Complex Integration

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42.2

c.

I will do this two ways, first recall that $\frac{d}{dt}e^{zt} = ze^{zt}$.

Therefore $\frac{d}{dt}e^{2it} = 2ie^{2it}$ and so $\frac{d}{dt}\frac{1}{2i}e^{2it} = e^{2it}$.

Further recall that if $\frac{d}{dt}W(t) = w(t)$ then:

$$\int_a^b w(t)dt = W(t)\Big|_a^b = W(b) - W(a)$$

Then we have:

$$\int_0^{\frac{\pi}{6}} e^{2it}dt = \frac{1}{2i}e^{2it}\Big|_0^{\frac{\pi}{6}} = -\frac{i}{2}(e^{i\frac{\pi}{3}} - e^0) = \frac{i}{2}(1 - (cos(\frac{\pi}{3}) + i\,sin(\frac{\pi}{3}))) = \frac{i}{2}(1 - \frac{1}{2} - i\frac{\sqrt{3}}{2}) = \frac{\sqrt{3}}{4} + i\frac{1}{4}$$

Now for the second way:

Recall that for $\theta \in \mathbb{R}$ we know $e^{i\theta} = cos(\theta) + i\,sin(\theta)$.

Further recall that if $w(t) = u(t) + iv(t)$ then:

$$\int_a^b w(t)dt = \int_a^b u(t) + iv(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

So if $f(t) = e^{2it}$ for $t \in \mathbb{R}$ then we can write $f(t) = cos(2t) + i\,sin(2t)$.

Now we are evaluating:

$$\int_0^{\frac{\pi}{6}} e^{2it}dt = \int_0^{\frac{\pi}{6}} cos(2t) + i\,sin(2t)dt = \int_0^{\frac{\pi}{6}} cos(2t)dt + i\int_0^{\frac{\pi}{6}} sin(2t)dt = \frac{1}{2}sin(2t)\Big|_0^{\frac{\pi}{6}} + i(-\frac{1}{2}cos(2t))\Big|_0^{\frac{\pi}{6}} =$$

$$\frac{1}{2}(sin(\frac{\pi}{3}) - sin(0) - i\,cos(\frac{\pi}{3}) + i\,cos(0)) = \frac{1}{2}(\frac{\sqrt{3}}{2} + i(1 - \frac{1}{2})) = \frac{\sqrt{3}}{4} + i\frac{1}{4}$$

Consistent with out answer from before.

Therefore we have:

$$\int_0^{\frac{\pi}{6}} e^{2it}dt = \frac{\sqrt{3}}{4} + i\frac{1}{4}$$

$\square$

42.3

Let $m, n \in \mathbb{Z}$ be arbitrary. Then notice that we can write:

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta = \int_0^{2\pi} e^{i(m-n)\theta}d\theta$$

Recall that if $\frac{d}{dt}W(t) = w(t)$ then:

$$\int_a^b w(t)dt = W(t)\Big|_a^b = W(b) - W(a)$$

- If $m = n$:

Then we get:

$$\int_0^{2\pi} e^{i(m-n)\theta}d\theta = \int_0^{2\pi} d\theta = \theta\Big|_0^{2\pi} = 2\pi$$

This is due to the fact that $\frac{d}{d\theta}\theta = 1$.

- If $m \neq n$:

Then we know that $\frac{d}{d\theta}e^{i(m-n)\theta} = i(m-n)e^{i(m-n)\theta}$ and since $m - n \neq 0$ we know $\frac{1}{m-n}$ is well defined and $\frac{d}{d\theta}\frac{1}{i(m-n)}e^{i(m-n)\theta} = e^{i(m-n)\theta}$.

So we get:

$$\int_0^{2\pi} e^{i(m-n)\theta}d\theta = \frac{1}{i(m-n)}e^{i(m-n)\theta}\Big|_0^{2\pi} = -\frac{i}{m-n}(e^{2(m-n)\pi i} - e^0)$$

Then since $m, n \in \mathbb{Z}$ we know $k = m - n \in \mathbb{Z}$. So $2k = 2(m-n)$ is an even integer.

Therefore $-\frac{i}{m-n}(e^{2k\pi i} - 1) = -\frac{i}{m-n}(cos(2k\pi) + i\ sin(2k\pi) - 1) = -\frac{i}{m-n}(1-1) = 0$.

