



IME-USP

Universidade de São Paulo - Instituto de Matemática e Estatística
Bachelor's Programme in Computer Science

HOT GAMES

Temperature, advantage and numbers

by

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Abstract (The Abstract is a short summary of what your thesis is about. It accurately reflects the content of the thesis providing information about the research problem, research aims, methods and procedures, results and implications. It is a short section. Abstracts give readers the opportunity to quickly see the main contents of the paper and enable them to decide whether the paper is of particular interest to their needs. This section will be one of the last sections that you write. No subheadings are used in an abstract.)

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Undergraduate Thesis

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It is usual, but not compulsory, to thank those who have been of particular help to you in completing the thesis.

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1

Introduction

“I learned very quickly that playing games and working on mathematics were closely intertwined activities for him, if not actually the same activity. His attitude resonated with and affirmed my own thoughts about math as play, though he took this attitude far beyond what I ever expected from a Princeton math professor, and I loved it.”

Manjul Bhargava ¹

It is no surprise that avid players of games that resemble a logical or mathematical puzzle, like checkers, develop an intuition that allows them to calculate faster, by asserting bad moves fast, recognizing winning and losing patterns ahead of time or many other strategies. Many times, it is not easy to understand or explain how does it work, while others, after seeing a move being played, it is easily possible to explain it by the principles that rule the game. A most essential, and sometime very hard, component of playing well any of these mathematical games is being able to know if you are ahead or behind in a given position.

While evaluating whether a position is winning or losing is already a hard task, with, for example, modern chess engines not agreeing in some positions, it is possible take this problem to higher standards. Imagine you are playing a variant of the game of chess. In this variant, each player is given a set of board positions, and each should choose one board to play as white. During this game, play will take place in each board in parallel, and, whoever checkmates the opponent faster, wins the game. If one wants to be a great player of such game, the knowledge of asserting if a position is winning or losing in a regular chess game is not enough, nor is the ability to play regular chess perfectly.

The most important ability for this game and for the games that will follow in this paper is to score each position. The ability to calculate the precise advantage a player has over the other is the object of interest of the field called Combinatorial Game Theory, inaugurated in **TODO: 1970** by the paper **TODO: ONAG**. The author of that paper, Jonh H. Conway, as found in the epigraph that starts this text, was, as many, an avid player of such games, however, unlike all others, was one of the brightest minds in the history of mathematics.

Dr. Conway realized that some games, to be defined in the next chapter, behave like numbers in every aspect. While, initially, that seems very useful to calculate the advantages, and win in the presented variation of chess, this realization means much

¹Fields medalist commenting on John H. Conway’s passing

more than this tool. It is true that some way of modeling the problem of assigning values to positions was the initial challenge and this would be a great step towards it; however, what came out from Conway's work was a **TODO: match** between positions and numbers, not a measure of the position.

This **TODO: match**, that is further explained in the following chapters and the main topic in this field of study, is built upon a new set of numbers. This special set of numbers was created, or discovered, when piecing together the necessary element to analyze any particular game: its game tree. Of course a computer program could traverse a game tree and tell who is winning the game, but it would not be able to easily tell who would be winning if two game trees were added. **TODO: citation!** Even defining how to meaningfully merge game trees is problematic.

Not only does the **TODO: system** created by the **TODO: magical genius**², Conway, gives the ability to add, but it also defines a method to multiply games. However, in order to define such methods, the definition of that new set of numbers was necessary. The Surreal Numbers, name given by Donald Knuth³, **TODO: for starters**, were required as, in this **TODO: system**, the number **TODO: matching** a position might not belong to the set of real numbers.

As some might understand from the title alone, however, the focus of this text are in the non-numbers. It is possible for games to be **TODO: unwritable** as a surreal number, although every surreal number has correspondent⁴ games. The concept of temperate, by consequence the concepts of hotness and coolness, however, does not forego the understanding of numbers.

The rest of this chapter relies on the next chapters. Skipping for now.

History of field of study - Quem inaugurou - Conway. Onag (paper que iniciou a area) - Conway. WW (winning ways – referencia canonica). Morreu esse ano - Combinatorial game theory - Aron S. (abordagem diferente – coisas q n tem no WW; especialmente em relacao a classificacao dos jogos) - Principais nomes da area - Conway + 2 - dialogar com titulo: hot (nao eh conceito inicial da area) / games (nao eh definicao obvia) - hot: temperatura/topico avancado: Conway. - Traduzir intuicao em definicoes matematicas - Definir o que eh games: games = mathematical play. Especificar. - Numbers

(The Introduction chapter should contain background information as appropriate, plus definitions of all special and general terms. Your topic should be: clearly stated and defined; have a clear overall purpose; and have clear, relevant and coherent aims and objectives. It is also informative to give a brief description of the contents of the remaining chapters of the thesis. This alerts the reader and prepares them for the rest of the thesis.)

