Differentiation of Power Series

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72.5

Let $f(z) = \frac{cos\ z}{z^2-(\frac{\pi}{2})^2} = \frac{cos\ z}{(z-\frac{\pi}{2})(z+\frac{\pi}{2})}$ when $z \neq \pm\frac{\pi}{2}$ and let $f(z) = -\frac{1}{\pi}$ when $z = \pm\frac{\pi}{2}$.

Note that $f(-z) = \frac{cos(-z)}{(-z)^2-(\frac{\pi}{2})^2} = \frac{cos\ z}{z^2-(\frac{\pi}{2})^2} = f(z)$ when $z \neq \pm\frac{\pi}{2}$ and $f(-\frac{\pi}{2}) = -\frac{1}{\pi} = f(\frac{\pi}{2})$. So $f(-z) = f(z)$ for all $z \in \mathbb{C}$.

We already know for $|w| < \infty$ that:

$$sin\ w = \sum_{n=0}^{\infty} \frac{(-1)^n w^{2n+1}}{(2n+1)!} = w - \frac{w^3}{3!} + \frac{w^5}{5!} - ...$$

Similarly we already know for $|w| < 1$ that:

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n = 1 + w + w^2 + ...$$

Letting $w = z - \frac{\pi}{2}$ we know for $|z - \frac{\pi}{2}| < \infty$ (or equivalently for $|z| < \infty$) that:

$$cos\ z = -sin(z - \frac{\pi}{2}) = -\sum_{n=0}^{\infty} \frac{(-1)^n (z-\frac{\pi}{2})^{2n+1}}{(2n+1)!} = -(z-\frac{\pi}{2}) + \frac{(z-\frac{\pi}{2})^3}{3!} - \frac{(z-\frac{\pi}{2})^5}{5!}...$$

Letting $w = \frac{-(z-\frac{\pi}{2})}{\pi}$ we know for $|\frac{-(z-\frac{\pi}{2})}{\pi}| = \frac{|z-\frac{\pi}{2}|}{\pi} < 1$ (or equivalently $|z-\frac{\pi}{2}| < \pi$) that:

$$\frac{1}{z+\frac{\pi}{2}} = \frac{1}{\pi + (z-\frac{\pi}{2})} = \frac{1}{\pi}\left(\frac{1}{1-\frac{-(z-\frac{\pi}{2})}{\pi}}\right) = \frac{1}{\pi}\sum_{n=0}^{\infty}\left(\frac{-(z-\frac{\pi}{2})}{\pi}\right)^n = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n(z-\frac{\pi}{2})^n}{\pi^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(z-\frac{\pi}{2})^n}{\pi^{n+1}}$$

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Therefore for $0 < |z - \frac{\pi}{2}| < \pi$ we know:

$$f(z) = \frac{cos\ z}{(z-\frac{\pi}{2})(z+\frac{\pi}{2})} = \left(-\sum_{n=0}^{\infty} \frac{(-1)^n (z-\frac{\pi}{2})^{2n+1}}{(2n+1)!} \right) \left(\sum_{n=0}^{\infty} \frac{(-1)^n (z-\frac{\pi}{2})^n}{\pi^{n+1}} \right) \left(\frac{1}{z-\frac{\pi}{2}} \right)$$

$$= \left(\frac{1}{z-\frac{\pi}{2}} \Big(-(z-\frac{\pi}{2}) + \frac{(z-\frac{\pi}{2})^3}{3!} - \frac{(z-\frac{\pi}{2})^5}{5!} + ... \Big) \right) \left(\frac{1}{\pi} - \frac{z-\frac{\pi}{2}}{\pi^2} + \frac{(z-\frac{\pi}{2})^2}{\pi^3} - ... \right)$$

$$= \left(-1 + \frac{(z-\frac{\pi}{2})^2}{3!} - \frac{(z-\frac{\pi}{2})^4}{5!} + ... \right) \left(\frac{1}{\pi} - \frac{z-\frac{\pi}{2}}{\pi^2} + \frac{(z-\frac{\pi}{2})^2}{\pi^3} - ... \right)$$

When we evaluate the right hand side at $z = \frac{\pi}{2}$ we get $\Big(-1+0+0+... \Big) \Big(\frac{1}{\pi} + 0 + 0 + ... \Big) = -\frac{1}{\pi}$ since all terms except the constant ones evaluate to 0.

