and the fundamental equivalence gives $o(H+X) = o(G+X)$. In Chapter II we'll develop several tools for establishing identities like the one in Figure 1.6.

Other Kinds of Values

The definition of equality

$$G = H \quad \text{if} \quad o(G+X) = o(H+X) \text{ for all } X$$

is sensitive to three primitive parameters:

- the definition of the *outcome* $o(G)$;
- the definition of the *sum* $G + H$; and
- the domain of the *for all* quantifier over X.

In arriving at $\mathbb{G}$ we took $o(G)$ to be the *normal-play outcome* of G; $G + H$ to be the *disjunctive sum* of G and H; and *for all* to range over the set of short partizan games. But there are many other natural choices for all three primitives. For example, $o(G)$ might just as well be defined as the *misère-play outcome* of G, and we'll discuss variant meanings of "sum" and "for all" in Section 4.

Astonishingly, in virtually all cases—no matter how we choose to fix meanings for these three parameters—the fundamental equivalence remains the same:

$$G = H \quad \text{if} \quad o(G+X) = o(H+X) \text{ for all } X.$$

Each time we vary the meanings of "outcome," "sum," and "for all," the fundamental equivalence yields a different theory—but it is always a coherent theory. The fundamental equivalence is the glue that holds combinatorial game theory together, and we'll turn to it for guidance again and again throughout this book.

Exercises

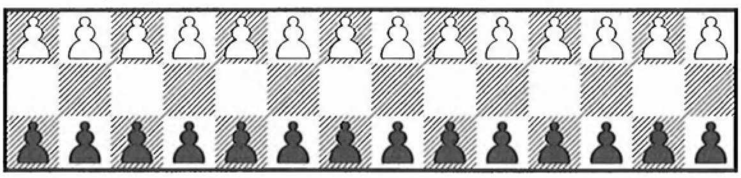
1.1 Determine all winning moves from the NIM position with heaps 18, 22, and 29.

1.2 Let G be a NIM position. Let n be the number of *distinct* winning moves on G. Prove that either $n = 0$ or n is odd.

1.3 Determine the strategy for misère NIM.

1.4 Let G be an impartial game. Suppose that there exists an option H of G such that every option J of H is also an option of G. Prove that G must be an $\mathscr{N}$-position.

1.5 DAWSON'S CHESS is played with two rows of pawns on a $3 \times n$ board:



The pawns move and capture like ordinary CHESS pawns, except that capture is *mandatory.* (If a choice of captures is available, then the player may select either one.) Show that DAWSON'S CHESS on a $3 \times n$ board is isomorphic to DAWSON'S KAYLES with $n + 1$ boxes, regardless of which play convention (normal or misère) is observed. (This is the original form of the game proposed by Dawson [**Daw73**].)

1.6 No position in BLUE–RED HACKENBUSH is an $\mathscr{N}$-position.

1.7 Determine the outcomes of 4×4, 5×4, and 5×5 DOMINEERING.

1.8 Let S_L and S_R be sets of positive integers. The **partizan subtraction game** on S_L and S_R is played with a single heap of n tokens. On her move, Left must remove k tokens for some $k \in S_L$; likewise, on his move Right removes k tokens for some $k \in S_R$. Denote by $o(H_n)$ the (normal-play) outcome of a heap of n tokens. Prove that the sequence $n \mapsto o(H_n)$ is periodic, i.e., there is some $p > 0$ such that $o(H_{n+p}) = o(H_n)$ for all sufficiently large n.

2. HACKENBUSH: A Detailed Example

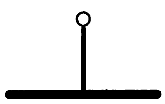
Many of the ideas and principles of combinatorial game theory are neatly illustrated by various situations that arise in HACKENBUSH. The simplest position is the game with no options at all,

which is called 0, the **empty game**. We'll write

$$0 = \{ \mid \}$$

to indicate that there are no options for either player.

Introducing a single blue edge gives the game

 $= \{$ $\mid \} = \{0 \mid \}$

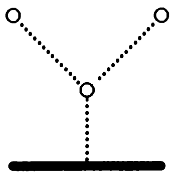
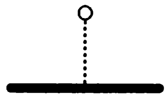
which is an $\mathscr{L}$-position, since Left can win immediately by moving to 0, while Right has no move at all. We'll call it 1, as it behaves like one spare move for Left.

What can one say about positions like the one in Figure 2.1, in which every disjoint component is monochromatically red or blue?

Clearly Left will always play to remove just one blue edge; there is nothing to be gained by deleting more than one at a time. Likewise Right will always play to delete a single red segment. Each player is powerless to disrupt the other's plans, so the win will ultimately go to whoever started with more edges.

We can therefore replace every blue component with a positive integer number of edges, and every red component with a negative integer, and total

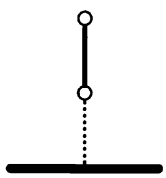
In BLUE–RED HACKENBUSH we could compute the value of *monochromatic* positions just by counting up the number of edges. This doesn't work in GREEN HACKENBUSH, though, and it's worth taking a moment to confirm that

 $+$  $= 0$

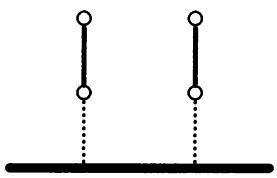
so that the value of the first component is $*1$, even though it has three edges. Fortunately there's a simple rule you can use to work out the values of arbitrary GREEN HACKENBUSH trees (see Exercise 2.1).

Tricolor HACKENBUSH

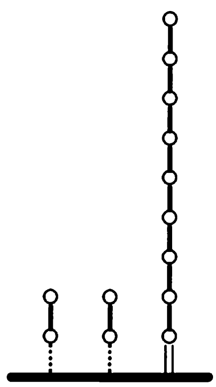
Let's consider one more example, the position

$G =$  .

G is a first-player win, but it's clearly more favorable to Left than $*$, and in fact two copies of G make an $\mathscr{L}$-position. For on

$G + G =$ 

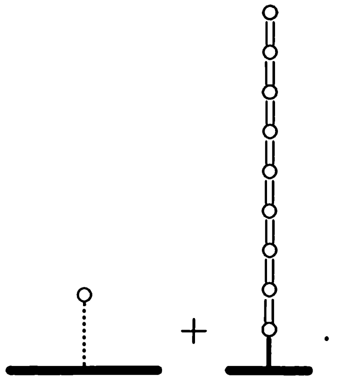
the *first* player to chop a green edge will lose, and Left can ensure this will always be Right. But $G + G$ is still infinitesimal: Right can win the sum

$G + G - 2^{-n} =$ 

using the familiar strategy of chopping green edges preferentially.

Exercises

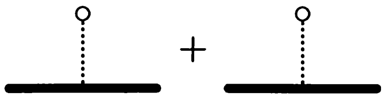
2.1 A HACKENBUSH position G is a **tree** if there is a unique path from the ground to each node. Let G be a tree that is completely green. Suppose that v is a vertex of G and the subtree above v consists of simple paths extending from v (i.e., every vertex above v has valence ≤ 2). Let $a_1, \ldots, a_k$ denote the lengths of these paths. Show that the subtree above v can be replaced by a single path either player may move to 0. Consider the sum $G + 2^{-n}$:



Left can win this game by following a simple strategy: chop the green edge preferentially or, if it's already gone, move to 0. In fact G is dominated by even the smallest numbers, and we might say that

$$\frac{1}{2^n} > G > -\frac{1}{2^n}$$

for all n. But $G \neq 0$, since it's a first-player win! Moreover,

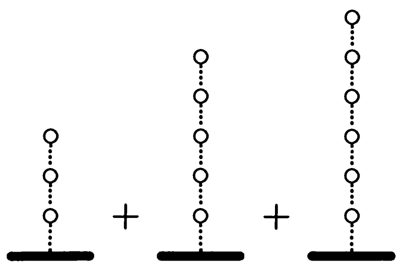


is a second-player win, so that $G + G = 0$, or equivalently $G = -G$. The value of G isn't a number at all, and in fact it's **infinitesimal** with respect to numbers like 2^{-n}. It has the special name

$$*$$

(pronounced "star"), and in Chapter II we'll see that it plays a central role in the theory of combinatorial games.

Now $*$ is obviously isomorphic to a nim-heap of size 1, and if we play Green Hackenbush with long stalks, it's clear that we're just playing Nim. For example, Bouton's Theorem tells us that



is a second-player win. We previously assigned these components **nim values** of 3, 5, and 6, respectively; but in the more general context of **game values** we need to call them something else, to distinguish them from ordinary integers. So we'll use the notation $*m$ (pronounced "star m") to denote a nim-heap of size m, and we'll write (for example)

$$*3 + *5 + *6 = 0.$$

It's clear that *any* nim-heap is dwarfed by 2^{-n}.

evaluates a position G as follows. For each potential option G^L, the computer plays out several thousand random games all the way to the end and assigns a score to G^L based on the *probability* of a favorable outcome. The computer then selects the option that achieved the most probable win. This basic approach can be enhanced in various ways, for example, by starting random games from a deeper level of the search tree, or by introducing a heuristic bias into the random plays.

Monte Carlo algorithms are far superior to classical search algorithms at the strategic level. However they are vulnerable to tactical weakness: if the opponent has just one good reply to a particular G^L, for example, there is some risk that the randomizer will simply overlook it. Since 2006, a hybrid approach known as UCT search (Upper Confidence bounds applied to Trees) has found spectacular success and cemented the dominance of Monte Carlo methods in computer GO. In the 2011 Computer Olympiad, the top three GO programs all used a version of Monte Carlo tree search; such programs are now able to beat professional players at reasonable handicaps.

Exercises

4.1 The outcomes for NIMANIA given on page 39 are correct.

4.2 In DIVISORS a position is a finite set $\mathcal{N} \subset \mathbb{N}^+$ that is divisor-closed (if a is an element of $\mathcal{N}$, then so are all divisors of a). A move consists of selecting an integer $a \in \mathcal{N}$ and removing a from the set, together with all multiples of a. Whoever removes 1, necessarily leaving the set empty, loses.

Prove that CHOMP is isomorphic to DIVISORS played on integers of the form $2^a 3^b$.

4.3 (a) All of the operations in Figure 4.9 are associative.
(b) Which ones are commutative? For which of them is 0 an identity?

4.4 *Selective sums.* If G and H are short impartial games, then $o(G \vee H) = \mathscr{P}$ if and only if $o(G) = \mathscr{P}$ and $o(H) = \mathscr{P}$ (normal play). Conclude that there are just two normal-play short impartial values for selective sums. How about misère play?

4.5 *Conjunctive sums.* The **remoteness** $\mathscr{R}(G)$ of a short impartial game G is defined by

$$\mathscr{R}(G) = \begin{cases} 0 & \text{if } G \cong 0\text{; otherwise:} \\ 1 + \min\{\mathscr{R}(G') : \mathscr{R}(G') \text{ is even}\} & \text{if some } \mathscr{R}(G') \text{ is even;} \\ 1 + \max\{\mathscr{R}(G') : \mathscr{R}(G') \text{ is odd }\} & \text{if every } \mathscr{R}(G') \text{ is odd.} \end{cases}$$

(a) $o(G) = \mathscr{P}$ if and only if $\mathscr{R}(G)$ is even.
(b) $\mathscr{R}(G \wedge H) = \min\{\mathscr{R}(G), \mathscr{R}(H)\}$ for all G and H.
(c) Define the **misère remoteness** $\mathscr{R}^-(G)$ of G by interchanging "odd" and "even" in the definition for $\mathscr{R}(G)$, and show that it works for determining the misère outcome of conjunctive sums.

4.6 Research the references for conjunctive and selective sums in the notes section (below), and draw the analogous tables to Figure 4.7.

4.7 TIC-TAC-TOE is a win for Maker when played under the Maker–Breaker convention.

Notes

FOX AND GEESE features prominently in *Winning Ways*. The first edition incorrectly asserted that Figure 4.1 has value $1 + \textbf{over}$; the correct answer $2 + \textbf{over}$ was later given by Jonathan Welton and verified by Berlekamp and Siegel. Berlekamp also proved that for $n \geq 9$, the $n \times 8$ starting position has the exact value $1+2^{-(n-8)}$; this analysis was incorporated into the second edition of *Winning Ways* in 2003.

The combinatorial theory of GO endgames was introduced in Wolfe's thesis [**Wol91**], building on Berlekamp's earlier discoveries in the theory of DOMINEERING [**Ber88**]. Further applications have been explored by Kim [**Kim95, BK96**], Landman [**Lan96**], Nakamura [**BN03**], Takizawa [**Tak02**], and others.

The temperature theory of GO was introduced in a seminal article by Berlekamp [**Ber96**]; many examples were subsequently found by Berlekamp, Müller, and Spight [**BMS96**]. The theory has been further pursued by Fraser [**Fras02**], Kao [**Kao97**], and Spight [**Spi99, Spi02, Spi03**].

Nakamura [**Nak09**] has recently introduced a beautiful new theory of GO capturing races, in which liberty counts are represented by partizan game values. Nakamura's theory bears a striking resemblance to the atomic weight theory (described in Section II.7 of this book).

ENTREPRENEURIAL CHESS was invented by Berlekamp and Pearson. It was inspired by the following CHESS problem, originally due to Simon Norton and mentioned in Guy's 1991 list of unsolved problems [**Guy91c**]: from the starting position in Figure 4.12, "what is the smallest board that White can win on if Black is given a win if he walks off the North or East edges of the board?"

Berlekamp and Pearson carried out a detailed temperature analysis of ENTREPRENEURIAL CHESS, showing (for example) that Figure 4.12 has mean value 17, among many other results [**BP03**]. Pearson also used related methods to solve Norton's original problem: the answer is 8×11.

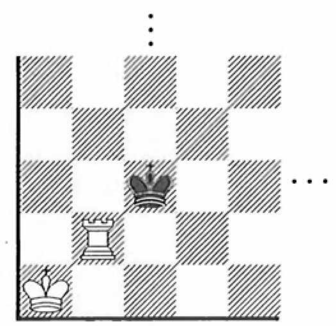


Figure 4.12. Simon Norton's CHESS problem.

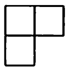
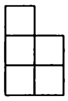
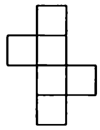
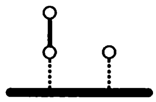
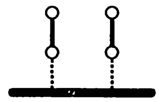
every Left incentive of G_1 is $\leq$ a Left incentive of some other G_i. Then in order to find a winning move for Left on G, one can discard G_1 from consideration. Indeed, it will often be the case that the incentives of a single component G_i dominate all the others, in which case there must be a winning move on G_i (assuming that one exists at all).

Exercises

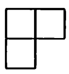
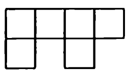
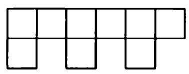
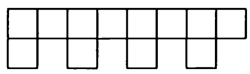
1.1 Determine all partial-order relations among the following games:

$$* \qquad \uparrow \qquad \downarrow \qquad \{1 \mid \uparrow\} \qquad \{\Uparrow \mid \downarrow\} \qquad \{1 \parallel 0 \mid -1\}$$

1.2 Confirm the following HACKENBUSH and DOMINEERING identities:

 $= *$  $= \{1 \mid -1\}$  $=$  $= \uparrow$  $= \Uparrow$

1.3 Show that the DOMINEERING positions

    $\cdots$

are equal to

$$* \qquad 0 \qquad * \qquad 0 \qquad \cdots$$

1.4 Prove the *Gift Horse Principle*: If $H \lhd\!\vert\, G$, then $G = \left\{G^L, H \;\middle|\; G^R\right\}$.

1.5 Suppose that for every subposition H of G, either H has options for both players; or else every option of H is dicotic. Prove that $G < 2$.

1.6 $\tilde{\mathrm{b}}(G) = \max\{1 + \tilde{\mathrm{b}}(G^L),\ 1 + \tilde{\mathrm{b}}(G^R)\}$, for all $G \ncong 0$.

1.7 $\tilde{\mathrm{b}}(G + H) = \tilde{\mathrm{b}}(G) + \tilde{\mathrm{b}}(H)$, for all G and H.

1.8 (a) Every Left incentive of $G + H$ is equal to a Left incentive either of G or of H.

(b) If $G \cong G_1 + G_2 + \cdots + G_k$ and every Left incentive of G_1 is $\leq$ a Left incentive of some other G_i, then Left need never move on G_1 from G.

1.9 Give examples of short games G and H such that $G = H$, but G and H have distinct Left incentives.

Notes

The axiomatic construction of partizan games was invented by Conway, who was also the first to notice that such games form a partially ordered Abelian group. Most of the material in this section originally appeared in ONAG and has since become standard.

Theorem 1.20 is sometimes adopted as the *definition* of $\geq$. The advantages of using the fundamental equivalence instead (cf. Definition 1.16) are that it is better-motivated and also generalizes seamlessly to wider classes of games for which the recursive characterization fails.

The term *dicotic*, proposed by Michael Weimerskirch, is new. Such games are traditionally called **all-small**, because (as we'll see in Section 4) all their subpositions are infinitesimal. "Dicotic" is preferred when (as here) we wish to emphasize

This is canonical: there are no dominated options, since $0 \not\gtrless *$; and $*$ can't reverse through 0, since $\uparrow + *$ is a first-player win.

The sum $\uparrow + *$ is usually abbreviated $\uparrow*$, pronounced "up-star."

Exercises

2.1 Determine the canonical forms of $\{a \cdot \uparrow \mid b \cdot \uparrow\}$, for all $a, b \in \mathbb{Z}$.

2.2 Let G be a short game with canonical form K. Show that:
  (a) For every K^L, there is a G^L such that $G^L \geq K^L$.
  (b) For every Left incentive $\Delta^L(K)$, there is a $\Delta^L(G) \geq \Delta^L(K)$.
  (c) For every subposition L of K, there is a subposition H of G such that $H = L$.

2.3 If G is dicotic, then so is its canonical form.

2.4 If K is in canonical form, then $\tilde{\mathrm{b}}(K) = \mathrm{b}(K)$.

2.5 $\mathrm{b}(G + H) \leq \mathrm{b}(G) + \mathrm{b}(H)$, for all games G and H. Give an example to show that the inequality might be strict.

2.6 A Left option G^L is **sensible** if $G^L \geq K^L$ for some K^L, where K is the canonical form of G. Otherwise G^L is **senseless**. Show that senseless options can be removed without changing the value of G.

2.7 A short game G is **even** if every option of G is odd; it is **odd** if $G \not\cong 0$ and every option of G is even. A value is even (resp. odd) if *any* of its representatives is even (resp. odd).
  (a) 0 is even and $*$ is odd, but $\uparrow$ is neither even nor odd.
  (b) If $n \in \mathbb{Z}$, then this definition agrees with the ordinary arithmetic definitions of "even" and "odd."
  (c) If G is even (resp. odd), then so is its canonical form.
  (d) Let $\mathcal{E} = \{G \in \mathbb{G} : G \text{ is even}\}$ and $\mathcal{O} = \{G \in \mathbb{G} : G \text{ is odd}\}$. Prove that $\mathcal{E}$ and $\mathcal{E} \cup \mathcal{O}$ are subgroups of $\mathbb{G}$, and show that $\mathcal{E}$ has index 2 in $\mathcal{E} \cup \mathcal{O}$.

2.8 (a) For each game G, there exists a game $H = G$ such that no subposition of H has more than three options.
  (b) If every subposition of H has at most two options, then $H \neq \uparrow*$. (So "three" in part (a) cannot be improved to "two.")

Notes

All of the material in this section is standard. The Simplest Form Theorem and its proof were first published in ONAG.

Exercise 2.8: Michael Albert.

3. Numbers

In the study of HACKENBUSH in Section I.2, we observed that the game $G = \{0 \mid 1\}$ resembles the rational number $\frac{1}{2}$, in the sense that $G+G$ is equal

Exercises

3.1 Exhibit short games G and H such that

$$L(G) + L(H),\ L(G+H),\ \text{and}\ L(G) + R(H)$$

are all distinct.

3.2 The hypothesis "G is not equal to a number" is necessary in Theorems 3.13, 3.21, and 3.22.

3.3 Let $x \geq 0$ be a number. Prove that

$$\mathrm{b}(x) = \lceil x \rceil + n,$$

where $x = m/2^n$ in lowest terms.

3.4 For any two numbers $x \geq y$, there exist games G, H, and J with $\mathcal{C}(G) = [x, y]$, $\mathcal{C}(H) = [x, y[$, and $\mathcal{C}(J) =]x, y[$.

3.5 If G has no Right options, then G is equal to an integer.

3.6 Calculate the stops and mean value of each of the following games:

$$G = \{1 \mid 0\}; \qquad H = \{1 \parallel 0 \mid -100\}; \qquad J = \{1 \parallel 0 \mid -1\}.$$

3.7 Let $x_1 > x_2 > \cdots > x_n \geq 0$. Determine the Left stop, Right stop, and mean value of

$$\pm x_1 \pm x_2 \pm \cdots \pm x_n.$$

3.8 Suppose that G is not equal to a number, and assume that G is in canonical form. Show that $R(G^L) \geq L(G^R)$ for *every* G^L and G^R.

3.9 Every position in BLUE–RED HACKENBUSH is a number.

3.10 Suppose that x is equal to a number and G is not, and let H be an arbitrary game. If Left has a winning move on $G + H + x$ of the form $G + H + x^L$, then she also has a winning move of the form $G^L + H + x$. (This strengthens Theorem 3.22.)

3.11 An **adorned number** is a symbol of the form x_L or x_R, with $x \in \mathbb{D}$. The **adorned stops** of G are defined by

$$L^a(G) = \begin{cases} x_R & \text{if } G \text{ is equal to a number } x; \\ \max\limits_{G^L}\left(R^a(G^L)\right) & \text{otherwise;} \end{cases}$$

$$R^a(G) = \begin{cases} x_L & \text{if } G \text{ is equal to a number } x; \\ \min\limits_{G^R}\left(L^a(G^R)\right) & \text{otherwise.} \end{cases}$$

In taking max and min, adorned numbers are ordered lexicographically, so that $x_L > x_R > y_L > y_R$ whenever $x > y$. Prove that $\mathcal{C}(G)$ is completely determined by $L^a(G)$ and $R^a(G)$.

3.12 The following are equivalent:
(i) $G - H$ is infinitesimal.
(ii) $L(G + X) = L(H + X)$ and $R(G + X) = R(H + X)$ for every X.

3.13 Does there exist a G such that $n \cdot G \not\gtrless 0$ for all n?

3.14 If $G^L \leq G^R$ for every G^L and G^R, then G is numberish (cf. Definition 4.5 on page 84, below).

3.15 If every incentive of G is negative, then G is equal to a number.

3.16 G is **alternating** (or **consecutive move banned**) if, for every subposition H of G, there are no second options of the form H^{LL} or H^{RR}. Classify all possible values for an alternating game G.

