IME-USP

Universidade de São Paulo - Instituto de Matemática e Estatística
Bachelor's Programme in Computer Science

HOT GAMES

Temperature, advantage and numbers

by

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Abstract (The Abstract is a short summary of what your thesis is about. It accurately reflects the content of the thesis providing information about the research problem, research aims, methods and procedures, results and implications. It is a short section. Abstracts give readers the opportunity to quickly see the main contents of the paper and enable them to decide whether the paper is of particular interest to their needs. This section will be one of the last sections that you write. No subheadings are used in an abstract.)

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Introduction

> "I learned very quickly that playing games and working on mathematics were closely intertwined activities for him, if not actually the same activity. His attitude resonated with and affirmed my own thoughts about math as play, though he took this attitude far beyond what I ever expected from a Princeton math professor, and I loved it."
>
> Manjul Bhargava [1]

It is no surprise that avid players of games that resemble a logical or mathematical puzzle, like checkers, develop an intuition that allows them to calculate faster, by asserting bad moves fast, recognizing winning and losing patterns ahead of time or many other strategies. Many times, it is not easy to understand or explain how does it work, while others, after seeing a move being played, it is easily possible to explain it by the principles that rule the game. A most essential, and sometime very hard, component of playing well any of these mathematical games is being able to know if you are ahead or behind in a given position.

While evaluating whether a position is winning or losing is already a hard task, with, for example, modern chess engines not agreeing in some positions, it is possible take this problem to higher standards. Imagine you are playing a variant of the game of chess. In this variant, each player is given a set of board positions, and each should choose one board to play as white. During this game, play will take place in each board in parallel, and, whoever checkmates the opponent faster, wins the game. If one wants to be a great player of such game, the knowledge of asserting if a position is winning or loosing in a regular chess game is not enough, nor is the ability to play regular chess perfectly.

The most important ability for this game and for the games that will follow in this paper is to score each position. The ability to calculate the precise advantage a player has over the other is the object of interest of the field called Combinatorial Game Theory, inaugurated in **TODO: 1970** by the paper **TODO: ONAG**. The author of that paper, Jonh H. Conway, as found in the epigraph that starts this text, was, as many, an avid player of such games, however, unlike all others, was one of the brightest minds in the history of mathematics.

Dr. Conway realized that some games, to be defined in the next chapter, behave like numbers in every aspect. While, initially, that seems very useful to calculate the advantages, and win in the presented variation of chess, this realization means much

[1] Fields medalist commenting on John H. Conway's passing

more than this tool. It is true that some way of modeling the problem of assigning values to positions was the initial challenge and this would be a great step towards it; however, what came out from Conway's work was a **TODO: match** between positions and numbers, not a measure of the position.

This **TODO: match**, that is further explained in the following chapters and the main topic in this field of study, is built upon a new set of numbers. This special set of numbers was created, or discovered, when piecing together the necessary element to analyze any particular game: its game tree. Of course a computer program could traverse a game tree and tell who is winning the game, but it would not be able to easily tell who would be winning if two game trees were added. **TODO: citation!** Even defining how to meaningfully merge game trees is problematic.

Not only does the **TODO: system** created by the **TODO: magical genius**[2], Conway, gives the ability to add, but it also defines a method to multiply games. However, in order to define such methods, the definition of that new set of numbers was necessary. The Surreal Numbers, name given by Donald Knuth[3], **TODO: for starters**, were required as, in this **TODO: system**, the number **TODO: matching** a position might not belong to the set of real numbers.

As some might understand from the title alone, however, the focus of this text are in the non-numbers. It is possible for games to be **TODO: unwritable** as a surreal number, although every surreal number has correspondent[4] games. The concept of temperate, by consequence the concepts of hotness and coolness, however, does not forego the understanding of numbers.

The rest of this chapter relies on the next chapters. Skipping for now.

History of field of study - Quem inaugurou - Conway. Onag (paper que iniciou a area) - Conway. WW (winning ways – referencia canonica). Morreu esse ano - Combinatorial game theory - Aron S. (abordagem diferente – coisas q n tem no WW; especialmente em relacao a classificacao dos jogos) - Principais nomes da area - Conway + 2 - dialogar com titulo: hot (nao eh conceito inicial da area) / games (nao eh definicao obvia) - hot: temperatura/topico avancado: Conway. - Traduzir intuicao em definicoes matematicas - Definir o que eh games: games = mathematical play. Especificar. - Numbers

(The Introduction chapter should contain background information as appropriate, plus definitions of all special and general terms. Your topic should be: clearly stated and defined; have a clear overall purpose; and have clear, relevant and coherent aims and objectives. It is also informative to give a brief description of the contents of the remaining chapters of the thesis. This alerts the reader and prepares them for the rest of the thesis.)

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[2]blabla

[3]The original name was...

[4]Any surreal number is **TODO: matched** by an infinite number of games

2

What to do with pen, paper and a friend

The Rules

In order to study mathematical plays and answer the many questions they raise, a new mathematical field of study was developed and many new terms were created. The phrase “mathematical play” is in itself a new term, for instance. While the most common term is ”Combinatorial Games”, the canonical reference for this field *Winning Ways for Your Mathematical Plays* 1981, uses the former, not the latter. As more is said about the topic, more meaning the term “mathematical play” is going to acquire.

