



IME-USP

Universidade de São Paulo - Instituto de Matemática e Estatística
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HOT GAMES

Temperature, advantage and numbers

by

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Abstract

Combinatorial games are called hot when both players want to make the next move and, typically, hot games lead to interesting play. Before studying this class of games, the text will present the fundamental ideas. The first four chapter of the thesis presents and explains the concepts involved with the analysis of combinatorial games, focused in short partisan games. In this first half, there is great concern with the correspondence between games and numbers and how games might not be numbers.

The following chapter make use of all content to provide insight into more advanced ideas and discuss basic applications of the definitions. The sixth chapter, then, works the way up to the proof of a most important theorem and provides its result for a game that permeates the text. The text finishes by providing some of the possible next steps and pointing to a few references.

Resumo

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The following chapter make use of all content to provide insight into more advanced ideas and discuss basic applications of the definitions. The sixth chapter, then, works the way up to the proof of a most important theorem and provides its result for a game that permeates the text. The text finishes by providing some of the possible next steps and pointing to a few references.

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1

Introduction

“I learned very quickly that playing games and working on mathematics were closely intertwined activities for him, if not actually the same activity. His attitude resonated with and affirmed my own thoughts about math as play, though he took this attitude far beyond what I ever expected from a Princeton math professor, and I loved it.”

Manjul Bhargava ¹

It is no surprise that avid players of games that resemble logical or mathematical puzzles, like checkers, develop an intuition that allows them to calculate faster. This intuition comes in many forms like asserting bad moves fast and recognizing losing, drawing and winning patterns. A most essential, and sometime very hard, component of playing well any of these mathematical games is being able to know if you are ahead or behind in a given position.

While evaluating which player a position favors is already a hard task, this ability is not enough to play well in the games this text showcases. Consider the following variant of the game of chess: each player is given a set of board positions, and each should choose one board to play as white. During this game, play will take place in each board in parallel, and, whoever checkmates the opponent faster, wins the game. If one wants to be a great player of this variant, asserting if a position is winning or losing in a regular chess game is not enough, nor is the ability to play regular chess perfectly.

The most important ability for this variant and for the games that will follow in this text is to score each position. Scoring a position is different from spotting which one is better from a range of options. To simplify, for these first few pages, the reader should assume that scoring a position is the same as labeling it with a real number. If one can label different position in chess, playing the proposed variation becomes easy. In this day and age, making a classifier that performs well is chess, or the proposed variant, and many other games of this sort is reality and that is largely undisputed. However, a method to make a perfect classifier or at least proving a position is better than another is not widespread.

The ability to precisely calculate the advantage a player has in a position is the object of interest of Combinatorial Game Theory. This theory provides a means of labeling all positions in games, not just in chess, but in all combinatorial games. A

¹Fields medalist commenting on John H. Conway’s passing

position in Go and a position Checkers, two different games, might have the same label and that means they are both equally good or bad. The modern approach to combinatorial games was inaugurated in 1976 in the book *On Numbers And Games*, but there are studies that date back from the 1930s [?]. The author of that book, Jonh Horton Conway, as found in the epigraph that starts this text, was, as many, an avid player of such games. In fact, Conway tells that the event that led to him invent, or discover, this theory was watching two Go players playing an endgame.

Conway realized that some positions behaved like numbers, in every aspect. While, initially, that seems very useful to evaluate them, Conway went ahead and proposed that some positions **are** numbers. Later the reader will be presented to the idea that all numbers are games, not just the reals. This **TODO: match** is further explained in the following chapters and is the main topic in this field of study.

As stated before, there are games that are numbers, but not real. In fact the games that are numbers formed a new set, to be analyzed in the next sections. These numbers will not look like numbers at first. However, after visiting how they add up together in the most natural way, how they form an enormous set from extremely simple rules and other great characteristics the reader might start appreciating them. The Surreal Numbers², name given by Donald Knuth in *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness*.

As some might understand from the title alone, however, the focus of this text are in the non-numbers. It is possible for games to not behave like surreal number, although every surreal number has correspondent³ games. The concept of temperature, and by consequence all the simpler concepts like hotness and coolness, however, does not forego the understanding of numbers. The reader will find that all games become numbers after some moves, so understanding them is paramount.

After a vista on both numbers and non-numbers, this text has two chapters targeted on exercising the concepts learned, visiting fun games to play and proofs on classes of games, including a new result from 2019 which is the first of its kind. The text as a whole will make the case that Combinatorial Game Theory is built upon extremely simple but powerful concepts. The few concepts are considered powerful because they not only provide a vast field of problems but allow simple proofs for them as well.

The next three sections will present the basics of Combinatorial Game Theory, describing numbers and non-numbers. The style of these sections will be similar to that of the book *Winning Ways for your Mathematical Plays*[?]. It means that concepts, notation and theorems are not highlighted or enumerated and, instead, their meaning are present in the text. This style fits well a text in Combinatorial Game Theory because most most of them are extremely simple and few of them are abstract. In this field, it is easy to write examples of the concepts so there is no necessity of abstracting as much as other areas of algebra or combinatorics.

The style of the book is also widely known for being non-rigorous where it does not have to be. The book will commonly use images of games that satisfy assertion and provide, or not, a logic why other examples also satisfy that. This is not followed as much in this text, justifications may be provided where not required.

On the other side of the spectrum there is *Combinatorial Game Theory*[?]. Siegel is likely the most active researcher of the field nowadays and created the

²Originally Conway only called them numbers, but greatly appreciated the name given by Knuth

³There are infinite games that are equal to any Surreal Number

most developed tool used to analyze them, CGSuite. Unlike the authors of *Winning Ways for your Mathematical Plays*, he gave a great deal of form to the field. The book contains hundreds of definitions, notations, theorems and lemmas which are all numerated and highlighted. One of the results is that his standards became widely used and that helps reading current papers. In general the book is more advanced and is the primary reference for chapters 5 and 6.

Between the two there is the book *On Numbers and Games*[?][?], and possibly all others. The first edition of the book, as stated before, gave birth to the modern approach of Combinatorial Game Theory. The book is much more theoretic, containing much more algebra and number theory, than the successors but is also extremely descriptive. There are other great books, but the content of this text references only to those books.

Lastly, it may be worth observing that the text does not contain, nor does it intend to, everything there is in the field. In fact, it does not even shows everything there is about hot games. It may, however, serve as reference to short games and will bring all the fundamental concepts of this class of games. There is a list of content that was omitted and approaches that were not discussed in the final chapter. Notice, however, that short games form a massive class of games and many of the fun games are short, so studying them first is typical and hopefully, it sparks interest in the remaining areas of the field.

2

What to do with pen, paper and a friend

The Rules

In order to study mathematical plays and answer the many questions they raise, a new mathematical field of study was developed and many new terms were created. The phrase “mathematical play” is in itself a new term, for instance. While the most common term is “Combinatorial Games”, the canonical reference for this field *Winning Ways for Your Mathematical Plays* 1981, uses the former, not the latter. As more is said about the topic, more meaning the term “mathematical play” is going to acquire.

The name “Combinatorial Game” might bring to light some information. It, at least, means that this field will deal with games, as in, an instance of a Game Theory problem, and, more specifically, a subset of those games. It also brings to light that the use of counting, finite structures and, most likely, graph representations are omnipresent. However, a definition of the object of interest becomes possible with the name mathematical play.

To play something mathematically could be understood as to engage in an activity in which the better use of mathematical ability, such as counting and logic, would result in advantage over its poor use. However it could be detailed further to an activity in which mathematical ability is the single defining factor. The later might make more sense because there are games, like poker, that do require some counting ability. However, luck and reading behavior skills are much more valuable to a successful game and this is something the definition would be better off forbidding.

Chance moves, like throwing a dice or flipping a card, are not fit for mathematical plays. Even with their removal, however, there are possibilities that would not be comfortably called mathematical plays. The nature of a mathematical plays is that both players can engage the same activity and generate advantages out of “good play”. For instance, it would be hard to agree that two people playing rock-paper-scissors are battling a mathematical fight, even though there are no chance moves.

It is very important that all players have **complete information** of the position. Games like rock-paper-scissors, in which players take action simultaneously, block complete information. Therefore, players must **move alternately**. The last

concerning factor in discerning mathematical from non-mathematical plays during this analysis is the number of players.

When each player has more than one opponent a goal greater than gaining advantage arises. When playing with over two people it is frequent that the best move is not the one that brings a better position but one that prevents any of the opponents from gaining an winning advantage. While that can be very mathematical, there is a clear distinction between sticking to two player games and allowing any number of players. Notice that one can consider soccer as a two player game - even though there are multiple agents in a team. In order to focus on the mathematical ability to make the best move, the option to allow only **two players** is the most interesting.

The only remaining criteria of this definition, as established in [TODO: WW], is related to the term play, and not the term mathematical. The rules of the game must guarantee that from any starting position, **TODO: play should always end because a player will not have moves available**. If following “normal play” convention, a player that cannot move is lost. It is correct to assume normal play, unless specified otherwise, in this field of study.

The foundations of mathematical plays, highlighted, give light to a complex and rich set of problems, although, at the same time, other complex and rich problems are left behind. The game of chess, for example, does not meet the ending condition and, therefore, is left out. Fortunately, games like chess might benefit from these studies with adaptations or additional rules (although they do not consist of good examples of combinatorial games).

Let us consider the following game:

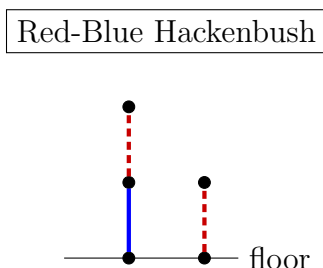


Figure 2.1

In RB-Hackenbush, a move is made by taking a single colored edge of the image and removing any edges that become disconnected from the floor. The player called **Left** can only remove **bLue and soLid**, edges, and the other, called **Right**, **Red and dashED** edges. It is a common practice to assume that all games are played between Left (blue) and Right (red), although, occasionally, new colors will be presented. In this text, every color also has a matching style.

The RB-Hackenbush is a game by the routine definitions of game, meaning it has a clear ruleset and potential to be fun. It is also a game by the definition above, which will be the new “routine” one used going forward. However, the instance of this game drawn above is also called a game. In the future, when analyzing a board state for example, the word position may seem to make more sense, but the correct word is game. The meaning of the word must be deduced from context.

The **game tree** term refers to the routine meaning, for example. The image below is the game tree that arises from the game, instance, presented above.

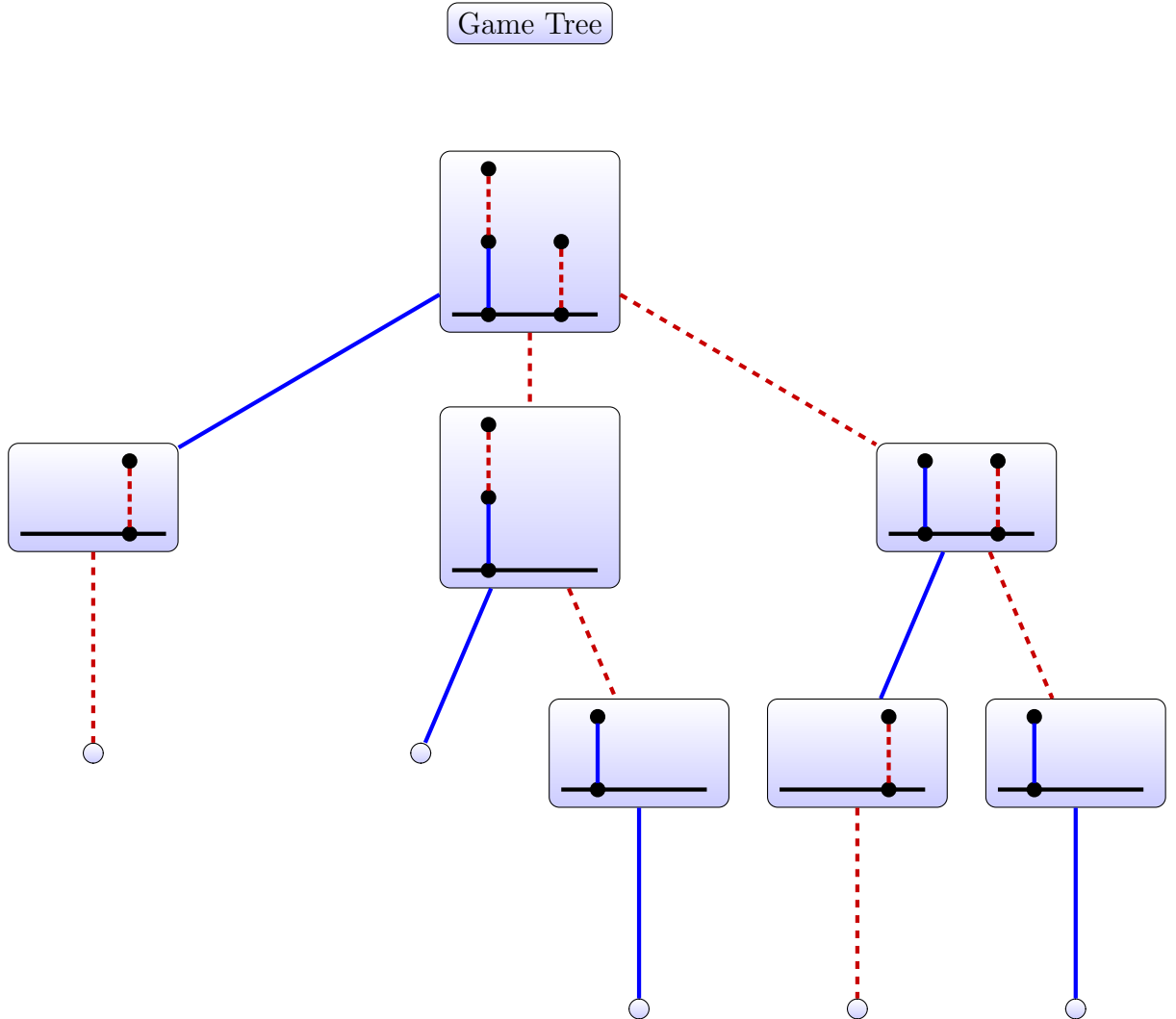


Figure 2.2

In the tree above the styled edges, in the same pattern as before, between configurations tells which player made a move. The game trees used in **TODO: CGT** may be slightly different as they, from each configuration, develop the moves from both players. This is a most important characteristic because it allows the sum of trees.

As noted previously, the game tree contains all the information required in order to calculate the number, or non-number, a game is equal to. The Surreal Numbers **TODO: KNUTH** have a recursive nature that can be completely separated from games. However, as they were created analyzing a position like the one above **TODO: ONAG**, its definition will be presented in terms of moves.

The model Conway, Berlekamp and Guy created to analyze games is based on finding the advantage a player has over the other. The calculation, in this model, of this advantage is given in terms of spare moves. For now, the reader may find this weird because the advantage might actually rely on the ability of the players, but when analyzing games it is important to expect both players play perfectly. Since a player loses if he or she cannot move, counting spare moves is counting how many sequential moves a player can make before reaching **equity** in the position.

