Imaginary Numbers

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2.2

Let $z = x + iy$, then $Re\, z = x$ and $Im\, z = y$. Recall that $i^2 = -1$ and $(-1)z = -z$ for any complex number z.

a.

Since multiplication is distributive for complex numbers we get $iz = i(x + iy) = ix + i^2 y = ix + (-1)y$.

Since addition is commutative for complex numbers we get $iz = ix + (-1)y = (-1)y + ix = -y + ix$.

So $Re(iz) = Re(-y + ix) = -y = -Im\, z$ $\square$

b.

Since multiplication is distributive for complex numbers we get $iz = i(x + iy) = ix + i^2 y = ix + (-1)y$.

Since addition is commutative for complex numbers we get $iz = ix + (-1)y = (-1)y + ix = -y + ix$.

So $Im(iz) = Im(-y + ix) = x = Re\, z$ $\square$

2.11

Let $z = (x, y)$ and $z^2 + z + 1 = 0$.

We get $z^2 + z + 1 = (x, y)(x, y) + (x, y) + (1, 0) = (x^2 - y^2, 2xy) + (x, y) + (1, 0) = (x^2 + x - y^2 + 1, 2xy + y) = (0, 0)$.

Since $z_1 = z_2$ if and only if $Re\, z_1 = Re\, z_2$ and $Im\, z_1 = Im\, z_2$ we get the simultaneous equations:

$x^2 + x - y^2 + 1 = 0$ and $2xy + y = 0$.

If $y = 0$ the second equation is satisfied and we are left to solve $x^2 + x + 1 = 0$ for $x \in \mathbb{R}$, this can also be seen as if $y = 0$ then z is purely real and we have the equation we started with.

If $x \in \mathbb{R}$ then $x^2 + x + 1 = x^2 + x + \frac{1}{4} + \frac{3}{4} = (x + \frac{1}{2})^2 + \frac{3}{4} > 0$ since $\frac{3}{4} > 0$ and $t^2 \geq 0$ for all $t \in \mathbb{R}$.

Therefore we are left with no solutions if $y = 0$. So assume $y \neq 0$.

Then from $2xy + y = 0$ we get $y(2x + 1) = 0$ and since $y \neq 0$ this means $2x + 1 = 0$ and hence $x = -\frac{1}{2}$.

Using $x = -\frac{1}{2}$ we get $x^2 + x - y^2 + 1 = \frac{1}{4} - \frac{1}{2} - y^2 + 1 = \frac{3}{4} - y^2 = 0$ giving $y^2 = \frac{3}{4}$ and hence $y = \pm\frac{\sqrt{3}}{2}$.

Therefore from the equation $z^2 + z + 1 = 0$ we get the solutions $z = (-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}) = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ $\square$

3.1

Recall that for complex numbers $z = x + iy$, $\frac{1}{z} = z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ and $i^2 = -1$.

a.

We are considering

$$z = \frac{1+2i}{3-4i} + \frac{2-i}{5i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)} + \frac{(2-i)i}{5i^2} = \frac{3+4i+6i+8i^2}{9+12i-12i-16i^2} + \frac{2i-i^2}{-5} = \frac{3-8+10i}{9+16} - \frac{1+2i}{5} = \frac{-5+10i}{25} - \frac{1+2i}{5} = \frac{-1+2i}{5} - \frac{1+2i}{5} = -\frac{2}{5} \ \square$$

b.

We are considering $z = \frac{5i}{(1-i)(2-i)(3-i)} = \frac{5i}{(2-1-i-2i)(3-i)} = \frac{5i}{(1-3i)(3-i)} = \frac{5i}{3-3-i-9i} = \frac{5i}{-10i} = \frac{5i^2}{-10i^2} = \frac{-5}{10} = -\frac{1}{2}$.

c.

We are considering $z = (1-i)^4 = (1-i)^2(1-i)^2$.

It is easier to first examine $(1-i)^2 = (1-i)(1-i) = 1 - 1 - 2i = -2i$.

Now we have $z = (1-i)^2(1-i)^2 = (-2i)(-2i) = 4i^2 = -4 \ \square$

5.6

Let $S = \{z \in \mathbb{C} : |z-1| = |z+i|\} = \{z \in \mathbb{C} : |z-1| = |z-(-i)|\}$.

