GROUP WORK PROJECT # 1 MScFE 622: Stochastic Modeling

GROUP NUMBER: 4186

FULL LEGAL NAME	LOCATION (COUNTRY)	EMAIL ADDRESS	MARK X FOR ANY NON-CONTRIBUTING MEMBER
Marin Stoyanov	Bulgaria	azonealerts@gmx.com	
Panagiotis Lois	Grenoble, France	loispanagiotis@outlook.com	

Statement of integrity: By typing the names of all group members in the text boxes below, you confirm that the assignment submitted is original work produced by the group (excluding any non-contributing members identified with an "X" above).	
Team member 1	Marin Stoyanov
Team member 2	Lois Panagiotis
Team member 3	

Use the box below to explain any attempts to reach out to a non-contributing member. Type (N/A) if all members contributed. **Note:** You may be required to provide proof of your outreach to non-contributing members upon request.

Step 1

a. Heston (1996) calibration

Since the client wants a short maturity derivative (15 days), this is why after loading the data we filter it, so that we have calls and puts data but only with 15 days maturity. Then we start preparing the inputs for the Heston 93 model without jumps using the Lewis (2000) approach. We initialize the variables like: SM = $232.90, $T = \frac{15}{250}$, $r = 0.015$. These are the predefined values that are given to us. The rest of the variables like: kappa_v, theta_v, sigma_v, rho, v0 are just initialized and their value for now is not important because they will be evaluated by the calibration process and this is how we will arrive at their values.

Before we continue to the calibration it is important to say that Heston model is using formulas that describe the life (movements) of a stock price (or other asset), where the price and the volatility follow a Brownian processes [1]:

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_{1t}$$

$$dV_t = k(\theta - V_t)dt + \boldsymbol{\sigma}\sqrt{V_t} dW_{2t}$$

Where:

- S_t: *the asset price at time t*
- r: *risk − free interest rate − the theoretical interest rate on an asset that carries no risk*
- $\sqrt{V_t}$: *volatility (standard deviation) of the asset price*
- $\boldsymbol{\sigma}$: *volatility of the volatility* $\sqrt{V_t}$
- θ: *long − term price variance*
- k: *rate of reversion to the long − term price variance*
- dt: *indefinitely small positive time increment*
- W_{1t}: *Brownian motion of the asset price*
- W_{2t}: *Brownian motion of the asset's price variance*
- ρ: *Correlation coefficient for* W_{1t} *and* W_{2t}

It is also important to say that the Brownian motion is a type of random process W_t, $t \in [0, T]$ that do have the following properties [2]:

- $W_0 = 0$
- W_0 *has independent movements*
- W_t *is continuous in t*

- $The\ increments\ W_t\ -\ W_0\ have\ a\ normal\ distribution\ with\ mean\ zero\ and\ variance\ |t\ -\ s|.$
  $(W_t\ -\ W_0) \sim N(0,\ |t\ -\ s|)$

Another important topic that is used in the Heston 93 model is the characteristic function (for short let's call it CF). The logic behind the the CF (which is often depicted like this: $\varphi_X(u)$) is that every real valued random variable X has a CF that describes the probability density function $(PDF\ for\ short)$ of a model (in this case we are talking about a stochastic volatility model).
It is defined as:

$$\varphi_X(u)\ =\ \mathbb{E}[e^{iuX}]\ ,\ with\ \ i = \sqrt{-\ 1}$$

And this can be rewritten like this:

$$\varphi_X(u)\ =\ \int_{-\infty}^{\infty} e^{iuX} dF(x)\ =\ \int_{-\infty}^{\infty} e^{iuX} f(x)dx$$

$where\ F(x)\ and\ f(x)\ denote\ the\ CDF\ and\ the\ PDF\ of\ X,\ respectively$

The Heston 93 model CF looks like this:

$$\varphi^H(u,\ T)\ =\ e^{H_1(u,T)\ +H_2(u,T)v_0}$$

Where:

$$H_1(u,T) \cong r_0 uiT\ +\ \frac{c_1}{c_v^2}\{(k_v\ -\ \rho\sigma_v ui\ +\ c_2)T\ -\ 2log[\frac{1 - c_3 e^{c_2 T}}{1 - c_3}]\}$$

$$H_2(u,T) \cong \frac{k_v - \rho\sigma_v ui + c_2}{\sigma_v^2}\ [\frac{1 - e^{c_2 T}}{1 - c_3 e^{c_2 T}}]\}$$

$$c_2(u,T) \cong k_v \theta_v$$

$$c_2 \cong k_v \theta_v$$

$$c_3 \cong \frac{k_v - \rho\sigma_v ui + c_2}{k_v - \rho\sigma_v ui - c_2}$$

Something else, which is important for Heston 93 model, is that there are 5 parameters to be calibrated and they are: Mean Reversion Speed (k_v) , Long-Run Average Volatility (θ_v) , Volatility of Volatility (σ_v), Correlation (ρ) , and the Initial Volatility (v_0)