²blabla

³The original name was...

⁴Any surreal number is **TODO: matched** by an infinite number of games

2

Pen, Paper, a Friend and the Rules

The Rules

In order to study mathematical plays and answer the many questions they raise, a new mathematical field of study was developed and many new terms were created. The phrase “mathematical play” is in itself a new term, for instance. While the most common term is “Combinatorial Games”, the canonical reference for this field *Winning Ways for Your Mathematical Plays* 1981, uses the former, not the latter. As more is said about the topic, more meaning the term “mathematical play” is going to acquire.

The name “Combinatorial Game” might bring to light some information. It, at least, means that this field will deal with games, as in, an instance of a Game Theory problem, and, more specifically, a subset of those games. It also brings to light that the use of counting, finite structures and, most likely, graph representations are omnipresent. However, a definition of the object of interest becomes possible with the name mathematical play.

To play something mathematically could be understood as to engage in an activity in which the better use of mathematical ability, such as counting and logic, would result in advantage over its poor use. However it could be detailed further to an activity in which mathematical ability is the single defining factor. The later might make more sense because there are games, like poker, that do require some counting ability. However, luck and reading behavior skills are much more valuable to a successful game and this is something the definition would be better off forbidding.

Chance moves, like throwing a dice or flipping a card, are not fit for mathematical plays. Even with their removal, however, there are possibilities that would not be comfortably called mathematical plays. The nature of a mathematical plays is that both players can engage the same activity and generate advantages out of “good play”. For instance, it would be hard to agree that two people playing rock-paper-scissors are battling a mathematical fight, even though there are no chance moves.

It is very important that all players have **complete information** of the position. Games like rock-paper-scissors, in which players take action simultaneously, block complete information. Therefore, players must **move alternately**. The last

concerning factor in discerning mathematical from non-mathematical plays during this analysis is the number of players.

When each player has more than one opponent a goal greater than gaining advantage arises. When playing with over two people it is frequent that the best move is not the one that brings a better position but one that prevents any of the opponents from gaining an winning advantage. While that can be very mathematical, there is a clear distinction between sticking to two player games and allowing any number of players. Notice that one can consider soccer as a two player game - even though there are multiple agents in a team. In order to focus on the mathematical ability to make the best move, the option to allow only **two players** is the most interesting.

The only remaining criteria of this definition, as established in [TODO: WW], is related to the term play, and not the term mathematical. The rules of the game must guarantee that from any starting position, **TODO: play should always end because a player will not have moves available**. If following “normal play” convention, a player that cannot move is lost. It is correct to assume normal play, unless specified otherwise, in this field of study.

The foundations of mathematical plays, highlighted, give light to a complex and rich set of problems, although, at the same time, other complex and rich problems are left behind. The game of chess, for example, does not meet the ending condition and, therefore, is left out. Fortunately, games like chess might benefit from these studies with adaptations or additional rules (although they do not consist of good examples of combinatorial games).

Let us consider the following game:

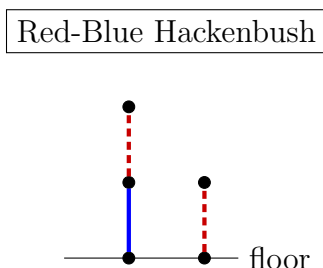


Figure 2.1

In RB-Hackenbush, a move is made by taking a single colored edge of the image and removing any edges that become disconnected from the floor. The player called **Left** can only remove **bLue and soLid**, edges, and the other, called **Right**, **Red and dashED** edges. It is a common practice to assume that all games are played between Left (blue) and Right (red), although, occasionally, new colors will be presented. In this text, every color also has a matching style.

The RB-Hackenbush is a game by the routine definitions of game, meaning it has a clear ruleset and potential to be fun. It is also a game by the definition above, which will be the new “routine” one used going forward. However, the instance of this game drawn above is also called a game. In the future, when analyzing a board state for example, the word position may seem to make more sense, but the correct word is game. The meaning of the word must be deduced from context.

The **game tree** term refers to the routine meaning, for example. The image below is the game tree that arises from the game, instance, presented above.