3.17 A BLUE–RED HACKENBUSH position is a **spider** if it has just one nonground node (the **center**) incident to more than two edges. The center is connected to the ground via several paths (the **legs**). In a **redwood spider** such as Figure 3.2, each leg consists of a single blue edge touching the ground, followed by a path of red edges leading up to the center.

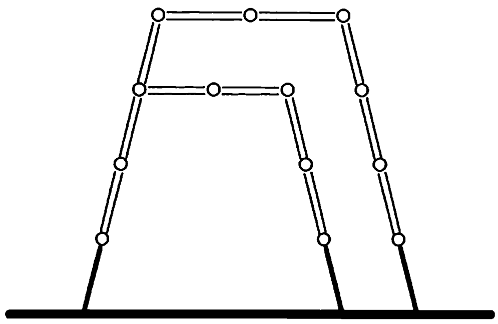


Figure 3.2. A redwood spider with three legs: $a_1 = 5$, $a_2 = 3$, $a_3 = 1$.

Let G be a redwood spider with n legs. Let

$$a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq 0$$

and assume that the the i^{th} leg of G has exactly one blue edge and $a_i + 1$ red edges, for $1 \leq i \leq n$. Show that $G = 2^{-r}$, where

$$r = \left\lfloor \frac{a_1}{2^1} + \frac{a_2}{2^2} + \cdots + \frac{a_{n-2}}{2^{n-2}} + \frac{a_{n-1} + a_n + 1}{2^{n-1}} \right\rfloor.$$

Notes

As early as the 1950s, Milnor [**Miln53**] and Hanner [**Han59**] considered sums of games whose terminal positions are assigned numerical scores. It was Conway who recognized that the explicit assignment of scores is not always necessary, because numbers themselves can be represented as particular game positions. $\mathbb{D}$ is just a fragment of a much larger number system, the **surreal numbers**, that is part of Conway's original vision; surreal numbers are described in Chapter VIII of this book.

Most of the material in this section is standard and is drawn from ONAG and *Winning Ways*. The Mean Value Theorem was proved independently in various forms by Hanner, Berlekamp, and Conway; the particularly elegant proof given here is due to Norton.

Exercise 3.16: Paul Ottaway.

Exercise 3.17: Berlekamp.

4. Infinitesimals

The theory of numbers presented in Section 3 provides a useful window into the structure of $\mathbb{G}$, but it is certainly not the whole story. For example, is "several moves away"; the longer it takes for Right to make the threat, the greater the advantage to Left. The atomic weight theory in Section 7 will help to make all of these notions more precise.

Exercises

4.1 Draw the game tree of $\uparrow n$ when n is even, and compare with Figure 4.2 on page 86. Do the same for $\uparrow^{[n]}$ and $\uparrow^{[n]}*$.

4.2 Prove Proposition 4.12.

4.3 Determine the uptimal confusion intervals of $\{\uparrow \mid \downarrow *\}$ and $\{\uparrow \mid *, \uparrow\}$.

4.4 (a) Show that if x and y are numbers and $x > y > 0$, then $+_x \ll +_y$. Is the same true if x and y are replaced by arbitrary games?
(b) State necessary and sufficient conditions on $x, y, z \in \mathbb{D}$ such that
$$\{x \mid y\} + +_z = \{x + +_z \mid y + +_z\}.$$
(c) Let G and H be games with $G > H$ and $R(H) > 0$. State necessary and sufficient conditions on G and H such that $+_G \ll +_H$. Does $+_G < +_H$ necessarily imply $+_G \ll +_H$?

4.5 $\{* \mid *\} : n = \uparrow^{[n]}$ for all n.

4.6 If $G > 0$, then for some n we have $G > +_n$ and $G \mathrel{|\!\triangleright} \{0 \mid +_n\}$.

4.7 True or false:
(a) If $G > 0$ and $G \ll 1$, then G is infinitesimal.
(b) If $G > 0$ is dicotic and $R(H) > 0$, then $G > +_H$.
(c) If $G > 0$ is dicotic, then $G > \uparrow^n$ for some n.
(d) If $G > \uparrow^{[n]}$ for all n, then necessarily $G > *$.

4.8 Determine the canonical forms of $k \cdot \uparrow^{[n]}$, for all $k, n \geq 1$.

4.9 (a) Show that the queenside of Figure I.4.3 (page 31) has value $\uparrow$ and the kingside has value $\Downarrow *$. Conclude that the position is a first-player win.
(b) Compute the values of each component of Figure 4.6, and determine its outcome class.

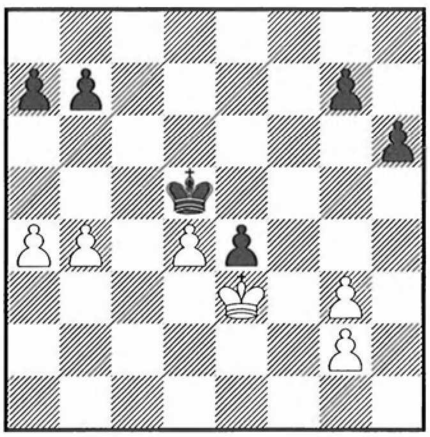


Figure 4.6. Popov–Dankov, Albena 1978.

4.10 Find all G such that $+_{+_{+_G}} = +_G$.

4.11 Let $G = \{0^k \mid +_x\}$, with $x > 0$ and $k \geq 1$. Show that for all n,
$$0 < G - k \cdot \left(\uparrow^{[n]}*\right) \ll \uparrow^n.$$

4.12 Let $y > x > 0$ and fix $k \geq 0$. Show that for every positive uptimal U,
$$0 < \{0^k \mid +_x\} - \{0^k \mid +_y\} < U.$$

4.13 Let G be a short game. Suppose that $H^L \leq H^R$ for every subposition H of G and every H^L and H^R. Determine all possible values for G.

4.14 Let Γ be the partizan subtraction game with $S_L = \{1\}$, $S_R = \{2, 3\}$ (cf. Exercise I.1.8 on page 15). Determine the value of a heap of n tokens, for all $n \geq 0$.

4.15 True or false: There exist distinct games $G_1, \ldots, G_k$, with each $R(G_i) > 0$, such that 0 can be expressed as a nontrivial linear combination
$$0 = a_1 \cdot +_{G_1} + a_2 \cdot +_{G_2} + \cdots + a_k \cdot +_{G_k} \quad (\text{each } a_i \in \mathbb{Z}, \text{ each } a_i \neq 0).$$

4.16 Express $(\uparrow m)^n$, $(\uparrow m)^{[n]}$, $(\uparrow^m)^n$, and $(\uparrow^m)^{[n]}$ as uptimals, for all $m, n \geq 1$.

4.17 (a) Prove the *Dicotic Translation Theorem*: If $L(G) > R(G)$ and H is dicotic, then $G + H = \left\{G^L + H \mid G^R + H\right\}$.

(b) Is the same true if H is an infinitesimal, but not necessarily dicotic?

4.18 Let G be a FLOWERS position with weight $w(G) = 0$. Show that:

(a) There exists an integer $m_0 \geq 0$ such that
$$G + *m \mathrel{|\!\!>} 0 \quad \Leftrightarrow \quad m \geq m_0, \quad \text{for all } m \geq 0.$$

(b) If Right has a flower of stem length 1 and $G \mathrel{|\!\!>} 0$, then chopping the flower of stem length 1 must be a winning move for Left.

(c) If Right has the (strictly) longest flower, then necessarily $G \mathrel{|\!\!>} 0$.

(d) If Right has a flower of stem length $m = 2^k$ or $2^k + 1$, all of Left's flowers are shorter than m, and there are no green flowers, then $G > 0$. Is this necessarily true for any other values of m?

4.19 Let G_m be the FLOWERS position $\left(*(m+1) : 1\right) - \left(*m : 1\right)$. Draw the partial order of $\{G_m : 1 \leq m \leq 15\}$. For which m and n is $G_m \ll G_n$?

4.20 Let $\mathcal{S} \subset \mathbb{N}$ be a nonempty finite set. The **superstar** $\uparrow_{\langle\mathcal{S}\rangle}$ is defined by
$$\uparrow_{\langle\mathcal{S}\rangle} = \{0, *, *2, \ldots, *m \mid *a\}_{a \in \mathcal{S}}$$
where $m = \mathrm{mex}(\mathcal{S})$ and a ranges over the elements of $\mathcal{S}$. (Here $\mathrm{mex}(\mathcal{S})$ denotes the **m**inimal **ex**cluded value of $\mathcal{S}$, as in Definition IV.1.1 on page 180.) Then a position in SUPERNIM is a sum of superstars and their negatives.

(a) $\uparrow_{\langle\mathcal{S}\rangle} \mathrel{\not\gtrless} *a$ for $a \in \mathcal{S}$, and $\uparrow_{\langle\mathcal{S}\rangle} > *a$ for $a \notin \mathcal{S}$.

(b) Every FLOWERS position occurs in SUPERNIM, and the Two-Ahead Rule generalizes (with positive and negative superstars taking the place of blue and red flowers).

(c) If $\mathcal{T} \subset \mathbb{N}$ and there is some integer m such that $\mathcal{T} = \mathcal{S} \oplus m$, then for the *least* such m we have $\uparrow_{\langle\mathcal{T}\rangle} = \uparrow_{\langle\mathcal{S}\rangle} + *m$.

4.21 A **flowering tree** is a HACKENBUSH tree (cf. Exercise I.2.1 on page 21) that is completely green, except for a single blue or red leaf (the **blossom**). The path from the ground to the blossom is the **trunk**. Any move on the trunk yields a nimber, and the associated set of nim values is the **trunk set**.

Let G be a flowering tree with a blue blossom and trunk set $\mathcal{S}$. Prove that $G \mathrel{\not\gtrless} *a$ for each $a \in \mathcal{S}$, and $G > *a$ for each $a \notin \mathcal{S}$. Prove furthermore that if no option of G has trunk set $\mathcal{S}$, then in fact $G = \uparrow_{\langle\mathcal{S}\rangle}$ (cf. Exercise 4.20).

4.22 What is the outcome of Figure I.1.3 on page 6? (Use Exercise 4.21.)

4.23 *Generalized uptimals.* Let $G = \{0 \mid G^R\}$ be a positive infinitesimal (so that each $G^R \mathrel{|\!\rhd} 0$). The infinitesimals G^n and $G^{[n]}$ are given by

$$G^{[0]} = 0; \quad G^{[n+1]} = \{G^{[n]} \mid G^R\}; \quad G^{n+1} = \{0 \mid G^R - G^{[n]}\}.$$

(a) $G^{n+1} = G^{[n+1]} - G^{[n]}$ for all $n \geq 0$.
(b) $0 < G^{n+1} \ll G^n$ for all n.
(c) If $x > y > 0$, then $\boldsymbol{+}_x \ll \boldsymbol{+}_y^n$ for all n.
(d) What is the canonical form of $G^R + G^{[n]}$ (expressed recursively in n)?
(e) True or false: If $G \geq H$, then $G^{[n]} \geq H^{[n]}$ and $G^n \geq H^n$.

4.24 *Fractional uptimals.* For all positive $x \in \mathbb{D}$, define

$$\uparrow^{[x]} = (* : x) - * \quad \text{and} \quad \uparrow^{x+1} = \{0 \mid \downarrow_{[x]}*\}.$$

(a) If $x > y \geq 1$, then $\uparrow^x \ll \uparrow^y$.
(b) If $x > 1$ and $m \geq 1$, then $\uparrow^x \not\gtrless *m$.
(c) $\uparrow^{x+1} = \uparrow^{[n]} - \uparrow^{[x]}$, where $n = \lfloor x \rfloor + 1$.
(d) Part (c) shows that $\uparrow^{x+1} \neq \uparrow^{[x+1]} - \uparrow^{[x]}$, except when x is an integer. Investigate the differences $D_x = \uparrow^{[x+1]} - \uparrow^{[x]}$. How does D_x compare with each $\uparrow^y$?
(e) Show that $\uparrow^{[x]} = \uparrow : (x-1)$ for $x \geq 1$. If $0 \leq x < 1$, show that $\uparrow^{[x]}$ is also of the form $\uparrow : y$ for some y. Determine y as a function of x.
(f) Let $\mathcal{U}^+$ be the group generated by $\{\uparrow^x : x \geq 1\}$. Show that $\mathcal{U}^+$ is totally ordered and free on the generators $\uparrow^x$.

4.25 A position in TOPPLING DOMINOES consists of a row of dominoes, each colored bLue or Red, represented as a string of L's and R's. On her turn, Left may "topple" any bLue domino; this removes the toppled domino, along with all dominoes to the east or to the west (Left's choice). For example, from LRLRR Left could move to either LR (toppling the middle domino eastward), RR (westward), RLRR (toppling the end domino westward), or the empty position (eastward). Right's moves are similar, toppling an R domino in either direction.
(a) $\mathrm{L}^n = n$; $(\mathrm{LR})^n = *n$ $(n \geq 0)$; and $\mathrm{LR}^{n+2}\mathrm{L} = \boldsymbol{+}_n$ $(n \geq 0)$.
(b) Every $x \in \mathbb{D}$ is represented by a TOPPLING DOMINOES position. (For $x > 0$, define $G_x = G_y\mathrm{LR}G_z$, where $x = \{y \mid z\}$ in canonical form.)
(c) Let X be a position whose value is a number $x > 0$. Prove that X contains no two consecutive R's.
(d) Fix a position X, and let Y be obtained by replacing every instance of R in X with RL. For example, if $X = \mathrm{LRR}$, then $Y = \mathrm{LRLRL}$. Prove that if X has value G, then LY has value $1 : G$.
(e) Prove that the representation in part (b) is unique. (Assume $x > 0$, show that $x = 1 : y$ for a unique number y, and proceed by induction, using (c) and (d).)
(f) Show furthermore that this position is a palindrome.

We conclude that the thermograph of G is entirely determined by G^{L_1} and the symmetric Right option G^{R_1}. Therefore:

$$G \quad \text{has the same thermograph as} \quad \pm 9.$$

The argument generalizes easily to show that the analogous position on a $2 \times n$ board has the same thermograph as $\pm(2n-5)$, for all $n \geq 3$. This is a nice illustration of the power of the temperature theory: the canonical forms of such positions explode in complexity as n increases, but a thermographic analysis provides simple and clear insights.

Exercises

5.1 Let $G = \{51 \pm 50 \parallel \{0, \pm 50 \mid -200\}\}$. Describe the thermograph of $n \cdot G$ for $n = 1, 2, \ldots, 51$.

5.2 Let $\mu = \frac{1}{2}(L(G) + R(G))$ and $\tau = \frac{1}{2}(L(G) - R(G))$.
(a) If G is not equal to a number, then $t(G) \geq \tau$ and $|m(G) - \mu| \leq t(G) - \tau$.
(b) If $t(G) = \tau$, then $m(G) = \mu$; if $m(G) = L(G)$, then $t(G) \geq 2\tau$.

5.3 (a) Let $G = \{4 \mid 1\}$. Calculate $\tilde{G}_t$ for all t. For which t does $\tilde{G}_t = G_t$?
(b) For all G, there is a number $\delta > 0$ such that
$$\tilde{G}_t = m(G) \quad \text{whenever} \quad t(G) < t \leq t(G) + \delta.$$
(c) Does $G = H$ necessarily imply $\tilde{G}_t = \tilde{H}_t$?

5.4 If G is not equal to an integer, then there exist G^L and G^R such that
$$m(G^L) \geq m(G) + t(G) \quad \text{and} \quad m(G^R) \leq m(G) - t(G).$$

5.5 If G is not equal to an integer and $t = t(G)$, then G_t is not equal to a number.

5.6 For any two numbers $x \geq y \geq 0$, there exist games G and H such that $t(G) = t(H) = x$ and $t(G + H) = y$.

5.7 (a) Suppose that $G = \{A \mid B\}$ and $t(A), t(B) < \frac{1}{2}(m(A) - m(B))$. Then
$$t(G) = \frac{m(A) - m(B)}{2} \quad \text{and} \quad m(G) = \frac{m(A) + m(B)}{2}.$$
(b) If the hypothesis is weakened to allow $t(A) = \frac{1}{2}(m(A) - m(B))$, do either (or both) of the conclusions still hold?
(c) What if we allow both $t(A) = t(B) = \frac{1}{2}(m(A) - m(B))$?

5.8 Suppose that the only numbers that occur as subpositions of G are integers.
(a) Show that if $m(G) = a/2^b$ in lowest terms, then $t(G) \geq 1 - 1/2^b$.
(b) Determine all possible values for $t(G)$.

5.9 Let a be the slope of $L_t(G)$ just below $t = 0$. Prove that the values of $L_t(G)$ for $t \leq 0$ are completely determined by a and $L(G)$.

5.10 $L_t(G) \leq \lceil L(G) \rceil$ and $R_t(G) \geq \lfloor R(G) \rfloor$ for all $t \geq -1$ (cf. Theorem 5.11(d)).

5.11 Every noninteger G has at least one Left incentive Δ such that $t(\Delta) \geq t(G)$.

5.12 (a) If $t \geq 0$ and G is not equal to a number, then $t\left(\int^t G\right) = t + t(G)$.
(b) If $G = H$, then $t\left(\int^t G - \int^t H\right) < t$.
(c) If $G = H$ and G and H are dicotic, then $\int^t G = \int^t H$.

5.13 Let $s, t \geq 0$. Then s is a critical temperature of G if and only if $s + t$ is a critical temperature of $\int^t G$. Moreover, $\int^t G$ has no critical temperatures on the interval $[0, t[$.

5.14 Suppose that $t(G) = t(H) = t$, and let ϵ and ζ be the first particles of G and H, respectively. Show that $t(G + H) = t$ if and only if $\epsilon + \zeta \neq 0$.

5.15 Is the operator $\oint_{1*}^{1}$ a homomorphism? Is it invariant of form? How about $\oint_{1}^{1}$?

5.16 (a) Does the value of $\int^T G$ depend on the form of T?
(b) Does the value of $\oint_S^T G$ depend on the form of S and/or T?

5.17 True or false: If x and $t \geq 0$ are numbers, $H = x + G$, and G and H are in canonical form, then $\int^t H = x + \int^t G$.

5.18 (a) Show that $\oint_0^0 G$ is dicotic for all G.
(b) If x is a number, show that $\oint_0^0 x$ is an uptimal, and express it in uptimal notation. (Reduce to the case $0 < x < \frac{1}{2}$, and consider the binary expansion of x.)
(c) YELLOW–BROWN HACKENBUSH is a restricted form of ordinary BLUE–RED HACKENBUSH. Left may remove any yeLLow edge, provided that her move disconnects *at least one* brown edge and *no other* yellow edges. Right may remove any bRown edge, with the converse restrictions. (In particular, neither player can move from any *monochromatic* component.) Determine the complete solution for YELLOW–BROWN HACKENBUSH stalks.

5.19 (a) If $L_t(G)$ or $R_t(G)$ changes slope at temperature t, then t is necessarily a critical temperature of G.
(b) Give an example of a game G with the same thermograph as ± 2, such that $t = 1$ is a critical temperature for G.

5.20 True or false:
(a) If $t = t(H)$ for some subposition H of the canonical form of G, then t is necessarily a critical temperature of G.
(b) If $G = H$, then G and H have the same critical temperatures.

Notes

In the 1950s, Milnor [**Miln53**] and Hanner [**Han59**] studied two-player games in which the terminal positions are assigned explicit scores, and proved an early version of the Mean Value Theorem. Later, once Conway recognized that his axiomatic system of combinatorial games intrinsically incorporates a number system (see Notes to Section 3 on page 82), the *Winning Ways* authors were quick to develop an appropriate temperature theory in the new context.

Most of the material in this section is classical and appeared (in various forms) in ONAG and *Winning Ways*. Generalized overheating operators (Definition 5.25 on page 115) were introduced later by Berlekamp, in conjunction with the study of DOMINEERING [**Ber88**]; the notation $\oint$ used here is new and is intended to distinguish from ordinary heating when both operators are used in the same context.

Left can win playing second: either by reverting a move G^R to some $a' \geq a$; or by responding on G if Right moves from $\Uparrow *$ to $\uparrow$. (By the Number Avoidance Theorem, we needn't consider moves on $\mathrm{rcf}(G)$.)

Next suppose $\mathrm{rcf}(G) = \{a \mid b\}$, so that $G = \{a, G^L \mid b, G^R\}$. Now on the sum H, Right's opening move on $\mathrm{rcf}(G)$ has an immediate counter on G; likewise Right's opening move from G to b.

If instead Right moves to some other G^R, then Left can revert to some $a' \geq a$, as before; while if Right opens by moving on $\Uparrow *$, Left can respond by moving immediately from G to a. In either case, after Left's move the position has the form

$$a' - \{a \mid b\} + \Uparrow * \quad (a' \geq a),$$

from which Right's best move is to $a' - a + \Uparrow * \geq 0$.

This shows that $G - \mathrm{rcf}(G) \leq \Uparrow *$; the other inequality is by symmetry. □

The bounds in Theorem 6.26 can be strengthened (see Exercise 6.15).

Exercises

6.1 (a) The hypothesis "G is not a number" is necessary in Lemma 6.5.
(b) The hypothesis "G is hot" is necessary in Theorem 6.9 and cannot be weakened to "G is not a number."

6.2 $G \mapsto \mathrm{rcf}(G)$ is *not* a group homomorphism of $\mathbb{G}$.

6.3 The $*$-**projection** of G, denoted by $p(G)$, is defined by

$$p(G) = \begin{cases} x & \text{if } G = x \text{ or } x* \text{ for some number } x; \\ \{p(G^L) \mid p(G^R)\} & \text{otherwise.} \end{cases}$$

Prove that $p(G_*) = \mathrm{rcf}(G)$. This gives an alternative construction for $\mathrm{rcf}(G)$.

6.4 Prove that $G_{0+} = \mathrm{rcf}(G)$ for all G. Conclude that G_t is reduced for all but finitely many t. This gives another alternative construction for $\mathrm{rcf}(G)$.

6.5 Let G be a game with canonical form K. Prove that if G is even-tempered, then so is K.

6.6 For which numbers s and t $(0 \leq s \leq t)$ is $\int^s \uparrow + \int^t \uparrow$ reduced?

6.7 Assume that G is not equal to a number, and suppose G' is obtained from G by replacing every Left option G^L with a new option $G^{L'} \geqq G^L$, and every Right option G^R with a $G^{R'} \geqq G^R$. Show that $G' \geqq G$. (Why does this *not* follow from inductive application of Lemma 6.5?)