Therefore we have that:

If $m = n$:

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta = 2\pi$$

And if $m \neq n$:

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta = 0$$

$\square$

42.5

Recall that for a function $w(t) = u(t) + iv(t)$ defined on $[a, b]$:

$$\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt \qquad (a \le c \le b)$$

Furthermore:

$$\int_a^b w(t)dt = -\int_b^a w(t)dt$$

Let $w(t) = u(t) + iv(t)$ be defined on $-a \le t \le a$.

a. Assume $w(t)$ is even on $[-a, a]$, that is $w(t) = w(-t)$ for all $t \in [-a, a]$.

Then we have that:

$$\int_{-a}^a w(t)dt = \int_{-a}^0 w(t)dt + \int_0^a w(t)dt$$

After using the substitution $\tau = -t$ (where $d\tau = -dt$) in the first integral we have:

$$\int_{-a}^a w(t)dt = \int_{-a}^0 w(t)dt + \int_0^a w(t)dt = -\int_a^0 w(\tau)d\tau + \int_0^a w(t)dt = \int_0^a w(\tau)d\tau + \int_0^a w(t)dt = 2\int_0^a w(t)dt$$

So if $w(t)$ is even on $[-a, a]$ then:

$$\int_{-a}^a w(t)dt = 2\int_0^a w(t)dt$$

$\square$

b. Assume $w(t)$ is odd on $[-a, a]$, that is $w(-t) = -w(t)$ for all $t \in [-a, a]$.

Then we have that:

$$\int_{-a}^a w(t)dt = \int_{-a}^0 w(t)dt + \int_0^a w(t)dt = -\int_0^{-a} w(t)dt + \int_0^a w(t)dt = \int_0^{-a} w(-t)dt + \int_0^a w(t)dt$$

After using the substitution $\tau = -t$ (where $d\tau = -dt$) in the first integral we have:

$$\int_{-a}^a w(t)dt = \int_0^{-a} w(-t)dt + \int_0^a w(t)dt = -\int_0^a w(\tau)d\tau + \int_0^a w(t)dt = \int_0^a w(t)dt - \int_0^a w(t)dt = 0$$

So if $w(t)$ is odd on $[-a, a]$ then:

$$\int_{-a}^a w(t)dt = 0$$

$\square$

43.5

Recall that if a function $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$ then $f'(z) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.

Further recall that for a function $w(t) = x(t) + iy(t)$ we know $w'(t) = x'(t) + iy'(t)$.

Assume that $f(z) = u(x,y) + iv(x,y)$ is analytic at a point $z_0 = x_0 + iy_0 = x(t_0) + iy(t_0) = z(t_0)$ on a smooth arc $z(t) = x(t) + iy(t)$ where $a \leq t \leq b$.

Then we know at $z_0 = x_0 + iy_0$ the Cauchy Riemann equations are satisfied.

That is $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

Now define $w(t) = f(z(t)) = u(x(t), y(t)) + iv(x(t), y(t))$ for $a \leq t \leq b$.

Then we have that $\frac{d}{dt}w(t) = \frac{d}{dt}\Big(u(x(t), y(t)) + iv(x(t), y(t))\Big) = \frac{d}{dt}u(x(t), y(t)) + i\frac{d}{dt}v(x(t), y(t))$.

Then using the chain rule we have:

$$\frac{d}{dt}w(t) = \frac{du}{dx}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt} + i(\frac{dv}{dx}\frac{dx}{dt} + \frac{dv}{dy}\frac{dy}{dt}) = u_x x'(t) + u_y y'(t) + i(v_x x'(t) + v_y y'(t))$$

Then at t_0 (and hence at $z(t_0) = z_0$) we can use the Cauchy Riemann equations to write:

$$w'(t_0) = u_x(x(t_0), y(t_0))\Big(x'(t_0) + iy'(t_0)\Big) + v_x(x(t_0), y(t_0))\Big(ix'(t_0) - y'(t_0)\Big) =$$

$$u_x(x(t_0), y(t_0))\Big(x'(t_0) + iy'(t_0)\Big) + iv_x(x(t_0), y(t_0))\Big(x'(t_0) + iy'(t_0)\Big) =$$

$$\Big(u_x(x(t_0), y(t_0)) + iv_x(x(t_0), y(t_0))\Big)\Big(x'(t_0) + iy'(t_0)\Big) = f'(z(t_0))z'(t_0)$$

This was true for arbitrary $t_0 \in [a,b]$ and is therefore true for all $t_0 \in [a,b]$.

