# Length n Probability Generation

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## Summary

The goals of this document are:

- Explain what an probability vector of length $n$ is and what properties such a vector has.
- Illustrate how to generate a random probability vector of length $n$.
- Analyze how to bias such a probability vector.
- Examine the distribution of the norms of such vectors.

## Importing Libraries

```
library(tidyverse)
library(latex2exp)
```

## Generator

```r
prob_generator <- function(N, m = 1000, shape = NA){
  shape <- ifelse(is.na(shape), rep(1, N), shape)
  gams <- matrix(rgamma(n = N*m, shape = shape), nrow = N)
  probs <- t(t(gams)/colSums(gams))
  return(probs)
}
```

This just uses the Dirichlet distribution for generating the probabilities.

The intuition is that we generate $X_j \sim \text{Gamma}(\alpha_j, \lambda)$ for $j \in \{1, ..., n\}$ Then we consider the distribution of $p = (p_1, ..., p_n) = \left(\frac{X_1}{S}, ..., \frac{X_n}{S}\right)$ where $S = \sum_{j=1}^{n} X_j$.

Note that this is an $n-1$ dimensional vector because we know $\sum_{j=1}^{n} p_j = \sum_{j=1}^{n} \frac{X_j}{S} = \frac{1}{S}\sum_{j=1}^{n} X_j = \frac{S}{S} = 1$. Quick note: once we know $n-1$ of the $p_j$ we know the last one.

Furthermore we also know $X_j \geq 0$ for each $j \in \{1, ..., n\}$ (since $X_j$ follows a gamma distribution) so $S = \sum_{j=1}^{n} X_j > X_j \geq 0$ for each $j \in \{1, ..., n\}$ as well. Which then tells us that $p_j = \frac{X_j}{S} \geq 0$ and $p_j = \frac{X_j}{S} \leq 1$.

So $p = (p_1, ..., p_n)$ is indeed a probability vector of length $n$.

The support of the Dirichlet distribution is the standard $n-1$ simplex. Which is given by $\Delta_{n-1} = \left\{(p_1, ..., p_n) \in \mathbb{R}^n : \forall_{j \in \{1,...,n\}} 0 \leq p_j \leq 1, \sum_{j=1}^{n} p_j = 1\right\}$.

Furthermore the density can be given by:

$$f_{p_1,...,p_n}(p_1, ..., p_n) = \frac{1}{B(\alpha)} \prod_{j=1}^{n} x_j^{\alpha_j - 1} \text{ for } (p_1, ..., p_n) \in \Delta_{n-1}$$

Where $B(\alpha) = \frac{\Gamma(\alpha_1)...\Gamma(\alpha_n)}{\Gamma(\alpha_1 + ... + \alpha_n)}$.

The special case where $\alpha_j = 1$ for each $j \in \{1, ..., n\}$ is essentially a uniform distribution over the standard $k-1$ simplex as the density is given by:

$$f_{p_1,...,p_n}(p_1, ..., p_n) = \frac{1}{B(\alpha)} \prod_{j=1}^{n} x_j^{\alpha_j - 1} = \frac{1}{B(\alpha)} \prod_{j=1}^{n} x_j^{1-1} = \frac{1}{B(\alpha)} = \frac{\Gamma(\alpha_1 + ... + \alpha_n)}{\Gamma(\alpha_1)...\Gamma(\alpha_n)}$$

$$= \frac{\Gamma(n)}{(\Gamma(1))^n} = \frac{(n-1)!}{(0!)^n} = (n-1)! \text{ for } (p_1, ..., p_n) \in \Delta_{n-1}$$

It is uniform since clearly the density does not depend on $p = (p_1, ..., p_n)$.

The above result uses the fact that $\Gamma(k) = (k-1)!$ for $k \in \{1, 2, ...\}$ which I will prove below:

$$\Gamma(k) = \int_0^\infty t^{k-1} e^{-t}\, dt$$

This clearly converges for $k - 1 \geq 0$ i.e. for $k \geq 1$.

Now let $u(t) = t^{k-1}$ and $\frac{dv}{dt} = e^{-t}$ so that $\frac{du}{dt} = (k-1)t^{k-2}$ and $v(t) = -e^{-t}$. Then:

$$\Gamma(k) = \int_0^\infty t^{k-1} e^{-t}\, dt = u(t)v(t)\Big|_0^\infty - \int_0^\infty v(t)\frac{du}{dt} dt = -t^{k-1}e^{-t}\Big|_0^\infty + \int_0^\infty (k-1)t^{k-2}e^{-t}\, dt$$

$$= \lim_{t\to\infty} -\frac{t^{k-1}}{e^t} + 0 + (k-1)\int_0^\infty t^{(k-1)-1}e^{-t}\, dt = (k-1)\Gamma(k-1)$$

Which again converges for $k - 1 \geq 1$ or equivalently $k \geq 2$.

Therefore if $k \geq 2$ then $\Gamma(k) = (k-1)\Gamma(k-1)$ which seems like a factorial, now we need our base case:

$$\Gamma(1) = \int_0^\infty t^{1-1}e^{-t}\, dt = \int_0^\infty e^{-t}\, dt = \int_{-\infty}^\infty f_T(t)\, dt = 1 = 0! \quad \text{where } T \sim \text{Exponential}(1)$$

Therefore if $k \geq 1$ we know for

$$\Gamma(k) = (k-1)\Gamma(k-1) = ... = (k-1)...(2)\Gamma(2) = (k-1)...(2)(1)\Gamma(1) = (k-1)...(2)(1) = (k-1)! \ \square$$

# Testing that this does indeed produce length $n$ probability vectors.

There is always the issue of machine precision for floating point numbers so the sums can not be exactly one but we can make a simple tolerance based on machine epsilon.

```
tol <- 4.75*.Machine$double.eps
tol
```

```
## [1] 1.054712e-15
```

```
for (i in 2:1000){
  test <- prob_generator(N = i)
  stopifnot(
    min(test) >= 0,
    max(abs(colSums(test) - 1)) <= tol
  )
}
```

As there are no errors we can see that all of these have non-negative probabilities and that they sum to 1 (within our tolerance based on machine imprecision) and hence define length $n$ probability vectors.