6.8 Assume that K is in reduced canonical form and K is not a number. Let Δ be an incentive of K. Prove that $\Delta \not\leqq 0$.

6.9 Suppose that $G^{L_1 R_1} \leq G$, and assume that G is not equal to a number (but is not necessarily hot). Prove that we can bypass G^{L_1} through $G^{L_1 R_1}$, as in

Theorem 6.9, *provided that* $G^{L_1R_1}$ is not equal to a number. (This might be false when $G^{L_1R_1}$ is a number; cf. Exercise 6.1(b).)

6.10 Let G be a game with reduced canonical form K (cf. Exercise 2.2 on page 68).
  (a) Show that for every K^L, there is a G^L such that $G^L \geqq K^L$.
  (b) Show that for every subposition T of K, there is a subposition H of G such that $H \equiv T$.

6.11 A Left option G^L is Inf-**sensible** if $G^L \geqq K^L$ for some K^L, where K is the reduced canonical form of G. Otherwise G^L is Inf-**senseless**. Show that removing Inf-senseless options changes the value of G by at most an infinitesimal (cf. Exercise 2.6 on page 68).

6.12 Show that an Inf-sensible option need not be sensible. Is the converse true?

6.13 Let $\Gamma = \textsc{Subtraction}(1, n \mid n{-}1, n)$, for fixed $n \geq 3$. Determine the reduced canonical forms of all Γ-heaps.

6.14 If G is dicotic and hereditarily transitive, then $G = * : H$ for some hereditarily transitive game H.

6.15 If G is hereditarily transitive, then $-U < G - \operatorname{rcf}(G) < U$ for any uptimal U of the form

$$U = \uparrow^{[n]} + \uparrow^{n} + * \quad (n \geq 1).$$

(This is proved for $n = 1$ as Theorem 6.26 on page 133.)

6.16 Let $\mathcal{E} = \{G \in \mathbb{G} : G \text{ is even-tempered}\}$ and $\mathcal{O} = \{G \in \mathbb{G} : G \text{ is odd-tempered}\}$. Prove that $\mathcal{E}$ and $\mathcal{E} \cup \mathcal{O}$ are subgroups of $\mathbb{G}$, and show that $\mathcal{E}$ has index 2 in $\mathcal{E} \cup \mathcal{O}$.

6.17 Let $\varepsilon \mathrel{|\!\!>} 0$ be an infinitesimal, and define

$$G_\varepsilon = \begin{cases} G & \text{if } G \text{ is a number;} \\ \{G^L_\varepsilon - \varepsilon \mid G^R_\varepsilon + \varepsilon\} & \text{otherwise.} \end{cases}$$

Note that this generalizes our definition of G_*. Prove that:
  (a) $G_\varepsilon \equiv G$, and if $G \equiv 0$, then $G_\varepsilon = 0$.
  (b) If $G = H$, then $G_\varepsilon = H_\varepsilon$.
  (c) $(G{+}H)_\varepsilon = G_\varepsilon{+}H_\varepsilon$. (Hint: First show that $L(G_\varepsilon) = L(G)$ and $R(G_\varepsilon) = R(G)$. Use this to prove the special case when H is a number. Then prove the full assertion by induction.)
  (d) If G_ε is not a number, then G_ε has at least one Left incentive and one Right incentive that are $\geq -\varepsilon$.
  (e) The *Dicotic Avoidance Theorem*: Let $\varepsilon \mathrel{|\!\!>} 0$ be dicotic, and assume that G_ε is not a number. Then G_ε has at least one Left incentive and one Right incentive that are $\geq$ every incentive (Left or Right) of ε.

Notes

Reduced canonical form was introduced in a 1996 paper by Dan Calistrate [**Cal96, Cal98**], who gave the construction in the form of Exercise 6.3. Inf-dominated and Inf-reversible moves first appeared earlier, in a paper by David Moews [**Moe91**], but in a totally different context (the group structure of $\mathbb{G}_3$; cf. Section III.3 of this book) that didn't involve the Inf-Simplest Form Theorem.

Now if $b = 0$, then $\mathrm{aw}(X_{ab}) = 0$. If $b > 0$, then X_{ab} has the Right option

$$X_{ab}^R = (\boxed{\blacktriangle\triangle})^a + \boxed{\blacktriangle\triangle} + (\boxed{\blacktriangle\triangle})^{b-1},$$

and by induction $\mathrm{aw}(X_{ab}^R) = 0$. This shows that $\mathrm{aw}(X_{ab}) \le 1$.

Likewise, if $a = 0$, then $\mathrm{aw}(Y_{ab}) = 0$ or 1, depending on whether $b \le 1$ or $b > 1$. If $a = 1$, then $\boxed{\blacktriangle\triangle\blacktriangle\blacktriangle} = {\downarrow}{*}$, so $\mathrm{aw}(Y_{ab}) = -1$ or 0. If $a = 2$, then we have $Y_{ab}^R = \boxed{\blacktriangle\triangle\triangle} + \boxed{\triangle}\,(\boxed{\blacktriangle\triangle})^b$; and since $\boxed{\blacktriangle\triangle\triangle} = \downarrow$, $\mathrm{aw}(Y_{ab}^R) = -1$ or 0. Finally, if $a > 2$, then Y_{ab} has the Right option

$$Y_{ab}^R = (\boxed{\blacktriangle\triangle})^{a-1}\boxed{\blacktriangle} + \boxed{\triangle\blacktriangle} + \boxed{\triangle}\,(\boxed{\blacktriangle\triangle})^b$$

with $\mathrm{aw}(Y_{ab}^R) = -1$ or 0. So in all cases, $\mathrm{aw}(Y_{ab}) \le 1$.

We have shown that $\mathrm{aw}(G^L) \le 1$ for every G^L. Furthermore, since $\boxed{\blacktriangle\blacktriangle\triangle} = \uparrow$, we have $\mathrm{aw}(X_{n-2,b}) = 1$, so this bound is attained. By symmetry every $\mathrm{aw}(G^R) \ge -1$, and this bound is also attained. By Lemma 7.17, we must have $\mathrm{aw}(G) = 0$.

Now consider $H = \boxed{\triangle}\,(\boxed{\blacktriangle\triangle})^n$. For $n = 2$ it is easily checked that $H = \uparrow^{[2]}$, so assume $n > 2$. Then every H^L has the form

$$Z_{ab} = \boxed{\triangle}\,(\boxed{\blacktriangle\triangle})^a + \boxed{\blacktriangle}\,(\boxed{\blacktriangle\triangle})^b .$$

If $b = 0$, then $\mathrm{aw}(Z_{ab}) \le 1$, while if $b > 0$, we have

$$Z_{ab}^R = \boxed{\triangle}\,(\boxed{\blacktriangle\triangle})^a + \boxed{\blacktriangle\triangle} + (\boxed{\blacktriangle\triangle})^{b-1}$$

with $\mathrm{aw}(Z_{ab}^R) \le 1$. Therefore $\mathrm{aw}(Z_{ab}) \le 2$, and since $n > 2$, this bound is clearly met when $b = 1$.

Finally, every H^R has the form

$$W_{ab} = \boxed{\triangle}\,(\boxed{\blacktriangle\triangle})^a\,\boxed{\triangle} + (\boxed{\blacktriangle\triangle})^b .$$

When $a = 0$, we have $\mathrm{aw}(W_{ab}) = 0$. When $a = 1$, we have $\boxed{\triangle\blacktriangle\triangle\triangle} = {\uparrow}{*}$ and hence $\mathrm{aw}(W_{ab}) = 1$. If $a = 2$, then $W_{ab}^L = \boxed{\triangle\blacktriangle\triangle\triangle} + (\boxed{\blacktriangle\triangle})^b$ has atomic weight 1. Finally, if $a > 2$, then there is the Left option

$$W_{ab}^L = \boxed{\triangle}\,(\boxed{\blacktriangle\triangle})^{a-1} + \boxed{\blacktriangle\triangle} + (\boxed{\blacktriangle\triangle})^b$$

with $\mathrm{aw}(W_{ab}^L) = 1$. In all cases, $\mathrm{aw}(W_{ab}) \ge 0$, and the bound is attained when $a = 0$.

We have shown that $\mathrm{aw}(H^L) \le 2$ for all H^L, that $\mathrm{aw}(H^R) \ge 0$ for all H^R, and that there exist options that attain both bounds. By Lemma 7.17, we have $\mathrm{aw}(H) = 1$.

Exercises

7.1 Determine the atomic weights of $\uparrow^n$, $\uparrow^{[n]}$, $+_x$, and $\{0^n \mid +_x\}$, for all $x > 0$ and $n \ge 1$.

7.2 If $\mathrm{aw}(G) = 0$, then $G \ll \Uparrow$.

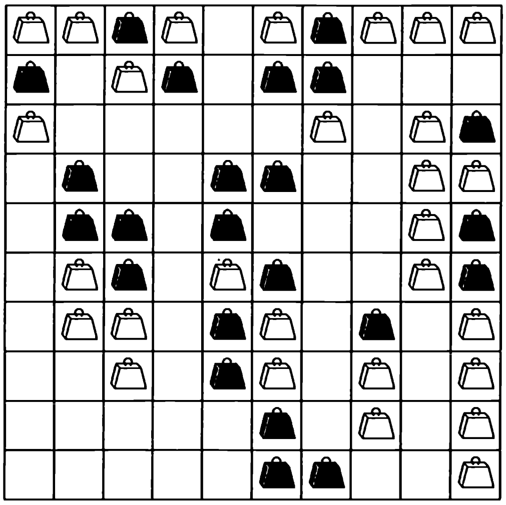


Figure 7.3. A tricky CLOBBER problem (Duffy and Kolpin, 2002).

7.3 (a) If $H \le 0$, then there exists a $G \le 0$ with $\mathrm{aw}(G) = H$.
(b) If $H \mathrel{\lhd\!|} 2$, then there exists a $G \mathrel{\lhd\!|} 0$ with $\mathrm{aw}(G) = H$.

7.4 If $R(\mathrm{aw}(G)) > 0$, then $n \cdot G > 0$ for some n.

7.5 True or false:
(a) If $G > 0$ and $G > *$, then $\mathrm{aw}(G) \ne 0$.
(b) If $\mathrm{aw}(G) = 0$, then $G \mathrel{\lhd\!|} \uparrow^x$ for some number $x > 1$ (cf. Exercise 4.24).
(c) If $\mathrm{aw}(G) > 0$, then $n \cdot G \ge 0$ for some n.

7.6 Exhibit a sequence of games $G_0 < G_1 < G_2 < \cdots$ of atomic weight 0 such that whenever $\mathrm{aw}(H) = 0$, we have $H < G_n$ for some n.

7.7 Show that G is atomic if and only if there exist dicotic games H and J such that $H \ge G \ge J$ and $\mathrm{aw}(H) = \mathrm{aw}(J)$.

7.8 If $G^L \le G^R$ for every G^L and G^R, then $G - m(G)$ is atomic with integer atomic weight. (This strengthens Exercise 3.14 on page 81.)

7.9 The value of every HACKENBUSH position has the form $x + G$, where x is a number and G is dicotic with integer atomic weight.

7.10 If $G \sim H$, then $G \equiv H$ (mod Inf). Conclude that the general theory of $\sim$ reduces to the case where G and H are infinitesimals.

7.11 If G is hereditarily transitive, then $G - \mathrm{rcf}(G)$ is atomic with atomic weight 1, 0, or -1.

7.12 The **bynumbers** B_n are given recursively by $B_n = \{0, B_a + B_{n-a} \mid *\}_{1 \le a < n}$. (In particular $B_1 = \{0 \mid *\} = \uparrow$.) Show that $\mathrm{aw}(B_n) = 1$ for all $n \ge 1$.

7.13 Analyze the CLOBBER sequences ()^n and ()^n.

7.14 Determine the outcome class of the CLOBBER position in Figure 7.3. (Hint: The western component has value $*2$. Compute the atomic weights of the other components and invoke Theorem 7.13, presuming $*2$ to be remote.)

7.15 Let $U > 0$. The **generalized Norton product** of G by U is given by

$$G{\cdot}U = \begin{cases} n \cdot U & \text{if } G \text{ is equal to an integer } n; \\ \{G^L \cdot U + U + \Delta \mid G^R \cdot U - U - \Delta\} & \text{otherwise,} \end{cases}$$

where Δ ranges over all Left and Right incentives of U.

(a) Prove Proposition 7.7 with $\uparrow$ replaced by arbitrary $U > 0$.

(b) Show that if $U \lhd\!\!| \; 0$, then Proposition 7.7(b) and (c) both fail.

(c) Show that if U is infinitesimal (resp. dicotic), then so is $G \cdot U$.

(d) True or false: If $U > 0$ and $H \cdot U > 0$, then $(G \cdot H) \cdot U = G \cdot (H \cdot U)$.

(e) True or false: $U = V$ implies $G \cdot U = G \cdot V$.

7.16 Let G be dicotic. The **positive galvanized atomic weight** $\mathrm{aw}^+(G)$ is defined as follows. Put

$$\tilde{v}^+(G) = \{\mathrm{aw}^+(G^L) - 2 \mid \mathrm{aw}^+(G^R) + 2\}.$$

Then $\mathrm{aw}^+(G) = \tilde{v}^+(G)$ *unless* $\tilde{v}^+(G)$ is an integer. In that case, $\mathrm{aw}^+(G)$ is the *largest* integer n such that

$$\mathrm{aw}^+(G^L) - 2 \lhd\!\!| \; n \lhd\!\!| \; \mathrm{aw}^+(G^R) + 2$$

for all G^L and G^R. The **negative galvanized atomic weight** $\mathrm{aw}^-(G)$ is defined the same way, but we choose the *smallest* integer n.

(a) Show that $G = H$ does *not* imply $\mathrm{aw}^+(G) = \mathrm{aw}^+(H)$.

(b) In the **galvanized sum** $G \circledast H$, the winner is determined based on the *component* in which the last move is made. Left wins if the last move is made on G, and Right wins if it is made on H, *regardless of who actually makes the last move.* Prove that the outcome of

$$(G_1 + G_2 + \cdots + G_m) \circledast (H_1 + H_2 + \cdots + H_n)$$

is exactly the same as the (normal-play) outcome of

$$\mathrm{aw}^+(G_1) + \cdots + \mathrm{aw}^+(G_m) + \mathrm{aw}^-(H_1) + \cdots + \mathrm{aw}^-(H_n) + ☆.$$

(c) Show that if G is any impartial game, then $G \circledast G$ is a first-player win.

Notes

The atomic weight theory was invented by Simon Norton in an extraordinary display of mathematical imagination. The theory was sketched in ONAG and first appeared completed in *Winning Ways.* Its curious nature and peculiar relationship to the nimbers remain deeply mysterious.

The atomic weight theory obviously has little to say about the vast hierarchy of infinitesimals $G \ll \uparrow$. Conway wrote in ONAG:

> Perhaps we can hope for an extended version of this theory which would enable us to measure still smaller games in terms of $\uparrow^2$, $\uparrow^3$, and so on. But the curious and complicated nature of the atomic weight algorithm suggests that any such theory will be very difficult to find. What do we expect to play the role of the remote stars, which enter so mysteriously and essentially into our theory? [**Con01**]

Scant progress has been made in understanding this problem, so we leave it as a question.

Question. Can the atomic weight theory be generalized to higher-order uptimals?

Exercises

1.1 If $G \in \mathbb{G}_{n+2}$ and $G > 0$, then $G \geq +_n$.

1.2 For $n \geq 2$, there is a $T \in \mathbb{G}_n^0$ such that if $G \in \mathbb{G}_n^0$ and $G > 0$, then $G \geq T$. (This is the dicotic analogue of Exercise 1.1.)

1.3 True or false: If $G \in \mathbb{G}_{n+1}$ is infinitesimal, then either $\uparrow n \geq G \geq \downarrow n$ or else $\uparrow n* \geq G \geq \downarrow n*$. (Compare to Theorem 1.5.)

1.4 If there are k hereditarily transitive games born by day n, then there are $k+1$ dicotic hereditarily transitive games born by day $n+1$. (Use Exercise II.6.14 on page 135.)

1.5 Enumerate the 67 dicotic games born by day 3, and arrange them into a table like Figure 1.1.

1.6 Draw the partial-order structure of $\mathbb{G}_2/\mathrm{Inf}$.

1.7 Fix an integer $n \geq 0$, and for each $G, H \in \mathbb{G}$ define

$$G \equiv_n H \quad \text{if} \quad o(G+X) = o(H+X) \quad \text{for each } X \in \mathbb{G}_n.$$

Prove that there is a one-to-one correspondence between elements of $\mathbb{G}_{n+1}$ and equivalence classes modulo $\equiv_n$.

Notes

The enumeration of $\mathbb{G}_2$ is a classical result. Figure 1.2 was given in the form shown here by Calistrate, Paulhus, and Wolfe [**CPW02**], improving upon earlier efforts by Guy [**Guy91d**].

The number of games born by day 3 was first isolated by Dean Hickerson and Robert Li in 1974, by a direct hand calculation. The count of day 4 games remains well (and perhaps permanently) out of reach.

The counts of dicotic and reduced games in Figure 1.5 were obtained using *cgsuite*; the hereditarily transitive statistics were first calculated by Angela Siegel [**Sie11**] (day ≤ 3) and Neil McKay (day 4).

Exercise 1.4: Neil McKay.

Exercise 1.7: Geoff Cruttwell.

2. Lattice Structure

Figure 1.2 (page 155) suggests that $\mathbb{G}_2$ has a rich poset structure. We'll now prove that *every* poset $\mathbb{G}_n$ has the structure of a distributive lattice. Throughout the following discussion, n will be a fixed positive integer.

Definition 2.1. For $G \in \mathbb{G}_n$ define

$$\lceil G \rceil = \{X \in \mathbb{G}_{n-1} : X \mathrel{|\!\!>} G\},$$
$$\lfloor G \rfloor = \{X \in \mathbb{G}_{n-1} : X \mathrel{<\!\!|} G\}.$$

Since σ is an automorphism, it must fix $\mathcal{A}$. By Theorem 2.6 we have

$$\mathcal{J}_{n+1} = \mathbb{G}_n \cup \Big\{ \{G \mid -n\} : G \in \mathbb{G}_n \Big\}$$

and by symmetry

$$\mathcal{M}_{n+1} = \mathbb{G}_n \cup \Big\{ \ \{n \mid G\} \ \ : G \in \mathbb{G}_n \Big\}.$$

It follows that $\mathcal{A} = \mathbb{G}_n \cup \{\pm n\}$. Moreover, every $G \in \mathbb{G}_n$ is comparable with n and $-n$, while $\pm n$ is not. Therefore $\pm n$ is the unique element of $\mathcal{A}$ incomparable with all other elements of $\mathcal{A}$, so it must be fixed by σ.

It follows that σ fixes $\mathbb{G}_n$, and therefore $\mathcal{J}_{n+1} \setminus \mathbb{G}_n$ as well. By induction, either σ is the identity on $\mathbb{G}_n$ or else $\sigma(J) = J^c$ for all $J \in \mathbb{G}_n$. The same must be true of $\mathcal{J}_{n+1} \setminus \mathbb{G}_n$ as well, since

$$\mathbb{G}_n \cong \mathcal{J}_{n+1} \setminus \mathbb{G}_n$$

via the isomorphism $J \mapsto \{J \mid -n\}$, which is companion-preserving by Lemma 2.12.

Now σ cannot be the identity on both parts of $\mathcal{J}_{n+1}$, by hypothesis; and if it's the identity on neither part, then we are done. To complete the proof, we must show that it can't be the identity on one and the companion automorphism on the other.

But if $\sigma(0) = *$ while $\sigma(\{0 \mid -n\}) = \{0 \mid -n\}$, then we have a contradiction, since

$$\{0 \mid -n\} \leq * \quad \text{but} \quad \{0 \mid -n\} \not\leq 0.$$

A similar contradiction occurs when $\sigma(0) = 0$ while $\sigma(\{0 \mid -n\}) = \{* \mid -n\}$. ☐

Exercises

2.1 Show that $\mathbb{G}$ is not a lattice (under the usual partial order).

2.2 Determine the companions of $\uparrow n$, $\uparrow^n$, $\uparrow^{[n]}$, $*m$, and $+_x$.

2.3 Determine all lonely elements of $\mathbb{G}_2$.

2.4 True or false: $(G + H)^c = G^c + H^c$ for all G and H.

2.5 G^c is infinitesimally close to G, for all G.

2.6 (a) If G is dicotic, then so is G^c, and $\mathrm{aw}(G) = \mathrm{aw}(G^c)$.
(b) Is the same true with "dicotic" replaced by "atomic"?

2.7 If $L(G) \neq 0$ and $R(G) \neq 0$, then G is lonely.

2.8 If G is an uptimal, then $G^c = G + *$. How does this extend to generalized uptimals (cf. Exercise II.4.23 on page 99)?

2.9 Every finite poset $\mathcal{P}$ is order-isomorphic to a subposet of $\mathbb{G}$.

2.10 Define lattices $\mathcal{L}_n$ as follows. $\mathcal{L}_0 = \{0\}$. For $n \geq 1$, let $\mathcal{K}_{n+1}$ be the set
$$\mathcal{K}_{n+1} = \mathcal{L}_n \times \{0,1\},$$
with the following partial order: $(x,0) \geq (y,0) \Leftrightarrow x \geq y \Leftrightarrow (x,1) \geq (y,1)$; $(x,0) \geq (y,1) \Leftrightarrow x \not\leq y$; and $(x,1) \not\geq (y,0)$ for all x and y. Then let $\mathcal{L}_{n+1}$ be the lattice of order-ideals of $\mathcal{K}_{n+1}$. Prove that $\mathcal{L}_n \cong \mathbb{G}_n$ for all n (as posets).

2.11 Let $\mathcal{A} \subset \mathbb{G}$ be hereditarily closed (cf. Definition 3.1 on page 168). Prove that $\mathcal{L} = \mathrm{ch}(\mathcal{A})$ is a distributive lattice. (This generalizes Theorem 2.5.) Show also that if $\mathcal{A}$ is closed under companionship, then $G \mapsto G^c$ is an automorphism of $\mathcal{L}$.