The name “Combinatorial Game” might bring to light some information. It, at least, means that this field will deal with games, as in, an instance of a Game Theory problem, and, more specifically, a subset of those games. It also brings to light that the use of counting, finite structures and, most likely, graph representations are omnipresent. However, a definition of the object of interest becomes possible with the name mathematical play.

To play something mathematically could be understood as to engage in an activity in which the better use of mathematical ability, such as counting and logic, would result in advantage over its poor use. However it could be detailed further to an activity in which mathematical ability is the single defining factor. The later might make more sense because there are games, like poker, that do require some counting ability. However, luck and reading behavior skills are much more valuable to a successful game and this is something the definition would be better off forbidding.

Chance moves, like throwing a dice or flipping a card, are not fit for mathematical plays. Even with their removal, however, there are possibilities that would not be comfortably called mathematical plays. The nature of a mathematical plays is that both players can engage the same activity and generate advantages out of “good play”. For instance, it would be hard to agree that two people playing rock-paper-scissors are battling a mathematical fight, even though there are no chance moves.

It is very important that all players have **complete information** of the position. Games like rock-paper-scissors, in which players take action simultaneously, block complete information. Therefore, players must **move alternately**. The last concerning factor in discerning mathematical from non-mathematical plays during this analysis is the number of players.

When each player has more than one opponent a goal greater than gaining advantage arises. When playing with over two people it is frequent that the best move is not the one that brings a better position but one that prevents any of the opponents from gaining an winning advantage. While that can be very mathematical, there is a clear distinction between sticking to two player games and allowing any number of players. Notice that one can consider soccer as a two player game - even though there are multiple agents in a team. In order to focus on the mathematical ability to make the best move, the option to allow only **two players** is the most interesting.

The only remaining criteria of this definition, as established in [**TODO: WW**], is related to the term play, and not the term mathematical. The rules of the game must guarantee that from any starting position, **TODO: play should always end because a player will not have moves available**. If following "**normal play**" convention, a player that cannot move is lost. It is correct to assume normal play, unless specified otherwise, in this field of study.

The foundations of mathematical plays, highlighted, give light to a complex and rich set of problems, although, at the same time, other complex and rich problems are left behind. The game of chess, for example, does not meet the ending condition and, therefore, is left out. Fortunately, games like chess might benefit from these studies with adaptations or additional rules (although they do not consist of good examples of combinatorial games).
Let us consider the following game:

Red-Blue Hackenbush

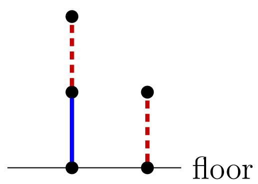


Figure 2.1

In RB-Hackenbush, a move is made by taking a single colored edge of the image and removing any edges that become disconnected from the floor. The player called **Left** can only remove **bLue and soLid**, edges, and the other, called **Right**, **Red and dashED** edges. It is a common practice to assume that all games are played between Left (blue) and Right (red), although, occasionally, new colors will be presented. In this text, every color also has a matching style.

The RB-Hackenbush is a game by the routine definitions of game, meaning it has a clear ruleset and potential to be fun. It is also a game by the definition above, which will be the new "routine" one used going forward. However, the instance of this game drawn above is also called a game. In the future, when analyzing a board state for example, the word position may seem to make more sense, but the correct word is game. The meaning of the word must be deduced from context.

The **game tree** term refers to the routine meaning, for example. The image below is the game tree that arises from the game, instance, presented above.

