MSc. Music and Acoustic Engineering

Musical Acoustics - A.Y. 2020/2021

H4 - Impedance maxima of a compound horn

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1 Impedance maxima of the pipe

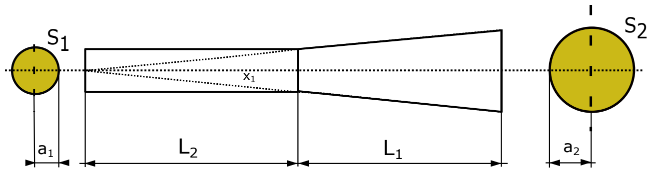

Figure 1: Scheme of the compound horn system. Beside each end a drawing of the corresponding cross section can be seen.

In order to compute the acoustic input impedance of a pipe, one needs to account not only for the air contained in the pipe as an ideal fluid but also for two phenomena occurring around it: the radiation load at the end and the wall losses along its length. The latter is relevant for narrow pipes in particular, but in our case, since the ratio between the pipe's length and the radius of its cross section is $L_1/a_1 = 10$, it is clear that we are not dealing with this condition. Moreover, a common way to characterize the influence of the viscous drag is the ratio of the pipe radius with the boundary layer thickness r_v. If this parameter is greater than 10, it is usually considered safe to neglect the wall losses. Since $r_v \propto \sqrt{f}$, this might not be true at low frequencies; however, in our case, we can see that r_v increases very rapidly from zero and at 1 Hz is already ~ 30. We assume, therefore, that it is safe for us to ignore such losses in the computation of the impedance.

Concerning the radiation, instead, we know that its influence on an open-ended pipe is usually not negligible. At low frequency, the strategy that is commonly used to address the study of radiation is to add an end correction Δ^{open} . At first order in ka the radiation impedance of an open end is equal to the input impedance of an open pipe of length Δ^{open}. We take the value $\Delta^{open} = 0.61a$; this holds well under the assumption $ka \ll 1$. However, for us $ka = 1$ at ~ 1 kHz, which is well inside the range of frequencies we are interested in. At higher frequencies it is still possible to use this model[1], but the end correction decreases with ka. This means that the values for the frequencies of the impedance maxima we will compute will be slightly underestimated in the higher range.

The input impedance of the pipe is therefore, in our model:

$$Z_{in} = jZ_0 \tan kL \qquad \text{, with } L = L_2 + \Delta^{open} = L_2 + 0.61a_1$$

The corresponding maxima are at

$$f_n = \left(n - \frac{1}{2}\right)\frac{c}{2L}, \qquad n \in \mathbb{N}$$

In Fig. 2 we can see the difference between the curve computed with and without the end correction, as well as the first four maxima for the former.

[1] As long as the wavefronts are planar, i. e. for $ka < 3.83$.

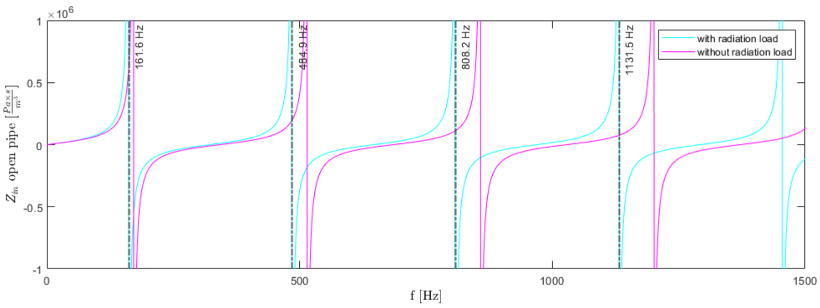

Figure 2: Imaginary part of the input impedance for the pipe. The two curves are computed with two different values of the length of the tube, one of which includes an end correction to account for the radiation.

2 Compound horn

We know that the expression for the input impedance of the pipe in terms of the load impedance Z_L at the far end is

$$Z_{in} = Z_0 \frac{Z_L \cos kL_2 + jZ_0 \sin kL_2}{jZ_L \sin kL_2 + Z_0 \cos kL_2} \tag{1}$$

with $Z_0 = \frac{\rho c}{S_1}$ being the characteristic impedance of the pipe. Characterizing the frequency-domain acoustic behavior of the compound system, therefore, is just a matter of writing the input impedance of the horn. Once again, this is quite straightforward for a conical horn terminating in an ideal open end:

$$Z_c = \frac{j\rho c}{S_1}\left(\cot kL_1 + \frac{1}{kx_1}\right)^{-1} \tag{2}$$

However, if we want to include the radiation impedance, we run into the same problem as before, except even worse, since the radius of the open end of the horn is double that of the pipe. In this case we get $ka = 1$ at ~ 545 Hz: as we will see shortly, the first maxima and minima of the impedance do not surpass this value by a large margin, so we can assume that using a constant value of the end correction doesn't have a devastating impact on our estimation of their position. When considering higher frequencies, this won't be true anymore.

