GROUP WORK PROJECT # 3
GROUP NUMBER: 3892

MScFE 620: Derivative Pricing

FULL LEGAL NAME	LOCATION (COUNTRY)	EMAIL ADDRESS	MARK X FOR ANY NON-CONTRIBUTING MEMBER
CHI CUONG NGUYEN	Australia	anthony.nguyen.dn5@gmail.com	
Marin Stoyanov	Bulgaria	azonealerts@gmx.com	

Statement of integrity: By typing the names of all group members in the text boxes below, you confirm that the assignment submitted is original work produced by the group (excluding any non-contributing members identified with an "X" above).	
Team member 1	**CHI CUONG NGUYEN**
Team member 2	**Marin Stoyanov**
Team member 3	

Use the box below to explain any attempts to reach out to a non-contributing member. Type (N/A) if all members contributed. **Note:** You may be required to provide proof of your outreach to non-contributing members upon request.
N/A

Introduction.
The Black-Sholes model applies the assumptions that the drift μ and the volatility σ are constants, while in the real world, they are not [7]. This is where the Heston model arrives which is an evolution of the Black-Sholes model but this time it takes into consideration that the volatility is stochastic and mean-reverting [6].

The main important differences with the Black-Scholes model are:

- It suggests that there is a correlation between volatility itself and the stock price [8] [9].
- It suggests that the volatility has a mean-reverting capability [8] [9].
- It gives a closed-form solution (analytical solution) which is derived from very specific mathematical calculations (stochastic calculus) [8] [9].
- It does not require, unlike the Black-Scoles model, that the underlying instrument price follows a lognormal distribution [8] [9].

The basic Heston model assumes that the price of the underlying asset is determined by a stochastic process [10]:

$dS_t = \mu S_t dt + \sqrt{v_t} S_t dZ_1$ eq. (1)

Where v_t is the instantaneous variance, is given by a Feller square-root or CIR process (Cox–Ingersoll–Ross model was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of the Vasicek model.)

$dv_t = k(\theta - v_t)dt + \sigma\sqrt{v_t}dZ_2$ eq.(2)

And Z_1, Z_2 are Wiener processes with correlation: ϱ

In equation 1 (stochastic differential equation for the underlying asset behavior under a risk-neutral measure) μ can be replaced with r because μ=r

In equation 2 (stochastic differential equation for the variance) the variables are as follow:

θ is the long run average variance

k is the rate of the mean-reversion

σ is the volatility of volatility (in short you can see it like: vol of vol)

Considering all the things said above we arrive at the following equations:

$$S_t = S_{t-1} e^{(r - \frac{v_t}{2})dt + \sigma\sqrt{v_t}dZ_1}$$

$$v_t = v_{t-1} + k(\theta - v_{t-1})dt + \sigma\sqrt{v_{t-1}}dZ_2$$

Q.5 to 7.

In particular, delta is implemented to measure the changes in option prices relative to changes in the underlying stock. The formula of the Greek delta could be presented in the following formulas:

$$Call\ Options\ \Delta = \frac{dC}{dS} = N(d_1)$$

$$Put\ Options\ \Delta = \frac{dP}{dS} = N(d_1) - 1$$

On the other hand, gamma (Γ) is the greek letter that is used for the sensitivity to changes in underlying asset price and the formula looks like this:

$$\Gamma = \frac{d^2V}{dS^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

Measures of vega and delta for call and put options could be found in the following tables (Table 1 and 2).

	Heston European Call	Heston European Put
Correlation Coefficicent	-0.30	-0.30
Option Price	2.98	2.79
Greek Delta	-0.17	0.06
Gamma	-0.52	0.08

Table 1. Comparison of Heston dynamics results for European call and put options and corresponding Delta and Gamma with ρ = -0.30.

Regarding the delta values in this question 5, the European call option displays a negative Greek Delta (**Δ** = -0.17), meaning a 1$ increase in the underlying asset price will be associated with a 0.17$ decrease in the call option payoff (and price). The put option reports a positive Geek Delta (**Δ** = 0.06), meaning a 1% increase in the underlying asset price will be associated with a 0.06$ increase in the put option payoff (and price). Meanwhile, as gamma is utilized to describe the difference in delta relative to the change in the underlying price, we can observe that there will be a 0.52 decrease in delta with the European Call when there is a 1% underlying price shock (with 1% increase and 1% decrease scenario). Regarding the European Put, Delta will increase by 0.08 when there is a 1% underlying price shock.