Therefore we have that $f(z) = \Big(-1 + \frac{(z-\frac{\pi}{2})^2}{3!} - \frac{(z-\frac{\pi}{2})^4}{5!} + ... \Big) \Big(\frac{1}{\pi} - \frac{z-\frac{\pi}{2}}{\pi^2} + \frac{(z-\frac{\pi}{2})^2}{\pi^3} - ... \Big)$ for $|z - \frac{\pi}{2}| < \pi$ since $f(\frac{\pi}{2}) = -\frac{1}{\pi}$ and the right hand side evaluates to the same at $z = \frac{\pi}{2}$.

So we know that $f(z)$ can be written as the product of power series with positive powers and hence as a power series with positive powers itself in the neighborhood $|z - \frac{\pi}{2}| < \pi$.

This means that $f(z)$ is analytic inside that neighborhood and hence analytic at $z = \frac{\pi}{2}$ since it can be represented as a power series.

Using the same process you get the same result for $z = -\frac{\pi}{2}$ but this can also be seen using $f(-z) = f(z)$ for all $z \in \mathbb{C}$.

Therefore we have shown that $f(z)$ is analytic at $z = \pm\frac{\pi}{2}$.

Since $f(z) = \frac{cos\ z}{(z-\frac{\pi}{2})(z+\frac{\pi}{2})}$ we know already that it is analytic for all $z \neq \pm\frac{\pi}{2}$ since the numerator and denominator are entire and the denominator is 0 if and only if $z = \pm\frac{\pi}{2}$.

Therefore we know that $f(z)$ is analytic for all $z \in \mathbb{C}$ and is hence entire $\square$

72.9

Recall that power series may be differentiated term by term when the series converges.

Let $f(z)$ be a function with a power series representation around z_0 inside some circle $|z - z_0| = R$ given below.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Let $S = \{n \in \{0, 1, 2, ...\} : f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k}(z - z_0)^k \text{ when } |z - z_0| < R\}$.

- Base case ($n = 0$):

We know that whenever $|z - z_0| < R$:

$$f^{(0)}(z) = f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} \frac{(0 + k)!}{k!} a_{0+k} (z - z_0)^k$$

Therefore we know $0 \in S$.

- Inductive step (n implies $n + 1$):

Assume that $n \in S$, then:

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n + k)!}{k!} a_{n+k} (z - z_0)^k$$

The above is a power series representation for $f^{(n)}(z)$ which converges when $|z - z_0| < R$ and hence we know we may differentiate it term by term when $|z - z_0| < R$. So we have:

$$\frac{d}{dz} f^{(n)}(z) = f^{(n+1)}(z) = \sum_{m=0}^{\infty} \frac{d}{dz} \frac{(n + m)!}{m!} a_{n+m} (z - z_0)^m = \sum_{m=1}^{\infty} m \frac{(n + m)!}{m!} a_{n+m} (z - z_0)^{m-1}$$

Then letting $k = m - 1$ (which gives $m = k + 1$) we have for $|z - z_0| < R$:

$$f^{(n+1)}(z) = \sum_{m=1}^{\infty} m \frac{(n + m)!}{m!} a_{n+m} (z - z_0)^{m-1} = \sum_{k=0}^{\infty} (k + 1) \frac{(n + k + 1)!}{(k + 1)!} a_{n+k+1} (z - z_0)^k$$

$$= \sum_{k=0}^{\infty} \frac{((n + 1) + k)!}{(k + 1)!/(k + 1)} a_{(n+1)+k} (z - z_0)^k = \sum_{k=0}^{\infty} \frac{((n + 1) + k)!}{k!} a_{(n+1)+k} (z - z_0)^k$$

Therefore we know $n + 1 \in S$

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Since we know $0 \in S$ and we know $n \in S$ implies $n+1 \in S$ we have that

$S = \{n \in \{0,1,2,...\} : f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k}(z-z_0)^k \text{ when } |z-z_0| < R\} = \{0,1,2,...\}$.