Equity is found in **zero positions**. Zero position are those in which the first player to move loses. The idea to call such positions zero made sense for Conway, and, therefore, in his new set of numbers, if a game G is in a zero position, $G = 0$. If left **can win** regardless of who starts, we call it positive, and, in the other way around, negative. In more special positions, a hint on the topic of this text, in which the first to play can win, we call them fuzzy.

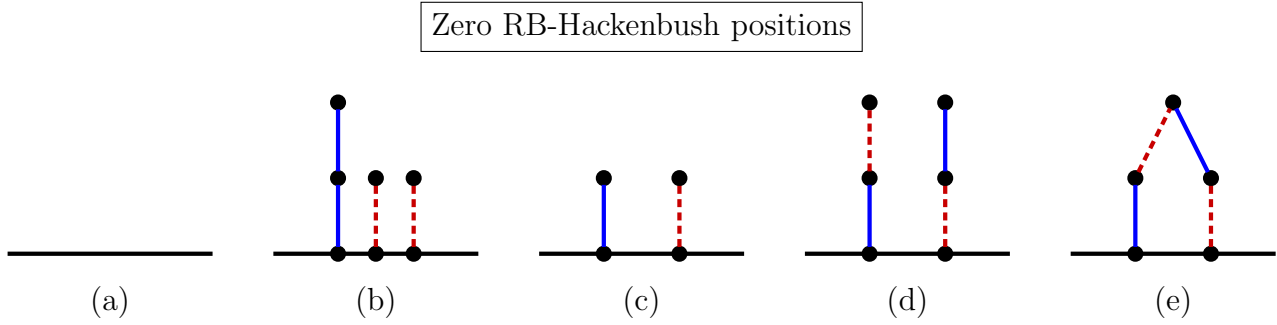


Figure 2.3

Although all games have the same value, the games are not the same. The way to represent a game derives from its game tree. The game is composed of two sets of games and $|$ is used as delimiter. The game (a) is one where neither player has available moves and, because of that, the game $(a) = 0 = \{0 | 0\}$. (c), on the other hand, is $\{\{ | \{ | \}\} | \{\{ | \}\} | \{\}\}$, that simplifies to $\{\{ | 0\} | \{0 | \}\}$. The games that form each of left and right sets are the configurations reachable by left and right, in a recursive definition.

The notations is exactly the same for numbers, but they should not be confused. For a game $G = \{x_1, \dots, x_n | y_1, \dots, y_m\}$ to be a number, it must be true that:

$$\forall x_i \in X, \forall y_j \in Y, x_i < y_j$$

The process of finding the number a game is equal to is the theme of the next section. With that said, some new number already showed up in the figure above. The number $\{0 | \}$, for example. What would be a pleasant real number for it? 1, because in this game, left has exactly one move to spare.

For the next few concepts a new game must be presented as RB-Hackenbush is incapable of generating fuzzy positions. However, a proof that all such games are numbers is due, and will be provided in the next section.

Domineering

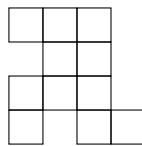


Figure 2.4

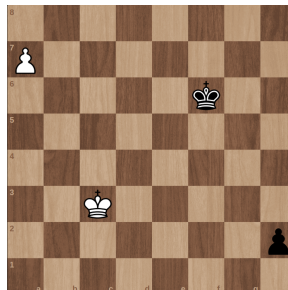
Domineering is played by placing, or marking, a 2x1 rectangle on the board, or drawing. Left plays in the vertical and right in the horizontal. The reader is invited to play the position above a few times, and realize that the player starting it is always able to win. In fact, with perfect play, both players can win with a move to spare. The quest for calculating the advantage a player has, so far, can be summarized by the question: “Who is ahead, by how much?”. However, in positions such as the one above, a new question is more important: “How big is the next move”. **TODO: Berlekamp 1.**

This second question is answered in temperature theory. Temperature theory must follow a detailed explanation and clear understanding of numbers and games, and, therefore, a good explanation will only take place in section 4. However, as this is the main tool used to develop the topic of this text, an idea will follow.

Temperature measures the activity of the position. The activity can be understood as the importance of the next moves. In chess, for example, closed positions would be colder than endgames positions. In the position below, being the next player to move is not that important. Both players would start improving the position of the pieces until one find a break-through. Being a move behind means less piece development but the game will progress slowly, reducing the impact of that.



In **hot** positions, opposed to **cold** position, the next moves are paramount for **both** players. In the board below, the player who moves first has a clear path to victory. It might be time to re-iterate that chess in itself is not a combinatorial game, although the following position can be considered one, as there are no drawing chances. In this specific position an subset of chess that removes stalemate from the ruleset is equivalent to chess, so it is possible to make it fit the definition.



It is correct to imagine that when putting a hot game G in the notation $G = \{X \mid Y\}$, then, $\exists x_i, y_j \mid x_i \geq y_j$, making it obvious that G is not a number. On the closed position, the game might be cooler, but it will definitely heat up after a few moves. The endgame position above however, is becoming a number after the next move. Because it is becoming a number, it receives the special name **switch**.

Switches are the most basic non-numbers. A switch is a non-numbers G that both left's and right's best moves are numbers, but $G^L \geq G^R$. It is extremely easy to find the **temperature** and **bias** of such games.

A Simple Switch in Domineering

$$G = \begin{array}{|c|c|c|} \hline & \square & \square \\ \hline \square & \square & \\ \hline \end{array}$$

Figure 2.5

In this game left has a move that leads to a zero position and right has two moves that lead to the same game with value -1. Therefore, $G = \{0 \mid -1\}$. The bias is the average of G^L, G^R that is equal to -0.5 in this case. The temperature is how much the game differs from the bias that is 0.5 in this case. A better way of writing this switch is $G = -0.5 \pm 0.5$. In general, a switch $H = \{x|y\}$ can be written as $H = (x + y)/2 \pm (x - y)/2$.

Calculating these values for general non-numbers is not simple enough to be completely conveyed at this point; Not because it is too hard or that describing it does not fit in the introductory pages, but because better understanding of numbers and arithmetic is required.

3

Numbers are games. The reals, the ordinals and many others.

“And behold! When the numbers had been created for infinitely many days, the universe itself appeared. And the evening and the morning were the \aleph day. And Conway looked over the rules he had made for numbers, and saw that they were very, very good. And he commanded them to be for signs, and series, and quotients, and root. Then there sprang up an infinite number less than infinity. And infinities of days brought forth multiple orders of infinities”

Donal Ervin Knuth, Surreal Numbers ¹

Conway, by counting the advantage, a player has over the other in a combinatorial game, unveiled a new way to discover numbers. It is clear at this point that numbers are not enough to represent games, but it should in no way discourage anyone interested on them. Until this point, the text showed some instances of the zero and hinted at 1 and -1, and the reader may also know how to build any integer in RB-Hackenbush and Domineering. This chapter shows that knowing that is no more than scratching the surface of surreal numbers.

For the first parts of this section a number $\{x_1, x_2, \dots \mid y_1, y_2, \dots\}$ might be called like a real number: 2, 5, 100000, $\frac{1}{3}$, $\sqrt{10}$, π , but there is no reason believe this equality yet. There is also no reason to believe that $1 < 3$ or that $1 + 1 = 2$ yet. However, first some numbers will be labeled and only then the proofs are shown.

Numbers Generated in Finite Steps

It is known that $G = \square\square = \{ \mid \{ \mid \} \} = \{ \mid 0 \} = -1$. What about $H = \begin{array}{c} \square \\ \square \end{array} \square$? If it were to be labeled by a real number, which one should it be?

We can find that by calculating $G + H + H$. To do that, one would usually find the game tree of $G + H + H \begin{array}{c} \square \quad \square \\ \square \quad \square \end{array}$, and fill the known values bottom-up. However, it is simpler in this case. $G + 2H = 0$, because whoever starts loses. Since $G = 1$, $H = 1/2$. Therefore $H = \{-1, 0 \mid 1\} = \{0 \mid 1\} = 1/2$. It might seem weird

¹**TODO: styke**

As one gets used to this area of mathematics, it becomes clear that analyzing the game $\{1 \mid \}$ is the same as analyzing $\begin{array}{|c|c|} \hline & \\ \hline \end{array}$, but the former is not reliant on a specific game. Rather, any combinatorial game has an instance equal to $\{1 \mid \}$. However, the rules, which have not been presented yet, come from the mathematical plays. The first practical rule in this text is finding the value of $\{n \mid \}$ and $\{ \mid -n\}$, with n natural.

Of course $\underbrace{\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \cdots \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array}}_{n+1} = n+1 = \{n|\}$. Although this is very simple looking at

Proving that $n + 1 = \{n|\}$, $\forall n \in \mathbb{N}$:

if $n = 0$:

$1 = \{0|\}$, already shown.

if $n = k$:

Suppose it is valid until $n = k - 1$.

$$k + 1 = (k) + 1 = (\{k - 1|\}) + 1 = \{k|\}.$$

In the last line of the proof above, the addition $(\{k-1\}) + 1$ was used. The notation was not defined yet, but that is a simple game addition showed multiple times already. Up until this point, the integers and $1/2$ are defined. The remaining numbers fall into two categories: the multiples of powers of two and the rest. Of course the integers and $1/2$ fall into the first category as well, but the process to find integers is different **TODO: and $1/2$ was a good example.**

In the case of $\frac{1}{2} = \{0|1\}$, it is true that $G^L < G < G^R$. Is that always true? Think of moves made in numbers: if left makes a move from G to G^L , is it true that left has fewer spare moves in G than in G^L ? Before, as a side note, it was said that all possible RB-Hackenbush games numbers, and the reason may help explain that.

Suppose a game $G = \{x_1, x_2, \dots | y_1, y_2, \dots\}$ in which $\forall x \in X$ x is a number and $\forall y \in Y$ y is a number. Assume that x_i and y_j are best moves for left and right respectively. Suppose that G is not a number, then $x_i \geq y_j$. A move in G corresponds to removing a colored edge from a tree and all the edges that become disconnected

to the floor. That means that $x_i = \text{blue_dot_diagram}$ and $y_j = \text{red_dot_diagram}$. Is it possible that $x_i \geq y_j$? A careful reader spotted immediately that $x_i > 0 \wedge y_j < 0$. That means that $G^{x_i} < G < G^{y_j}$, which is a contradiction with the statement that $x_i \geq y_j$.

Other phrasing for “all RB-Hackenbush games are numbers” is “it is not possible to make a move that improves your position in RB-Hackenbush”, or, “left cannot make a movement that increases the value of G ”. Now, it may be clearer why, in numbers, $G^L < G < G^R$. Knowing this, however, is not enough to find the value of G .

If $G = \{3|10\}$, it is clear that $3 < G < 10$ but what is the value of G ? The **simpler** number that fits the interval, 4. There are some equivalent ways to check which number is simpler. A good one is figuring out which one require less effort to

write in their recursive and simplified form. For example, the simplified, recursive form for 4 is $\{\{\{\{\{\{\}\}\}\}\}\}\}$. Another good way is deciding which one is younger in the number tree represented below.

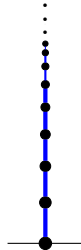
It is very recommended to understand that the **simplicity principle** is indeed finding the simplest number from a range of possibilities. An integer with smallest modulo is simple than one with greater modulo, and an irreducible fraction with denominator 2 is simpler than one with denominator 4. The formula for the simplicity principle, however, is

$$G = \begin{cases} 0, & \text{if } G^L < 0 < G^R \\ n + 1, & \text{if } G = \{n \mid \} \\ -n - 1, & \text{if } G = \{ \mid -n \} \\ \frac{2p+1}{2q+1}, & \text{if } G = \{\frac{p}{2q} \mid \frac{p+1}{2q}\} \end{cases}$$

Some examples are $\frac{1}{4} = \{0.1 \mid 0.3\}$, $\frac{1}{8} = \{\frac{1}{9} \mid 0.2\}$, $\frac{-3}{4} = \{-1 \mid -0.6\}$.

Numbers Generated After Infinite or More Steps

The formula above only allows G to have an infinite amount of values, but it is not closes to what was stated in the beginning of the section. The remaining numbers are hidden in the end of infinite games, in the sense that the board size is infinite, because, as explained before, it is extremely unpleasant to play something that does not end.

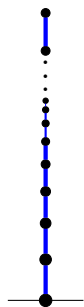


In the game above, left has an infinite number of possible moves, but **TODO: his/her** move always leads to an integer. What is the value of this game? $\{1, 2, \dots \mid \}$. It happens that this number has been baptized much earlier in mathematics of ω , the first ordinal number. It might not be as straight forward as finitely generated numbers, but the principle to calculate the value is the same.

For any finite number of red edges in G , if you add the game above to G , the result will be positive. However, it is also very simple to verify that ω is by far not the largest possible advantage. One simple example of that is the following.

The game above is an infinite stack of blue edges with another one on the top. One could wrongly argue that this additional edge on the top is simply part of the infinite stack. This is a wrong argument because this additional edge on the top allows Left to move to ω , and therefore, has the value of $\omega + 1$, making the games different. This detail is paramount to understanding the statement found on the epigraph of this section.

A hypothetical John Horton Conway wrote that phrase near a cave close to the edge of the Indian Ocean, where a couple of future mathematicians went to find



themselves. The words attributed to Conway by Knuth say that in the \aleph th day the universe appeared. That is because Knuth realizes that in that specific days all real numbers are generated.

However, again, accordingly to the hypothetical Conway, days kept passing by and more numbers were generated. The number $\omega + 1$ is one of the infinitely many surreal numbers that are not real numbers. However, this is not in the slightest the end of story. In fact, even considering that $\omega + \omega + \dots$ is also generated in the same format, the important part might again be missed. That is because only looking at large numbers is not way forward.

As seen before $\frac{1}{2} = \{0 \mid 1\}$, $\frac{1}{4} = \{0 \mid \frac{1}{2}\}$, \dots , $\frac{1}{2^{n+1}} = \{0 \mid \frac{1}{2^n}\}$. Therefore, the same way that $\omega = \{1, 2, \dots \mid\}$, $\epsilon = \{0 \mid 1, \frac{1}{2}, \frac{1}{4}, \dots\}$. A game with value ϵ may be simply the game of RB-Hackenbush starting with a blue edge and following with an infinite number of red edges. First notice that ϵ still have positivity, because Left wins no matter who starts, and then notice that its positivity is smaller than that of any real number. Now that ϵ and ω are known members of the \aleph day, only the remaining real numbers remain.

The fact that real numbers are generated in the \aleph th day and how to generate them with possible games is part of the section 5. The reason for that is that although it is easy generating reals with RB-Hackenbush games, it might also be worth it to know how to generate them with other games, and in fact, any other games.

The Surreal Numbers

This text presents only few characteristics of the surreal numbers as they are not the focus. It is also not necessary to know every algebraic property of numbers to go forward with the study of temperature this text focus on. However, it so happens that with very few construction rules, the numbers contain all the real, ordinal and infinitesimal numbers.

