RIT

Online Ramsey Theory in Planar Graphs
Independent Study Final Report

by

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Independent Study Report
CSCI0599 - Computer Science Undergraduate Independent Study

Directed to professor Stanisław P. Radziszowski and professor Elouise Oyzon

Report submitted in partial fulfillment of the
requirements for the independent study course

at

Rochester Institute of Technology
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Introduction

The work done during the semester was split into the development of a visualization tool for simulations involving graphs and the study of Ramsey Theory. The time dedicated to each part was about half of the academic semester, but more details can be found on the activity rundown on Appendix A and hosted in this independent study's main page[1]. The final state of the application developed and a copy of this report are also found on the same web page.

The next two sections bring an executive summary of Online Ramsey Theory on Planar Graphs, with what was studied the most, and a description of the simulation tool. The last section is a retrospective of the semester, including expectations and results. The references section mentions papers that are not quoted in the text but were studied and helped better understanding some ideas.

Overview

- **Course:** CSCI0599 - Computer Science Undergraduate Independent Study.

- **Credit count:** 3 Advanced Elective credits.

- **Term:** Fall of 2021.

- **Advisor:** Stanisław P. Radziszowski.

- **Meeting:** 11 meetings throughout the semester.

- **Topic:** Online Ramsey Theory in Planar Graphs.

- **Web page:** https://people.rit.edu/∼mtl9706/is.

- **Github:** https://github.com/MLaurentys/IS_ramsey.

- **Approved by:** Elouise Oyzon, department of Interactive Games and Media.

Objectives

The main objective of the study was bringing together algebraic results of Ramsey Theory and a graph visualization tool. The choice to focus on planar graphs is tied to the fact that they provide easier visualization with plane embedding. The secondary goal was getting a deeper understanding of the online version of Ramsey type problems, and, more specifically, when restricting the problems to the class of planar graphs.

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[1]Located on https://people.rit.edu/∼mtl9706/is.

Online Ramsey Theory

Ramsey Theory follows an interesting theorem stating that for any positive integers r and s, there is a least positive integer $R(r,s)$ such that every blue-red edge coloring of $K_{R(r,s)}$ has a blue clique with r vertices or a red clique with s vertices. One extensively studied problem is calculating such $R(r,s)$ which proves to be very hard, even for very small values of r and s.

The relation of any blue-red edge coloring of a graph G having either a blue clique of size r or a red clique of size s is denoted by $G \to (r,s)^e$. Similarly defined, if every edge coloring of G has either a blue copy of H or a red copy of F, then $G \to (H,F)^e$. In a special case, where H is isomorphic to F, the *Ramsey Arrowing* is denoted by $G \to (H)^e$. Another set of problems arise when considering the online version of the problem.

On-line Ramsey Theory[1]

The word "online" means that, when checking if $G \to (H)^e$, the edge coloring of G takes place before the entirety of G is known. The typical way, introduced by *Online Ramsey Theory*[1], to present the online version of the problem is through a combinatorial game between painter and builder:

> Online Ramsey Game: Take an unbounded set of vertices with no edges and a target graph H. Builder selects two disconnected vertices and adds an edge between them. Following, Painter colors the introduced edge. If a monochromatic copy of H is ever created, then the Builder wins. Painter wins if it provides a strategy to avoid H indefinitely.

If the builder provides the edges of G, and $G \to (H)^e$, then the builder wins, because there is no possible way to paint the edges of G while avoiding H, as defined above. However, it is not true that if the painter cannot prevent H, when given edges of G, then $G \to (H)^e$. The mismatch comes from an additional challenge the painter faces on the online version. The added challenge of the online version comes from the lack of information when the Painter takes its decisions.

Consider now a slightly different version. In this version, the builder can only create graphs belonging to a class of graphs $\mathcal{G}$. Considering that $H \in \mathcal{G}$, H is called unavoidable if the builder can force the painter to create a monochromatic copy of H, and called avoidable otherwise. If every graph of a class is unavoidable, then the class is called self-unavoidable.

