Arguments and Branches of Functions

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30.6

Recall that for $z_1, z_2 \in \mathbb{C}$ we know $|z_1 z_2| = |z_1||z_2|$.

Also recall that for $t_1, t_2, \theta \in \mathbb{R}$ we know that if $t_1 \le t_2$ then $e^{t_1} \le e^{t_2}$ since e^t is an increasing function and $|e^{i\theta}| = 1$.

Let $z = x + iy$ then consider $e^{(z^2)} = e^{(x+iy)^2} = e^{x^2+2ixy-y^2} = e^{x^2-y^2}e^{2ixy}$.

Then we have that $|e^{(z^2)}| = |e^{x^2-y^2}e^{2ixy}| = |e^{x^2-y^2}||e^{2ixy}| = |e^{x^2-y^2}| = e^{x^2-y^2}$ since $x, y \in \mathbb{R}$.

Furthermore we know $x^2 - y^2 \le x^2 + y^2$, so $e^{x^2-y^2} \le e^{x^2+y^2} = e^{|z|^2}$ $\square$

30.8

c. We want to find all $z \in \mathbb{C}$ such that $e^{2z-1} = 1$. Let $z = x + iy$, then $e^{2z-1} = e^{2(x+iy)-1} = e^{(2x-1)+2iy}$.

So we want to solve $e^{(2x-1)+2iy} = 1$. We have $e^{(2x-1)+2iy} = e^{2x-1}e^{2iy} = e^{2x-1}(cos(2y) + i\ sin(2y))$.

Setting this equal to 1 we get $e^{2x-1}(cos(2y) + i\ sin(2y)) = e^{2x-1}cos(2y) + ie^{2x-1}sin(2y) = 1 + 0i$.

Therefore we must have the simultaneous equations $e^{2x-1}cos(2y) = 1$ and $e^{2x-1}sin(2y) = 0$.

From the second equation ($e^{2x-1}sin(2y) = 0$):

Since $e^{2x-1} \neq 0$ for any $x \in \mathbb{R}$ we must have that $sin(2y) = 0$.

Therefore $2y = n\pi$, hence $y = \frac{n\pi}{2}$ for $n \in \mathbb{Z}$.

From the first equation ($e^{2x-1}cos(2y) = 1$):

Now we have restricted $y = \frac{n\pi}{2}$ and so $2y = n\pi$ for $n \in \mathbb{Z}$.

However if $n\, mod\, 2 = 1$ (meaning n is odd) then $cos(n\pi) = -1$ and this would give $e^{2x-1}cos(2y) = -e^{2x-1} = 1$.

This clearly has no solutions as $e^{2x-1} > 0$ so it can not be that $e^{2x-1} = -1$.

So we must further restrict y so that both equations are satisfied.

We need $cos(2y) > 0$ and $2y = n\pi$ where $n \in \mathbb{Z}$, therefore we must have that n is an even integer.

So $y = \frac{n\pi}{2}$ for $n \in \{m \in \mathbb{Z} : m\, mod\, 2 = 0\} = \{0, \pm 2, \pm 4, \pm 6, ...\}$.

Now we know $cos(2y) = cos(n\pi) = 1$ since n is even. So $e^{2x-1}cos(2y) = e^{2x-1}$.

Setting this equal to 1 we get $e^{2x-1} = 1$ and hence $ln(e^{2x-1}) = 2x - 1 = ln(1) = 0$. Consequently $x = \frac{1}{2}$.

So $e^{2z-1} = 1$ if and only if $z = \frac{1}{2} + i\frac{n\pi}{2}$ where $n \in \{m \in \mathbb{Z} : m\, mod\, 2 = 0\} = \{0, \pm 2, \pm 4, \pm 6, ...\}$ $\square$

33.5

a. Recall that for $\theta \in \mathbb{R}$ we know $|e^{i\theta}| = 1$. Also recall that $log(z) = ln|z| + i\, arg\, z$.

As seen previously we know that the n distinct nth roots of a complex number z are given by $c_0, c_1, ..., c_{n-1}$.

Where $c_k = \sqrt[n]{r}(e^{i\frac{\theta}{n}} w_n^k)$, $w_n = e^{i\frac{2\pi}{n}}$, and $k \in \{0, 1, ..., n-1\}$ (here we take $r = |z|$ and $\theta = Arg\, z$).

