Integration

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7.2.3

a. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.

First assume there exists a sequence of partitions (P_n) of $[a, b]$ such that $lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$.

Recall that for any partition P of $[a, b]$ we have $U(f, P) \geq L(f, P)$.

So for all $n \in \mathbb{N}$ we have $U(f, P_n) - L(f, P_n) \geq 0$ and therefore $|U(f, P_n) - L(f, P_n)| = U(f, P_n) - L(f, P_n)$.

Then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for $n \geq N$ we have $|U(f, P_n) - L(f, P_n) - 0| = U(f, P_n) - L(f, P_n) < \epsilon$.

So for any $\epsilon > 0$ let $P_\epsilon = P_N$ then we have found a partition such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

Therefore f is integrable on $[a, b]$ by the integrability criterion.

Now assume that f is integrable on $[a, b]$.

Then by the integrability criterion for all $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.

So for each $n \in \mathbb{N}$ let P_n be such that $U(f, P_n) - L(f, P_n) < \frac{1}{n}$. Such a P_n exists by the integrability criterion.

Recall that for any partition P of $[a, b]$ we have $U(f, P) \geq L(f, P)$.

So for all $n \in \mathbb{N}$ we have $0 \leq U(f, P_n) - L(f, P_n) < \frac{1}{n}$.

Therefore $lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$ by the squeeze theorem since $(\frac{1}{n}) \to 0$ and $(0) \to 0$.

So we have found a sequence of partitions (P_n) of $[a, b]$ such that $lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$.

Assume we have such a sequence of partitions (P_n) of $[a, b]$, then by the above proof f is integrable on $[a, b]$.

Furthermore note that for any partition P we have that $L(f, P) \leq L(f) = \int_a^b f = U(f) \leq U(f, P)$.

Therefore for all $n \in \mathbb{N}$ we have $L(f, P_n) \leq \int_a^b f \leq U(f, P_n)$.

So for all $n \in \mathbb{N}$ we have $0 \leq \int_a^b f - L(f, P_n) \leq U(f, P_n) - L(f, P_n)$.

So by the squeeze theorem $lim_{n\to\infty} \int_a^b f - L(f, P_n) = \int_a^b f - lim_{n\to\infty} L(f, P_n) = 0$ and hence $lim_{n\to\infty} L(f, P_n) = \int_a^b f$.

Since $lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$ we have by the algebraic limit theorem that

$$lim_{n\to\infty} U(f, P_n) = lim_{n\to\infty} L(f, P_n) = \int_a^b f.$$

Therefore f is integrable on $[a, b]$ if and only if there exists a sequence of partitions (P_n) of $[a, b]$ such that

$lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$, and in this case $lim_{n\to\infty} L(f, P_n) = lim_{n\to\infty} U(f, P_n) = \int_a^b f$ $\square$

7.3.5

a. Let $A = [a,b] \cap \mathbb{Q}$. Since $\mathbb{Q}$ is countable we know that A is also countable so we may write $A = \{a_1, a_2, a_3, ...\}$.

For $n \in \mathbb{N}$ let $A_n = \{a_1, a_2, ..., a_n\}$ then let $f_n(x) = 1$ if $x \in A_n$ and $f_n(x) = 0$ otherwise.

As $n \to \infty$ clearly $A_n \to A$ and so $(f_n(x)) \to f(x)$ where $f(x) = 1$ is $x \in A$ and $f(x) = 0$ otherwise.

This is Dirichlet's function restricted to the domain of $[a,b]$.

Clearly each f_n has only finitely many discontinuities.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we know for all $x \in [a,b]$ that in every $V_\epsilon(x)$ there is some element of A.

Therefore $U(f,P) = 1$ for any partition P. Similarly, the irrationals are dense in $\mathbb{R}$ so $L(f,P) = 0$ for any partition P.

So we have that $U(f) = 1$ and $L(f) = 0$ so f is not integrable.

So this is such an example of a sequence of functions $(f_n) \to f$ such that each f_n has at most finitely many discontinuities but f is not integrable.

b. This is not possible. Let $(f_n) \to f$ uniformly with each f_n having at most finitely many discontinuities.

Let the discontinuities of f_n be $D_n = \{d_1, d_2, ..., d_{m_n}\}$, and let these be in increasing order.