Together with the formulation of these problems the authors show[1] that the class of forests is self-unavoidable, also mentioned in the second section. For this proof, check the visualization featured in the application or in the original paper. An important comment on this self-unavoidability and the difference between the online and offline version of the problem is that forests are clearly avoidable in the offline version.

On the offline version, in fact, it is possible to avoid P_4 by following a simple strategy. Elect roots for all the trees in the forest. Color blue all the edges with odd distance to the root and red those with even distance. This strategy does not work in the online version because the distances to a root are not known at the time they are created by the builder.

When concluding their paper, the authors conjecture that the class of outerplanar graphs is exactly the class of unavoidable graphs when playing with $\mathcal{G}$ as the planar graphs.

Online Ramsey Theory for Planar Graphs[3]

Ten years later, Šárka Petříčková[3], proved the conjecture wrong. She showed that the builder can force all outerplanar graphs, but can also force some graphs that are not outerplanar. She proved that all outerplanar graphs are unavoidable, but, then, showed a class of graphs, that is not outerplanar, but that is unavoidable when playing on the planar graphs.

Although the proofs contain too many details to be adapted here, it is worth bringing counter examples of the conjecture. First she shows that even cycles are unavoidable. Then, she shows a strategy, based on structures consisting of even cycles, and adds edges and vertices to the graph in a way that the target graph is reached. The class of non-outerplanar graphs used by her is the class of $\theta_{i,j,k}$ with i, j and k even. A graph denoted by $\theta_{i,j,k}$ is composed of three internally disjoint paths of lengths i, j and k. The graphs $\theta_{2,2,2}$ and $\theta_{2,2,4}$:
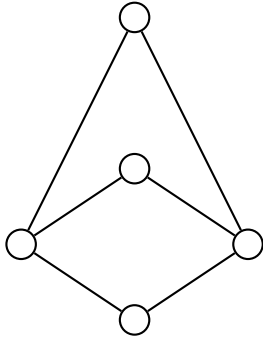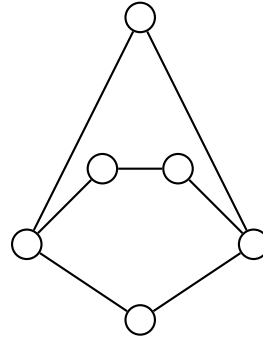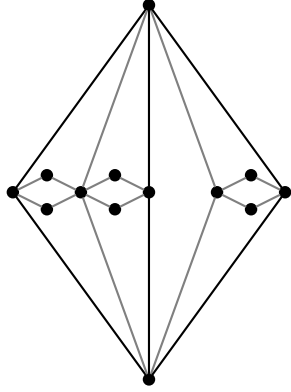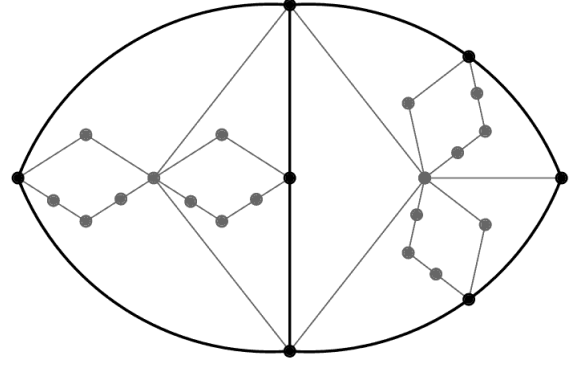

$\theta_{2,2,2}$[6] $\theta_{2,2,4}$[3]

Are clearly not outerplanar, however it is possible for the builder to force a monochromatic copy of it on the class of planar graphs. Starting from a base structure composed of even-length cycles, showed in gray, the builder just have to build the edges of the graphs below, in black, extracted from Petříčková's work.