Therefore for $n = 2$ we have the two distinct roots $c_0 = \sqrt{r}e^{i\frac{\theta}{2}}$ and $c_1 = \sqrt{r}e^{i\frac{\theta}{2}}e^{i\pi} = \sqrt{r}e^{i(\frac{\theta}{2}+\pi)}$.

We know that $i = e^{i\frac{\pi}{2}}$ since $Arg\, i = \frac{\pi}{2}$ and $|i| = 1$, so we use $\theta = \frac{\pi}{2}$ and $r = 1$.

Therefore the two roots of i are given by $i^{\frac{1}{2}} = \{e^{i\frac{\pi}{4}}, e^{i(\frac{\pi}{4}+\pi)}\} = \{e^{i\frac{\pi}{4}}, e^{i\frac{5\pi}{4}}\}$.

Then we know $Arg\, e^{i\frac{\pi}{4}} = \frac{\pi}{4}$ and hence $arg\, e^{i\frac{\pi}{4}} = \{\theta = \frac{\pi}{4} + 2n\pi : n \in \mathbb{Z}\}$. Also $|e^{i\frac{\pi}{4}}| = 1$.

So $log(e^{i\frac{\pi}{4}}) = ln|e^{i\frac{\pi}{4}}| + i\, arg\, e^{i\frac{\pi}{4}} = ln\, 1 + i\, arg\, e^{i\frac{\pi}{4}} = i\, arg\, e^{i\frac{\pi}{4}} = \{\theta = \frac{\pi}{4} + 2n\pi : n \in \mathbb{Z}\} = \{\theta = \pi(\frac{1}{4} + 2n) : n \in \mathbb{Z}\}$.

Similarly we know $\pi(\frac{1}{4} + 1) = \frac{5\pi}{4} \in arg\, e^{i\frac{5\pi}{4}}$ and hence $arg\, e^{i\frac{5\pi}{4}} = \{\theta = \frac{\pi}{4} + \pi + 2n\pi : n \in \mathbb{Z}\}$. Also $|e^{i\frac{5\pi}{4}}| = 1$.

So $log(e^{i\frac{5\pi}{4}}) = ln|e^{i\frac{5\pi}{4}}| + i\, arg\, e^{i\frac{5\pi}{4}} = ln\, 1 + i\, arg\, e^{i\frac{5\pi}{4}} = i\, arg\, e^{i\frac{5\pi}{4}} = \{\theta = \pi(\frac{1}{4} + (2n+1)) : n \in \mathbb{Z}\}$.

Clearly $\{2n : n \in \mathbb{Z}\} = \{0, \pm 2, \pm 4, \pm 6, ...\}$ is the set of even integers.

Similarly $\{2n+1 : n \in \mathbb{Z}\} = \{\pm 1, \pm 3, \pm 5, ...\}$ is the set of odd integers.

Therefore $\{2n : n \in \mathbb{Z}\} \cup \{2n+1 : n \in \mathbb{Z}\} = \{n : n \in \mathbb{Z}\} = \mathbb{Z}$.

This results in the fact that $\{\frac{1}{4} + 2n : n \in \mathbb{Z}\} \cup \{\frac{1}{4} + 2n + 1 : n \in \mathbb{Z}\} = \{\frac{1}{4} + n : n \in \mathbb{Z}\}$.

Consequently $\{\pi(\frac{1}{4} + 2n) : n \in \mathbb{Z}\} \cup \{\pi(\frac{1}{4} + (2n+1)) : n \in \mathbb{Z}\} = \{\pi(\frac{1}{4} + n) : n \in \mathbb{Z}\}$.

Now since the square roots of i are given by $i^{\frac{1}{2}} = \{e^{i\frac{\pi}{4}}, e^{i\frac{5\pi}{4}}\}$ and we know:

$log(e^{i\frac{\pi}{4}}) = \{\theta = \pi(\frac{1}{4} + 2n) : n \in \mathbb{Z}\}$ and $log(e^{i\frac{5\pi}{4}}) = \{\theta = \pi(\frac{1}{4} + (2n+1)) : n \in \mathbb{Z}\}$.