Since we are using the Lewis (2001) approach let's define its formula for pricing a call option:

$$C_0 = S_0 - \frac{\sqrt{S_0 K}e^{-rT}}{\pi}\int_0^\infty Re[e^{-izk}\varphi(z - \frac{i}{2})]\frac{dz}{z^2+\frac{1}{4}}$$

$$where\ \varphi(z - \frac{i}{2})\ is\ the\ characteristic\ function\ of\ the\ Heston\ 93\ model$$

For calculating the integral value you should use the quadrature method.

After we have the necessary methods you should also know that the Heston 93 model celebration procedure is to:

1. "Brute force" the parameters (permutations) and then find reasonable values or a range of values for the parameters
2. Dig deeper in these ranges for better calibration

The ranges that we tried for the "brute force" part are:

1. k_v (Kappa_v) : starting from 1 and ending at 10.6, with iteration step = 1
2. θ_v (Theta_v) : starting from 0.01 and ending at 0.41, with iteration step = 0.01
3. σ_v (Sigma_v) : starting from 0.01 and ending at 0.251, with iteration step = 0.1
4. ρ (rho) : starting from -0.75 and ending at 0.01, with iteration step = 0.1
5. v_0 (Initial Volatility) : starting from 0.01 and ending at 0.031, with iteration step = 0.01

The error function that we are using is the Mean Squared Error (MSE) of option market prices [7]:

$$MSE = min_\alpha \frac{1}{N}\sum_{n=1}^{N}(C_n^* - C_n^{Model}(\alpha))^2$$

The results of Bates 93 model calibration without jumps looks like this:

1. k_v (Kappa_v) : 50.527
2. θ_v (Theta_v) : 0.005
3. σ_v (Sigma_v) : 0.711
4. ρ (rho) : -0.895
5. v_0 (Initial Volatility) : 0.282