With that being said, let us now discuss the maxima and minima of the impedance at low frequencies. For the impedance of the horn we can take (2) and substitute L_1 with $L_1 + 0.61a_2$. We can then use this value of Z_c in lieu of the impedance load Z_L in (1). This gives us an equation for the maxima and one for the minima: the former is obtained by requesting that the denominator in (1) be zero, while the latter results from doing the same thing with the numerator. Explicitly:

maxima: $$\tan kL_2 - \cot kL_1 + \frac{1}{kx_1} = 0$$

minima: $$\cot kL_2 + \cot kL_1 + \frac{1}{kx_1} = 0$$

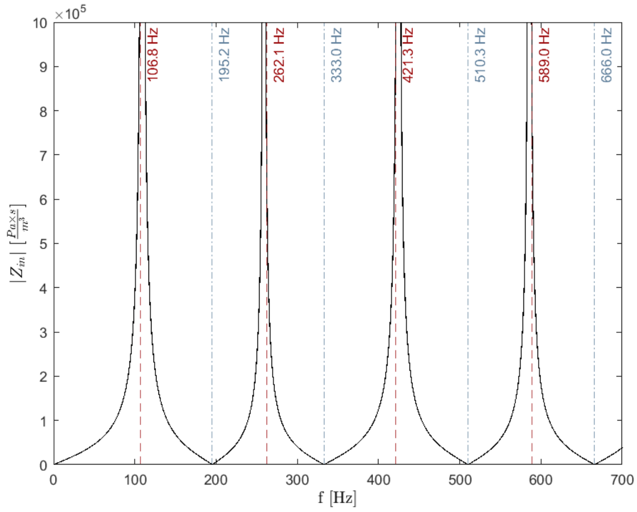

Figure 3: Magnitude of the impedance of the compound horn. The red vertical lines correspond to the maxima, while the blue ones to the minima.

The results are shown together with a graph of the magnitude of the impedance in Fig. 3.

Finally, in Fig. 4 we report the graph for the two limiting cases of the impedance (low and high frequencies) from 0 up to 4 kHz. We expect of course the previous approximation to not give satisfactory results for higher values of ka, which goes up to more than 7 in this case. However we know that for increasing frequency the imaginary part of the radiation impedance tends to $Z_R \simeq Z_0$, so we used this value as the load Z_L in the expression for the input impedance of the finite conical horn, which is:

$$Z_{\text{IN}}^{\text{horn}} = \frac{\rho c}{S_1} \left\{ \frac{jZ_L[\sin(kL-\theta_2)/\sin\theta_2] + (\rho c/S_2)\sin kL}{Z_L[\sin(kL+\theta_1-\theta_2)/\sin\theta_1\sin\theta_2] - (j\rho c/S_2)[\sin(kL+\theta_1)/\sin\theta_1]} \right\}$$
$$\text{where } \theta_{1/2} = \tan^{-1}\left(kx_{1/2}\right)$$

Again, substituting this value in (1) gives us the input impedance of the compund horn in this limiting case.

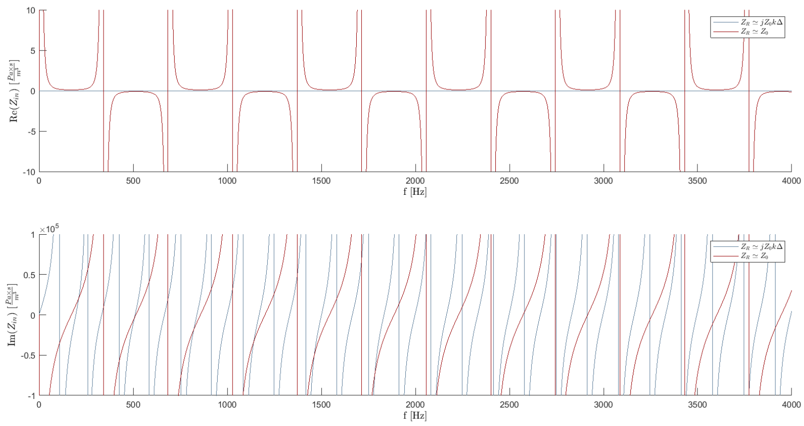

Figure 4: Input impedance for the compund horn. The two curves represent the approximations at low (blue) and high (red) frequencies.