	Heston European Call	Heston European Put
Correlation Coefficicent	-0.70	-0.70
Option Price	2.17	3.42
Greek Delta	-0.13	0.08
Gamma	-0.39	0.11

Table 2. Comparison of Heston dynamics results for European call and put options and corresponding Delta and Gamma with ρ = -0.70.

Regarding the delta values in question 6, the European call option displays a negative Greek Delta (**Δ** = -0.13), meaning a 1$ increase in the underlying asset price will be associated with a 0.13$ decrease in the call option payoff (and price). The put option reports a positive Geek Delta (**Δ** = 0.08), meaning a 1$ increase in the underlying asset price will be associated with a 0.08$ increase in the put option payoff (and price). Meanwhile, as gamma is utilized to describe the difference in delta relative to the change in the underlying price, we can observe that there will be a 0.39 decrease in delta with the European Call when there is a 1% underlying price shock (with 1% increase and 1% decrease scenario). Regarding the European Put, Delta will increase by 0.11 when there is a 1% underlying price shock.

Question 8 & 10:

The Merton (1976) model is the next evolution of the Black-Scoles model where this time much more importance is given to the drift component as in real-world stock prices do constantly experience jumps due to fundamental reasons like good/bad earnings, sudden news of bankruptcy or missed profit warnings etc [11].

The Merton (1976) model is expressed by a stochastic differential equation which includes both a Brownian motion and jump factor that is defined by a Poisson process with λ (intensity) and μ_j (an average jump size). The formula looks like this:

$$dS_t = (r - r_j)S_t dt + \sigma S_t dZ_t + J_t S_t dN_t$$

Where:

$r_j = \lambda(e^{\mu_j + \frac{\delta^2}{2}}) - 1$ this is used for the "correction" of the drift term so that to maintain the risk-neutral measure

J_t is the jump at time t and it does have the following distribution:

$$log(1 + J_t) \approx Normal(\, log(1 + \mu_j) - \frac{\delta^2}{2}, \delta^2)$$

Thanks to the Merton (1976) model we can arrive at the following formula:

$S_t = S_{t-1}(e^{(r - r_j - \frac{\sigma^2}{2})dt + \sigma\sqrt{dt}z_t^1} + (e^{\mu_j + \delta z_t^2} - 1)y_t)$

Here in this equation we have 3 sources of randomness:

1. z_t^1 the stock price diffusion which has a standard normal distribution
2. z_t^2 the size of the jump which has a standard normal distribution
3. y_t the timing of the jump which has a Poisson distribution

The jumps in the Merton (1976) model follow a Poisson process which is a counting process of rare random and independent events [12].

The probability mass function of a Poisson distributed random variable X is

$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$

Where:

$\lambda > 0$

$E(x) = Var(X) = \lambda$

The following contents describe the results of the European call and put pricing under the Monte Carlo approach via the Merton model.

	Merton European Call	Merton European Put
Jump Intensity	0.75	0.75
Option Price	8.32	7.26
Greek Delta	0.10	0.09
Gamma	0.00	0.00

Table. 3. Comparison of Merton model results for European call and put options and corresponding Delta and Gamma with $\lambda = 0.75$.

Regarding the delta values in question 8, the European call option displays a positive Greek Delta (**Δ** = 0.10), meaning a 1$ increase in the underlying asset price will be associated with a 0.10$ increase in the call option payoff (and price). The put option reports a positive Geek Delta (**Δ** = 0.09), meaning a 1$ increase in the underlying asset price will be associated with a 0.09$ increase in the put option payoff (and price). Meanwhile, as gamma is utilized to describe the difference in delta relative to the change in the underlying price, we can observe that there will be no change in delta with the European Call and Put when the underlying price changes according to the calculation results.

Question 9 & 10:

The following contents describe the results of the European call and put pricing under the Monte Carlo approach via the Merton model (with a change in the jump intensity value, from 0.75 to 0.25).

	Merton European Call	Merton European Put
Jump Intensity	0.25	0.25
Option Price	6.84	5.79
Greek Delta	0.09	0.07
Gamma	0.00	0.00

Table. 4. Comparison of Merton model results for European call and put options and corresponding Delta and Gamma with $\lambda = 0.25$.

Regarding the delta values in question 8, the European call option displays a positive Greek Delta ($\mathbf{\Delta} = 0.09$), meaning a 1$ increase in the underlying asset price will be associated with a 0.09$ increase in the call option payoff (and price). The put option reports a positive Geek Delta ($\mathbf{\Delta} = 0.07$), meaning a 1$ increase in the underlying asset price will be associated with a 0.07$ increase in the put option payoff (and price). Meanwhile, as gamma is utilized to describe the difference in delta relative to the change in the underlying price, we can observe that there will be no change in delta with the European Call and Put when the underlying price changes according to the calculation results.