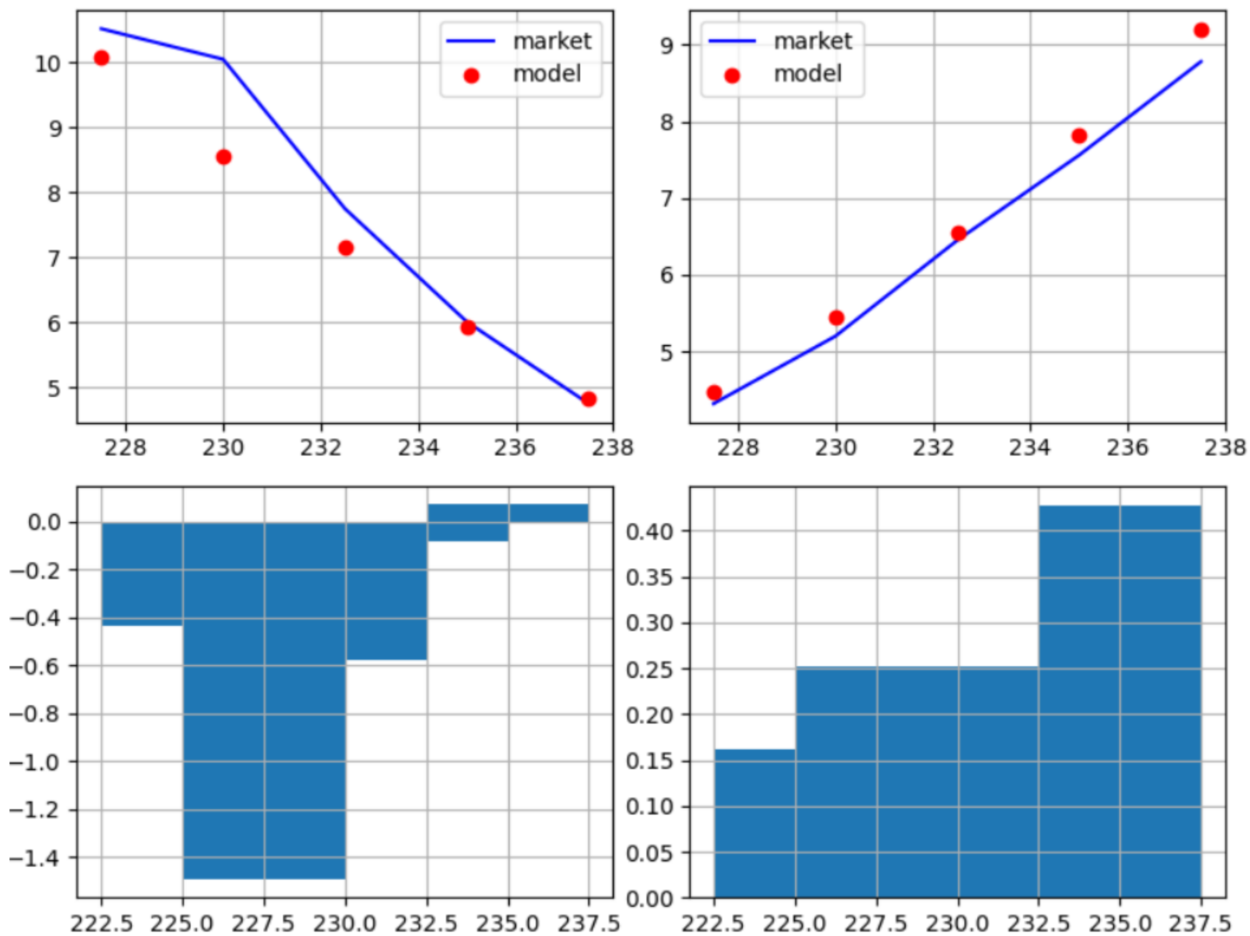

Fig.1. Calls and Puts market prices vs Heston 93 model calibration results (15 day maturity and NO jumps)

It is clearly visible that the calibration is very good because the model estimations (in red dots) are very close to the real market prices. There are negligible differences that are around \$0.5 to \$1.

b. Price an Asian Option

We have provided to the client an Asian call option. To obtain a trustworthy price we have used Monte-Carlo simulations, in a risk-natural setting, that allowed us to create a sufficient number of paths. Our methodology also included the calibration of the Heston (1996) model, under Lewis (2001) approach.

Let's look at the pricing procedure more precisely:

Step 1: Find the optimum parameters calibrating the Heston's (1996) model.

We begin by calibrating the model using past prices of the call option, the current price of the SM Energy company, and its strike price. The parameters are calibrated using an optimization technique to minimize the difference between the model's predictions and observed market prices. These parameters are: the

Long-Run Average Volatility (θ), Mean Reversion Speed (κ), Volatility of Volatility (σ), Correlation (ρ), and the Initial Volatility (V0). To measure the difference between the Heston's parameters and the observed data we used the method of Mean-Square-Error (MSE).

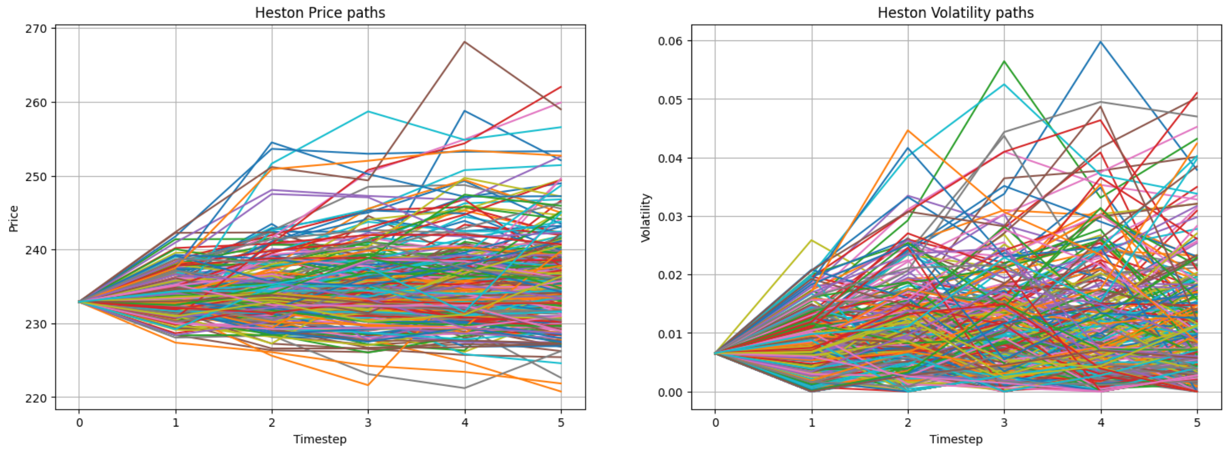

Fig.2. : Simulated price paths (left), and volatility paths (right) using Monte-Carlo simulations.