Because of its extremely simple definition and big expressiveness, it is an extremely interesting topic. The idea that the number of spare moves a player has might not be a real number might not be confusing, but it should be somewhat hard to accept. It is true that this fact is based on the construction Conway made and it is not necessarily true for all ways to analyze games, but the straight forward way of using game trees to build numbers lead to this characteristic.

Because the creation/discovery of surreals is very recent, it definitely makes people apprehensive as it is not clear if its properties are good or bad. The mathematician Phillip Ehrlich is an eloquent participant in this discussion. He makes

the point that the surreals do not have an intrinsic problem and that they show an unifying nature between paths in mathematics. In one of his papers, Ehrlich proposes that, while the real numbers form, on his words, an arithmetic continuum, the surreals form the absolute arithmetic continuum.

However, it is still a problem converting the domain of typical studies, such as calculus, from the reals to the surreals. Integrals of functions in the surreals are particularly hard to define. The 2015 revision of the paper ANALYSIS ON SURREAL NUMBERS, by SIMON RUBINSTEIN-SALZEDO AND ASHVIN SWAMINATHAN, for example, gave a definition of integration that solved a problem with previous integration but it is clear that others persist.

As Salzedo and Swaminathan point-out: “The ‘Conway-Norton’² integral failed to have standard properties of real integration, however, such as translation invariance: $\int_a^b f(x)dx = \int_{a-t}^{b-t} f(x+t)dx$, for any surreal function f and $a, b, t \in \mathbf{No}$. While Fornasiero fixed this issue in [For04]³, the new integral, like its predecessor, yields $\exp(\omega)$ instead of the desired $\exp(\omega) - 1$ for $\int_0^\omega \exp(x)dx$ ”. In the “Open Questions” section of the paper, the authors tell that there are still problems with their definition, meaning that the conversion of domains in integral studies is not completed, at least by 2015.

While many questions remain open, however, it is still possible to work with a subset of the surreals that only contain the reals for example. The remaining of the text does not require profound knowledge of anything that is not mentioned in regards to numbers.

²In ONAG 2nd edition page 228, Conway tells that Norton’s definition of integrals do not have desired properties.

³Antongiulio Fornasiero Phd. Thesis: *Integration on surreal numbers*.

4

Heating things up

This sections will provide new tools and ideas to analyze games and to do that, the same strategy of the previous section will be followed. Initially, the reader is presented with an intuitive idea and preliminary formulation of the problem. The first part contains many examples and its intent is to provide the reader enough to use the more mathematical heavy part to in fact gain advantage in any game played.

Now that enough is known about numbers, it is possible to work with non-numbers. The only known non-numbers at this point are the switches. But is knowing what they are enough to playing them? The game simpler cashing cheques will tell.

In this game there is a table with purple cheques. Each cheque has two numbers written on top, and, in each player's turn they will either pay one coin or cash a cheque that will grant him a number of coins equal to the correspondent associated integer. What is the best move for Left?

Simpler Cashing Cheques



Definitely the move is not paying, as Left can earn money in his turn. A good thing to grasp from this example is that you should never play in a number, paying a coin in this case, if there are non-numbers, cashing a purple cheque in this case. Should Left cash 8, 5 or 1? 1, of course. The reader is encouraged to play as Left and trying to find the best possible outcome, but the answer is playing the hottest switch. Although the game above is not a switch, it is a sum of switches, and, because of that can benefit of the simplified notation discussed earlier.

$$\begin{aligned}
 G &= \left(\frac{8-4}{2} \pm \frac{8+4}{2} \right) + \left(\frac{5-5}{2} \pm \frac{5+5}{2} \right) + \left(\frac{1-13}{2} \pm \frac{1+13}{2} \right) \\
 &= (2 \pm 6) + (0 \pm 5) + (-6 \pm 7) \\
 &= -4 \pm 7 \pm 6 \pm 5
 \end{aligned}$$

If you analyze the result above, it becomes clear that Left must play on the rightmost component as, although it will not provide many coins, it will prevent

right from cashing a huge amount. It is very possible to build scenarios where a player would even pay for cashing a cheque if that prevented the opponent from getting rich. Now that playing a simpler cashing cheques became easy, a more challenging task will rise. How to play Domineering well?

Adding a number with a temperature in a simplified position, like the expression above, should be acceptable by anyone following up to this point. Following, in the other hand, numbers will be added together with non-numbers just like number are added together and this might cause confusion. However, understanding that this sum is possible and intuitive is simple.

Playing a sum of games is just like playing a game with a set of independent rulesets and components.

$$G = \square\square\square + \textcircled{8 \mid 4} + \textcolor{blue}{|},$$

for example, is a game where each player makes a move¹ in any of the components and loses if cannot make a move. In other words, G is a game like every other, except for the more complex ruleset.

The result of this sum is obvious if all components are also numbers or switches. In the case of playing numbers and general non-numbers, sensible players will always play in non-numbers first. With this in mind other facts become clear. The first is that the temperature of a non-number added to a number is unaltered.

The second is that such a sum is actually a sum of non-numbers, added together with a number after they cool out. The sum of general non-numbers is thoroughly discussed in the remaining of the chapter, however, it worth noticing what the goal of this discussion is.

By the end of this section it will be thought how to convert any game in a 2D graphic composed of two **TODO: curves** that collapse to one line at some point, whose axes is number x cooling factor. The purpose of all this is that if the ending point falls in the positive side, Left gets an advantage, and if it falls in the positive, right does.

To build this graphic one is required to traverse the game tree, so the effort may seem fruitless as the game tree itself provides the winning strategy by itself. However, in cases where the game tree resulting from the sum of games is too large or expensive for a computer to run, there is a good strategy to playing this sum without knowing the complete game tree. In order to build the thermograph and play the **thermostrat**² correctly, there are a lot of minor concepts not discussed yet.

Other than the bias, playing a game like Domineering well involves the concepts of **Left/Right stops**, **toenail**, **ambient temperature**, **freezing point**, **cooling**, **heating** and a few others. To put all that together and provide a clear visualization of the best strategy, the thermograph.

$$\text{Given } G = \begin{array}{|c|c|c|c|} \hline \square & & & \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \square & \square & \square & \square \\ \hline \end{array}, \text{ how to proceed?}$$

¹There are other ways of playing G that will not be discussed

²This text will not present the thermostrat

Given $G = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}$, how to proceed?

G is definitely not a switch nor a sum of switches. It is possible to say the temperature in G is going to stay high for quite some time, because hotness is a term used to define the importance of the next move. A good place to start is writing out the game tree and building a temperature graphic of how it builds up from simpler positions until the more complicated ones.

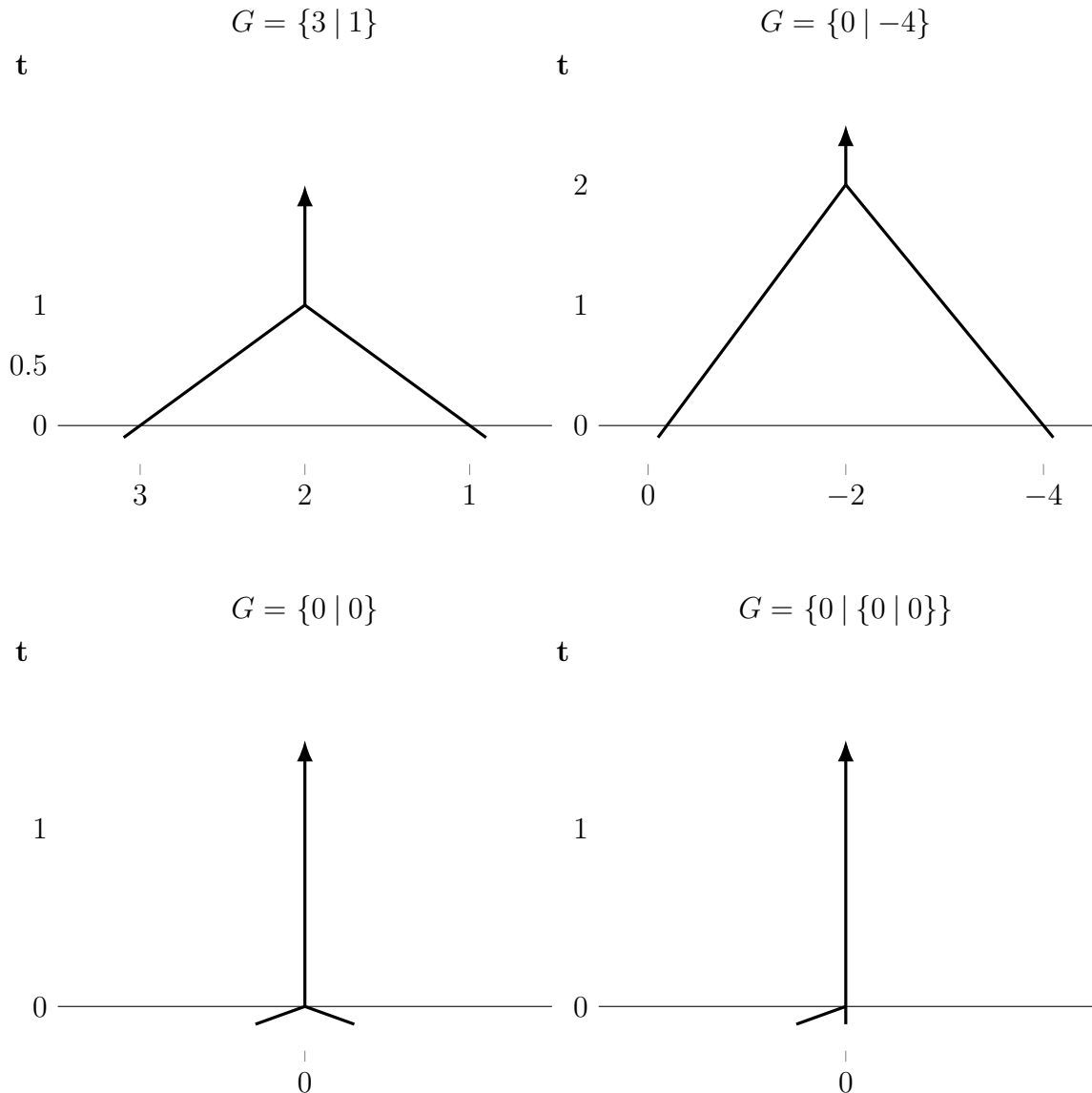
To decrease the confusion that builds up in complicated positions, there is the idea of a cooling factor. The non-number G cooled by t degrees is represented by G_t and is defined by:

$$G_t = \{G_t^L - t \mid G_t^R + t\} \quad \forall t \leq t'$$

$$G_t = x \quad \forall t > t'$$

Given t' is the smallest cooling factor such that $G_{t'}$ is infinitesimally close to a number x ,

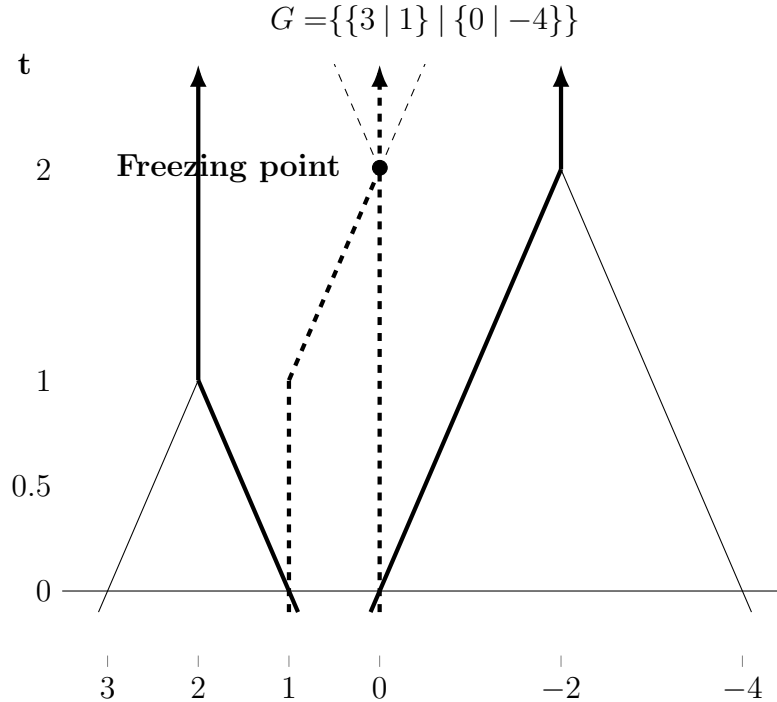
The temperature $t(G)$ is equal to t' . Now that both axes are defined, some examples of thermographs:



Some characteristics might be immediately apparent. The first is that the x-axis is reversed. The reason for that is to keep Right's movements to the right and Left's to the left. The second characteristic may be that all the thermographs end with a vertical **mast**. The mast begins at t' and indicates that G is a number from that point forward. The last one is that the graphic continues past the $y = 0$ line. It is worth noticing that the difference between the last two thermographs is below the $y = 0$ line.

Toenails, the segments below the $y = 0$ line, are important and may seem different, but they are actually simple extensions of the graphic. The reason for the last two toenails to be different is that cooling is applied to all the Left and Right alternatives, but in opposite directions. It is important to remember $G_t = \{G_t^L - t \mid G_t^R + t\}$, because it explains the difference. Both Left's and the first Right's toenail came from cooling 0, but the second Right's toenail came from cooling $\{0 \mid 0\}$.

The next example, the second of non-switch hot games, shows how cooling L and R alternatives work.