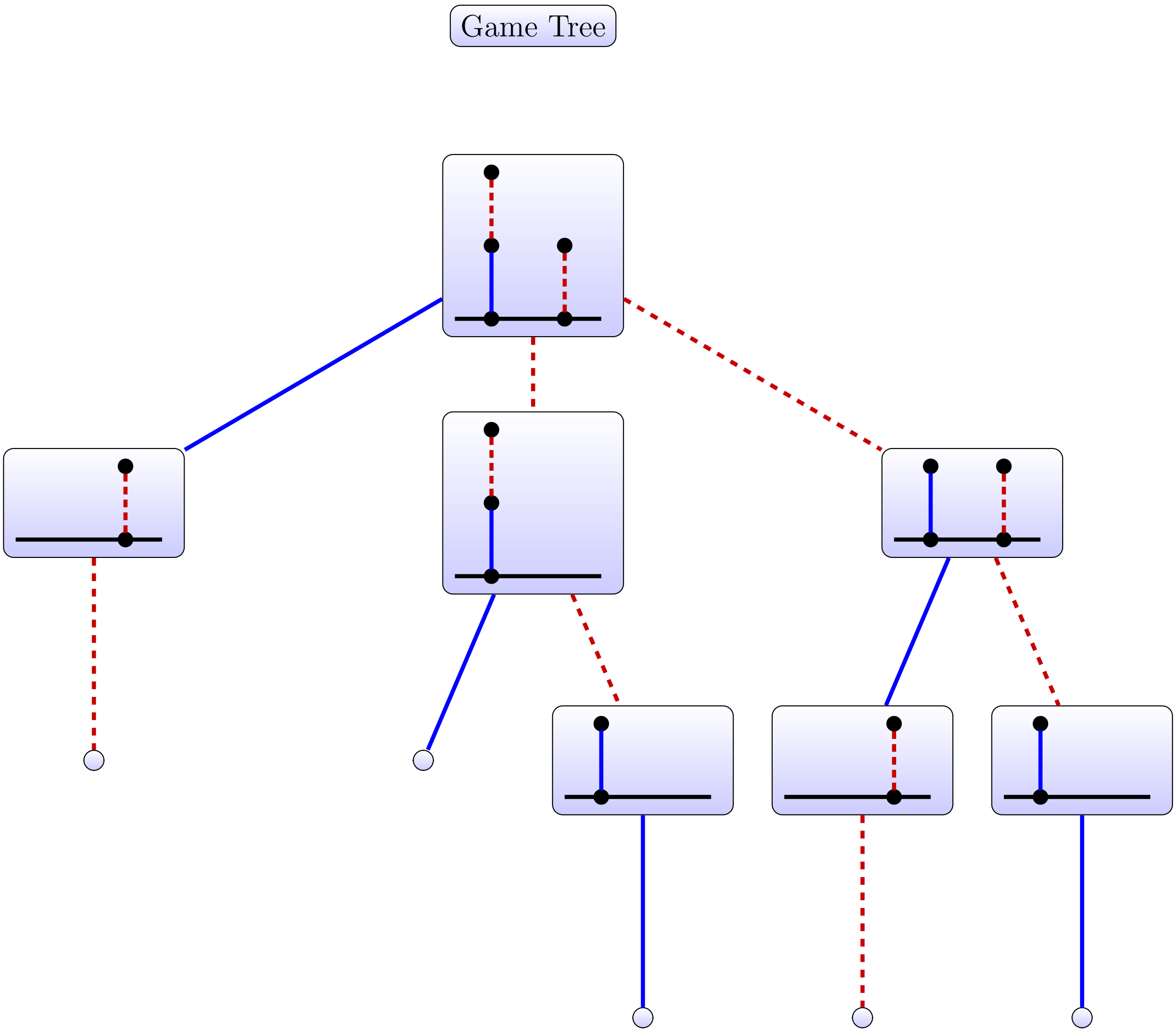


Figure 2.2

In the tree above the styled edges, in the same pattern as before, between configurations tells which player made a move. The game trees used in **TODO: CGT** may me slightly different as they, from each configuration, develop the moves from both players. This is a most import characteristic because it allows the sum of trees.

As noted previously, the game tree contains all the information required in order to calculate the number, or non-number, a game is equal to. The Surreal Numbers **TODO: KNUTH** have a recursive nature that can be completely separated from games. However, as they were created analyzing a position like the one above **TODO: ONAG**, its definition will be presented in terms of moves.

The model Conway, Berlekamp and Guy created to analyze games is based on finding the advantage a player has over the other. The calculation, in this model, of this advantage is given in terms of spare moves. For now, the reader may found this weird because the advantage might actually rely on the ability of the players, but when analyzing games it is important to expect both players play perfectly. Since a player loses if he or she cannot move, counting spare moves is counting how many sequential moves a player can make before reaching **equity** in the position.

Equity is found in **zero positions**. Zero position are those in which the first player to move loses. The idea to call such positions zero made sense for Conway, and, therefore, in his new set of numbers, if a game $G$is in a zero position, $G = 0$. If left **can win** regardless of who starts, we call it positive, and, in the other way around, negative. In more special positions, a hint on the topic of this text, in which the first to play can win, we call them fuzzy.

Zero RB-Hackenbush positions

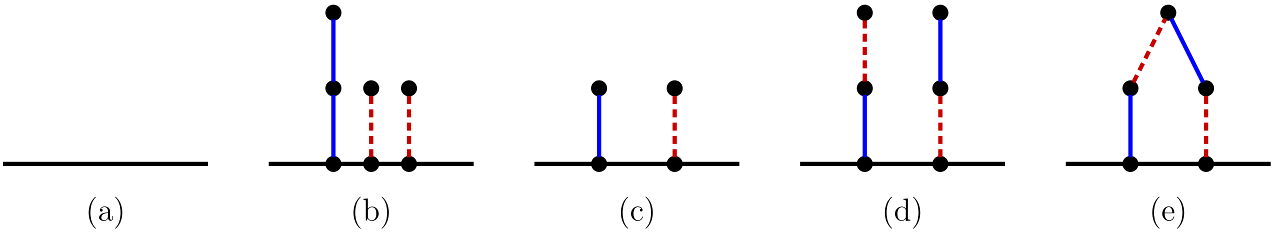


Figure 2.3

Although all games have the same value, the games are not the same. The way to represent a game derives from its game tree. The game is composed of two sets of games and | is used as delimiter. The game (a) is one where neither player has available moves and, because of that, the game $(a) = 0 = \{0 \mid 0\}$. (c), on the other had, is $\{\{\,\mid\{\,\mid\,\}\}\mid\{\{\,\mid\,\}\mid\,\}\}$, that simplifies to $\{\{\,\mid 0\}\mid\{0\mid\,\}\}$. The games that form each of left and right sets are the configurations reachable by left and right, in a recursive definition.