2.12 *Lattices of dicotic games.*
- (a) For $n \geq 1$, show that $\mathbb{G}_n^0$ is not a lattice.
- (b) Let $\mathcal{L}_n^0 = \mathbb{G}_n^0 \cup \{\triangledown, \vartriangle\}$, where $\triangledown$ and $\vartriangle$ are new symbols defined so that $\triangledown \geq G \geq \vartriangle$ for all $G \in \mathbb{G}_n^0$. Prove that $\mathcal{L}_n^0$ is a distributive lattice.
- (c) Characterize the join-irreducibles of $\mathcal{L}_{n+1}^0$ in terms of $\mathcal{L}_n^0$.
- (d) Prove that $\mathcal{L}_n^0$ admits exactly one nontrivial automorphism.

2.13 *Lattices of reduced game values.* Consider the poset for $\mathbb{G}_n/\mathrm{Inf}$ with order given by $\geqq$ (cf. Section II.6).
- (a) Prove that $\mathbb{G}_n/\mathrm{Inf}$ is a lattice.
- (b) Show that $\mathbb{G}_n/\mathrm{Inf}$ is not distributive.
- (c) Let $\mathcal{A} = \{\mathrm{rcf}(G) : G \in \mathbb{G}_n \text{ and } G \text{ is not numberish}\}$. Prove that $\mathcal{A}$ is a distributive sublattice of $\mathbb{G}_n/\mathrm{Inf}$.

Notes

The distributive lattice structure of $\mathbb{G}_n$ was first described by Calistrate, Paulhus, and Wolfe in a 2002 paper [**CPW02**]. This breakthrough was followed by a flurry of activity. The structure of join-irreducibles was described in [**FHW05**]. Shortly afterward, Fraser and Wolfe used these results to obtain bounds on the number of games born by day n. In [**FW04**] they showed that, writing g_n for the cardinality of $\mathbb{G}_n$,
$$2^{g_n/2g_{n-1}} \leq g_{n+1} \leq g_n + 2^{g_n} + 2$$
and in particular gave a (fairly weak) bound of $g_4 \geq 3 \cdot 10^{12}$. They also proved that
$$g_n = 2^{g_{n-1}^{\alpha(n)}}$$
where $0.51 < \alpha(n) < 1$ and $\alpha(n) \to 1$ as $n \to \infty$.

Open Problem. Improve upon the Fraser–Wolfe bounds for the cardinality of $\mathbb{G}_n$ (and in particular $\mathbb{G}_4$).

Several other natural classes of games are also known to have a lattice structure. $\mathbb{G}_n^0$ has many of the same properties as $\mathbb{G}_n$: although $\mathbb{G}_n^0$ is not a lattice (for example, $\uparrow(n-1)$ and $\uparrow(n-1)*$ have no join), a distributive lattice structure can be recovered by adjoining "synthetic" top and bottom elements [**Sie05**]. This is explored in Exercise 2.12.

More recently, Albert and Nowakowski found a sweeping generalization of results of this type [**AN11**]. If $\mathcal{A}$ is any hereditarily closed set of values, then the set

Write $m = 2^a b$ with b odd. By Theorem 3.16, $2^a \cdot h^k(Y) \in \mathcal{A}_n$. Since $Y \notin \mathcal{A}_n$ and since $2^k \cdot h^k(Y) = Y$, it follows that $a > k$. Therefore

$$2^a \cdot h^k(Y) = 2^{a-k} \cdot Y.$$

Now since $\mathcal{A}_n$ is 2-divisible, there must be some $A \in \mathcal{A}_n$ such that

$$2^{a-k} \cdot Y = 2^{a-k} \cdot A.$$

In particular,

$$2^{a-k} \cdot (Y - A) = 0.$$

But $Y - A \in \mathcal{A}_n + X_n$, so the minimality assumption on the order of Y gives $2^{a-k} \cdot Y = 0$, and therefore $G = 0$.

This proves that every $\mathcal{A}_n \cong \mathbb{D}^p \times (\mathbb{D}/\mathbb{Z})^q$ for some $p, q \in \mathbb{N}$. Moreover, the birthdays of the X_n must be unbounded (since there are just finitely many games of bounded birthday), so the minimality assumptions on the X_n imply $\mathbb{G} = \bigcup_n \mathcal{A}_n$. Therefore $\mathbb{G} \cong \mathbb{D}^p \times (\mathbb{D}/\mathbb{Z})^q$ for some $p, q \in \mathbb{N} \cup \{\omega\}$.

To complete the proof, we must show that neither p nor q is finite. But if q were finite, then $\mathbb{G}$ would have just 2^q unequal elements of order 2, and the infinite list

$$*, *2, *3, *4, \ldots$$

shows this to be false. Likewise, the games

$$\uparrow, \uparrow^2, \uparrow^3, \uparrow^4, \ldots$$

represent an infinite list of linearly independent torsion-free elements of $\mathbb{G}$ (cf. Theorem II.4.24 on page 94), so p cannot be finite. □

Exercises

3.1 In the definition of $h(G)$ on page 169, we put $n = 3 \cdot \mathrm{b}(G)$. Show that this can be improved to $n = \left\lceil \frac{3}{2} \cdot \mathrm{b}(G) \right\rceil$. Give an example to show that in general, this is the smallest choice of n for which Lemma 3.5 can still be proved.

3.2 Determine the group structure of $\langle \mathbb{G}_2^0 \rangle$.

3.3 Determine the partial-order structure of $\langle \mathbb{G}_2 \rangle$. (Which elements are ≥ 0?)

3.4 There is no short game G with $3 \cdot G = \uparrow$.

3.5 (a) If G has finite order, then for every n, either $n \cdot G = 0$ or $n \cdot G \not\gtrless 0$.
(b) If $G \mathrel{|\!\!\triangleright} 0$ has infinite order, then $\{n \in \mathbb{N} : n \cdot G \geq 0\}$ is isomorphic to a submonoid of $\mathbb{N}$ (as a partially ordered monoid).
(c) Let G have infinite order and put $\mathcal{A} = \{n \cdot G : n \in \mathbb{Z}\}$. Prove that the isomorphism type of $\mathcal{A}$ is determined by the partially ordered monoids $\{n \cdot G : n \geq 0\}$ and $\{n \cdot G : n \leq 0\}$.

Exercises

1.1 Let G be a short partizan game, and suppose that every option of G is equal to a nim-heap, so that

$$G = \{*a_1, *a_2, \ldots, *a_k \mid *b_1, *b_2, \ldots, *b_l\}.$$

Prove that if $m = \operatorname{mex}\{a_1, a_2, \ldots, a_k\} = \operatorname{mex}\{b_1, b_2, \ldots, b_l\}$, then $G = *m$.

1.2 Prove uniqueness in Theorem 1.6.

1.3 In NIM_k a move is to remove any number of tokens from each of j heaps, for some $j \le k$. It is not necessary to remove the same number of tokens from each heap. (So NIM_1 is just NIM.) Determine the $\mathscr{P}$-positions of NIM_k.

1.4 MARK is played with heaps of tokens. A player may either remove one token from a heap or reduce the heap size by half (rounding *down* to the new heap size). Thus for $n > 0$ we have $H_n = \{H_{n-1}, H_{\lfloor n/2 \rfloor}\}$. Determine the nim value of H_n for all n.

1.5 For all integers $a, b \ge 0$, define

$$a \circledast b = \operatorname{mex}\left(\begin{array}{c} \{a' \circledast b \ : a' < a\} \cup \{a \circledast b' : b' < b\} \\ \cup \ \{a' \circledast b' : a' < a, \ b' < b, \ \text{and } a' \circledast b = a \circledast b'\} \end{array} \right).$$

Prove that $(\mathbb{N}, \circledast)$ is an Abelian group isomorphic to $\mathbb{Z}_3^\omega$, with each copy of $\mathbb{Z}_3$ corresponding to one digit in the ternary representation of $\mathbb{N}$.

1.6 The **sequential sum** $G \to H$ was described in Section I.4 (cf. Figure I.4.9 on page 41). Assume throughout this exercise that G, H, X, and Y are short impartial games. Prove that:

(a) $\mathscr{G}(*m \to *n) = n$ if $n > m$; or $n - 1$ otherwise.

(b) If $\mathscr{G}(Y) = m$, then $\mathscr{G}(G \to Y) = \mathscr{G}(G \to *m)$. Conversely, $\mathscr{G}(X) = m$ need *not* imply that $\mathscr{G}(X \to G) = \mathscr{G}(*m \to G)$.

Now, for each impartial G we define the **signature** $\chi_G : \mathbb{N} \to \mathbb{N}$ by

$$\chi_G(m) = \mathscr{G}(G \to *m).$$

(c) If $\chi_G = \chi_H$, then $\mathscr{G}(X \to G \to Y) = \mathscr{G}(X \to H \to Y)$ for all X and Y.

(d) $\chi_G(m) = m$ if $G \cong 0$, and $\operatorname{mex}\{\chi_{G'}(m) : G' \in G\}$ otherwise.

(e) $\chi_{G \to H} = \chi_G \circ \chi_H$.

1.7 *Turning games.* Let $\mathcal{F}$ be a collection of nonempty finite subsets of $\mathbb{N}^+$. $\text{TURNING}(\mathcal{F})$ is played with a row of n coins, labelled $C_1, \ldots, C_n$ from left to right, each of which may be arranged either heads-up or tails-up. Each position can be represented by a vector $(c_1, \ldots, c_n) \in \mathbb{Z}_2^n$, with $c_i = 1$ if and only if C_i is heads-up. A legal move is to select any $\mathcal{A} \in \mathcal{F}$ and turn over coins C_i for all $i \in \mathcal{A}$, provided that the *rightmost* coin is turned heads-to-tails. We write $\text{TURNING}(d)$ as shorthand for the special case where $\mathcal{F}$ consists of all nonempty sets of cardinality $< d$. Prove that:

(a) For any $\mathcal{F}$, every position in $\text{TURNING}(\mathcal{F})$ can be expressed as a disjunctive sum of single-heads positions.

(b) Every position in $\text{TURNING}(2)$ has nim value 0 or 1.

(c) $\text{TURNING}(3)$ reduces to NIM, with $c_i = 1$ corresponding to the presence of a heap of size i.

(d) The *Mock Turtle Theorem.* Fix an even integer $d \geq 2$. Then the following are equivalent: (i) $(c_1, \ldots, c_n)$ is a $\mathscr{P}$-position for TURNING(d); and (ii) $c_1 \oplus \cdots \oplus c_n = 0$ and $(c_2, \ldots, c_n)$ is a $\mathscr{P}$-position for TURNING($d-1$).

1.8 *Lexicodes.* For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_2^n$, the **Hamming distance** $\delta(\mathbf{a}, \mathbf{b})$ is equal to the number of indices at which $\mathbf{a}$ and $\mathbf{b}$ differ:

$$\delta(\mathbf{a}, \mathbf{b}) = \left|\{i \leq n : a_i \neq b_i\}\right|.$$

The **binary lexicographic code** (or **lexicode**) of distance d and length n is the subset $\mathcal{L} \subset \mathbb{Z}_2^n$, defined as follows by $<_{\text{lex}}$-induction: $\mathbf{a} \in \mathcal{L}$ if and only if $\delta(\mathbf{a}, \mathbf{b}) \geq d$, for all $\mathbf{b} <_{\text{lex}} \mathbf{a}$ with $\mathbf{b} \in \mathcal{L}$. Elements of $\mathcal{L}$ are known as **codewords**. (Note that each codeword is the lexicographically least element of $\mathbb{Z}_2^n$ that is "sufficiently distant" from all prior codewords.)

(a) Show that $\mathcal{L}$ coincides with the set of $\mathscr{P}$-positions of TURNING(d) with n coins. Conclude that $\mathcal{L}$ is necessarily a subgroup of $\mathbb{Z}_2^n$.

The **dimension** of $\mathcal{L}$ is the unique k such that $\mathcal{L} \cong \mathbb{Z}_2^k$. Show that:

(b) The binary lexicode with $d = 3$ and $n = 2^r - 1$ has dimension $2^r - r - 1$. Write out a list of all 16 codewords for $n = 7$. (This is the **Hamming code**, a perfect n-bit one-error-correcting code.)

(c) $d = 7$ and $n = 23$ has dimension 12. (This is the celebrated **Golay code**, a perfect three-error-correcting code.)

(d) If $d \geq 2$ is even, then the dimension for distance d, length n is the same as for distance $d-1$, length $n-1$. (Use Exercise 1.7(d). The cases $d = 4$, $n = 2^r$ and $d = 8$, $n = 24$ are the **extended Hamming** and **extended Golay codes**.)

1.9 The *Lexicode Theorem.* Let $s \geq 2$. The **base-s lexicode** of distance d and length n is the subset $\mathcal{L} \subset \mathbb{Z}_s^n$, defined just as in Exercise 1.8 (with "2" replaced by "s"). Prove that if s is a 2-power, then $\mathcal{L}$ is necessarily closed under componentwise nim-sum. (Consider the game TURNING$_s$(d), played with s-sided "coins" and otherwise analogous to TURNING(d).)

Notes

The Sprague–Grundy Theorem was first proved in the 1930s, independently by Roland Sprague and Patrick Grundy, and came to prominence with the work of Richard Guy a decade later. Its history is discussed in detail in Appendix C.

Nim values have surprising connections to the theory of error-correcting codes. Some of these connections are explored in Exercises 1.8 and 1.9, and in Exercise 5.9 on page 221. For further results, see [**CS86, Con90, Ple91, Fra96b, FR03**].

Exercise 1.3: E. H. Moore.

Exercise 1.4: Aviezri Fraenkel.

Exercise 1.5: Simon Norton; it has been generalized to base n by François Laubie [**Lau99**].

Exercise 1.6: Stromquist and Ullman.

Exercise 1.7: Hendrik Lenstra.

Exercises 1.8 and 1.9: Conway and Sloane.

given for $n \geq 1$ by
$$\mathscr{G}(n) = \left\lfloor \frac{n-1}{3} \right\rfloor .$$
Now a legal move of Γ is to remove a single token and split the remainder into exactly three heaps. Therefore the moves from H_n are to

$$H_a + H_b + H_c, \quad \text{with } a, b, c \geq 1 \text{ and } a + b + c = n - 1.$$

But

$$\begin{aligned}\mathscr{G}(a) \oplus \mathscr{G}(b) \oplus \mathscr{G}(c) &\leq \mathscr{G}(a) + \mathscr{G}(b) + \mathscr{G}(c) \\ &\leq \frac{a-1}{3} + \frac{b-1}{3} + \frac{c-1}{3} = \frac{n-4}{3}.\end{aligned}$$

Since
$$\frac{n-4}{3} < \left\lfloor \frac{n-1}{3} \right\rfloor ,$$
it follows that $\lfloor (n-1)/3 \rfloor$ is not an excludent of H_n. To complete the argument we must show that every smaller number *is* an excludent; that is, if $m < \lfloor (n-1)/3 \rfloor$, then H_n has an option of nim value m.

Fix such m. Since $\lfloor (n-1)/3 \rfloor > m$, we have $\lfloor (n-1)/3 \rfloor \geq m + 1$, and hence $n - 3m \geq 4$. If $n - 3m$ is even, then put $a = (n - 3m - 2)/2$ and note that

$$a + a + (3m + 1) = n - 1 \quad \text{and} \quad \mathscr{G}(a) \oplus \mathscr{G}(a) \oplus \mathscr{G}(3m + 1) = 0 \oplus m = m.$$

Moreover $a \geq 1$, so H_n has the option $H_a + H_a + H_{3m+1}$ of value m.

Similarly, if $n - 3m$ is odd, then put $a = (n - 3m - 3)/2$ and note that

$$a + a + (3m + 2) = n - 1 \quad \text{and} \quad \mathscr{G}(a) \oplus \mathscr{G}(a) \oplus \mathscr{G}(3m + 2) = 0 \oplus m = m.$$

Moreover $a \geq 1$, so H_n has the option $H_a + H_a + H_{3m+2}$ of value m. This exhausts all cases.

Exercises

2.1 Let $\mathcal{S} = \{a, b\}$, with $a, b \geq 1$. Prove that SUBTRACTION($\mathcal{S}$) is purely periodic with period $a + b$.

2.2 *Ferguson's Pairing Property.* Let Γ be a subtraction game with subtraction set $\mathcal{S}$. Then for all $n \geq 0$,
$$\mathscr{G}(n) = 0 \quad \text{if and only if} \quad \mathscr{G}(n + s) = 1,$$
where s is the smallest element of $\mathcal{S}$.

2.3 Let Γ and Γ' be impartial rulesets. We say that Γ and Γ' have **identical positions** if every position of Γ is isomorphic to a position of Γ', and vice versa. Prove that for every impartial ruleset Γ that is closed under disjunctive sum, there is a heap game Γ' with identical positions.

2.4 Show that DAWSON'S CHESS (Exercise I.1.5 on page 14) is **0.137**.

2.5 Analyze the games $\mathbf{0.d}_1$, where $\mathbf{d}_1 = 2^k$. (Show that $\mathbf{0.4}$ is equivalent to DAWSON'S KAYLES. For $k \geq 3$, generalize the solution for $\mathbf{0.8}$.)

2.6 *Duplicate Kayles, etc.* Let $\Gamma = \mathbf{0.7}^k$ $(k \geq 2)$. The game
$$\Gamma' = \mathbf{0.0}^{t-1}\mathbf{7}^{k'} \qquad (t \geq 1),$$
with $k' = (k-2)t + 2$, is known as **t-plicate** Γ. Prove that
$$\mathscr{G}'(tn) = \mathscr{G}'(tn+1) = \cdots = \mathscr{G}'(tn + (n-1)) = \mathscr{G}(n) \text{ for all } n.$$
(Here $\mathscr{G}'(n)$ denotes the nim value of a Γ'-heap of size n.)

2.7 *Generalized Periodicity Theorem.* Let $\Gamma = \mathbf{d}_0.\mathbf{d}_1\mathbf{d}_2\ldots\mathbf{d}_k$ be a take-and-break game of finite length k. Let $t \geq 0$ be such that $\mathbf{d}_i < 2^{t+1}$ for all i. Suppose that there exist $n_0 \geq 0$ and $p \geq 0$ such that
$$\mathscr{G}(n+p) = \mathscr{G}(n) \quad \text{for all } n \text{ with } n_0 \leq n < tn_0 + (t-1)p + k.$$
Prove that
$$\mathscr{G}(n+p) = \mathscr{G}(n) \quad \text{for all } n \geq n_0.$$

2.8 *All-but subtraction games.* Let $\mathcal{S} \subset \mathbb{N}^+$ and suppose that $\mathbb{N}^+ \setminus \mathcal{S}$ is finite. Prove that SUBTRACTION($\mathcal{S}$) is arithmetic periodic.

2.9 Let $\Gamma = \mathbf{0.d}_1\mathbf{d}_2\mathbf{d}_3\ldots$. Suppose that there exist $i, j \geq 1$, with i odd, j even, and $\mathbf{d}_i, \mathbf{d}_j \geq 4$. Prove that Γ has just finitely many single-heap $\mathscr{P}$-positions.

2.10 Let $\Gamma = \mathbf{0.d}_1\mathbf{d}_2\mathbf{d}_3\ldots$. Suppose that there exist $i, j \geq 1$, with i odd, j even, and $\mathbf{d}_i, \mathbf{d}_j \geq 8$. Prove that $\lim_{n\to\infty} \mathscr{G}(n) = \infty$.

2.11 $\mathbf{0.3F}$ is arithmetic periodic with saltus 3.

2.12 *Austin's Theorem.* For $n \in \mathbb{N}$, denote by $f(n)$ the cardinality of the set
$$\{a \oplus b : a, b \in \mathbb{N} \text{ and } a + b = n\}.$$
Prove that:
(a) For all n, we have $f(2n+1) = f(n)$ and $f(2n) = f(n) + f(n-1)$.
(b) For all n, either $f(n) = f(a)$ for some $a \leq \frac{1}{2}n$ or else $f(n) = f(a) + f(b)$ for some $a \leq \frac{1}{2}n$ and $b \leq \frac{1}{4}n$.
(c) There is a real number $\alpha < 1$ such that $f(n) \leq \frac{5}{4}n^\alpha$ for all n.
(d) If Γ is an octal game with finite length, then Γ cannot be arithmetic periodic with nonzero saltus.

Notes

A general theory of subtraction games remains elusive, despite their apparent simplicity. There is an easy solution for two-element subtraction sets (Exercise 2.1), and three-element sets of the form $\{a, b, a+b\}$ were analyzed in *Winning Ways*. Several other special cases were recently solved by Ho [**Ho**].

Question. Does there exist a complete theory of finite subtraction games?

Octal games were introduced by Richard Guy and Cedric Smith in a seminal 1956 paper [**GS56**]. The sparse space theory is due to Berlekamp, who in 1973 discovered the bifurcation of GRUNDY'S GAME into rare and common values. Since then there's been little more to say about the normal-play *theory* of octal games, although advances in computing technology have gradually expanded its scope, at exist $b_0 > 0$ and $p > 0$ such that

$$\mathcal{R}(a, b_0) = \mathcal{R}(a, b_0 + p).$$

A straightforward induction now shows that $\mathcal{R}(a, b) = \mathcal{R}(a, b + p)$ for all $b \geq b_0$, so the sequence

$$b \mapsto f(a, b)$$

is periodic with period $\leq p$. But $\mathcal{G}(a, b) = f(a, b) + b$, so the sequence

$$b \mapsto \mathcal{G}(a, b)$$

is arithmetic periodic with period and saltus $\leq p$. $\square$

Exercises

3.1 Prove the converse of Lemma 3.3: if $\lfloor n\alpha \rfloor$ and $\lfloor n\beta \rfloor$ are complementary, then α and β are irrational and $1/\alpha + 1/\beta = 1$.

3.2 Determine all misère $\mathcal{P}$-positions of WYTHOFF.

3.3 Every $\mathcal{G}$-value m appears exactly once on every row of the WYTHOFF plane.

3.4 For all m, there is exactly one a with $\mathcal{G}(a, a) = m$. Moreover, this value of a satisfies $m/2 \leq a \leq 2m$.

3.5 A position in EUCLID is an ordered pair (a, b) of positive integers. A move is to decrease the larger of a and b by any positive multiple of the smaller, provided that it remains ≥ 1. Prove that:
   (a) (a, b) is a $\mathcal{P}$-position if and only if either $a/b < \phi$ or $b/a < \phi$ (where ϕ is the golden ratio).
   (b) Every EUCLID position can be written as the sequential sum of nim-heaps (see Figure I.4.9 on page 41 for the definition of sequential sum).

3.6 Let $\mathcal{A} \subset \mathbb{N}^2$. Define a ruleset $\Gamma(\mathcal{A})$, played with two heaps of tokens, as follows. A player may either: (i) remove any number of tokens from a single heap; or (ii) remove a tokens from one heap and b from the other, where $(a, b) \in \mathcal{A}$. Prove that if $\mathcal{A}$ contains no elements of the form (a, a), then $\Gamma(\mathcal{A})$ has the same $\mathcal{P}$-positions as NIM.

