IME-USP

# HOT GAMES

**Temperature, advantage and numbers**

by

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# Abstract

Combinatorial games are called hot when their position is active, meaning that both players want to make the next move. Hot games are the ones that typically lead to an interesting play. Before focusing in this class of games, the text will present fundamental ideas for their study. The first three chapters present and explain the concepts involved with the analysis of combinatorial games. In this first half, there is great concern with the correspondence between games and numbers and how games might or might not be numbers.

The fourth chapter formalizes the idea of temperature and provides methods to handle hot games. The following chapter makes use of all content to provide insight into more advanced ideas and discusses basic applications of the definitions. The sixth chapter, then, works the way up to the proof of a most important theorem and provides its result for a game that permeates the text, Domineering. The text finishes by providing some of the possible next steps, areas that were not discussed and pointing to a few references.

# Resumo

Jogos combinatórios são chamados quentes quando suas posições são ativas, o que significa que ambos os jogadores querem fazer o próximo movimento. Jogos quentes são aqueles que geralmente proporcionam partidas interessantes. Antes de discutir essa classe de jogos, o texto apresentará ideias fundamentais para seu estudo. Os primeiros três capítulos apresentam e explicam os conceitos envolvidos com análise de jogos combinatórios. Nessa primeira metade, há grande foco na correspondência entre jogos e números e como jogos podem ser ou não números.

O quarto capítulo formaliza a ideia de temperatura e contém métodos para lidar com jogos quentes. O próximo capítulo usa de todo o conteúdo a fim de tratar de problemas mais avançados e discutir aplicações das definições. O sexto capítulo capítulo toma os conteúdos até então apresentados para provar um teorema importante e prover esse resultado para um jogo que permeia o texto, o Domineering. O texto acaba elencando possíveis próximos passos e áreas que não foram discutidas, apontando para algumas outras referências.

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# 1

# Introduction

> "I learned very quickly that playing games and working on mathematics were closely intertwined activities for him, if not actually the same activity. His attitude resonated with and affirmed my own thoughts about math as play, though he took this attitude far beyond what I ever expected from a Princeton math professor, and I loved it."
>
> Manjul Bhargava [1]

It is no surprise that avid players of games which resemble logical or mathematical puzzles, like checkers, develop an intuition that allows them to calculate faster. This intuition comes in many forms like asserting bad moves fast and recognizing losing, drawing and winning patterns. A most essential, and sometimes very hard, component of playing well any of these mathematical games is being able to know if you are ahead or behind in a given position.

While asserting which player a position favors is already a hard task, this ability is not enough to play well in the games this text showcases. Consider the following variant of the game of chess: each player is given a set of board positions, and each should choose one board to play as white. During this game, play will take place in each board in parallel, and, whoever checkmates the opponent faster, wins the game. If one wants to be a great player of this variant, asserting if a position is winning or loosing in a regular chess game is not enough, nor is the ability to play regular chess perfectly.

The most important ability for this "parallel" variant, and for the games that this text focuses on, is to score each position. Scoring a position is different from spotting which one is better from a range of options. To simplify, for these first few pages, the reader can regard scoring as labeling a position with a real number. If one can label different position in chess, and these labels reflect advantage, then playing the proposed variation becomes easy. In this day and age, making a classifier that performs well in chess, or the proposed variant, and many other games of this sort is a reality. However, a method for perfectly classifying, or, at least, proving that position is better than another, is not widespread.

The ability to precisely calculate the advantage a player has in a position is the object of interest of Combinatorial Game Theory. This theory provides means of labeling all positions in games, not just in chess, but in all combinatorial games. A position in Go and a position Checkers, two different games, might have the same label and that means they are both equally good or bad. The modern approach to combinatorial games was

[1] Fields medalist commenting on John Horton Conway's passing.

inaugurated in 1976 in the book *On Numbers And Games* [? ], but there are studies that date back from the 1930s [? ]. The author of that book, Jonh Horton Conway, as found in the epigraph that starts this text, was, as many, an avid player of such games. In fact, Conway tells that the event that led to him invent, or discover, this theory was watching two Go players playing an endgame.

Conway realized that some positions behaved like numbers, in every aspect. While, initially, that seems very useful to evaluate games, Conway went ahead and proposed that some positions are numbers.

There is a difference between labeling an object as a number and an object *being* that number. The key point in Conway's ideia is that one can define operations as addition such that the sum of two games *is* the sum of the numbers they are equal to. The beautiful thing about games is that the sum is defined in the most natural way.

Later the reader will be presented to the idea that integer numbers are games, fractions are games, reals are games, and many more numbers are also games. This match is further explained in the following chapters and it is a fundamental topic in Combinatorial Game Theory.

As stated before, there are games that are numbers, but not real numbers. In fact the games that are numbers formed a new class, to be analyzed in the next sections. These numbers will not look like numbers at first. However, after visiting how they add up together in the most natural way, how they form an enormous class from extremely simple rules and other great characteristics, like being a completely ordered set, the reader might start appreciating them. These are the Surreal Numbers[2], name given by Donald Knuth in *Surreal Numbers: how two ex-students turned on to pure mathematics and found total happiness* [? ].

As some might understand from the title *Hot Games* alone, however, the focus of this text are in the non-numbers. It is possible for games to not behave like any of the surreal numbers, although every surreal number has correspondent[3] games. The understanding of game temperature, and by consequence all the simpler concepts like hotness and cooling, however, does not forego a good grasp of numbers. The reader will find that all games either are numbers or become numbers after some moves, so understanding them is paramount.

After a vista on both numbers and non-numbers, this text has two chapters targeted on exercising the concepts learned, visiting fun games to play and proofs on classes of games, including a new result from 2019 which is the first of its kind. The text as a whole will make the case that Combinatorial Game Theory is built upon extremely simple but powerful concepts. The few concepts are considered powerful because they not only provide a vast field of problems but also allow simple proofs for them as well.

The next three sections will present the basics of Combinatorial Game Theory, describing numbers and non-numbers. The style of these sections will be similar to that of the book *Winning Ways for your Mathematical Plays* [? ]. It means that concepts, notation and theorems are not highlighted or enumerated and, instead, their meaning are presented in the regular paragraphs. This style fits well a text in Combinatorial Game Theory because most of them are extremely simple and only a few of them are abstract. In this field, it is easy to write examples of the concepts so there is no necessity of abstracting as much as other areas of algebra or combinatorics.

[2]Originally, Conway simply called them numbers, but greatly appreciated the name given by Knuth afterwards.

[3]There are infinite games that are equal to any Surreal Number.

The style of the book is also widely known for being non-rigorous where it does not have to be. The book will commonly use images of games that satisfy assertion and provide, or not, a logic to why other games also satisfy that, without a rigorous proof. This is not completely followed in this text as justifications may be provided where not required.

On the other side of the spectrum there is *Combinatorial Game Theory* [? ]. Siegel is likely the most active researcher of the field nowadays and created the most developed program used to analyze them, `CGSuite`. `CGSuite` is a very useful tool and more about it may be found in appendix A. Unlike the authors of *Winning Ways for your Mathematical Plays*, he gave a great deal of form to this field. The book contains hundreds of definitions, notations, theorems and lemmas which are all enumerated and highlighted. One of the results is that his standards became widely used and that helps reading current papers. In general the book is more advanced and it is the primary reference for chapters 5 and 6.

Between the two there is the book *On Numbers and Games* [? ? ], and possibly all others. The first edition of the book, as stated before, gave birth to the modern approach of Combinatorial Game Theory. The book is much more theoretical and focused on pure mathematics, containing much more algebra and number theory than the successors, but is also extremely descriptive. There are other great books, but almost all the content of this text references only these three books.

Lastly, it may be worth observing that the text does not, nor does it intend to, contain everything there is in the field. In fact, it does not even shows everything there is about hot games. It may, however, serve as reference to short partisan games and will bring all the fundamental concepts of this class of games. There is a list of content that was omitted and approaches that were not discussed in the final chapter. Notice, however, that short games form a massive class of games and many of the fun games are short, so studying them first is typical, and hopefully it sparks interest in the remaining areas of the field.

# 2

# What to do with pen, paper and a friend

## The Rules

In order to study mathematical plays and answer the many questions they raise, a new mathematical field of study was developed and many new terms were created. The phrase "mathematical play" is in itself a new term, for instance. While the most common term is "Combinatorial Games", the most classic reference for this field, the book *Winning Ways for your Mathematical Plays*, favors the former.

The name "Combinatorial Game" does bring to light some important information. It, at the very least, tells that the use of counting, finite structures and graph representations are heavily used. However, it is possible to understand more of the object of interest considering the name mathematical play.

To play something mathematically could be understood as to engage in an activity in which the better use of mathematical ability, such as counting and logic, would result in advantage over its poor use. However it could be detailed further to an activity in which mathematical ability is the single defining factor. The latter might make more sense because there are games, like poker, that do require some counting ability, but that would not be likely called mathematical, since luck and reading behavior skills are much more valuable to a successful game.

Chance moves, like throwing a dice or flipping a card, are not fit for mathematical plays. Even with their removal, however, there are possibilities that would not be comfortably called mathematical plays. The nature of a mathematical plays is that both players can engage the same activity and generate advantages only out of good play. For instance, it would be hard to agree that two people playing rock-paper-scissors are battling a mathematical fight, even though there are no chance moves.

It is very important that all players have complete information of the position. Games like rock-paper-scissors, in which players take action simultaneously, block complete information. Therefore, players must move alternately. The last concerning factor in discerning mathematical from non-mathematical plays during this analysis is the number of players.

When each player has more than one opponent a goal greater than gaining advantage arises. When playing with over two people it is frequent that the best move is not the one that brings a better position but one that prevents any of the opponents from gaining an winning advantage. While that can be very mathematical, there is a clear distinction between sticking to two player games and allowing any number of players. In order to focus on the mathematical ability to make the best move, the option to allow only two players in a game is the most interesting.

As said in the introduction, this text only deals with short partisan games, so there are a few remaining criteria for the definition of game used in the text. Short games are finite and loopfree games. A finite game is a game which possibly leads to a finite number of subpositions and a loopfree game is one that a position does never lead to itself. Together the two additional restrictions make it so that short games always end after a finite number of steps because a player will eventually not be able to make a move.

When a player cannot move, he/she loses if the game is played under normal play and loses if playing in a misére version. This text only considers games under normal play. Lastly a partisan game is a game where some moves are only available to specific players. One example of partisan game is chess, where the black pieces are reserved for black and the white pieces for white. From now on, when using game to refer to a combinatorial game, it will, in fact, refer to a short partisan game.

The foundations of mathematical plays give light to a complex and rich set of problems, although, at the same time, other complex and rich problems are left behind. The game of chess, for example, does not meet the ending condition and, therefore, is left out. Fortunately, games like chess might benefit from these studies with adaptations or additional rules (although they do not consist of good examples of combinatorial games).

## The Game

Consider the following game:

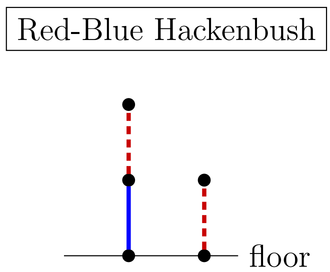



*Figure 2.1: The first instance of a combinatorial game*

In Red-Blue Hackenbush, or RB-Hackenbush, a move is made by removing a single colored edge of the image and any other edges that become disconnected from the floor. The player called Left can only remove bLue and soLid, edges, and the other, called Right, RED and dashED edges. It is a common practice to assume that all games are played between Left and Right and that if necessary the same color scheme will be used, although, occasionally, new colors will be presented.

The RB-Hackenbush is a game by the routine definition of game, meaning it has a clear ruleset and potential to be fun. It is also a game by the definition above, which will be the new routine one used going forward. However, the instance of this game drawn above is also called a game. In the future, when analyzing a board state for example, the word position is used interchangeably with the word game. Deciding which meaning the word game has must be deduced from context.

The following image depicts the fundamental process for analyzing a game with the lenses of combinatorial game theory.

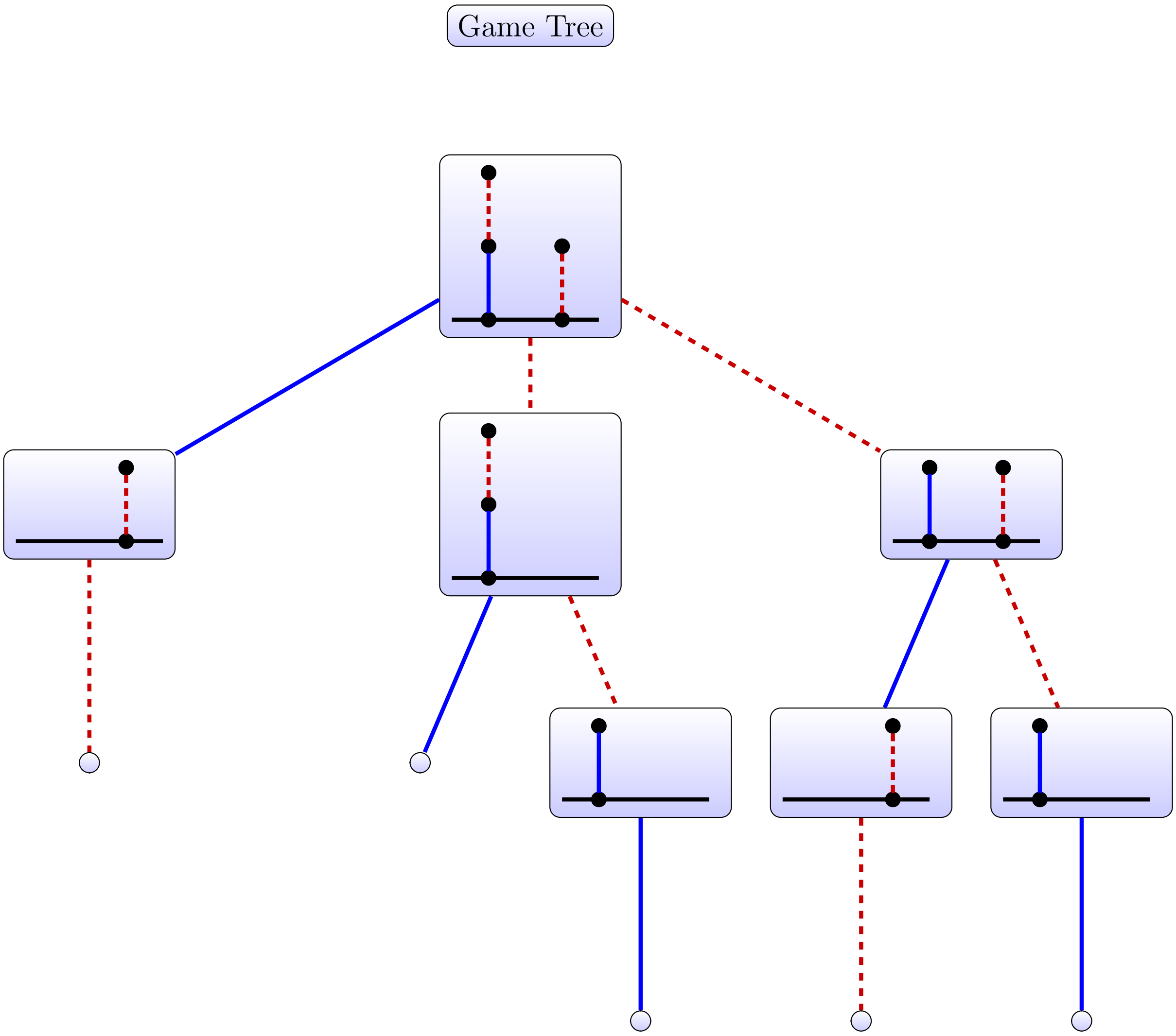
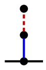


*Figure 2.2: Method used to build a game tree*

The game tree is a term commonly used to refer to a graph containing all the possible outcomes of each move. The tree used may be slightly different as in each configuration both players' move are considered, regardless of who is the next to play. The image above is the game tree that arises from the game presented in *Figure 2.1*.

In the tree above the styled edges, in the same pattern as before, between configurations tell which player made a move. As noted previously, the game tree contains all the information required to analyze games. Consider that analyzing a game is the same as calculating the number, or non-number, a game is equal to, like suggested in the introduction.

Again from the introduction, it is known that there is a most natural way to add two games together. This possibility comes from the fact that it is possible to easily merge together the game trees, as they are used in the figure. The addition of two games is equal to the merge of the two game trees. This merge is done by making the left and right options equal to the union of the left and right options of the summands.

Notice that in the case of the figure it is possible to consider that the whole game is actually the sum of two separate games. Each component is a graph connected to the floor, namely  and , because they are both disconnected. Also, notice how the game tree is in fact the merge of the components' game trees.

To complete the analysis of the game above, the next step is to decide if it is a number or not. The definition of Surreal Numbers, the only class that contains all games that are numbers, has a recursive nature that can be completely separated from games. However, as they were created analyzing a position like the one above [**?** ], it makes sense to present it in the original terms, counting spare moves.

The model Conway, Berlekamp and Guy created to analyze games is based on finding the advantage a player has over the other. According to this model, the calculation of this advantage is given in terms of spare moves. When analyzing games it is important to expect that both players play perfectly, so the number of spare moves is calculated considering that both parts take optimal decisions. Since a player loses if he or she cannot move, counting spare moves is counting how many sequential moves a player can make before reaching equity in the position.

Equity is found in zero positions. Zero positions are those in which the first player to move loses. The idea to call such positions zero made sense for Conway, and, therefore, in his new set of numbers, if a game $G$ is in a zero position, $G = 0$. If Left can win regardless of who starts, the number is called positive, and, if Right wins, negative. In more special positions, a hint on the topic of this text, in which the first to play can win, positions are called fuzzy.