Question 13:

Prices of the American Call and options via the Heston and Merton models could be found in the table 5 and 6. Technically, under the framework of American options, the Monte-Carlo approach will implement simulation at each time step of price evolution paths and compare the expected payoffs between the scenario of exercising the option or not.

	Heston Amercian Call	Heston European Call
Option Price	2.16	2.98

Table. 5. Comparison of Heston model results for American versus European call options.

	Merton Amercian Call	Merton European Call
Option Price	6.94	8.32

Table. 6. Comparison of Merton model results for American versus European call options.

In comparison to the original European Call price generated in questions 5 and 8, American Call Options computed via both the Heston and Merton models are relatively lower than their European counterparts. This could be attributed to the difference in exercise approaches between the American and European derivatives, while the American option holder can exercise the right before the expiration date, the European option, however, does not display this characteristic.

Question 14:

The barrier options are derivative products that rely on the underlying asset price level, which is pre-set with the limit to a barrier/ knock-out level [5]. Specifically, an up-and-in option will come to exist only when the underlying stock price passes and increases above a price barrier. The price of the Up-and-In European Call and Simple European Call (via Monte Carlo Simulation) is measured in the following table (Table 7).

	Simple European Call	Up-and-In European Call
Option Price	0.01	0.89

Table. 7. Comparison of Heston model results for simple European Call versus Up-and-In European call

In comparison to the Simple Monte Carlo European Call price, Up-and-In Call Options computed via is relatively higher. This could be explained by the price barrier preference when entering the position. Regarding a call option, the buyers are more likely to prefer the uptrend outlook of the underlying asset to a drop scenario, therefore, a position entering when the price is increasing provides a higher possibility of profit for the Up-and-In Call.

Question 15:

The price of the Down-and-In European Put and Simple European Put (via Monte Carlo Simulation) is measured in the following table (Table 8). In contrast to the Up-and-In Call, the DAI Put will only come to exist when the stock price drops to a certain level of low price (the barrier price).

	Simple European Put	Down-and-In European Put
Option Price	0.17	2.8

Table. 8. Comparison of Heston model results for simple European Call versus Up-and-In European call

In comparison to the Simple Monte Carlo European Put price, the Down-and-In Put Options computed via is relatively higher. This could be explained by the price barrier preference when entering the position. Regarding a put option, the buyers are more likely to prefer the downtrend outlook of the underlying asset to a drop scenario, therefore, a position entering when the price is decreasing provides a higher possibility of profit for the Down-and-In Put.

References:

1. Tompaidis, Stathis; Pricing American-style options by Monte Carlo simulation: alternatives to ordinary least squares; The Journal of Computational Finance, 18(1), pp. 121-143
2. Kyle Siegrist; Central Limit Theorem; Probability, Mathematical Statistics, and Stochastic Processes; University of Alabama in Huntsville; Last updated: Apr 24, 2022
3. Kyle Siegrist; The Law of LargeNumbers; Probability, Mathematical Statistics, and Stochastic Processes; University of Alabama in Huntsville; Last updated: Apr 24, 2022
4. Schumacher, M. Johannes; Introduction to Financial Derivatives: Modeling, Pricing and Hedging; ISBN: 978-94-6240-612-4; pp. 123
5. Tian, M., Yang, X. and Zhang, Y., 2019. Barrier option pricing of mean-reverting stock model in uncertain environment. *Mathematics and Computers in Simulation*, *166*, pp.126-143.
6. Heston, Steven L. (1993). "A closed-form solution for options with stochastic volatility with applications to bond and currency options". Review of Financial Studies. 6 (2): pp.327–343
7. Schumacher, M. Johannes; Introduction to Financial Derivatives: Modeling, Pricing and Hedging; ISBN: 978-94-6240-612-4; pp. 76
8. Akhilesh Ganti, Somer Anderson, Timothy Li; Heston Model: Meaning, Overview, Methodology; Investopedia
9. Corporate Finance Institute, "Heston Model", https://corporatefinanceinstitute.com/resources/derivatives/heston-model/
10. Wilmott, P. (2006), *Paul Wilmott on Quantitative Finance* (2nd ed.), p. 861
11. Merton, Robert C. "Option pricing when underlying stock returns are discontinuous." Journal of Financial Economics, vol. 3, no. 1–2, 1976, pp. 125–144.
12. Sung Nok Chiu; Dietrich Stoyan; Wilfrid S. Kendall; Joseph Mecke (27 June 2013). Stochastic Geometry and Its Applications. John Wiley & Sons. ISBN 978-1-118-65825-3