3.7 Fix $r > 0$. r-WYTHOFF is played with two heaps of tokens. On her turn, a player may either: (i) remove any number of tokens from any one heap; or (ii) remove a tokens from one heap and b from the other, where $|a - b| < r$. Prove that the n^{th} $\mathcal{P}$-position of r-WYTHOFF is given by

$$(a_n, b_n) = (\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor)$$

where

$$\alpha = \frac{1}{2}\left(2 - r + \sqrt{r^2 + 4}\right) \quad \text{and} \quad \beta = \alpha + r.$$

3.8 Fix $k > 0$. 2^k-NIMHOFF is played with n heaps of tokens. On her turn, a player may either: (i) remove any number of tokens from any one heap; or (ii) remove exactly 2^k tokens from each of two distinct heaps. Prove that $\mathcal{G}(a_1, a_2, \ldots, a_n) = a_1 \circledast a_2 \circledast \cdots \circledast a_n$, where

$$a \circledast b = a \oplus b \oplus a^k b^k.$$

3.9 *Adjoining $\mathscr{P}$-positions as moves.* WYTHOFF^2 is played with two heaps of tokens. On her turn, a player may either: (i) move as in ordinary WYTHOFF; or (ii) remove a tokens from one heap and b from the other, where (a, b) is a WYTHOFF $\mathscr{P}$-position. Determine the $\mathscr{P}$-positions of WYTHOFF^2.

Notes

The $\mathscr{P}$-positions of WYTHOFF were first isolated in a 1907 paper by the Dutch mathematician Willem Wythoff [**Wyt07**]. More than a century after Wythoff's initial discovery, it remains unknown how to compute the $\mathscr{G}$-values of arbitrary coordinates efficiently. However, there have been promising advances in understanding the large-scale geometry of its $\mathscr{G}$-values.

Theorem 3.7, showing that every row of WYTHOFF is arithmetic periodic, is due to Norbert Pink [**Pin93, DFP99**]. The proof used here was given by Landman [**Lan02**].

Blass and Fraenkel showed that the positions of $\mathscr{G}$-value exactly 1 lie within a bounded distance of the lines with slope ϕ and $1/\phi$ [**BF90**]. Specifically, they showed that

$$8 - 6\phi < a_n - \phi n < 6 - 3\phi \qquad \text{and} \qquad -3\phi < b_n - \phi^2 n < 8 - 3\phi$$

where (a_n, b_n) is the n^{th} position with $\mathscr{G}$-value 1. They also gave an algorithm for calculating (a_n, b_n) and proved that this algorithm is polynomial-time *provided that* a certain regularity hypothesis (the *convergence conjecture*) is true.

Miraculously, Gabriel Nivasch has recently generalized the Blass–Fraenkel results to positions of $\mathscr{G}$-value exactly m, for arbitrary m [**Niv09**]. Nivasch showed that all such positions lie within a bounded distance of the lines with slope ϕ and $1/\phi$, with the bound depending only on m, and generalized the Blass–Fraenkel algorithm to arbitrary m.

Nivasch has also shown that every m appears exactly once along every WYTHOFF diagonal (where a **diagonal** is a set of the form $\{(a, a+k) : a \in \mathbb{N}\}$). This generalizes Exercise 3.4, which is the case $k = 0$.

An intriguing new approach was recently proposed by Eric Friedman and Adam Landsberg, inspired by renormalization techniques from experimental physics [**FL09**]. Rather than try to compute the $\mathscr{G}$-values for WYTHOFF explicitly, Friedman and Landsberg studied their large-scale geometric structure empirically and showed that it can be explained by certain recurrence relations.

Generalizations of WYTHOFF. Aviezri Fraenkel has been an active force in the development of variant rulesets that generalize WYTHOFF. Exercises 3.7 through 3.9 are all due to Fraenkel [**Fra82, FL91, FO98**]; he has also analyzed the misère version of r-WYTHOFF (Exercise 3.7) and discovered some surprising connections to continued fractions and Fibonacci sequences [**Fra84**].

Fraenkel has also proposed the following ruleset, n-HEAP WYTHOFF. On her turn, a player may either:

- remove any number of tokens from any one heap; or
- remove a_i tokens from heap i, provided that $a_1 \oplus a_2 \oplus \cdots \oplus a_n = 0$.

We denote by $a \oslash b$ the unique integer c such that $c \otimes b = a$. Likewise, we denote by $a^{\textcircled{n}}$ the n^{th} power of a in $(\mathbb{N}, \oplus, \otimes)$:

$$a^{\textcircled{n}} = \overbrace{a \otimes a \otimes \cdots \otimes a}^{n \text{ times}}.$$

Theorem 5.7 guarantees the existence of $a \oslash b$ and $a^{\boxed{1/2}}$, and it's also possible to describe them by explicit recursive constructions; see Exercise 5.7.

Exercises

5.1 The **Conway product** of G and H, denoted by $G \times H$, is defined by

$$G \times H = \{G' \times H + G \times H' + G' \times H' : G' \in G \text{ and } H' \in H\}.$$

(a) Prove the *nim-multiplication rule*: $*a \times *b = *c$, where $c = a \otimes b$.
(b) Prove that $\mathscr{G}(G \times H) = \mathscr{G}(G) \otimes \mathscr{G}(H)$ for all G and H.

5.2 Let $\Gamma =$ TURNING$(\mathcal{E})$ and $\Lambda =$ TURNING$(\mathcal{F})$ (cf. Exercise 1.7 on page 182). The **cross product** $\Gamma \times \Lambda$ is played with a two-dimensional array of $m \times n$ coins, labeled $C_{i,j}$ for $1 \le i \le m$ and $1 \le j \le n$. A legal move is to select any $\mathcal{A} \in \mathcal{E}$ and $\mathcal{B} \in \mathcal{F}$ and turn over coins $C_{i,j}$ for all $i \in \mathcal{A}$ and $j \in \mathcal{B}$, provided that the *northeast-most* coin is turned heads-to-tails. Prove that:
(a) Every position in $\Gamma \times \Lambda$ can be expressed as a disjunctive sum of single-heads positions.
(b) If coin C_i in Γ has nim value a and coin C_j in Λ has nim value b, then coin $C_{i,j}$ in $\Gamma \times \Lambda$ has nim value $a \otimes b$.
(c) In particular, the nim values of NIM $\times$ NIM are exactly given by the nim-multiplication table (Figure 5.1 on page 217).

5.3 If $k \ge 2$, then $x^{\textcircled{3}} = 2^{2^k}$ has exactly three solutions in $\mathbb{N}$.

5.4 If a is a 2-power, then the sequence

$$\langle a^{\textcircled{n}} : n \text{ is a Fermat 2-power}\rangle$$

is monotonically nonincreasing with limit a.

5.5 If n is a Fermat 2-power, then

$$n^{\textcircled{n}} = n \oplus 1 \quad \text{and} \quad (n^2)^{\textcircled{n}} = n^2 \oplus n \oplus e, \text{ for some } e \in \{0, 1\}.$$

(It is an open problem to determine e as a function of n.)

5.6 If n is a Fermat 2-power, then the $(n+1)^{\text{st}}$ nim-roots of unity are given by

$$\left(a^{\textcircled{2}} \oplus n \oplus \tfrac{n}{2}\right) \oslash \left(a^{\textcircled{2}} \oplus a \oplus \tfrac{n}{2}\right) \quad \text{for} \quad 0 \le a < n.$$

5.7 *Explicit inverses and square roots.* For all $a \in \mathbb{N}$:
(a) $1 \oslash a = \text{mex}(\mathcal{B})$, where $\mathcal{B}$ is the closure of $\{0\}$ under the mapping

$$b \mapsto (1 \oplus (a \oplus a') \otimes b) \oslash a',$$

with a' ranging over all integers $< a$.
(b) $a^{\boxed{1/2}} = \text{mex}(\mathcal{C})$, where $\mathcal{C}$ is the closure of $\left\{(a')^{\boxed{1/2}} : a' < a\right\}$ under the mapping

$$(b_1, b_2) \mapsto (b_1 \otimes b_2 \oplus a) \oslash (b_1 \oplus b_2),$$

in which (b_1, b_2) ranges over pairs of *unequal* elements of $\mathcal{C}$.

5.8 For every 2-power a, write $q(a) = (a \otimes a)/a$. Lemma 5.6(f) shows that if a is a Fermat 2-power, then $q(a) = 3/2$. Prove the following for every 2-power a:
  (a) If a is *not* a Fermat 2-power, then $q(a) > 3/2$.
  (b) If a has the form 2^{2^k+1}, then $q(a) = 13/8$.
  (c) If $a = 2^{2^k+j}$ and $a' = 2^{2^{k'}+j}$, with $j < 2^k, 2^{k'}$, then $q(a) = q(a')$. This defines a sequence indexed by j (and independent of choices of k) whose first two values are $3/2$ and $13/8$. Determine the next few values in this sequence.

5.9 If s is a Fermat 2-power, then every base-s lexicode $\mathcal{L}$ is necessarily closed under componentwise nim-product. Therefore $\mathcal{L}$ is a vector space over the finite field of order s. (See Exercises 1.8 and 1.9 for the definition of lexicode.)

Notes

Nim-product was discovered by Conway in the context of the transfinite Field $\mathbf{ON}_2$ (cf. Section VIII.4). It has been studied extensively by Lenstra [**Len78**].

Much of Lemma 5.6 emerges as a consequence of the more general theory presented in Section VIII.4. The proof given in this section is due to Knuth and is instructive for its elementary nature.

Exercises 5.2 through 5.6: Lenstra.

Exercise 5.7(b): Clive Bach.

Exercise 5.9: Conway and Sloane.

Example. Let G be tame of genus a^b. Since $*2$ is tame and firm, the addition rules for tame games give (for all $n \geq 1$)

$$\mathscr{G}^{\pm}(G + n \cdot *2) = \begin{cases} a^a & \text{if } n \text{ is even;} \\ (a \oplus 2)^{a \oplus 2} & \text{if } n \text{ is odd.} \end{cases}$$

Therefore if G is tame, then $\mathscr{G}^*(G)$ is completely determined by $\mathscr{G}^{\pm}(G)$ and we have

$$\mathscr{G}^*(G) = 0^{12} \text{ or } 1^{03} \text{ or } a^a$$

when

$$\mathscr{G}^{\pm}(G) = 0^1 \text{ or } 1^0 \text{ or } a^a,$$

respectively.

Exercises

2.1 For any integers $a, b \geq 0$, there is a game G with $\mathscr{G}^{\pm}(G) = a^b$.

2.2 Every position in **0.56** is tame.

2.3 (a) If G is restive, then $\mathscr{G}^*(G)$ is completely determined by $\mathscr{G}^{\pm}(G)$. Determine all possible extended genera for G.
(b) Give examples of restless games G and H with $\mathscr{G}^{\pm}(G) = \mathscr{G}^{\pm}(H)$, but $\mathscr{G}^*(G) \neq \mathscr{G}^*(H)$.
(c) Determine all possible extended genera for a restless game G.

2.4 If G and H are generally tame, then so is $G + H$.

2.5 If G is generally restive, then $G + G$ is generally tame with genus 0^0. Also, $G + G + G$ is generally tame with genus a^a, where $a = \mathscr{G}^+(G)$.

2.6 Theorem 2.9 is false if G is permitted to be an arbitrary tame game (rather than specifically a NIM position).

2.7 Theorem 2.9 remains true if R is generally restive.

2.8 Suppose R is generally restive of genus a^b, and let m be an integer with $1 < m < \min\{b, b \oplus 1\}$. Prove that $R + *m$ is generally tame, and determine its genus.

2.9 *The Noah's Ark Theorem.* Let G be restless. Assume that:
(i) $G + G$ is a misère $\mathscr{N}$-position; and
(ii) for every $a \geq 2$, if G has an option of genus a^a, then G also has an option of genus $(a \oplus 1)^{(a \oplus 1)}$.
Prove that $o(2n \cdot G + T) = o(T)$ for every $n \geq 0$ and tame game T. Prove furthermore that $o\big((2n+1) \cdot G + T\big) = o(G + T)$.

Notes

Almost all of the material in this section is drawn from ONAG and *Winning Ways*, but our terminology is slightly different. Conway defined the **genus** of G to be $\mathscr{G}^*(G)$, rather than $\mathscr{G}^{\pm}(G)$. This has the unfortunate effect of disguising the central point of the genus theory, which is that $\mathscr{G}^{\pm}(G)$ (rather than $\mathscr{G}^*(G)$) exactly

0	$*2_\#$	$*2_\#0$	$*2_\#20$	$*2_\#210$
$*$	$*3_\#$	$*2_\#1$	$*2_\#21$	$*2_\#320$
$*2$	$*32$	$*2_\#2$	$*2_\#30$	$*2_\#321$
$*3$	$*2_{\#\#}$	$*2_\#3$	$*2_\#31$	$*2_\#3210$
$*4$			$*2_\#32$	

Figure 3.2. The 22 distinct misère impartial games born by day 4.

Next suppose that G has birthday 3. Then G must have $*2$ as an option, and its remaining options must be among 0 and $*$. Furthermore, by the misère mex rule, the options of G must contain *neither* or *both* of 0 and $*$. This leaves just two possibilities

$$*3 = \{0, *, *2\} \quad \text{and} \quad *2_\# = \{*2\}.$$

Finally, suppose that G has birthday 4. Then G must have either $*3$ or $*2_\#$ as an option (or both). If $*2_\#$ is not an option of G, then by the misère mex rule the only possibilities are

$$*4 = \{0, *, *2, *3\}, \quad *3_\# = \{*3\}, \quad \text{and} \quad *32 = \{*2, *3\}.$$

Sixteen possibilities remain: those in which $*2_\#$ is an option of G, and G's remaining options comprise one of the 16 subsets of $\{0, *, *2, *3\}$.

Which of these are canonical? Suppose that such a G simplifies to H. Then at least one option of G reverses through H, so that $\mathrm{b}(H) \leq 2$. In particular, since $*2_\#$ has birthday 3, it cannot be an option of H and so must have H as an option. This implies $H = *2$.

Now G's options are a superset of H's, so 0 and $*$ are options of G. Moreover, $*2$ cannot be an option of G, since it is neither an option of H nor contains H as an option. We conclude that among the 16 possibilities, the only *non*-canonical games are $*2_\#10$ and $*2_\#310$. Thus there are 14 birthday-4 games with $*2_\#$ as an option. □

The full list of games born by day 4 is summarized in Figure 3.2.

Exercises

3.1 (a) Prove that $*2_\#$ is not equal to any NIM position.
(b) Exhibit a specific game X such that $o(*2_\# + X) \neq o(*2_2 + X)$.

3.2 (a) Give an example to show that $G = H$ need *not* imply $G^- = H^-$.
(b) Show that for *every* G, there is an H with $G = H$ but $G^- \neq H^-$.

3.3 True or false: If $G =^- H$, then necessarily $G =^+ H$.

3.4 Determine which games born by day 4 are tame, restive, and restless.

3.5 There are precisely 4,171,780 distinct games born by day 5. (There are 2^{22} games to consider. Subtract from 2^{22} one term for each of the five games H born by day 3. Carefully observe the proviso in the $H = 0$ case.)

3.6 If $G + H = 0$, then either $G = H = 0$ or $G = H = *$.

3.7 The *Misère Periodicity Theorem.* Let $\Gamma = \mathbf{d}_0.\mathbf{d}_1\mathbf{d}_2\ldots\mathbf{d}_k$ be an octal game of finite length k. Suppose that there exist $n_0 \geq 1$ and $p \geq 1$ such that

$$H_{n+p} = H_n \quad \text{for all } n \text{ with } n_0 \leq n < 2n_0 + p + k.$$

Then it follows that

$$H_{n+p} = H_n \quad \text{for all } n \geq n_0.$$

Notes

Grundy and Smith undertook the first systematic effort to develop a misère theory in a 1956 paper [**GrS56**]. They understood the proviso and knew about reversible moves, and they correctly calculated the number of games born by day 5 (4,171,780). Without the Simplest Form Theorem, however, they were unable to prove this answer correct.

In the 1970s Conway finally proved the Simplest Form Theorem. In *On Numbers and Games*, he wrote of this theorem:

> It was formerly known to some people as *Grundy's conjecture*, although Professor Smith informs me that in fact Grundy conjectured no such thing, and firmly believed the opposite! [**Con01**]

Conway's breakthrough verified the Grundy–Smith count of day 5 games, and using similar techniques he calculated the number of games born by day 6. The original count published in ONAG was slightly inaccurate; the corrected answer was given by Chris Thompson in 1999 [**Tho99**] and independently verified by Hoey and Siegel. For the record, there are exactly

$$\begin{aligned}2^{4171780} &- 2^{2096640} - 2^{2095104} - 2^{2094593} - 2^{2094080} - 2^{2091523} - 2^{2091522}\\ &- 2^{2088960} - 2^{2088705} - 2^{2088448} - 2^{2088193} - 2^{2086912} - 2^{2086657}\\ &- 2^{2086401} - 2^{2086145} - 2^{2085888} - 2^{2079234} + 2^{1960962} + 21\end{aligned}$$

games born by day 6. (A derivation of this result can be found in [**CS**].)

The Simplest Form Theorem also makes possible an abstract structure theory for misère impartial games (analogous to the theory of normal-play partizan games in Chapter III). Conway proved several key results in this direction. Denote by $\mathcal{M}$ the monoid of misère impartial game values; then:

Theorem (Conway's Cancellation Theorem). *$\mathcal{M}$ is cancellative: $G + J = H + J$ implies $G = H$, for all $G, H, J \in \mathcal{M}$.*

He also showed that for each G, there are just finitely many games H and J with $G = H + J$. Proofs of these results are surprisingly tricky; they can be found in Allemang's thesis [**All84**] and in [**CS**].

In [**CS**] it is shown that if $n \cdot G = n \cdot H$, then either $G = H$ or $G = H + *$. This in turn implies that $*$ is the unique torsion element of the group of fractions

Therefore

$$\Phi(a)\Phi(b-p) = \Phi(a)\Phi(b).$$

Now $b-p \neq 0$ (since $n_0 \geq 1$), so $H_a + H_{b-p}$ is an option of H_n, and it follows that $x \in \Phi[H_n]$. A similar argument works for the converse direction. □

Exercises

4.1 Determine the outcome of $K_9 + K_{15} + 3 \cdot K_{25} + K_{118}$ (where K_n denotes a KAYLES heap of size n).

4.2 A **bipartite monoid** is a pair $(\mathcal{Q}, \mathcal{P})$, where $\mathcal{Q}$ is a commutative monoid and $\mathcal{P} \subset \mathcal{Q}$ is an arbitrary subset of $\mathcal{Q}$. A **homomorphism** $f : (\mathcal{Q}, \mathcal{P}) \to (\mathcal{Q}', \mathcal{P}')$ is a monoid homomorphism $f : \mathcal{Q} \to \mathcal{Q}'$ such that $x \in \mathcal{P}$ iff $f(x) \in \mathcal{P}'$.

If there exists a surjective homomorphism $f : (\mathcal{Q}, \mathcal{P}) \to (\mathcal{Q}', \mathcal{P}')$, then we say $(\mathcal{Q}', \mathcal{P}')$ is a **quotient** of $(\mathcal{Q}, \mathcal{P})$. A bipartite monoid $(\mathcal{Q}, \mathcal{P})$ is **reduced** if it has no proper quotients. We write r.b.m. as shorthand for reduced bipartite monoid. Prove that:

(a) Every bipartite monoid $(\mathcal{Q}, \mathcal{P})$ has a unique reduced quotient (up to isomorphism). This r.b.m. is called the **reduction** of $(\mathcal{Q}, \mathcal{P})$.

(b) If $\mathscr{A}$ is a set of games, then $(\mathscr{A}, \mathscr{A} \cap \mathscr{P})$ is a bipartite monoid whose reduction is $\mathcal{Q}(\mathscr{A})$.

(c) If $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$ and $\mathscr{B} \subset \mathscr{A}$, then there is a submonoid $\mathcal{R} < \mathcal{Q}$ such that $\mathcal{Q}(\mathscr{B})$ is the reduction of $(\mathcal{R}, \mathcal{R} \cap \mathcal{P})$.

(d) If $\mathscr{A}$ is a closed set of games and $(\mathcal{Q}, \mathcal{P})$ is an r.b.m., then the following are equivalent: (i) $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$; (ii) there exists a surjective monoid homomorphism $\Phi : \mathscr{A} \to \mathcal{Q}$ such that for all $G \in \mathscr{A}$,

$$\Phi(G) \in \mathcal{P} \quad \text{if and only if} \quad G \not\cong 0 \text{ and } \Phi(G') \notin \mathcal{P} \text{ for all } G' \in G.$$

Notes

Conway's work on misère theory in the 1970s led to easy solutions for several tame octal games, including **0.56**. At the same time, other results seemed only to confirm the intrinsic complications of the general case. In ONAG Conway undertook an analysis of misère GRUNDY'S GAME, computing the outcome $o(G_n)$ up to $n = 50$ by brute-force application of the genus theory. His conclusion was rather discouraging:

> It would become intolerably tedious to push this sort of analysis much further, and I think there is no practicable way of finding the outcome of G_n for much larger n. [**Con01**]

Nonetheless others continued to press forward. Dean Allemang, in conjunction with his *generalized genus theory* (see Notes to Section 2 on page 241), formally introduced the localized misère equivalence relation (Definition 4.1) and proved a form of the Quotient Periodicity Theorem (Theorem 4.6) [**All84, All01**]. Using these and other techniques, he found solutions to several wild misère octal games, including **0.26**, **0.53**, and **4.7**.

The solution to misère KAYLES was discovered by William Sibert in 1973 by brute-force analysis. Sibert didn't make his solution public until the late 1980s, so

Exercises

5.1 Prove that if $\mathcal{K}$ is the kernel of $(\mathcal{Q}, \mathcal{P})$, then $\mathcal{K} \cap \mathcal{P} \neq \emptyset$.

5.2 $(\mathcal{Q}, \mathcal{P})$ is **regular** if $\mathcal{K} \cap \mathcal{P}$ is a singleton. Prove that Theorems 5.8 and 5.9 remain true if "normal" is weakened to "regular."

5.3 Describe the transition algebras and mex functions for $\mathrm{cl}(*2)$ and $\mathrm{cl}(*2_\#)$.

5.4 If $G \subset \mathscr{A}$ and $\Phi[G] = \mathcal{M}_x$ for some $x \in \mathscr{A}$, then $\mathcal{Q}(\mathscr{A}[G]) \cong \mathcal{Q}(\mathscr{A})$ and $\Phi(G) = x$.