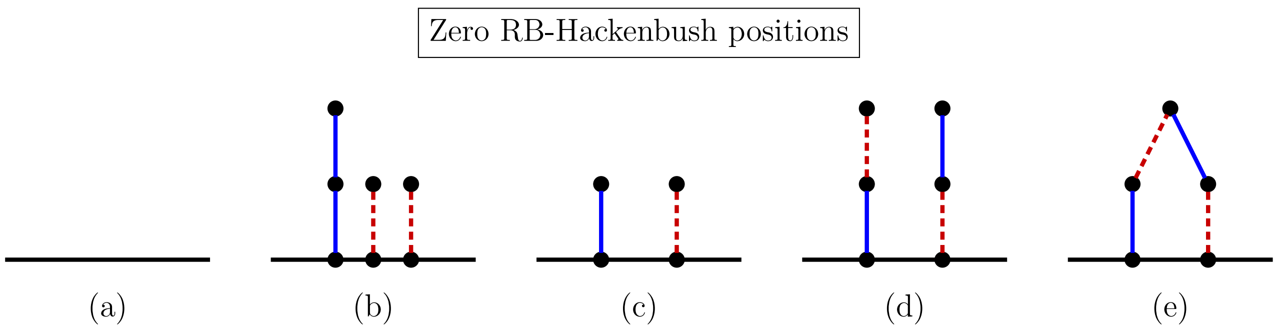



*Figure 2.3: Several instances of 0 positions in RB-Hackenbush*

Although all games have the same value, the games are not the same because their game tree is different. Moving forward, there is a new way to represent a game that derives directly from its game tree. The game is composed of two sets of games and $|$ is used as delimiter between the sets. The game (a) is one where neither player has available moves and, because of that, the game $(a) = 0 = \{\,|\,\}$. On the other had, $(c) = 0 = \{\{\,|\,\{\,|\,\}\}\,|\,\{\{\,|\,\}\,|\,\}\}$, that simplifies to $\{\{\,|\,0\}\,|\,\{0\,|\,\}\}$. The games that form each of the left and right sets are the configurations reached after Left and Right make a move, in a recursive definition.

The notation is exactly the same for numbers, but they should not be confused. For a game $G = \{X \mid Y\}$ to be a number, it must be true that $X < Y$, meaning:

$$x < y, \text{for all } x \text{ in } X \text{ and } y \text{ in } Y$$

The process of finding the number a game is thoroughly discussed to is the theme of the next section. With that said, some new numbers already showed up in the figure above. The number $\{0 \mid \}$, for example. What would be a pleasant real number for it? 1, because in this game, left has exactly one move to spare.

For the next few concepts a new game must be presented as RB-Hackenbush is incapable of generating fuzzy positions, the reason why all such games are numbers will be provided in the next section. It is worth for the reader trying to create any instance of fuzzy position in this game to accept it for th meantime.

Domineering

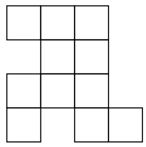


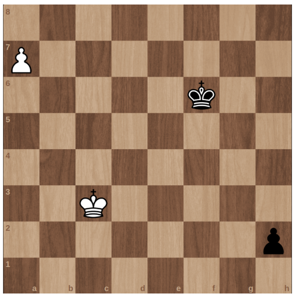
*Figure 2.4: The first game that is hot*

Domineering is played by placing, or marking, a $2 \times 1$ rectangle on the board, or drawing. Left plays in the vertical and right in the horizontal. The reader is invited to play the position above a few times, and realize that the player starting it is always able to win. In fact, with perfect play, both players can win with a move to spare. The quest for calculating the advantage a player has, so far, can be been summarized by the question: "Who is ahead, by how much?". However, in positions such as the one above, a new question is more important: "How big is the next move" [**?** ].

This second question is answered in temperature theory. Temperature theory must follow a detailed explanation and clear understanding of numbers and games, and, therefore, a good explanation will only take place in section 4. However, as this is the main tool used to develop the topic of this text, the intuition or idea follows in the remaining of this chapter. The examples will use the game of chess because it is a widely known combinatorial game and serves well the purpose. Chess, however, is not a short game, as, although it is a finite game, it is not loopfree.

Temperature measures the activity of a position. Activity should be understood as the importance of the next move. To make an analogy, in chess, for example, closed positions would be colder than endgame positions. In the position below, being the next player to move is not that important. Both players would start improving the position of the pieces until one finds a break-through. Being a move behind means less piece development but the game will progress slowly, reducing the impact of that.

In hot positions, opposed to cold position, the next moves are paramount for both players. In the board below, the player who moves first has a clear path to victory. In case chess rules are not known, in the position below White can move the pawn one square forward and that results in a promotion. White promotes to a queen and regardless of Black's move, White can soon capture the pawn, resulting in a straight-forward win for White. The same can be said for Black if he/she starts.



It is correct to imagine that when writing hot game $G$ in the notation $G = \{X \mid Y\}$, then, $X \not\leq Y$, which implies that there are left and right options $G^L$ and $G^R$ such that $G^L \geq G^R$, making it obvious that $G$ is not a number. On the closed position, the game might be cooler, but it will definitely heat up after a few moves. The endgame position above however, is cooling down completely, becoming a number, after the next move. Because both Left and Right options are numbers, it receives the special name of switch.

Switches are the most basic non-numbers. A switch is a non-numbers $G$ that both left's and right's best moves are numbers, but $G^L \geq G^R$. It is extremely easy to find the temperature and bias of such games. The process is shown here and will be used in again in the future.

A Simple Switch in Domineering

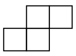
G = 

*Figure 2.5: The first game that is a switch*

In this game left has a move that leads to a zero position and right has two moves that lead to the same game with value -1. Therefore, $G = \{0 \mid -1\}$. The bias is the average of $G^L$ and $G^R$, which is equal to $-0.5$ in this case. The temperature is how much the left and right values differ from the bias, which is 0.5 in this example. A better way of writing this switch is $G = -0.5 \pm 0.5$. In general, a switch $H = \{x \mid y\}$ can be written as $H = (x + y)/2 \pm (x - y)/2$.

Calculating these values for general non-numbers is more demanding. The temperature of a general hot game is calculated based on its subpositions temperatures and the process of cooling them. This process is visualized through a temperature graphic called thermograph.

Although building thermographs is simple and cooling sounds harder than it is, these concepts are saved for later. They do not fit introductory pages because better understanding of numbers and arithmetic is required. With that said, there is one important detail that might have gone unnoticed until this point.

It was said that numbers can be added together, because the game trees are able to be merged together. However, merging game trees is more general than adding numbers, so the reality is that not just the numbers, but all games can be summed using the same procedure. If $G$ and $H$ are games, $G + H$ is a game and it is defined by

$G + H = \{G^{L_I} + H, G + H^{L_I} \mid G^{R_I} + H, G + H^{R_I}\}$, where $G^{L_I}$ is used to denote all possible Left's moves in $G$ and the remaining is defined in a similar fashion.

Notice that game addition is used to define addition itself, but that is a consequence of the recursive nature of games. After a finite number of steps, all of $G^{L_I}, H^{L_I}, G^{R_I}, H^{R_I}$ become empty, finishing the recursive structure. Therefore it is possible to add together numbers and non-numbers, but the properties of this operation are only described and used in the next chapters.

# 3

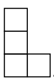
# Numbers are games. The reals, the ordinals and many others

> "And behold! When the numbers had been created for infinitely many days, the universe itself appeared. And the evening and the morning were the $\aleph$[1] day. And Conway looked over the rules he had made for numbers, and saw that they were very, very good. And he commanded them to be for signs, and series, and quotients, and root. Then there sprang up an infinite number less than infinity. And infinities of days brought forth multiple orders of infinities"
>
> Donald Ervin Knuth, Surreal Numbers [2]

Conway, by counting the number of spare moves a player has in combinatorial games, unveiled a new way to discover numbers. It is clear at this point that although numbers are not enough to represent games, they are the building blocks of all games, including the non-numbers. Until this point, the text showed some instances of the zero and hinted at 1 and $-1$, and the reader may also guess on how to build any integer in RB-Hackenbush and Domineering. This chapter shows that knowing that is no more than scratching the surface of surreal numbers.

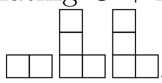
For the first part of this section a number $\{x_1, x_2, \dots \mid y_1, y_2, \dots\}$ might be called like a real number: $2, 5, 100000, \frac{1}{3}, \sqrt{10}, \pi$, but there is no reason to believe this equality yet. There is also no reason to believe that $1 < 3$ or that $1 + 1 = 2$ yet. Up until this point, numbers are labels to games that trivially translate the number of spare moves a player has. However, first a few more numbers will be labeled and only then the operations and comparisons will be defined.

## Numbers Generated in Finite Steps

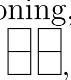
It is known that $G = \square\square = \{ \mid \{ \mid \}\} = \{ \mid 0\}$ and $G$ was labeled $-1$. What would be a good label for $H =$ , given that it must be meaningful like described previously? If it were to be labeled by a real number, which one should it be?

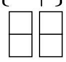
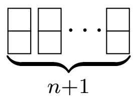
[1] In order to match the the cardinal number notation, it would be more precise to write $\aleph_0$. The text will use the book notation $\aleph$.

[2] Excerpt of the book *Surreal Numbers: how two ex-students turned on to pure mathematics and found total happiness* that precedes the moment the couple define a rule for addition of numbers.

It is possible to find out a candidate by calculating $G + H + H$. To do that, one would usually find the game tree of $G + H + H =$ , and fill the known values bottom-up. However, it is simpler in this case. $G + 2H = 0$, because whoever starts loses. Because $G = -1$, a good label for $H$ is $\frac{1}{2}$. Therefore it is true that $H = \{-1, 0 \mid 1\} = \{0 \mid 1\} = \frac{1}{2}$.

The second equality is true because Left would not move to $-1$ since it is a strictly worse move than moving the game to 0. It might not seem natural for a player to be half a move up in a game, if he/she always plays one move at a time, but if it is desired that $\frac{1}{2} + \frac{1}{2} = 1$, that label makes sense for addition. It might be valuable to reiterate that $H$ is definitely positive because Left wins no matter who starts, but $H < 1$, as Left does not have any spare moves, because $H + G = H - 1 < 0$.

As one gets used to this kind of reasoning, it becomes clear that analyzing the game $\{1 \mid \}$ is the same as analyzing the game , but the former is not reliant on a specific ruleset. Rather, any combinatorial game has an instance equal to $\{1 \mid \}$. However, the rules used to calculate the value of $G = \{X \mid Y\}$, which are presented in the remaining of this section, help understand it better. The first practical rule in this text is finding the value of $\{n \mid \}$ and $\{ \mid -n\}$, for any natural number $n$.

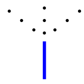
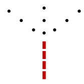
Since  $= \{1 \mid \}$, is it correct to assume that  $= \{n \mid \}$? Yes, it is correct and the recursive nature of combinatorial games show the reader why. A $2 \times 1$ board is equal to 1 since Left has a move to spare. Therefore, two $2 \times 1$ boards is equal to 2 since Left has a move to spare before reaching 1. It is simple enough to visualize it in domineering boards. The fact is it should be simple to vizualise it in any combinatorial game. For any integer $k$, the game $\{k \mid \}$ has the same value of the sum $\{k - 1 \mid \} + 1$ if Left's best option $k$ is positive. If Left's best option is zero, then the game is equal to 1, as already shown. If Left's best option is negative the result is zero, because Right cannot play, and Left's move would result in the game $k$, which is negative, showing that whoever starts loses.

In the case $k$ is positive, the addition $\{k - 1 \mid \} + 1$ was used. The notation is not the one used before, but it is possible replace 1 with $\{0 \mid \}$, as seen before. $\{k - 1 \mid \} + \{0 \mid \}$ is an instance of addition described in the previous chapter.

$$
\begin{aligned}
\{k-1 \mid \} + \{0 \mid \} &\overset{def}{=} \{\{k-2 \mid \} + \{0 \mid \}, \{k-1 \mid \} + \{ \mid \} \mid \} \\
&= \{\{k-2 \mid \} + \{0 \mid \}, \{k-1 \mid \} \mid \} \\
&\overset{def}{=} \{\{\{k-3 \mid \} + \{0 \mid \} \mid \}, \{\{k-2 \mid \} + \{ \mid \} \mid \}, \{\{k-2 \mid \} \mid \} \mid \} \\
&= \{\{\{k-3 \mid \} + \{0 \mid \} \mid \}, \{\{k-2 \mid \} \mid \}, \{\{k-2 \mid \} \mid \} \mid \} \\
&= \{\{\{k-3 \mid \} + \{0 \mid \} \mid \}, \{\{k-2 \mid \} \mid \} \mid \} \\
&\overset{def}{\vdots} \\
&= \underbrace{\{\{\ldots\{\{}_{k+1} \underbrace{\mid \} \mid \} \ldots \} \mid \}}_{k+1} \\
&= \underbrace{\{\{\ldots\{\{}_{k} 0 \underbrace{\mid \} \mid \} \ldots \} \mid \}}_{k} \\
&= \{k \mid \} = k + 1
\end{aligned}
$$

Every point discussed for $\{X \mid\}$ is valid for $\{\mid Y\}$, through a similar argument. Up to this point, the integers and $\pm\frac{1}{2}$ are defined. The remaining numbers fall into three categories: the dyadic rationals - numbers of the form $\frac{a}{2^k}$, the numbers created in exactly infinite, $\aleph$, amount of steps, and the ones generated after more than infinite amount of steps. Of course the integers and $\frac{1}{2}$ fall into the first category.

In the case of $G = \frac{1}{2} = \{0 \mid 1\}$, it is true that $G^L < G < G^R$. Is that always true? That is restricted for numbers, since in non-numbers $G^R > G^L$. If Left makes a move from $G$ to $G^L$, is it true that Left has fewer spare moves in $G^L$ than in $G$? Before, as a side note, it was said that all possible RB-Hackenbush games are numbers, and the reason may help explain that it is true.

Suppose a game $G = \{X \mid Y\}$ in which for all $x$ in $X$, $x$ is a number and for all $y$ in $Y$, $y$ is a number. Assume that $x_0$ in $X$ and $y_0$ in $Y$ are best moves for Left and Right respectively. If $G$ is not a number, then $x_0 \geq y_0$. Now consider $H$ an instance of RB-Hackenbush. A move in $H$ corresponds to removing a colored edge from a tree and all the edges that become disconnected to the floor. Assume again for $H$ that $x_0$ and $y_0$ are edges correspondent to the best moves for Left and Right respectively. That means that $x_0$ is similar to  and $y_0$ is similar to  . Is it possible that $G^{x_i} \geq G^{y_j}$?

Consider the game built by connecting $x_0$ to the floor. The game is definitely positive, as Left wins no matter who starts. If Right starts, he/she may have an available move in an edge above the one connected to the floor. Whether it is Left playing second or first, he/she may simply remove the blue edge connected to the floor, then Right has no moves remaining and loses the game. An analogous argument shows that $y_0$ is negative. Since $x_0$ is positive, the game $H^{x_0}$ resulting from removing this positive branch is less positive, or more negative, than $H$. In the same way, $H^{y_0}$ is more positive or less negative than $H$. This fact, in turn, means that $H$ is a number.

Other phrasing for "all RB-Hackenbush games are numbers" is "it is not possible to make a move that improves your position in RB-Hackenbush", or, "Left cannot make a movement that increases the value of $G$". Visualizing it brings the general idea why, in numbers, $G^L < G < G^R$. A more general approach is to consider, by induction, that $G^{LL} < G^L < G^{LR}$ and $G^{RL} < G^R < G^{RR}$. $G^{LL}$ translates to any possible Left move in $G^L$ and the others are defined similarly.

Consider the sum $G - G^L$. Before proceeding, the subtraction in this field means adding the negation. The negation of $\{X \mid Y\}$ is $\{-Y \mid -X\}$. A good way to think of negation of a game $G$ as the same game $G$, but with roles reversed. In the case of RB-Hackenbush for example, in $-G^L$, Left plays the red edges. If Left starts, he/she can move to $G^L - G^L = 0$ and win the game. If Right starts, the game may move to $G^R - G^L$ or $G - G^{LR}$. In the first case, it is clear that $G^R - G^L > 0$ because, from the definition of numbers, $G^R > G^L$. In the second case Left can move to $G^L - G^{LR}$, which is positive due to the induction hypothesis. Therefore, regardless of who starts in $G - G^L$, Left wins, so the game is positive, meaning $G > G^L$. A similar argument shows that $G < G^R$.

Knowing this, however, is not enough to find the value of G. If $G = \{3|10\}$, it is clear that $3 < G < 10$ but what is the value of G? The simpler number that fits this interval, 4. What is called simpler is the minimal-birthday number that fits the interval. The word birthday comes from the initial analogy used to describe the creation of all the numbers. Before visiting the birthday tree and finishing explaining the simplicity principle, it is necessary to visit the definition of the comparison $\leq$.

Just like the definition of addition, the comparison is also done by using game trees so it is the same for numbers and non-numbers. Consider the games $G$ and $H$. $G \leq H$ if they are equal or $H$ is more positive than $G$. This condition is met if $H > G^L$ and $G < H^R$. These restrictions enforce that $G = \{G^L \mid G^R, H^R\}$ and $H = \{G^L, H^L \mid H^R\}$, which guarantee the desired relation.

The birthday tree is the name given to the hierarchical structure that contains the definition of all surreal numbers. The tree is composed of layers, called generations, and each layer is a set of all the numbers formable with the available symbols. The zeroth generation is defined to be the set $\{0\} = \{\{ \mid \}\}$. The three first generations are:

| Generation | Labels | Elements |
|---|---|---|
| $0th$ | $\{0\}$ | $\{\{ \mid \}\}$ |
| $1st$ | $\{-1, 0, 1\}$ | $\{\{ \mid 0\}, \{ \mid \}, \{0 \mid \}\}$ |
| $2nd$ | $\{-2, -1, \frac{-1}{2}, 0, \frac{1}{2}, 1, 2\}$ | $\{\{ \mid -1\}, \{ \mid 0\}, \{-1 \mid 0\}, \{ \mid \}, \{0 \mid 1\}\{0 \mid \}, \{1 \mid \}\}$ |

Because 0 is first found on the $0th$ generation it is called a zeroth generation number even though it belongs to all following ones. The same way, 1 and $-1$ are called first generation numbers.

An important aspect of the generations is that, although labeling is not defined yet, they are fully ordered. Ordering is necessary for the simplicity rule because, as said before, $G^L < G < G^R$. In order to find the value of $G$, it is necessary to find a generation that contains both $G^L$ and $G^R$, order it using the previously defined $\leq$ operation and find the oldest number lying strictly between $G^L$ and $G^R$.

It is possible, however, to find cases where there are no numbers between $G^L$ and $G^R$. In these cases, one could look for a fitting number in the next generation. Another way to solve this is to notice that there are no numbers between $G^L$ and $G^R$ in a generation if and only if both options are from the same generation and are consecutive in that generation. This condition implies that $G^L = \frac{p}{2^q}$ and $G^R = \frac{p+1}{2^q}$. In these cases, the number in the next generation that lies strictly between them is the number $G$ and this number is labeled $\frac{2p+1}{2^{q+1}}$.

The rules and definitions up until this point add up to:

$$G = \begin{cases} 0, & \text{if } G^L < 0 < G^R \\ n+1, & \text{if } G = \{n \mid \} \\ -n-1, & \text{if } G = \{ \mid -n\} \\ \frac{2p+1}{2^{q+1}}, & \text{if } G = \{\frac{p}{2^q} \mid \frac{p+1}{2^q}\} \\ \text{search the tree} & \text{otherwise} \end{cases}$$

Some examples are $\frac{1}{4} = \{\frac{1}{10} \mid \frac{3}{10}\}$, $\frac{1}{8} = \{0 \mid \frac{1}{4}\}$, $10 = \{9 \mid \}$, $1 = \{\frac{1}{2} \mid \}$. The first and last examples require some manipulation as they would require a search on the birthday tree, as described previously. This would be a problem because both options of the first example would require an infinite number of generations to be created. However, since the embedded real numbers ought to be ordered the same and since surreals numbers are generated successively, the first number that fits the interval is the simplest that fits. Since $\frac{1}{10} < \frac{1}{4} < \frac{3}{10}$ and no older number older fits the interval, the equality stands.

