Irrationality and Sets

Matthew Seguin

1.2.1

a. For the sake of contradiction assume $\sqrt{3}$ is not irrational, that is assume it can be represented as a fraction of integers.

Then $\sqrt{3} = \frac{p}{q}$ for some p, q $\in \mathbb{Z}$

We can also assume that p and q are relatively prime because if they aren't we simply reduce the fraction until the numerator and denominator are relatively prime. Therefore so far we have that the only common divisor of p and q is 1.

Squaring both sides gives $3 = \frac{p^2}{q^2}$

Multiplying both sides by q^2 gives $p^2 = 3q^2$

This means that 3 divides p^2, and since 3 is not a perfect square this also means that 3 divides p.

So we can write p as $p = 3k$ for some k $\in \mathbb{Z}$

Substituting into the previous equation gives $p^2 = (3k)^2 = 9k^2 = 3q^2$

Dividing both sides by 3 gives $3k^2 = q^2$

This means that 3 divides q^2, and since 3 is not a perfect square this also means that 3 divides q.

So we have reached a contradiction because we have shown that 3 divides both p and q, hence p and q have the common divisor 3. However, our assumption was that p and q had no common divisor besides 1 since they are relatively prime. Thus our assumption that $\sqrt{3}$ can be represented as a fraction of integers must be false. So $\sqrt{3}$ is irrational. $\square$

A similar argument works for showing that $\sqrt{6}$ is irrational.
You get $6q^2 = p^2$ and hence p^2 is divisible by 6 which is not a perfect square so p is also divisible by 6. The rest follows similarly.

b. You can not use the same process in an attempt to say that $\sqrt{4}$ is irrational.

If you follow the same process you get $\sqrt{4} = \frac{p}{q}$

Squaring both sides gives $4 = \frac{p^2}{q^2}$

Multiplying both sides by q^2 gives $p^2 = 4q^2$

This means 4 divides p^2, but that does not imply that 4 divides p. Instead it only implies that 2 divides p.

So we can write p as $p = 2k$ for some k $\in \mathbb{Z}$

Substituting into the previous equation gives $p^2 = (2k)^2 = 4k^2 = 4q^2$

Dividing both sides by 4 gives $k^2 = q^2$

There is no contradiction here since k and q can both be 1 and hence p = 2 so $p^2 = 4$

So the same process clearly does not work for attempting to say that $\sqrt{4}$ is irrational.

1.2.5

Let U be an arbitrary universal set.
Recall that for any set S, $S^c = \{x \in U : x \notin S\}$
For any two sets X and Y, $X \cup Y = \{x \in U : x \in X \; or \; x \in Y\}$
And for any two sets X and Y, $X \cap Y = \{x \in X : x \in Y\} = \{y \in Y : y \in X\}$

a. Let A and B be subsets of $\mathbb{R}$.

If $x \in (A \cap B)^c$ then by definition $x \notin A \cap B$.

So if $x \notin A \cap B$ then by definition $x \notin A$ or $x \notin B$.

And since $x \notin A$ or $x \notin B$ by definition $x \in A^c$ or $x \in B^c$.

So by definition $x \in A^c \cup B^c$.

Therefore if an element is in $(A \cap B)^c$ then it is also in $A^c \cup B^c$, so $(A \cap B)^c \subseteq A^c \cup B^c$.

b. Let A and B be subsets of $\mathbb{R}$.

If $x \in A^c \cup B^c$ then by definition $x \in A^c$ or $x \in B^c$.

So if $x \in A^c$ or $x \in B^c$ then by definition $x \notin A$ or $x \notin B$.

And since $x \notin A$ or $x \notin B$ by definition $x \notin A \cap B$.

So by definition $x \in (A \cap B)^c$.

Therefore if an element is in $A^c \cup B^c$ then it is also in $(A \cap B)^c$, so $A^c \cup B^c \subseteq (A \cap B)^c$. $\square$

Since $(A \cap B)^c \subseteq A^c \cup B^c$ and $A^c \cup B^c \subseteq (A \cap B)^c$ every element in $(A \cap B)^c$ is also in $A^c \cup B^c$ and vice versa.

Therefore $(A \cap B)^c = A^c \cup B^c$

c. Let A and B be subsets of $\mathbb{R}$.

- If $x \in (A \cup B)^c$ then by definition $x \notin A \cup B$.

  So if $x \notin A \cup B$ then by definition $x \notin A$ and $x \notin B$.

  And since $x \notin A$ and $x \notin B$ by definition $x \in A^c$ and $x \in B^c$.

  So by definition $x \in A^c \cap B^c$.