Step 2: Using Monte-Carlo to create simulations
To obtain an accurate estimate, we run 100,000 simulations using the selected 20-days maturity. We use simulations to create random paths for the stock price, and stochastic volatility.

Step 3: Pricing
To compute the price of the instrument we used the average payoff of the Asian call option we simulated during the 20 days. We have added a 4% fee to the obtained price, as a compensation for the company.

In conclusion, the pricing methodology used Heston's model calibrated parameters, to eliminate the difference of the parameters with the historical data, and a large number of simulations, to obtain an accurate price estimation.

Results:
The Asian call price using Monte-Carlo simulations has been estimated to be 0.14 $. Adding 4% as a fair compensation, the client has to be invoiced 0.146 $ per call option.

Step 2

a. Calibrating Heston model with jumps, Bates (1996)

The Bates model is a variation of the stochastic volatility Heston (1996) pricing model that uses Jump-diffusion parameters. Similar to the Heston model, we are using calibration to its jump parameters to reflect the underlying asset and volatility. More precisely, the calibration of the Bates model in Python includes the following steps:

1) Calibrate the stochastic volatility parameters using Heston, for a 60-days maturity. The procedure of Heston model calibration is explained in Step 1 / question a).
2) Then we will create an error function for the jump parameters. The function measures the difference between the model parameters and market prices. We have used the Mean-Square Error function (MSE) for this purpose.
3) After we have defined both the parameters of the stochastic volatility and the jump component, we are ready to compute the price of the options. The characteristic function in the Bates model is an extension of the Heston model that incorporates the parameters of stochastic volatility, and jump-diffusion, which can better predict the future price. The characteristic function in Bates model is simply calculated by the characteristic function of Heston (1996), and Merton (1976) models.

$$\varphi_0^{B69}(u,T) = \varphi_0^{H93} \times \varphi_0^{M76J}(u,T)$$

$$\varphi_0^{M76J}(u,T) = e^{(iu\omega+\lambda(e^{iu\mu j-u^2\delta^2/2}-1)\ T}$$

$$\varphi_0^{H93}(u,T) = e^{H1(u,T)+H2(u,T)\nu 0}$$

Parameters obtained from the calibration in a risk-natural setting:

<u>Stochastic volatility parameters:</u>
Mean-Reversion Speed (κ) = 3.531
Long-Run Average Volatility (θ) = 0.036
Volatility of Volatility (σ) = 0.505
Correlation (ρ) = -1
Initial Volatility (V0) = 0.017

<u>Jump-parameters:</u>
Jump in Asset Prices (λ) = 0.0001286
Jump mean (μ) = -0.583
Jump volatility (δ) = 0.000546