Which means that for all $n \in \{0,1,2,...\}$ we know for $|z-z_0| < R$:

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k}(z-z_0)^k$$

- Showing that the power series representation for $f(z)$ is the Taylor series representation.

Now we know that for all $n \in \{0,1,2,...\}$ when $|z-z_0| < R$:

$$f^{(n)}(z) = \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} a_{n+k}(z-z_0)^k = n!\, a_n + (n+1)!\, a_{n+1}(z-z_0) + \frac{(n+2)!}{2!} a_{n+2}(z-z_0)^2 + ...$$

Therefore when we evaluate both sides at $z = z_0$ we get the following when $|z-z_0| < R$:

$$f^{(n)}(z_0) = n! a_n + (n+1)!\, a_{n+1}(z_0-z_0) + \frac{(n+2)!}{2!} a_{n+2}(z_0-z_0)^2 + ... = n!\, a_n + 0 + 0 + ... = n!\, a_n$$

So we have shown that for every $n \in \{0,1,2,...\}$ that $f^{(n)}(z_0) = n!\, a_n$ which means that $a_n = \frac{f^{(n)}(z_0)}{n!}$.

This is the nth term of the Taylor series for $f(z)$, so the power series representation for $f(z)$ is the Taylor series. This was true for an arbitrary function $f(z)$ and an arbitrary power series representation for $f(z)$ and hence is true for all power series representations of any function $f(z)$ that has a power series representation with the circle of convergence $|z-z_0| < R$.

Therefore if $f(z)$ has a power series representation for $|z-z_0| < R$ then that power series is the Taylor series $\square$

73.1

We already know that for $|z| < \infty$:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...$$

We also already know that for $|z| < 1$:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + ...$$

Therefore if $|-z^2| = |z|^2 < 1$ (or equivalently $|z| < 1$) we know:

$$\frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n} = 1 - z^2 + z^4 - z^6 + ...$$

Then we know for $0 < |z| < 1$:

$$\frac{1}{z(1+z^2)} = \frac{1}{z}\Big(\frac{1}{1+z^2}\Big) = \frac{1}{z}\left(\sum_{n=0}^{\infty} (-1)^n z^{2n}\right) = \sum_{n=0}^{\infty} \frac{1}{z}(-1)^n z^{2n} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1} = \frac{1}{z} - z + z^3 - z^5 + ...$$

So using multiplication of series we get for $0 < |z| < 1$:

$$\frac{e^z}{z(1+z^2)} = e^z\Big(\frac{1}{z(1+z^2)}\Big) = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} (-1)^n z^{2n-1}\right) = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + ...\right)\left(\sum_{n=0}^{\infty} (-1)^n z^{2n-1}\right)$$

$$= 1\sum_{n=0}^{\infty} (-1)^n z^{2n-1} + z\sum_{n=0}^{\infty} (-1)^n z^{2n-1} + \frac{z^2}{2!}\sum_{n=0}^{\infty} (-1)^n z^{2n-1} + \frac{z^3}{3!}\sum_{n=0}^{\infty} (-1)^n z^{2n-1} + ...$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n-1} + \sum_{n=0}^{\infty} z(-1)^n z^{2n-1} + \frac{1}{2!}\sum_{n=0}^{\infty} z^2(-1)^n z^{2n-1} + \frac{1}{3!}\sum_{n=0}^{\infty} z^3(-1)^n z^{2n-1} + ...$$

$$= \sum_{n=0}^{\infty} (-1)^n z^{2n-1} + \sum_{n=0}^{\infty} (-1)^n z^{2n} + \frac{1}{2!}\sum_{n=0}^{\infty} (-1)^n z^{2n+1} + \frac{1}{3!}\sum_{n=0}^{\infty} (-1)^n z^{2n+2} + ...$$

$$= \Big(\frac{1}{z} - z + z^3 - z^5 + ...\Big) + \Big(1 - z^2 + z^4 - z^6 + ...\Big) + \frac{1}{2!}\Big(z - z^3 + z^5 - z^7 + ...\Big) + \frac{1}{3!}\Big(z^2 - z^4 + z^6 - z^8 + ...\Big)$$