For $\theta_{2,2,2}$

For $\theta_{2,2,4}$

Petříčková finishes her paper by bringing to light the question whether the class of planar graphs is self-unavoidable, but conjecturing that K_4 is avoidable. Of course that if her conjecture is right, then the class of planar graphs is not self-unavoidable. Since 2014, however, no direct progress has been made when it comes to planar graphs, but there are some developments studying other classes.

Online Ramsey theory for a triangle on F-free graphs[3]

In this paper there is a small shift from what has been highlighted so far. Firstly, it deals with non-planar planar graphs. Secondly, and more importantly, instead of taking a class and deciding whether it is self-unavoidable, the authors take a structure, C_3 and study the classes where it is unavoidable. The paper is concerned with classes of graphs where a determined structure is not a subgraph.

First, the authors show that C_3 is unavoidable in the class of K_4-minor-free graphs. A graph G is a minor of another graph H if, after a sequence of vertex deletions and edge removals or contractions, G is obtained from H. This way, K_4-minor-free graphs are those for which K_4 is never obtained after a sequence of such operations. The result obtained by the authors directly extend the result proved on *On-line Ramsey Theory*.

In the paper from 2004, the authors show that C_3 is unavoidable in the class of outerplanar graphs. This means they showed C_3 is unavoidable in the class of $(K_{2,3}, K_4)$-minor-free graphs. This extension shows that $K_{2,3}$ is not necessary to force a triangle.

Following up, the paper presents a total of five graphs $X_1, X_2.X_3, X_4$ and X_5, the larger ones having six vertices. The authors claim and prove that, if a graph F has no isolated vertex and is not isomorphic to X_5, then:

Painter wins the Online Ramsey Game for C_3 on the class of F-free graphs

$$\Longleftrightarrow$$

F is isomorphic to a subgraph of either X_1, X_2, X_3 or X_4[2]

Then, they end their paper by highlighting the only missing result for C_3 for such classes of graphs in Online Ramsey Games: "Who wins the Online Ramsey game for C_3 on X_5-free graphs"?

One more problem

When changing the offline problem to the online version, the Painter becomes much weaker. To balance that, restrictions are added to the Builder as well, in the form of

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[2]This is specially nice because these graphs have between five and six vertices.

restricting its moves to certain classes of graphs. Another option, however, would be making the painter stronger.

For this version, instead of using an unbound set of vertices, the builder has a finite number in its disposal. Additionally, consider that instead of coloring each edge in sequence, the Builder builds k edges in each turn. When the Painter has to make its decisions, it has more knowledge of the overall structure. However, it is not clear whether it is meaningful or not. It actually turns out that this boost is not very meaningful[7]. *Ramsey games against a one-armed bandit* analyzes the problem thoroughly and, although very different from the algebraic approach presented in the other references, complements nicely the study on Online Ramsey Theory.

Visualization Tool

The tool developed during the semester draws simulations using graphs, specified in JSON format, in a canvas. The drawing and canvas interaction part of the application are accomplished using the cytoscapejs library while the interface and simulation functionality were developed specifically for this project. The tool has limited user customization options, but allows the drawing of any graph using a pre-defined layout algorithm.

Simulation specification

The simulation used by the application is specified via JSON using the following schema:

```json
{
    "title": "Title displayed on page",
    "description": "Description showed together with title",
    "steps": [
        {
            "graph": "String representation of graph using graph6[3]
                format.",
            "colors": "color of each edge in the graph, in the order
                used in the upper triangle of the 'graph'. Specified
                with a sequence of characters between 'r' (red),
                'b' (blue), 'g' (gray) and 'w' (white).",
            "title": "Step title",
            "description": "Step description.",
            "duration": [0-9]+ //seconds the current step is displayed
        },
        ...
    ]
}
```

The app uses the schema above to create a visualization of the theorem proving that forests are self-unavoidable in the Online Ramsey Game[4]. The JSON string of this simulation is found on the side menu of the application.