## How to Bias The Generated Probabilities

First recall that the standard $\alpha = (\alpha_1, ..., \alpha_n) = (1, ..., 1)$ will produce uniform probabilities over $\Delta_{n-1}$ (which is just $\Omega$ for length $n$ probabilities). For several values of $n$ let us examine the joint distribution of $p_i$ and $p_j$ under the standard $\alpha$.

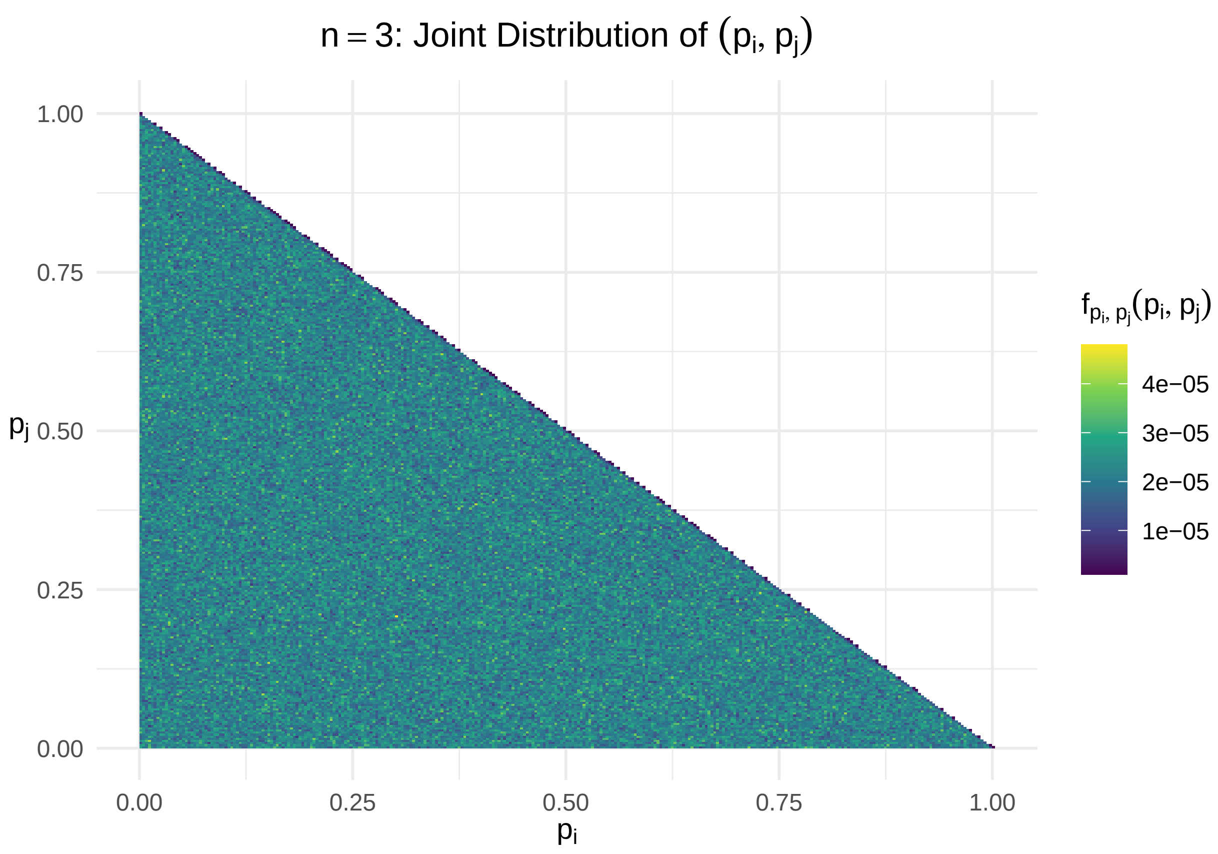
### Distribution from the standard $\alpha = (\alpha_1, ..., \alpha_n) = (1, ..., 1)$

The $n = 2$ case is trivial as we will just have the line $(p, 1-p)$ so I will skip that case.

- For $n = 3$:

```
probs <- prob_generator(N = 3, m = 1000000)
p_comp <- data.frame(pi = probs[1,], pj = probs[3,])

p_comp %>%
  ggplot(aes(x = pi, y = pj)) +
  geom_bin2d(aes(fill = after_stat(density)),
             bins = 300) +
  scale_fill_viridis_c() +
  labs(x = TeX("$p_i$"),
       y = TeX("$p_j$"),
       fill = TeX("$f_{p_i,p_j} (p_i,p_j)$"),
       title = TeX("$n=3$: Joint Distribution of $(p_i, p_j)$")) +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5),
        axis.title = element_text(color = "black"),
        axis.title.y = element_text(angle = 0, vjust = 0.5))
```