The remaining problematic example is $1 = \{\frac{1}{2} \mid \}$. Consider the games, which happen to be numbers, $G = \{0 \mid \}$ and $H = \{\frac{1}{2} \mid \}$. $G = 1$, directly from the simplicity rules. It is said that $H = 1$, but that is not a direct implication. It is possible, however, to show $G \leq H \leq G$. $H \leq G$, because 0.5 is the only Left option of $H$, there are no Right options in $G$ and $0.5 < G = 1$. $G \leq H$, because 0 is the only Left option of $G$, there are no Right options in $H$ and $0 < H$. The last inequality, $0 < H$, is true because in the game $H$, Left wins no matter who starts, and this means the game is positive.

It is possible to think of many other cases that require other manipulations to fall into the precisely defined rules. The reality is that the first four rules and the $\leq$ comparison are enough to label all the numbers. The fuzzy notion of simplicity brought in the last few paragraphs helps finding and understanding the reason for labels given but it is not necessary. The 'Simplicity Rules' are actually just the first four rules.

$$G = \begin{cases} 0, & \text{if } G^L < 0 < G^R \\ n+1, & \text{if } G = \{n \mid \} \\ -n-1, & \text{if } G = \{ \mid -n\} \\ \frac{2p+1}{2^{q+1}}, & \text{if } G = \{\frac{p}{2^q} \mid \frac{p+1}{2^q}\} \end{cases}$$

## Numbers generated after infinite steps

The formula above only allows $G$ to have an infinite amount of values, but the dimension of this infinity might not approach the stated in the beginning of the section yet. The remaining numbers are hidden in the end of infinite games. Infinite games, in this section, indicate board sizes that infinite, not infinite play, because, as explained before, this text only deals with short games.

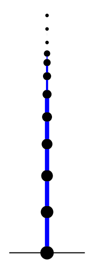


*Figure 3.1: The first infinite game*

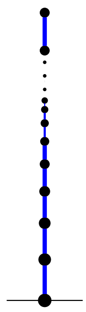
In the game above, Left has an infinite number of possible moves, but his/her move always leads to an integer. What is the value of this game? $\{1, 2, ... \mid \}$, the one more than the largest natural number. This number has been baptized much earlier in mathematics. It was called $\omega$, the first ordinal number, and the label is kept in the surreal numbers. It might not be as straight forward as finitely generated numbers, but the principle to calculate the value is the same.

For any finite number of red edges in $G$, if you add the game above to $G$, the result will be positive. However, it is also very simple to verify that $\omega$ is by far not the largest possible advantage. One simple example for that is the following section.

$\omega$ is one of numbers generated in the $\aleph$ generation. Another is the number $\{0 \mid \frac{1}{1}, \frac{1}{2}, \ldots\}$, labeled $\epsilon$. This number is positive, as Left wins no matter who starts, but it is smaller than any positive real number. A game with value $\epsilon$ may be simply the game of RB-Hackenbush starting with a blue edge and following with an infinite number of red edges.

These numbers are special in the sense that they are the first non-real numbers to show up, but they are not the only non-reals and they are not the only members of their generation. The integers and dyadic rationals show up in earlier generations, but the remaining reals all show up in the $\aleph$ generation. Chapter 5 will develop this topic a little further but, in general, it is not paramount to the topic of temperature.

## Numbers generated after more than infinite steps



The game above is an infinite stack of blue edges with another one on the top. One could wrongly argue that this additional edge on the top is simply part of the infinite stack. This is a wrong argument because this additional edge on the top allows Left to move to $\omega$, and therefore, has the value of $\omega + 1$, making the games different. The reason why in this case you can move to $\omega$ is that you can move on the top-most edge, while before, there was no top-most edge. This detail is paramount to understanding the statement found on the epigraph of this section.

A hypothetical John Horton Conway wrote the paragraph found on the epigraph near a cave close to the edge of the Indian Ocean, where a couple of future mathematicians went to find themselves. The words attributed to Conway by Knuth say that in the $\aleph$th day the universe appeared. That is because Knuth realizes that in that specific day all real numbers are generated.

However, again, accordingly to the hypothetical Conway, days kept passing by and more numbers were generated. The number $\omega + 1$ is one of the infinitely many surreal numbers that are not real numbers. However, this is not the end of the story. In fact, even considering that $\omega + \omega + ...$ and $\omega^\omega$ are also generated in the same format, the important part might again be missed. That is because looking at large numbers is not the only way forward.

Now that $\epsilon$ is a known member of the $\aleph$ generation, it is possible to make numbers like $\{0 \mid \epsilon\}$ and $\{-\epsilon \mid 0\}$ and also games like $\{1 \mid 1 + \epsilon\}$ and $\{1 - \epsilon \mid 1\}$. As the generations keep getting created it is easy to see that there are infinite numbers that are positive but less than any positive real number. In fact if taking the example around 1, it is possible to notice that between any two real numbers there are infinite non-real numbers. These non-real numbers are called infinitesimals.

There are still more non-real number that are not infinitesimals nor cardinals. A good example is the number $\sqrt{\omega} = \{1, 2, 3, \ldots \mid \omega, \frac{\omega}{2}, \frac{\omega}{4}, \ldots\}$. To verify that the label $\sqrt{\omega}$ is proper one would need to learn the definition of multiplication and verify that $\sqrt{\omega} \cdot \sqrt{\omega} = \omega$. As this text makes no use of multiplication this is not going to be verified. This example serves only to contribute to the point that there are truly many numbers originating from the simplicity rules.

## The Surreal Numbers

This text presents only a few characteristics of the surreal numbers as they are not the key point of the text. It is also not necessary to know every algebraic property of numbers to go forward with the study of temperature this text focuses on. However, it so happens that with very few construction rules, the surreal numbers, simply called numbers in the text, contain all the reals, the ordinals and the infinitesimals numbers.

Because of its extremely simple definition and big expressiveness, it is an extremely interesting topic. The idea that the number of spare moves a player has might not be a real number might not be confusing, but it should be somewhat hard to accept. It is true that this fact is based on the construction Conway made and it is not necessarily true for all ways to analyze games, but the straight-forward way of using game trees to build numbers lead to this characteristic.

Because the creation/discovery of surreals is very recent, it definitely makes people apprehensive as it is not clear if their properties are good or bad. The mathematician Phillip Ehrlich is an eloquent participant in this discussion. He makes the point that the surreals do not have an intrinsic problem and that they show an unifying nature between paths in mathematics. In one of his papers, Ehrlich proposes that, while the real numbers form, on his words, an arithmetic continuum, the surreals form the absolute arithmetic continuum[**?** ].

However, it is still a problem converting the domain of typical studies, such as calculus, from the reals to the surreals. Integrals of functions in the surreals are particularly hard to define. In the 2015 revision of their paper [**?** ], Salzedo and Swaminathan made important contributions. For reference, they proved the intermediate value theorem in **No**, how the class of all surreal numbers is called. However, for example, they used a definition of integration that, although solving a problem with previous definitions, they make it clear that other problems persist.

As the same authors point-out: "The 'Conway-Norton'[3] integral failed to have standard properties of real integration, however, such as translation invariance: $\int_a^b f(x)dx = \int_{a-t}^{b-t} f(x+t)dx$, for any surreal function $f$ and $a, b, t \in \mathbf{No}$. While Fornasiero fixed this issue [**?** ], the new integral, like its predecessor, yields $exp(\omega)$ instead of the desired $exp(\omega) - 1$ for $\int_0^\omega exp(x)dx$". In the "Open Questions" section of the paper, the authors tell that there are still problems with their definition, meaning that the conversion of domains in integral studies is not completed, at least by 2015.

While many questions remain open, however, it is still possible to work with a subset of the surreals that only contain the reals for example. The remaining of the text does not require profound knowledge of anything that is not mentioned in regards to numbers.

[3]In ONAG 2nd edition page 228, Conway tells that Norton's definition of integrals do not have desired properties.

# 4

# Heating things up

This section will provide new tools and ideas to analyze games and to do that, the same strategy of the previous section will be followed. Initially, the reader is presented with an intuitive idea and preliminary formulation of the problem. The first part contains many examples and its intent is to provide the reader enough to use the more mathematical heavy part to in fact gain advantage in any game played.

Now that enough is known about numbers, it is possible to work with non-numbers. The only known non-numbers at this point are the switches. But is knowing what they are enough to playing them? The game simpler cashing cheques will tell.

## Temperature

In this game there is a table with purple cheques. Each cheque has two numbers written on top, and, in each player's turn they will either pay one coin, blue or red circles, or cash a cheque that will grant him/her a number of coins equal to the correspondent associated integer. The number may be negative and that indicates the opposing player will granted coins. What is the best move for Left?

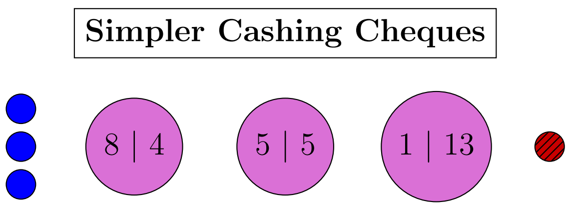



*Figure 4.1: A sum of switches and numbers*

Definitely the move is not paying, as Left can earn money in his turn. A good thing to grasp from this example is that you should never play in a number, paying a coin in this case, if there are non-numbers, cashing a purple cheque in this case, available. Should Left cash 8, 5 or 1? The reader is encouraged to play as Left and trying to find the best possible outcome, but the answer is playing the hottest switch. After trying different combination of moves, it is clear that caching 1 is the best move. Although the game above is not a switch, it is a sum of switches, and, because of that, it can also benefit of the simplified notation discussed earlier, in chapter 2.

$$
\begin{aligned}
G =& \left(\frac{8-4}{2} \pm \frac{8+4}{2}\right) + \left(\frac{5-5}{2} \pm \frac{5+5}{2}\right) + \left(\frac{1-13}{2} \pm \frac{1+13}{2}\right) \\
=&(2 \pm 6) + (0 \pm 5) + (-6 \pm 7) \\
=& -4 \pm 7 \pm 6 \pm 5
\end{aligned}
$$

If you analyze the result above, it becomes clear that Left must play on the rightmost component as, although it will not provide many coins, it will prevent right from cashing a huge amount. It is very possible to build scenarios where a player would even pay for cashing a cheque if that prevented the opponent from getting rich. Now that playing a simpler cashing cheques became easy, a more challenging task will rise. How to play Domineering well?

Adding a number with a temperature in a simplified position, like the expression above, should be acceptable by anyone following up to this point. Following, in the other hand, numbers will be added together with non-numbers just like number are added together and this might cause confusion. However, understanding that this sum is simple. As discussed in chapter 2, adding two games is the same as merging their game trees.

Playing a sum of games is just like playing a game with a set of independent rulesets and components.

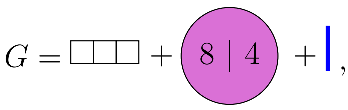



*Figure 4.2: A sum of games with different rulesets*

for example, is a game where each player makes a move[1] in any of the components and loses if cannot make a move. In other words, $G$ is a game like every other, except for the more complex ruleset.

The result of this sum is obvious if all components are also numbers or switches. In the case of playing numbers and general non-numbers, sensible players will always play in non-numbers first. With this in mind other facts become clear. One is that the temperature of a non-number added to a number is unaltered.

The other is that such a sum of games can be seen as sum of non-numbers, added together with a number after they cool out. The sum of general non-numbers is thoroughly discussed in the remaining of the chapter, however, it is worth noticing what the goal of this discussion is.

## Thermographs

By the end of this section it will be thought how to convert any game in a 2D graphic composed of two trajectories that collapse to one line at some point, whose axes are number and cooling factor. The thermograph condenses multiple and important information from the game tree. It is used for multiple purposes, but, in particular, it helps finding the best move to make and tells the average of a game, who the game favors regardless of who moves next.

[1]There are other ways of playing sum of games that will not be discussed

To build this graphic one is required to traverse the game tree, so the effort may seem fruitless as the game tree provides the winning strategy by itself. However, in cases where the game tree resulting from the sum of games is too large or expensive for a computer to run, there is a good strategy to playing this sum without knowing the complete game tree. In order to build the thermograph and play the thermostrat[2] correctly, there are a lot of minor concepts not discussed yet.

Other than the bias, playing a game like Domineering well involves the concepts of Left/Right stops, toenail, ambient temperature, freezing point, cooling, heating and a few others. To put all that together and provide a clear visualization of the best strategy, the thermograph is used.

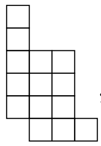
Given $G =$  , how to proceed?

*Figure 4.3: The first game that remains hot after a few moves*

G is definitely not a switch nor a sum of switches. It is possible to say the temperature in G is going to stay high for quite some time, because hotness is a term used to define the importance of the next move. A good place to start is writing out the game tree and building a temperature graphic of how it builds up from simpler positions until the more complicated ones.

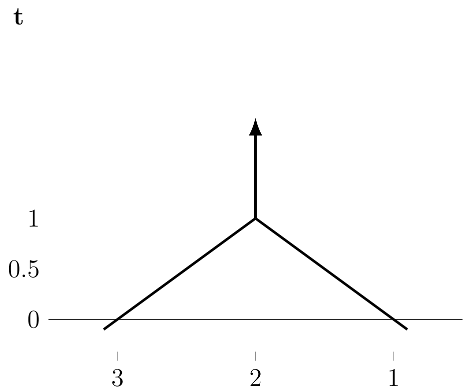
To decrease the confusion that builds up in complicated positions, there is the idea of a cooling factor. The non-number $G$ cooled by $t$ degrees is represented by $G_t$ and is defined by:

$$
\begin{aligned}
G_t =& \{G_t^L - t \mid G_t^R + t\} && \text{for all } t \leq t' \\
G_t =& \, x && \text{for all } t > t'
\end{aligned}
$$

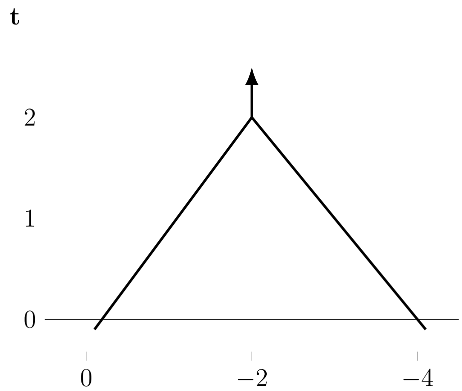
Given $t'$ is the smallest cooling factor such
that $G_{t'}$ is infinitesimally close to a number $x$,

The temperature $t(G)$ is equal to $t'$. Now that both axes are defined, some examples of thermographs are:

$G = \{3 \mid 1\}$




$G = \{0 \mid -4\}$




[2]Thermostrat is a strategy to playing sum of hot games. It provides an excellent approximation to the optimal strategy.

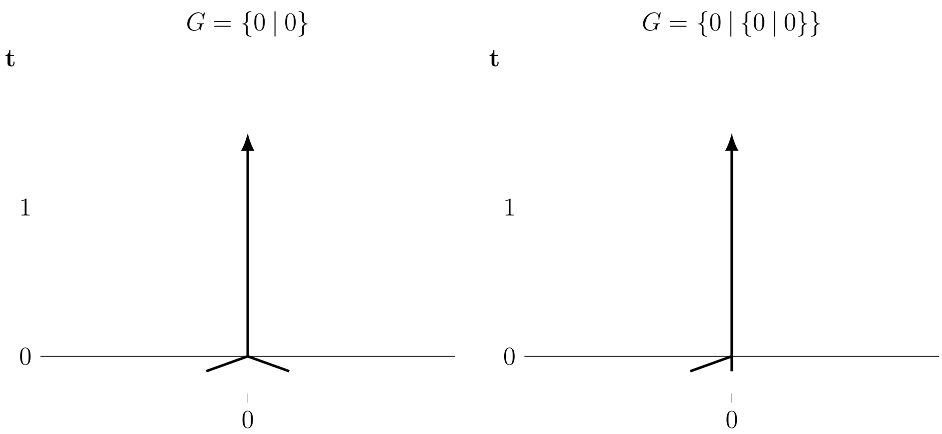



*Figure 4.4: Some games and their thermographs*

Some characteristics might be immediately apparent. The first is that the x-axis is reversed. The reason for that is to keep Right's movements to the right and Left's to the left. The second characteristic may be that all the thermographs end with a vertical mast. The mast begins at $t'$ and indicates that $G$ is a number from that point forward. The last one is that the graphic continues past the $y = 0$ line. It is worth noticing that the difference between the last two thermographs is below the $y = 0$ line.

Toenails, the segments below the $y = 0$ line, are important and may seem different, but they are actually simple extensions of the graphic. The reason for the last two toenails to be different is that cooling is applied to all the Left and Right alternatives, but in opposite directions. It is important to remember $G_t = \{G_t^L - t \mid G_t^R + t\}$, because it explains the difference. Both Left's and the first Right's toenail came from cooling 0, but the second Right's toenail came from cooling $\{0 \mid 0\}$.

The next example, the second of non-switch hot games, shows how cooling $L$ and $R$ alternatives work.

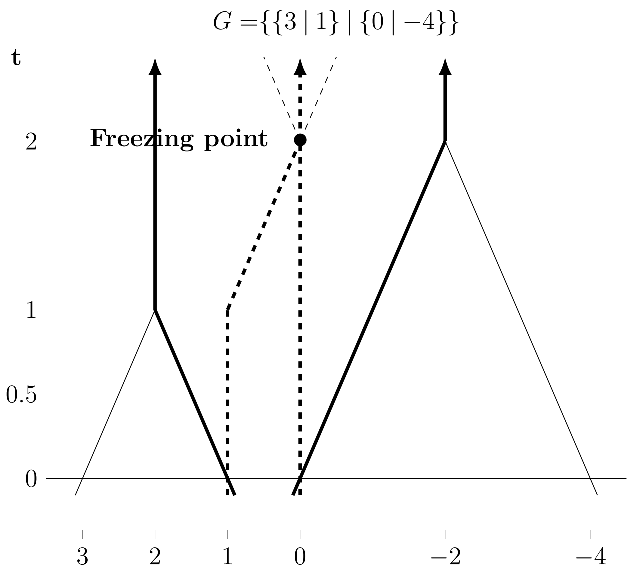
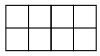



*Figure 4.5: The dissection of a thermograph*

In the example above, the thick dashed thermograph is the thermograph of G. The light dashed segments are illustrative extensions of the cooling of $G^L$ and $G^R$. The bold lines were used to show what part of L's and R's trajectories are taken into consideration: Right's slant is used to build Left's slant and vice-versa. The reason for this is that after either player makes a move, it will be the opponent's turn to move, and, in this way, the opposing slant is the important one.