5.5 Suppose that $(\mathcal{Q}, \mathcal{P})$ is normal with kernel $\mathcal{K}$. If $G \subset \mathscr{A}$ and $\Phi[G] \subsetneq \mathcal{K}$, then $\mathcal{Q}(\mathscr{A}[G]) \cong \mathcal{Q}(\mathscr{A})$ and $\Phi(G) \in \mathcal{K}$.

5.6 The *Mex Interpolation Principle.* Let $G \subset \mathscr{A}$, and suppose that there is some $H \in \mathscr{A}$ such that $\Phi[H] \subset \Phi[G] \subset \mathcal{M}_x$, where $x = \Phi(H)$. Prove that $\mathcal{Q}(\mathscr{A}[G]) \cong \mathcal{Q}(\mathscr{A})$ and $\Phi(G) = x$. Show furthermore that $\mathscr{A}$ and $\mathscr{A}[G]$ have identical mex functions.

Notes

This section largely follows the developments in [**PS08a**], with some material drawn from [**Sie13b**].

The classification theory has been extended by Siegel. With some effort, it is possible to show that $\mathcal{R}_8$ is the only quotient of order 8 and $\mathcal{T}_3$ is the only quotient of order 10. There are multiple quotients of order 12, and thereafter the quotients proliferate rapidly. The details of these and other results are found in [**Sie13b**].

Many questions about finite misère quotients remain open, including:

Question. Let $(\mathcal{Q}, \mathcal{P}) = \mathcal{Q}(\mathscr{A})$, with quotient map $\Phi : \mathscr{A} \to \mathcal{Q}$, and let $G, H \in \mathscr{A}$. Does $\Phi(G) = \Phi(H)$ necessarily imply $\mathscr{G}(G) = \mathscr{G}(H)$?

Question. If $(\mathcal{Q}, \mathcal{P})$ is a misère quotient and $\mathcal{A}$ is a maximal subgroup of $\mathcal{Q}$, must $\mathcal{A} \cap \mathcal{P}$ be nonempty?

Although the theory of finite quotients remains incomplete, still less is known about the structure of infinite misère quotients. Such quotients are commonplace: for example, a proof that $\mathcal{Q}(*(2_\#0)0)$ is infinite can be found in [**PS08a**]. Now every finitely generated monoid is finitely presented (cf. Rédei's Theorem, Theorem B.4.8 on page 472), so every finitely generated set of games $\mathscr{A}$ necessarily has a finitely presented misère quotient. In particular, this is true for every partial quotient $\mathcal{Q}_n(\Gamma)$ of every heap ruleset Γ. However, a more general structure theory for such quotients remains elusive. Even a computational finiteness test remains out of reach:

Open Problem. Specify an algorithm to determine whether or not $\mathcal{Q}(\mathscr{A})$ is infinite, given a finitely generated set of impartial games $\mathscr{A}$.

(The input to such an algorithm is a finite set of generators for $\mathscr{A}$.) The existing algorithms for calculating misère quotients [**PS08b**] *assume* that $\mathcal{Q}(\mathscr{A})$ is finite, terminating gracefully when a finite quotient is found but going into an infinite loop when $\mathcal{Q}(\mathscr{A})$ is infinite. Here's a more difficult problem:

Exercises

6.1 *Games born by day n.* Prove that:
   (a) All four games born by day 1 are pairwise incomparable.
   (b) If G and H are born by day 2 and $G \not\cong H$, then $G \neq H$. Conclude that there are 256 pairwise unequal games born by day 2. (Contrast with normal play: Theorem III.1.2 on page 154.)
   (c) There exist nonisomorphic games born on day 3 that are equal.

6.2 True or false: If $G \geq H$ in misère play, then $G \geq H$ in normal play.

6.3 Write $G \geq_S H$ if the following four conditions hold: (i) every H^L is equal to some G^L; (ii) every G^L is equal to some H^L; (iii) if H is a Left end, then so is G; (iv) if G is a Right end, then so is H. Prove that for all G and H:
   (a) If $G \geq_S H$, then $G \geq H$.
   (b) If G and H are born by day 2 and $G \geq H$, then $G \geq_S H$.
   (c) There exist G and H born on day 3 with $G \geq H$, but $G \not\geq_S H$.

6.4 Let x and y be numbers in normal-play canonical form. If $x > y$ in normal play, then $* : x > * : y$ in misère play.

6.5 Let $\mathscr{A} = \mathrm{cl}(1, \bar{1})$. Show that $\mathcal{Q}(\mathscr{A}) \cong \mathbb{Z}$ as a partially ordered monoid.

6.6 *Dicotic games in misère play.* Let $\mathscr{A} = \tilde{\mathbb{G}}^0$, the set of all short dicotic games (as defined in Section II.1).
   (a) $* + * \equiv 0 \pmod{\mathscr{A}}$.
   (b) If G is dicotic and $H + \overline{H}$ is a misère $\mathscr{N}$-position for every subposition H of G, then $G + \overline{G} \equiv 0 \pmod{\mathscr{A}}$.
   (c) A game G is **binary** if every nonempty subposition of G has *exactly* one move for each player. Suppose G is binary, and assume every alternating run of every subposition of G has length at most 3. Show that $G + \overline{G} \equiv 0 \pmod{\mathscr{A}}$. (Use (b).)

6.7 G is said to be a **dead end** if either: every subposition of G is a Left end; or every subposition of G is a Right end; or G is not an end and every proper subposition of G is a dead end. Let $\mathscr{A}$ be the set of all dead ends. Prove that:
   (a) $\mathscr{A}$ is closed under disjunctive sum.
   (b) The following are dead ends: every dicotic game; every HACKENBUSH position; every DOMINEERING position.
   (c) If $G \in \mathscr{A}$ and G is an end, then $G + \overline{G} \equiv 0 \pmod{\mathscr{A}}$.

Notes

For decades, partizan games in misère play were considered essentially intractable. Then in 2007, Mesdal and Ottaway [**MO07**] showed that every nonempty game is distinct from 0 (our Corollary 6.7), suggesting that a coherent theory is possible. Later the same year, the full theory was isolated by Aaron Siegel [**Sie13a**] in essentially the form presented here. Siegel also obtained an exact count of 256 partizan misère games born by day 2 (Exercise 6.1) and an upper bound of 2^{183} games born by day 3.

Exercises

1.1 If $G \geq 0$, then Left can win G playing second (resolving the asymmetry in Theorem 1.14 in the case $H = 0$).

1.2 If G is not equal to any loopfree game, then either some $G^L \geq G$ or else some $G^R \leq G$. (Therefore Theorem II.1.30 holds if and only if G is loopfree.)

1.3 A Left strategy σ for G is a **complete survival strategy** if, whenever Left can survive a subposition X of G playing first, then $\sigma(X)$ is a survival move and σ is a survival strategy for $\sigma(X)$. Prove that every game G admits a complete survival strategy.

1.4 For loopy games G and H, define

$$G \geq^{\sharp} H \quad \text{if} \quad \text{Left can win } G - H \text{ playing second,}$$
$$G \geq^{\flat} H \quad \text{if} \quad \text{Left can survive } G - H \text{ playing second.}$$

(Ordinary $\geq$ could be called $\geq^{\natural}$ in this context.) Show that:
(a) $\geq^{\sharp}$ is transitive but not reflexive, and $\geq^{\flat}$ is reflexive but not transitive.
(b) If G is loopfree and $G \geq^{\flat} H \geq^{\sharp} J$, then $G \geq^{\sharp} J$.
(c) G^{L_1} is **strongly dominated** if $G^{L_2} \geq^{\sharp} G^{L_1}$ for some G^{L_2}. If G' is obtained from G by removing a strongly dominated option, then $G' =^{\sharp} G$.
(d) G^{L_1} is **strongly reversible** if $G^{L_1 R_1} \leq^{\sharp} G$ for some $G^{L_1 R_1}$. If G' is obtained from G by bypassing a strongly reversible option, then $G' =^{\sharp} G$.
(e) $G \geq^{\sharp} G$ if and only if G is equal to a loopfree game. (Apply (c) and (d).)

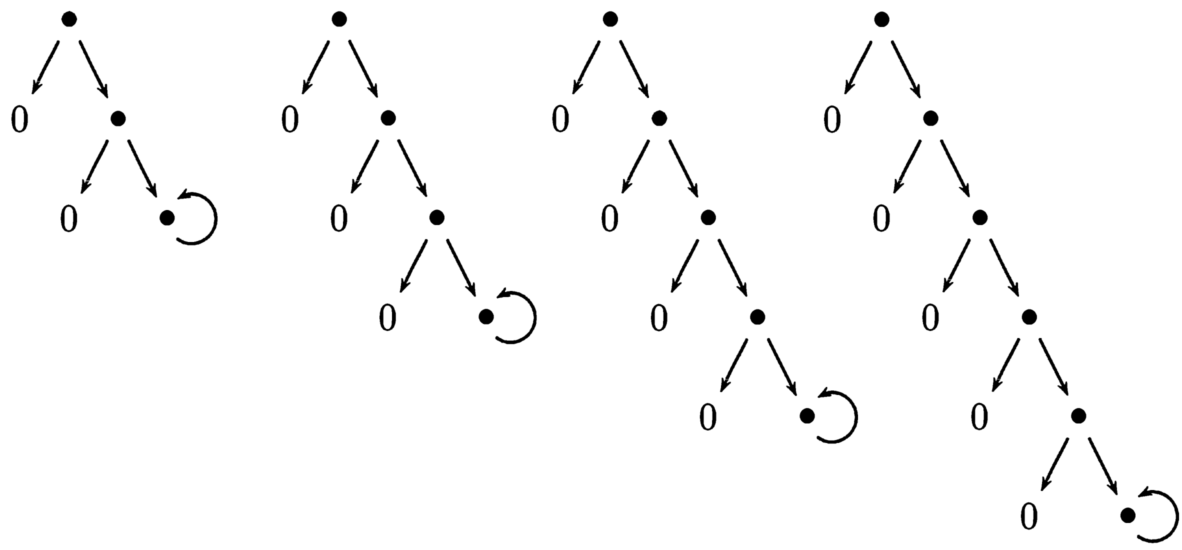
Notes

Almost immediately after Conway introduced the axiomatic theory of loopfree partizan games, efforts were underway to extend it to loopy games. The asymmetry in Theorem 1.14 was recognized early on. Fraenkel and Tassa [**FT82**] addressed this problem by defining two separate relations, as in Exercise 1.4. The Fraenkel–Tassa approach yields some useful theorems (such as Exercise 1.4(e)), but its applicability is limited.

The first hints of a more general theory appeared in a groundbreaking paper by Robert Li [**Li76**]. Li's paper inspired Conway to develop a more general theory, with the help of his students, Simon Norton and Clive Bach. These developments will be discussed in subsequent sections.

2. Stoppers

A loopy game G is a **stopper** if no subposition of G admits an infinite alternating run. For example, every position in FOX AND GEESE is necessarily a stopper: the geese can only make a bounded number of moves throughout the game, constraining the length of any alternating run.



$+_{\mathbf{on}}$ **ace** **deuce** **trey**

Figure 2.5. The games **ace**, **deuce**, **trey**, ...

and

$$*.11 < \mathbf{ace} < *.12,$$

so that $\mathrm{aw}(\mathbf{ace}) = 1$ and therefore $\mathrm{aw}(\mathbf{pip}_n) = n$. In fact it's not hard to show that

$$\mathbf{ace} - \uparrow^{[\mathbf{on}]}* = \uparrow^{\mathbf{on}},$$

so that **ace** and $\uparrow^{[\mathbf{on}]}*$ are quite close indeed.

These sorts of one-off results are easy enough to obtain, and they suggest a more general theory.

Open Problem. Extend the atomic weight calculus to (an appropriate subclass of) stoppers.

Exercises

2.1 Give examples to show that if G and H are stoppers, then $G + H$ might be any of the nine outcome classes in Figure 1.3 (page 282).

2.2 Draw game trees (as in Figure 2.3 on page 297) to illustrate the limits $\uparrow^{[\mathbf{on}]}* = \sup_n(\uparrow^{[n]}*)$ and $\uparrow^{\mathbf{on}}* = \inf_n(\uparrow^{n}*)$.

2.3 If $G > 0$ and G is dicotic, then $G \geq +_{\mathbf{over}}$.

2.4 Exhibit a decreasing sequence G_n of loopfree games with $\uparrow^{[\mathbf{on}]} = \inf_n G_n$, and an increasing sequence H_n with $\uparrow^{\mathbf{on}} = \sup_n H_n$.

2.5 Exhibit a stopper G such that $G = \sup_n(*2 : n)$. Express G as the inf of another sequence.

2.6 If G and H are loopfree and $G \simeq H$, then $G \cong H$.

2.7 True or false: There exists a stopper G such that
$$G = \sup\{0.1,\ 0.101,\ 0.10101,\ 0.1010101,\ \ldots\}.$$

2.8 A stopper P is a **pseudonumber** if for every subposition Q of P and every Q^L and Q^R, we have $Q^L < Q^R$.

(a) Assuming P to be in canonical form, prove that P is a pseudonumber if and only if $P^L \le P \le P^R$ for every P^L and P^R.

(b) Prove the *Pseudonumber Avoidance Theorem*: Suppose that P is equal to a pseudonumber and G is not. If Left has a winning (resp. survival) move on $G+P$, then she has a winning (resp. survival) move of the form $G^L + P$.

(c) Show that every pseudonumber is either a number, **on**, **off**, or a game of the form $x + \mathbf{over}$ or $x + \mathbf{under}$, where x is a number. Conclude that the pseudonumbers are totally ordered.

2.9 *Stops.* For G a finite stopper, define
$$L(G) = \begin{cases} G & \text{if } G \text{ is equal to a pseudonumber;} \\ \max\limits_{G^L}\big(R(G^L)\big) & \text{otherwise;} \end{cases}$$
$$R(G) = \begin{cases} G & \text{if } G \text{ is equal to a pseudonumber;} \\ \min\limits_{G^R}\big(L(G^R)\big) & \text{otherwise.} \end{cases}$$

Prove that:

(a) $L(G)$ and $R(G)$ are well-defined (the recursion necessarily terminates) and are independent of the form of G.

(b) $L(G) \ge R(G)$ for all G.

(c) If $L(G) = \mathbf{off}$, then $G = \mathbf{off}$; if $R(G) = \mathbf{on}$, then $G = \mathbf{on}$.

(d) The *Pseudonumber Translation Theorem*: If P is equal to a pseudonumber and G is not, then $G + P = \big\{G^L + P \bigm| G^R + P\big\}$.

2.10 *Pseuduptimals.* For all positive pseudonumbers P (cf. Exercise 2.8), we define
$$\uparrow^{[P]} = \big(* : P\big) - * \quad \text{and} \quad \uparrow^{1+P} = \big\{0 \bigm| \downarrow_{[P]}*\big\}.$$

These form a common generalization of $\uparrow^{[\mathbf{on}]}$ and the fractional uptimals from Exercise II.4.24 on page 99.

(a) Determine the canonical forms of $\uparrow^{[\mathbf{over}]}$ and $\uparrow^{1+\mathbf{over}}$.

(b) Express $\uparrow^{[\mathbf{over}]}$ and $\uparrow^{1+\mathbf{over}}$ as the sup and inf of sequences of loopfree games. (See Exercise II.4.24 on page 99.)

(c) Prove that if $P > Q \ge 1$, then $\uparrow^P \ll \uparrow^Q$.

(d) Prove that $\uparrow^{[1+P]} - \uparrow^{[P]} = \uparrow^{1+P}$, except when P is loopfree.

(e) Which pseuduptimals are absorbed by $\uparrow^P$?

Notes

Stoppers were introduced by Conway in a 1978 paper titled simply *Loopy games* [**Con78**]. The theory was expanded in *Winning Ways*, and most of the material in this section derives from one of those two sources. Note that $\uparrow^{[\mathbf{on}]}$ and $+_{\mathbf{on}}$ were called **upon** and **tiny**, respectively, in *Winning Ways*.

Exercise 2.8(c): Robert Li.

Likewise, it cannot be the case that $G_{i+1}^R \le G_i$, since this would imply that G_{i+1} is reversible through G_{i+1}^R (as a Left option of G_i). So Left must also have a winning move from $G_{i+1}^R - G_i^L$. □

Theorem 3.9 can be extended to cycles that are "almost monochromatic," having just a single move for Right.

Theorem 3.10. *Suppose that G is a stopper in canonical form. Then G contains no bichromatic cycles with just a single move for Right.*

Proof. Suppose (for contradiction) that G has such a cycle, say,

$$G_0, G_1, G_2, \ldots, G_n \quad (n \ge 1),$$

with each $G_{i+1} = G_i^L$ and $G_0 = G_n^R$. We first show that

$$G_0 \le G_1 \le G_2 \le \cdots \le G_n.$$

The proof is identical to the proof of Theorem 3.9, except when Right moves from $G_n - G_{n-1}$ to $G_n - G_n$, in which case Left must respond to $G_n - G_0$. Now if Right moves to $G_0 - G_0$, Left continues around the cycle just as before; any other move for Right loses outright, since G_n has no dominated or reversible moves.

This shows, in particular, that $G_0 \le G_{n-1}$. But $G_0 = G_{n-1}^{LR}$, contradicting the assumption that G_{n-1} has no reversible moves. □

Theorem 3.11. *If G is a stopper in canonical form, then every cycle of G is either a* 1*-cycle or has length* ≥ 4.

Proof. By Theorems 3.9 and 3.10, every cycle of G is either a 1-cycle or contains at least two moves for each player. □

Stoppers with canonical 4-cycles exist; an example is shown in Figure 3.2. (In this diagram, the arrows are carefully arranged so that moves for Left always point to the Left, but the edges are additionally labeled with "L" or "R" for clarity.) With more effort it's possible to construct examples with more elaborate cycles (see Exercise 3.5, Figure 3.3, and the notes for this section).

Exercises

3.1 In the statement of the Fusion Lemma, it is assumed that no subposition of G has any dominated or reversible options. Show by example that both hypotheses are necessary.

3.2 Prove that fusion does not introduce any dominated or reversible options.

3.3 Determine the atomic weight of each position in Figure 3.2 on the next page.

3.4 Express each position in Figure 3.2 as the sup and inf of loopfree games.

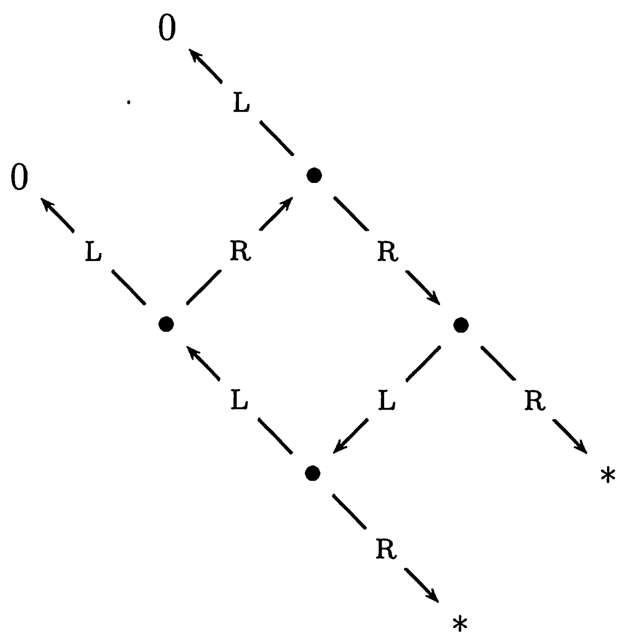


Figure 3.2. A stopper with a 4-cycle in canonical form.

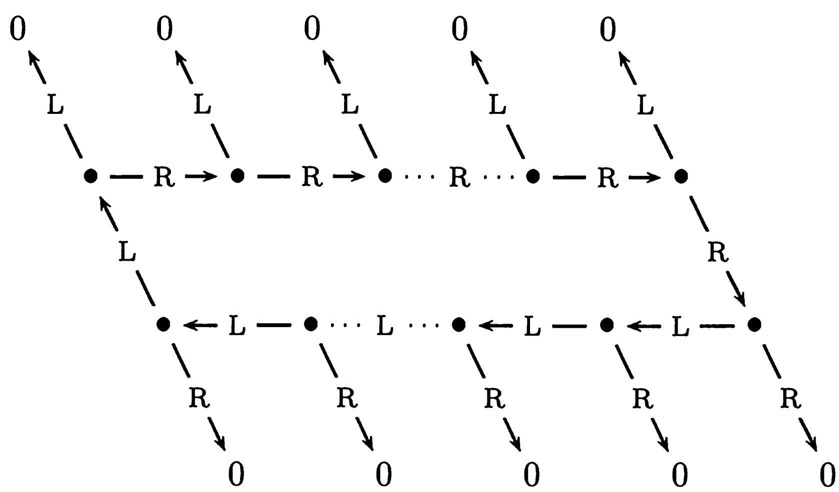


Figure 3.3. A schema for arbitrarily long canonical cycles (cf. Exercise 3.5).

3.5 Suppose that G contains a cycle with m consecutive Left edges, followed by n consecutive Right edges (so that the cycle alternates just once between Left and Right). Theorems 3.9 and 3.10 show that if G is canonical, then necessarily $m \geq 2$ and $n \geq 2$. Figure 3.2 gives an example with $m = n = 2$.

(a) Figure 3.3 illustrates a schema of such G. The dotted edges indicate variable numbers of edges, allowing for arbitrary choices of m and n. Prove that Figure 3.3 is canonical if and only if $m \geq 3$ and $n \geq 3$.

(b) Give an example of a canonical stopper exhibiting a cycle with $m = 3$ and $n = 2$.