For $z \in \mathbb{C}$, $|z-1|$ represents the distance of z from 1 and $|z+i| = |z-(-i)|$ represents the distance of z from $-i$.

This means that S consists of all points in $\mathbb{C}$ that are equidistant from the points $z_1 = 1$ and $z_2 = -i$.

If you have two circles of radius r centered at 1 and $-i$ they will only ever intersect at most twice and the real value of these two intersections will be different. Therefore if $z_1 \in S$ and $Re\, z_2 = Re\, z_1$ then $z_2 \in S$ if and only if $z_1 = z_2$.

As $Re\, z$ varies the rate of change of the distance from 1 must be equal to the rate of change of the distance from $-i$ otherwise subsequent points in S can't remain equidistant to both 1 and $-i$.

This equality of rates of change along different axes is the nature of lines, so it is safe to say that S defines a line.

The line defined by S must be perpendicular to the line segment connecting 1 and $-i$, otherwise we get different rates of change for the distances from 1 and $-i$, which again we can't have.

Simply said, the line must be the perpendicular bisector of the line segment connecting 1 and $-i$.

In $\mathbb{C}$ the vector going from z_1 to z_2 is given by $z_2 - z_1$. For our example this is $1 - (-i) = 1 + i$.

To get a perpendicular vector for the purpose of defining a line we must rotate this 90^o (direction of the rotation doesn't matter here it will define the same line later), multiplying by i will rotate this vector 90^o counterclockwise.

So we can use the vector $i(1+i) = i + i^2 = -1 + i$ to define our line, this gives us the necessary slope but we still need a point on the line. We could use the midpoint of the line segment but there is an easier one.

Notice that $|0 - (-i)| = |i| = 1 = |1| = |0 - 1|$ so we have that $0 \in S$ and hence on our line.

Therefore the line S defines is represented with the parametric equation $0 + t(-1+i) = t(-1+i)$ for $t \in \mathbb{R}$.

In vector notation this is given by $(-t, t)$ for $t \in \mathbb{R}$ and hence $y = -x$ as desired.

Therefore $|z-1| = |z+i|$ defines the line through the origin with slope -1 $\square$

What I am doing by multiplying the vector defining our slope by t is allowing the length of our vector to cover all of $\mathbb{R}$ as t varies while still keeping the same direction thus constructing a line. Adding 0 then gives our line a starting point.

Note however that you could make the starting point any point on the line for example $\frac{1}{2} - i\frac{1}{2}$, the midpoint of the segment connecting 1 and $-i$.

Another way to think of this is to consider two expanding circles whose radii are always equal (initially 0) that are centered at 1 and $-i$, then S is the set of all points where these circles intersect at any radius in $\mathbb{R}$.

You can see this algebraically as well. Let $z = x + iy$.

Then $|z-1| = |x+iy-1| = |x-1+iy| = \sqrt{(x-1)^2+y^2}$ and $|z+i| = |x+iy+i| = |x+i(1+y)| = \sqrt{x^2+(1+y)^2}$.

So if $|z-1| = |z+i|$ we have $\sqrt{(x-1)^2+y^2} = \sqrt{x^2+(1+y)^2}$ and hence $(x-1)^2+y^2 = x^2+(1+y)^2$.

So $x^2 - 2x + 1 + y^2 = x^2 + 1 + 2y + y^2$ and $-2x = 2y$, therefore $y = -x$ as desired.

5.7

Note that in this problem I use the result of 5.9 to say that $|z^n| = |z|^n$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ for $a_0, a_1, ..., a_n \in \mathbb{C}$ and $a_n \neq 0$.

Now let $w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \frac{a_2}{z^{n-2}} + ... + \frac{a_{n-1}}{z}$ for $z \neq 0$, then $z^n w + a_n z^n = z^n(w + a_n) = P(z)$ for $z \neq 0$.

We also get that $wz^n = a_0 + a_1 z + a_2 z^2 + ... + a_{n-1} z^{n-1}$ and by the triangle inequality

$|wz^n| = |w||z|^n \leq |a_0| + |a_1 z| + |a_2 z^2| + ... + |a_{n-1} z^{n-1}| = |a_0| + |a_1||z| + |a_2||z|^2 + ... + |a_{n-1}||z|^{n-1}$.