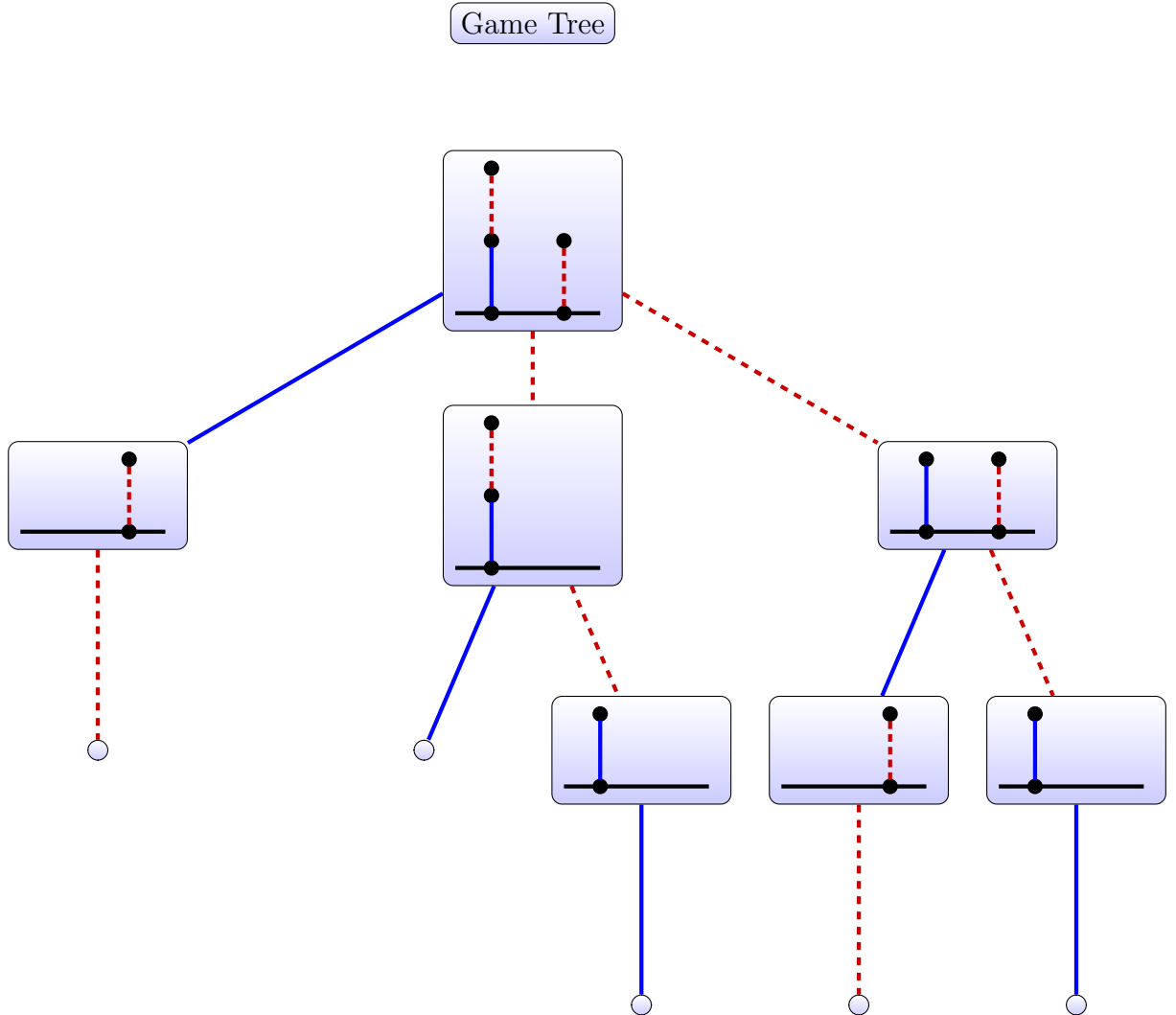


Figure 2.2

In the tree above the styled edges, in the same pattern as before, between configurations tells which player made a move. The game trees used in **TODO: CGT** may be slightly different as they, from each configuration, develop the moves from both players. This is a most important characteristic because it allows the sum of trees.

As noted previously, the game tree contains all the information required in order to calculate the number, or non-number, a game is equal to. The Surreal Numbers **TODO: KNUTH** have a recursive nature that can be completely separated from games. However, as they were created analyzing a position like the one above **TODO: ONAG**, its definition will be presented in terms of moves.