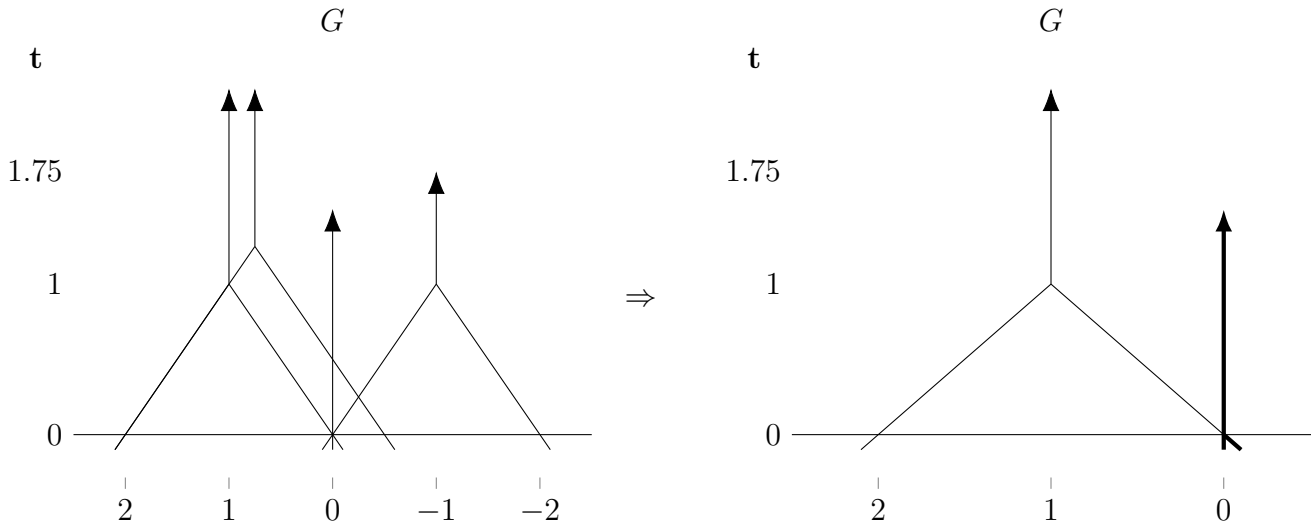
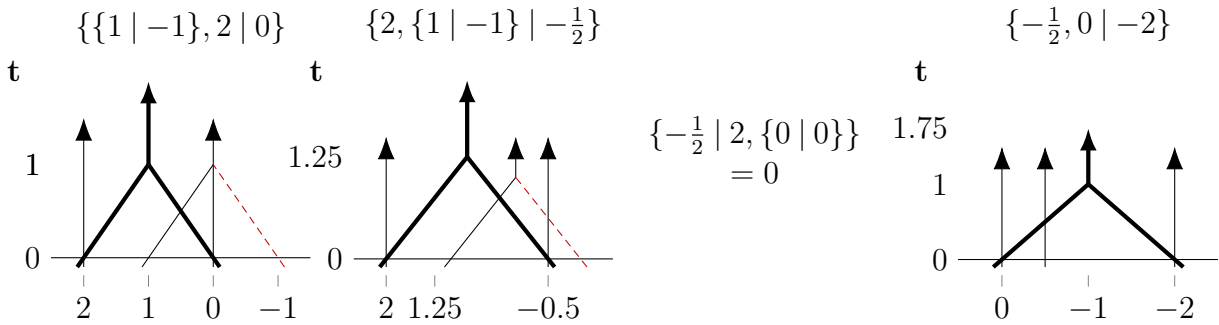
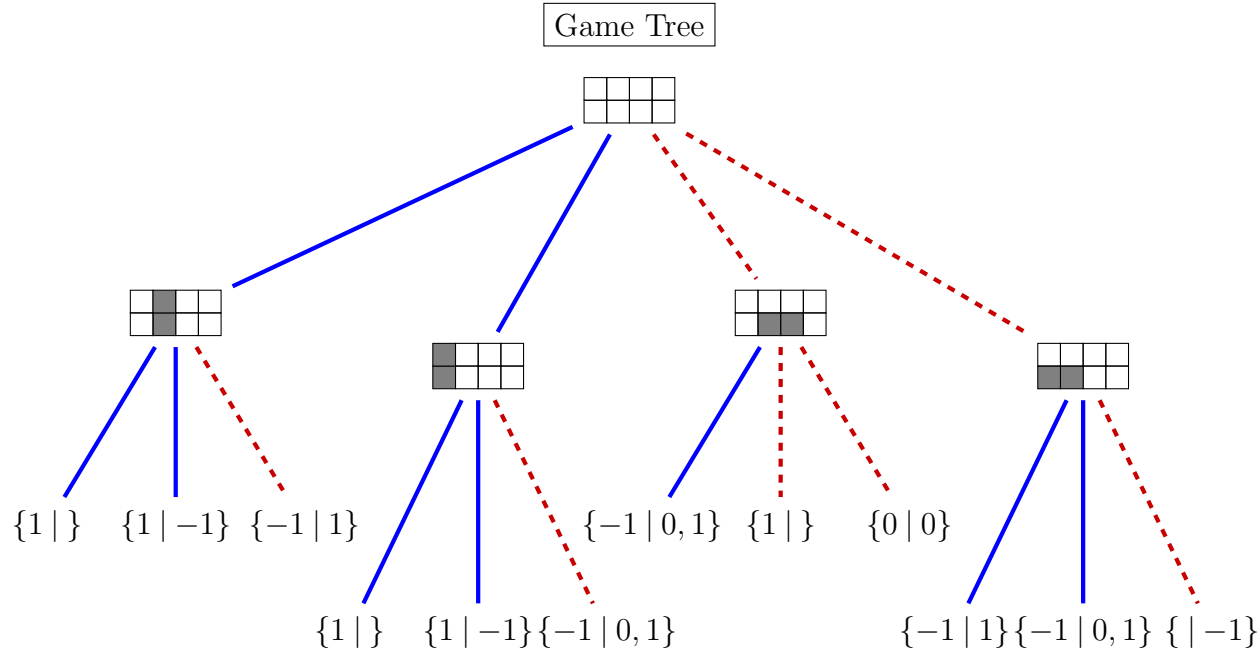


In the example above, the thick dashed thermograph is the thermograph of G . The light dashed segments are illustrative extensions of the cooling of L and R . The bold lines were used to show what part of L and R are taken into consideration: Right's slant is used to build Left's slant and vice-versa. The reason for this is that after either player makes a move, it will be the opponent's turn to move, and, this way, the opposing slant is the important one.

It is also worth mentioning that a freezing point will always be reached after an equal amount of move by each player. In the example above, the freezing point is the same as the junction point where the right slant bends, but that is not always the case. Before visiting **TODO: siegel's** formulation of the problem, one that brings good notation and formalism, an example of a real scenario is due. In a fun

game each player has more than one option for their moves and that has not been addressed yet.

What is the thermograph of the game $G = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$?



The thermograph of G and $\{\{0 \mid 0\} \mid 0\}$ are the same, although they are different games. From the steps taken to build that thermograph one can see that not all available moves contribute to build the temperature. Because we can calculate that some moves are strictly worse than others it makes sense that they should be ignored, but the thermograph shows exactly why.

When Left moves, he/she should avoid giving Right greater advantage over getting him/herself a possible better outcome. In the first thermograph, it is clear that 2 is greater, meaning it is more positive, or better for Left, than $\{1 \mid -1\}$. The reason for that is because the red right slant is more to the right than 2, not because the left slant is more to the right than 2.

The comment that G 's thermograph is the same as that of the game $\{\{0 \mid 0\} \mid 0\}$, but they are different games is surprisingly paramount to playing games well. It is so important that games of the form $\{\{x \mid 0\} \mid 0\} = -_x$ and $\{0 \mid \{0 \mid -x\}\} = +_x$ receive special names, minies and tinies, respectively. The game above is negative, of course, but Right's advantage really small.

In the game above, $G = -_2$, Right's advantage is much smaller than that of $-_1$. In fact, for any number $x, y \mid x > y \geq 0$, any multiple of $-_x$ is less negative than $-_y$. It may come of as a surprise because usually multiplication is only made between numbers, but, in fact, this observation makes sense when a real example is provided.

One should think of minies and tinies as late fees. In the case of tinies, if Left makes a move, he/she is fine. However, if he/she does not, Right may send a warning e-mail that tells Left to do so. If, after the e-mail, Left makes the move, he/she is fine again, but if Left does not, Right can charge a late fee from Left.

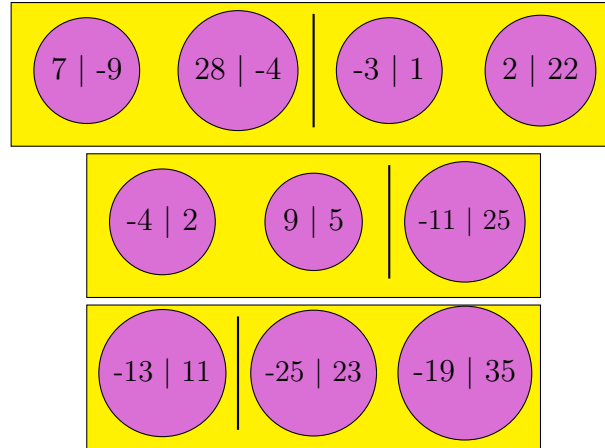
The reason for any multiple of $+_x$ remain smaller than $+_y$ is just that. Since there is the warning phase in the game, players will always be in time to reply. The only case where a reply is not worth is when there are more important moves to make. Because, the only factor, in this case, that makes a move more or less worth is the value of x , it does not matter how many $+_x$ there are. When playing situations where one can choose between quantity of minies/tinies over quality, the choice for quality is always the correct one.

One important situation, however, did not happen in the game above. There will be cases where there is no clear best move for a component C , and that will depend on the ambient temperature of the game. The reader can see that if $G = C$, then there are best first move to make. In the example above, $G^L = \{2 \mid 0\}$ and $G^R = 0$.

However, if other components are added together with C , the best first move in C may change. The reader can imagine the situation where it Left turn to play, but Right has an important move to make next. Knowing that, Left makes a greedy move, that allows the opponent to acquire more advantage than other options, but allows him/herself more advantages if Right does not capitalize. Right is cornered between making the previous important move and this new option Left allowed.

For example, check the following game of extended simpler cashing cheques, where the previous rules apply, but the players may also play on yellow squares. The yellow squares allows players to select any purple correspondent purple cheque before disappearing. Purple cheques might also show negative values, indicating the opponent receives the promised coins.

Extended Simpler Cashing Cheques



It is a good thing now to consider the game above as the sum of three components, A, B, C from top to bottom. It is completely clear what the best moves for left and right are in each of the components. However, from the thermograph of the components, one will find that no moves are dominated, they all influence the resulting thermograph. As seen before, this means that depending on the ambient temperature the best move in each component will vary.

In the last example, the best move for Left is actually to play in B and make the move to $\{9 \mid -5\}$. In this case, the reader should notice that $A + B - 5$ is less negative than the original game, but also that Right will not necessarily move in $\{9 \mid -5\}$. Left's move, is an example of taking advantage of the fact the opponent will be distracted moving elsewhere to make the best of, in this case, a bad situation.

5

It's all about finding the calm in the chaos

Although the concepts and definitions of combinatorial game theory have impact in general mathematics, it is a relatively new field of study that has many open problems and not so many resources or implementations available. The purpose of this section is to bring examples, pieces of code and discuss a particular software that implements most of the theory discussed up to this point. The code fragments featured in the text follow C++ syntax, with the intent of being as simple as possible.

The examples and code fragments also serve the purpose of showing possible readers that the theory is actually quite simple, not requiring great mathematical skills, and that applying it is also not difficult. With them, some concepts presented will become clearer and a few facts enunciated will be shown. A second part of this section will introduce a few new games so that the reader might find other interesting games to play. In this second part, the reader will also see that it is not hard to make up fun games on the spot. This section can work as a midpoint between presenting the concepts and a shift in focus to handle the problem of temperature bounding, and may serve to verify if the concepts are clear.

The Numbers

A very important and recurrent theme in the early parts of the text is the correspondence between games and surreal numbers. It is correct to say that all real numbers are also games, but it may not be clear how to make a particular real number. For example, the reader may not be able to make a game with value equals to π . In fact, the reader may not know what numbers are easy or hard to do. For now, the first two rules of the simplicity principle should be clear enough.

```
using std::max, std::floor, std::vector, std::unique_ptr;

// Initial class structure
class SurrealNumber {
public:
    float toFloat ();
    ...
private:
    vector<unique_ptr<SurrealNumber>> left;
```

```

        vector<unique_ptr<SurrealNumber>> right;
        ...
};

float SurrealNumber::toFloat () {
    float ret;
    if (left.empty())
        if (right.empty())
            ret = 0.0f;
        else
            ret = min(floor(-1 + minRight()->toFloat()), 0.0f);
    else if (right.empty())
        ret = max(floor(1 + maxLeft()->toFloat()), 0.0f);
    // other cases
    ...
    return ret;
}

```

While easy and hard is relative, every number that is a dyadic rational, a number that is of the form $\frac{a}{2^b}$, $a \in \mathbb{Z}, b \in \mathbb{N}$ is easy to form. A good method to make the representation of $z = \frac{a}{2^b}$, $z \geq 0$ in $\{x \mid y\}$ is:

- 1) Calculate $d \in \mathbb{Z} \mid 0 \leq z - d < 1$.
- 2) If $z = d$ then $x = z - 1, y = z + 1$, stop.
- 3) Binary search for any $w = z - d \in (0, 1)$
- 4) x is d added to the oldest number to the left.
- 6) y is d added to the oldest number to the right.

For example, $\frac{89}{16} = 5 + \frac{9}{16} = \{5 + \frac{1}{2} \mid 5 + \frac{5}{8}\}$, because the binary search for $\frac{9}{16} = \frac{89-80}{16}$ follows the path $\frac{1}{2} \xrightarrow{R} \frac{3}{4} \xrightarrow{L} \frac{5}{8} \xrightarrow{L} \frac{9}{16}$. In RB-Hackenbush, building $\frac{89}{16}$ is now easy. Take a pile with five blue edges and add it to a blue-red-blue-blue pile. The red-blue-blue pile on top of that derives from the right-left-left path on the binary search for $\frac{9}{16}$, with a right turn corresponding to a red edge and vice-versa. Therefore, it is possible to complement the featured snippet, repeating the idea, but with the objective of finding $z = \frac{a}{2^b}$ given $\{X \mid Y\}$.

```

//other cases
else {
    float maxL = maxLeft();
    float minR = minRight();
    if (maxL < 0 && 0 < minR)
        return 0.0f;
    float d = floor(maxL);
    maxL -= d; minR -= d;
    float fact = 1.0f;
    while (fact > 1.0f/4096.0f)
        if (x < fact)
            if (y > fact)
                break;
}

```

```

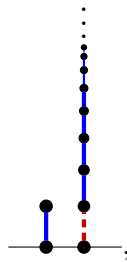
        else
            fact *= 0.5f;
        else
            fact *= 1.5f;
    ret = d + fact;
}

```

All the remaining numbers are hard in the sense that they require an infinite number of steps to define. A simple code like the one above will not handle that. That is because $\frac{2}{3}$ and π are both generated in the \aleph th day (section 3), like any other non-dyadic fraction. However, these are not equally hard. Because $\frac{2}{3}$ is a periodic number when using binary representation, its path in the numbers tree is well-defined.

It is true that both $\frac{2}{3}$ and π are given by something equivalent to the following: $n = \{x \in S_* : x < n \mid y \in S_* : y > n\}$, where S_* is the set of numbers generated until day \aleph . However, $\frac{2}{3} = 0.\overline{10}_2$ can also be defined through the infinite right-left-right-left... path in the binary tree, because a 1 in the binary representation is a step do the right in the tree and vice-versa. One way to think of this fact is that, when doing a binary search for x between the dyadics in $(0,1)$, the starting question is $x > 1/2$. The 1 indicates a yes, so the following question is $x > 3/4$, and so on. Since numbers smaller than x go on the left set and vice-versa, $\frac{2}{3} = \{1/2, 5/8, \dots \mid 3/4, 11/16, \dots\}$, showing that the numbers visited in the binary search alternate in the left and right sets. Repeating decimals, in RB-Hackenbush, are extremely easy to spot because they are always a sequence of reds and blues followed by a finite pattern repeated infinitely.

