Problem 1

Part (a)

The consumer solves

$$\max_{c_0,c_1} [\log c_0 + \beta \log c_1] \quad \text{s.t.} \quad c_t \le a_t, \quad a_1 = a_0 - c_0.$$

Since $c_1 = a_1$ (it is optimal to consume all the remaining cake in the last period), we can reformulate this as

$$\max_{a_1} [\log(a_0 - a_1) + \beta \log a_1] .$$

To get the first-order condition, just set the derivative of the objective with respect to a_1 equal to 0:

$$-\frac{1}{a_0 - a_1} + \beta \frac{1}{a_1} = 0.$$

Solving for a_1 gives an optimal savings (call it a_1^*) of

$$a_1^* = \frac{\beta a_0}{1+\beta}. \tag{1}$$

Then optimal consumption in each period is

$$c_0^* = a_0 - a_1^* = \frac{a_0}{1+\beta}, \quad c_1^* = a_1^* = \frac{\beta a_0}{1+\beta}$$

and the value function is

$$v_2(a_0) = \log(c_0^*(a_0)) + \beta \log(c_1^*(a_0)) = \log\left(\frac{a_0}{1+\beta}\right) + \beta \log\left(\frac{\beta a_0}{1+\beta}\right) = k_2 + (1+\beta)\log a_0, \tag{2}$$

where k_2 is a constant which isn't all that important to us.

Part (b)

Now the consumer solves

$$\max_{c_0,c_1,c_2} \left[\log c_0 + \beta \log c_1 + \beta^2 \log c_2\right] \quad \text{s.t.} \quad c_t \le a_t, \quad a_{t+1} = a_t - c_t.$$

Notice that this is equivalent to

$$\max_{c_0,c_1,c_2} [\log c_0 + \beta (\log c_1 + \beta \log c_2)] = \max_{a_1} [\log(a_0 - a_1) + \beta v_2(a_1)] ,$$

since the consumer's problem at $t = 1$ is equivalent to his problem at $t = 0$ in part (a). Given that we know $v_2(a) = k_2 + (1+\beta)\log a$ from (2), the FOC (wrt a_1) is

$$-\frac{1}{a_0 - a_1} + \beta(1+\beta)\frac{1}{a_1} = 0,$$

which has solution

$$a_1^* = \frac{\beta(1+\beta)a_0}{1+\beta(1+\beta)}.$$

Observe that $t = 2$ in this problem is analogous to $t = 1$ in part (a), with a_1^* taking the place of a_0 and a_2^* taking the place of a_1^*. Then, from (1), we know that

$$a_2^* = \frac{\beta a_1^*}{1+\beta} = \frac{\beta^2 a_0}{1+\beta(1+\beta)}$$

and

$$c_0^* = a_0 - a_1^* = \frac{a_0}{1+\beta(1+\beta)}, \quad c_1^* = a_1^* - a_2^* = \frac{\beta a_0}{1+\beta(1+\beta)}, \quad c_2^* = a_2^* = \frac{\beta^2 a_0}{1+\beta(1+\beta)}.$$

The value function is

$$\begin{aligned} v_3(a_0) &= \log\left(\frac{a_0}{1+\beta(1+\beta)}\right) + \beta \log\left(\frac{\beta a_0}{1+\beta(1+\beta)}\right) + \beta^2 \log\left(\frac{\beta^2 a_0}{1+\beta(1+\beta)}\right) \\ &= k_3 + (1+\beta+\beta^2)\log a_0. \end{aligned} \tag{3}$$

Part (c)

Our first goal is to find the a' that solves

$$\max_{a' \in [0,a]} \left[\log(a - a') + \beta v(a')\right] = \max_{a' \in [0,a]} \left[\log(a - a') + \beta(\gamma_0 + \gamma_1 \log a')\right],$$

where the second expression comes from plugging in the conjectured function for $v(a')$. The FOC (wrt a') is

$$-\frac{1}{a-a'} + \beta\gamma_1 \frac{1}{a'} = 0,$$

which has solution

$$a' = \frac{\beta\gamma_1 a}{1+\beta\gamma_1}; \tag{4}$$

this is in the interval $[0, a]$ as long as $\gamma_1 > 0$, which, intuitively, it is. Plugging this solution into the Bellman equation, $v(a) = \max_{a'} [\log(a - a') + \beta v(a')]$, we get

$$\begin{aligned} v(a) &= \log\left(a - \frac{\beta\gamma_1 a}{1+\beta\gamma_1}\right) + \beta v\left(\frac{\beta\gamma_1 a}{1+\beta\gamma_1}\right) \\ &= \log\left(\frac{a}{1+\beta\gamma_1}\right) + \beta\left(\gamma_0 + \gamma_1 \log\left(\frac{\beta\gamma_1 a}{1+\beta\gamma_1}\right)\right) \\ &= \log a - \log(1+\beta\gamma_1) + \beta\left(\gamma_0 + \gamma_1 \log\left(\frac{\beta\gamma_1}{1+\beta\gamma_1}\right) + \gamma_1 \log a\right) \\ &= \underbrace{-\log(1+\beta\gamma_1) + \beta\left(\gamma_0 + \gamma_1 \log\left(\frac{\beta\gamma_1}{1+\beta\gamma_1}\right)\right)}_{\gamma_0'} + \underbrace{(1+\beta\gamma_1)}_{\gamma_1'} \log a, \end{aligned} \tag{5}$$

which means that $v(a)$ does in fact have the conjectured log form. Finally, we solve for γ_0 and γ_1 by equating them to γ_0' and γ_1'. Starting with γ_1:

$$\gamma_1 = \gamma_1' = 1 + \beta\gamma_1 \Rightarrow \gamma_1 = \frac{1}{1-\beta}.$$

This makes sense: the coefficient on the $\log a$ term is $1+\beta$ in $v_2(a)$ and $1+\beta+\beta^2$ in $v_3(a)$, so for $v(a) = v_\infty(a)$ we would expect it to be

$$1 + \beta + \beta^2 + \beta^3 + \cdots = \frac{1}{1-\beta},$$

which bears out. As for γ_0, we get

$$\gamma_0 = \gamma_0' = -\log \gamma_1 + \beta\gamma_0 + \beta\gamma_1 \log \beta,$$

where I used the fact that $1 + \beta\gamma_1 = \gamma_1$ to simplify the expression. It follows that

$$\begin{aligned}(1-\beta)\gamma_0 &= \log(1-\beta) + \frac{\beta}{1-\beta}\log\beta \\ \Rightarrow \gamma_0 &= \frac{1}{1-\beta}\log(1-\beta) + \frac{\beta}{(1-\beta)^2}\log\beta.\end{aligned}$$

Part (d)

Using the expression for the optimal a' from the last part, (4), and the fact that $1 + \beta\gamma_1 = \gamma_1$, we get

$$a' = g(a) = \frac{\beta\gamma_1 a}{1+\beta\gamma_1} = \beta a.$$

This means that in every period the consumer saves a fraction β of the remaining stock of cake, which he brings into the next period; of course, this also means he consumes a fraction $1-\beta$ of the remaining stock in every period. Hence the stock of cake remaining at the beginning of period t is just $a_t = \beta^t a_0$, while period-t consumption is $c_t = (1-\beta)a_t = (1-\beta)\beta^t a_0$. The consumer technically never runs out of cake, since $\beta^t a_0 > 0$ as long as $\beta, a_0 > 0$, but $\lim_{t\to\infty} a_t = 0$ so eventually the poor soul is reduced to scraping crumbs off the floor.