Notice that only finitely many of the series have a given term z^n so for $0 < |z| < 1$ we get the following:

$$\frac{e^z}{z(1+z^2)} = \frac{1}{z} + 1 + z\Big(-1 + \frac{1}{2!}\Big) + z^2\Big(-1 + \frac{1}{3!}\Big) + z^3\Big(1 - \frac{1}{2!} + \frac{1}{4!}\Big) + z^4\Big(1 - \frac{1}{3!} + \frac{1}{5!}\Big)$$

$$= \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \frac{13}{24}z^3 + \frac{101}{120}z^4 - ...$$

73.5

Recall that the coefficients for a Laurent series about z_0 are given by:

$$c_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

Let C be the positively oriented circle $|z| = 1$, then clearly $z_0 = 0$ is inside C.

We are already given the Laurent series below for $0 < |z| < \pi$:

$$\frac{1}{z^2 sinh\ z} = \frac{1}{z^3} - \frac{1}{6}\Big(\frac{1}{z}\Big) + \frac{7}{360}z + ...$$

Since C is positively oriented and $z_0 = 0$ is inside C we know for the Laurent series about $z_0 = 0$ of $f(z) = \frac{1}{z^2 sinh\ z}$:

$$-\frac{1}{6} = c_{-1} = \frac{1}{2\pi i}\int_C f(z)dz = \frac{1}{2\pi i}\int_C \frac{1}{z^2 sinh\ z}dz$$

Therefore we have that:

$$\int_C \frac{1}{z^2 sinh\ z}dz = 2\pi i\Big(-\frac{1}{6}\Big) = -\frac{\pi i}{3}$$

$\square$

73.8

Let $f(z)$ be an entire function with the series representation $f(z) = z + a_2z^2 + a_3z^3 + ...$ for $|z| < \infty$.

Recall that power series may be differentiated term by term in their radius of convergence to get the total derivative.

a. Let $g(z) = f(f(z))$, then $g(0) = f(f(0)) = f(0) = 0$.

We know $g(z)$ is entire so it has a series representation for $|z| < \infty$:

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n = g(0) + g'(0)z + \frac{g''(0)}{2!} z^2 + \frac{g'''(0)}{3!} z^3 + ... = g'(0)z + \frac{g''(0)}{2!} z^2 + \frac{g'''(0)}{3!} z^3 + ...$$

Let us first find the series expansions for $f'(z), f''(z)$, and $f'''(z)$ about $z_0 = 0$ whenever $|z| < \infty$:

$$f'(z) = \frac{d}{dz}\Big(z + a_2z^2 + a_3z^3 + ... + a_nz^n + ...\Big) = \frac{d}{dz}z + \frac{d}{dz}a_2z^2 + \frac{d}{dz}a_3z^3 + ... + \frac{d}{dz}a_nz^n + ... = 1 + 2a_2z + 3a_3z^2 + ... na_nz^{n-1} + ...$$

$$f''(z) = \frac{d}{dz}\Big(1 + 2a_2z + 3a_3z^2 + ... + na_nz^{n-1} + ...\Big) = 2a_2 + 6a_3z + ... + n(n-1)a_nz^{n-2} + ...$$

$$f'''(z) = \frac{d}{dz}\Big(2a_2 + 6a_3z + 12a_4z^2 ... + n(n-1)a_nz^{n-2} + ...\Big) = 6a_3 + 24a_4z + ... + n(n-1)(n-2)a_nz^{n-3} + ...$$

We know that $g'(z) = \frac{d}{dz} f(f(z)) = f'(z)f'(f(z))$.