  Therefore if an element is in $(A \cup B)^c$ then it is also in $A^c \cap B^c$, so $(A \cup B)^c \subseteq A^c \cap B^c$.

- If $x \in A^c \cap B^c$ then by definition $x \in A^c$ and $x \in B^c$.

  So if $x \in A^c$ and $x \in B^c$ then by definition $x \notin A$ and $x \notin B$.

  And since $x \notin A$ and $x \notin B$ by definition $x \notin A \cup B$.

  So by definition $x \in (A \cup B)^c$.

  Therefore if an element is in $A^c \cap B^c$ then it is also in $(A \cup B)^c$, so $A^c \cap B^c \subseteq (A \cup B)^c$.

Since $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$ every element in $(A \cup B)^c$ is also in $A^c \cap B^c$ and vice versa.

Therefore $(A \cup B)^c = A^c \cap B^c$ $\square$

1.2.7

Let f be a function $f : S \to T$, and $A \subseteq S$.
Define $f(A) = \{f(x) \in T : x \in A\}$
Recall that a real interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$

a. Let $f(x) = x^2$, A = [0, 2], and B = [1, 4].

Since $f(x)$ is strictly increasing on $[0, \infty)$, the endpoints of the intervals A and B will provide the minimum and maximum values of $f(A)$ and $f(B)$.

- Therefore $f(A) = [0, 4]$ and $f(B) = [1, 16]$.

- Furthermore $A \cap B = [1, 2]$, so $f(A \cap B) = [1, 4]$.
  And $f(A) \cap f(B) = [1, 4]$, so $f(A \cap B) = f(A) \cap f(B)$ here.

- Furthermore $A \cup B = [0, 4]$, so $f(A \cup B) = [0, 16]$.
  And $f(A) \cup f(B) = [0, 16]$, so $f(A \cup B) = f(A) \cup f(B)$ here.

b. Let $f(x)$ be as before, $A = [-1, 0]$, and $B = [0, 1]$.

As before $f(x)$ is strictly increasing on $[0, \infty)$, but since I am introducing a negative domain I will point out that $f(x)$ is strictly decreasing on $(-\infty, 0)$. So as before the endpoints of the intervals A and B will provide the minimum and maximum values of $f(A)$ and $f(B)$.

So $f(A) = [0, 1] = f(B)$ and therefore $f(A) \cap f(B) = [0, 1]$.

Furthermore $A \cap B = \{0\}$ so $f(A \cap B) = \{f(0)\} = \{0\}$.

Clearly $1 \in f(A) \cap f(B)$ but $1 \notin f(A \cap B)$, so $f(A \cap B) \neq f(A) \cap f(B)$ here.

c. Let g be an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ and A and B be arbitrary subsets of $\mathbb{R}$.

Let $x \in g(A \cap B)$. Then $g(y) = x$ for some $y \in A \cap B$.

For this y since $y \in A \cap B$, $y \in A$ and $y \in B$.

So $g(y) = x \in g(A)$ and $g(y) = x \in g(B)$, therefore $g(y) = x \in g(A) \cap g(B)$.

Therefore if $x \in g(A \cap B)$ then $x \in g(A) \cap g(B)$, so $g(A \cap B) \subseteq g(A) \cap g(B)$ for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ $\square$

d. Let g be an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ and A and B be arbitrary subsets of $\mathbb{R}$.

I claim that $g(A \cup B) = g(A) \cup g(B)$ for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ and arbitrary subsets A and B of $\mathbb{R}$.

- Let $x \in g(A \cup B)$. Then $g(y) = x$ for some $y \in A \cup B$.

  For this y since $y \in A \cup B$, $y \in A$ or $y \in B$.

  So $g(y) = x \in g(A)$ or $g(y) = x \in g(B)$, therefore $g(y) = x \in g(A) \cup g(B)$.

  Therefore if $x \in g(A \cup B)$ then $x \in g(A) \cup g(B)$, so $g(A \cup B) \subseteq g(A) \cup g(B)$.

- Let $x \in g(A) \cup g(B)$. Then $x \in g(A)$ or $x \in g(B)$.

  So $g(y) = x$ for some $y \in A$ or $g(z) = x$ for some $z \in B$.

  So $g(w) = x$ for some $w \in A \cup B$, therefore $x \in g(A \cup B)$.

  Therefore if $x \in g(A) \cup g(B)$ then $x \in g(A \cup B)$, so $g(A) \cup g(B) \subseteq g(A \cup B)$.