The notations is exactly the same for numbers, but they should not be confused. For a game $G = \{x_1, \ldots, x_n \mid y_1, \ldots, y_m\}$ to be a number, it must be true that:

$$\forall x_i \in X, \forall y_j \in Y, x_i < y_j$$

The process of finding the number a game is equal to is the theme of the next section. With that said, some new number already showed up in the figure above. The number $\{\ 0 \mid\ \}$, for example. What would be a pleasant real number for it? 1, because in this game, left has exactly one move to spare.

For the next few concepts a new game must be presented as RB-Hackenbush is incapable of generating fuzzy positions. However, a proof that all such games are numbers is due, and will be provided in the next section.

Domineering

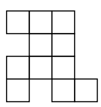


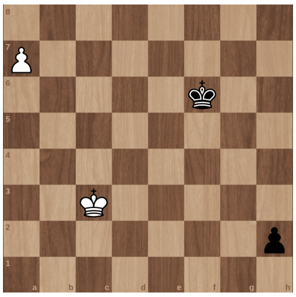
Figure 2.4

Domineering is played by placing, or marking, a 2x1 rectangle on the board, or drawing. Left plays in the horizontal and right in the vertical. The reader is invited to play the position above a few times, and realize that the player starting it is always able to win. In fact, with perfect play, both players can win with a move to spare. The quest for calculating the advantage a player has, so far, can be been summarized by the question: "Who is ahead, by how much?". However, in positions such as the one above, a new question is more important: "How big is the next move". **TODO: Berlekamp 1**.

This second question is answered in temperature theory. Temperature theory must follow a detailed explanation and clear understanding of numbers and games, and, therefore, a good explanation will only take place in section 4. However, as this is the main tool used to develop the topic of this text, an idea will follow.

Temperature measures the activity of the position. The activity can be understood as the importance of the next moves. In chess, for example, closed positions would be colder than endgames positions. In the position below, being the next player to move is not that important. Both players would start improving the position of the pieces until one find a break-through. Being a move behind means less piece development but the game will progress slowly, reducing the impact of that.

In **hot** positions, opposed to **cold** position, the next moves are paramount for **both** players. In the board below, the player who moves first has a clear path to victory. It might be time to re-iterate that chess in itself is not a combinatorial game, although the following position can be considered one, as there are no drawing chances. In this specific position an subset of chess that removes stalemate from the ruleset is equivalent to chess, so it is possible to make it fit the definition.



It is correct to imagine that when putting a hot game G in the notation $G = \{X \mid Y\}$, then, $\exists x_i, y_j \mid x_i \geq y_j$, making it obvious that G is not a number. On the closed position, the game might be cooler, but it will definitely heat up after a few moves. The endgame position above however, is becoming a number after the next move. Because it is becoming a number, it receives the special name **switch**.

Switches are the most basic non-numbers. A switch is a non-numbers G that both left's and right's best moves are numbers, but $G^L \geq G^R$. It is extremely easy to find the **temperature** and **bias** of such games.

A Simple Switch in Domineering

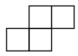
G = 

Figure 2.5

In this game left has a move that leads to a zero position and right has two moves that lead to the same game with value -1. Therefore, $G = \{0 \mid -1\}$. The bias is the average of G^L, G^R that is equal to -0.5 in this case. The temperature is how much the game differs from the bias that is 0.5 in this case. A better way of writing this switch is $G = -0.5 \pm 0.5$. In general, a switch $H = \{x|y\}$ can be written as $H = (x+y)/2 \pm (x-y)/2$.

Calculating these values for general non-numbers is not simple enough to be completely conveyed at this point; Not because it is too hard or that describing it does not fit in the introductory pages, but because better understanding of numbers and arithmetic is required.

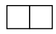
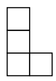
3

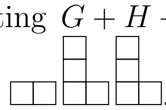
Numbers are games. The reals, the cardinals and many others.

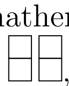
Conway, by counting the advantage, a player has over the other in a combinatorial game, unveiled a new way to discover numbers. It is clear at this point that numbers are not enough to represent games, but it should in no way discourage anyone interested on them. Until this point, the text showed some instances of the zero and hinted at 1 and -1, and the reader may also know how to build any integer in RB-Hackenbush and Domineering. This chapter shows that knowing that is no more than scratching the surface of surreal numbers.

For the first parts of this section a number $\{x_1, x_2, ... \mid y_1, y_2, ...\}$ might be called like a real number: $2, 5, 100000, \frac{1}{3}, \sqrt{10}, \pi$, but there is no reason believe this equality yet. There is also no reason to believe that $1 < 3$ or that $1 + 1 = 2$ yet. However, first some numbers will be labeled and only then the proofs are shown.