Other than some real numbers being hard to draw, drawing non-reals is also not trivial. It was shown that there are infinitely many numbers between any two real numbers, in section 3. This might seem hard to understand or accept initially, because the reals form the complete ordered field. However, at least based on Conway surreal numbers, the fact that it is possible to build a game in which the advantage a player has is strictly between two real numbers gives some light to the fact. To do that, simply add an infinitesimal to a real number. For example:

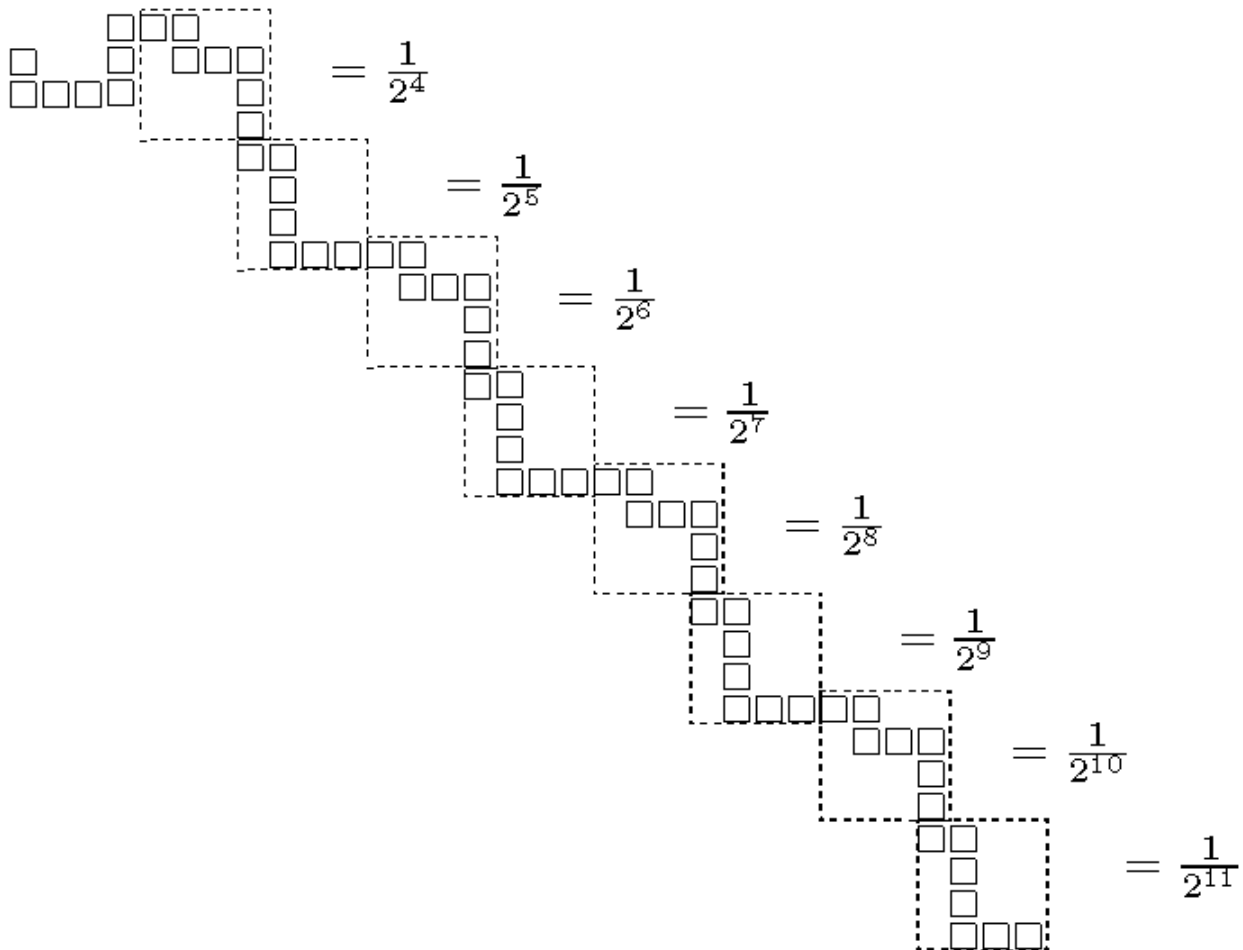


has value $1 - \epsilon$, a number smaller than 1, but larger than any $x \in \mathbb{R}, x < 1$. Although the non-reals might be new, the effort of writing hard reals and non-reals as RB-Hackenbush is very similar.

RB-Hackenbush is an extremely good example to understand numbers, as finding dyadics and repeating decimals are very easy. However, not all games are like this.

It was shown that $\begin{array}{|c|} \hline \square \\ \hline \square \end{array} = \frac{1}{2}$, but how would one build an instance of $\frac{1}{4}$, or $\frac{1}{2^n}$, for $n \in \mathbb{N}$? It turns out that it is not trivial at all.

In fact, only in 1996 a partial solution was given. One of the results of **TODO: ref** was the existence of arbitrarily small values of domineering games, before that, it was unknown whether or not they existed. It was only in 2015 **TODO: ref** that a method to create all dyadic rationals was introduced. Following the strategy presented in 1996, the so-called Yonghoan Kim's snakes were created, named after the author. The snake's representation is found below, copied from **TODO: gong 477**:



In this method of finding dyadics, there is an initial structure, that is not marked by a dashed rectangle. To this structure, additional ones are appended. As shown, each additional structure alternated between two others. Of course that the resulting game is not the only game with the given values.

In fact, there are special games with the same value as the snakes. The 2015 paper is based on these special games. These special games are based on concatenating and imploding bridges in domineering games. Bridges are cells that do not change the value of a game. For example, $\square\square$ and $\square\square\square$ have the same value, so one of the extremities is a bridge. The strategy is built upon two theorems that are beautiful because they are powerful yet very simple.

The *Bridge Splitting Theorem for Domineering* was introduced by Conway's. It reads: If the value of $G\square$ is the same as G , then the value of $G\square H$ is the sum of the values of G and $\square H$, given G and H do not intersect. The proof is:

$$G \square H \leq G + \square H = G \square + \square H \leq G \square H$$

The first inequality since splitting a horizontal always favors Right, and, the second inequality is true because merging horizontal squares also do always favor Left. The second important theorem was created by the authors.

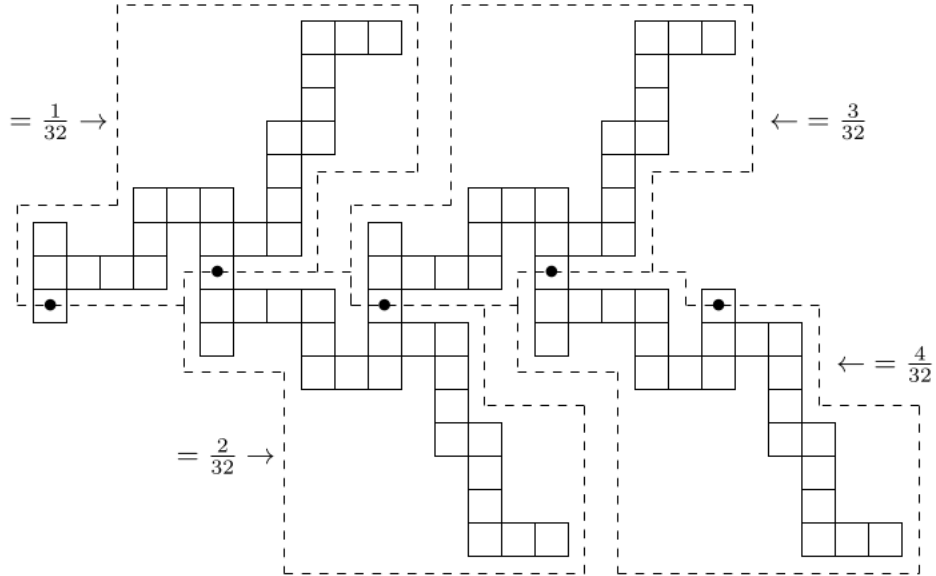
Bridge Destroying Theorem for Domineering

If the value of $G \square$, \square^H , $\square I$ and \square_J are the same of the values of G , H , I , J , then the value of $G \square_J^H I$ is the same as the sum of G, H, I and J , provided that neither of the games have common edges. The proof is also simple:

$$\begin{aligned} G \square_J^H I &\leq G + \square_J^H I \leq G + I + \square_J^H = G + H + I + J = \\ &= H + J + G \square I \leq G \square_J^H I \end{aligned}$$

The first two inequalities are true because they both split a horizontal line, favoring right. The two equalities are true because they are applications of the *Bridge Splitting Theorem for Domineering*. The last inequality is true because is a linking of a vertical line, which can only favor Left.

An example of applying this is figuring out that $\begin{array}{c} \square \\ \square \end{array} = \begin{array}{c} \square \\ \square \end{array} = \frac{1}{2}$, therefore it is true that $\begin{array}{c} \square \square \\ \square \square \end{array} = 1$. Another, much more beautiful example is given on 2015 paper:



By combining the two theorems with Kim's snakes, it would be possible to create any dyadic in a single connected component, if not for two problems. The first is the problem of creating a bridge. However, that is a non-problem because the theorems mentioned above are enough to provide explosive squares. An explosive square that

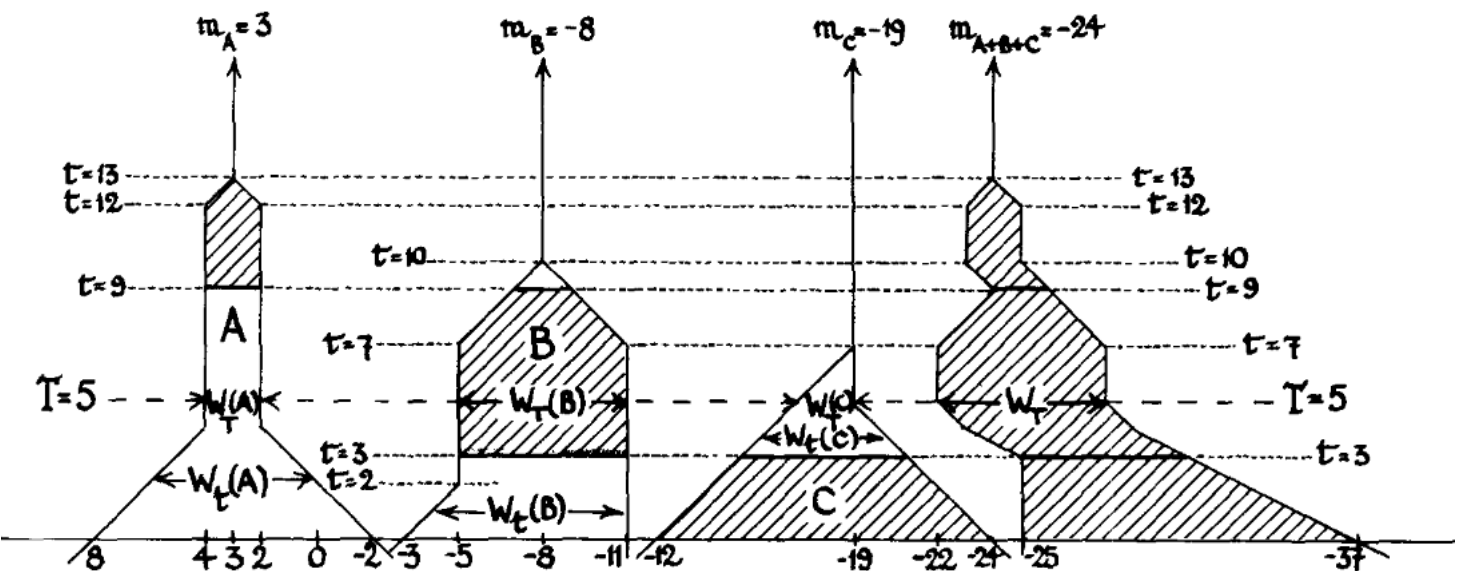
is always there is a square below the two vertical squares to the left of the base configuration of Kim's snakes. Since the value of the snake is equal to the values of the sum of the bridged components, the base component may be interchanged with the one with an explosive square and the values of the snake would remain the same.

The second problems is that of making sure ever component will not share an edge with the others. The solution of this second problem is fond on the aforementioned paper. The domineering discussion above serves to show how some games are good to analyse and create examples with, while other not so much. Another examples of that was the game "Extended Simpler Cashing Cheques (ESCC)".

The non-numbers

When the topic changed from numbers to non-numbers, an example of ambient temperature, and how it affected play, was required. In order to keep the results to the boundaries of the literature, an instance of this game was showed. This instance was made by reverse engineering the example of the same topic found in **TODO: WW** in this convenient game. Normally, it is extremely hard to generate a configuration with any temperature value, but ESCC allows it, and this is why its creation in this text was important.

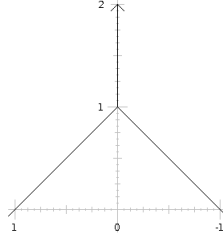
ESCC allows an easy conversion from a $\{X | Y\}$ to a configuration, mainly because any switch is trivial to write down. The thermograph found in domineering is copied below:



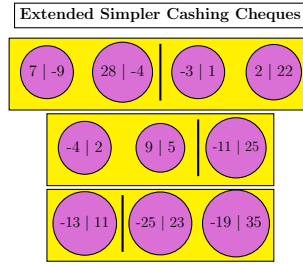
To verify that the resulting compounded thermograph of the example featured in the previous section is indeed the one found above, it is possible to use Aaron Siegel's CGSuite. CGSuite is an implementation of almost all of the methods found in Combinatorial Game Theory. This text will not go in detail on the programming language used in the engine, but will bring to light some information.

Between other practical and interesting features, it is possible to create any games and calculate and plot their thermographs. For example:

```
> g := {1 | -1}
±1
> g.Thermograph.Plot()
```

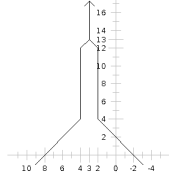


The game from last section could, then, be converted to the following:

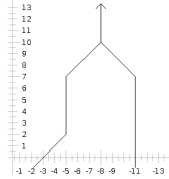


⇓

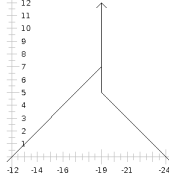
```
> A := {{7|9}, {28|4} | {-3| -1}, {2| -22}}
{8, {28|4} | 2, {2| -22}}
> B := {{-4| -2}, {9| -5} | {-11| -25}}
{-3, {9| -5} | -11| -25}
> C := {{-13| -11} | {-25| -23}, {-19| -35}}
{-12| -24, {-19| -35}}
> A.Thermograph.Plot()
```



```
> B.Thermograph.Plot()
```



```
> C.Thermograph.Plot()
```



Therefore the game from last section is indeed the same as the one featured in **TODO: WW**. The steps taken to reverse engineer the games whose thermographs were the ones indicated in the book, might be of useful to speed up the process of calculating and exercising the process of building thermographs by head. This process uses a theorem not yet mentioned.