The model Conway, Berlekamp and Guy created to analyze games is based on finding the advantage a player has over the other. The calculation, in this model, of this advantage is given in terms of spare moves. For now, the reader may find this weird because the advantage might actually rely on the ability of the players, but when analyzing games it is important to expect both players play perfectly. Since a player loses if he or she cannot move, counting spare moves is counting how many sequential moves a player can make before reaching **equity** in the position.

Equity is found in **zero positions**. Zero position are those in which the first player to move loses. The idea to call such positions zero made sense for Conway, and, therefore, in his new set of numbers, if a game G is in a zero position, $G = 0$. If left **can win** regardless of who starts, we call it positive, and, in the other way around, negative. In more special positions, a hint on the topic of this text, in which the first to play can win, we call them fuzzy.

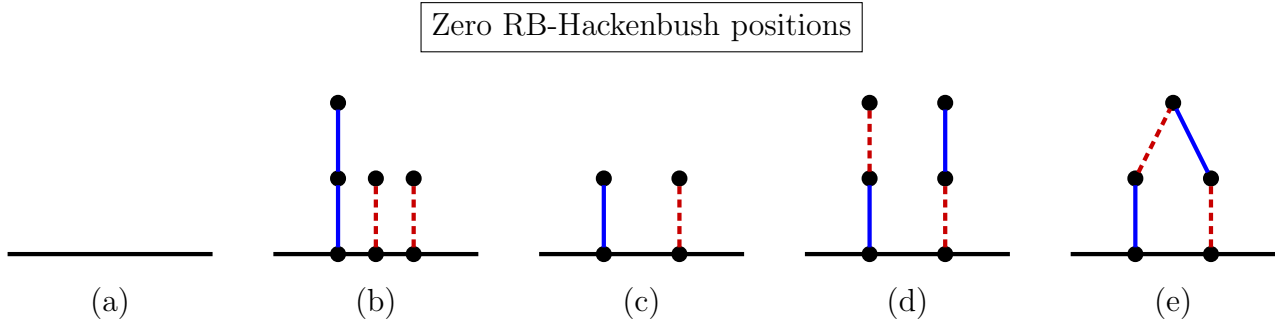


Figure 2.3

Although all games have the same value, the games are not the same. The way to represent a game derives from its game tree. The game is composed of two sets of games and $|$ is used as delimiter. The game (a) is one where neither player has available moves and, because of that, the game $(a) = 0 = \{ | \}$. (c), on the other had, is $\{ \{ | \{ | \} \} | \{ \{ | \} | \} \}$, that simplifies to $\{ \{ | 0 \} | \{ 0 | \} \}$. The games that form each of left and right sets are the configurations reachable by left and right, in a recursive definition.

The notations is exactly the same for numbers, but they should not be confused. For a game $G = \{x_1, \dots, x_n | y_1, \dots, y_m\}$ to be a number, it must be true that:

$$\forall x_i \in X, \forall y_j \in Y, x_i < y_j$$

The process of finding the number a game is equal to is the theme of the next section. With that said, some new number already showed up in the figure above. The number $\{ 0 | \}$, for example. What would be a pleasant real number for it? 1, because in this game, left has exactly one move to spare.

For the next few concepts a new game must be presented as RB-Hackenbush is incapable of generating fuzzy positions. However, a proof that all such games are numbers is due, and will be provided in the next section.

Domineering

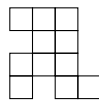


Figure 2.4

Domineering is played by placing, or marking, a 2x1 rectangle on the board, or drawing. Left plays in the horizontal and right in the vertical. The reader is

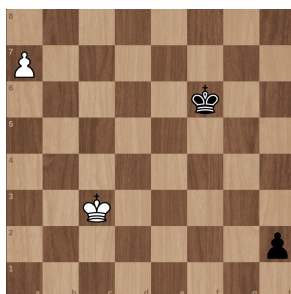
invited to play the position above a few times, and realize that the player starting it is always able to win. In fact, with perfect play, both players can win with a move to spare. The quest for calculating the advantage a player has, so far, can be summarized by the question: “Who is ahead, by how much?”. However, in positions such as the one above, a new question is more important: “How big is the next move”. **TODO: Berlekamp 1.**

This second question is answered in temperature theory. Temperature theory must follow a detailed explanation and clear understanding of numbers and games, and, therefore, a good explanation will only take place in section 4. However, as this is the main tool used to develop the topic of this text, an idea will follow.

Temperature measures the activity of the position. The activity can be understood as the importance of the next moves. In chess, for example, closed positions would be colder than endgames positions. In the position below, being the next player to move is not that important. Both players would start improving the position of the pieces until one find a break-through. Being a move behind means less piece development but the game will progress slowly, reducing the impact of that.



