The Mandelbrot Set

Sean Kinnally, Mark Lavrentyev

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Definition (Complex Dynamics)

The study of how the iterates of a holomorphic function behave. For a function f, we denote $f \circ \ldots \circ f$ (k times) as f^k.

Example: $f(z) = z^2 + 2$

- $f(0) = 2$
- $f^2(0) = 6$
- $f^3(0) = 38$
- $f^4(0) = 1446$
- $f^5(0) = 2090918$

Example: $f(z) = z^2 - \frac{1}{2}$

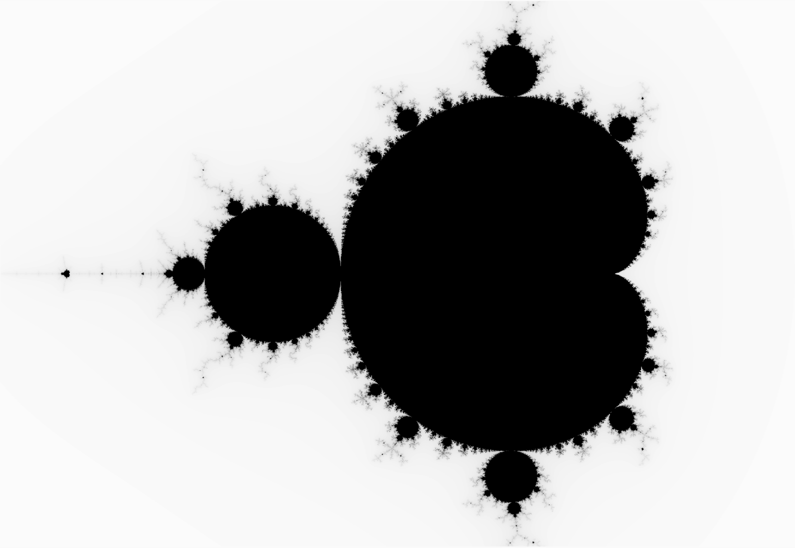
- $f(0) = -1/2$
- $f^2(0) = -1/4$
- $f^3(0) = -7/16 \approx -1/2$
- $f^4(0) = -79/256 \approx -1/4$
- $f^5(0) = 26527/65536 \approx -1/2$

Mandelbrot set definition

Definition (Mandelbrot set)

For $P_c(z) = z^2 + c$, the Mandelbrot set $\mathcal{M}$ is the set of values $c \in \mathbb{C}$ such that $P_c^k(0)$ is bounded for all k.

Mandelbrot set definition



Demo

Properties of $\mathcal{M}$

Theorem (boundedness)

$\forall c \in \mathcal{M}, |c| \leq 2$ *i.e.* $\mathcal{M}$ *is contained in a disk of radius 2 around the origin.*

Theorem (closedness)

The Mandelbrot set is closed.

Observe that while some properties of the Mandelbrot set are visually obvious, proving them is non-trivial, given the fractal nature of it.

Boundedness Proof

- Suppose $|c| > 2$, if $|z| = |c|$, then $|z^2 + c| \geq |c|^2 - |c| = (|c| - 1)|z|$
- By the maximum principle, applied to $z/(z^2 + c)$ on the exterior of the circle, this estimate $|P_c(z)| = |z^2 + c| \geq (|c| - 1)|z|$ persists for all z satisfying $|z| > |c|$
- By iterating, starting with $P_c(0) = c$, we obtain successively $|P_c^2(0)| = (|c| - 1)|c|$, $|P_c^3(0)| \geq (|c| - 1)^2|c|$, and eventually $|P_c^k(0)| \geq (|c| - 1)^{k-1}|c|$
- Thus $|P_c^k(0)| \to \infty$, so $c \notin \mathcal{M}$. It follows that the $\mathcal{M}$ is a subset of the closed disk $\{|c| \leq 2\}$

Julia sets

Julia sets

Definition (Filled-in Julia set)

For a polynomial q, the filled-in Julia set $\mathcal{K}_q$ is the set of $z \in C$ such that iterating q is bounded.