Proposition 4.17 (A, B, C Property). *For all stoppers A, B, and C,*

$$A \mathbin{\curlywedge} B \geq C \quad \textit{if and only if} \quad A \geq \overline{B} \mathbin{\curlyvee} C.$$

Proof. If $A \mathbin{\curlywedge} B \geq C$, then certainly $A + B \mathrel{\hat{\gtrsim}} C$. So

$$\hat{o}(A + B + \overline{C}) \geq \hat{o}(C + \overline{C})$$

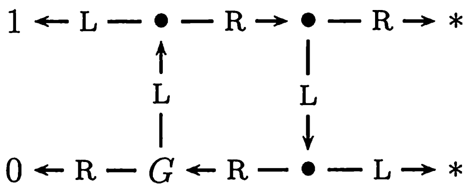
and it follows that Left can survive $A + B + \overline{C}$, playing second. Since A is a stopper, her play must concentrate on $B + \overline{C}$, so by Theorem 4.4

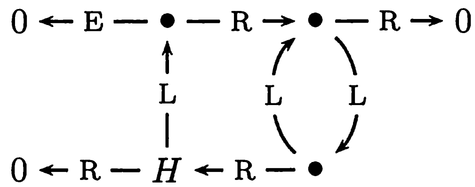
$$A \mathrel{\check{\gtrsim}} \overline{B} + C.$$

Therefore $A \geq \overline{B} \mathbin{\curlyvee} C$. The converse is given by the same argument in reverse, using the fact that $\overline{C}$ is a stopper. □

Exercises

4.1 Determine the onside and offside of the following games G and H. Express all answers in canonical form.





4.2 The **height** of a plumtree G is the longest sequence of nonpass moves proceeding from G. Determine the number of distinct plumtrees of height ≤ 2.

4.3 If G and H are plumtrees, then $G + H$ is stopper-sided.

4.4 G is a **zugzwang game** if $G^L < G < G^R$ for every G^L and G^R. Prove that if G is a zugzwang game, then $G = x \mathbin{\&} y$ for some $x, y \in \mathbb{D}$.

4.5 G is a **weak zugzwang game** if $G^L \leq G \leq G^R$ for every G^L and G^R. Prove that if G is a weak zugzwang game, then $G = S \mathbin{\&} T$ for some pseudonumbers S and T (cf. Exercise 2.8 on page 300).

4.6 (Cf. Propositions 2.6 and 4.12) Let G be an arbitrary game and let T be a stopper. Suppose that $T \geq H$ for every subposition H of G. Show that $\Sigma_n^T(G) \geq G$ for all n. (Show that any strategy for $G + X$ can be converted into one for $\Sigma_n^T(G) + X$.)

4.7 *Grafting plumtrees.* Let S and T be stopper plumtrees, with $S \geq T$. Let

$$R = \left\{ S^L,\ \hat{d}(T^L) \,\middle|\, \check{d}(S^R),\ T^R \right\}$$

where for all G,

$$\hat{d}(G) = \{G^L \mid G^R, \mathbf{dud}\} \quad \text{and} \quad \check{d}(G) = \{G^L, \mathbf{dud} \mid G^R\}.$$

Prove that $R = S \mathbin{\&} T$.

4.8 *Onside simplification.* A Left option G^{L_1} is **onside-dominated** if $G^{L_2} \mathrel{\hat{\gtrsim}} G^{L_1}$ for some other Left option G^{L_2}. Likewise, a Right option G^{R_1} is **onside-dominated** if $G^{R_2} \mathrel{\hat{\lesssim}} G^{R_1}$ for some other Right option G^{R_2}, *provided that* no alternating cycle contains the move $G \to G^{R_2}$.

G^{R_1} is **onside-reversible** if $G^{R_1L_1} \hat{\geq} G$ for some $G^{R_1L_1}$. Likewise, G^{L_1} is **onside-reversible** if $G^{L_1R_1} \hat{\leq} G$ for some $G^{L_1R_1}$, *provided that* no alternating cycle contains the sequence of moves $G \to G^{L_1} \to G^{L_1R_1}$.

(a) If G' is obtained from G by eliminating an onside-dominated option from some subposition of G, then $G' \hat{=} G$.

(b) If G' is obtained from G by bypassing an onside-reversible option from some subposition of G, then $G' \hat{=} G$.

(c) State and prove the corresponding definitions and results for offside simplification.

(d) Suppose G' is obtained from G by fusing Y to X, where $X \hat{=} Y$ and there is no even-length alternating path from X to Y. Show that $G' \hat{=} G$.

Notes

Decomposition into sides was first recognized by Robert Li [**Li76**]. Li studied only **zugzwang games** in which it is a disadvantage to move (Exercises 4.4 and 4.5), generalizing ordinary numbers to the loopy context.

On reading Li's paper, Conway suspected a more general theory. Together with Norton and Bach, he developed the theory of stoppers and showed that they naturally represent the sides of many games. These results were published in a groundbreaking paper titled simply *Loopy games* [**Con78**] and also in *Winning Ways*. Later, Moews generalized the sidling technique and applied it to Go [**Moe93, Moe96b**].

The proof of the Sidling Theorem given here is previously unpublished (as far as the author knows) and is somewhat simpler than the proof in *Winning Ways*. In many cases, however, the Sidling Theorem fails to isolate the sides of G, even when G is stopper-sided. There are several additional techniques that are more successful in practice, such as *onside simplification* (Exercise 4.8) and *unraveling* [**Sie05, Sie09c**], but none is known to be fully general.

Open Problem. Give an algorithm to determine whether an arbitrary loopy game G is stopper-sided and to calculate its sides if it is.

Games that have been successfully analyzed using this theory include Backsliding Toads and Frogs [**Sie09a**] (a loopy variant of Toads and Frogs described in *Winning Ways*) and Hare and Hounds [**Sieb**]. Albert and Siegel developed an effective implementation of sided simplification [**Sie05, Sie09c**] that works in *most* cases and is incorporated into *cgsuite*.

Conway's version of the theory is somewhat different from ours. In Conway's framework, a game G comes equipped with a mapping p that associates to each infinite play $\vec{G}$ an outcome $p(\vec{G}) \in \{\mathscr{L}, \mathscr{D}, \mathscr{R}\}$. Therefore infinite plays are not necessarily draws, but depend on the annotation p. An (annotated) game is **free** if $p(\vec{G}) = \mathscr{D}$ for every infinite play $\vec{G}$, and **fixed** if $p(\vec{G}) \neq \mathscr{D}$ for every such $\vec{G}$. Then we define

$$G = H \quad \text{if} \quad o(G + X) = o(H + X) \text{ for all } X,$$

as usual, but G, H, and X range over all *annotated* games, and $o(G)$ observes the annotation of G in the case of infinite plays. We define G^+ (resp. G^-) to be the

Proof. Uniqueness is clear, since Theorem 5.12 shows that the value of a stable G depends only on its class and variety.

For existence, fix a stable class $[G]$ and put

$$H = (G \mathbin{\hat{+}} D) \mathbin{\check{+}} V.$$

Then certainly $[H] = [G]$, since $G \mathbin{\hat{+}} D \geq H \geq \overline{G} \mathbin{\check{+}} D$. Moreover, from Theorem 5.13 we have

$$V(H) = (V(G) \mathbin{\hat{+}} D) \mathbin{\check{+}} V = D \mathbin{\check{+}} V = V,$$

so that H has class $[G]$ and variety V. $\square$

Exercises

5.1 *More varieties of* $\uparrow^{\mathbf{on}}$.
   (a) Using *cgsuite*, investigate all sums of the four varieties
$$\{1 + G \mid \uparrow^{\mathbf{on}} \,\|\, \downarrow_{\mathbf{on}} \mid -1 + H\}$$
   as G and H range over $\uparrow^{\mathbf{on}}$ and $\downarrow_{\mathbf{on}}$.
   (b) Let $G = \{0 \mid \uparrow^{\mathbf{on}} \,\|\, \downarrow_{\mathbf{on}}\}$. Determine its variety V. Show that
$$G,\ G \mathbin{\hat{+}} G,\ G \mathbin{\hat{+}} G \mathbin{\hat{+}} G,\ \ldots$$
   exhibits a sequence of strictly increasing varieties of $\uparrow^{\mathbf{on}}$,
$$V < V \mathbin{\hat{+}} V < V \mathbin{\hat{+}} V \mathbin{\hat{+}} V < \cdots.$$

5.2 Classify all *dicotic* varieties of **over**.

5.3 If D is an idempotent and $D < \mathbf{on}$, then $D \leq \mathbf{over}$.

5.4 If $G^{\circ} = 0$, then G is necessarily equal to a loopfree game. (Apply Exercise 1.4(e) on page 289.)

5.5 (a) $G \mathbin{\check{+}} \mathbf{+}_{\mathbf{on}} = G$ for all G, unless G is equal to a loopfree game. (Use Exercise 5.4.)
   (b) $G \mathbin{\check{+}} \mathbf{+}_{\mathbf{over}} = G$ for all dicotic G, unless G is equal to a loopfree game.

5.6 Every game of degree $\mathbf{+}_{\mathbf{on}}$ has the form $G + \mathbf{+}_{\mathbf{on}}$ or $G + \mathbf{-}_{\mathbf{on}}$, with G loopfree.

5.7 For each $m \geq 2$, let $\mathbf{star}_m = \{0 \,\|\, 0, *m \mid 0, \mathbf{pass}\}$. Show that:
   (a) $\mathbf{star}_m$ is an idempotent that absorbs $*m$ and $\uparrow^2$, but it is confused with $*a$ for every $a \geq 1$ with $a \neq m$.
   (b) If G is an idempotent and $G \geq \mathbf{star}_m$ for every $m \geq 2$, then $G \geq \mathbf{over}$.

5.8 Let $D \geq 0$ be an idempotent and suppose that $G^{\circ} \leq D$ (but is not necessarily equal to D). Then G is **D-stable** if $(G \mathbin{\hat{+}} D) \mathbin{\check{+}} (\overline{G} \mathbin{\hat{+}} D) \geq 0$. If G is D-stable, then we say $\hat{V}_D(G)$ and $\check{V}_D(G)$ are the **upsum** and **downsum varieties** of G, respectively (here we write the subscript D for clarity, to emphasize the dependence on D).
   (a) $\hat{V}_D(G)$ and $\check{V}_D(G)$ are necessarily varieties of degree D.
   (b) If G is D-stable, then $\hat{V}_D(G) \leq \check{V}_D(G)$, but the inequality might be strict.

(c) If G and H are D-stable and $H^\circ = D$, then
$$\hat{V}(G \mathbin{\hat{+}} H) = \hat{V}_D(G) \mathbin{\hat{+}} \hat{V}(H) \quad \text{and} \quad \check{V}(G \mathbin{\check{+}} H) = \check{V}_D(G) \mathbin{\check{+}} \check{V}(H).$$
So G behaves like it has variety $\hat{V}_D(G)$ in upsums with games of degree D, or variety $\check{V}_D(G)$ in downsums.

Notes

Almost all of the material in this section is based upon *Winning Ways*, where it was first introduced. Almost none of it was actually proved there, and I know of no other sources that exist at the time of this writing.

It's a fascinating theory—but, like the theory of sides in Section 4, it is incomplete. For one thing, it is unknown whether the sum of finite stoppers is necessarily stopper-sided—so that all of the applications of upsum and downsum in this section are, in the most general case, suspect. On top of this, there is of course the Stability Conjecture itself. Until these questions are resolved, the best that we can offer is a rich structure theory for stable games whose degree is known to be a stopper.

Within this context, there are nonetheless many interesting questions worth pursuing. The specific class and variety structure of various idempotents deserves more attention. For some idempotents, such as **on** and $+_{\mathbf{on}}$, there are simple answers (see Exercise 5.6, for example); but most appear to have a more intricate structure.

Open Problem. Investigate the class and variety structure of various idempotents such as $\uparrow^{\mathbf{on}}$, $\mathbf{star}_m$, and $\spadesuit$.

Here $\mathbf{star}_m$ is defined in Exercise 5.7, and $\spadesuit$ is given by
$$\spadesuit = \{0^2 \;\|\; \{\mathbf{on} \mid 0^4\}\}.$$
A partial analysis of the structure of $\spadesuit$ appears in Chapter 11 of *Winning Ways*, but there is doubtless more to be discovered. This is a theory that's ripe for further exploration.

Proof of Theorem 1.10. When $t = -1$, this reduces to Proposition 1.11. For $t > -1$ the proof is identical to the $t > 0$ case from Lemma 1.7. □

Exercises

1.1 If x is a number, then $L_{\text{full}}(x) = \lceil x \rceil$ and $R_{\text{full}}(x) = \lfloor x \rfloor$.

1.2 Fix G and $\delta > 0$. Suppose $t(H)$ is an integer multiple of 2δ, for every subposition H of G. Prove that $L_t^\delta(G) = L_t(G)$ and $R_t^\delta(G) = R_t(G)$ whenever t is an integer multiple of 2δ. (In particular, the limits in Theorem 1.6 converge to a constant for all sufficiently small δ.)

Notes

The orthodox theory was introduced by Berlekamp in a 1996 paper, *The economist's view of combinatorial games* [**Ber96**]. Berlekamp sought a playable Go variant in which orthodox play can be enforced. The classical temperature theory, defined in terms of cooling rates set by auction, proved to be unsuitable for this purpose—professional Go players balked at the idea of paying a "tax" in order to move. Berlekamp's insight was that paying a tax of t points is essentially equivalent to declining a coupon of temperature t, leading to the equivalent formulation of the temperature theory in terms of coupon stacks.

A subsequent paper on AMAZONS [**Ber00b**] presented a more formal exposition of the orthodox theory, including a proof of Theorem 1.6. The proof given here is a modification of a simpler proof due to Bewersdorff [**Bew04**].

The term *fullstop* is new to this book.

2. Orthodoxy

Theorem 1.6 (and its generalization, Theorem 1.10) serves as the essential basis of the orthodox theory: optimal play of G in the "typical" environment $\mathscr{E}_t^\delta$ coincides exactly with the classical thermography of Section II.5, with an error of at most δ. The central ideas of the temperature theory can therefore be used to define the main ingredients of orthodoxy.

It will sometimes be tedious to keep track of the δ terms explicitly, so we'll use the shorthand $\mathscr{E}_t$ for the limiting value as $\delta \to 0$. For example,

$$L(G + \mathscr{E}_t) \quad \text{is shorthand for} \quad \lim_{\delta \to 0} L(G + \mathscr{E}_t^\delta).$$

Definition 2.1. Let G be a short game. A Left option G^L is said to be **orthodox at temperature** t (or simply t**-orthodox**) if

$$R_t(G^L) - t = L_t(G).$$

is guaranteed the orthodox forecast following **sentestrat**; and by symmetry neither can hope to do better.

Exercises

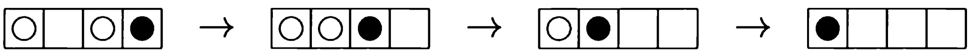
2.1 (a) Give an example of a game G with an orthodox option G^L that is canonically senseless (cf. Exercise II.2.6 on page 68).
(b) Show that for every t there is a Left option of G that is both sensible *and* t-orthodox.
(c) True or false: There exists a G with an orthodox option G^L that is Inf-senseless (cf. Exercise II.6.11 on page 135).

Notes

Most of the material in this section is due to Berlekamp [**Ber96, Ber02**]. Details of its origin are described in the notes to Section 1; a summary of its extensions to loopy games can be found in Sections 3 through 5.

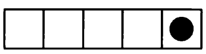
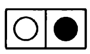
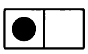
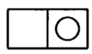
3. Generalized Temperature

PUSH is played with black and white tokens on a $1 \times n$ strip. Left may slide any black token one space to the left. If the space is occupied by another token, then it is pushed to the left along with any tokens immediately behind it. Tokens pushed off the board are removed from the game. In the example below, Left can push the white tokens off the board in three moves:

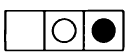
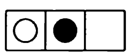


Likewise, Right may slide any white token one space to the right, pushing any tokens in its immediate path.

PUSH exhibits some familiar values. A single black token at the far end of a $1 \times n$ strip obviously has value n, and it's easy to construct a ± 1:

 $= 5,$  $= \Big\{$  $\Big|$  $\Big\} = \pm 1.$

There are also loopy values. Consider the following position:

$K =$  $= \Big\{$  $\Big| -1 \Big\} = \{1 \mid K \parallel -1\}.$

Here K has an option K^L from which Right can return to K. This situation is called a **ko**. We'll draw the game graph of K using an arc to illustrate

(Either $t-\delta$ is a hill temperature, so that $L_{t-\delta}(G) = \tilde{\lambda}_{t-\delta}(G)$, or else $\lambda_t(G) > L_{t-\delta}(G)$, in which case we can apply Proposition 4.8(d).) Therefore

$$\begin{aligned} L_t^\delta(G) &\geq \tilde{\lambda}_t(G) - 2k(n-1)\delta \\ &\geq \lambda_t(G) - 2kn\delta. \end{aligned}$$

(iii) $L_t^\delta(G) \geq \tilde{\lambda}_t(G) - 2kn\delta$ whenever $n = \tilde{s}^L(G)$. Fix a Left option G^L such that $\tilde{\lambda}_t(G) = \rho_t(G^L \setminus G) - t$; then

$$\begin{aligned} L_t^\delta(G) &\geq R_t^\delta(G^L \setminus G) - t \\ &\geq \rho_t(G^L \setminus G) - t + 2kn\delta && \text{((i), inductively)} \\ &= \tilde{\lambda}_t(G). \end{aligned}$$ □

In light of Theorem 4.11, we'll drop the notation $\lambda_t(G)$ and $\rho_t(G)$ in favor of $L_t(G)$ and $R_t(G)$. We'll also write $\tilde{L}_t(G)$ and $\tilde{R}_t(G)$ in place of $\tilde{\lambda}_t(G)$ and $\tilde{\rho}_t(G)$. The main consequences of the thermographic calculus are summarized as follows:

Theorem 4.16 (cf. Theorem II.5.11 on page 108). *For all simple G:*

(a) *$L_t(G)$, $R_t(G)$, $m(G)$, and $t(G)$ are well-defined.*

(b) *$L_t(G)$ and $R_t(G)$ are piecewise linear in t. Every segment has integer slope.*

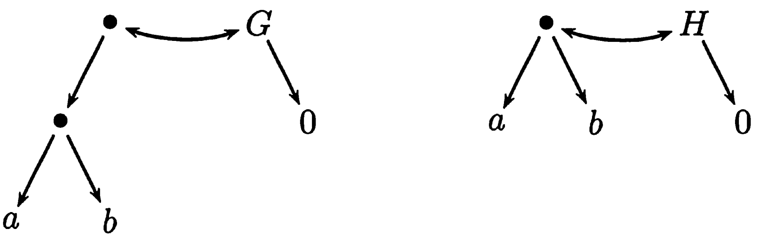
(c) *$L_t(G) \geq R_t(G)$ for all t.*

Proof. These are all immediate consequences of Theorem 4.11. □

Exercises

4.1 For each example on pages 356–360, use the thermographic calculus to prove that its thermograph matches the asserted diagram.

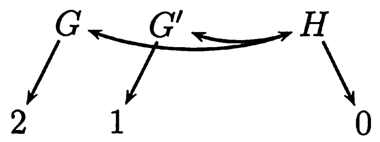
4.2 Let $a, b \in \mathbb{D}$ with $a > b > 0$. Determine mast values and temperatures for the games G and H shown below, in terms of a and b:



4.3 Exhibit a game G whose scaffolds match Figure 4.2(a) on page 367.

4.4 *Multiple ko from the same position.*

(a) Compute thermographs of the games G, G', and H shown at right (cf. Figure 3.10).



(b) What if the exits to 2 and 1 are replaced by arbitrary simple games X and Y? Extend to a general analysis of such positions.

4.5 Prove that the conditions in Proposition 4.8(a)–(d) uniquely characterize the thermographic intersection.

4.6 (a) For every $x \in \mathbb{Q}$ and every $t > -1$, there is a simple game G with $m(G) = x$ and $t(G) = t$.

(b) However, if G is any game with $t(G) = -1$, then $m(G)$ is an integer.

4.7 Fix a cave temperature t of the scaffold $S = (l, r)$. For $x \in [l_t, r_t]$, define the **quenching temperature** at x, denoted $q(x)$, as the *smallest* temperature such that (t', x) is still in the cave, for all $t' \in [q(x), t]$. (Envision a ball dropped into the cave from (t, x). The ball travels straight down through the cave, and $q(x)$ is the temperature at which it finally collides with the cave wall.)

Prove that in the thermographic intersection of S, the mast value at t is equal to the unique x of lowest quenching temperature, except possibly when some $q(x) = -1$.

Notes

Generalized thermography was introduced by Berlekamp [**Ber96**] and has been explored by Berlekamp, Müller, and Spight [**BMS96**], Spight [**Spi99, Spi02**], and others, mainly in the context of specific applications to Go.

There is obvious room for improvement in the scope of the theory. The calculus presented here applies only to simple games; but $L_t(G)$ and $R_t(G)$ are *defined* for all G, so it is a well-defined problem to extend the calculus to encompass more complicated cycles. William Fraser made partial progress in his doctoral thesis [**Fras02**], but no fully general method is known.

Open Problem. Develop a thermographic calculus that works for all loopy games.

The generalized temperature theory was invented in order to better understand Go. Other examples of games with koban—Push (introduced in Section 3) and the related game Woodpush (described in *Lessons in Play*)—seem rather contrived. While advances in Go are major achievements, it would be valuable to have some other examples to which the temperature theory can be meaningfully applied.

Open Problem. Invent some new loopy games that play well under a koban restriction, as a source of further examples for the generalized temperature theory.

5. Komaster Thermography

The thermographic calculus presented in Section 4 correctly determines $L_t(G)$ and $R_t(G)$, and therefore $m(G)$ and $t(G)$, for all simple games G.

the key battle occurs on K—and no matter how large the value of n, Left will have just finitely many threats with which to fight this battle, whereas Right effectively has infinitely many. We see that

$$L_t^{\sharp\sharp}(H+K) = R_t^{\sharp\sharp}(H+K) = 4$$

so the thermograph of $H+K$ is always the same, regardless of the threat environment! The presence of H gives Right a sort of "super-komonster" status that is resilient even against actual komonster assumptions.

However, this status only works at certain temperatures, for suppose we replace K by a hotter variant and consider the sum

$$H + K', \quad \text{where} \quad K' = \{8 \mid K' \parallel 0\}.$$

In this case, if Left opens by moving to $H + (K')^L$ and Right then plays to $H^R + (K')^L$, Left will simply ignore the threat on H and complete the ko on K'. The infinitely many threats granted by H all have bounded temperature, and in this respect they are *weaker* than the threats in a true komonster environment. This type of situation is still not fully understood.