We have that $log(i^{\frac{1}{2}})$ must be the union of both of the above sets because those are the log sets for each of the only two square roots of i. Clearly each of those above sets is a subset of $log(i^{\frac{1}{2}})$ and since they are the log sets for each of the only square roots of i we must have that if $z \in log(i^{\frac{1}{2}})$ then z is in one of those two sets. Hence the union of the two sets above is a subset of $log(i^{\frac{1}{2}})$ and $log(i^{\frac{1}{2}})$ is a subset of the union of the two sets above (meaning they are equal).

Therefore $log(i^{\frac{1}{2}}) = \{\theta = \pi(\frac{1}{4} + 2n) : n \in \mathbb{Z}\} \cup \{\theta = \pi(\frac{1}{4} + (2n+1)) : n \in \mathbb{Z}\} = \{\theta = \pi(\frac{1}{4} + n) : n \in \mathbb{Z}\}$ $\square$

34.2

b. Recall that $log(z) = ln|z| + i\,arg\,z$. Now for $z \neq 0$ write $z = re^{i\theta}$ where $r = |z|$ and $\theta \in arg\,z$ is arbitrary.

Then we have $\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$. So we know that $-\theta \in arg\frac{1}{z}$.

Therefore after fixing some $\theta \in arg\,z$ we can write:

$$arg\frac{1}{z} = \{-\theta + 2n\pi : n \in \mathbb{Z}\} = \{-\theta - 2n\pi : n \in \mathbb{Z}\} = \{-(\theta + 2n\pi) : n \in \mathbb{Z}\}.$$

Since $arg\,z = \{\theta + 2n\pi : n \in \mathbb{Z}\}$ we get that $arg\frac{1}{z} = \{-(\theta + 2n\pi) : n \in \mathbb{Z}\} = \{-\phi : \phi \in arg\,z\} = -arg\,z$.

We also know that $|\frac{1}{z}| = \frac{1}{|z|} = \frac{1}{r}$ where $r = |z| > 0$.

Therefore $log\frac{1}{z} = ln|\frac{1}{z}| + i\,arg\frac{1}{z} = ln\frac{1}{|z|} - i\,arg\,z = ln(|z|^{-1}) - i\,arg\,z = -ln|z| - i\,arg\,z = -(ln|z| + i\,arg\,z) = -log\,z$.

Now recall the fact that for $z_1, z_2 \neq 0$ we know $log(z_1 z_2) = log\,z_1 + log\,z_2$.

Finally for $z_1, z_2 \neq 0$ we have that $log(\frac{z_1}{z_2}) = log(z_1\frac{1}{z_2}) = log\,z_1 + log\frac{1}{z_2} = log\,z_1 - log\,z_2$ $\square$

36.1

a. We already know for $z, c \in \mathbb{C}$ we can write $z^c = e^{c\,logz}$.

Recall that $log(z) = ln|z| + i\,arg\,z$.

Then we have $(1+i)^i = e^{i\,log(1+i)}$.

We may write $log(1+i) = ln|1+i| + i\,arg(1+i) = ln|\sqrt{2}| + i\{\frac{\pi}{4} - 2n\pi : n \in \mathbb{Z}\}$.

Therefore $i\,log(1+i) = i\,ln|\sqrt{2}| + i^2\{\frac{\pi}{4} - 2n\pi : n \in \mathbb{Z}\} = i\,ln\sqrt{2} + \{-(\frac{\pi}{4} - 2n\pi) : n \in \mathbb{Z}\} = i\frac{ln\,2}{2} + \{-\frac{\pi}{4} + 2n\pi : n \in \mathbb{Z}\}$.

So for $n \in \mathbb{Z}$ we have $e^{i\,log(1+i)} = e^{i\frac{ln\,2}{2} - \frac{\pi}{4} + 2n\pi} = e^{i\frac{ln\,2}{2}}e^{-\frac{\pi}{4} + 2n\pi}$.

This was true for arbitrary $n \in \mathbb{Z}$ and is therefore true for all $n \in \mathbb{Z}$.

So for $n \in \mathbb{Z}$ we are left with:

$$(1+i)^i = e^{i\frac{ln\,2}{2}}e^{-\frac{\pi}{4} + 2n\pi} \ \square$$

36.6

We already know that $|e^{i\phi}| = 1$ for all $\phi \in \mathbb{R}$, furthermore we know for $z, c \in \mathbb{C}$ we can write $z^c = e^{c\,logz}$.