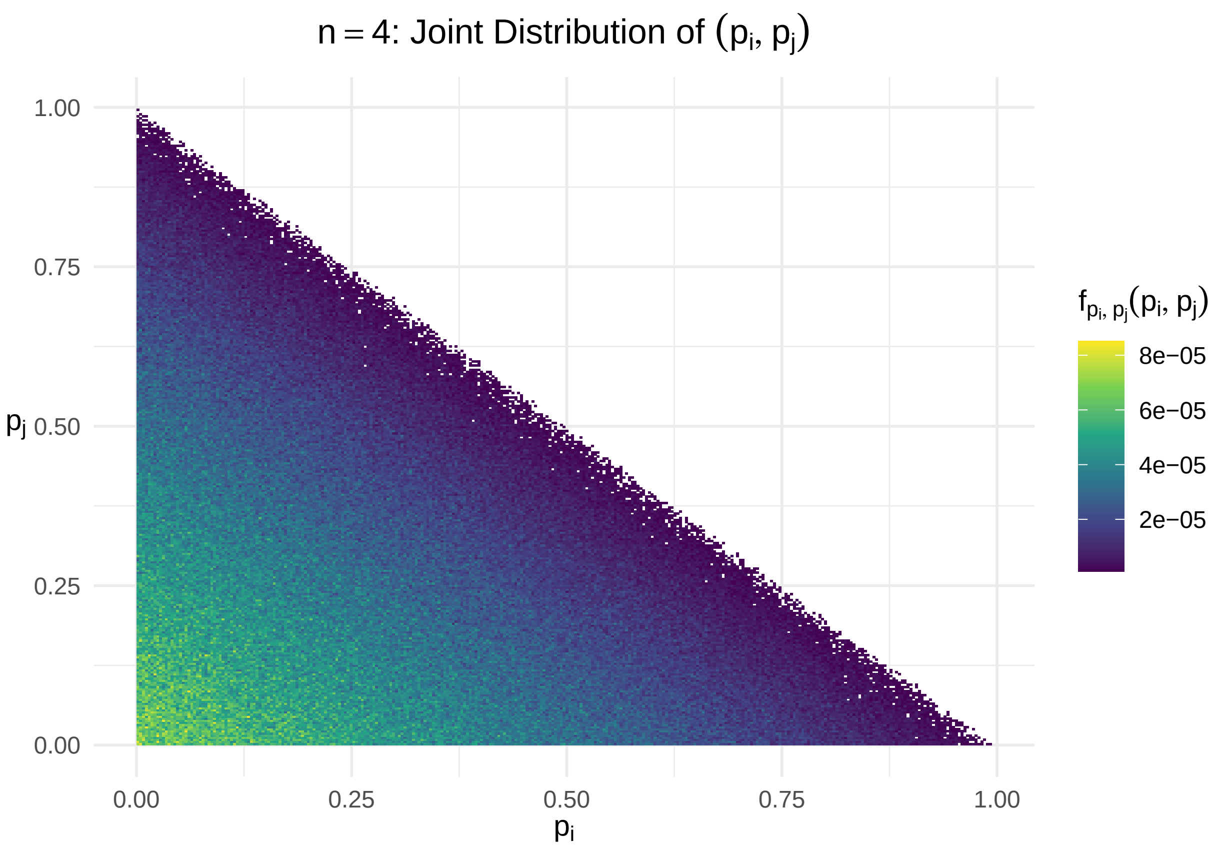



This is exactly what we would expect. Since knowing $p_i$ and $p_j$ entirely determines the final probability and we know $p = (p_1, p_2, p_3)$ is uniform over $\Delta_{n-1} = \Delta_2$ we should see that this is a uniform distribution which it is here.

- For $n = 4$:

```
probs <- prob_generator(N = 4, m = 1000000)
p_comp <- data.frame(pi = probs[1,], pj = probs[4,])

p_comp %>%
  ggplot(aes(x = pi, y = pj)) +
  geom_bin2d(aes(fill = after_stat(density)),
             bins = 300) +
  scale_fill_viridis_c() +
  labs(x = TeX("$p_i$"),
       y = TeX("$p_j$"),
       fill = TeX("$f_{p_i,p_j} (p_i,p_j)$"),
       title = TeX("$n=4$: Joint Distribution of $(p_i, p_j)$")) +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5),
        axis.title = element_text(color = "black"),
        axis.title.y = element_text(angle = 0, vjust = 0.5))
```

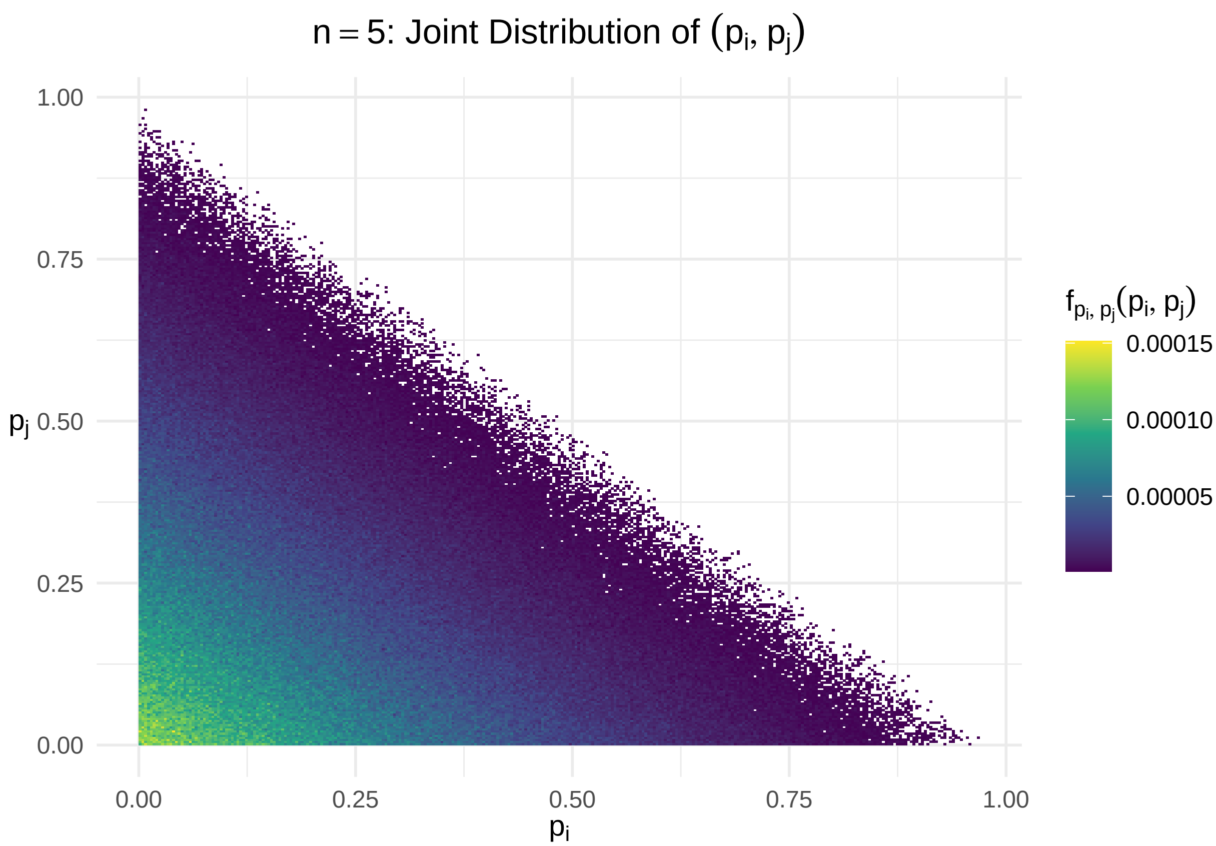



Again this is exactly what we would expect. $p = (p_1, ..., p_4)$ is now uniform over $\Delta_{n-1} = \Delta_3$. Given Here we are more likely to see smaller values of $p_i$ and $p_j$ since there are now 2 more probabilities we have to account for. So when we assign one of the probabilities the remaining ones must now sum to a value smaller than 1 and so we often see smaller values.