In **hot** positions, opposed to **cold** position, the next moves are paramount for **both** players. In the board below, the player who moves first has a clear path to victory. It might be time to re-iterate that chess in itself is not a combinatorial game, although the following position can be considered one, as there are no drawing chances. In this specific position an subset of chess that removes stalemate from the ruleset is equivalent to chess, so it is possible to make it fit the definition.



It is correct to imagine that when putting a hot game G in the notation $G = \{X \mid Y\}$, then, $\exists x_i, y_j \mid x_i \geq y_j$, making it obvious that G is not a number. On the closed position, the game might be cooler, but it will definitely heat up after a few moves. The endgame position above however, is becoming a number after the next move. Because it is becoming a number, it receives the special name **switch**.

Switches are the most basic non-numbers.

3

Theory

This chapter should outline, compare and discuss key ideas, explanations, concepts, models and theories. You should present these ideas in a systematic, well-structured and logical sequence. It is expected that you use prominent and up-to-date books and articles. All literature should be referenced, not just for quotations, but also for ideas and information/knowledge drawn from the works of others. Refer to the [Teaching and Learning platform](#) for guidance on how to incorporate references into your text

3.1 Previous Research

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3.2 Theoretical Approach

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4

Data

This Chapter should demonstrate that you have conducted a thorough and critical investigation of relevant sources. Apart from a presentation of the sources of your data, this chapter allows you to critically discuss the data (whatever these data are, ‘quantitative’ or ‘qualitative’, primary or secondary), which is proof of good research. You can even do good research with poor data but you must demonstrate that you are aware of the data quality and accordingly are careful in your interpretations. Essentially, there are three aspects to consider:

1. Reliability, which, for example, could depend on whether they are estimates or more direct evidence;
2. Representativity, which is about how typical the data are; for example, you may have arguments why the very few cases are typical or you may carry out statistical tests;
3. Validity, which is about the relevance of the data for your case. Strictly speaking, sometimes no valid data are available but one may argue that there are other data which could be used as ‘proxies’.)

4.1 Source Material

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5

Methods

In this Chapter, you present in more concrete terms the method(s) you are going to apply. And as always in research, it is good to demonstrate awareness of the weaknesses or limitations of the method you use. It makes no difference if you work with interviews, econometric models, or a comprehensive analysis of data from various sources. Transparency should be the guideline: make it possible for your readers to follow, or even repeat, your analysis!

5.1 The Approach [or Model]

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6

Empirical Analysis

This chapter covers three areas: analysis of the data; discussion of the results of the analysis; and how your findings relate to the literature. The analysis of the data can be discussed here but the details of any analysis, such as statistical calculations, should be shown in the appendices. You should present any discussion clearly and logically and it should be relevant to your research questions/hypotheses or aims and objectives. Insert any tables or figures that you decide are important in a relevant part of the text not in the appendices, and discuss them fully. Make sure that you relate the findings of your primary research to your literature review. You can do this by comparison: discussing similarities and particularly differences. If you think your findings have confirmed some literature findings, say so and say why. If you think your findings are at variance with the literature, say so and say why.

6.1 Results

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When placing tables (??) within the body of the text, the citation is placed above the table.

6.2 Discussion

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When placing figures (illustrations, pictures, graphs, diagrams, charts, maps etc.) within the body of the text, the citation is placed below the figure (??)

7

Conclusion

State the main conclusions of your study. State explicitly how and to what extent you have fulfilled your aims and objectives/answered your research questions/proved your hypotheses (whichever is appropriate). Your conclusions should follow logically from your findings and not contain any new material.

7.1 Research Aims

7.2 Research Objectives

7.3 Practical Implications

7.4 Future Research

7.5 Chapter Summary

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XKDC. The economic argument. Web page, 2015. URL <https://xkcd.com/808>. Accessed 2017/01/25.

Refer to LUSEM's Harvard referencing guidelines in the Teaching and Learning platform. [Lusem.lu.se/asks](http://lusem.lu.se/asks)

Appendix A

(Appendix A title)

The final sections of your thesis are the appendices. Each appendix should be lettered (A, B, etc.,) and should consist of detailed information that is interesting but not essential to the main thrust of your findings section.

The appendices should be in the order that they are referred to in the main text. For instance, if Appendix A refers to something on page 25 and Appendix B refers to something on page 15, the appendices need to be re-lettered. This inconsistency occurs when text is moved around or inserted.)

Appendix B

(Appendix B title)