Therefore since $g(A \cup B) \subseteq g(A) \cup g(B)$ and $g(A) \cup g(B) \subseteq g(A \cup B)$, $g(A \cup B) = g(A) \cup g(B)$ for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$ $\square$

1.2.12

Recall that a relation $>$ on a set S is transitive if given $a, b, c \in S$ where $a > b$ and $b > c$ then $a > c$.
Let $y_1 = 6$ and for each $n \in \mathbb{N}$ define $y_{n+1} = \frac{(2y_n - 6)}{3}$.

a. Let $S = \{n \in \mathbb{N} : y_n > -6\}$.

- Base Case:

  We know $y_1 = 6 > -6$, so $1 \in S$.

- Inductive Step:

  Assume that $n \in S$, then $y_n > -6$

  Multiplying both sides by 2 we get $2y_n > -12$

  Subtracting 6 from both sides we get $2y_n - 6 > -18$

  Dividing both sides by 3 we get $\frac{2y_n - 6}{3} > -6$

  Since $y_{n+1} = \frac{(2y_n - 6)}{3}$, we have that $y_{n+1} > -6$ and therefore $n + 1 \in S$.

  So for any $n \in \mathbb{N}$ if $n \in S$ then $n + 1 \in S$.

Therefore since $1 \in S$ and if $n \in S$ then $n + 1 \in S$, $S = \mathbb{N}$. So $y_n > -6$ for all $n \in \mathbb{N}$ $\square$

b. Let $S = \{n \in \mathbb{N} : y_n > y_{n+1}\}$.

- Base Case:

  We know $y_1 = 6$, so $y_2 = \frac{(2y_1 - 6)}{3} = 2$

  Therefore since $y_1 = 6 > 2 = y_2$, $1 \in S$.

- Inductive Step:

  Assume that $n \in S$, then $y_n > y_{n+1}$

  Multiplying both sides by 2 we get $2y_n > 2y_{n+1}$

  Subtracting 6 from both sides we get $2y_n - 6 > 2y_{n+1} - 6$

  Dividing both sides by 3 we get $\frac{2y_n - 6}{3} > \frac{2y_{n+1} - 6}{3}$

  Since $y_{n+1} = \frac{(2y_n - 6)}{3}$ and $y_{n+2} = \frac{(2y_{n+1} - 6)}{3}$, we have that $y_{n+1} > y_{n+2}$ and therefore $n + 1 \in S$.

  So for any $n \in \mathbb{N}$ if $n \in S$ then $n + 1 \in S$.

Therefore since $1 \in S$ and if $n \in S$ then $n + 1 \in S$, $S = \mathbb{N}$. So $y_n > y_{n+1}$ for all $n \in \mathbb{N}$.

Since $>$ is a transitive relation on $\mathbb{R}$ we have that if $n, m \in \mathbb{N}$ and $n > m$ then $y_n < y_m$.

So $y_1 > y_2 > y_3 > ...$, and therefore the sequence $(y_1, y_2, y_3, ...)$ is decreasing $\square$

1.2.13

Recall from problem 1.2.5 that $(A \cup B)^c = A^c \cap B^c$.

a. Let $S = \{n \in \mathbb{N} : (A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c \ for\ arbitrary\ sets\ A_1, A_2, ...A_n\}$

- Base Case:

  We know for a set A_1 that $(A_1)^c = A_1^c$, so $1 \in \mathbb{N}$.

- Inductive Step:

  Assume that $n \in S$, then $(A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$.

  Consider another arbitrary set A_{n+1}. Let $B = A_1 \cup A_2 \cup ... \cup A_n$.

  Then $(B \cup A_{n+1})^c = B^c \cap A_{n+1}^c$ by the result of problem 1.2.5.

  Since $B = A_1 \cup A_2 \cup ... \cup A_n$ and $B^c = (A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$, we have

  that $(A_1 \cup A_2 \cup ... \cup A_n \cup A_{n+1})^c = (B \cup A_{n+1})^c = B^c \cap A_{n+1}^c = A_1^c \cap A_2^c \cap ... \cap A_n^c \cap A_{n+1}^c$. Therefore $n+1 \in S$.

  So for any $n \in \mathbb{N}$ if $n \in S$ then $n+1 \in S$.

Therefore since $1 \in S$ and if $n \in S$ then $n+1 \in S$, $S = \mathbb{N}$. So $(A_1 \cup A_2 \cup ... \cup A_n)^c = A_1^c \cap A_2^c \cap ... \cap A_n^c$ for all $n \in \mathbb{N}$ $\square$

b. Let $B_1 = \{\frac{1}{x} \in \mathbb{Q} : x \in \mathbb{N},\ x \geq 1\}$, $B_2 = \{\frac{1}{x} \in \mathbb{Q} : x \in \mathbb{N},\ x \geq 2\}$, ... , $B_n = \{\frac{1}{x} \in \mathbb{Q} : x \in \mathbb{N},\ x \geq n\}$, ...