- For $n = 5$:

```
probs <- prob_generator(N = 5, m = 1000000)
p_comp <- data.frame(pi = probs[1,], pj = probs[5,])

p_comp %>%
  ggplot(aes(x = pi, y = pj)) +
  geom_bin2d(aes(fill = after_stat(density)),
             bins = 300) +
  scale_fill_viridis_c() +
  labs(x = TeX("$p_i$"),
       y = TeX("$p_j$"),
       fill = TeX("$f_{p_i,p_j} (p_i,p_j)$"),
       title = TeX("$n=5$: Joint Distribution of $(p_i, p_j)$")) +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5),
        axis.title = element_text(color = "black"),
        axis.title.y = element_text(angle = 0, vjust = 0.5))
```




Again this is exactly what we would expect. $p = (p_1, ..., p_5)$ is now uniform over $\Delta_{n-1} = \Delta_4$ and by a similar argument we will see smaller probabilities in this joint graph generally.

- This pattern of smaller probabilities being more likely continues for all $n \in \mathbb{N}$ again by the same reasoning as before.

### Distribution when we change $\alpha$

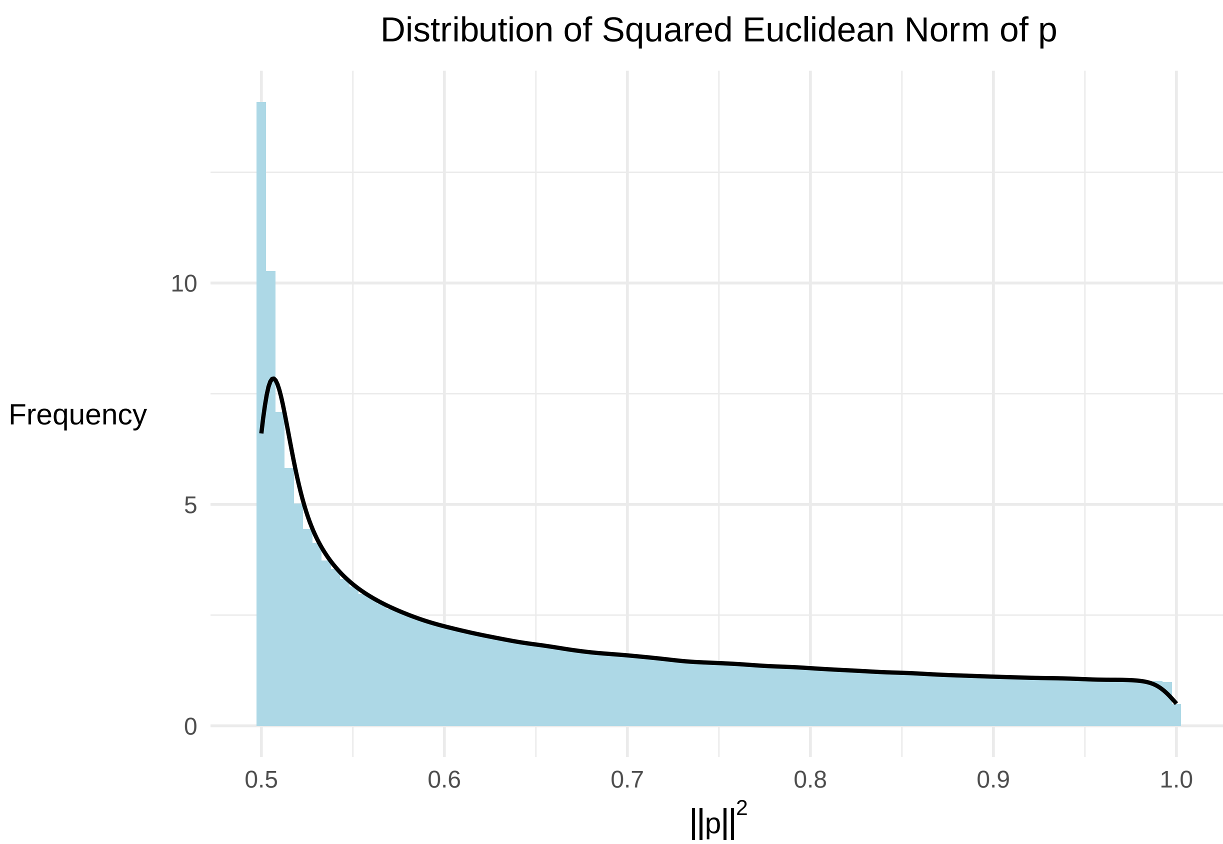
## Examining Distribution of the Square of the Euclidean Norm

Below are some graphs to analyze the distribution of $||p||^2$.

