Exercise 3.2.3

Prove that if Q satisfies $(Qx, Qy) = (x, y)$ for all $x, y \in R^n$, then Q is orthogonal.

Answer

Notice that $(., .)$ means inner product.

$$(x, y) = x^t y \tag{1}$$

$$(Qx, Qy) = (Qx)^t (Qy) = x^t Q^t Q y \tag{2}$$

Show that: $x^t y = x^t Q^t Q y \rightarrow Q$ orthogonal

Contradiction: assuming $Q^t Q = M \neq I$

$$x^t y = x^t Q^t Q y$$
$$x^t y = x^t (My)$$
$$x^t y = x^t \hat{y}$$
$$\|x^t y\| = \|x^t \hat{y}\|$$