In the example above, the freezing point is the same as the junction point where the right slant bends, but that is not always the case. Before visiting Siegel's formulation of the problem, one that brings good notation and formalism, an example of a real scenario is due. In a fun game each player has more than one option for their moves and that is addressed in the following example, using the domineering as example.

What is the thermograph of the game $G =$  ?

First develop the game tree until reaching sufficiently small games. In this stage one must reach end positions that are all number or switches, as these are the only simple thermographs to draw.

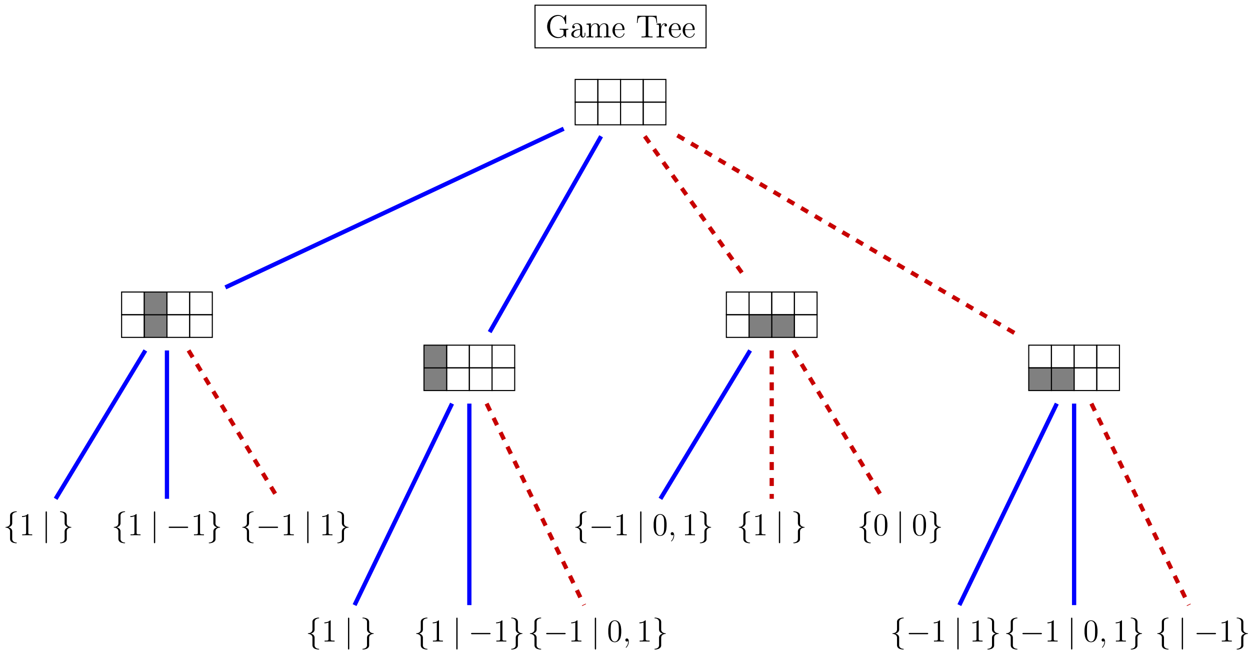



*Figure 4.6: Partial game tree of a hot game*

Draw the thermograph for each of the sufficiently small games. In this stage there might be several options for left and right but most of them could be from dominated[3] moves and can be ignored. In the following thermographs, the bold lines are of the actual thermograph and not the options.

[3]Dominated moves are moves strictly worse than other options for all cooling factors.

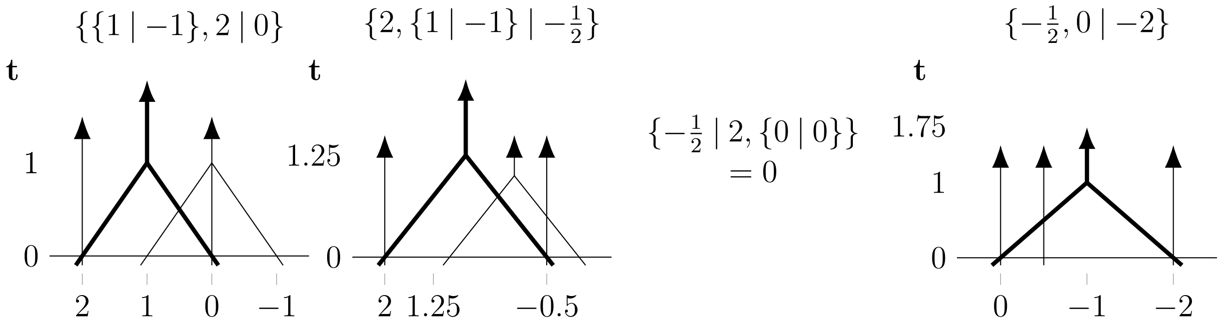
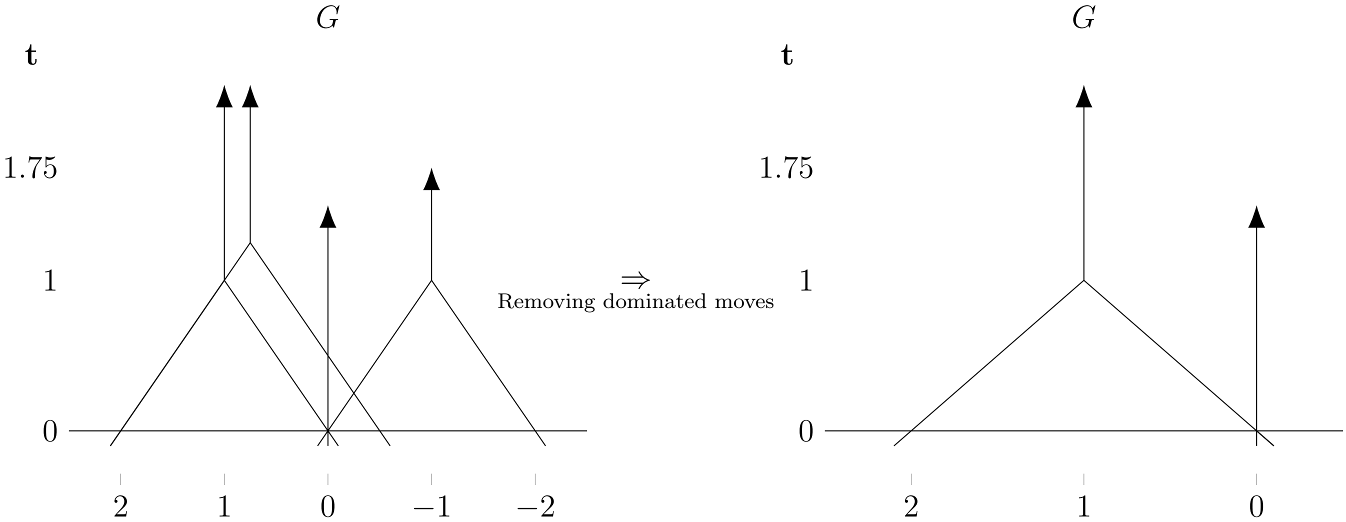
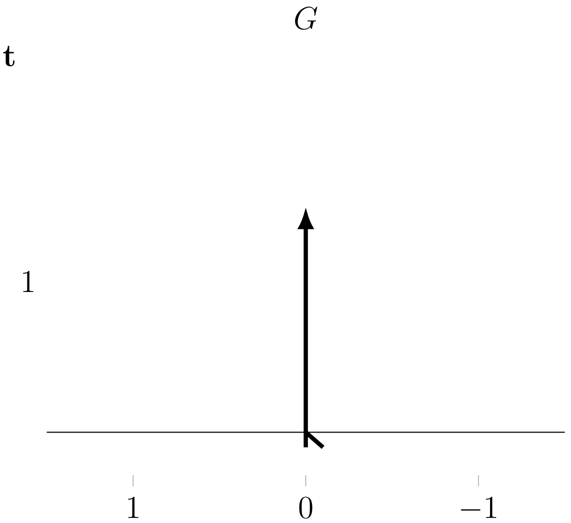



*Figure 4.7: Thermographs with several left and right options.*

The remaining of the process consists of building the thermographs of the more complicated games based on the already built thermographs. In this specific case, the only complex game is that of the original game $G$. This step may require several iterations.



The thermograph of $G$ and $\{\{0 \mid 0\} \mid 0\}$ are the same, although they are different games. From the steps taken to build that thermograph one can see that not all available moves contribute to build the temperature. Because we can calculate that some moves are strictly worse than others it makes sense that they should be ignored, but the thermograph shows exactly why.

When Left moves, he/she should avoid giving Right greater advantage over getting him/herself a possible better outcome. In the first thermograph, it is clear that 2 is greater, meaning it is more positive, or better for Left, than $\{1 \mid -1\}$. The reason for that is because the red right slant is more to the right than 2, not because the left slant is more to the right than 2.

## Minies and tinies advantages

The comment that $G$'s thermograph is the same as that of the game $\{\{0 \mid 0\} \mid 0\}$, but they are different games is surprisingly paramount to playing games well. It is so important that a game of the form $\{\{x \mid 0\} \mid 0\}$ or $\{0 \mid \{0 \mid -x\}\}$, with $x$ a natural number, is denoted by $-_x$ or $+_x$ and receives the special name miny or tiny, respectively. The game $\{\{0 \mid 0\} \mid 0\}$ is negative, of course, but Right's advantage is small. However, it is not the smallest possible.

In the game $G = -_2 = \{\{2 \mid 0\} \mid \}$, Right's advantage is much smaller than that of $-_1$. In fact, for any numbers $x, y$ such that $x > y \geq 0$, any multiple of $-_x$ is less negative than $-_y$. Multiplication is usually made between numbers, but multiplying a game with an integer is just adding that many copies of the game. The observation that any multiple of $-_x$ is less negative than $-_y$ makes sense when a real example is provided.

One should think of minies and tinies as late fees. In the case of tinies, if Left makes a move, he/she is fine. However, if he/she does not, Right may send a warning e-mail that tells Left to do so. If, after the e-mail, Left makes the move, he/she is fine again, but if Left does not, Right can charge a late fee from Left.

The reason for any multiple of $+_x$ remain smaller than $+_y$ is just that. Since there is the warning phase in the game, players will always be in time to reply. The only case where a reply is not worth it is when there are more important moves to make. Because, the only factor, in this case, that makes a move more or less valuable is the value of $x$, it does not matter how many $+_x$ there are. When playing situations where one can choose between quantity of minies/tinies over quality, the choice for quality is always the correct one.

## Non-dominated moves

One important situation, however, did not happen in the game above. There will be cases where there is no clear best move for a component $C$, and that will depend on the ambient temperature of the game. The reader can see that if $G = C$, then there are few best first moves to make and they are all equal. In the example found in *Figure 4.6*, $G^{L_*} = \{2 \mid 0\}$ and $G^{R_*} = 0$ are the definitely best moves for Left and Right, independently of adding it to other components. In that game, the other moves are dominated.

However, taking a different component $C$, where there are non-dominated options, the best first move in $C$ may change, for either player, if it is added to other components. The reader can imagine the situation where it is Left's turn to play, but Right has an important move to make next. Knowing that, Left makes a greedy move, that allows the opponent to acquire more advantage than other options, but also allows him/herself more advantages if Right does not capitalize. Right is cornered between making the previous important move and this new option Left allowed.

For example, check the following game of extended simpler cashing cheques, where the previous rules apply, but the players may also play on yellow rectangles. The yellow rectangles allow players to select any correspondent purple cheque and then discard the yellow square.

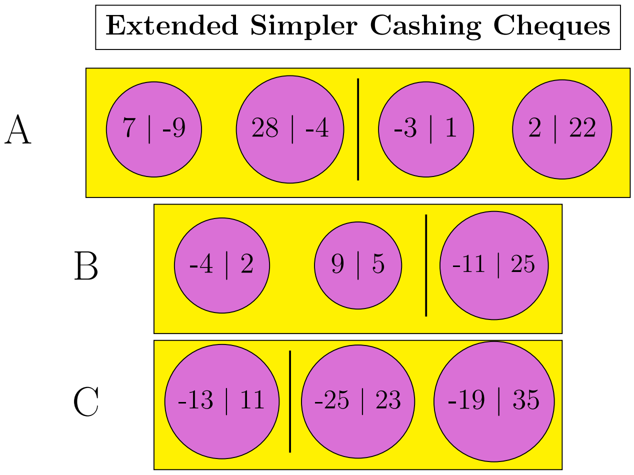


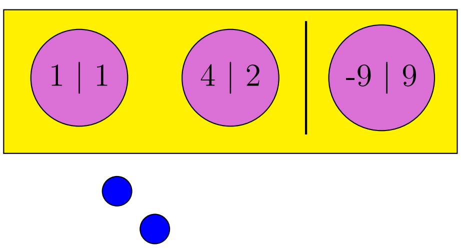

*Figure 4.8: Sum of hot components without dominated moves*

It is a good thing now to consider the game above as the sum of three components, $A, B, C$ from top to bottom, as shown in *Figure 4.8*. It is completely clear what the best moves for left and right are in each of the components. However, from the thermograph of the components, one will find that no moves are dominated, they all influence the resulting thermograph. As seen before, this means that depending on the ambient temperature the best move in each component will vary.

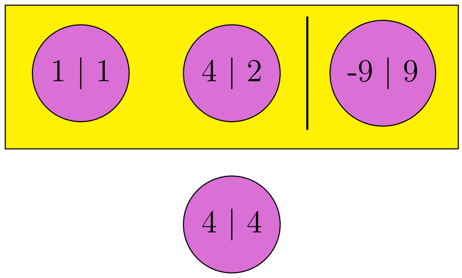
In this example, the best move for Left is actually to play in $B$ and make the move to $\{9 \mid -5\}$. In this case, the reader should notice that $A + B - 5$ is less negative than the original game, but also that Right will not necessarily move in $\{9 \mid -5\}$. Left's move is an example of taking advantage of the fact that the opponent will be distracted moving elsewhere to make the best of, in this case, a bad situation.

Consider, now, a simpler example.

The best move for Left is selecting the purple cheque equal to $\{1 \mid -1\}$



The best move for Left is selecting the purple cheque equal to $\{4 \mid -2\}$



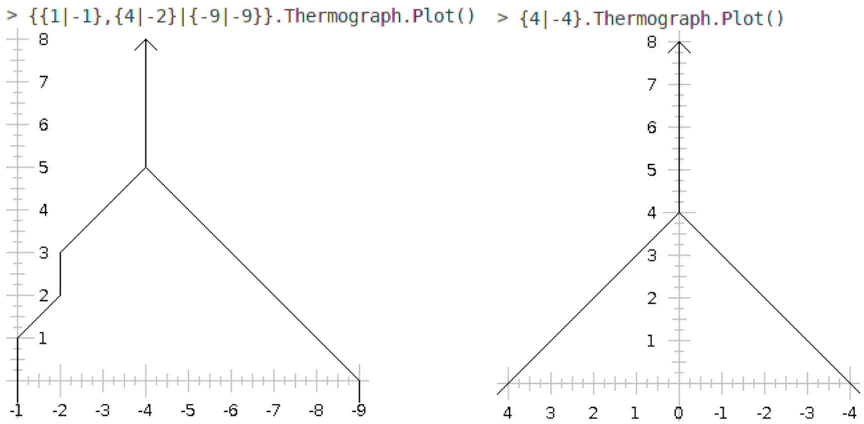




*Figure 4.9: Complete example of the best move changing*

The reader can verify that the only winning moves are the highlighted best moves. The thermograph for the second game, where the best move is selecting the purple cheque $\{4 \mid -2\}$ from the yellow rectangle, is also drawn. The thermograph shows that the temperature higher at the first component. However, it also shows that the temperature of the second component is larger than that of any of the left options of the first component. This indicates that Right will move in the second component on his/her move, and this allows Left to make de greed choice.

# 5

# It's all about finding calm in the chaos

Although the concepts and definitions of combinatorial game theory have impact in general mathematics, it is a relatively new field of study that has many open problems and not so many resources or implementations available. The purpose of this section is to bring examples, pieces of code and discuss a particular software that implements most of the theory discussed up to this point. The code fragments featured in the text use `C++` syntax, with the intent of being as simple as possible.

The examples and code fragments also serve the purpose of showing possible readers that the theory is actually quite simple, not requiring great mathematical skills, and that applying it is also not difficult. With them, some concepts presented will become clearer and a few facts enunciated will be shown. A second part of this section will introduce a few new games so that the reader might find other interesting games to play. In this second part, the reader will also see that it is not hard to make up fun games on the spot. This section can work as a midpoint between presenting the concepts and a shift in focus to handle the problem of temperature bounding, and may serve to verify if the concepts are clear.

## The Numbers

A very important and recurrent theme in the early parts of the text is the correspondence between games and surreal numbers. It is correct to say that all real numbers are also games, but it may not be clear how to make a particular real number. For example, the reader may not be able to make a game with value equals to $\pi$. In fact, the reader may not know what numbers are easy or hard to do. The first two rules of the simplicity principle should be clear enough.

```
class SurrealNumber {
  public:
    float ToFloat ();
    ...
  private:
      vector<SurrealNumber*> left;
      vector<SurrealNumber*> right;
      ...
```

```
};

float SurrealNumber::ToFloat () {
    float ret;
    if (left.empty())
        if (right.empty())
            ret = 0.0f;
        else
            ret = min(floor(-1 + minRight()->toFloat()), 0.0f);
    else if (right.empty())
        ret = max(floor(1 + maxLeft()->toFloat()), 0.0f);
    // other cases
    ...
    return ret;
}
```

While easy and hard is relative, every number that is a dyadic rational, a number that is of the form $\frac{a}{2^b}, a \in \mathbb{Z}, b \in \mathbb{N}$ is easy to form. A good method to make the representation of $z = \frac{a}{2^b}, z \geq 0$ in $\{x \mid y\}$ is:

1) Calculate $d \in \mathbb{Z} \mid 0 \leq z - d < 1$.
2) If $z = d$ then $x = z - 1$ and $y = \emptyset$, stop.
3) Binary search for oldest dyadic rational $w$, with $d < w < 1$
4) Save the steps taken in the search
5) $x$ is $d$ added to the oldest number to the left.
6) $y$ is $d$ added to the oldest number to the right.