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[3]Compact representation of graphs via strings. Format created by Brendan McKay. Find the specification at https://users.cecs.anu.edu.au/∼bdm/data/formats.txt.

[4]More details on the next section.

Technologies used

The app was developed using `Nodejs`, is implemented in `TypeScript` and is based on `React` and `Redux` frameworks. The conversion to static HTML and JavaScript is done via `webpack`. The code has the most typical view/component/store architecture. There is no connection to backend services or databases. The app is available on this study's github page[5].

Future developments

One of the important steps forward is improving the graph definition on the simulation JSON file. Using graph6 or another common string formatting is adequate but there is no good online tools for drawing graphs and getting their strings. *House of graphs*[6] has one built-in but it is limited is some regards. It only shows a limited number of character of the string conversion (being useless for graphs with over 20 vertices) and the querying of graphs drawing destroys the current canvas, meaning similar queries involve a lot of repeated drawing.

A good way to improve the graph definition is developing the graph6 functionality of the application further. Allowing a similar drawing style to that of House of Graphs but with real-time graph6 conversions would solve most of the issues creating new simulations. However, the problem of specifying the edge colors would remain. To solve that, the app should also display the order of edges or allow for coloring via canvas, with a similar style of real-time conversion.

Another necessity is improving the layout algorithm choices. Neither cytoscapejs nor other graph drawing frameworks have layout algorithms that guarantee drawings of planar graphs without crossing edges. Implementing such algorithms for the main libraries would be a good step forward to make the simulation better. Another related improvement is allowing the user to pick a layout in the simulation specification.

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[5]https://github.com/MLaurentys/IS_ramsey.
[6]Find more on https://hog.grinvin.org/

Lessons Learned

I wrote the Undergraduate Thesis for my bachelors degree during 2020. It was my first formal research/scientific essay. In the beginning of the year, I decided to give it my best and experience the most, because I would probably not have another chance to try it out, specially with an advisor. I was lucky enough to choose a topic I ended up loving. I spent around five months studying very hard and implementing a lot of ideas I read about. I got a rough sketch of everything I wanted to discuss in my essay in June and started working on the essay soon after.

By November I had a decent looking draft which had all the content I wanted to develop and it was around 80 pages long. I spent the next 2 months revising and improving the text and, although I ended up adding some more content, I got to shrink it to 71 pages. I loved working on it and was very happy that my advisor and the grading committee liked it as well. During the work, that lasted 13 months, I do not remember getting stuck or unfocused.

Leading up to this semester I wanted to do more of that. I reached out to a few professors at RIT and I was very hopeful of working with my current advisor, Stanisław P. Radziszowki. Due to academic requirements, the initial research plan was revised to reserve half of the semester to developing a visual tool, which was not exactly what I wanted to do, but I did not mind that.

Focusing on the second half of the semester, I believe that a first mistake I made was in my expectations. I remember thinking that I had a very easy time writing and, unlike my thesis, I would not be starting from scratch: I already knew the proper tools, how to draft a plan that works for me, how to organize references, take good notes an so on. I thought the only missing piece was the knowledge, so after starting learning, I would be ready to start writing.

There were two anticipated text deliveries this semester, but the first one got merged into this final report due to my inability to write it. I cannot pinpoint exactly what the problem was but I will list what I believe major factors were. Unlike with Combinatorial Game Theory, that has didactic materials and books that cover all the developments in the field except for some bleeding edge changes, Ramsey Theory is much more opaque.

In one side I had *Winning Ways*, a gem containing every basic idea there is to know presented in a brilliant manner, and on the other *Combinatorial Game Theory* by Aaron Siegel, which formalized every topic and had the notation used essentially everywhere else. Not only that but there was also CGSuite, by Siegel again, that gave visuals and calculations for every prototype or idea I had. This semester was a different process, where I was reaching out for information scattered across papers, that used different notations and terminologies.