Then we know f_n is continuous and therefore integrable on (d_k, d_{k+1}) for each $k \in \{1, 2, ..., m_n - 1\}$.

For each k fix some $z_k \in (d_k, d_{k+1})$ then we know f_n is integrable on $[x, z_k]$ for all $x \in (d_k, z_k)$.

Therefore f_n is integrable on $[d_k, z_k]$.

Similarly we know f_n is integrable on $[z_k, y]$ for all $y \in (z_k, d_{k+1})$.

Therefore f_n is integrable on $[z_k, d_{k+1}]$ and is hence integrable on $[d_k, d_{k+1}]$.

This was for each $[d_k, d_{k+1}]$ and therefore we have that f_n is integrable on its domain.

This was for arbitrary f_n and is therefore true for each f_n, so each f_n is integrable on its domain.

Since uniform convergence preserves integrability we have that f is also integrable on its domain.

So if $(f_n) \to f$ uniformly with each f_n having at most finitely many discontinuities then f is integrable $\square$

c. Let $A = [a,b] \cap \mathbb{Q}$. Then for $n \in \mathbb{N}$ let $f_n(x) = \frac{1}{n}$ if $x \in A$ and $f_n(x) = 0$ otherwise.

This is a modified version of Dirichlet's function restricted to the domain of $[a,b]$.

Consider some arbitrary f_n.

As before we have that since $\mathbb{Q}$ is dense in $\mathbb{R}$ we know for all $x \in [a,b]$ that in every $V_\epsilon(x)$ there is some element of A.

Therefore $U(f_n, P) = \frac{1}{n}$ for any partition P. Similarly, the irrationals are dense in $\mathbb{R}$ so $L(f_n, P) = 0$ for any partition P.

So we have that $U(f_n) = \frac{1}{n}$ and $L(f_n) = 0$ so f_n is not integrable.

This was for arbitrary f_n and is therefore true for all f_n, so each f_n is not integrable.

Now let $\epsilon > 0$ then let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \epsilon$.

Then for $n \geq N$ we have $|f_n(x) - 0| = |f_n(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$. So $(f_n) \to f = 0$ uniformly.

Since constant functions are continuous they are also integrable.

So this is such an example of a sequence of functions $(f_n) \to f$ uniformly where each f_n is not integrable but f is integrable.

7.5.2

a. False, derivatives do not necessarily conserve continuity.

I don't remember which sample work it was but we looked at $f(x) = x^2 cos(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$.

We concluded that $f'(x) = 2x\, cos(\frac{1}{x}) + sin(\frac{1}{x})$ for $x \neq 0$ and $f'(0) = 0$.

However f' is clearly not continuous at 0 as $2x\, cos(\frac{1}{x})$ grows arbitrarily small as we get close to 0 but $sin(\frac{1}{x})$ grows extremely oscillatory as we get close to 0. So the limit of f' does not exist at 0 and hence f' is not continuous at 0.

b. True, this is a result of the fundamental theorem of calculus.

If g is continuous on $[a, b]$ then it is also integrable on $[a, b]$.

By defining the function $G(x) = \int_a^x g$ we get that G is continuous on $[a, b]$, and differentiable on $[a, b]$ since g is continuous on $[a, b]$. Consequently, $G' = g$.

So every continuous function is the derivative of some function.

c. False, the converse is true but this statement is not true.

Let h be Thomae's function on $[0, a]$ which we have seen previously is discontinuous at every rational number.

Now consider any partition P of $[0, a]$. You can always find an irrational number in any of the segments of the partition since the irrationals are dense in $\mathbb{R}$ so $L(h, P) = 0$ for any partition P. Hence $L(h) = 0$.

Furthermore for any point $y \in \mathbb{R}$ we have seen that $lim_{x \to y} h(x) = 0$. So by refining your partitions repeatedly you can make $U(h, P)$ arbitrarily small hence $inf\{U(h, P), P \in \mathcal{P}\} = 0$.

So $\int_0^a h = 0$ and this was for an arbitrary $a \in \mathbb{R}$ and is therefore true for all $a \in \mathbb{R}$.

So choose any $x \in \mathbb{Q}$, then $H(x) = \int_0^x h = 0$ is constant and therefore differentiable. However $h(x)$ is discontinuous.

So the differentiability of H does not imply the continuity of h.