For example, $\frac{89}{16} = 5 + \frac{9}{16} = \{5 + \frac{1}{2} \mid 5 + \frac{5}{8}\}$, because the binary search for $\frac{9}{16} = \frac{89-80}{16}$ follows the path $(1 \xrightarrow{L})\frac{1}{2} \xrightarrow{R} \frac{3}{4} \xrightarrow{L} \frac{5}{8} \xrightarrow{L} \frac{9}{16}$. In RB-Hackenbush, building $\frac{89}{16}$ is now easy. Take a pile with five blue edges and add it to a blue-red-blue-blue pile. The red-blue-blue pile on top of that derives from the right-left-left path on the binary search for $\frac{9}{16}$, with a right turn corresponding to a red edge and vice-versa.

A similar idea is used in the following featured snippet, but with the objective of finding $z = \frac{a}{2^b}$ given $\{X \mid Y\}$.

```
    //other cases
    else
        ret = FindSimplesFittingNumber();
    return ret;
}
```

```
float SurrealNumber::FindSimplesFittingNumber {
    float maxL = maxLeft();
    float minR = minRight();
    if (maxL < 0 && 0 < minR)
        return 0.0f;
    float d = floor(maxL);
    float x = maxL - d;
    float y = minR - d;
    float fact = 1.0f;
    while (fact > EPSILON)
        if (x < fact)
            if (y > fact)
                break;
            else
                fact *= 0.5f;
        else
            fact *= 1.5f;
    return d + fact;
}
```

All the remaining numbers are hard in the sense that they require an infinite number of steps to define. A simple code like the one above will not handle that. That is because $\frac{2}{3}$ and $\pi$ are both generated in the ℵth day (section 3), like any other non-dyadic fractions. However, these are not equally hard. Because $\frac{2}{3}$ is a periodic number when using binary representation, its path in the numbers tree is well-defined.

It is true that both $\frac{2}{3}$ and $\pi$ are given by something equivalent to the following: $n = \{x \in S_* : x < n \,|\, y \in S_* : y > n\}$, where $S_*$ is the set of numbers generated until day ℵ. However, $\frac{2}{3} = 0.\overline{10}_2$ can also be defined through the infinite right-left-right-left... path in the binary tree, because a 1 in the binary representation is a step do the right in the tree and vice-versa. One way to think of this fact is that, when doing a binary search for $x$ between the dyadics in (0,1), the starting question is $x > 1/2$. The 1 indicates a yes, so the following question is $x > 3/4$, and so on. Since numbers smaller than $x$ go on the left set and vice-versa, $\frac{2}{3} = \{1/2, 5/8, \ldots \,|\, 3/4, 11/16, \ldots\}$, showing that the numbers visited in the binary search alternate in the left and right sets. Repeating decimals, in RB-Hackenbush, are extremely easy to spot because they are always a sequence of red and blue edges followed by a finite pattern repeated infinitely.

Other than some real numbers being hard to draw, drawing non-reals is also not trivial. It was shown that there are infinitely many numbers between any two real numbers, in section 3. This might seem hard to understand or accept initially, because the reals form the complete ordered field. However, at least based on Conway's surreal numbers, the fact that it is possible to build a game in which the advantage a player has is strictly between two real numbers gives some light to the fact. To do that, simply add an infinitesimal to a real number, as seen before.

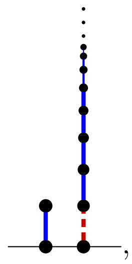
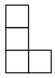


*Figure 5.1: Drawing non-reals is not hard*

The game in the figure has value $1 - \epsilon$, a number smaller than 1, but larger than any $x \in \mathbb{R}, x < 1$. Although the non-reals might be new, the effort of writing hard reals and non-reals as RB-Hackenbush is very similar.

RB-Hackenbush is an extremely good example to understand numbers, as finding dyadics and repeating decimals are very easy. However, not all games are like this. It was shown that  $= \frac{1}{2}$, but how would one build an instance of $\frac{1}{4}$, or $\frac{1}{2^n}$, for $n \in \mathbb{N}$? It turns out that it is not trivial at all.

In fact, only in 1996 a partial solution was given. One of the results of the paper "New Values in Domineering" [**?** ] was the existence of arbitrarily small values of domineering games. Before that, it was unknown whether or not they existed. It was only in 2015 [**?** ] that a method to create all dyadic rationals in a single component was introduced. Following the strategy presented in 1996, the so-called Yonghoan Kim's snakes were created, named after the author. The snake's representation is found in *Figure 5.2*, copied from Richard K. Guy's list of unresolved problems in *Games of no Chance* [**?** ].

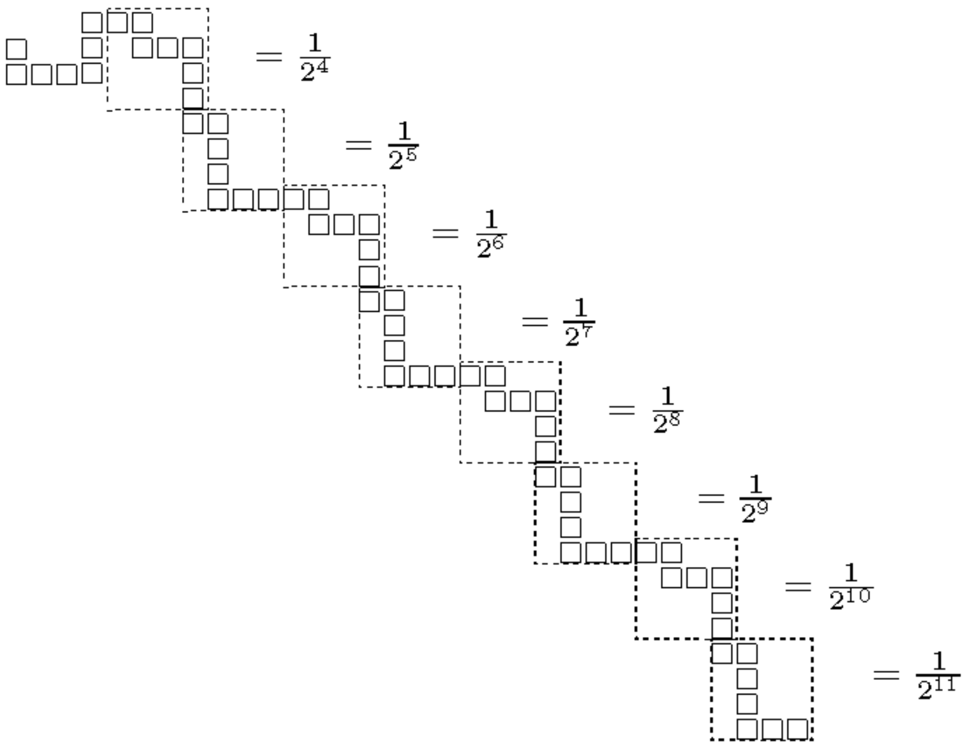



*Figure 5.2: Kim's snakes*

In this method of finding dyadic rationals, there is a initial structure, that is not marked by a dashed rectangle. To this structure, additional ones are appended. As shown, each additional structure alternated between two others. Of course that the resulting game is not the only game with the given values.

In fact, there are special games with the same value as the snakes. The 2015 paper is based on these special games. These special games are based on concatenating and imploding bridges in domineering games. Bridges are cells that do not change the value of a game, and are also called explosives squares. For example, $\square\square$ and $\square\square\square$ have the same value, so one of the extremities is a bridge. The strategy is built upon two theorems that are beautiful because they are powerful yet very simple.

The *Bridge Splitting Theorem for Domineering* was introduced by Conway. It reads: If the value of $G\square$ is the same as $G$, then the value of $G\square H$ is the sum of the values of $G$ and $\square H$, given $G$ and $H$ do not intersect. The proof is:

$$G\square H \leq G + \square H = G\square + \square H \leq G\square H$$

The first inequality is valid since splitting a horizontal always favors Right, and, the second inequality is true because merging horizontal squares always favors Left. The second important theorem was created by the authors of the aforementioned paper.

*Bridge Destroying Theorem for Domineering*

If the value of $G\square$, $\begin{array}{c}H\\ \square\end{array}$, $\square I$ and $\begin{array}{c}\square\\ J\end{array}$ are the same of the values of $G$, $H$, $I$, $J$, then the value of $\begin{array}{c}H\\ G\square I\\ J\end{array}$ is the same as the sum of $G, H, I$ and $J$, provided that neither of the games have common edges. The proof is also simple:

$$\begin{array}{c}H\\ G\square I\\ J\end{array} \leq G + \begin{array}{c}H\\ \square I\\ J\end{array} \leq G + I + \begin{array}{c}H\\ \square\\ J\end{array} = G + H + I + J =$$

$$= H + J + G\square I \leq \begin{array}{c}H\\ G\square I\\ J\end{array}$$

The first two inequalities are true because they both split a horizontal line, favoring right. The two equalities are true because they are applications of the *Bridge Splitting Theorem for Domineering*. The last inequality is true because it is a linking of a vertical line, which can only favor Left.

An example of applying this is figuring out that $\begin{array}{l}\square\\ \square\\ \square\square\end{array} = \begin{array}{c}\square\\ \square\\ \square\square\square\end{array} = \frac{1}{2}$, therefore it is true that $\begin{array}{l}\ \ \square\ \ \ \square\\ \ \ \square\ \ \ \square\\ \square\square\square\square\square\end{array} = 1$. Another, much more beautiful example is copied here from the same paper from 2015:

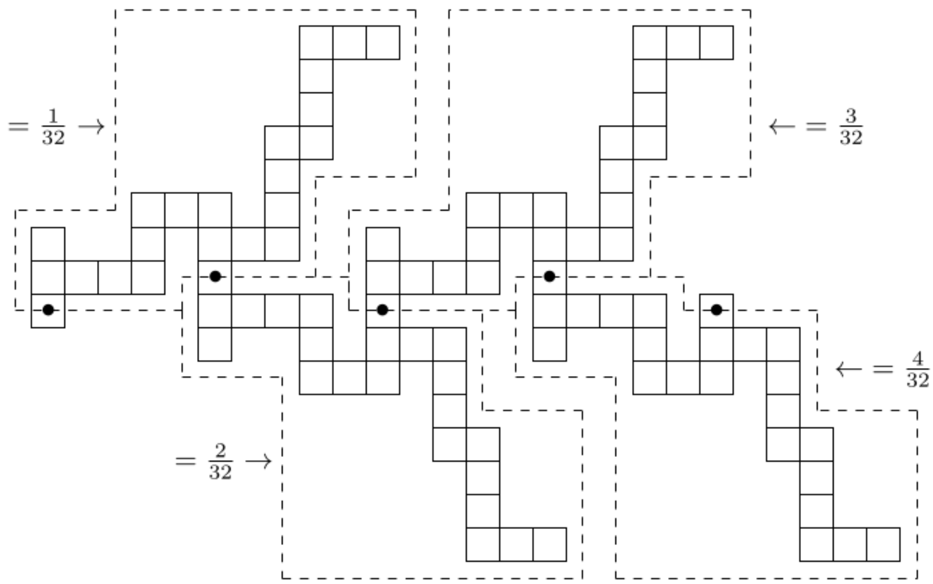



*Figure 5.3: How to merge Kim's snakes to create other dyadic rationals*

By combining the two theorems with Kim's snakes, it would be possible to create any dyadic in a single connected component, if not for two problems. The first is the problem of creating a bridge. However, that is a non-problem because Kim's snakes always have explosive squares. An explosive square that is always there is a square below the two vertical squares to the left of the base configuration of Kim's snakes. Since the value of the snake is equal to the values of the sum of the bridged components, the base component may be interchanged with the one with an explosive square and the values of the snake would remain the same.

The second problem is that of making sure that ever component will not share an edge with the others. The solution of this second problem is handled and solved in the paper. The domineering discussion above serves to show how some games are good to analyse and create examples with, while other not so much. RB-Hackenbush is a great game for working with numbers and "Extended Simpler Cashing Cheques" (ESCC) is a great game for studying temperature.

## The non-numbers

When the topic changed from numbers to non-numbers, an example of ambient temperature, and how it affected play, was required. In order to keep the results to the boundaries of the literature, an instance of such game was shown. This instance was made by reverse engineering the example of the same topic found on *Winning Ways* [**?** ] using a convenient game. Normally, it is extremely hard to generate a configuration with any given temperature value, but ESCC allows it, and this is why its creation in this text was important.

ESCC allows an easy conversion from a $\{X \mid Y\}$ to a configuration, mainly because any switch is trivial to write down. The original thermograph, that was reverse engineered, is found in the book *Winning Ways*, *2nd* edition, on page 160, is copied below:

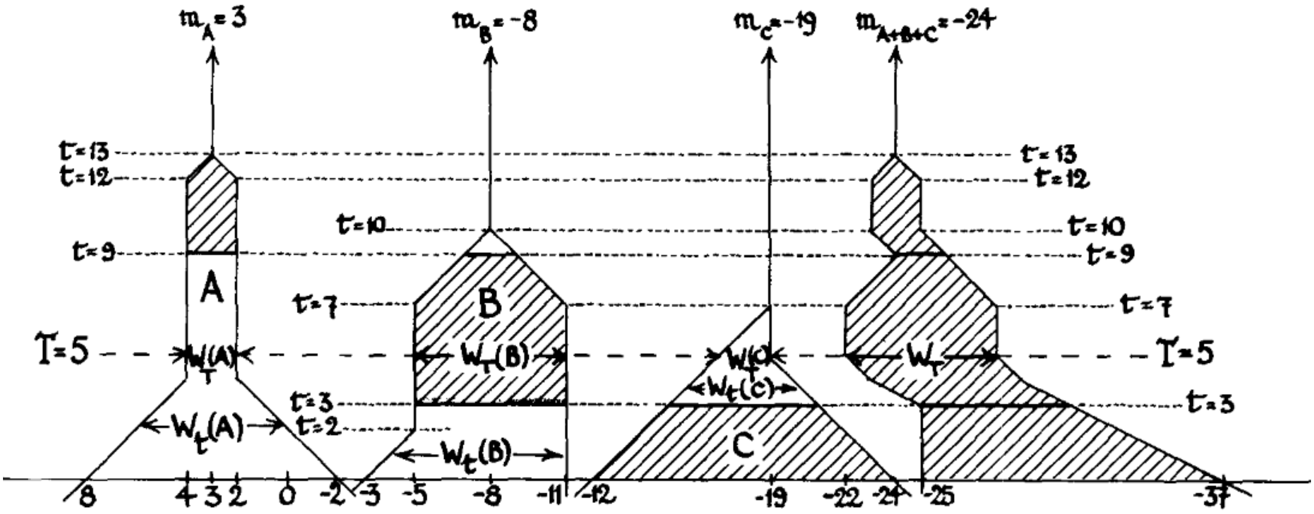
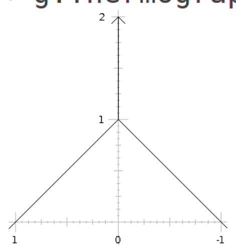


*Figure 5.4: Figure extracted from the book Winning Ways*

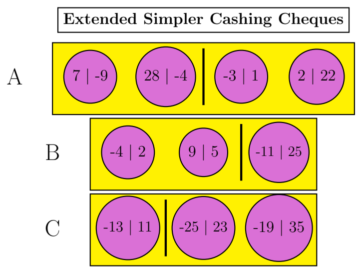
To verify that the resulting compounded thermograph of the example featured in the previous section is indeed the one found above, it is possible to use Aaron Siegel's `CGSuite`. `CGSuite` is an implementation of almost all of the methods found in Combinatorial Game Theory. This text will not go in detail on the software package, but Appendix A brings to light some additional information.

Between other practical and interesting features, it is possible to create any games and calculate and plot their thermographs. For example:

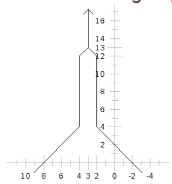
```
>g := {1 | -1}
±1
>g.Thermograph.Plot()
```



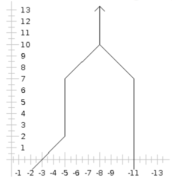

The game below, copied from the previous chapter could, then, be converted to the following thermographs in *Figure 5.5*:



```
>A := {{7|9},{28|4}|{-3|-1},{2|-22}}
{8,{28|4}|-2,{2|-22}}
>B := {{-4|-2}, {9|-5} | {-11|-25}}
{-3,{9|-5}||-11|-25}
>C := {{-13|-11}|{-25|-23},{-19|-35}}
{-12|-24,{-19|-35}}
>A.Thermograph.Plot()
```




```
>B.Thermograph.Plot()
```




```
>C.Thermograph.Plot()
```

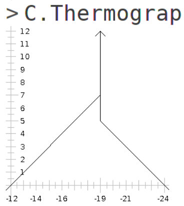



*Figure 5.5: Thermographs of A, B and C components plotted with* ***CGSuite***

Therefore the game from last section has indeed the same thermograph as the one featured in the book. The steps taken to reverse engineer the thermographs from the book, might be useful to fully grasp, speed up and exercise the process of building thermographs. To do that, first there is need for one more theorem.

This theorem is about the structure of thermographs. It states that every line in a thermograph has slope equals to $-1$, $0$ or $1$ and masts have slope $0$. In order to match the common notation, consider that a slope $0$ line is a vertical line and $-1, 1$ slope lines to be lines with slopes $1, -1$, respectively. Using the notation created by Aaron [**?** ], let $G$ be a game and $\lambda_t(G), \rho_t(G)$ its thermograph's left and right trajectories, respectively. Also let $\lambda_t\tilde{(}G), \rho_t\tilde{(}G)$ be the drafts of left and right trajectories. They are defined as following:

$$\begin{aligned} \widetilde{\lambda}_t(G) &= \max_{G^L}(\rho_t(G^L) - t) \\ \widetilde{\rho}_t(G) &= \min_{G^R}(\lambda_t(G^R) + t) \end{aligned}$$

$$\lambda_t(G) = \widetilde{\lambda}_t(G),\ \rho_t(G) = \widetilde{\rho}_t(G) \text{ if } t \text{ bellow the freezing point}$$

$$\lambda_t(G) = \rho_t(G) = x \text{ if } t \geq t' \text{ with } x = \widetilde{\lambda}_{t'}(G) = \widetilde{\rho}_{t'}(G)$$

Observing the thermographs presented so far, for any temperature above the freezing point $t'$, the values of the left and right slants are the same and do not change. Notice how the definitions do indeed match the drawings. Now, to prove the theorem, consider the following partial results and the proofs that follow.

(a) Every segment of $\lambda_t(G)$ has slope $0$ or $-1$.
(b) Every segment of $\rho_t(G)$ has slope $0$ or $1$.
(c) Both trajectories have masts of slope $0$, with the same value.

The proof is not complicated. Following is an adaptation of the proof given in Aaron's book, which is more direct. First the proof of (a) and (b), given via induction:

*Base:* (a) and (b) are true if $G$ is a number, as the slope is always zero.