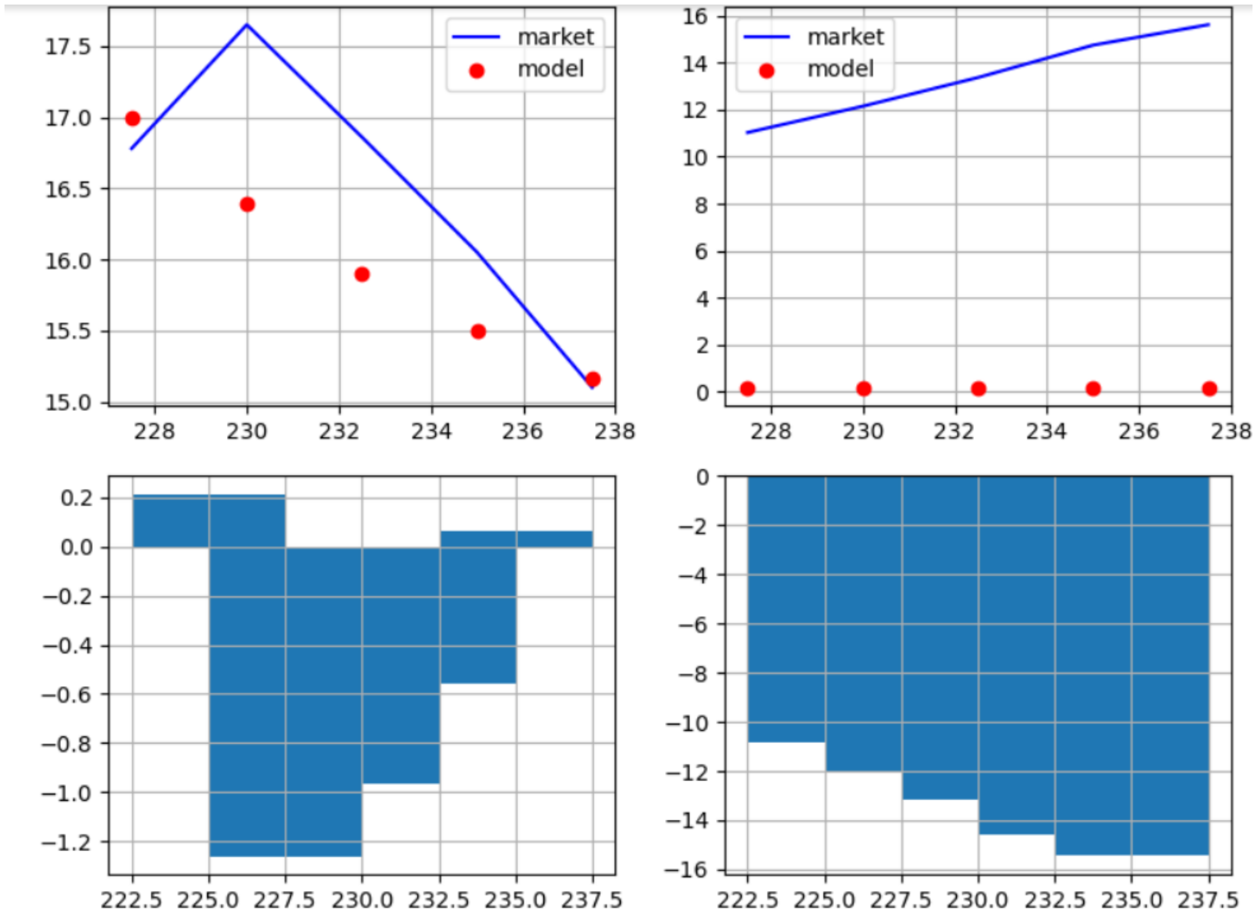

Fig.3. Calls and Puts market prices vs Heston 93 model calibration results (60 day maturity and WITH jumps)

Again the model estimations for Call options are close to the real market prices with some differences of around $1. There is something else interesting and this is the Put options estimations

b. Carr-Madan (1999) approach to Bates (1996)

Carr-Madan (1999) approach to Bates(1996), is an alternative solution to Lewis (2001). This solution relies on Fast-Fourier pricing (FFT). Similarly with the Lewis approach, we would need to calculate the error function, and integral value for the specific case of Bates (1996) characteristic function.

The call price in Carr-Madan (1999) model is given by:

$$C_0 = \frac{e^{-\alpha\kappa}}{\pi}\int_0^{\infty} e^{-i\nu\kappa}\,\frac{e^{-rT}\varphi^{B96}(\nu-(\alpha+1)i,T)}{\alpha^2+\alpha-\nu^2+i(2\alpha+1)\nu}\,d\nu$$

Where:

φ^{B96} = Characteristic function of Bates, and

α = The continuous dividend yield.

The parameters obtained from the Bates calibration using the Carr-Madan approach are:

<u>Jump-parameters:</u>
Jump in Asset Prices (λ) = 0.0001286
Jump mean (μ) = -0.583
Jump volatility (δ) = 0.000546

The calibrated parameters are similar to those obtained from the calibration of the Lewis (2001) model. We have found that the FFT procedure to the stochastic volatility jump-diffusion model of Bates (1996) can be used to the more simple Fourier-based techniques of the Lewis model.

Step 3

a. Calibrate a CIR (1985) model

The process of calibrating a CIR model is straight forward and by implementing the following algorithm:

1. Gather data: In our case we already have the data so we move to the next step
2. Using forward rates and interpolation by cubic spline

The term structure looks like this:

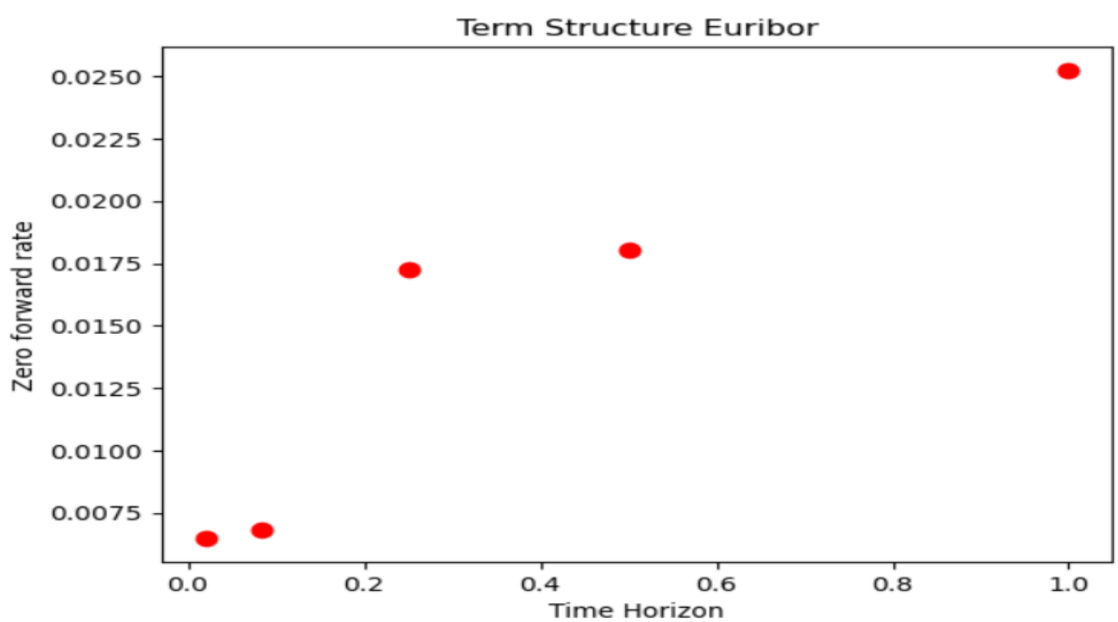

Fig 4. Term structure of interest rates

After using the cubic split and we do the interpolation looks like this:

Fig.5.

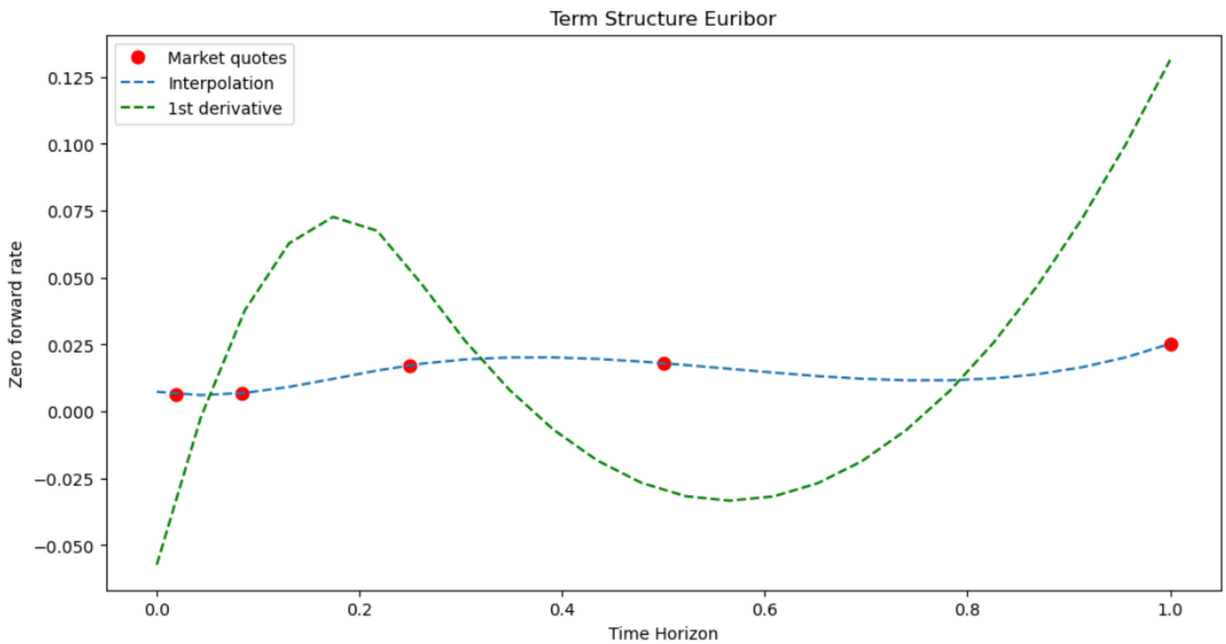

Fig 5. Interpolation of interest rates with cubic spline

3. Using error function: we are using mean squared error (MSE)

4. Optimization function: we are optimizing with the fmin optimization from the scipy.interpolate package and thus doing the calibration

After the calibration we are getting the following values for:

1. k_v (Kappa_v) : 0.69 . This means that the mean reversion speed is growing so reverting back to the long-term average could be at a faster rate.
2. θ_v (Theta_v) : 0.19 . So the long term average volatility is expected to be 0.19
3. σ_v (Sigma_v) : 0.51 . About the volatility of volatility of 0.51. This means that some turbulence could be expected and the volatility could be changing quite rapidly and sharply. Which suggests that the risk will grow.