Although I believe I learned a lot and was introduced to interesting and fun subjects, the content did not have time to sink in. When writing, this played the role of making me scared of producing false statements or incorrectly quoting authors. I second guessed

some decisions I took and ideas I had. It definitely slowed down the process of writing the essay.

For the same reason it was difficult to produce steady progress. There were times when I understood a new idea, maybe in a proof for a theorem, and that just threw a piece I was developing out. This was progress, in the sense that I learned, but was a slow down in the sense that I did not produce anything. This happened once when I was studying the problem of making the painter stronger by increasing the number of edges before coloring and again when I described the work by Šárka Petříčková.

The problems making progress during the middle of the semester led me to the biggest issue I had. Writing the status report became a burden. I found myself constantly thinking about it, but when I sat down to produce I did not have the words or the ideas necessary to make something of reasonable quality. I want to highlight that my advisor never pressured me. He was always understanding and supportive, at times even reassuring about some ideas I wanted to discard. However, I really wanted to have something to show for.

While having disappointments, I know a bit more now about myself and my relation with writing scientific texts. A first lesson is how valuable a general solid understanding of a field is. Usually, because I write about what I know best, I overlooked the fact that, knowing some theorems and notations does not mean you can write a cohesive and interesting text about a subject. There tends to be more to the ideas connecting theorems than the sum of their results.

A second lesson I take is that I am bad at writing on demand. I am usually great with deadlines and become more productive the more urgent something becomes, but it seems I write the best when I am calm. Following up there an advice my advisor gave me early on but which I did not follow. When developing the app he told me to start writing, but I, instead wanted to finish the app as soon as possible. If I had started earlier I would bump into the knowledge holes earlier and get more time to mature my questions.

It does seems starting to write as early as possible is good because, like in software development, you only notice you have questions when you try to do it yourself. I feel weird now about not following the advice, because it does seem very obvious. A fourth lesson is the value a good book has. I always appreciated books, because I am very focused and get a lot from every one I read. Not having them, though, makes it even more evident. There are other lessons, but I think I brought to light the most meaningful to me right now.

To finalize the text I believe I should say I am happy to have taken this class. Although it came with quite some mental burden, which I described abundantly, I loved the experience and I am forever glad I had this experience sooner rather then later. The end results are not what I hoped for, in the writing department, but I am happy about the effort I put into figuring some problems out. Without a doubt, the difficulties became lessons due to the help of my advisor. I think he did a great job incentivising and reassuring me every time I needed that.

Bibliography

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[3] Hojin Choi, Jisu Choi, Ilkyooand Jeong, and Sang-il Oum. Online ramsey theory for a triangle on f-free graphs. *Journal of Graph Theory*, 2019.

[4] Pavel Dvořák. Online ramsey theory. Master's thesis, Charles University, 2015.

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[6] Šárka Petříčková. Online ramsey games on planar graphs. Unpublished. Slide presentation. https://faculty.math.illinois.edu/∼mlavrov/seminar/2014-petrickova.pdf, 2014.

[7] E. Friedgut, Y. Kohayakawa, V. Rödl, A. Ruciński, and P. Tetali. Ramsey games against one-armed bandit. *Combinatorial Probability Computing*, 2013.

Appendix A

Log of tasks

Week	Proposal	Work Done	Hours
Total hours	• Should have 135 hours dedicated to the project • ~9 hours a week	• Week 16 total: 135 hours • Hours per week until end of semester: 0h	135 / 135
16	• Write a report of the work done during the semester • Outline main challenges faced both during the implementation of the proof visualization and the writing of the report	• Revised report with content and recent inclusions	8
15	• Write a report of the work done during the semester • Outline main challenges faced both during the implementation of the proof visualization and the writing of the report	• Revised report after evaluation, adding more content • Made the website for IS home page	16
14	• Write a report of the work done during the semester • Outline main challenges faced both during the implementation of the proof visualization and the writing of the report	• Finished the APP • Wrote final report • Wrote self-evaluation	30
13	• Write a report of the work done during the semester • Outline main challenges faced both during the implementation of the proof visualization and the writing of the report	• TODO: Finish the app • Stopped working on the status report: idea seems to optimistic to current progress • Wrote bibliography for the final report • Researched structure for the text and prepared files	4