*Induction Step:* Let $\lambda_t(G^R)$ satisfying (a) and $\rho_t(G^L)$ satisfying (b). Since, by definition, $\widetilde{\lambda}_t(G) = \underset{G^L}{max}(\rho_t(G^L) - t)$, the slope of $\widetilde{\lambda}_t(G)$ is either 0 or 1 translated by $-t$. The same goes for $\widetilde{\rho}_t(G)$. Since $\widetilde{\rho}_t(G) = \underset{G^R}{min}(\lambda_t(G^R) + t)$, the slope of $\widetilde{\lambda}_t(G)$ is either $-1$ or 0 translated by $+t$. This way, (a) and (b) are true.

Now, the proof of (c), also via induction. Notice that this part proves the existence of a freezing point, and the mast follows by definition.

*Base:* (c) is true if $G$ is a number, as the slope is always zero.

*Induction Step:* Let the slopes of $\lambda_t(G^R)$ and $\rho_t(G^L)$ be zero. Therefore, the masts of $\widetilde{\lambda}_t(G)$ and $\widetilde{\rho}_t(G)$ are $-1$ and $+1$ respectively. Since $\widetilde{\lambda}_t(G) \leq \widetilde{\rho}_t(G)$, the left and right trajectory cross and, therefore, there is a freezing point. By definition, after the freezing point, the slope is zero.

The reader can go through the thermographs in this text and verify that this is indeed true, in all cases. Trying to build a counter-example for the previous slope theorem may help understand the proof. Inevitably, the recursive definition of the thermograph always leads to the base case of number's thermographs, as all the leafs of all game trees are zero, a number. Now that these rules are properly stated, it is possible to understand how to write a game $G = \{X \mid Y\}$ based on an existing thermograph.

Starting by the thermograph of $C$ of *Figure 5.5*, it is easy to spot that there is only one viable option for Left. Since the left slant, $\widetilde{\lambda}_t(G)$, of that graphic has slope $-1$, then $\rho_t(G)$ has slope 0. Since the slant crosses the $x$-axis on $x = -12$, then a possibility to this Left option is the number $-12$. The option then, is the canonical form of this number, adapted to fit the rules of the cashing cheques game. The right trajectory is a bit more complicated.

Starting with the inclined slant, the procedure described above helps one finding out that the number $-24$ is enough. The vertical line between the bent line and the freezing point must come from a non-number. Because a bent line added to the cooling factor $t$ is a straight line, then the left slant of this non-number must cross the x-axis in $x = -19$. Another requirement is that this game must be hot until $t = 7$, because the straight line goes until this point. With this in place, it is possible to create the game $\{-19 \mid -35\}$, because $-35 = -19 - 7 - 7$, in which the sevens come from the $t = 7$ and are used twice because the cooling happens in both directions. In this example, the right option could be any number smaller than $-35$, because the boiling point would remain the same

The thermograph of $B$ is more difficult. The first bent line of the left trajectory is the same as before, and, in this case, the number $-3$ is enough. The straight line, just like before, comes from a non-number whose right option is $-5$. Unlike the non-number from the previous example, however, the freezing point of this case must be exactly 7, so the distance between the right and left stops must be 14, unlike the previous example. With this information, it becomes possible to build the game $\{9 \mid -5\}$. The last bent segment is the mast of the non-number.

The right trajectory of $B$ is the simplest $\{-11 \mid -25\}$, which has already been explained. The process to build the game $A$ is a repetition of the left trajectory of $B$ for both the left and right trajectories. By going through this, hopefully it became clear that thermographs are not hard to build. Also, by using the same example as the book, the reader might also benefit from the remarks found there.

## Numbers and temperatures in Domineering

The brief description of how to build dyadic rationals in domineering and how long it took for the process to be created, showed that domineering is not a good game to study numbers. Up to this point domineering has been used to exemplify the study of temperature, but the truth is is not so good to study temperature with. Temperature in Domineering is limited, both in already calculated instances and in current knowledge of the game.

In fact, in 2005 a paper [**?** ] brought to light that Elwyn Berlekamp conjectured, in 2004, that the maximum temperature in domineering is 2. That is yet to be proven or disproved, although computed instances also lead to the same conclusion. It may not be all that clear yet, but many common games cannot have arbitrarily high temperatures. In a domineering position with temperature two, whoever plays first finishes with two spare moves. Before trying to create such a position, the intricacies of such a position might not be clear.

In 2004 an instance of that position was found, and that was the only one, in a connected board of course, up to this day. While this conjecture is a typical open problem in combinatorial game theory that many researches know about, the problem is hard to solve. The next chapter is dedicated to the Ph.D. thesis of Svenja Huntemann [**?** ] that gives an upper bound for classes of games. One of the applications is to bound the temperature of certain configurations in domineering. Although that does not answer the question, it might be a good step.

## It is simple but not easy

The initial code fragment of the section only dealt with simple surreal numbers, even making a lot of assumptions about left and right, without verifying anything. It might make sense to think about how one would implement, in practice, the numerical part of a library like `CGSuite`. Considering only the `SurrealNumber` class, if the only available method of defining numbers were the left and right sets, how would be the simple number $3 = \{2 \mid \}$ be stored?

Of course that, with only that available, it had to be stored constrained to the recursive structure of games, so to save the number 3 the computer would have to store the game tree of $\{\{\{\{\mid\}\mid\}\mid\}\mid\}$. Although correct according to the theory, implementing number like that, only, would cause problems because evaluating a number would no be a trivial operation. For instance, telling whether $2.5 = \{2 \mid 3\} < 3 = \{2 \mid \}$ would require traversing both number's left and right trees. The canonical representation of the number 3, for example, would be something like *Figure 5.6*:

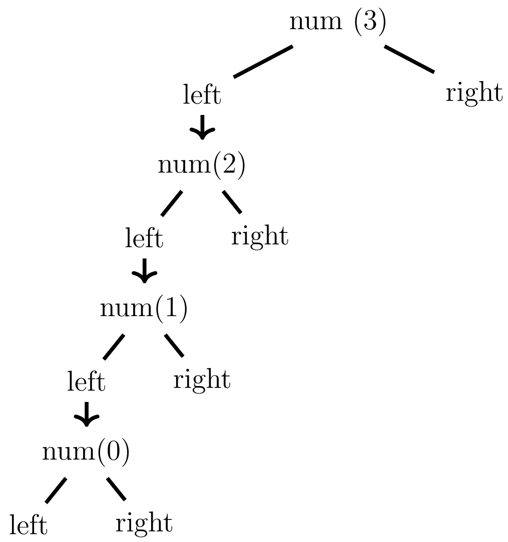



*Figure 5.6: Representation of a simple number in a computer*

This form of representing simple number is a problem both in terms of memory consumption and processing speed. A viable implementation of combinatorial games must look for alternative representations. Some possibilities are allowing surreal numbers to be represented with integers and floats or to store results of the tree traversals to minimize the number of times the complex tree structure must be analyzed. When looking at numbers strictly, the tree that resulted in a float or another chosen numerical representation may be discarded because it does not contain special information.

However, when changing the domain from surreal numbers to games, the tree cannot be always discarded. The reason for that is that the game tree may contain relevant information for gameplay. One could try keeping only the best move for each position in order to save space. However, as seen before, adding games may change which move is best so not all branches can be discarded.

Consider the following draft of a combinatorial game class declaration:

```
class CombinatorialGame {
  public:
    CombinatorialGame (const vector<CombinatorialGame*>& l,
                       const vector<CombinatorialGame*>& r);

    CombinatorialGame (vector<CombinatorialGame*>&& l,
                       vector<CombinatorialGame*>&& r);
    ...
  private:
      // Calculated and cached during initialization
      vector<CombinatorialGame*> left;
      vector<CombinatorialGame*> right;
      bool isSurreal;
      bool isSwitch;
      float realValue;
      float bias;
      float temperature;
      Thermograph thermograph;
};
```

Implementing the draft above woulds help with space requirements and unnecessary calculations but does not completely solve any of the more complex issues. It would, for example, avoid repeated calculations, unlike the version where the GetFloat method went through the game tree calculating it on demand.

However, for one, an implementation of this draft would be incapable of handling infinite games, as those cannot be stored in vectors. Again, a more complete version can be found in `CGSuite`'s github repository, but understanding and tackling the difficulties may be a worth exercise. Even that package used by many researches has it issues and does not deal well with infinity either.

## Other Games To Play

Now that vocabulary, methods and tools were introduced, this section brings some games to put your knowledge to proof.

**The Amazons** is a typical game that, like domineering, is fun and only requires pen and paper. Amazons is typically played in a 10x10 chess board with each player having four initial pieces in the fixed initial position depicted below. Each piece works just like a chess queen. In each turn, the player selects one of his/her queens and moves in an arbitrarily number of squares, like the regular chess queen. After moving, the amazon has to shoot an arrow in any direction however many cells desired. The arrow burns the cell. The amazon and the arrow cannot go through any burned square, and this is what makes the game interesting.

A common variation is one where every square the amazon (queen) goes through is then burned from the game, making it so that no queen can move over it. Burning the path either amazon goes through is a replacement for shooting the arrows and there are many other possibilities, like shooting two arrows or shooting and burning the path, or choosing either of those.

In either variation, like in most short games, the initial position is unimportant and the players can choose any starting position they find interesting.

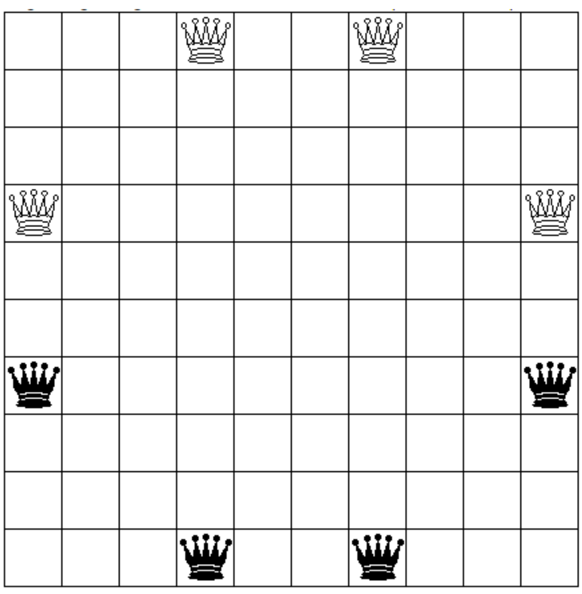


*Figure 5.7: Most typical starting positions in Amazons*

For example, play may take place in an 8x8, a chessboard for convenience, with each player having two queen in opposing corners. Since, in this example, player Left is a beginner an Right is a seasoned player, Left has one additional queen next to one of his/her queens. It serves to show that there are no restrictions to the initial position - games do not have to be mathematically balanced to be fun.

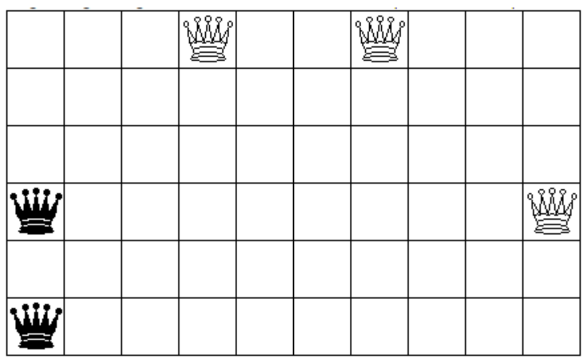


*Figure 5.8: Unbalanced and fun position in Amazons*

**The Angel Game** is more of a mathematical challenge more than a game, because it is not meant to actually be played. This game is played in a infinite squared board, with an Angel being located in one of its cells. Each turn the Angel flies over $k$ squares and lands, depending on the desired rules. When it is the Devil's turn to play, it possesses a square from the infinite board. The challenge is to know whether, and for what values of $k$, the Devil wins, given the Angel may not lend in a possessed square. It is clear that this game, depending on the parameters and variations, may not be a short game.

This game led to a lot of research until it was solved. Although it may not interest someone looking for a fun game it serves to show that even games on infinite boards are part of this theory. An important aspect of this game is that the position, after every move, becomes better and better for the Devil, but still, sometimes it does not improve enough for the Devil to win, even after an arbitrarily large number of moves.

**Snorts** is yet another fun game to be played with pen and paper. Snorts is a picture coloring game. Two herdsman, Bob, that has a herd of bulls, and Chad, that has a herd of cows, compete to buy properties in a large open field. They are both interested in acquiring as many properties as possible, regardless of their sizes, but they respect each others space. Since cows cannot live next to bulls and vice-versa, Bob and Chad cannot buy any property next to each other.

The initial position for playing snorts any collection of shapes. The initial position is somewhat problematic to draw, because may allow the first player to have easier time playing. However, by making a large initial open space, it is hard to make a boring game. A good method to draw the board is to make a circle-like shape and start partitioning it randomly, making sure that there are no very large partitions. The picture below is an example of a snorts game in progress.

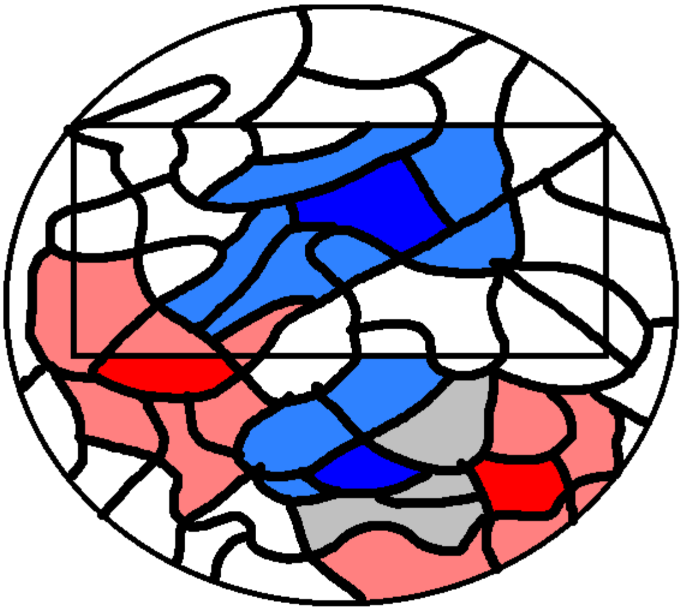


*Figure 5.9: Ongoing match of snorts*

**The Knights** is another possible variation of Amazons. The game has the same rules but uses chess knights instead of chess queens. This game may, for experienced chess players, seem less fun that The Amazons, as it is much more predictable. It may be true, but, like the other variations of Amazons it is worth trying.

# 6

# An amazing result and a dive into algebra

As the name suggests, this chapter has a different focus compared to the previous ones. This chapter is concerned with classifying a few games, both in terms of gameplay and in terms of algebraic structure. This is relevant because it highlights important characteristics in underlying structure and in possible values.

This sections features a few theorems that end up proving an upper bound for the boiling point of classes of games, called here *the result*.

## Some more thermograph characteristics

The temperature's upper bound results from a thorough analysis of the thermograph. The paper *Bounding Game Temperature Using Confusion Intervals* [**?** ] brings the first of such results, although the content featured first in Svenja Huntemann's PhD thesis [**?** ] in 2018, a year before. The proof of the boiling point requires a few smaller results and all the required content is summarized in the remaining of this subsection.

The confusion interval of a game is the range of numbers with which it is confused. A game is confused with a number if it is neither greater, smaller nor equal to that number. Any game with temperature greater than 0 is confused with numbers. Take any instance of such cases from previous sections to verify that their confusion interval includes all numbers between the points where the thermograph intersects the $t = 0$ line. Some intervals also include either one or both of the intersection points, depending on what the toenail of the thermograph looks like.

However, the reader can also verify that some non-numbers with temperature 0 are also confused with numbers. Section four shows that the thermograph of $\{0 \,|\, 0\}$ is different to that of 0, although above $t = 0$ they are the same. However, if using the surreal $\leq$ operation defined in *Chapter 3*, one notices that $\{0 \,|\, 0\} \nleq 0$, which shows 0 is confused with $\{0 \,|\, 0\}$. However, not all non-numbers behave this way, because some fall in gaps, not being confused with any number. Gaps is another delicate subject related to numbers that will not be discussed. $\{0 \,|\, \{0 \,|\, 0\}\}$ is a game inside a gap.

The confusion interval of a game $G$ is usually indicated by $\ell(G)$ and the points where the left and right trajectories, $\lambda_t(G), \rho_t(G)$, intersect the $t = 0$ line, are usually denoted by $LS(G)$ and $RS(G)$, and called left and right stops. Therefore $\ell(G) = LS(G) - RS(G)$. Another important definition is the thermic version of a game. A thermic version of a game $G$ is a game $\tilde{G}$ with the same temperature of $G$ but with only one left and one right option, that are also left and right options for $G$.

The existence of thermic versions follows: if $G$ is hot, then there exists left and right options of $G$, $G^{L_*}$ and $G^{R_*}$, such that $t(G) = t(\{G^{L_*} \mid G^{R_*}\})$. The proof is simple. The mast of $G$ begins where the cooled thermographs of some of the left and right options first intersect. There are possibly many left and right options that intersect on the same point. By taking any one of them, $G^{L_+}$ and $G^{R_+}$, it is clear that the thermic version of a game exists, because $t(G) = t(\{G^{L_+} \mid G^{R_+}\})$.

Finding in practice the thermic version of a game is not at all simple. In fact, without additional information of $G$, it would be needed to build the entire thermograph, or equivalent information, beforehand. This is an important consideration, that is also made on Huntemann's thesis, but one that is not problematic. Soon, the bound will be created for $G$ directly, so that finding the thermic version is not necessary.

It is trivial but it is worth highlighting that the thermograph of $G$ and that of $\tilde{G}$ may not be the same. In particular $\lambda_t(\tilde{G}) \geq \lambda_t(G)$ and $\rho_t(\tilde{G}) \leq \rho_t(G)$. If it is not clear why the observation is true: for any temperature $t$, the trajectories of the thermograph were either the maximum/minimum of all the corresponding options. The selected left and right options for the thermic version of $G$ are only the minimum/maximum for temperatures close to the boiling point. Therefore the initial part of the trajectory might be different, but is always lesser/greater or equal to the thermic trajectories.

Using the notation built so far it may be simpler. $\lambda_t(G) = LS(G_t)$ and, by definition, $\lambda_t(G) = max_{G^L}(RS(G^L_t) - t)$. Since $G^{L_+} \in G^L$, it is clear that $\lambda_t(\tilde{G}) \geq \lambda_t(G)$. An equivalent statement can be made for the right trajectory.