It looks like the values that are shown on fig.6 from both the market prices and the model estimations are showing that a rise could be expected in the near future. Yes there are some differences between the model and the market values but this is expected and is clearly visible in the chart below.

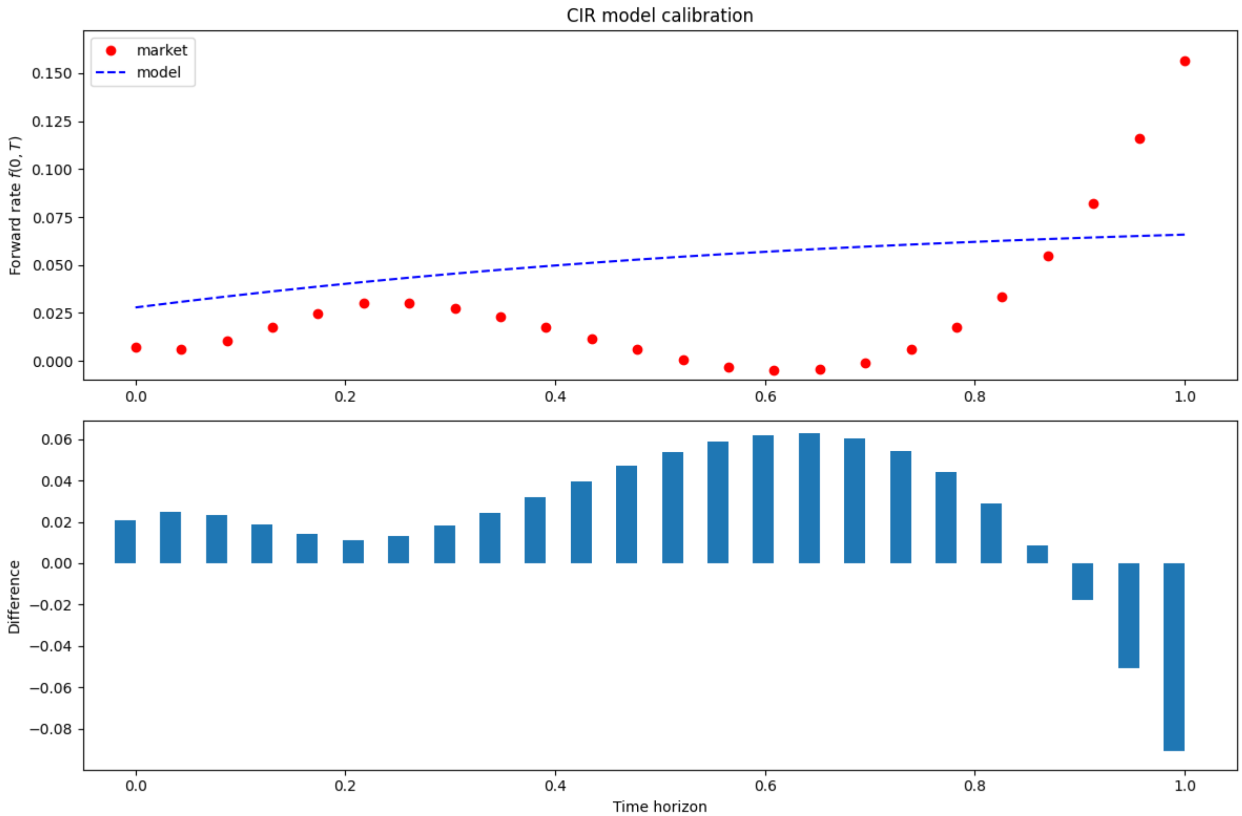

Fig.6 Plotting the CIR calibration results and the differences

b. Simulate Euribor 12-months rate with the CIR parameters obtained using Monte-Carlo simulations.