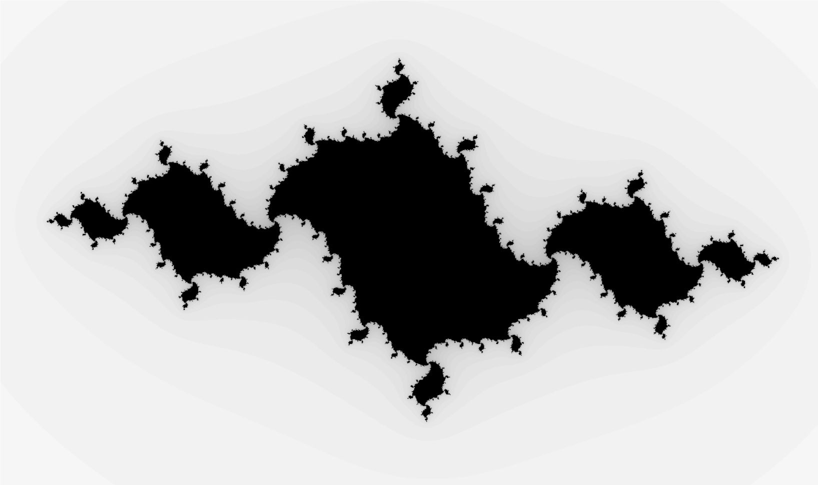
Definition (Julia set)

The Julia set is the boundary of the filled-in Julia set i.e. $\mathcal{J}_q = \partial\mathcal{K}_q$.

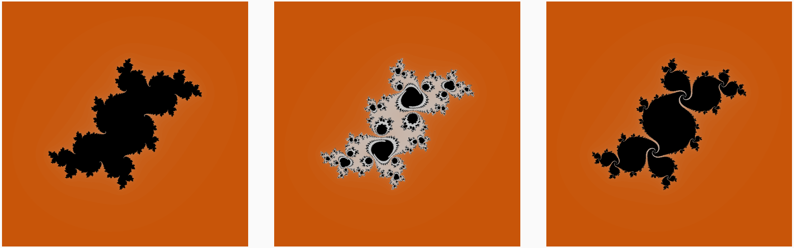
Remark

We can extend the definition of Julia sets to all rational functions in the same way.

Example: $f(z) = z^2 + (-0.973 - 0.177i)$



Unpredictable behavior



$\mathcal{K}_c$ for $c = -0.1 + 0.64i, -0.1 + 0.65i$, and $-0.1 + 0.66i$.

Connection to Mandelbrot set

Theorem

The Julia set for P_c, written $\mathcal{J}_c$ is connected iff $c \in \mathcal{M}$. If $c \notin \mathcal{M}$, then $\mathcal{J}_c$ is totally disconnected.

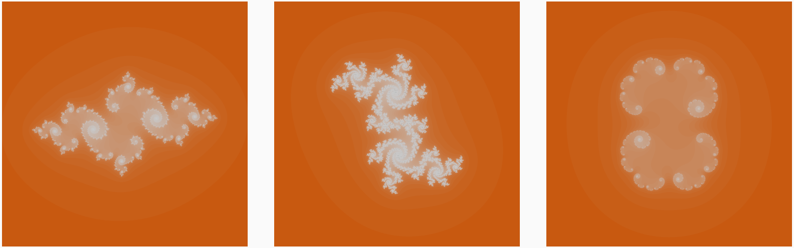
Totally disconnected: definition

Definition (totally disconnected)

A set $X \subset \mathbb{C}$ is totally disconnected if there are no connected components consisting of more than just one point.

Example: the Cantor set $\mathcal{C}$ is totally disconnected. We construct this by starting with $[0, 1]$ and iteratively removing the middle third of every component.

Totally disconnected Julia sets



$\mathcal{K}_c$ for $c = -0.77 + 0.18i, 0.23 - 0.55i$, and $0.31 - 0.01i$.

Open questions

- Is $\mathcal{M}$ *locally connected*? i.e. for every $z \in \mathcal{M}$, is there some small connected neighborhood around z?
- Does the boundary $\partial\mathcal{M}$ have non-zero Lebesgue measure ("area")?

Questions?

References

[1] M. Frame, B. Mandelbrot, and N. Neger. *Fractal Geometry*. URL: https://users.math.yale.edu/public_html/People/frame/Fractals/.

[2] T. Gamelin. “Complex Analysis”. In: Undergraduate Texts in Mathematics. Springer New York, 2003. Chap. 12. ISBN: 9780387950693.

[3] C. McMullen. “Complex Dynamics and Renormalization”. In: Princeton University Press, 1994. Chap. 4. ISBN: 9780691029825.