- First $n = 2$:

```r
probs <- prob_generator(N = 2, m = 1000000)
sq_norms <- data.frame(sq_norm = colSums(probs^2))

sq_norms %>%
  ggplot(aes(x = sq_norm)) +
  geom_histogram(aes(y = after_stat(density)),
                 fill = "lightblue",
                 bins = 100) +
  geom_density(col = "black",
               linewidth = 0.75) +
  labs(x = TeX("$||p||^2$"),
       y = "Frequency  ",
       title = TeX("Distribution of Squared Euclidean Norm of $p$")) +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5),
        axis.title = element_text(color = "black"),
        axis.title.y = element_text(angle = 0, vjust = 0.5))
```

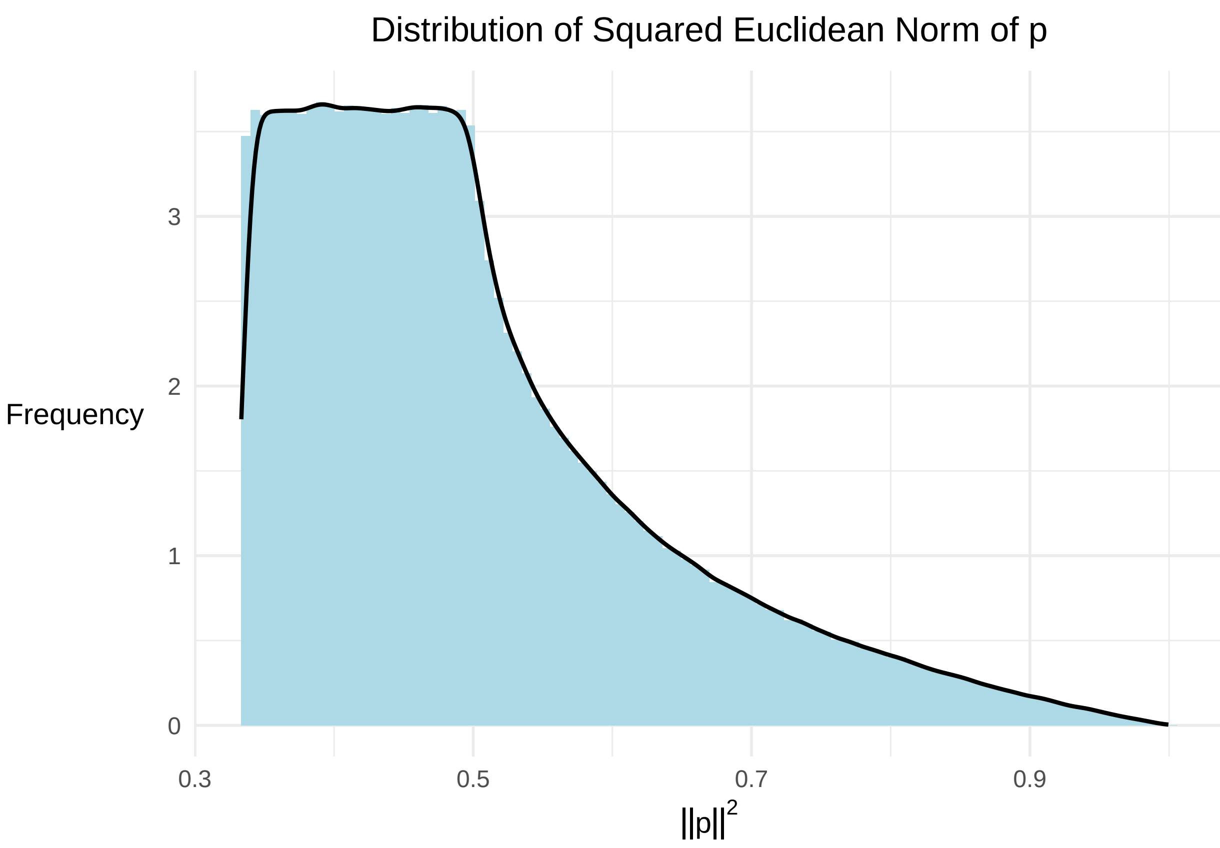



Comment.

- Now $n = 3$:

```
probs <- prob_generator(N = 3, m = 1000000)
sq_norms <- data.frame(sq_norm = colSums(probs^2))

sq_norms %>%
  ggplot(aes(x = sq_norm)) +
  geom_histogram(aes(y = after_stat(density)),
                 fill = "lightblue",
                 bins = 100) +
  geom_density(col = "black",
               linewidth = 0.75) +
  labs(x = TeX("$||p||^2$"),
       y = "Frequency  ",
       title = TeX("Distribution of Squared Euclidean Norm of $p$")) +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5),
        axis.title = element_text(color = "black"),
        axis.title.y = element_text(angle = 0, vjust = 0.5))
```