12	• Gather all papers that discuss Online Ramsey Games played on Planar Graphs • Write a report that tells the status and the important milestones in this area of study	• TODO: Finish the app • TODO: Fish writing status report	0
11	• Gather all papers that discuss Online Ramsey Games played on Planar Graphs • Write a report that tells the status and the important milestones in this area of study	• TODO: Finish the app • Wrote about the extension of the Ramsey Game studied • Finished selecting topics for section 2 of status report	8
10	• Gather all papers that discuss Online Ramsey Games played on Planar Graphs • Write a report that tells the status and the important milestones in this area of study	• TODO: Finish the app • Revised introduction • Studied an extension of the game where Right is more powerful	6
9	• Gather all papers that discuss Online Ramsey Games played on Planar Graphs • Write a report that tells the status and the important milestones in this area of study	• TODO: Finish the app • Made a list of sources and marked content to use in my report • Wrote most of the introduction to check styles with supervisor	2
8	• Use a visualization tool to generate a proof that every forest is unavoidable in the class of forests • Show that if all trees with n vertices are unavoidable, then so are all trees with $n+1$ vertices • This would be done by following the inductive proof provided on Online Ramsey Theory [1]	• TODO: Added more styling and github link to page • TODO: Added fixed position support for simulations to compensate the fact no built-in styles meet planar requirements • Decided some of what will go on the status report. Also defined the rough structure • Made a list of sources and marked content to use in my report • Wrote some of the introduction	7
7	• Use a visualization tool to generate a proof that every forest is unavoidable in the class of forests • Show that if all trees with n vertices are unavoidable, then so are all trees with $n+1$ vertices • This would be done by following the inductive proof provided on Online Ramsey Theory [1]	• TODO: Added more styling and github link to page • TODO: Added fixed position support for simulations to compensate the fact no built-in styles meet planar requirements • TODO: Decide on what will go on the status report. Also define structure • Started writing the simulations. Blocked by missing a good graph6 tool to use • Added coloring support for simulations • Improved layout for simulations	5

6	• Develop a program that simulates a builder-painter game for planar graphs. It draws the planar embedding of the graphs generated during play	• Finished graph6 converting • Tested other layouts. Discovered that planar support is not there. It is also not implemented in D3 nor visjs, which are the other two main graph frameworks • Started reading nauty code related to graph6. Difficult • Studied other styles available in js. None really good • Finished simulation base functionality	12
5	• Develop a program that simulates a builder-painter game for planar graphs. It draws the planar embedding of the graphs generated during play	• Added webpack to the project in order to produce static pages to export • Learned Fary's Theorem • Revisited other basic graph theory problems	7
4	• Develop a program that simulates a builder-painter game for planar graphs. It draws the planar embedding of the graphs generated during play	• Made a sketch of app end result, and left the gameplay part out. It will not be necessary for simulations • Started developing graph6 <=> cytoscape converter • Started developing simulations based on graph6	7
3	• Develop a program that simulates a builder-painter game for planar graphs. It draws the planar embedding of the graphs generated during play	• Added Bootstrap CSS to the project. Added basic styling • Studied graph6 format • Tried to find some graph6 <=> cytoscape converter but could not	5
2	• Develop a program that simulates a builder-painter game for planar graphs. It draws the planar embedding of the graphs generated during play	• Created a base web app to further test cytoscape • Started a React project to continue development	9
1	• Find and test visualizations tools for planar graphs • Decide what tool to use and if a new one is necessary	• Tried Cytoscape, vis and D3 for js • Found interesting tools like house of graphs, might use in the future • Tried some browser tools, but none got close to the level of customizability I was looking for. Decided to move on with development	9