This theorem is about the structure of thermographs. It states that every line in a thermograph has slope equals to -1 , 0 or 1 and masts have slope 0 . Using the same

notation as **TODO: CGT**, let G be a game and $\lambda_t(G), \rho_t(G)$ its thermograph's left and right trajectories, respectively. It means that:

$$\tilde{\lambda}_t(G) = \max_{G^L}(\rho_t(G^L) - t)$$

$$\tilde{\rho}_t(G) = \min_{G^R}(\lambda_t(G^R) + t)$$

$$\lambda_t(G) = \tilde{\lambda}_t(G), \rho_t(G) = \tilde{\rho}_t(G) \text{ if } t \text{ below the freezing point.}$$

As seen before, for any temperature above the freezing point, the values of the left and right slants are the same and do not change.

(a) Every segment of $\lambda_t(G)$ has slope 0 or -1 .

(b) Every segment of $\rho_t(G)$ has slope 0 or 1.

(c) Both trajectories have masts of slope 0, with the same value.

The proof is not complicated. First the proof of (a) and (b), given via induction:

Base: (a) and (b) are true if G is a number, as the slope is always zero.

Induction Step: Let $\lambda_t(G^R)$ satisfying (a) and $\rho_t(G^L)$ satisfying (b). Since $\tilde{\lambda}_t(G) = \max_{G^L}(\rho_t(G^L) - t)$, the slope of $\tilde{\lambda}_t(G)$ is either 0 or 1 translated by $-t$.

The same goes for $\tilde{\rho}_t(G)$. Since $\tilde{\rho}_t(G) = \min_{G^R}(\lambda_t(G^R) + t)$, the slope of $\tilde{\rho}_t(G)$ is either -1 or 0 translated by $+t$. This way, (a) and (b) are true.

Now, the proof of (c), also via induction.

Base: (c) is true if G is a number, as the slope is always zero.

Induction Step: Let the slopes of $\lambda_t(G^R)$ and $\rho_t(G^L)$ be zero. Therefore, the masts of $\tilde{\lambda}_t(G)$ and $\tilde{\rho}_t(G)$ are -1 and $+1$ respectively. Since $\tilde{\lambda}_t(G) \leq \tilde{\rho}_t(G)$, the left and right trajectory cross and, therefore, there is a freezing point. By definition, after the freezing point, the slope is zero.

If the reader goes through the thermographs in this text, he/she will notice that this is indeed true for the examples. Trying to find a counter-example may help understand the proof, as inevitably, the recursive definition of the thermograph always leads to the base case of number's thermographs. Now that this rules are properly stated, it is possible to understand how to write a game $G = \{X \mid Y\}$ based on an existing thermograph.

Starting by the thermograph of C , it is easy to spot that there is only one viable option for Left. Since the left slant, $\tilde{\lambda}_t(G)$, of that graphic has slope -1 , then $\rho_T(G)$ has slope 0. Since the slant crosses the x-axis in $x = -12$, then an option to this Left option is the number -12 . The option then, is the canonical form of this number, adapted to fit the rules of the cashing cheques game. The right trajectory is a bit more complicated.

Starting with the inclined slant, the procedure described above helps one finding out that the number -24 is enough. The vertical line between the bent line and the freezing point must come from a non-number. Because a bent line added to the cooling factor t is a straight line, then the left slant of this non-number must cross the x-axis in $x = -19$. Another requirement is that this game must be hot until $t = 7$, because the straight line goes until this point. With this in place, it is possible to create the game $\{-19 \mid -35\}$, because $-35 = -19 - 7 - 7$, in which the sevens come from the $t = 7$ and are used twice because the cooling happens in both

directions. In this example, the right option could be any number smaller than -35 , because the boiling point would remain the same

The thermograph of B is more difficult. The first bent line of the left trajectory is the same as before, and, in this case, the number -3 is enough. The straight line, just like before, comes from a non-number whose right option is -5 . Unlike the non-number from the previous example, however, the freezing point of this case must be exactly 7, so the distance between the right and left stops must be 14, unlike the previous example. With this information, it becomes possible to build the game $\{9 \mid -5\}$. The last bent segment is the mast of the non-number.

The right trajectory of B is the simplest $\{-11 \mid -25\}$, which has already been explained. The process to build the game A is a repetition of the left left trajectory of B for both the left and right trajectories. By going through this, hopefully it became clear that thermographs are not hard to build. Also, by using the same example as the book, the reader might also benefit from the remarks found there.

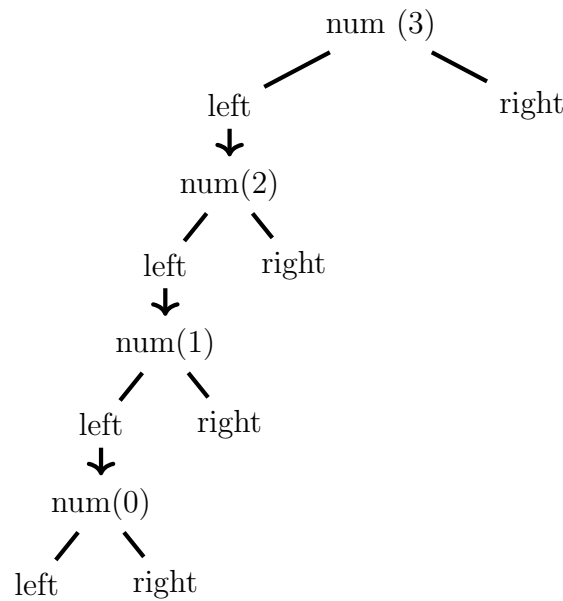
Just like RB-Hackenbush is a very good game to build example of numbers, Extended Simpler Cashing Cheques is a very good game to build examples of temperature. The brief description of how to build dyadics in domineering and how long it took for the process to be created, showed that domineering is not a good game to study numbers. Up to this point domineering has been used to exemplify the study of temperature, but the truth is temperature in domineering is limited, both in numerical values and in current knowledge of the game.

In fact, in 2005 two researches brought to light that Elwyn Berlekamp conjectured, in 2004, that the maximum temperature in domineering is 2. That is yet to be proven or disproved, although databases lead to this result. It may not be all that clear yet, but many common games cannot have arbitrarily high temperatures. In a domineering position with temperature two, whoever plays first finishes with two spare moves. Before trying to create such a position, the intricacies of such a position might not be clear.

In 2004 an instance of that position was found, and that was the only one, in a connected board of course, up to this day. While this conjecture is a typical open problem in combinatorial game theory, there are not many attempts to solve this. The next section is dedicated a PhD thesis that gave an upper bound for certain configurations in domineering, although that does not answer the question.

The initial code fragment of the section only dealt with simple surreal numbers, even making a lot of assumptions about left and right, without verifying anything. It might make sense to think about how one would implement, in practice, the numerical part of a library like CGSuite. Considering only the SurrealNumber class, if the only available method of defining numbers were the left and right sets, how would be the simple number $3 = \{2 \mid \}$ be written?

Of course that, with only that available, the recursive structure of surreal numbers $\{\{\{\{\mid\}\mid\}\mid\}\}$. Although correct according to the theory, implementing number like that, only, would cause problems because evaluating a number would no be a trivial operation. For instance, telling whether $2.5 = \{2 \mid 3\} < 3 = \{2 \mid \}$ would require traversing both number's left and right trees. The canonical representation of the number 3, for example, would be something like this:



Given this problems, a viable implementation of combinatorial games must look for alternative representations. Some possibilities are allowing surreal numbers to be represented with integers and floats or to store results of the tree traversals to minimize the number of times the complex tree structure must be analyzed. When looking at numbers strictly, the tree that resulted in a float or another chosen numerical representation may be discarded because it does not contain special information.

However, when changing the domain from surreal numbers to games, the tree cannot be always discarded. The reason for that is that the game tree may contain relevant information for gameplay. Although one could try keeping only the best move for each position in order to save space, the problem of adding games, that changes which move is best makes it difficult.

Consider the following draft:

```

// CombinatorialGame.hpp
#include "everything_required.hpp"
using namespace std;

class CombinatorialGame {
public:
    CombinatorialGame (vector<CombinatorialGame*> l,
                      vector<CombinatorialGame*> r);
    ...
private:
    vector<CombinatorialGame*> left;
    vector<CombinatorialGame*> right;
    bool isSurreal;
    bool isSwitch;
    float realValue;
    float bias;
    float temperature;
};

```

```

//CombinatorialGame.cpp
#include "path/to/CombinatorialGame.hpp"

CombinatorialGame::CombinatorialGame (
    vector<CombinatorialGame*> l,
    vector<CombinatorialGame*> r) {
    float maxL, minR;
    bool lFlag, rFlag;
}

```

The draft handles some of the hard aspects of evaluation but a completed version contains solutions to many additional cases. The draft, for one, is incapable of handling infinite games, since there is no way interpretation for infinite structures. A more complete version can be found in CGSuite github repository, but the more important details are addressed.

Other Games To Play

Now that vocabulary, methods and tools were introduced, this section brings some games to put your knowledge to proof.

The Amazons is a typical game that, like domineering, is fun and only requires pen and paper. Amazons is typically played in a 10x10 chess board with each player having four initial pieces in the fixed initial position depicted below. Each piece works just like a chess queen. In each turn, the player selects one of his/her queens and moves in an arbitrarily number of squares. Every square the amazon (queen) goes through is then burned from the game, making it so that no queen can move over it. The initial position is unimportant and the players can choose any starting position they find interesting.

TODO: image of typicall position

For example, play may take place in an 8x8, a chessboard for convenience, with each player having two queen in opposing corners. Since, in this example, player Left is a beginner and Right is a seasoned player, Left has one additional queen next to one of his/her queens. It serves to show that there are no restrictions to the initial position - games do not have to be mathematically balanced to be fun.

TODO: image of given posi

The Angel Game is more of a mathematical challenge more than a game, because it is not meant to actually be played. This game is played in an infinite squared board, with the Angel being located in one of its cells. Each turn the Angel flies over k squares and lands, depending on the desired rules. When it is the devil's turn to play, it possesses a square from the infinite board. The challenge is to know whether, and for what values of k , the devil wins, given the Angel may not land in a possessed square.

This game led to a lot of research until it was solved. Although it may not interest someone looking for a fun game it serves to show that even games on infinite boards are part of this theory. An important aspect of this game is that the position, after every move, becomes better and better for the devil, but still, sometimes it does not improve enough for the devil to win, even after an arbitrarily large number of moves.

Snorts is yet another fun game to be played with pen and paper. Snorts is a picture coloring game. Two herdsman, Bob that has a herd of bulls and Chad that had a herd of cows, compete to buy properties in a large open field. They are both interested in acquiring as many properties as possible, regardless of their sizes, but they respect each others space. Since cows cannot live next to bulls and vice-versa, Bob and Chad cannot buy any property next to each other.

The initial position for playing snorts is a partitioned shape. The initial position is somewhat problematic to draw, because may cause the first player to have easier tie playing. However, by making a large initial open space, it is harder to make a boring game. A good method to draw the board is to make a circle-like shape a start subdividing it randomly, making sure that there are not very large partitions. The picture below is an example of a snorts game in progress.

TODO: image

The Markswoman Amazons is the first example of this section on how to create a game. This game is a variation of The Amazons, with one additional rule. After any movement is made, the Amazon that moved shoots an arrow in any direction that flies any number of squares, hitting and burning an open square. The arrow cannot hit other Amazons, but may target an already burned square. This serves to show how easy it is to adpat existing games.

The Knights is another possible variation of Amazons. The game has the same rules but uses chess knights instead of chess queens. This game may, for experienced chess players, seem less fun that the amazons, as it is much more predictable. It may be true, but, like the markswoman amazons may be worth trying.

.....

6

An amazing result and a dive into algebra

As the name suggests, this section has a different focus compared to the previous. This section is concerned with classifying a few games, both in terms of gameplay and in terms of algebraic structure. This is important because it highlights important characteristics in underlying structure and in possible values.

This section features a few theorems that end up proving an upper bound for the boiling point of classes of games.

The Result

The temperature's upper bound results from a thorough analysis of the thermograph. **TODO: quote** brings the first of such results, although the content featured first in Svenja's PhD thesis in 2018. The proof of the boiling point requires a few smaller results and all the required content is summarized in the remaining of this subsection.

The confusion interval of a game is the range of numbers with which it is confused. A game is confused with a number if it is neither greater, smaller nor equal to that number. Any game with temperature greater than 0 is confused with numbers. Take any instance of such cases from previous sections to verify that their confusion interval includes all numbers between the points where the thermograph intersects the $t = 0$ line. Some intervals also include either one or both of the intersection points, depending on what the thermograph looks below that line.

However, the reader can also verify that some non-numbers with temperature 0 are also confused with numbers. Section four shows that the thermograph of $\{0 \mid 0\}$ is different to that of 0, although above the $y = 0$ they are the same. However, if using the surreal $<$ operation defined in chapter 3, one notices that $\{0 \mid 0\} \not\leq 0$. However, not all non-numbers behave this way, because some fall in gaps, as seen in section 3. $\{0 \mid \{0 \mid 0\}\}$ is such a game and it is said that its confusion interval does not contain any numbers.

The confusion interval of G is usually indicated by $\ell(G)$ and the intersection points are usually denoted by $LS(G)$ and $RS(G)$, and called left and right stops. Therefore $\ell(G) = LS(G) - RS(G)$. Another important definition is the thermic version of a game. The thermic version of a game G is a game \tilde{G} with the same temperature of G but with only one left and one right option, that are also left and

right options for G . **TODO: maybe rephrase.**

Theorem: If G is hot, then there exists a G^L and a G^R such that $t(G) = t(\{G^L \mid G^R\})$. The proof is simple: the mast of G begins where the cooled thermographs of some of the left and right options first intersect. There are possibly many left and right options that intersect on the same point. By taking any one of them, G^{L+} and G^{R+} , it is clear that the thermic version of a game exists, because $t(G) = t(\{G^{L+} \mid G^{R+}\})$.

Finding in practice the thermic version of game is not at all simple. In fact, without additional information of G , it would be needed to build the entire thermograph, or equivalent information, beforehand. This is an important consideration, that is also made on Dr. Huntemann thesis, but one that is not problematic. Soon, the bound will be created for G directly, so that finding the thermic version is not necessary.

It is trivial but it is worth highlighting that the thermograph of G and that of \tilde{G} may not be the same. In particular $\lambda_t(\tilde{G}) \geq \lambda_t(G)$ and $\rho_t(\tilde{G}) \leq \rho_t(G)$. If it is not clear why the observation is true: for any temperature t , the trajectories of the thermograph were either the maximum/minimum of all the corresponding options. The selected left and right options for the thermic version of G are only the minimum/maximum for temperatures close to the boiling. Therefore the initial trajectory might be different, but is always lesser/greater or equal to the thermic trajectories.

Using the notation described above it may be simpler. $\lambda_t(G) = LS(G_t) = \max_{G^L} (RS(G_t^L) - t)$. Since $G^{L+} \in G^L$, it is clear that $\lambda_t(\tilde{G}) \geq \lambda_t(G)$. An equivalent statement can be made for the right trajectory.