We have performed 100,000 simulations to generate possible interest rate paths of the Euribor rate. The calibrated parameters of the CIR model have been used to calculate the daily change in interest rates. The interest rates simulation are graphically presented below:

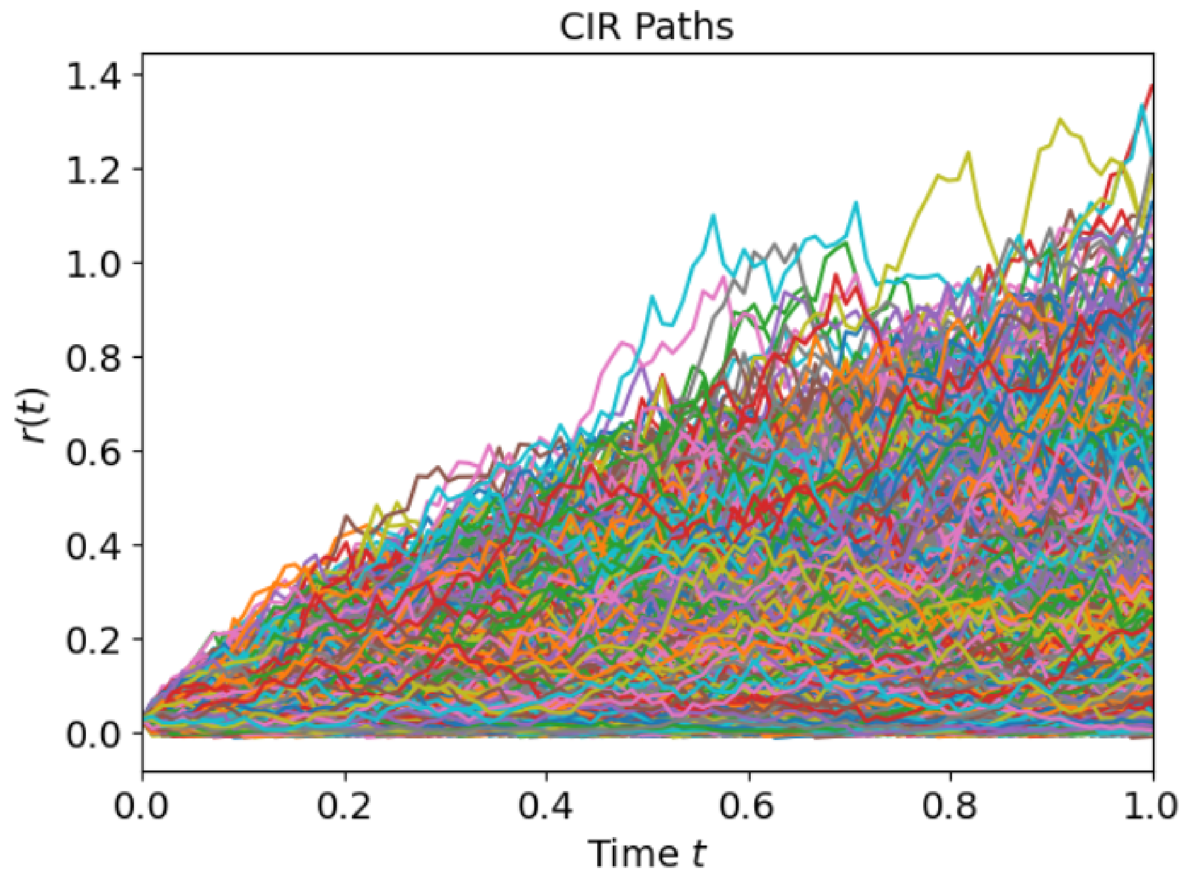

Fig 7: 1-year Interest rate paths using CIR inputs, and Monte-Carlo simulations.

<u>i) Range (max and min) that the 12-month Euribor in 1 year?</u>

We can find the minimum, and maximum 12-month Euribor, by comparing the estimated interest rates at the final step of the simulation. We use a 95% confidence interval to visualize it.

Min(rate) = -0.010

Max(rate) = 1.373

<u>ii) Expected value of the 12-month Euribor in 1 year?</u>

We calculate the average interest rate of the final path of all simulated paths. The estimated 12-month Euribor rate after one year is 0.1088.

<u>iii) How will this expected number affect the pricing of your products in 1 year versus the current 12-month Euribor rate</u>?

We have found that the expected value of the 12-month interest rate is higher than the current 12-month rate. We can draw the conclusion that the cost of the 12-month Euribor will increase after one year, consequently affecting the borrowing rates for investors, and borrowers.

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