Problem 1

Let's start by writing down a (general) Bellman equation for this problem, letting i denote the current state:

$$v(k, z_i) = \max_{k' \in [0, F(k,z_i)]} \left\{ U\left(F(k, z_i) - k'\right) + \beta E\left[v(k', z_{i'})|z_i\right] \right\}.$$

$E\left[v(k', z_{i'})|z_i\right]$ is the expected value next period given investment decision k' and conditional on current productivity z_i. With only two possible states, $i \in \{g, b\}$, this becomes

$$\begin{aligned} E\left[v(k', z_{i'})|z_i\right] &= \Pr\left[z_{i'} = z_g|z_i\right] v(k', z_g) + \Pr\left[z_{i'} = z_b|z_i\right] v(k', z_b) \\ &\equiv \pi_{ig} v_g(k') + \pi_{ib} v_b(k'), \end{aligned}$$

following the notation of the problem.

The first step is to find the optimal decision rule for k' as a function of model parameters (including the ones we guessed for the value functions). As with the cake-eating problem, we get the first-order condition by differentiating the objective function with respect to k' and setting the derivative equal to 0:

$$-U'\left(F(k, z_i) - k'\right) + \beta\left(\pi_{ig} v_g'(k') + \pi_{ib} v_b'(k')\right) = 0.$$

Now's a good time to bring in the simplifying and parametric assumptions: as stated in the problem, we have $U(c) = \log c$, $F(k, z) = zk^\alpha$, $\pi_{gg} = \pi_{bb} = \pi$, $\pi_{gb} = \pi_{bg} = 1 - \pi$, and $v_i(k) = \gamma_{0i} + \gamma_1 \log k$ (though this is technically a guess that we will verify). Plugging these in, the FOC becomes

$$-\frac{1}{z_i k^\alpha - k'} + \beta\left(\pi\gamma_1 \frac{1}{k'} + (1 - \pi)\gamma_1 \frac{1}{k'}\right) = -\frac{1}{z_i k^\alpha - k'} + \beta\gamma_1 \frac{1}{k'} = 0.$$

Solving for k', we get

$$k_i' = \frac{\beta\gamma_1}{1 + \beta\gamma_1} z_i k^\alpha,$$

where I use the subscript i to make it clear that the optimal choice of k' depends on the current state. Notice that this is similar to the cake-eating result, with zk^α taking the place of the remaining cake stock.

Now that we know k' we can rewrite the Bellman equation (with all of the assumptions) as

$$\begin{aligned} v_i(k) = \log&\left(z_i k^\alpha - \frac{\beta\gamma_1}{1 + \beta\gamma_1} z_i k^\alpha\right) \\ &+ \beta\left(\pi\left(\gamma_{0i} + \gamma_1 \log\left(\frac{\beta\gamma_1}{1 + \beta\gamma_1} z_i k^\alpha\right)\right) + (1 - \pi)\left(\gamma_{0j} + \gamma_1 \log\left(\frac{\beta\gamma_1}{1 + \beta\gamma_1} z_i k^\alpha\right)\right)\right) \end{aligned}$$

where j denotes the *other* state (not i). Simplifying, we get

$$\begin{aligned} v_i(k) &= \log\left(\frac{1}{1 + \beta\gamma_1} z_i k^\alpha\right) + \beta\left(\pi\gamma_{0i} + (1 - \pi)\gamma_{0j} + \gamma_1 \log\left(\frac{\beta\gamma_1}{1 + \beta\gamma_1} z_i k^\alpha\right)\right) \\ &= \underbrace{\log\left(\frac{1}{1 + \beta\gamma_1} z_i\right) + \beta\left(\pi\gamma_{0i} + (1 - \pi)\gamma_{0j} + \gamma_1 \log\left(\frac{\beta\gamma_1}{1 + \beta\gamma_1} z_i\right)\right)}_{\gamma_{0i}'} + \underbrace{(\alpha + \beta\gamma_1\alpha)}_{\gamma_1'} \log k, \end{aligned}$$

which confirms that v has the log form that we guessed (with γ_1 *not* depending on the state). As we did for the cake problem, we solve for γ_{0i} and γ_1 by equating them to the γ'_{0i} and γ'_1 above, starting with the latter:

$$\gamma_1 = \gamma'_1 = \alpha + \alpha\beta\gamma_1 \Longrightarrow \gamma_1 = \frac{\alpha}{1-\alpha\beta}.$$

This implies that the optimal decision rule is

$$k'_i = g_i(k) = \alpha\beta z_i k^\alpha,$$

which means that the economy saves (invests) a fraction $\alpha\beta$ of the output in each period (and consumes the remaining $1-\alpha\beta$). Notice that if we set $\alpha = z_i = 1$, we're back to the γ_1 and decision rule we found for the cake problem (why?).