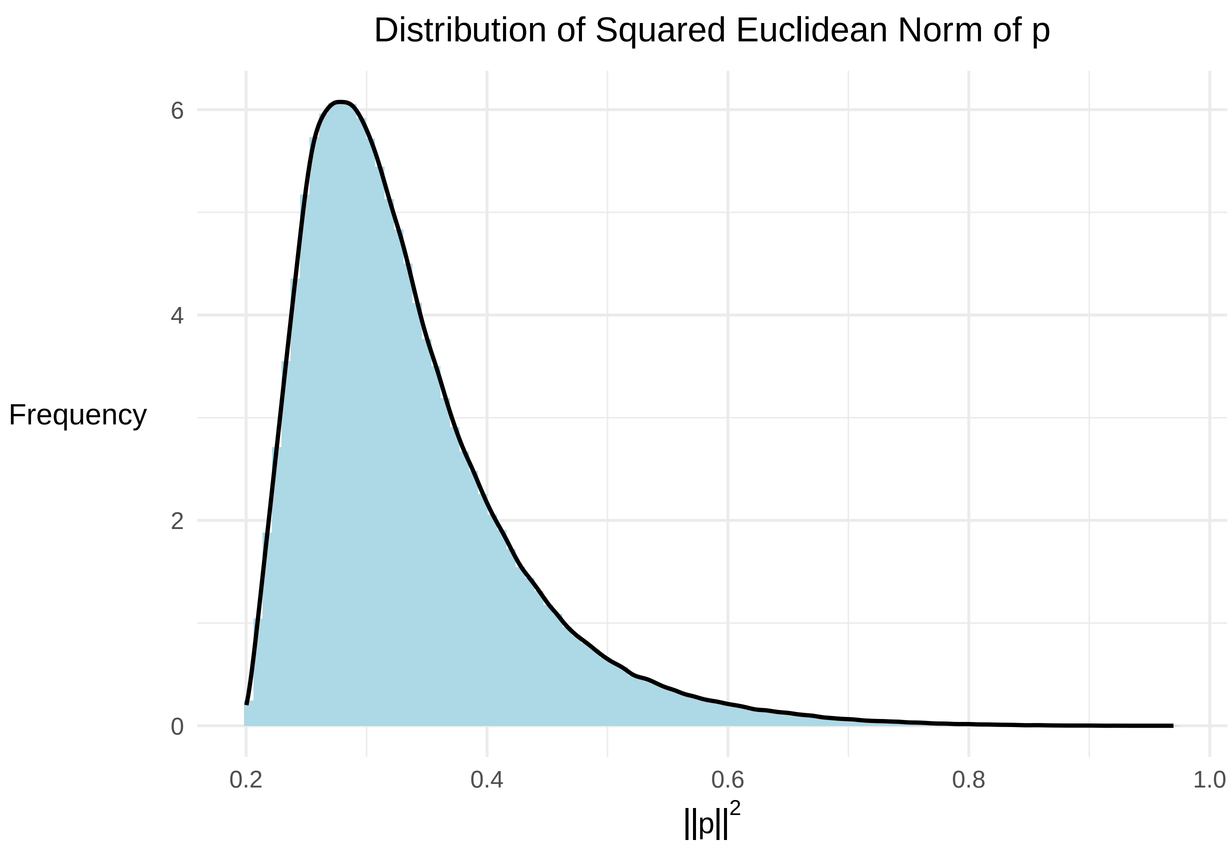



Comment.

- Now $n = 5$:

```
probs <- prob_generator(N = 5, m = 1000000)
sq_norms <- data.frame(sq_norm = colSums(probs^2))

sq_norms %>%
  ggplot(aes(x = sq_norm)) +
  geom_histogram(aes(y = after_stat(density)),
                 fill = "lightblue",
                 bins = 100) +
  geom_density(col = "black",
               linewidth = 0.75) +
  labs(x = TeX("$||p||^2$"),
       y = "Frequency  ",
       title = TeX("Distribution of Squared Euclidean Norm of $p$")) +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5),
        axis.title = element_text(color = "black"),
        axis.title.y = element_text(angle = 0, vjust = 0.5))
```

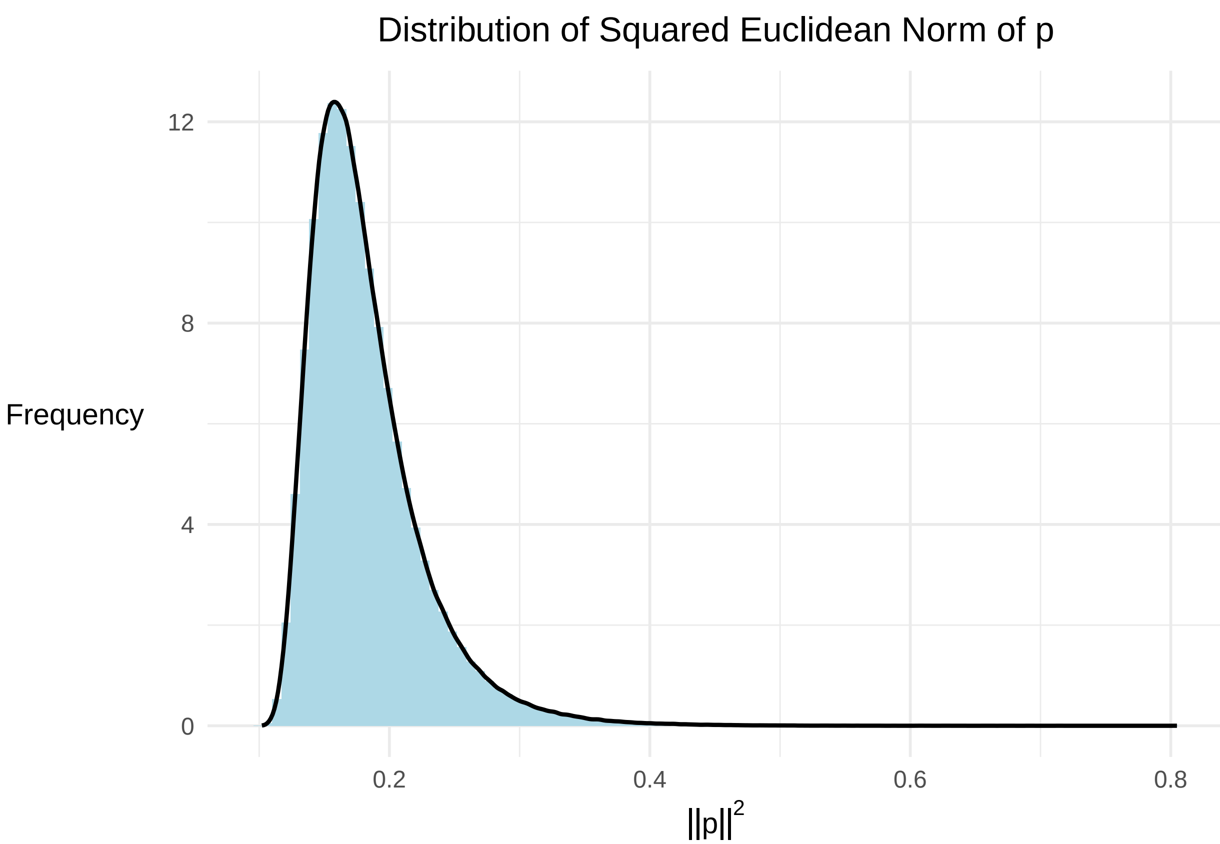



Comment.