So for $a \in \mathbb{R} \subseteq \mathbb{C}$ we have $z^a = e^{a\,log\,z}$ where $log\,z = ln|z| + i\,arg\,z$.

Then we know $a\,log\,z = a\,ln|z| + ia\,arg\,z$.

The set $arg\,z = \{Arg\,z + 2n\pi : n \in \mathbb{Z}\}$, so $ia\,arg\,z = \{i(a\,Arg\,z + 2na\pi) : n \in \mathbb{Z}\}$.

Then $e^{ia\,arg\,z} = \{e^{i(a\,Arg\,z + 2na\pi)} : n \in \mathbb{Z}\}$ and so $|e^{ia\,arg\,z}| = \{|e^{i(a\,Arg\,z + 2na\pi)}| : n \in \mathbb{Z}\} = \{1\}$

Therefore $|z^a| = |e^{a\,log\,z}| = |e^{a\,ln|z| + ia\,arg\,z}| = |e^{a\,ln|z|}||e^{ia\,arg\,z}| = e^{a\,ln|z|} = |z|^a$.

Where $|z|^a$ is the principle argument of $\{|z|^a e^{2in\pi} : n \in \mathbb{Z}\}$.

This was true for arbitrary $z \in \mathbb{C}$ and $a \in \mathbb{R}$ and is therefore true for all $z \in \mathbb{C}$ and $a \in \mathbb{R}$.

So for $z \in \mathbb{C}$ and $a \in \mathbb{R}$ we have shown that $|z^a| = |z|^a$ $\square$

38.9

b. Recall that for $z = x + iy$ we know $|sin\ z|^2 = sin^2 x + sinh^2 y$ and $|cos\ z|^2 = cos^2 x + sinh^2 y$.

Further recall that $cos(z_1 + z_2) = cos\ z_1\ cos\ z_2 - sin\ z_1\ sin\ z_2$, $cos\ iz = cosh\ z$, and $sin\ iz = isinh\ z$.

Note that for $t \in \mathbb{R}$ we have $sinh\ t = \frac{e^t - e^{-t}}{2} \leq \frac{e^t + e^{-t}}{2} = cosh\ t$ since $e^{-t} > 0$.

So $cos(x + iy) = cos\ x\ cos\ iy - sin\ x\ sin\ iy = cos\ x\ cosh\ y - i\ sin\ x\ sinh\ y$.

Therefore $|cos(x + iy)|^2 = |cos\ x\ cosh\ y - i\ sin\ x\ sinh\ y| = cos^2 x\ cosh^2 y + sin^2 x\ sinh^2 y \leq cos^2 x cosh^2 y + sin^2 x cosh^2 y =$

$cosh^2 y(cos^2 x + sin^2 x) = cosh^2 y$.

So we know that $|cos\ z|^2 \leq cosh^2 y$ and hence $|cos\ z| \leq |cosh\ y| = cosh\ y$ (since $cosh\ y > 0$).

Furthermore we know $sinh^2 y = |cos\ z|^2 - cos^2 x$ and since $cos^2 x \geq 0$ we know $sinh^2 y \leq |cos\ z|^2$.

Consequently we know that $|sinh\ y| \leq |cos\ z|$.

So we have shown that $|sinh\ y| \leq |cos\ z| \leq cosh\ y$ □

38.14

Recall that if $z = x + iy$ we know $cos\ z = cos\ x\ cosh\ y - i\ sin\ x\ sinh\ y$ and $sin\ z = sin\ x\ cosh\ y + i\ cos\ x\ sinh\ y$.

Further recall that for $t \in \mathbb{R}$ (and also $t \in \mathbb{C}$) $cos(-t) = cos\ t$ and $sin(-t) = -sin\ t$.

a.

So $cos\ iz = cos(ix - y) = cos(-y)\ cosh\ x - i\ sin(-y)\ sinh\ x = cos\ y\ cosh\ x + i\ sin\ y\ sinh\ x$.

Therefore:

$$\overline{cos\ iz} = \overline{cos\ y\ cosh\ x + i\ sin\ y\ sinh\ x} = cos\ y\ cosh\ x - i\ sin\ y\ sinh\ x =$$

$$cos(ix + y) = cos(i(x - iy)) = cos(i(\overline{x + iy})) = cos(i\overline{z})$$

This was true for arbitrary $z \in \mathbb{C}$ and is therefore true for all $z \in \mathbb{C}$.