Therefore if $f(z)$ is analytic at $z_0 = z(t_0)$ and $w(t) = f(z(t))$ then $w'(t_0) = f'(z(t_0))z'(t_0)$ $\square$

46.4

Let $f(z) = 1$ if $Im\, z < 0$ and $f(z) = 4Im\, z$ if $Im\, z > 0$.

Then let C be the contour from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$ $(-1 \leq x \leq 1)$.

We can write $C = C_1 + C_2$ where C_1 is the arc from $-1 - i$ to 0 and C_2 is the arc from 0 to $1 + i$ (both along $y = x^3$).

Clearly along C_1 we know $f(z) = 1$ since $x < 0$ and hence $Im\, z = y = x^3 < 0$.

Similarly along C_2 we know $f(z) = 4Im\, z$ since $x > 0$ and hence $Im\, z = y = x^3 > 0$.

Our path for each is therefore $z(t) = t + it^3$. This gives $z'(t) = 1 + 3it^2$.

For C_1 we know $-1 \leq t \leq 0$ and for C_2 we know $0 \leq t \leq 1$.

Therefore we have that:

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz = \int_{-1}^{0} (1)(1 + 3it^2)dt + \int_0^1 (4t^3)(1 + 3it^2)dt =$$

$$\left(\int_{-1}^{0} dt + i\int_{-1}^{0} 3t^2 dt\right) + \left(\int_0^1 4t^3 dt + i\int_0^1 12t^5 dt\right) = t\Big|_{-1}^{0} + i\left(t^3\Big|_{-1}^{0}\right) + t^4\Big|_0^1 + i\left(2t^6\Big|_0^1\right) =$$

$$1 + i + 1 + 2i = 2 + 3i$$

□

46.7

Let $f(z)$ be the principle branch of z^{-1-2i}, that is $f(z) = e^{(-1-2i)Log\, z}$ (where $Log\, z = ln|z| + i\, Arg\, z$).

Then let C be the contour $z(\theta) = e^{i\theta}$ where $0 \leq \theta \leq \frac{\pi}{2}$.

For any point z on C we know $|z| = |e^{i\theta}| = 1$ and $Arg\, z = \theta$ since $0 \leq \theta \leq \frac{\pi}{2}$ which is a subset of the range for the principle argument of a complex number.

This means that for any point z on C we know $f(z) = e^{(-1-2i)Log\, z} = e^{(-1-2i)(ln|z|+i\, Arg\, z)} = e^{(-1-2i)i\theta} = e^{2\theta - i\theta}$.

We have also seen before that $\frac{d}{d\theta}e^{i\theta} = ie^{i\theta}$, so $z'(\theta) = ie^{i\theta}$.

Therefore we have that:

$$\int_C f(z)dz = \int_0^{\frac{\pi}{2}} (e^{2\theta - i\theta})(ie^{i\theta})d\theta = i\int_0^{\frac{\pi}{2}} e^{2\theta - i\theta + i\theta}d\theta = i\int_0^{\frac{\pi}{2}} e^{2\theta}d\theta = i\left(\frac{1}{2}e^{2\theta}\Big|_0^{\frac{\pi}{2}}\right) = \frac{i}{2}(e^{\pi} - e^0) = \frac{i}{2}(e^{\pi} - 1)$$

□

46.10

Let C be the unit circle $|z| = 1$ taken counter clockwise, we can parameterize this with $z(\theta) = e^{i\theta}$ where $0 \leq \theta \leq 2\pi$.

Now let $m, n \in \mathbb{Z}$ be arbitrary. Recall from a previous problem that we know:

If $m = n$:

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = 2\pi$$

And if $m \neq n$:

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = 0$$

Now notice that if $z = re^{i\theta} = r(cos(\theta) + i\,sin(\theta))$ then $\overline{z} = r(cos(\theta) - i\,sin(\theta)) = r(cos(-\theta) + i\,sin(-\theta)) = re^{-i\theta}$.

For any point z on C we then have that $z^m = e^{im\theta}$ and $\overline{z}^n = (e^{-i\theta})^n = e^{-in\theta}$. We also know that $\frac{d}{d\theta} z(\theta) = ie^{i\theta}$.

Therefore we have:

$$\int_C z^m \overline{z}^n dz = i \int_0^{2\pi} e^{im\theta} e^{-in\theta} e^{i\theta} d\theta = i \int_0^{2\pi} e^{i(m+1)\theta} e^{-in\theta} d\theta$$

Then since $m \in \mathbb{Z}$ we know $m + 1 \in \mathbb{Z}$, say $m + 1 = k$.