Getting the two γ_{0i}'s will require a bit more algebra, but we can simplify a few things now that we know the value of γ_1:

$$\begin{aligned}
\gamma'_{0i} &= \log\left(\frac{1}{1+\beta\gamma_1}z_i\right) + \beta\left(\pi\gamma_{0i} + (1-\pi)\gamma_{0j} + \gamma_1\log\left(\frac{\beta\gamma_1}{1+\beta\gamma_1}z_i\right)\right)\\
&= \log(1-\alpha\beta) + \log z_i + \beta\left(\pi\gamma_{0i} + (1-\pi)\gamma_{0j} + \frac{\alpha}{1-\alpha\beta}\left(\log(\alpha\beta) + \log z_i\right)\right)\\
&= \underbrace{\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)}_{\eta_0} + \underbrace{\frac{1}{1-\alpha\beta}}_{\eta_1}\log z_i + \beta\left(\pi\gamma_{0i} + (1-\pi)\gamma_{0j}\right).
\end{aligned}$$

I introduce the constants η_0 and η_1 to make the algebra a bit easier to follow (hopefully). Setting $\gamma'_{0i} = \gamma_{0i}$ for each $i \in \{g, b\}$ we get a system of two equations in two unknowns (γ_{0g} and γ_{0b}):

$$\begin{aligned}
\gamma_{0g} &= \eta_0 + \eta_1\log z_g + \beta\left(\pi\gamma_{0g} + (1-\pi)\gamma_{0b}\right) \text{ and}\\
\gamma_{0b} &= \eta_0 + \eta_1\log z_b + \beta\left(\pi\gamma_{0b} + (1-\pi)\gamma_{0g}\right).
\end{aligned}$$

Summing these equations, we find that

$$\begin{aligned}
\gamma_{0g} + \gamma_{0b} &= 2\eta_0 + \eta_1(\log z_g + \log z_b) + \beta(\gamma_{0g} + \gamma_{0b})\\
\Longrightarrow \gamma_{0g} + \gamma_{0b} &= \frac{1}{1-\beta}\left(2\eta_0 + \eta_1(\log z_g + \log z_b)\right),
\end{aligned}$$

and taking their difference, we get

$$\begin{aligned}
\gamma_{0g} - \gamma_{0b} &= \eta_1(\log z_g - \log z_b) + \beta(2\pi-1)(\gamma_{0g} - \gamma_{0b})\\
\Longrightarrow \gamma_{0g} - \gamma_{0b} &= \frac{1}{1-\beta(2\pi-1)}\eta_1(\log z_g - \log z_b),
\end{aligned}$$

from which we can immediately see that $\gamma_{0g} > \gamma_{0b}$ when $z_g > z_b$, as expected. Adding these two equations and dividing by 2, we get

$$\gamma_{0g} = \frac{1}{1-\beta}\left(\eta_0 + \frac{\eta_1}{2}(\log z_g + \log z_b)\right) + \frac{1}{1-\beta(2\pi-1)}\frac{\eta_1}{2}(\log z_g - \log z_b)$$

and, subtracting them and dividing by 2, we get

$$\gamma_{0b} = \frac{1}{1-\beta}\left(\eta_0 + \frac{\eta_1}{2}(\log z_g + \log z_b)\right) - \frac{1}{1-\beta(2\pi-1)}\frac{\eta_1}{2}(\log z_g - \log z_b).$$

Notice that in the case of perfectly persistent states, $\pi = 1$, these simplify to

$$\begin{aligned}
\gamma_{0g} &= \frac{1}{1-\beta}(\eta_0 + \eta_1 \log z_g) \text{ and} \\
\gamma_{0b} &= \frac{1}{1-\beta}(\eta_0 + \eta_1 \log z_b).
\end{aligned}$$

This is a good sanity check: if there's no switching between states, then only the current state's productivity should matter.