A result, when taking $t = 0$, is:

$$
\begin{aligned}
\ell(\tilde{G}) &\leq \ell(G) \\
\text{Since } \ell(\tilde{G}) = LS(\tilde{G}) - RS(\tilde{G}) &\leq LS(G) - RS(G) = \ell(G)
\end{aligned}
$$

The thermograph, as described in the previous section, is composed of straight lines and $\pm 45\,\deg$ oblique lines. The bound is based on this and the size of the confusion interval. Before proceeding, one additional definition.

Let $T^L$ be the sequence $(0, t_1, t_2, \ldots, t_k)$ of the temperatures of the turning points of $\lambda(\tilde{G})$. Let, now, $A^L$ be the sequence $(a_0, a_1, \ldots, a_k)$ of labels given by:

$$
\begin{cases}
a_i \text{ is } \textit{vertical} \text{ if } \rho(\tilde{G}^L_{t_{i+1}}) = \rho(\tilde{G}^L_{t_i}) \\
a_i \text{ is } \textit{oblique} \text{ if } \rho(\tilde{G}^L_{t_{i+1}}) < \rho(\tilde{G}^L_{t_i})
\end{cases}
$$

Lastly,

$$
T^L_{\mathrm{V}} = \sum_{\substack{i \mid a_i \text{ is} \\ \text{vertical}}} (t_{i+1} - t_i)
$$

$$
T^L_{\mathrm{O}} = \sum_{\substack{i \mid a_i \text{ is} \\ \text{oblique}}} (t_{i+1} - t_i)
$$

See *Figure 6.1* for a visual representation of the new definition. Notice that the right counterpart can be done similarly. Let $T^L_{\mathrm{V}}$ be the length of the vertical segments and let $T^L_{\mathrm{O}}$ be the length of the oblique segments. The temperature of the game $G$ can be given by a combination of the oblique and vertical parts of the thermograph like the following:

$$
t(G) = T^L_{\mathrm{V}} + T^L_{\mathrm{O}}
$$

```
> {10, {13|9}, {16|8}, {19|7}| 0}.Thermograph.Plot()
```

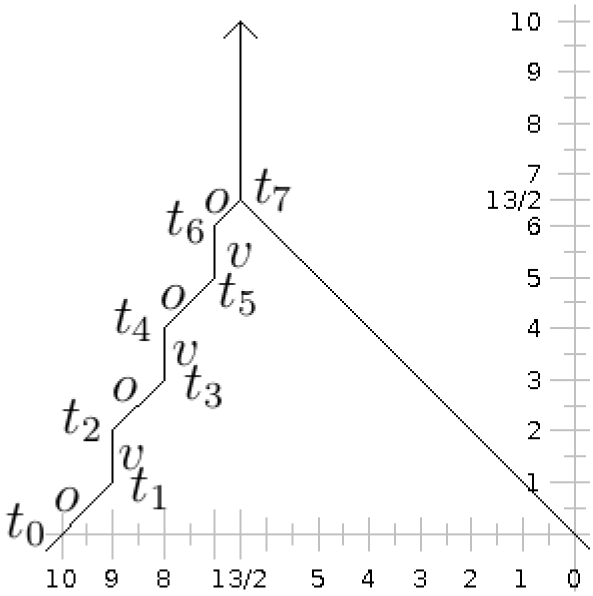



*Figure 6.1: Visual representation of how to calculate $T^L_V$ and $T^L_O$*

.

The length of the confusion interval of the thermic version of $G$ can also be given in terms of the oblique part of the thermograph:

$$\ell(\tilde{G}) = T^L_{\mathrm{O}} + T^R_{\mathrm{O}}.$$

## The theorem

*The theorem* discussed in the beginning of the chapter is the following:
For any game $G$ and $\tilde{G}$ its thermic version, it is true that

$$t(G) \leq \ell(H\,) + \frac{\ell(G)}{2}$$

$$\text{Where } \begin{cases} H = \tilde{G}^L & \text{if } T^L_V \geq T^R_V \\ H = \tilde{G}^R & \text{otherwise} \end{cases}$$

For simplicity, the proof considers that $H = \tilde{G}^L$, but again, an equivalent procedure proves the other case.

The length of $T^L_V$ is at most the length of the oblique segments of $\rho(\tilde{G}^L)$. Notice that it could be smaller because the left and right options might intersect before the mast of $\tilde{G}^L$. In any case, it is clear that the length of this oblique segment is at most the same size of the confusion interval of $\tilde{G}^L$. It could be smaller in case there are oblique segments in $\lambda(\tilde{G}^L)$.

Therefore, $T^L_V \leq \ell(\tilde{G}^L)$

Additionally, it is true that:

$$T_V^L + T_O^L = T_V^R + T_O^R$$

And, since in this case $T_V^L \geq T_V^R$:

$$T_O^L \leq T_O^R$$

Therefore:

$$\begin{aligned} 2 \times T_O^L &\leq T_O^L + T_O^R \\ &= \ell(\tilde{G}) \\ &\leq \ell(G) \\ T_O^L &\leq \frac{\ell(\tilde{G})}{2} \leq \frac{\ell(G)}{2} \end{aligned}$$

Resulting in:

$$t(G) = T_V^L + T_O^L \leq \ell(\tilde{G}^L) + \frac{\ell(G)}{2}$$

$T_V^L + T_O^L = T_V^R + T_O^R$ is straight forward, because what defines the height is the intersection point of left and right, which is the same for both trajectories. $2 \times T_O^L \leq T_O^L + T_O^R$ is a direct result of the previous equation with the given consideration that $T_V^L \geq T_V^R$.

Although longer and using more notation than other proofs in this text, it is yet another simple proof. Each of the steps are short and *Figure 6.1* helps with the visualization of the equations.

This theorem uses the thermic version of games, which, again, are hard to find in itself. However, from this version, another result that does not use the thermic versions is possible.

*The theorem* might be re-written as:

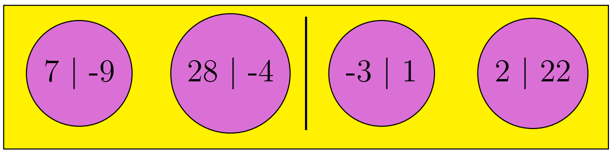
Let $n, m$ be two non-negative numbers. If $S$ is class of games such that for all $G \in S$, $\ell(G) \leq n$ and for left and right options $G^{L/R}$, $\ell(G^{L/R}) \leq m$, then:

$$BP(G) \leq \frac{n}{2} + m$$

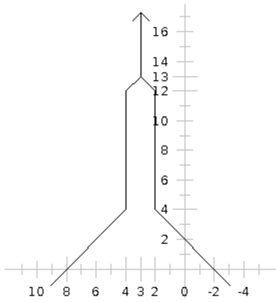
Although immediate, deciding whether the first version implies the version above might require an explanation. $t(G) \leq \ell(\tilde{G}^L) + \frac{\ell(G)}{2} \leq \frac{n}{2} + \ell(\tilde{G}^L)$ because for all $G \in S$, $\ell(G) \leq n$. Since $\ell(G^{L/R}) \leq m$, then $\ell(\tilde{G}^L) \leq m$, which implies the last inequality $t(G) \leq \frac{n}{2} + \ell(\tilde{G}^L) \leq \frac{n}{2} + m$.

Handling a class of games is not as straight-forward as using the definitions typically used in this text. A good way to use the theorem is to first create a class, based the restrictions - therefore having a bound for the confusion intervals of the games and its Left and Right options. Then one should prove a game or a pattern of games belongs to this class, to make use of this bound.

Let, for example, the class $S$ to be defined by $\{G : \ell(G) \leq 100, \ell(G^L) \leq 200, \ell(G^R) \leq 200\}$. Since $S$ meets the requirements it has the proposed boiling point. For all games $G \in S, BP(G) \leq 250$. The maximum length of the confusion intervals are so large that, in particular, all games of this text belong to $S$. The hottest game $G$ analyzed in this text is:




The thermograph of $G = \{\{7 \mid 9\}, \{28 \mid 4\} \mid \{-3 \mid -1\}, \{2 \mid -22\}\}$ is:




In this case, $\ell(G) = 10 \leq 100$ and $\ell(\{7 \mid 9\}) = \ell(\{-3 \mid -1\}) = 0 \leq 200$ and, finally, $\ell(\{28 \mid 4\}) = \ell(\{2 \mid -22\}) = 24 \leq 200$, so $G \in S$. The boiling point bound the theorem provided for $S$ is loose for $G$, since $\forall H \in S, BP(H) \leq 250$ and $BP(G) = 13$. However, the size of the confusion intervals that defined the class are also loose. In general cases, specially when confusion intervals start getting large, the bound will be loose, but that is because the bound itself is large, not because the theorem is inefficient.

Let $S_*$, now, be the class given by $\{G : \ell(G) \leq 10, \ell(G^L) \leq 24, \ell(G^R) \leq 24\}$. From the analysis of the previous paragraph, $G \in S_*$. However, in this case, the confusion intervals of $G$ are tight in respect to the ones that define the class. From *the theorem* for all games $G \in S_*,\ BP(G) \leq 29$. The bound is tighter for $G$ but is still loose, which may lead to the belief that this bound is loose in general.

In order to verify whether the bound is in fact loose, it would be required to show the following, or something similar:

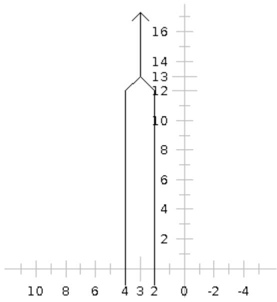
$$\forall n, m \in \mathbb{R}\text{[1]}, S = \{G : \ell(G) \leq n, \ell(G^L), \ell(G^R) \leq m\}$$
$$\exists x \in \mathbb{R},\ x > 0 \mid \max_{G \in S} BP(G) + x \leq \frac{n}{2} + m$$

To dismiss that idea, however, it could be shown:

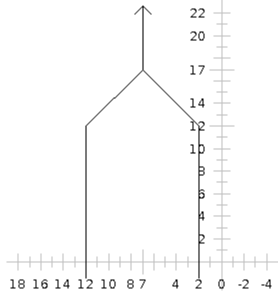
$$\exists G \in S_* \mid\ \nexists x \in \mathbb{R}, x > 0,\ BP(G) + x \leq 29$$

Starting out with the previous example, take $\tilde{G} = \{\{28 \mid 4\} \mid \{2 \mid -22\}\}$. The temperature is 13, the same as before, since, $t(G) = t(\tilde{G})$. However, the Left and Right slant may be moved, since $\ell(\tilde{G}) = 2$. Take $G_+ = \{\{36 \mid 12\} \mid \{2 \mid -22\}\}$, result of shifting the left slant to the left, which maintains the left and right confusion intervals, but increases the confusion interval for $\tilde{G}$ while maintaining that $G_+ \in S_*$.

[1] Convergence can be more complicated using surreal numbers. If instead $x \in \mathrm{SN}$ was used, an additional infinitesimal $y \in \mathrm{SN}$ would be required resulting in additional complexity in a topic that is not the focus of the text.




$\tilde{G}$

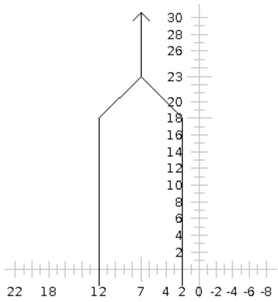



$G_+$

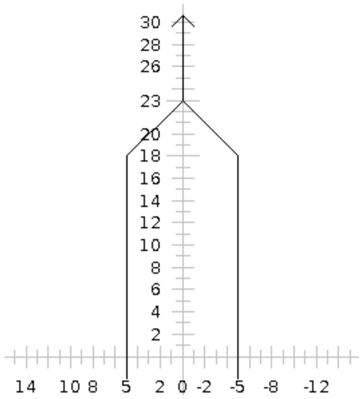
The temperature 17 is higher than the initial one but is still far away from the desired 29. However, it is still possible to increase it more. The boiling point of left and right slants are 12, and, for this reason, the oblique line in the thermograph start at $t = 12$. Making the left and right games hotter would increase the overall temperature, but increasing the confusion interval is not allowed as it is desired that the resulting game belongs to the class $S_*$.

Since the thermographs of $G_+^{L/R}$ below the boiling point are made of two oblique lines, making part of the lines straight would increase their temperature. The way to achieve higher temperature without increasing the confusion interval is changing $G_+^L$ so that $RS(G_+^L)$ remains in the same position but $T_O^R$ of $G_+^L$ is increased. Change, then $G_+^L$ to $\{\{60 \mid 36\} \mid 12\}$ and $G^R$ to $\{2 \mid \{-22 \mid -46\}\}$. The reason why $\{36 \mid 12\}$ was replaced by $\{60 \mid \{36 \mid 12\}\}$ as $G^L$ is to keep $\ell(H) = 24$, for $H$ any hot game in the left and right sub-trees of the game tree of $G_+$. This pattern will be kept and helps simplifying results.

Let $G_- = \{\{\{60 \mid 36\} \mid 12\} \mid \{2 \mid \{-22 \mid -46\}\}\}$ be the game resulting from this changes. The temperature of $G_-$ is 23. To simplify further analysis, take a game $G_1$, result of centering $G_-$ at 0. In this case, $G_1 = \{\{\{53 \mid 29\} \mid 5\} \mid \{-5 \mid \{-29 \mid -53\}\}\}$. Notice that $G_1 = \pm\{\{53 \mid 29\} \mid 5\}$.




$G_-$




$G_1$

Using the strategy of making one of the slants of $G^{L/R}$ straighter in order to increase their temperature, it is possible to keep increasing the temperature of $G$. Consider the following sequence of games is $S_*$:

| $i$ | $G_i$ | $t(G_i)$ |
|---|---|---|
| 0 | $\pm\{29 \mid 5\}$ | 17 |
| 1 | $\pm\{\{53 \mid 29\} \mid 5\}$ | 23 |
| 2 | $\pm\{\{\{77 \mid 53\} \mid 29\} \mid 5\}$ | 26 |
| 3 | $\pm\{\{\{\{101 \mid 77\} \mid 53\} \mid 29\} \mid 5\}$ | 27.5 |

$\dots$

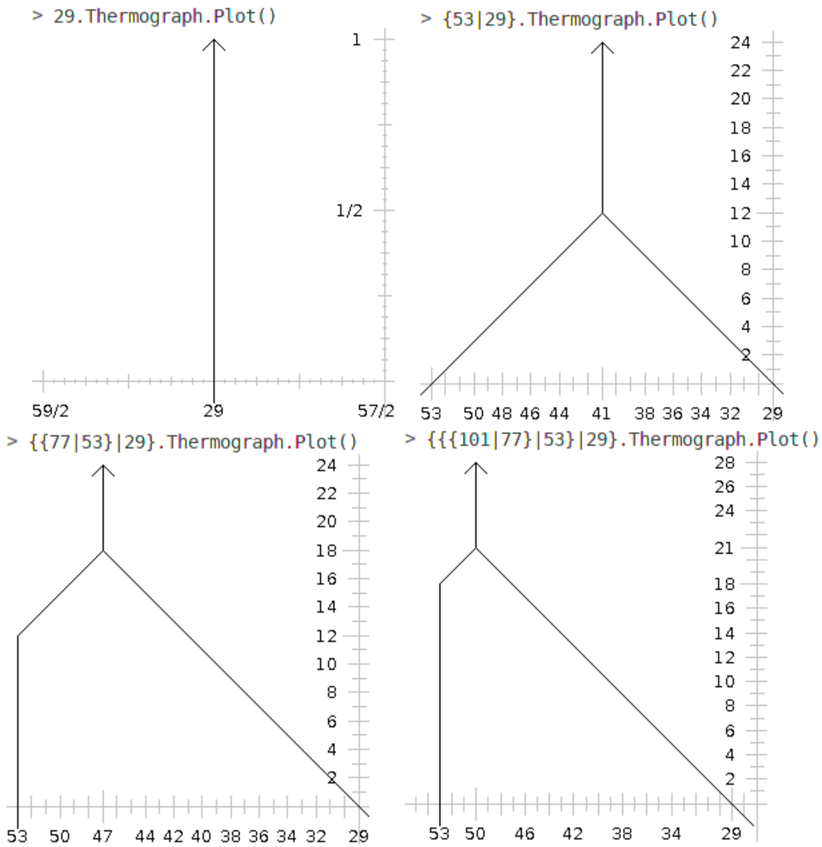



*Figure 6.2: Thermographs of the left option of each $G_i$ in the table*

The result that the temperature of games in this sequence increases and converges to 29 is required to prove the bound is tight, so it must be proved. Instead of only proving the convergence in this particular case, which is is sufficient, the following proof shows that the boiling point, given by Svenja, for all classes are tight. The following proof is general for all sequence of games made in the likes of the sequence above.

*Proof:*

Let $OB$ be the bound $\ell(G^{L/R})$ for the options of $G$ in the class $S$.

Let $B$ be the bound for $\ell(G)$ in the class $S$.

$\rightarrow$ Both $T_V^L$ and $T_V^R$ of $G$ are given by

$$\sum_{i=0}^{n} \frac{OB}{2^{i+1}}$$

The following proof only shows the case for the left option as the proof for the right is analogous.

Refer to *Figure 6.2* for a visualization of this statement.

[Via induction on $i$]

Base case: $n = 0$.

In this case, $G^L_i = G^L_0 = \{\frac{B}{2} + OB \mid \frac{B}{2}\}$. Since $G^L_0$ is a switch, both the trajectories are oblique. The trajectories, then, meet at $BP(G^L_0) = \frac{OB}{2}$. This way, the right trajectory of the thermograph of $G^L_0$ is oblique until $\frac{OB}{2}$ and vertical after that. Therefore the left trajectory of $G_0$ is vertical until $\frac{OB}{2}$ and oblique after that point.

As a result,

$$T^L_V \text{ of } G = \frac{OB}{2} = \sum_{i=0}^{0} \frac{OB}{2^{i+1}}$$

Considering now that $k > 0$.

$$G_k = \{\{\{\{\ldots\{\tfrac{B}{2} + (k+1)OB \mid \tfrac{B}{2} + kOB\} \mid \ldots\} \mid \tfrac{B}{2} + 2OB\} \mid \tfrac{B}{2} + OB\} \mid \tfrac{B}{2}\}$$

Let $b = \frac{B}{2} + OB$. It is possible to rewrite $G$:

$$G_k = \{\{\{\{\ldots\{b + kOB \mid b + (k-1)OB\} \mid \ldots\} \mid b + OB\} \mid b\} \mid \tfrac{B}{2}\}$$

Via inductive hypothesis on:

$$H = \{\{\{\ldots\{b + kOB \mid b + (k-1)OB\} \mid \ldots\} \mid b + OB\} \mid b\}$$

The total vertical length of the left trajectory of $H_* = \pm H$ is equal to $\sum_{i=0}^{k-1} \frac{OB}{2^{i+1}}$. Added to the fact that the right option of $H$ is a number, the total oblique length of the right trajectory of $H^L$ is equal to $\sum_{i=0}^{k-1} \frac{OB}{2^{i+1}}$ and it is oblique until the boiling point.