By the results of the previously mentioned problem we have:

If $m + 1 = k = n$:

$$\int_C z^m \overline{z}^n dz = i \int_0^{2\pi} e^{ik\theta} e^{-in\theta} d\theta = i(2\pi) = 2\pi i$$

And if $m + 1 = k \neq n$:

$$\int_C z^m \overline{z}^n dz = i \int_0^{2\pi} e^{ik\theta} e^{-in\theta} d\theta = i(0) = 0$$

This was true for arbitrary $m, n \in \mathbb{Z}$ and is therefore true for all $m, n \in \mathbb{Z}$. Therefore we have that:

If $m + 1 = n$:

$$\int_C z^m \overline{z}^n dz = 2\pi i$$

And if $m + 1 \neq n$:

$$\int_C z^m \overline{z}^n dz = 0$$

$\square$

46.13

Let C_0 be the circle centered at z_0 with radius R parameterized by $z(\theta) = z_0 + Re^{i\theta}$ where $-\pi \leq \theta \leq \pi$.

Now let $n \in \mathbb{Z}$ be arbitrary.

Similar to what we have seen before we then know $\frac{d}{d\theta} z(\theta) = Rie^{i\theta}$.

Also for any point z on C we have that $(z - z_0)^{n-1} = (z_0 + Re^{i\theta} - z_0)^{n-1} = R^{n-1}e^{i(n-1)\theta}$.

Now we have that:

$$\int_C (z - z_0)^{n-1} dz = i \int_{-\pi}^{\pi} (R^{n-1}e^{i(n-1)\theta})(Re^{i\theta}) d\theta = i \int_{-\pi}^{\pi} R^n e^{in\theta} d\theta$$

Now if $n = 0$ then we have:

$$\int_C (z - z_0)^{n-1} dz = i \int_{-\pi}^{\pi} d\theta = i(\pi - (-\pi)) = 2\pi i$$

If $n \neq 0$ then we have:

$$\int_C (z - z_0)^{n-1} dz = iR^n \int_{-\pi}^{\pi} e^{in\theta} d\theta = iR^n \left(\frac{1}{in} e^{in\theta} \Big|_{-\pi}^{\pi} \right) = \frac{R^n}{n}(e^{in\pi} - e^{-in\pi}) =$$

$$\frac{R^n}{n}(cos(n\pi) + i\,sin(n\pi) - cos(-n\pi) - i\,sin(-n\pi)) = \frac{R^n}{n}(cos(n\pi) + i\,sin(n\pi) - cos(n\pi) + i\,sin(n\pi)) =$$

$$\frac{2iR^n}{n} sin(n\pi) = 0$$

Therefore we have that:

If $n = 0$:

$$\int_C (z - z_0)^{n-1} dz = 2\pi i$$

And if $n \neq 0$:

$$\int_C (z - z_0)^{n-1} dz = 0$$

□

47.2

Recall that for a contour C and a function $f(z)$, if $|f(z)| \leq M$ for all $z \in C$ and the length of C is L then we know:

$$\left| \int_C f(z)dz \right| \leq ML$$

Let C be the line segment from $z = i$ to $z = 1$. Clearly the length of C is $L = |1 - i| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

As suggested in the problem notice that if $z \in C$, that is if $z(t) = i + (1-i)t$ for some $t \in [0, 1]$ then it's distance from the origin is greater than the distance of the midpoint $z(\frac{1}{2}) = \frac{1}{2} + \frac{1}{2}i$ (where $|z(\frac{1}{2})| = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.

One way to see this is by finding the minimum of the modulus (or equivalently the modulus squared) as a function of t.

We know $|z(t)|^2 = |i + (1-i)t|^2 = |t + i(1-t)|^2 = t^2 + (1-t)^2$ where $t \in [0, 1]$.

Then we get $\frac{d}{dt}|z(t)|^2 = \frac{d}{dt}(t^2 + (1-t)^2) = 2t - 2(1-t) = 4t - 2$.

Setting this equal to 0 we get $t = \frac{1}{2}$ is the only critical point. Furthermore $\frac{d^2}{dt^2}|z(t)|^2 = \frac{d}{dt}4t - 2 = 4 > 0$.

So this function is concave up and therefore $z(\frac{1}{2})$ must be the minimum.