Exercises

5.1 Determine the biased thermographs of each example on pages 356–360.

5.2 Convert the proof sketch of Theorem 5.19 into a formal induction.

5.3 If G is simple, then $t(G) \le \min\{t^\sharp(G), t^\flat(G)\}$.

5.4 *Orthodox accounting with one hyperactive component.*

(a) Let $G_1, \ldots, G_k$ be simple. Suppose that G_1 is hyperactive and each of $G_2, \ldots, G_k$ is placid. Suppose Left plays first (second) on the compound $G = G_1 + \cdots + G_k + \Theta$. Show that Left can guarantee a score of at least
$$z + \frac{1}{2}\sum_{i=1}^{m} \Delta t_i + \sum_{j=1}^{n} \Delta u_j,$$
where Δt_i are the signed temperature drops, Δu_j are the signed ko adjustments, and
$$z = m^\sharp(G_1) + m(G_2) + \cdots + m(G_k) \pm t/2,$$
with the sign of $t/2$ depending on who moves first. (Use the following modification of sentestrat: Left assumes that Right will win any ko on $G_2 + \cdots + G_k$, but she fights any ko on G_1 using threats from Θ. Use the fact that $m(G_i) = m^\flat(G_i)$ for all $i \neq 1$, just as in the proof of Theorem 5.16.)

(b) Conclude that on $G_1 + \cdots + G_k + \mathscr{E}_t^\delta + \Theta$, either player can achieve the biased orthodox forecast z, to within a bounded multiple of δ.

(c) Why doesn't this work if more than one component is hyperactive?

Lattice Structure of PG

The lattice structure of $\mathbb{G}_\alpha$ generalizes the structure of finite $\mathbb{G}_n$ in a straightforward way (Exercise 1.9). Conversely, the structure of $\mathbb{G}^0_\alpha$ is less clear. The preceding discussion of infinitesimals can be generalized to show that if $\alpha \geq \omega$, then $\mathbb{G}^0_\alpha$ has no maximal element (Exercise 1.10). Since for $1 \leq n < \omega$, $\mathbb{G}^0_n$ has exactly two maximal elements ($\uparrow(n{-}1)$ and $\uparrow(n{-}1)*$), this shows that the $\alpha \geq \omega$ case is fundamentally different.

Open Problem. Describe the partial-order structure of $\mathbb{G}^0_\alpha$, for $\alpha \geq \omega$.

Exercises

1.1 For every long game G, there is a K such that $K = G$ and K has no reversible options.

1.2 If $\alpha > \beta$, then $\boldsymbol{+}_\alpha \ll \boldsymbol{+}_\beta$.

1.3 Determine the smallest positive game born by day α, for all ordinals α. (There are three separate cases, depending on whether α is a limit ordinal; a limit ordinal $+ 1$; or an ordinal of the form $\beta + n$ for $n \geq 2$.)

1.4
 (a) $\uparrow^{[\alpha]} + \uparrow^{\alpha} + * = \{0 \mid \uparrow^{\alpha}\}$ for all $\alpha \in \mathbf{ON}$.
 (b) If $\alpha > \beta$, then $\uparrow^{\alpha} \ll \uparrow^{\beta}$, for all $\alpha, \beta \in \mathbf{ON}$.
 (c) Write out the options for $\uparrow^{[\omega]}*$ and $\uparrow^{\omega}*$.
 (d) $\uparrow^{[\alpha+1]} - \uparrow^{\alpha+1} = \uparrow^{[\alpha]}$ for all $\alpha \in \mathbf{ON}$. What can one say about $\uparrow^{[\omega]} - \uparrow^{\omega}$?

1.5 If α is an ordinal of the form ω^{β}, then $\bigcup_{\xi<\alpha} \mathbb{G}_\xi$ is a group.

1.6 Show that $\infty + \infty = \{\mathbb{N} \parallel \mathbb{N} \mid 0\}$.

1.7 A **Dedekind section** of **SN** is a pair of Classes $(\mathbf{X}, \mathbf{Y})$ with $\mathbf{X} \cup \mathbf{Y} = \mathbf{SN}$, such that $x > y$ for all $x \in \mathbf{X}$, $y \in \mathbf{Y}$. For $G \in \mathbf{PG}$, the **Left** and **Right sections** of G are defined by

$$\mathbf{L}(G) = \big(\{x \in \mathbf{SN} : x \geq G\}, \{x \in \mathbf{SN} : x \mathrel{\lhd\!\!\mid} G\}\big),$$
$$\mathbf{R}(G) = \big(\{x \in \mathbf{SN} : x \mathrel{\mid\!\!\rhd} G\}, \{x \in \mathbf{SN} : x \leq G\}\big).$$

A section $(\mathbf{X}, \mathbf{Y})$ is **numeric** if it has the form $\mathbf{L}(x)$ or $\mathbf{R}(x)$, for some surreal number x.
 (a) If $G \notin \mathbf{SN}$, then $\mathbf{L}(G) = \sup\big(\mathbf{R}(G^L)\big)$ and $\mathbf{R}(G) = \inf\big(\mathbf{L}(G^R)\big)$, where $\sup(\mathbf{X}_i, \mathbf{Y}_i) = (\bigcup \mathbf{X}_i, \bigcap \mathbf{Y}_i)$ and $\inf(\mathbf{X}_i, \mathbf{Y}_i) = (\bigcap \mathbf{X}_i, \bigcup \mathbf{Y}_i)$.
 (b) The Left and Right sections of every short game are numeric.
 (c) If x is equal to a number and G is not, *and* if the Left and Right sections of G are numeric, then $G + x = \{G^L + x \mid G^R + x\}$.
 (d) For all G, there is a *set* $\mathcal{A} \subset \mathbf{SN}$ such that $\mathbf{L}(G)$ can be expressed as either the sup or the inf of the sections of elements of $\mathcal{A}$.

1.8 Define

$$\int^{\mathbb{N}} G = \begin{cases} G & \text{if } G \text{ is equal to a number;} \\ \left\{ n + \int^{\mathbb{N}} G^L \;\middle|\; -n + \int^{\mathbb{N}} G^R \right\}_{n \in \mathbb{N}} & \text{otherwise.} \end{cases}$$

(a) True or false: If $G > 0$, then $\int^{\mathbb{N}} G > n$ for all $n \in \mathbb{N}$.
(b) Let $G = \int^{\mathbb{N}} \uparrow$. Investigate the relative size of $\infty - G$.

1.9 (a) $\mathbb{G}_\alpha$ is a complete distributive lattice, for all α.
(b) Describe the join-irreducibles of $\mathbb{G}_\alpha$ in terms of $\bigcup_{\beta<\alpha} \mathbb{G}_\beta$.

1.10 (a) Every element of $\mathbb{G}^0_\omega$ is $\leq {\uparrow}n$ for some n.
(b) Find an increasing sequence of games $U_n \in \mathbb{G}^0_{\omega+1}$ such that every element of $\mathbb{G}^0_{\omega+1}$ is $\leq U_n$ for some n.
(c) Generalize to higher $\mathbb{G}^0_\alpha$.

1.11 *Universal embedding property for Abelian groups.*
(a) The generalized Norton product (cf. Exercise II.7.15 on page 149) extends to long games, with exactly the same definition. Show that for all $r, s \in \mathbb{R}$ and all $U > 0$, we have $(r+s) \cdot U = r \cdot U + s \cdot U$, and $r \geq s$ if and only if $r \cdot U \geq s \cdot U$.
(b) Let $G = \frac{2}{n} \cdot \{2x \mid x\} - \frac{3x}{2}$, with $n \geq 2$ and $x \in \mathbf{SN}$. Show that $n \cdot G = 0$ and that $k \cdot G \mathrel{|\!\rhd} y$ for $1 \leq k < n$ and all $y < \frac{x}{n}$.
(c) Prove Theorem 1.27, cited below. (By Zorn's Lemma, it suffices to assume that $\mathcal{B}$ is generated by $\mathcal{A} \cup \{a\}$, for some a. If $na \notin \mathcal{A}$ for any $n > 0$, then define $g(na)$ to be a sufficiently large ordinal. Otherwise, let $n \in \mathbb{N}$ be least with $na \in \mathcal{A}$, fix an H with $n \cdot H = f(na)$, and define $g(na)$ to be $H + J$, where J is given by (b) for some sufficiently large α.)

1.12 *Surreal pseudonumbers.* A **transfinite loopy game** is a loopy game whose graph might have infinitely many vertices. A transfinite stopper G is a **pseudonumber** if for every subposition H of G and every H^L and H^R, we have $H^L < H^R$ (cf. Exercise VI.2.8 on page 300). Show that:
(a) If G is a pseudonumber, then $G^L \leq G \leq G^R$ for each G^L and G^R.
(b) The pseudonumbers are totally ordered by $\geq$.
(c) If $\mathcal{A}$ is a set of pseudonumbers, then $\hat{\mathcal{A}} = \{\mathcal{A} \mid \mathbf{pass}\}$ is a pseudonumber and is a least upper bound for $\mathcal{A}$.
(d) The ENTREPRENEURIAL CHESS position in Figure I.4.4(b) (on page 32) has value **on** & $\hat{\mathbb{Z}}$.

Notes

Most of this section essentially paraphrases ONAG. There has also been substantial further progress in characterizing the abstract structure of **PG**. First David Moews proved that **PG** has the universal embedding property for Abelian groups [**Moe02**]:

Theorem 1.27 (Moews). *Suppose that $\mathcal{B}$ is an Abelian group (whose domain is a set) and let $\mathcal{A} < \mathcal{B}$ be a subgroup. Then for any embedding $f : \mathcal{A} \to$ **PG** of Abelian groups, there is an embedding $g : \mathcal{B} \to$ **PG** extending f.*

It is straightforward to show that this characterizes the structure of **PG** as an Abelian group, providing a sort of high-level analogue to Theorem III.3.19

So

$$\begin{aligned}\mathcal{M}_{\omega+1} = \quad & \left(\mathbb{R} + \left\{-\tfrac{1}{\omega}, 0, \tfrac{1}{\omega}\right\}\right) \\ & \cup \left(\mathbb{D} + \left\{-\tfrac{2}{\omega}, -\tfrac{1}{2\omega}, \tfrac{1}{2\omega}, \tfrac{2}{\omega}\right\}\right) \\ & \cup \left(\{-\omega, \omega\} + \{-1, 0, 1\}\right).\end{aligned}$$

Exercises

2.1 If n is an integer in canonical form and G is any long game, then

$$G \times n = \overbrace{G + G + \cdots + G}^{n \text{ times}}.$$

2.2 $\mathcal{O}_\alpha$ is a group if and only if $\alpha = \omega^\beta$ for some β.

2.3 Determine all elements of $\mathcal{M}_{\omega+2}$.

2.4 There exist long games G, H, and J such that $G = H$, but $G \times J \neq H \times J$.

2.5 *Explicit square roots.* Let $x \geq 0$, and define the **square root** y of x by

$$y = \left\{ \sqrt{x^L},\ \frac{x + y^L y^R}{y^L + y^R} \;\middle|\; \sqrt{x^R},\ \frac{x + y^L y^{L'}}{y^L + y^{L'}},\ \frac{x + y^R y^{R'}}{y^R + y^{R'}} \right\},$$

where $\sqrt{x^L}$ and $\sqrt{x^R}$ are defined recursively and $(y^L, y^{L'})$ and $(y^R, y^{R'})$ range over pairs chosen so that the relevant denominator is nonzero. Prove that $y^2 = x$.

Notes

Conway arrived at the surreal numbers in the early 1970s, almost by accident; the fascinating story of their discovery is told in detail in Appendix C. They quickly captured the imagination of mathematicians around the world. In an interesting twist, they first appeared in print in a 1974 novella by Donald Knuth [**Knu74**], who also coined the term "surreal number." This is a rare instance of a substantial new mathematical theory first introduced in print by a work of fiction!

In 1976 Conway finally published ONAG, which includes much of the essential material on surreals and remains a standard reference. A more formal treatment was later given by Harry Gonshor [**Gon86**]. Norman Alling's book [**All87**] employs a different approach, with an emphasis on classical analytic and topological properties of **SN**.

The proof of Proposition 2.3 given here is due to Schleicher and Stoll [**SS06**]; the remainder of our treatment mostly follows ONAG. More recent advances are discussed in the notes to Section 3 on page 437.

Exercise 2.5: Clive Bach.

3. The Structure of Surreal Numbers

Having established that **SN** has a Field structure, we can proceed to investigate its algebraic and analytic properties. The key result is that every

Now let

$$g(x) = x^n + \sum_{i=0}^{n-2} x^i r_i.$$

Certainly $g(x)$ has a root $a \in \mathbb{R}$ (since $\mathbb{R}$ is real-closed), so we can write

$$(\dagger) \qquad g(x) = g_1(x)g_2(x),$$

with $g_1(x) = (x-a)^m$ and a not a root of $g_2(x)$. Since $c_{n-1} = 0$, it cannot be the case that $g(x) = (x-a)^m$, so $\deg(g_2) \geq 1$.

Next consider the ring $\mathcal{R} = \mathbb{R}[\zeta_0, \ldots, \zeta_{n-2}]$. The completion of $\mathcal{R}$ over the ideal $(\zeta_0, \ldots, \zeta_{n-2})$ is the ring

$$\hat{\mathcal{R}} = \mathbb{R}[[\zeta_0, \ldots, \zeta_{n-2}]]$$

consisting of all formal power series in the variables $\zeta_0, \ldots, \zeta_{n-2}$ over $\mathbb{R}$. By classical results in commutative algebra (cf. Hensel's Lemma [**Eis95**]), the factorization $(\dagger)$ of g lifts to a factorization of f,

$$f(x) = f_1(x)f_2(x),$$

such that $\deg(f_i) = \deg(g_i)$, and the coefficients of f_i are in $\hat{\mathcal{R}}$. But $\zeta_0, \ldots, \zeta_{n-2}$ are infinitesimals, so by Theorem 3.24, all their power series converge. Therefore the coefficients of f_i are in **SN**.

Since f has odd degree, one of f_1 or f_2, say f_1, must have odd degree. But $1 \leq \deg(f_1) < n$, so by induction f_1 has a root in **SN**. Therefore f has a root in **SN**. $\square$

Exercises

3.1 Express $\sqrt[3]{\omega+1}$ and $1/\sqrt[3]{\omega+1}$ in normal form.

3.2 **SN**$[i]$ is (up to isomorphism) the unique algebraically closed Field whose domain is a proper Class. (Here $\mathbf{SN}[i] \cong \mathbf{SN}[x]/(x^2+1)$.)

3.3 *Universal embedding property for totally ordered fields.* Suppose $\mathcal{F}$ is a totally ordered field (whose domain is a set) and let $\mathcal{E} < \mathcal{F}$ be a subfield. Then for any embedding $f : \mathcal{E} \to \mathbf{SN}$ of totally ordered fields, there is an embedding $g : \mathcal{F} \to \mathbf{SN}$ extending f.

3.4 *Universal embedding property for fields.* Suppose $\mathcal{F}$ is a field (whose domain is a set) and let $\mathcal{E} < \mathcal{F}$ be a subfield. Then for any embedding $f : \mathcal{E} \to \mathbf{SN}[i]$ of fields, there is an embedding $g : \mathcal{F} \to \mathbf{SN}[i]$ extending f.

3.5 *Surreal number theory.* x is an **omnific integer** if $x = \{x-1 \mid x+1\}$. The Class of omnific integers is denoted by **OZ**.

(a) The following are equivalent, for all $x \in \mathbf{SN}$: (i) $x \in \mathbf{OZ}$; (ii) $\pi_0(x)$ is an integer, and $\pi_y(x) = 0$ for all $y < 0$; (iii) $\sigma_x(\alpha) = \sigma_x(\alpha+1)$ for all $\alpha \in \mathbf{ON}$.

(b) **OZ** is a Subring of **SN**, and every $x \in \mathbf{SN}$ is the quotient of two omnific integers.

(c) *Euclidean algorithm.* For any $a, b \in \mathbf{OZ}$ with $b > 0$, there are unique $q, r \in \mathbf{OZ}$ with $a = bq + r$ and $0 \le r < b$.
(d) $(\omega, \omega\sqrt{2})$ is a nonprincipal ideal of $\mathbf{OZ}$.

3.6 A surreal number x is a **generalized epsilon number** if $\omega^x = x$ (cf. Definition B.3.16 on page 468). x is **reducible** if all the exponents in its normal form have strictly smaller birthday than x, and **irreducible** otherwise.
(a) The following are equivalent: (i) x is irreducible; (ii) x is a generalized epsilon number; (iii) $\mathrm{b}(x)$ is an epsilon number.
(b) For $y \in \mathbf{SN}$, define $\epsilon(y)$ recusively by
$$\epsilon(y) = \Big\{\alpha,\ \epsilon(y^L)+1,\ \omega^{\epsilon(y^L)+1},\ \omega^{\omega^{\epsilon(y^L)+1}},\ \ldots \ \Big|\ \epsilon(y^R)-1,\ \omega^{\epsilon(y^R)-1},\ \omega^{\omega^{\epsilon(y^R)-1}},\ \ldots\Big\}$$
where α ranges over all ordinals $< \epsilon_0$. Prove that x is a generalized epsilon number iff $x = \epsilon(y)$ for some y, and show that $\mathrm{b}(x) = \epsilon(\mathrm{b}(y))$.
(c) x is a **quasi-epsilon number** if $\omega^{-x} = x$. Show that every quasi-epsilon number is infinitesimal, and find a recursive enumeration $\epsilon^{-1}(y)$ of the quasi-epsilon numbers (analogous to $\epsilon(y)$ for ordinary epsilon numbers). What can we say about $\mathrm{b}(x)$?

Notes

Most of the material in this section is due to Conway, and our treatment largely recapitulates ONAG, with substantial elaboration. The proofs of Theorem 3.11 and Lemmas 3.18 and 3.22 (mostly glossed over by Conway) are due to Gonshor [**Gon86**].

Further progress has been sporadic. Notably, Martin Kruskal found a suitable definition for $\exp(x)$ that has the expected analytic properties, which was further explored by Gonshor (see [**Gon86**] for discussion). Writing $\log(x)$ for its inverse, one can then make the usual definition $x^y = \exp(y \times \log(x))$. Note that this is *not* the same notion of exponential as the ω-power used in the normal form.

Gonshor has also made the following conjecture:

Conjecture. $\mathrm{b}(x \times y) \le \mathrm{b}(x) \times \mathrm{b}(y)$ for all $x, y \in \mathbf{SN}$.

Although the analogue for addition ($\mathrm{b}(x+y) \le \mathrm{b}(x) + \mathrm{b}(y)$) is completely trivial, Gonshor's conjecture has proved to be surprisingly resistant to attack, and the best-known bounds for $\mathrm{b}(xy)$ are quite poor. Ehrlich and van den Dries have made partial progress, by proving Gonshor's conjecture in the case where x and y are both of the form $\omega^z \times r$ [**vdDE01**].

In the same paper, Ehrlich and van den Dries showed that $\mathcal{O}_\alpha$ is a ring if and only if $\alpha = \omega^{\omega^\beta}$ for some β; and a field if and only if α is an epsilon number. Moreover, each such field is closed under Kruskal–Gonshor exponentiation. The proofs of these results are surprisingly intricate, despite their deceptive similarity to Exercise 2.2 (which is elementary).

Exercises

4.1 Prove the identity $\alpha_7 = \omega + 1$. (First show that no field of order 2^{2^k} can contain an element of order 7, and use this to prove that every integer has a seventh nim-root in $\mathcal{P}_\omega$. Then show that $\mathcal{P}_4(\omega)$ has order exactly 64 and that ω has a seventh nim-root in this field, but $\omega + 1$ does not.)

4.2 If $\mathcal{P}_\gamma < \mathcal{P}_\delta$ are fields, then $\delta = \gamma^\alpha$ for some α. Moreover if $\alpha = 4$, then $\mathcal{P}_{\gamma^2}$ is also a field.

4.3 If $p = 2^{2^k} + 1$ is a Fermat prime, then $\alpha_p = 2^{2^k}$.

4.4 For each integer $n \geq 2$, let $\kappa_n = \min\{\alpha \in \mathcal{P}_\tau : n \text{ divides } \deg(\alpha)\}$, where $\deg(\alpha)$ denotes the degree of the irreducible polynomial of α over $\mathcal{P}_2$.

(a) Show that this definition is consistent with the earlier definition of κ_q (when q is a prime power).

(b) Let $n \geq 2$, let p be the smallest prime dividing n, and let q be the largest power of p dividing n. Prove that
$$\kappa_n = \begin{cases} \kappa_{n/q} & \text{if } q \text{ divides } \deg(\kappa_{n/q}); \\ \kappa_{n/q} + \kappa_q & \text{otherwise.} \end{cases}$$

(c) Conclude that every κ_n can be written uniquely as a finite sum
$$\kappa_n = \kappa_{q_1} + \kappa_{q_2} + \cdots + \kappa_{q_k},$$
where $q_1, q_2, \ldots, q_k$ are distinct maximal prime-power divisors of n.

4.5 For any $\alpha \in \mathcal{P}_\tau$, the multiplicative group of $\mathcal{P}_\omega(\alpha)$ is generated by
$$\{\alpha \oplus m : m < \omega\}.$$
(Let f be the irreducible nim-polynomial of α over P_ω. Let g be any nim-polynomial with coefficients in P_ω; if $g(\alpha) \neq 0$, then f and g are relatively prime. Use the theorem—you don't have to prove it!—that for every sufficiently large integer n, there exists an irreducible monic polynomial h of degree n with $g \equiv h \pmod{f}$. Show that if $n = 2^k$, then h splits over ω and therefore $g(\alpha)$ can be written as a product of ordinals of the form $\alpha \oplus m$.)

4.6 Let $f(p) = \deg(\zeta_p)$, where ζ_p is a primitive p^{th} nim-root of unity. Let κ_n be as in Exercise 4.4. Prove that for all primes p:

(a) $\alpha_p \geq \kappa_{f(p)}$. (First show that $\zeta_p \in \ker(h)$, where $h : \mathcal{P}_2(\alpha_p) \to \mathcal{P}_2(\alpha_p)$ is given by $h(x) = x^{\boxed{p}}$. Conclude that $f(p)$ divides $\deg(\alpha_p)$.)

(b) $\alpha_p \leq \kappa_{f(p)} \oplus m$ for some integer $m \geq 0$. (Show that some $\beta \in \mathcal{P}_2(\kappa_{f(p)})$ is not a p^{th} nim-power in κ_p, and use Exercise 4.5.)

(c) $\alpha_p = \kappa_{f(p)} \oplus m$ for some integer $m \geq 0$.

(d) The value of α_p can be effectively determined.

Notes

The Field $\mathbf{ON}_2$ was introduced by Conway, who proved the Simplest Extension Theorem along with Lemma 4.4. The treatment here essentially follows ONAG, though with some changes of notation.

Remarkably, the notion of "algebraic simplicity" inherent in $\mathbf{ON}_2$ closely parallels the historical cognitive development of mathematics: the ranking established by Theorem 4.3 is essentially the order in which each algebraic operation came to