Finitely Defined Numbers

It is known that $G =$  $= \{ \mid \{ \mid \}\} = \{ \mid 0\} =$ -1. What about $H =$  ?

We can find that by calculating $G + H + H$. To do that, one would usually find the game tree of $G + H + H$ , and fill the known values bottom-up. However, it is simpler in this case. $G + 2H = 0$, because whoever starts loses. Since $G = 1$, $H = 1/2$. Therefore $H = \{-1, 0 \mid 1\} = \{0 \mid 1\} = 1/2$. It might seem weird that a player may be half a move up in a game, if he or she only plays a move, but it is true. It might be valuable to reiterate that H is definitely positive because left wins no matter who starts, but $H < 1$, as left does not have remaining move at the end of the game.

As one gets used to this area of mathematics, it becomes clear that analyzing the game $\{1 \mid \}$ is the same as analyzing , but the former is not reliant on a specific game. Rather, any combinatorial game has an instance equal to $\{1 \mid \}$. However, the rules, which have not been presented yet, come from the the mathematical plays. The first practical rule in this text is finding the value of $\{n \mid \}$ and $\{ \mid -n\}$, with n natural.

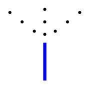
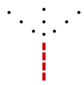
Of course $\underbrace{\square\square\cdots\square}_{n+1} = n+1 = \{n|\}$. Although this is very simple looking at domineering boards, one could achieve the result through other means.

Proving that $n+1 = \{n|\}, \forall n \in \mathbb{N}$:

$$\begin{aligned}
&\text{if } n = 0: \\
&\qquad 1 = \{0|\}, \text{already shown.} \\
&\text{if } n = k: \\
&\qquad \text{Suppose it is valid until } n = k-1. \\
&\qquad k+1 = (k)+1 = (\{k-1|\})+1 = \{k|\}.
\end{aligned}$$

In the last line of the proof above, the addition $(\{k-1|\})+1$ was used. The notation was not defined yet, but that is a simple game addition showed multiple times already. Up until this point, the integers and 1/2 are defined. The remaining numbers fall into two categories: the multiples of powers of two and the rest. Of course the integers and 1/2 fall into the first category as well, but the process to find integers is different **TODO: and 1/2 was a good example**.

In the case of $\frac{1}{2} = \{0|1\}$, it is true that $G^L < G < G^R$. Is that always true? Think of moves made in numbers: if left makes a move from G to G^L, is it true that left has fewer spare moves in G than in G^L? Before, as a side note, it was said that all possible RB-Hackenbush games numbers, and the reason may help explain that.

Suppose a game $G = \{x_1, x_2, ...|y_1, y_2, ...\}$ in which $\forall x \in X$ x is a number and $\forall y \in Y$ y is a number. Assume that x_i and y_j are best moves for left and right respectively. Suppose that G is not a number, than $x_i \geq y_j$. A move in G corresponds to removing a colored edge from a tree and all the edges that become disconnected to the floor. That means that $x_i =$  and $y_j =$ . Is it possible that $x_i \geq y_j$? A careful reader spotted immediately that $x_i > 0 \land y_j < 0$. That means that $G^{x_i} < G < G^{y_j}$, which is a contradiction with the statement that $x_i \geq y_j$.

Other phrasing for “all RB-Hackenbush games are numbers” is “it is not possible to make a move that improves your position in RB-Hackenbush”, or, “left cannot make a movement that increases the value of G”. Now, it may be clearer why, in numbers, $G^L < G < G^R$. Knowing this, however, is not enough to find the value of G.

If $G = \{3|10\}$, it is clear that $3 < G < 10$ but what is the value of G? The **simpler** number that fits the interval, 4. There are some equivalent ways to check which number is simpler. A good one is figuring out which one require less effort to write in their recursive and simplified form. For example, the simplified, recursive form for 4 is $\{\{\{\{\{|\}|\}|\}|\}|\}$. Another good way is deciding which one is younger in the number tree represented below.

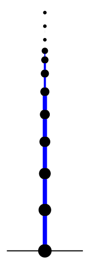
It is very recommended to understand that the **simplicity principle** is indeed finding the simplest number from a range of possibilities. An integer with smallest modulo is simple than one with greater modulo, and an irreducible fraction with denominator 2 is simpler than one with denominator 4. The formula for the simplicity principle, however, is

$$\text{G} = \begin{cases} 0, \text{ if } G^L < 0 < G^R \\ n+1, \text{ if G} = \{\text{n }|\} \\ -n-1, \text{ if G} = \{|\text{-n}\} \\ \frac{2p+1}{2^{q+1}}, \text{if G} = \{\frac{p}{2^q}|\frac{p+1}{2^q}\} \end{cases}$$

Some examples are $\frac{1}{4} = \{0.1|0.3\}$, $\frac{1}{8} = \{\frac{1}{9}|0.2\}$, $\frac{-3}{4} = \{-1|-0.6\}$.

Other Numbers

The formula above only allows G to have an infinite amount of values, but it is not closes to what was stated in the beginning of the section. The remaining numbers are hidden in the end of infinite games, in the sense that the board size is infinite, because, as explained before, it is extremely unpleasant to play something that does not end.