Therefore $|w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \frac{|a_2|}{|z|^{n-2}} + ... + \frac{|a_{n-1}|}{|z|}$.

Since each of $|a_0|, |a_1|, |a_2|, ..., |a_n|$ is finite we can let $a = max\{|a_0|, |a_1|, |a_2|, ..., |a_{n-1}|\}$.

Then let $R > max\{\frac{na}{|a_n|}, 1\}$, this exists because n, a, and $|a_n|$ are finite and $|a_n| \neq 0$ otherwise this problem simplifies to the case where $P(z)$ is a polynomial of degree $n - 1$.

Now if $|z| > R > \frac{na}{|a_n|}$ we have $|z| > \frac{na}{|a_n|} \geq \frac{n|a_k|}{|a_n|}$ for all $k \in \{0, 1, 2, ..., n-1\}$ by our definition of a.

Furthermore we get that $|z|^n > |z|^{n-1} > ... > |z|$ since $|z| > R > 1$.

So we have $\frac{|a_k|}{|z|^{n-k}} \leq \frac{|a_k|}{|z|} < \frac{|a_n|}{n}$ for each $k \in \{0, 1, 2, ..., n-1\}$ when $|z| > R$.

Using our expression we got from the triangle inequality we have:

$|w| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \frac{|a_2|}{|z|^{n-2}} + ... + \frac{|a_{n-1}|}{|z|} < (n-1)\frac{a_n}{n}$ when $|z| > R$.

Therefore $|a_n + w| \leq |a_n| + |w| < |a_n| + \frac{(n-1)}{n}|a_n| < 2|a_n|$ when $|z| > R$.

This gives $|P(z)| = |z|^n|w + a_n| < |z|^n(2|a_n|) = 2|a_n||z|^n$ when $|z| > R$

This was true for an arbitrary choice of $n \in \mathbb{N}$ and is therefore true for all $n \in \mathbb{N}$ $\square$

5.9

Let $S = \{n \in \mathbb{N} : \forall z \in \mathbb{C}, |z^n| = |z|^n\}$

Clearly $1 \in S$ since $z^1 = z$ and so $|z^1| = |z| = |z|^1$.

From problem 5.8 we have the result that for two complex numbers z_1, z_2 we know $|z_1 z_2| = |z_1||z_2|$.

Now assume that $n \in S$, then $|z^n| = |z|^n$.

Then $|z^{n+1}| = |z^n z| = |z^n||z| = |z|^n|z| = |z|^{n+1}$, so $n + 1 \in S$.

Therefore by induction $S = \mathbb{N}$ and so for all $n \in \mathbb{N}$, $|z^n| = |z|^n$ for all $z \in \mathbb{C}$ $\square$

Proof of the result from problem 5.8:

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then as seen before $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$.

Therefore $|z_1 z_2| = \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + y_1 x_2)^2} = \sqrt{x_1^2 x_2^2 - 2x_1 x_2 y_1 y_2 + y_1^2 y_2^2 + x_1^2 y_2^2 + 2y_1 y_2 x_1 x_2 + y_1^2 x_2^2} =$

$\sqrt{x_1^2 x_2^2 + y_1^2 y_2^2 + x_1^2 y_2^2 + y_1^2 x_2^2} = \sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2 y_2^2 + y_1^2 x_2^2} = \sqrt{x_1^2(x_2^2 + y_2^2) + y_1^2(y_2^2 + x_2^2)} = \sqrt{(x_2^2 + y_2^2)(x_1^2 + y_1^2)}$

We also know $|z_1| = \sqrt{x_1^2 + y_1^2}$ and $|z_2| = \sqrt{x_2^2 + y_2^2}$, so $|z_1||z_2| = (\sqrt{x_1^2 + y_1^2})(\sqrt{x_2^2 + y_2^2}) = \sqrt{(x_2^2 + y_2^2)(x_1^2 + y_1^2)} = |z_1 z_2|$.

Therefore for all $z_1, z_2 \in \mathbb{C}$ we have $|z_1 z_2| = |z_1||z_2|$ $\square$