Because of this, the draft of the left trajectory of $G^L_k$ is vertical until the boiling point of $H$. Since the right option of $G^L_k$ is a number, its right trajectory is oblique until the boiling point. Therefore, $BP(G^L_k) = \sum_{i=0}^{k} \frac{OB}{2^{i+1}}$.

Once again because the right slant of $G^L_k$ is oblique until $\sum_{i=0}^{k} \frac{OB}{2^{i+1}}$, it is true that:

$$T^L_V \text{ of } G = \sum_{i=0}^{k} \frac{OB}{2^{i+1}}$$

$\rightarrow$ If both the left option of $G^L$ and the right option of $G^R$ have their straight length increased by $l$, then so does $G$.

As seen before, $t(G) = T^L_V + T^L_O$. Since increasing both trajectories' straight segments does not affect the size point they intersect, it is possible that the total length of the oblique segments remains the same. Since, again, the size of the oblique segment does not change and the size of the vertical segments ins increased by $l$, then the temperature is increased by $l$.

→ The temperature of the games in this sequence converges to the proposed boiling point, while always being elements of the initial class $S$.

Notice that $RS(G^L)$ and $LS(G^R)$ remain the same across all iterations, and that so does $RS(G^L) - LS(G^R) = \ell(G)$. The very same fact is also true for $G^L, G^R$. This serves to show that every game in the sequence belongs to $S$.

The temperature of the game $G_k$, whose thermograph is symmetric, is given by:

$$t(G_k) = T_V^L + T_O^L = \sum_{i=0}^{k} \frac{OB}{2^{i+1}} + \frac{B}{2} = OB \sum_{i=1}^{k} (\frac{1}{2})^i + \frac{B}{2}$$

$\sum_{i=1}^{k} (\frac{1}{2})^i$ is a well-known convergent series and it converges to 1. It means that for any positive real number $x$, there exists an index $k$ of the sequence such that:

$$t(G_k) = OB \sum_{i=1}^{k} (\frac{1}{2})^i + \frac{B}{2} > OB + \frac{B}{2} - x$$

The sequence of games, then, shows that the given bound is not loose. In fact, it shows that the bound is optimal for some cases. It is worth mentioning that, although this section follows the aforementioned paper, the proof above is original. The paper does convey the strategy and the regard about the optimality of the bound but the proof of convergence is skipped. Although long and with many symbols, the proof is quite simple and might have been considered trivial.

An interesting way to complete and put to practice the analysis of the theorem is to show it can be used to characterize a pattern of domineering boards. To use the bound all games following the pattern must belong to a class, based on the confusion interval restrictions, like before.

When bounding '$2 \times n$ snakes' boards, the thesis and the featured papers diverge. Probably in the year between the publication of the thesis and the paper the technique was refined, resulting in the thesis bringing a bound of 5 to the temperature and the paper reducing it to 3. The proof brought to this text will show the point of divergence, but concerns with the bound of 3. Before showing the proof, another necessary definition and lemma are provided.

A 'snake' board, same adjective used in "Kim's snakes" from the previous chapter, is a board without any $2 \times 2$ subgrid. A '$2 \times n$' snake is a snake that fits in a $2 \times n$ board. It is possible to use the term 'snake fitting in a $2 \times n$ board', and this means that the board is a snake that can be folded such that it becomes a '$2 \times n$' snake with the same value.

The aforementioned lemma follows. Let $\epsilon$ be any infinitesimal and $G$ be a hot game. If for all left options $G^L$, $n$ is a number such that $G^L - G - n + \epsilon \leq 0$, then $\ell(G) \leq n$. The proof for this lemma is simple:

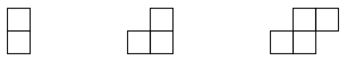
$$\begin{aligned} \ell(G) &= LS(G) - RS(G) \\ &= RS(G^L) + LS(-G) \\ &\leq LS(G^L - G) \\ &\leq LS(n) \\ &= n \end{aligned}$$

It follows by definition $RS(G) = -LS(-G)$ and it is clear that there exists a left option $G^L$ such that $LS(G) = RS(G^L)$ , but the second equation is there to help visualize the first inequation. The last inequation is true because $G^L - G + \epsilon \leq n \longrightarrow LS(G^L - G) \leq n$, explained amid the discussion in the beginning of this section. The first inequation is true because $G^L - G = \{G^{LL} - G^L \mid G^{LR} - G^R\}$, meaning that $LS(G^L - G)$ derives from a game $H = G^L_+ - G_-$. In particular, it is possible to take $G^L_+$ to be a game that defines the right stop for $G^L$ and $-G_-$ the game that defines the left stop for $-G$, and, therefore, the inequality holds.

The meaning of this lemma is that if it is possible to bound the value of the game to the left, or to positive side, then it is possible to bound the confusion interval. This is a weaker result compared to the previous one but it is one that connects the left and right stops with the confusion interval so it is extremely useful.

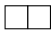
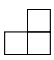
Now it is possible to characterize some domineering boards. *The temperature of snakes fitting a* $2 \times n$ *board is at most* 3. The proof consists of showing that $\ell(G), \ell(G^L), \ell(G^R) \leq 2$, by the means of showing that for $H = G \vee G^L \vee G^R \rightarrow H^L - H - 2 \leq 0$ and using the previous lemma.

Considering the case $H = G^R$, $H$ is a two component board, so it is possible to separate them in distinct games. Let $-H = -G^R = -G_1 - G_2$. $H^L$ on the other hand is $G^R$ with the addition of a left move. The only cases a left move does not exist in $G^R$ is if $G$ is one of the following:



In all the cases above, the temperature is below 3, so it is not a problem. Without losing generality, let $G_1$ be a component where there is a right move, and let the initial left move be any such move in $G_1$, therefore, $H^L = G^L_1 + G_2$. Now it is clear that this case can be reduced to the case $H = G_*$, with $G_*$ a smaller board than $G$:

$$H^L - H = G^L_1 + G_2 - G_1 - G_2 = G^L_1 - G_1$$

Repeating the process with $H = G^L$, $-H = -G^L = -G_1 - G_2$ and $H^L = G^L_1 + G_2$. The only cases $G^L_1$ does not exist is if $G =$  or  and both of them have temperature lesser than 3. Again it is clear that this case can also be reduced to the case $H = G_*$:

$$H^L - H = G^L_1 + G_2 - G_1 - G_2 = G^L_1 - G_1$$

Therefore, if $\ell(G) \leq 3$ then the confusion interval of both option are also smaller than 3. To prove that $G^L - G - 2 \leq 0$ it is shown a winning strategy for Right in $G^L - G - 2$, which shows that the position is negative. First notice that $-G$ is $G$ flipped so that it fits in a $n \times 2$ board; but a useful way, that is used in this proof, is to consider that $-G$ is Left playing on the horizontal and Right playing on the vertical.

To help visualize the strategy for this case, let the 'shadow' of a move be the same cells that the move occupied, but in the opposing board. Also the board resultant from play in $G^L$ is called the original board and the board resultant from play in $-G$ is called the opposing board. For instance:

In the game $G^L - G - 2$, if Right plays first, he/she simply has to mimic each and every move Left made by playing the shadow move, maintaining $-2$ intact, starting with the move already played in $G^L$. If, instead Left goes first, he/she can play one of two


*Orginal move* — *Original board*

*Shadow move* — *Opposing board*

*Figure 6.3: A move and its shadow on the negated board*

options. The first is to play a move that does not intersect the shadow of the previous move. In this case, right must play the shadow of either the first or the second move.

Essentially, this first case maintains the original conditions while reducing the board size. This case leads $G^{L_*} - G - 2$ to $G^{L_*L} - G^{L_*} - 2$, which, by taking $H = G^{L_*}$, is $H^L - H - 2$. Therefore the only worrisome Left move is one that intersects the shadow of the initial move. This move is not exactly the shadow move because in the opposing board Left moves in another orientation. After Left's move, Right must move on the $-2$ component, taking it to $-1$.

Now, Left can, again, play a move that does not intersect any of the shadows, but this would be met by the same strategy as before, so Left must play a move that intersects a shadow. However, Left cannot move intersecting the shadow of the previous move, because the initial move would be adjacent to this third move and since the mover are both vertical, a $2 \times 2$ subgrid would have to exist. The only shadow Left can intersect is that of the first move, playing on the opposing component again. After Left's third move, right plays on the $-1$ component.

At this time, Left can no longer play on any shadows, so he/she cannot avoid Right's strategy and will eventually lose. Therefore, if $G$ is a snake fitting in a $2 \times n$ grid, then $G^L - G - 2 < 0$. This means that $\ell(G), \ell(G^L), \ell(G^R) \leq 2$. By using the bound result, in turn, it means that $t(G) \leq 3$.

The result presented, from the 2019 paper [**?** ], is different, as stated before, from the 2018 thesis because the first part of the proof, that shows the cases $H = G^L, G^R$ can be reduced to the case $H = G$, was missing. Instead the original text only showed that the confusion intervals of $G^L, G^R$ were smaller than 4. This second result is clear because both $G^L$ and $G^R$ are made of two components that are snakes fitting in $2 \times n$ grids. As the text shows that such a snake has the confusion interval smaller or equal to 2, then two snakes have that interval smaller or equal to 4.

Although the result for the game of Domineering was improved it is still insufficient to prove Berlekamp's conjecture, that the maximum temperature is 2, even in a specific board pattern. Regardless of solving or not the original question, this theorem is an important progress as it only takes into account the thermic nature of games, visualized through the thermograph. It states that by the own nature of games there is a bound to the temperature, which is a very intuitive result, but one that took long to be formalized.

Although reaching important results from a very recent past, there is more to the topic. However, the study and work done in this chapter marks the end on the discussion of game temperature in this text. A good place to follow-up and continue learning is the end of chapter six of volume one of the book *Winning Ways*. Other options are going through Siegel's book and studying his implementation of `CGSuite`. Siegel's book is much shorter and direct, making it harder, but may be the best option if looking for more formalism.

# 7

# Unexplored games and final remarks

Combinatorial game theory is a new field in mathematics that was initially found in the blend of recreational mathematics, combinatorics and number theory. While the initial focus is finding strategies to play better and win games, the developments it prompted in other areas of mathematics have their own place, independent of their purpose when first created/discovered. The Surreal Numbers form a class with extremely interesting properties and found much enthusiasm for that reason, not because it was necessary to analyzing combinatorial games.

However, the surreal numbers inevitably carry the simplicity inherited from the mathematical plays it was created/discovered for. It is commonplace to learn how to breach the gap between rational and real numbers only in superior education, partially because any of the common constructions involves operations and procedures not used in school. Surreal Numbers give for free the construction of real numbers from its few and simple rules. Not only the reals, but a definition of positivity and negativity that dismisses comparison, a common method for the creation of all the numbers, new numbers and, at last, a direct way of visualizing each number with RB-Hackenbush games. Numbers are the main building block the text used to analyze games.

The non-numbers, a name used in this text for games with temperatures, are the remaining games and required more elements and concepts to be understood. The thermograph, or the form commonly used to visualize temperature, options, stops and boiling points, is extensively analyzed and the text presents one of the most recent related results.

The reader that understands all the featured topics and is able to replicate calculations and results to games other than the mentioned might carry the false belief that that he/she understands all of combinatorial game theory. This text, however, only describes the class of partisan games leaving many variations unmentioned. Not only the focus is specific, but also not all perspectives necessary to tackle the problem of finding the best move are visited.

The objective of this information is to show that there is much more to the field, not that little progress was made. In fact, it is correct to state that the classes of games discussed in the text are contain most of the ideas and many of the definitions necessary to studying other classes.

Both in the results discussed and the ones omitted there is a common trend in this field. The definitions are always simple, but powerful enough to allow questions to be solved and answered with simple proofs, although there are many open questions. The procedure to build all dyadic rationals in domineering boards, for example, is based on a theorem provable in two lines of text and applying it is the simplest of ways. Although its explanation is simple, it remained open for years and the resulting pattern is not something that would be found before the proof.

Finding the upper bound to temperature is another case where the question remained open for long, but the strategy used to answer is simplicity: it consists only of listing and manipulating properties of the thermographs.That is not to say that the bounding of temperature is completely solved but the progress made used simple steps.

From this perspective combinatorial game theory is a beautiful field in mathematics. The definitions it is based upon are extremely few in number and very simple in form, but they allow proving more complex problems in a uncomplicated manner. However, in this text, one of the unvisited places in the field is computational complexity. In practice, calculating values is not an easy problem.

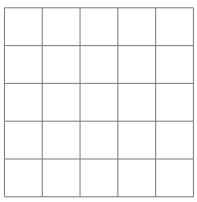
The other featured aspect skipped is the variety of forms combinatorial games might take, because the text only deals with partisan games played on normal conditions. The remaining contains a brief description of these two skipped problems and points toward texts on th respective subjects. This chapter, that finalizes the text, also addresses a few of the next steps studying or researching the field and ends with a personal reflection on the learning, implementing and writing experiences involved in this work.

## Undeveloped Pieces

### Complexity

The text repeatedly made the case that most concepts are simple in Combinatorial Game Theory. It said, for example, that the method used to find the number a game is equal to is straight-forward. However this hides an important factor.

Either calculating the number or building a thermograph requires, first, traversing the game tree. Traversing this tree and calculating the values is simple enough, but it is slow. And it is slow because this tree quickly becomes large. For example, consider the following game of Domineering.



There are, of course, 25 squares, which means that at most 12 moves might be made during a match and in addition, some of these moves might be equivalent, so they do not have to be computed multiple times. It happens that only 604 nodes, or different position, may be reached[**?** ]. In a $6 \times 6$ board it jumps to $17,232$ and in $7 \times 7$ and $8 \times 8$ board it goes to $408,260$ and $441,990,070$ nodes. Although these results are from 1998, results on larger boards took many years and only in 2016 a result for $11 \times 11$[**?** ] board was found, but abandoning a naive building of a game tree. The game tree of a $9 \times 9$ board has around 25 trillion nodes.

If not clear at first it should be now that there is no known form of calculating values for games in polynomial time is respect to their sizes. The question, then, is in what class of problems it fits. It happens that many of the games are known to be PSPACE-complete and some are known to be EXPTIME.

There is of course much more to be said an it is another very interesting topic. A good starting point is *Playing Games with Algorithms: Algorithmic Combinatorial Game Theory* [**?** ]. The thesis *Games, Puzzles, and Computation* [**?** ] is more extensive, covering different classes of games, but may be more complex. Regardless of the suggested texts, there is a large amount of great books on complexity theory.

### Other games

By looking at the definition of combinatorial games given in the beginning of section 2, it is easy to come up with games that do not fit well the class of games studied, the short games. One clear example is the game of chess, because there are ties and there are drawn positions.

However, games like chess are not the only class of games that behave differently. Consider G-Hackenbush, a variation of RB-Hackenbush, in which both blue and red edges are replaced by green ones. Green edges may be removed by either player, meaning that both players have the same available moves regardless of the position. The class of games that follow this property is called Impartial Games.

Impartial games were the first games studied, previously to the consolidation of the modern form of the field, by Conway. They have an interesting property of being reducible to an instance of a game of Nim. Typically they are simpler to study but that is not always true. Partisan games, in which players have different sets of available moves may contain components that are impartial, so most of the theory in this text applies to impartial games.

Games like chess, on the other hand, are not so simple. Since there are more possible outcomes than short games analyzing them requires additional elements. Such games benefit from knowing short game's theory but do require further concepts. A good place to start is the corresponding chapter in the book *Combinatorial Game Theory*.

## Post-Match

The choice of Combinatorial Game Theory as subject was an extremely fortunate decision. This field triggered the feeling of uncovering something beautiful hidden in plain sight. I believe the book *Winning Ways* played a role in that due to the presentation I became fond of, but I do consider there is inherent beauty in this field of study.

I believe the simplicity of everything about it is something to awe. This is the most important reason why this thesis ended up being about it and not some other subject. The first step in this direction was a great interest in the games of Chess, Shogi and Go. The first idea I had was to develop heuristics to find shortest paths in the game of chess, and possibly extend that to other games.

During that period I started reading articles and scrolling through books and that quickly led me into the field of Combinatorial Game Theory. The book *Winning Ways* is one of the canonical references and I luckily started with that. This choice led me to a year of great learning experiences, in many areas, and personal maturing.

Learning and writing about the topics of this text was the most important activity to conclude my Bachelor's Thesis, but there are other subjects I studied in the process that went unmentioned in the text. I am grateful for those too, as I got to develop my understanding of real analysis and complexity theory further than I would normally, for example. This year we faced a pandemic due to the corona virus widespread. I believe that enjoying the topic as much definitely helped me to focus, be proactive and productive in times of isolation and periods it was easier to get distracted.

I had only handful of on-campus classes and that required me, and most of the world, to adapt to new circumstances. While it did not affect the content of this text directly, it greatly impacted my mental state. The consistent work it took really helped keeping me on track. Another factor that cannot go unmentioned is the help of my advisor.

José Coelho kept regular meetings throughout the year with me. That helped a lot with keeping me in check and I am extremely thankful for the dozens of hours we spent talking and discussing the most varied subjects. In the times I did not have work to show he was understanding and his willingness to help pushed me further and further.

I will definitely continue my studies and work in this area. The field puts together elements of mathematics and computer science in a extremely visual way. I does not only allow one to study while having fun but also gives the sensation of discovery when you apply something or study something new.

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# Appendix A

# CGSuite

`CGSuite`, which stands for Combinatorial Game Suite, is a software suite used to study and research combinatorial games. The tool implements all the common methods in the field of study, which includes, but is not limited to, all the ones described in this text. `CGSuite` is widely used by people interested in Combinatorial Game Theory.

The package also includes a graphical interface and an interpreter, for its own scripting language, together with the game analysis functionality. `CGSuite` is an open source software and the development is made on github and releases are published on sourceforge. The github repository is found on https://github.com/aaron-siegel/cgsuite and the sourceforge distribution on http://cgsuite.sourceforge.net/. The core functionality is implemented in Java and recent updates are also developed in scala. The maintainer of the project is Aaron Nathan Siegel, the autor of *Combinatorial Game Theory* [**?** ].

The typical use of the software is through the GUI and consists of manipulating small games, in the sense that results can be calculated in reasonable time, drawing them and their thermographs. `CGSuite` also allows users to extend the provided games and implement new games while still making use of the tools available. Extending the package is usually done using the same scripting language.

The software is powerful enough to handle large cases sometimes, but in the cases such as evaluating a truly large instance or several large instances a personalized solution may be needed. Unfortunately there are no other complete implementations yet. There is ongoing development of another Surreal Numbers library, written in julia, but nothing that would replace a personalized solution.