In the game above, left has an infinite number of possible moves, but **TODO: his/her** move always leads to an integer. What is the value of this game? $G = \{\mathbb{N}|\}$, the number "the largest integer plus one". It happens that this number has been baptized much earlier in mathematics of ω.

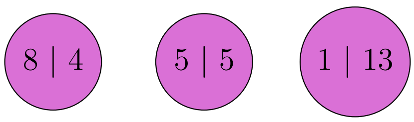
4

Heating things up

Now that enough is known about numbers, it is possible to work with non-numbers. The only known non-numbers at this point are the switches. But is knowing what they are enough to playing them? The game simpler cashing cheques will tell.

In this game there is a table with purple cheques. Each cheque has two numbers written on top, and, in each player's turn they will either pay one coin or cash a cheque that will grant him a number of coins equal to the correspondent associated integer. What is the best move for Left?

Simpler Cashing Cheques



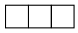
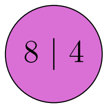
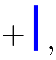
Definitely the move is not paying, as Left can earn money in his turn. A good thing to grasp from this example is that you should never play in a number, paying a coin in this case, if there are non-numbers, cashing a purple cheque in this case. Should Left cash 8, 5 or 1? 1, of course. The reader is encouraged to play as Left and trying to find the best possible outcome, but the answer is playing the hottest switch. Although the game above is not a switch, it is a sum of switches, and, because of that can benefit of the simplified notation discussed earlier.

$$
\begin{aligned}
G = & \left(\frac{8-4}{2} \pm \frac{8+4}{2}\right) + \left(\frac{5-5}{2} \pm \frac{5+5}{2}\right) + \left(\frac{1-13}{2} \pm \frac{1+13}{2}\right) \\
= & (2 \pm 6) + (0 \pm 5) + (-6 \pm 7) \\
= & -4 \pm 7 \pm 6 \pm 5
\end{aligned}
$$

If you analyze the result above, it becomes clear that Left must play on the rightmost component as, although it will not provide many coins, it will prevent right from cashing a huge amount. It is very possible to build scenarios where a player would even pay for cashing a cheque if that prevented the opponent from getting rich. Now that playing a simpler cashing cheques became easy, a more challenging task will rise. How to play Domineering well?

Adding a number with a temperature in a simplified position, like the expression above, should be acceptable by anyone following up to this point. Following, in the other hand, numbers will be added together with non-numbers just like number are added together and this might cause confusion. However, understanding that this sum is possible and intuitive is simple.

Playing a sum of games is just like playing a game with a set of independent rulesets and components.

G =  +  + ,

for example, is a game where each player makes a move[1] in any of the components and loses if cannot make a move. In other words, G is a game like every other, except for the more complex ruleset.

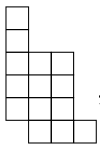
The result of this sum is obvious if all components are also numbers or switches. In the case of playing numbers and general non-numbers, sensible players will always play in non-numbers first. With this in mind other facts become clear. The first is that the temperature of a non-number added to a number is unaltered.

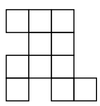
The second is that such a sum is actually a sum of non-numbers, added together with a number after they cool out. The sum of general non-numbers is thoroughly discussed in the remaining of the chapter, however, it worth noticing what the goal of this discussion is.

By the end of this section it will be thought how to convert any game in a 2D graphic composed of two **TODO: curves** that collapse to one line at some point, whose axes is number x cooling factor. The purpose of all this is that if the ending point falls in the positive side, Left gets an advantage, and if it falls in the positive, right does.

To build this graphic one is required to traverse the game tree, so the effort may seem fruitless as the game tree itself provides the winning strategy by itself. However, in cases where the game tree resulting from the sum of games is too large or expensive for a computer to run, there is a good strategy to playing this sum without knowing the complete game tree. In order to build the thermograph and play the **thermostrat**[2] correctly, there are a lot of minor concepts not discussed yet.

Other than the bias, playing a game like Domineering well involves the concepts of **Left/Right stops**, **toenail**, **ambient temperature**, **freezing point**, **cooling**, **heating** and a few others. To put all that together and provide a clear visualization of the best strategy, the thermograph.

Given G = , how to proceed?

Given G = , how to proceed?

G is definitely not a switch nor a sum of switches. It is possible to say the temperature in G is going to stay high for quite some time, because hotness is a

[1] There are other ways of playing G that will not be discussed

[2] This text will not present the thermostrat

term used to define the importance of the next move. A good place to start is writing out the game tree and building a temperature graphic of how it builds up from simpler positions until the more complicated ones.