A result, when fixating $t = 0$ is:

$$\ell(\tilde{G}) \leq \ell(G)$$

$$\text{Since } \ell(\tilde{G}) = LS(\tilde{G}) - RS(\tilde{G}) \leq LS(G) - RS(G) = \ell(G)$$

The thermograph, as described in the previous section, is composed of straight lines and $\pm 45^\circ$ oblique lines. The bound is based on this and the size of the confusion interval. Before proceeding, one additional definition.

Let T^L be the sequence $(0, t_1, t_2, \dots, t_k)$ of the temperatures of the turning points of $\lambda(\tilde{G})$. Let, now, A^L be the sequence (a_0, a_1, \dots, a_k) of labels given by:

$$\begin{cases} a_i \text{ is } \textit{vertical} & \text{if } \rho(\tilde{G}_{t_{i+1}}^L) = \rho(\tilde{G}_{t_i}^L) \\ a_i \text{ is } \textit{oblique} & \text{if } \rho(\tilde{G}_{t_{i+1}}^L) < \rho(\tilde{G}_{t_i}^L) \end{cases}$$

Lastly,

$$\begin{aligned} T_V^L &= \sum_{\substack{i \mid a_i \text{ is} \\ \text{vertical}}} (t_{i+1} - t_i) \\ T_O^L &= \sum_{\substack{i \mid a_i \text{ is} \\ \text{oblique}}} (t_{i+1} - t_i) \end{aligned}$$

Notice that the right counterpart can be done similarly. Essentially, T_V^L is the length of the vertical segments and T_O^L is the length of the oblique segments. Of course

$$t(G) = T_V^L + T_O^L.$$

and

$$\ell(\tilde{G}) = T_O^L + T_O^R.$$

The theorem is the following:

Given G and \tilde{G} its thermic version, then

$$t(G) \leq \ell(H) + \frac{\ell(G)}{2}$$

where $H = \tilde{G}^L$ if $T_V^L \geq T_V^R$, otherwise $H = \tilde{G}^R$. For simplicity, the proof considers that $H = \tilde{G}^L$, but again, an equivalent procedure proves the other case.

The length of T_V^L is at most the length of the oblique segments of $\rho(\tilde{G}^L)$. Notice that it could be smaller because the left and right options might intersect before the mast of \tilde{G}^L . In any case, it is clear that the length of this oblique segments is at most the same size of the confusion interval of \tilde{G}^L . It could be smaller in case there are oblique segments in $\lambda(\tilde{G}^L)$.

Therefore, $T_V^L \leq \ell(\tilde{G}^L)$

Additionally:

$$T_V^L + T_O^L = T_V^R + T_O^R$$

Since, in this case, $T_V^L \geq T_V^R$:

$$T_O^L \leq T_O^R$$

Therefore:

$$2 \times T_O^L \leq T_O^L + T_O^R = \ell(\tilde{G}) \leq \ell(G)$$

$$T_O^L = \frac{\ell(\tilde{G})}{2} \leq \frac{\ell(G)}{2}$$

Resulting in:

$$t(G) = T_V^L + T_O^L \leq \ell(\tilde{G}^L) + \frac{\ell(G)}{2}$$

Although longer than other proofs in this text, it is yet another simple proof. The equations and the many symbols might make it unclear, but the each of the steps can be visualized. For clarification, it follows an explanation of the starting points of each sequence or equations **TODO: improve clarification?**

$T_V^L + T_O^L = T_V^R + T_O^R$ is straight forward, because what defines the height is the intersection point of left and right, which is the same for both trajectories. $2 \times T_O^L \leq T_O^L + T_O^R$ is a direct result of the previous inequation.

This theorem uses the thermic version of games, which, again, are hard to find in itself. However, from this version, another result that does not use the thermic versions is possible.

The theorem might be re-written as:

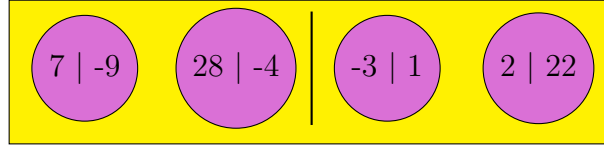
Let n, m two non-negative numbers. If S is class of games such that for all $G \in S$, $\ell(G) \leq n$ and for all options $\ell(G^{L/R}) \leq m$, then:

$$BP(S) \leq \frac{n}{2} + m$$

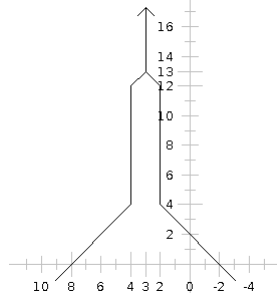
Although immediate, deciding whether the first version implies the version above might require an explanation. $t(G) \leq \ell(\tilde{G}^L) + \frac{\ell(G)}{2} \leq \frac{n}{2} + \ell(\tilde{G}^L)$ is true because $\forall G \in S$, $\ell(G) \leq n$. Since $\ell(G^{L/R}) \leq m$, then $\ell(\tilde{G}^L) \leq m$, which implies the last inequation $t(G) \leq \frac{n}{2} + \ell(\tilde{G}^L) \leq \frac{n}{2} + m$.

Using a class of games is not as straight forward as using the definitions typically used in this text. To idea to use the theorem is to first create a class, following the restrictions - therefore having a bound for the temperature. Then one should prove a game or a pattern of games belong to this class, to make use of this bound.

Let, for example, the class S to be defined by $\{G : \ell(G) \leq 100, \ell(G^L) \leq 200, \ell(G^R) \leq 200\}$. Since S meets the requirements it has the proposed boiling point. $\forall G \in S, BP(G) \leq 250$. The maximum length of the confusion intervals are so large that, in particular, all games of this text belong to S . It is possible to verify that all the temperatures of games instanced in the text are indeed smaller than 250 **TODO: maybe remove this comment**. In fact the hottest game G analyzed in this text was:



With, as shown before, the thermograph:



In this case, $\ell(G) = 10 \leq 100$ and $\ell(G^{L_1}) = \ell(G^{R_1}) = 0 \leq 200$ and $\ell(G^{L_2}) = \ell(G^{R_2}) = 24 \leq 200$. The boiling point bound the theorem provided for S is loose for G . However, it is unclear if that is because the confusion intervals that define the class are also too loose or if the bound itself is large.

Let S_* , now, be the class given by $\{G : \ell(G) \leq 10, \ell(G^L) \leq 24, \ell(G^R) \leq 24\}$. From the analysis of the previous paragraph, $G \in S_*$. However, in this case, the confusion intervals of G are tight in respect to the ones that define the class. From the theorem $\forall G \in S_*, BP(G) \leq 29$. The bound is tighter for G but is still loose, which may lead to the belief that this bound is loose in general.

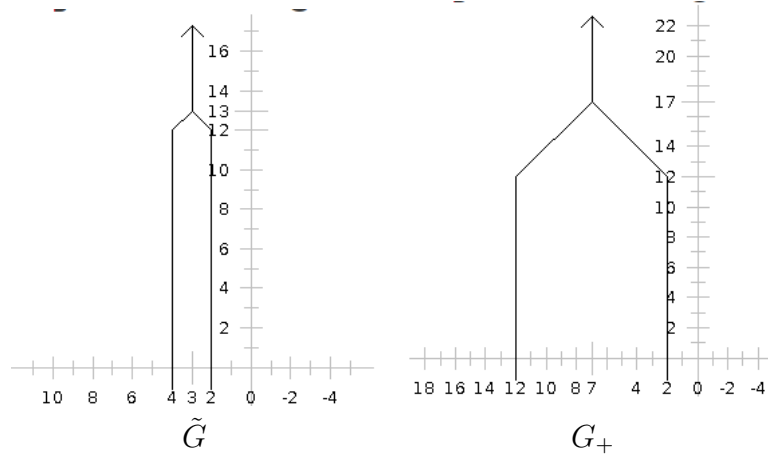
In order to verify whether the bound is in fact loose, it would be needed to show the following, or something similar:

$$\forall S, \exists x \in \mathbb{N}, x > 0 \mid \max_{G \in S} BP(G) \leq BP(S) + x$$

To dismiss the idea, however, it could be shown:

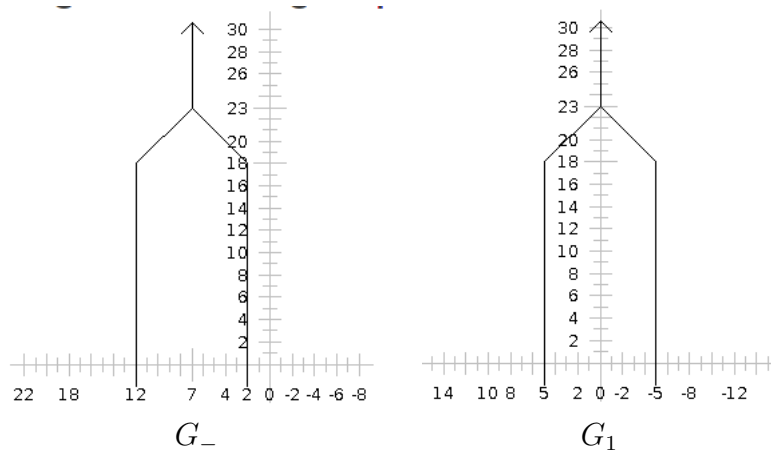
$$\exists G \in S_* \mid \nexists x > 0, BP(G) \leq BP(S) + x$$

Starting out with the previous example, take $\tilde{G} = \{\{28 \mid 4\} \mid \{2 \mid -22\}\}$. The temperature is, of course, the same as before, 13. However, the Left and Right slant may be moved, since $\ell(\tilde{G}) = 2$. Take $G_+ = \{\{36 \mid 12\} \mid \{2 \mid -22\}\}$, result of shifting the left slant to the left, while maintaining the left confusion interval.



The temperature 17 is higher than the initial one but is still far away from the desired 29. However, it is still possible to increase it more. The boiling point of left and right slants are 12, and, for this reason, the oblique line in the thermograph start at $t = 12$. Making the left and right games hotter would increase the overall temperature, but increasing the confusion interval is not allowed as it is desired that the resulting game belongs to the class S_* .

Since the thermographs of $G^{L/R}$ are made of two oblique lines, making part of the lines straight would increase their temperature. One way of doing it is changing G^L to $\{\{60 \mid 36\} \mid 12\}$. The temperature of $G_- = \{\{\{60 \mid 36\} \mid 12\} \mid \{2 \mid \{-22 \mid -46\}\}\}$ is 23. To simplify further analysis, take $G_1 = \{\{\{53 \mid 29\} \mid 5\} \mid \{-5 \mid \{-29 \mid -53\}\}\}$, result of centering G_- at 0. Notice that $G_1 = \pm\{\{53 \mid 29\} \mid 5\}$.



Following the idea of making one of the slants of $G^{L/R}$ straighter in order to increase their temperature, it is possible to keep increasing the temperature of G . Consider the following sequence of games is S_* :

i	G	$t(G)$
0	$\pm \{29 \mid 5\}$	17
1	$\pm \{\{53 \mid 29\} \mid 5\}$	23
2	$\pm \{\{\{77 \mid 53\} \mid 29\} \mid 5\}$	26
3	$\pm \{\{\{\{101 \mid 77\} \mid 53\} \mid 29\} \mid 5\}$	27.5
...		

The result that the temperature of games in this sequence increases and converges to 29 is desired, but it must be proved. The following proof is general for all sequence of games made in the likes of the sequence above.

Proof: **TODO: I made the proof – revise**

→ Each term of the sequence increases the length of the straight segment of the left option of G^L by $\frac{\ell(G^L)}{2^{i+1}}$.

[Via Induction]

Base case: $i = 1$. Let $G = \{\{a + 2\ell(G^L) \mid a + \ell(G^L)\} \mid a\}$. The right trajectory, ρ , of $\{a + 2\ell(G^L) \mid a + \ell(G^L)\}$ is oblique until $t = \frac{\ell(G^L)}{2}$. Therefore the left trajectory, λ , of G is straight until $\frac{\ell(G^L)}{2}$. At $t = \frac{\ell(G^L)}{2}$, the right trajectory is at the $a + \frac{\ell(G^L)}{2}$ x-coordinate. Since both trajectories are oblique above this temperature, they cross at the point $(a + \frac{3\ell(G^L)}{4}, \frac{3\ell(G^L)}{4})$. Therefore, temperature was increased by $\frac{\ell(G^L)}{2}$.

Induction Hypothesis: $i = k$. Let

$$G = \{\{\{\{\dots \{a + (k+1)\ell(G^L) \mid a + k\ell(G^L)\} \mid \dots\} \mid a + 2\ell(G^L)\} \mid a + \ell(G^L)\} \mid a\}$$

Let $b = a + \ell(G^L)$. It is possible to rewrite G so that:

$$G = \{\{\{\{\dots \{b + k\ell(G^L) \mid b + (k-1)\ell(G^L)\} \mid \dots\} \mid b + \ell(G^L)\} \mid b\} \mid a\}$$

Using the inductive hypothesis on:

$$H = \{\{\{\dots \{b + k\ell(G^L) \mid b + (k-1)\ell(G^L)\} \mid \dots\} \mid b + \ell(G^L)\} \mid b\}$$

The left slant of H^L has a straight segment of size $\sum_{i=1}^{k-1} \frac{\ell(G^L)}{2^{i+1}}$. Since the right

option of H^L is a number, $BP(H^L) = \sum_{i=1}^{k-1} \frac{\ell(G^L)}{2^{i+1}} + \frac{\ell(G^L) - \sum_{i=1}^{k-1} \frac{\ell(G^L)}{2^{i+1}}}{2}$. Therefore,

$BP(H^L) = \frac{\ell(G^L) + \sum_{i=1}^{k-1} \frac{\ell(G^L)}{2^{i+1}}}{2} = \sum_{i=1}^k \frac{\ell(G^L)}{2^{i+1}}$. Since the right slant of H^L is oblique with size equals to the given temperature, then the left slant of H is straight with the same size. Since $H = G^L$, then the inductive step is indeed correct.

→ Each term of the sequence increases the length of the straight segment of the right option of G^R by $\frac{\ell(G^R)}{2^{i+1}}$.

The proof is analogous.

→ If both the left option of G^L and the right option of G^R have their straight length increased by l , then so does G .

As seen before, $t(G) = T_V^L + T_O^L$. Since $\forall i, G_i = \tilde{G}_i$ and that increasing both trajectories straight segments does not affect the size of the oblique segments, it is possible to conclude that the temperature of the whole game will increase exactly l . Since, again, the size of the oblique segment does not change than the size of the straight segment increased by l .

→ The temperature of the games in this sequence converges to the proposed boiling point, while always being elements of the initial class S .

Notice that $RS(G^L)$ and $LS(G^R)$ remain the same across all iterations, and that $RS(G^L) - LS(G^R) = \ell(G)$, it is clear that $\forall i, \ell(G_i) = \ell(G)$.

Notice that the observation above is valid for G^L and G^R .

The previous two observations show that every game in the sequence belongs to S .