- Now $n = 10$:

```
probs <- prob_generator(N = 10, m = 1000000)
sq_norms <- data.frame(sq_norm = colSums(probs^2))

sq_norms %>%
  ggplot(aes(x = sq_norm)) +
  geom_histogram(aes(y = after_stat(density)),
                 fill = "lightblue",
                 bins = 100) +
  geom_density(col = "black",
               linewidth = 0.75) +
  labs(x = TeX("$||p||^2$"),
       y = "Frequency  ",
       title = TeX("Distribution of Squared Euclidean Norm of $p$")) +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5),
        axis.title = element_text(color = "black"),
        axis.title.y = element_text(angle = 0, vjust = 0.5))
```

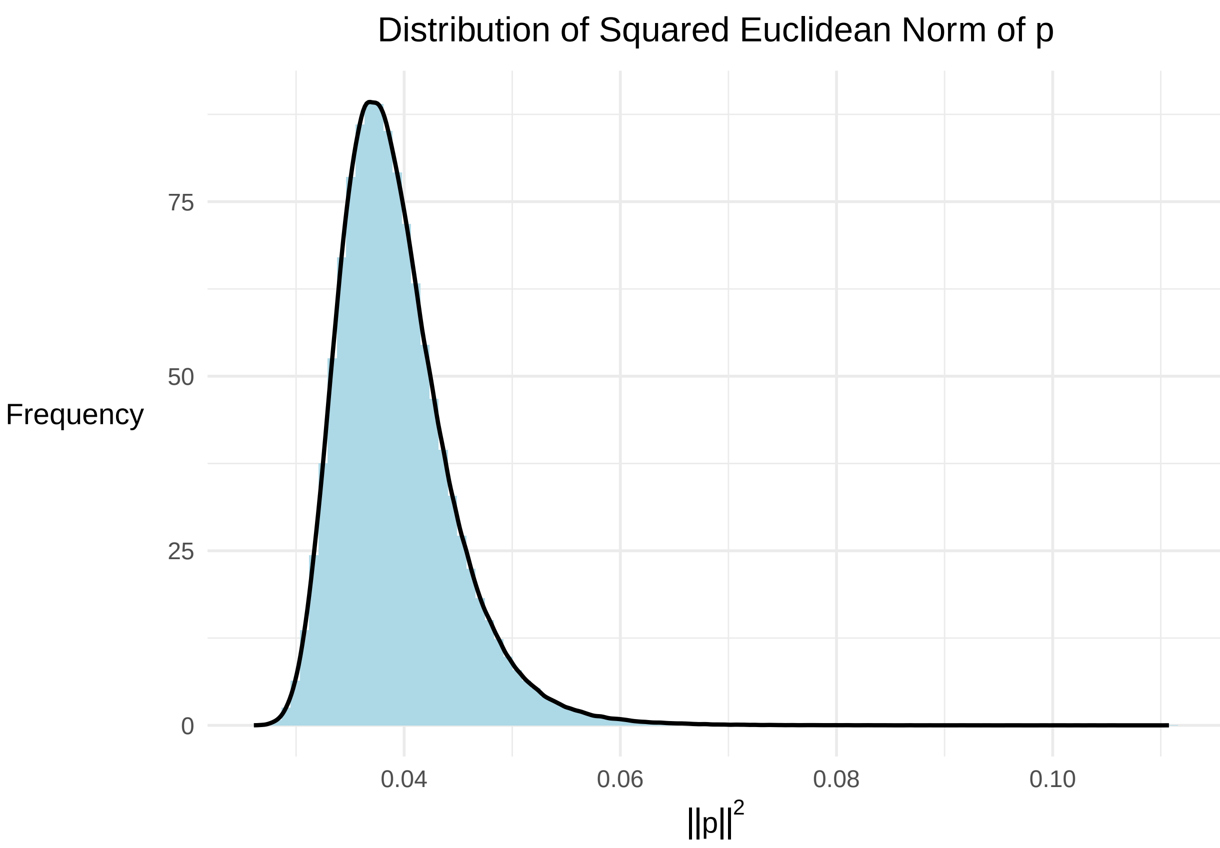



Comment.

- Now $n = 50$:

```
probs <- prob_generator(N = 50, m = 1000000)
sq_norms <- data.frame(sq_norm = colSums(probs^2))

sq_norms %>%
  ggplot(aes(x = sq_norm)) +
  geom_histogram(aes(y = after_stat(density)),
                 fill = "lightblue",
                 bins = 100) +
  geom_density(col = "black",
               linewidth = 0.75) +
  labs(x = TeX("$||p||^2$"),
       y = "Frequency  ",
       title = TeX("Distribution of Squared Euclidean Norm of $p$")) +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5),
        axis.title = element_text(color = "black"),
        axis.title.y = element_text(angle = 0, vjust = 0.5))
```