Since the midpoint is the closest to the origin we know $|\frac{1}{z^4}| = \frac{1}{|z|^4}$ is maximized over C at the midpoint since $|z(\frac{1}{2})| < |z(t)|$ for all $t \in [0, 1] \backslash \{\frac{1}{2}\}$ and hence $\frac{1}{|z(\frac{1}{2})|} > \frac{1}{|z(t)|}$ for all $t \in [0, 1] \backslash \{\frac{1}{2}\}$.

So we have that $|\frac{1}{z^4}| = \frac{1}{|z|^4} \leq \frac{1}{(\frac{\sqrt{2}}{2})^4} = (\sqrt{2})^4 = 4$.

Therefore we know:

$$\left| \int_C \frac{1}{z^4} dz \right| \leq 4\sqrt{2}$$

$\square$

47.4

Recall that for a contour C and a function $f(z)$, if $|f(z)| \leq M$ for all $z \in C$ and the length of C is L then we know:

$$\left| \int_C f(z)dz \right| \leq ML$$

Let C_R be the upper half of the circle $|z| = R$ taken in the counterclockwise direction where $R > 2$.

We know that the length of C_R is then $L = \pi R$.

Now we are considering the function $f(z) = \frac{2z^2-1}{z^4+5z^2+4} = \frac{2z^2-1}{(z^2+4)(z^2+1)}$ over C_R.

Since $|z| = R$ over C_R we know that $|f(z)| = |\frac{2z^2-1}{(z^2+4)(z^2+1)}| = \frac{|2z^2-1|}{|z^2+4||z^2+1|}$.

Then using the triangle inequalities $|z_1 + z_2| \leq |z_1| + |z_2|$ and $|z_1 + z_2| \geq ||z_1| - |z_2||$ we get:

$|2z^2 - 1| \leq |2z^2| + |-1| = 2|z|^2 + 1$ and $|z^2 + 4| \geq ||z^2| - |4|| = ||z|^2 - 4|$ and $|z^2 + 1| \geq ||z^2| - |1|| = ||z|^2 - 1|$.

Since $|z| = R > 2$ over C_R we know $R^2 - 1 > R^2 - 4 > 0$ so we have the following over C_R:

$|2z^2 - 1| \leq 2R^2 + 1$ and $|z^2 + 4| \geq |R^2 - 4| = R^2 - 4$ and $|z^2 + 1| \geq |R^2 - 1| = R^2 - 1$.

So over C_R we know $|f(z)| = \frac{|2z^2-1|}{|z^2+4||z^2+1|} \leq \frac{2R^2+1}{(R^2-4)(R^2-1)}$

Therefore we know:

$$\left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} dz \right| \leq \Big(\frac{2R^2+1}{(R^2-4)(R^2-1)} \Big) \Big(\pi R \Big) = \frac{\pi R(2R^2+1)}{(R^2-4)(R^2-1)}$$

Then as the problem suggests we can divide the numerator and denominator by R^4 to get:

$$\left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} dz \right| \leq \frac{\pi R(2R^2+1)}{(R^2-4)(R^2-1)} = \frac{\frac{\pi R(2R^2+1)}{R^4}}{\frac{(R^2-4)(R^2-1)}{R^4}} = \frac{\pi(\frac{2}{R} + \frac{1}{R^3})}{\frac{R^4-5R^2+4}{R^4}} = \frac{\pi(\frac{2}{R} + \frac{1}{R^3})}{1 - \frac{5}{R^2} + \frac{4}{R^4}}$$

Clearly $lim_{R\to\infty} \frac{2}{R} = 0$, $lim_{R\to\infty} \frac{1}{R^3} = 0$, $lim_{R\to\infty} \frac{-5}{R^2} = 0$, and $lim_{R\to\infty} \frac{4}{R^4} = 0$.

Therefore by the familiar limit theorems we know $lim_{R\to\infty} \pi(\frac{2}{R} + \frac{1}{R^3}) = 0$ and $lim_{R\to\infty} 1 - \frac{5}{R^2} + \frac{4}{R^4} = 1$.

And finally since $1 \neq 0$ we know:

$$\lim_{R\to\infty} \frac{\pi(\frac{2}{R} + \frac{1}{R^3})}{1 - \frac{5}{R^2} + \frac{4}{R^4}} = \frac{0}{1} = 0$$

Since $|z| \geq 0$ for all $z \in \mathbb{C}$ and we have just seen the limit as $R \to \infty$ of the above expression is 0 (where the expression is an upper bound for the modulus of the integral we examined before) we can say by the squeeze theorem:

$$\lim_{R\to\infty} \left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} dz \right| = 0$$

$\square$