The initial height is $\frac{\ell(G)}{2} + \frac{\ell(G^L)}{2}$. From the first three points of the proof, at each iteration i the straight segments of $\lambda(G)$ and $\rho(G)$ increase by $\frac{\ell(G^L)}{2^{i+1}}$, $\ell(G^L) = \ell(G^R)$. Since increasing the straight segments of $\lambda(G)$ and $\rho(G)$ does not affect the size of the oblique segments, then $t(G)$ increases by that amount at each iteration.

$$t(G_k) = \frac{\ell(G)}{2} + \frac{\ell(G^L)}{2} + \sum_{i=1}^k \frac{\ell(G^L)}{2^{i+1}} = \frac{\ell(G)}{2} + \sum_{i=1}^k \frac{\ell(G^L)}{2^i} = \frac{\ell(G)}{2} + \ell(G^L) - \frac{\ell(G^L)}{2^k} \leq \frac{\ell(G)}{2} + \ell(G^L).$$

The sequence of games, then, shows that the given bound is not loose. In fact, it shows that the bound is optimal for some cases. It is worth mentioning that, although this section follows the aforementioned paper, the proof above is original. The paper does convey the strategy and the regard about the optimality of the bound but the proof of convergence is skipped. Although long and with many symbols, the proof is quite simple and might be considered trivial.

An interesting way to complete and put to practice the analysis of the theorem is to show it can be used to characterize a pattern of domineering boards. To use the bound all games following the pattern must belong to a class, based on the confusion interval restrictions, like before.

When bounding ‘ $2 \times n$ snakes’ boards, the thesis and the featured papers diverge. Probably in the year between the publish of the thesis and the paper the technique was refined, resulting in the thesis bringing a bound of 5 to the temperature and the paper reducing it to 3. The proof brought to this text will show the point of divergence. Before showing the proof, another necessary definition and lemma are provided.

A ‘snake’ board, same adjective used in “Kim’s snakes” from the previous chapter, is a board without any 2×2 subgrid. A ‘ $2 \times n$ ’ snake is a snake that fits in a $2 \times n$ board. It is possible to use the term ‘snake fitting in a $2 \times n$ board’, and this means that the board is a snake that can be folded such that it becomes a ‘ $2 \times n$ ’ snake with the same value.

The lemma is: Let ϵ be any infinitesimal and G be a hot game. If for all left options G^L , n is a number such that $G^L - G - n + \epsilon \leq 0$, then $\ell(G) \leq n$. The proof is simple, again.

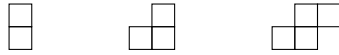
$$\begin{aligned}\ell(G) &= LS(G) - RS(G) \\ &= RS(G^L) + LS(-G) \\ &\leq LS(G^L - G) \\ &\leq n\end{aligned}$$

It is clear that $LS(G) = RS(G^L)$, $RS(G) = -LS(-G)$, but the second equation is there to help visualize the first inequation. The last inequation is true because $G^L - G + \epsilon \leq n \rightarrow LS(G^L - G) \leq n$, explained amid the discussion in the beginning of this section. The first inequation is true because $G^L - G = \{G^{LL} - G^L \mid G^{LR} - G^R\}$, meaning that $LS(G^L - G)$ derives from a game $H = G^L_+ - G_-$. In particular, it is possible to take G^L_+ to be a game that defines the right stop for G^L and $-G_-$ the game that defines the left stop for $-G$, and, therefore, the inequality is true.

The meaning of this lemma is that if it is possible to bound the value of the game to the left, or to positive side, then it is possible to bound the confusion interval. This is a weaker result compared to the previous but it is one that connects the stops with the confusion interval so it is extremely useful.

Now it is possible to characterize some domineering boards. *The temperature of snakes fitting a $2 \times n$ board is at most 3.* The proof consists of showing that $\ell(G), \ell(G^L), \ell(G^R) \leq 2$, by the means of showing that for $H = G \vee G^L \vee G^R \rightarrow H^L - H - 2 \leq 0$ and using the lemma above.

Considering the case $H = G^R$, H is a two component board, so it is possible to separate them in distinct games. Let $-H = -G^R = -G_1 - G_2$. H^L on the other hand is G^R with the addition of a left move. The only cases a left move does not exist in G^R is if G is one of the following:



In all the cases above, the temperature is below 3, so it is not a problem. Without losing generality, let G_1 be a component where there is a right move, and let the initial left move be any such move in G_1 , therefore, $H^L = G_1^L + G_2$. Now it is clear that this case can be reduced to the case $H = G_*$, with G_* a smaller board than G :

$$H^L - H = G_1^L + G_2 - G_1 - G_2 = G_1^L - G_1$$

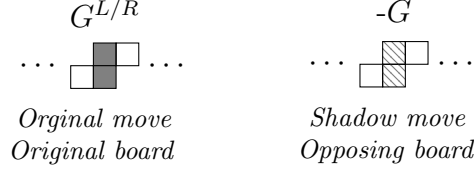
Repeating the process with $H = G^L$, $-H = -G^L = -G_1 - G_2$ and $H^L = G_1^L + G_2$. The only cases G_1^L does not exist is if $G = \square\square$ or $\square\square$ and both of them have temperature lesser than 3. Again it is clear that this case can also be reduced to the case $H = G_*$:

$$H^L - H = G_1^L + G_2 - G_1 - G_2 = G_1^L - G_1$$

Therefore, if $\ell(G) \leq 3$ then the confusion interval of both option are also smaller than 3. To prove that $G^L - G - 2 \leq 0$ it is shown a winning strategy for right in

$G^L - G - 2$, which shows that the position is negative. First notice that $-G$ is G flipped so that it fits in a $n \times 2$ board, but a useful way, that is used in this proof, to consider $-G$ is Left playing on the horizontal and Right playing on the vertical.

To help visualize the strategy for this case, let the ‘shadow’ of a move be the same cells that the move occupied, but in the opposing board. Also the board resultant from play in G^L is called the original board and the board resultant from play in $-G$ is called the opposing board. For instance:



In the game $G^L - G - 2$, if Right plays first, he/she simply has to mimic each and every move Left made by playing the shadow move, maintaining -2 intact, starting with the move already played in G^L . If, instead Left goes first, he/she can play one of two options. The first is to play a move that does not intersect the shadow of the previous move. In this case, right must play the shadow of either the first or the second move.

Essentially, this first case maintains the original conditions while reducing the board size. This case leads $G^{L*} - G - 2$ to $G^{L*L} - G^{L*} - 2$, which, by taking $H = G^{L*}$, is $H^L - H - 2$. Therefore the only worrisome Left move is one that intersects the shadow of the initial move. This move is not exactly the shadow move because in the opposing board Left moves in another orientation. After Left’s move, Right must move on the -2 component, taking it to -1 .

Now, Left can, again, play a move that does not intersect any of the shadows, but this would be met by the same strategy as before, so Left must play a move that intersects a shadow. However, Left cannot move intersecting the shadow of the previous move, because the initial move would be adjacent to this third move and since the mover are both vertical, a 2×2 subgrid would have to exist. The only shadow Left can intersect is that of the first move, playing on the opposing component again. After Left’s third move, right plays on the -1 component.

At this time, Left can no longer play on any shadows, so he/she cannot avoid Right’s strategy and will eventually lose. Therefore, if G is a snake fitting in a $2 \times n$ grid, then $G^L - G - 2 < 0$. This means that $\ell(G), \ell(G^L), \ell(G^R) \leq 2$. By using the bound result, in turn, it means that $t(G) \leq 3$.

This result from the 2019 paper is different, as stated before, from the 2018 thesis because the first part of the proof, that shows the cases $H = G^L, G^R$ can be reduced to the case $H = G$, was missing. Instead the original text only showed that the confusion intervals of G^L, G^R were smaller than 4. This second result is clear because both G^L and G^R are made of two components that are snakes fitting in $2 \times n$ grids. As the text shows that such a snake has the confusion interval smaller or equal to 2, then two snakes have that interval smaller or equal to 4.

7

Unexplored games and final remarks

Combinatorial game theory is a new field in mathematics that was initially found in the blend of recreational mathematics, combinatorics and number theory. While the initial focus is finding strategies to play better and win games, the developments it prompted in other areas of mathematics have their own place, independent of the initial purpose. The Surreal Numbers form a class with many interesting properties and found much enthusiasm for that reason, not because it was necessary to analyze games.

However, the surreal numbers inevitably carry the simplicity inherited from the mathematical plays it was created/discovered for. It is commonplace to learn how to breach the gap between rational and real numbers only in superior education, partially because any of the common constructions involves operations and procedures not used in school. Surreal Numbers give for free the construction of real numbers from its few and simple rules. Not only the reals, but a definition of positivity and negativity that dismisses comparison, a common method for the creation of all the numbers, new numbers and, at last, a direct way of visualizing each number with RB-Hackenbush games. Numbers are the main building block the text used to analyze games.

The non-numbers, a name used in this text for games with temperatures, are the remaining games and required more elements and concepts to be understood. The thermograph, or the form commonly used to visualize temperature, options, stops and boiling points, is extensively analyzed and the text presents one of the most recent related results.

The reader that understands all the featured topics and is able to replicate calculations and results to games other than the mentioned might carry the false belief that understands all of combinatorial game theory. This text, however, only describes the class of partisan games leaving many variations even unmentioned. Not only the focus is specific, but also not all perspectives necessary to tackle the problem of finding the best move are visited.

These information serves to show that there is much more to the field, not that little progress understanding it was made. In fact, it is fair to state that the classes of games discussed in the text are the more complicated ones and many of the definitions apply to other classes.

Both in the results discussed and the ones omitted there is a common trend in

this field. The definitions are always simple, but powerful enough to allow many questions, and, although there are many open questions, the proofs are and solutions are simple as well. The procedure to build all dyadic rationals in domineering boards, for example, is based on a theorem provable in two lines of text and applying it is the simplest of ways. Although its explanation is simple, it remained open for years and the resulting pattern is not something that would be found before the proof.

Finding the upper bound to temperature is another case where the question remained open for long, but the strategy used to answer is simplicity: it consists only of listing and manipulating properties of the thermographs. That is not to say that the bounding of temperature is completely solved but the progress made used simple steps.

From this perspective combinatorial game theory is a beautiful field in mathematics. The definitions it is based upon are extremely few in number and very simple in form, but they allow proving more complex problems in a uncomplicated manner. However, in this text, one of the unvisited places in the field is computational complexity. In practice, calculating values is not an easy problem.

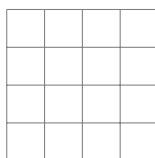
The other featured aspect skipped is the variety of forms combinatorial games might take, because the text only deals with partisan games played on normal conditions. The remaining contains a brief description of these two skipped problems and points toward texts on the respective subjects. This chapter, that finalizes the text, also addresses a few of the next steps studying or researching the field and ends with a personal reflection on the learning, implementing and writing experiences involved in this work.

Undeveloped Pieces

Complexity

The text repeatedly made the case that most concepts are simple in Combinatorial Game Theory. It said, for example, that the method used to find the number a game is equal to is straight-forward. However this hides an important factor.

Either calculating the number or building a thermograph requires, first, traversing the game tree. Traversing this tree and calculating the values is simple enough, but it is slow. And it is slow because this tree quickly becomes large. For example, consider the following game of Domineering.



There are, of course, 25 squares, which means that at most 12 moves might be made and in addition, some of these moves might be equivalent, so they do not have to be computed multiple times. It happens that only 604 nodes, or different position, may be reached[?]. In a 6×6 board it jumps to 17,232 and in 7×7 and 8×8 board it goes to 408,260 and 441,990,070 nodes. Although these results are from 1998, results on larger boards took many years and only in 2016 a result for 11×11 [?] board was found, but abandoning a naive building of a game tree. The game tree of a 9×9 board has around 25 trillion nodes.

If not clear at first it should be now that there is no known form of calculating values for games in polynomial time in respect to their sizes. The question, then, is in what class of problems it fits. It happens that many of the games are known to be PSPACE-complete and some are known to be EXPTIME.

There is of course much more to be said and it is another very interesting topic. A good starting point is *Playing Games with Algorithms: Algorithmic Combinatorial Game Theory*[?]. The thesis *Games, Puzzles, and Computation* is more extensive, covering different classes of games, but may be more complex. Regardless of the suggested texts, there is a large amount of great books on complexity theory.

Other games

By looking at the definition of combinatorial games given in the beginning of section 2, it is easy to come up with games that do not fit well the class of games studied, the short games. One clear example is the game of chess, because there are ties and there are drawn positions.

However, games like chess are not the only class of games that behave differently. Consider G-Hackenbush, a variation of RB-Hackenbush, in which both blue and red edges are replaced by green ones. Green edges may be removed by either player, meaning that both players have the same available moves regardless of the position. The class of games that follow this property is called Impartial Games.

Impartial games were the first games studied, previously to the consolidation of the modern form of the field. They have an interesting property of being reducible to an instance of a game of Nim.

Games like chess, on the other hand, are not so simple. there are more classes of results than short games, which allows games to become more complex. Such games benefit from knowing short game's theory but do require further concepts. A good place to start is the corresponding chapter in the book *Combinatorial Game Theory*.

Post-Match

The choice of Combinatorial Game Theory as subject was an extremely fortunate decision. This field triggered the feeling of uncovering something beautiful hidden in plain sight. I believe the book *Winning Ways* played a role in that due to the presentation I became found of, but I do considered there is inherent beauty in the field.

I believe the simplicity of everything about it is something to awe. This is the most important reason why this thesis ended up being about it and not some other subject. The first step in this directions was a great interest in the games of Chess, Shogi and Go. The first idea I had was to develop heuristics to find shortest paths in the game of chess, and possibly extend that to other games using more general graph theory.

During this period I started reading articles and scrolling through books and that quick led me into the field of Combinatorial Game Theory. The book *Winning Ways* is one of the canonical references and I luckily started with that. This choice led me to a year of great learning experiences, in many areas, and personal maturing.

Learning and writing about the topics of this text was the most important activity to conclude my Bachelor's Thesis, but there are other subjects I studied in

the process that went unmentioned in the text. I am grateful for those too, as I got to develop my understanding of real analysis and complexity theory further than I would normally, for example. This year was also the one of we had the corona virus widespread and, hopefully, the last year we have to partake isolation and health/economic insecurity.

I had only handful of on-campus classes and that required me, and most of the world, to adapt to new circumstances. While it did not effect the content of this text directly, it greatly impacted the world we live in. Frequently during the year I found myself struggling to concentrate and in a general bad mental state. The consistent work it took really helped keeping me on track. Another factor that can not go unmentioned is the help of my advisor.

José Coelho kept regularly meeting me almost every week during this year. That helped a lot keeping me in check and I am extremely thankful of the dozens of hours we spent talking and discussing the most varied subjects. In the times I did not have work to show he was understanding and his willingness to help pushed me further and further.

Bibliography

Appendix A

(Appendix A title)

The final sections of your thesis are the appendices. Each appendix should be lettered (A, B, etc.,) and should consist of detailed information that is interesting but not essential to the main thrust of your findings section.

The appendices should be in the order that they are referred to in the main text. For instance, if Appendix A refers to something on page 25 and Appendix B refers to something on page 15, the appendices need to be re-lettered. This inconsistency occurs when text is moved around or inserted.)

Appendix B

(Appendix B title)