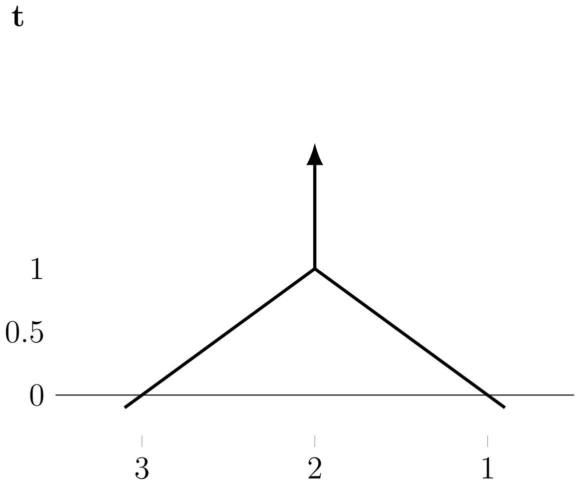
To decrease the confusion that builds up in complicated positions, there is the idea of a cooling factor. The non-number G cooled by t degrees is represented by G_t and is defined by:

$$G_t = \{G_t^L - t \mid G_t^R + t\} \ \forall t \leq t'$$
$$G_t = \mathrm{x} \ \forall t > t'$$

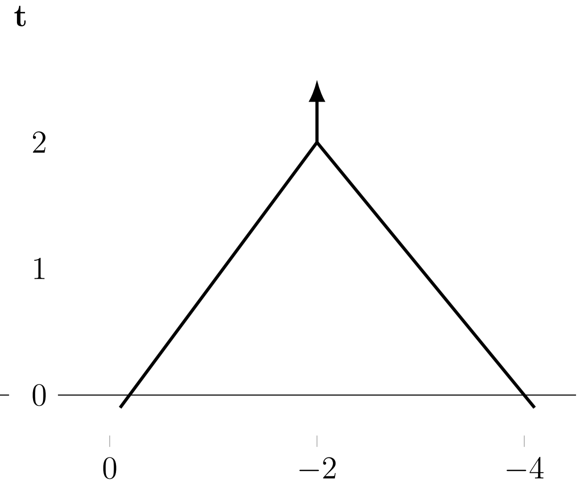
Given t' is the smallest cooling factor such
that $G_{t'}$ is infinitesimally close to a number x,

The temperature $t(G)$ is equal to t'. Now that both axes are defined, some examples of thermographs:

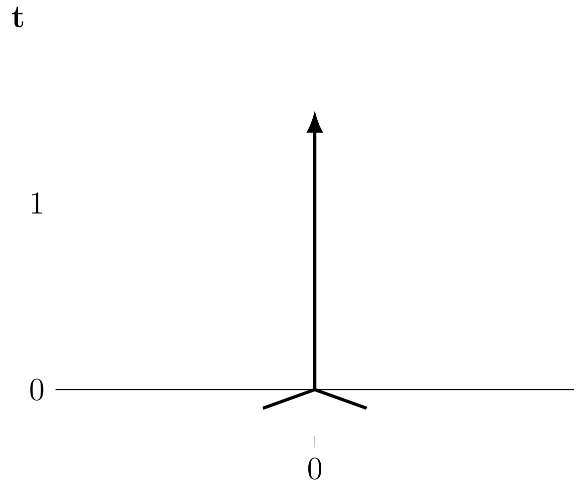
$G = \{3 \mid 1\}$



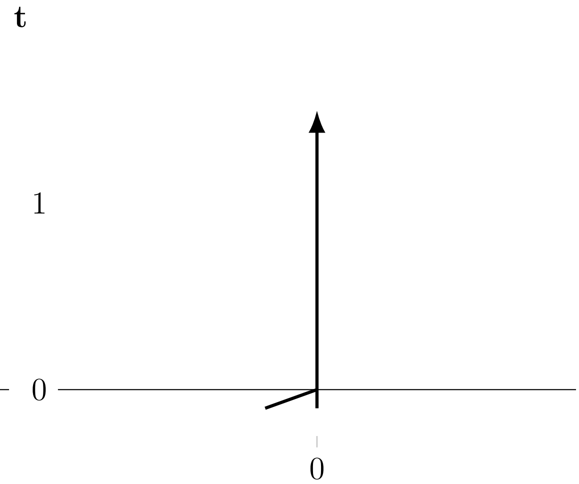
$G = \{0 \mid -4\}$



$G = \{0 \mid 0\}$



$G = \{0 \mid \{0 \mid 0\}\}$

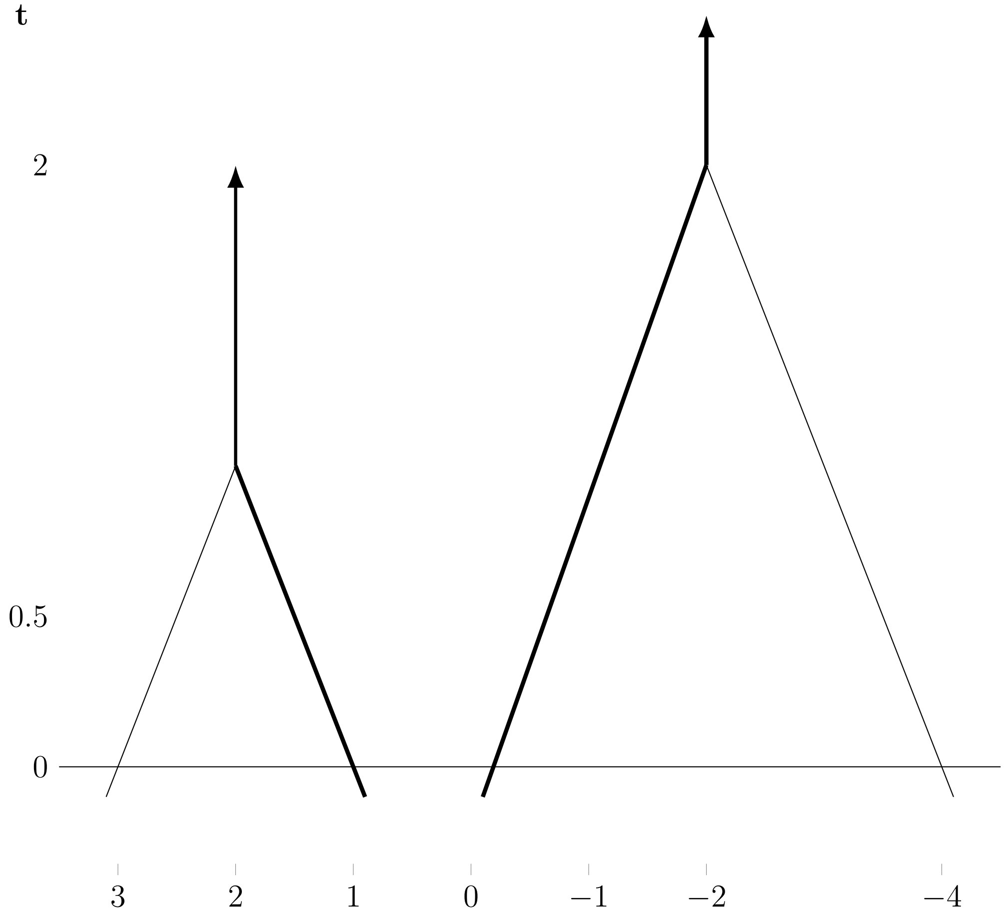


Some characteristics might be immediately apparent. The first is that the x-axis is reversed. The reason for that is to keep Right's movements to the right and Left's to the left. The second characteristic may be that all the thermographs end with a vertical **mast**. The mast begins at t' and indicates that G is a number from that point forward. The last one is that the graphic continues past the $y = 0$ line. It is worth noticing that the difference between the last two thermographs is below the $y = 0$ line.

Toenails, the segments below the $y = 0$ line, are important, but they are actually simple extensions of the graphic. The reason for the last two toenails to be different is that cooling is applied to all the Left and Right alternatives, but in opposite directions. It is important to remember $G_t = \{G_t^L - t \mid G_t^R + t\}$, because it explains the difference. Both Left's and the first Right's toenail came from cooling 0, but the second Right's toenail came from cooling $\{0 \mid 0\}$.

The next example, the first of non-switch hot games, shows how cooling L and R alternatives work.

$G = \{\{3 \mid 1\} \mid \{0 \mid -4\}\}$ xmin



5

Methods

In this Chapter, you present in more concrete terms the method(s) you are going to apply. And as always in research, it is good to demonstrate awareness of the weaknesses or limitations of the method you use. It makes no difference if you work with interviews, econometric models, or a comprehensive analysis of data from various sources. Transparency should be the guideline: make it possible for your readers to follow, or even repeat, your analysis!

5.1 The Approach [or Model]

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6

Empirical Analysis

This chapter covers three areas: analysis of the data; discussion of the results of the analysis; and how your findings relate to the literature. The analysis of the data can be discussed here but the details of any analysis, such as statistical calculations, should be shown in the appendices. You should present any discussion clearly and logically and it should be relevant to your research questions/hypotheses or aims and objectives. Insert any tables or figures that you decide are important in a relevant part of the text not in the appendices, and discuss them fully. Make sure that you relate the findings of your primary research to your literature review. You can do this by comparison: discussing similarities and particularly differences. If you think your findings have confirmed some literature findings, say so and say why. If you think your findings are at variance with the literature, say so and say why.

6.1 Results

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When placing tables (**??**) within the body of the text, the citation is placed above the table.

6.2 Discussion

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When placing figures (illustrations, pictures, graphs, diagrams, charts, maps etc.) within the body of the text, the citation is placed below the figure (**??**)

7

Conclusion

State the main conclusions of your study. State explicitly how and to what extent you have fulfilled your aims and objectives/answered your research questions/proved your hypotheses (whichever is appropriate). Your conclusions should follow logically from your findings and not contain any new material.

7.1 Research Aims

7.2 Research Objectives

7.3 Practical Implications

7.4 Future Research

7.5 Chapter Summary

Bibliography

Refer to LUSEM's Harvard referencing guidelines in the Teaching and Learning platform. `Lusem.lu.se/asks`

Appendix A

(Appendix A title)

The final sections of your thesis are the appendices. Each appendix should be lettered (A, B, etc.,) and should consist of detailed information that is interesting but not essential to the main thrust of your findings section.
The appendices should be in the order that they are referred to in the main text. For instance, if Appendix A refers to something on page 25 and Appendix B refers to something on page 15, the appendices need to be re-lettered. This inconsistency occurs when text is moved around or inserted.)

Appendix B

(Appendix B title)