So we have shown that $\overline{cos\ iz} = cos(i\overline{z})$ for all $z \in \mathbb{C}$ □

b.

So $sin\ iz = sin(ix - y) = sin(-y)\ cosh\ x + i\ cos(-y)\ sinh\ x = -sin\ y\ cosh\ x + i\ cos\ y\ sinh\ x$.

Therefore:

$$\overline{sin\ iz} = \overline{-sin\ y\ cosh\ x + i\ cos\ y\ sinh\ x} = -sin\ y\ cosh\ x - i\ cos\ y\ sinh\ x =$$

$$-(sin\ y\ cosh\ x + i\ cos\ y\ sinh\ x) = -sin(ix + y) = -sin(i(x - iy)) = -sin(i(\overline{x + iy})) = -sin(i\overline{z})$$

So if we want $\overline{sin\ iz} = sin(i\overline{z})$ then we get $-sin(i\overline{z}) = sin(i\overline{z})$.

Therefore $sin(i\overline{z}) = 0$ and since $sin(z_0) = 0$ if and only if $z_0 = n\pi$ for some $n \in \mathbb{Z}$, we get $sin(i\overline{z}) = 0$ if and only if $iz = n\pi$ and hence $z = -n\pi i$ for some $n \in \mathbb{Z}$ (which is equivalent to $z = n\pi i$ for some $n \in \mathbb{Z}$).

So we have shown $\overline{sin\ iz} = sin(i\overline{z})$ if and only if $z = n\pi i$ for some $n \in \mathbb{Z}$ □

39.6

b. Recall from a previous problem that for $z \in \mathbb{C}$ we know $|sinh(Im\, z)| \leq |cos\, z| \leq cosh(Im\, z)$.

Let $z \in \mathbb{C}$ be arbitrary with representation $z = x + iy$.

Let $w = iz = i(x + iy) = ix - y$ we may apply the above inequality to w.

We get $|sinh\, x| \leq |cos\, w| \leq cosh\, x$ where $w = iz$.

Then since $cos\, iz = cosh\, z$ we have $|sinh\, x| \leq |cosh\, z| \leq cosh\, x$.

This was true for arbitrary $z \in \mathbb{C}$ and is therefore true for all $z \in \mathbb{C}$.

So we have shown that $|sinh\, x| \leq |cosh\, z| \leq cosh\, x$ □

Problem 2

Recall that for $z \in \mathbb{C}$ we know $log\, z = ln|z| + i\, arg\, z$.

Further recall that for $n \in \mathbb{N}$ we know $z^{\frac{1}{n}} = \sqrt[n]{|z|}e^{i\frac{arg\, z}{n}}$.

When I say $\frac{arg\, z}{n}$ I mean $\frac{arg\, z}{n} = \{\frac{\theta}{n} : \theta \in arg\, z\}$.

Similarly, when I say $e^{i\frac{arg\, z}{n}}$ I mean $e^{i\frac{arg\, z}{n}} = \{e^{i\frac{\theta}{n}} : \theta \in arg\, z\}$.

This expression gives all the possible representations for $z^{\frac{1}{n}}$.

Therefore we have $|z^{\frac{1}{n}}| = \sqrt[n]{|z|} = |z|^{\frac{1}{n}}$ and $arg\, z^{\frac{1}{n}} = \frac{arg\, z}{n}$.

So we get:

$$log(z^{\frac{1}{n}}) = ln|z^{\frac{1}{n}}| + i\, arg\, z^{\frac{1}{n}} = ln(|z|^{\frac{1}{n}}) + i\frac{arg\, z}{n} = \frac{1}{n}ln|z| + i\frac{arg\, z}{n} = \frac{1}{n}(ln|z| + i\, arg\, z) = \frac{1}{n}log\, z.$$

This was true for arbitrary $n \in \mathbb{N}$ and is therefore true for all $n \in \mathbb{N}$.

This was also true for arbitrary $z \in \mathbb{C}$ and is therefore true for all $z \in \mathbb{C}$.

So for all $z \in \mathbb{C}$ we have that $log\, z^{\frac{1}{n}} = \frac{1}{n}log\, z$ for all $n \in \mathbb{N}$ □