Let $S = \{n \in \mathbb{N} : B_1 \cap B_2 \cap ... \cap B_n \neq \phi\}$

- Finite Case:

  - Base Case:

    $B_1 = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$ so $1 \in B_1$ but $1 \notin \phi$, so $B_1 \neq \phi$. Therefore $1 \in S$.

  - Inductive Step:

    Note that $B_1 \supset B_2 \supset B_3 \supset ...$, so $B_1 \cap B_2 \cap B_3 \cap ... \cap B_n = B_n$

    Assume that $n \in S$, then $B_1 \cap B_2 \cap B_3 \cap ... \cap B_n \neq \phi$.

    Then consider B_{n+1}, since $B_1 \cap B_2 \cap B_3 \cap ... \cap B_n \neq \phi$ we know $B_1 \cap B_2 \cap B_3 \cap ... \cap B_n = B_n$.

    And since $B_{n+1} \subset B_n$, $B_{n+1} \cap B_n = B_{n+1}$. So $\frac{1}{n+1} \in B_1 \cap B_2 \cap B_3 \cap ... \cap B_n \cap B_{n+1}$ while $\frac{1}{n+1} \notin \phi$.

    So $B_1 \cap B_2 \cap B_3 \cap ... \cap B_n \cap B_{n+1} \neq \phi$, and therefore $n+1 \in S$.

    So for any $n \in \mathbb{N}$ if $n \in S$ then $n+1 \in S$.

  Therefore since $1 \in S$ and if $n \in S$ then $n+1 \in S$, $S = \mathbb{N}$. So $B_1 \cap B_2 \cap B_3 \cap ... \cap B_n \neq \phi$ for all $n \in \mathbb{N}$. $\square$

- Infinite Case:

  Let $A = \cap_{j=1}^{\infty} B_j = B_1 \cap B_2 \cap B_3 \cap ...$, and let $n \in \mathbb{N}$.

  Say for the sake of contradiction that $x = \frac{1}{n} \in A$, then $x \in B_1 \cap B_2 \cap B_3 \cap ...$ so $x \in B_1$ *and* $x \in B_2$ *and* $x \in B_3$ *and* ...

  But consider $B_{n+1} = \{\frac{1}{n+1}, \frac{1}{n+2}, \frac{1}{n+3}, ...\}$, clearly $x = \frac{1}{n} \notin B_{n+1}$. So we have a contradiction.

  Therefore our assumption that $x = \frac{1}{n} \in A$ is false, and there are no elements in A since any element in A would be of the form $\frac{1}{n}$ for some $n \in \mathbb{N}$. So $A = \cap_{j=1}^{\infty} B_j = \phi$ $\square$

c. Let $A_1, A_2, A_3, ...$ be arbitrary sets.

$$\text{Consider } (\cup_{j=1}^{\infty} A_j)^c \text{ and } \cap_{j=1}^{\infty} A_j^c$$

- If $x \in (\cup_{j=1}^{\infty} A_j)^c$, then $x \notin \cup_{j=1}^{\infty} A_j$, so $x \notin A_1$ *and* $x \notin A_2$ *and* $x \notin A_3$ *and* ...

  Therefore $x \in A_1^c$ *and* $x \in A_2^c$ *and* $x \in A_3^c$ *and* ..., so $x \in A_1^c \cap A_2^c \cap A_3^c \cap ... = \cap_{j=1}^{\infty} A_j^c$

  So $(\cup_{j=1}^{\infty} A_j)^c \subseteq \cap_{j=1}^{\infty} A_j^c$

- If $x \in \cap_{j=1}^{\infty} A_j^c$, then $x \in A_1^c$ *and* $x \in A_2^c$ *and* $x \in A_3^c$ *and* ..., so $x \notin A_1$ *and* $x \notin A_2$ *and* $x \notin A_3$ *and* ...

  Therefore $x \notin A_1 \cup A_2 \cup A_3 \cup ...$, so $x \in (A_1 \cup A_2 \cup A_3 \cup ...)^c = (\cup_{j=1}^{\infty} A_j)^c$

  So $\cap_{j=1}^{\infty} A_j^c \subseteq (\cup_{j=1}^{\infty} A_j)^c$

Therefore every element in $(\cup_{j=1}^{\infty} A_j)^c$ is in $\cap_{j=1}^{\infty} A_j^c$ and vice versa.

So $(\cup_{j=1}^{\infty} A_j)^c = \cap_{j=1}^{\infty} A_j^c$ $\square$