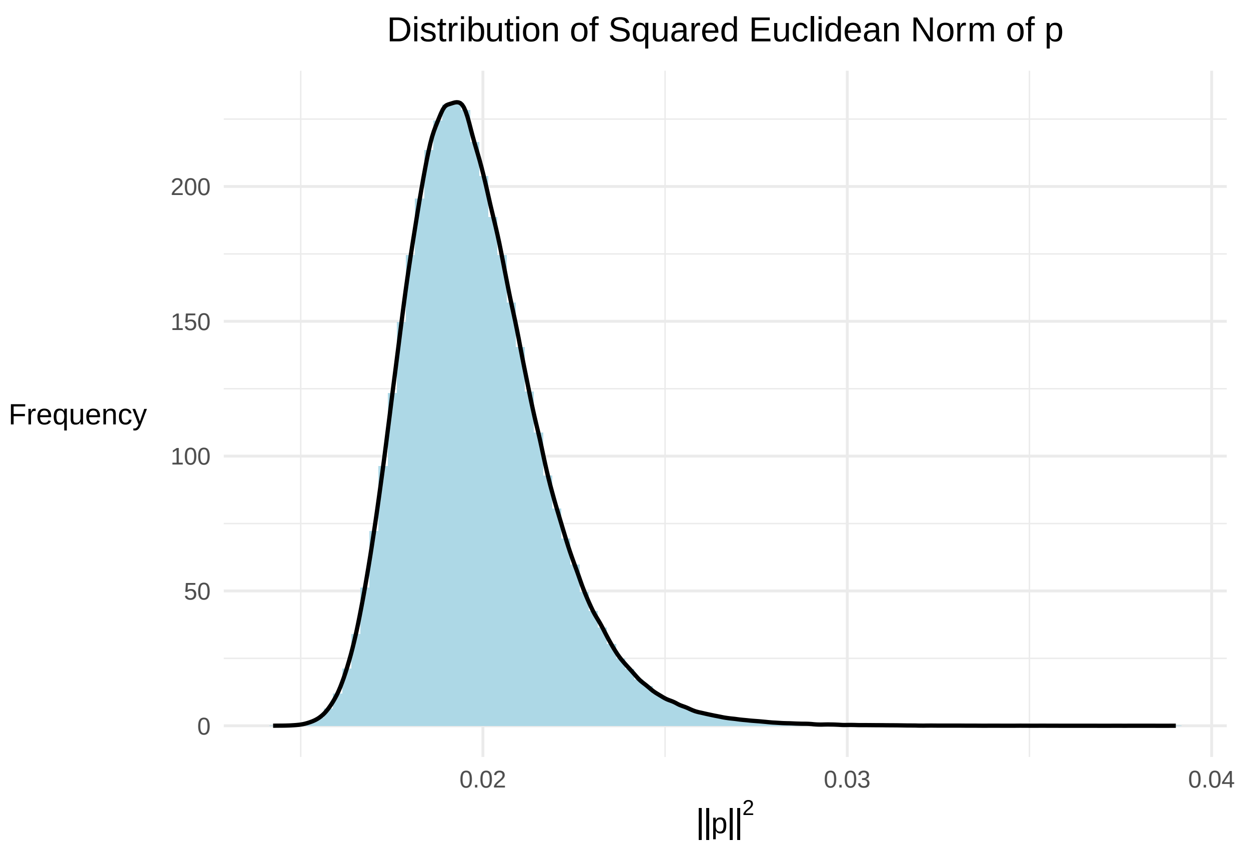



Comment.

- Now $n = 100$:

```
probs <- prob_generator(N = 100, m = 1000000)
sq_norms <- data.frame(sq_norm = colSums(probs^2))

sq_norms %>%
  ggplot(aes(x = sq_norm)) +
  geom_histogram(aes(y = after_stat(density)),
                 fill = "lightblue",
                 bins = 100) +
  geom_density(col = "black",
               linewidth = 0.75) +
  labs(x = TeX("$||p||^2$"),
       y = "Frequency  ",
       title = TeX("Distribution of Squared Euclidean Norm of $p$")) +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5),
        axis.title = element_text(color = "black"),
        axis.title.y = element_text(angle = 0, vjust = 0.5))
```

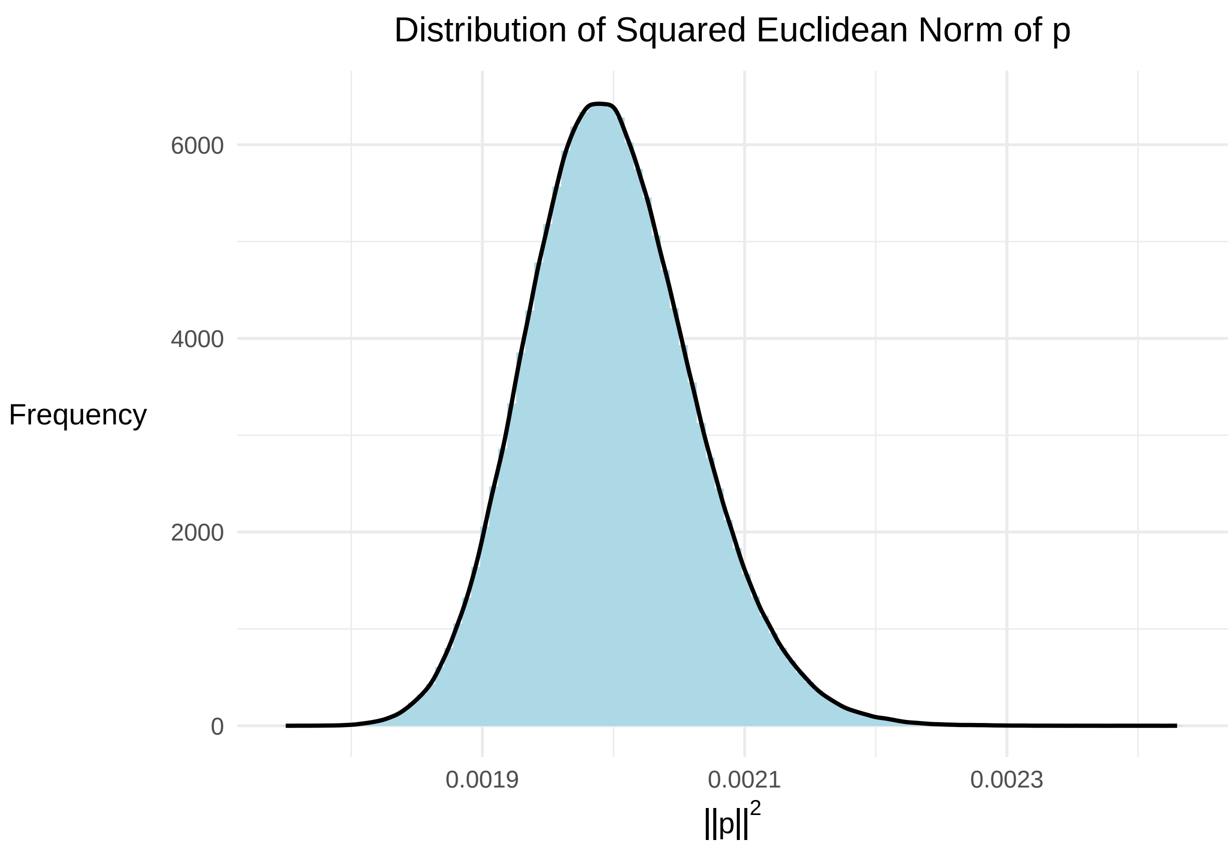



Comment.

- Now $n = 1000$:

```
probs <- prob_generator(N = 1000, m = 1000000)
sq_norms <- data.frame(sq_norm = colSums(probs^2))

sq_norms %>%
  ggplot(aes(x = sq_norm)) +
  geom_histogram(aes(y = after_stat(density)),
                 fill = "lightblue",
                 bins = 100) +
  geom_density(col = "black",
               linewidth = 0.75) +
  labs(x = TeX("$||p||^2$"),
       y = "Frequency  ",
       title = TeX("Distribution of Squared Euclidean Norm of $p$")) +
  theme_minimal() +
  theme(plot.title = element_text(hjust = 0.5),
        axis.title = element_text(color = "black"),
        axis.title.y = element_text(angle = 0, vjust = 0.5))
```




Comment.