Answer

a)
Remember: A is positive definite if it is *real, squared, symmetric,* and *satisfies* $x^T Ax > 0$ for any non-zero $x \in R^n$.
A is real and squared (given). Since all elements outside the main diagonal are 0, A is symmetric.
Without losing generality, $x^T = [x_1, x_2]$. Then,

$$\begin{aligned} x^T Ax &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 4x_1 & 9x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 4x_1^2 + 9x_2^2 \end{aligned} \tag{1}$$

Since $4x_1^2 + 9x_2^2 > 0$ *(1)*, $x^T Ax > 0$.
Hence A is positive definite.

b)
Remember: If $A = GG^T$, G is lower triangular with only positive numbers on the main diagonal, then G is the *Cholesky factor* of A.

Using the following recipes:

$$g_{ii} = +\sqrt{a_{ii} - \sum_{k=1}^{i-1} g_{ik}^2}$$

$$g_{ij} = \frac{(a_{ij} - \sum_{k=1}^{j-1} g_{ik} g_{jk})}{g_{jj}}, i > j$$

$g_{11} = +\sqrt{4} = 2$
$g_{21} = 0$
$g_{22} = +\sqrt{9} = 3$
$g_{12} = 0$
Hence, $G = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

c)
It is worth noticing that the following additional matrices are **not** Cholesky factors, which is unique.

$G_1 = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$
$G_1 = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$
$G_1 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

d)
Notice a_{ii} comes from the squared values of the main diagonal in G. Therefore, the sign is lost and does not have an effect on A.
A, *n-by-n,* has 2^n lower triangular matrices decompositions.