Then $g''(z) = \frac{d}{dz}\Big(f'(z)f'(f(z))\Big) = f''(z)f'(f(z)) + \big(f'(z)\big)^2 f''(f(z))$.

Finally $g'''(z) = \frac{d}{dz}\Big(f''(z)f'(f(z)) + \big(f'(z)\big)^2 f''(f(z))\Big) =$

$f'''(z)f'(f(z)) + f''(z)f'(z)f''(f(z)) + 2f'(z)f''(z)f''(f(z)) + \big(f'(z)\big)^3 f'''(f(z))$

Now note that by evaluating the power series found above $f'(0) = 1$, $f''(0) = 2a_2$, and $f'''(0) = 6a_3$.

Therefore the first three nonzero terms in the Taylor series for $g(z)$ about $z_0 = 0$ are:

$$b_1 = \frac{g'(0)}{1!} = f'(0)f'(f(0)) = \big(f'(0)\big)^2 = 1.$$

$$b_2 = \frac{g''(0)}{2!} = \frac{1}{2}\Big(f''(0)f'(f(0)) + \big(f'(0)\big)^2 f''(f(0))\Big) = \frac{1}{2}\Big(f''(0)f'(0) + \big(f'(0)\big)^2 f''(0)\Big) = \frac{1}{2}\Big(2a_2 + 2a_2\Big) = 2a_2$$

$$b_3 = \frac{g'''(0)}{3!} = \frac{1}{6}\Big(f'''(0)f'(f(0)) + f''(0)f'(0)f''(f(0)) + 2f'(0)f''(0)f''(f(0)) + \big(f'(0)\big)^3 f'''(f(0))\Big) =$$
$$\frac{1}{6}\Big(f'''(0)f'(0) + \big(f''(0)\big)^2 f'(0) + 2f'(0)\big(f''(0)\big)^2 + \big(f'(0)\big)^3 f'''(0)\Big) = \frac{1}{6}\Big(2f'''(0) + 3\big(f''(0)\big)^2\Big)$$
$$= \frac{1}{6}\Big(2(6a_3) + 3(2a_2)^2\Big) = 2(a_3 + a_2^2).$$

Therefore we have that for $|z| < \infty$:

$$g(z) = f(f(z)) = z + 2a_2z^2 + 2(a_3 + a_2^2)z^3 + ...$$

$\square$

c. Let $f(z) = sin\ z$ and $g(z) = sin(sin(z))$.

We already know for $|z| < \infty$:

$$f(z) = sin\ z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...$$

So we have that $a_2 = 0$ and $a_3 = -\frac{1}{3!}$ and $sin\ z$ has a power series of the form from part a.

Therefore we may apply our results from part a on $g(z) = f(f(z)) = sin(sin(z))$.

That is we know for $|z| < \infty$:

$$g(z) = f(f(z)) = sin(sin(z)) = z + 2a_2z^2 + 2(a_3 + a_2^2)z^3 + ... = z + 2(0)z^2 + 2(-\frac{1}{3!} + 0^2)z^3 + ... = z - \frac{z^3}{3} + ...$$

$\square$

Problem 2

Let $f(z)$ be analytic in the domain $|z| < 1$, such that $f(0) = 1$ and $f(z) = z + f(z^2)$.

Since $f(z)$ is analytic in $|z| < 1$ we know it has a power series representation for $|z| < 1$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Therefore for $|z^2| = |z|^2 < 1$ (or equivalently for $|z| < 1$):

$$f(z^2) = \sum_{n=0}^{\infty} a_n (z^2)^n = \sum_{n=0}^{\infty} a_n z^{2n}$$

Also note that since these series must be each the respective Taylor series we know $a_0 = f(0^2) = f(0) = 1$.

Therefore we have the following:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = 1 + \sum_{n=1}^{\infty} a_n z^n = z + \left(1 + \sum_{n=1}^{\infty} a_n z^{2n}\right) = z + f(z^2)$$

$$\sum_{n=1}^{\infty} a_n z^n = (a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 + ...) = (z + a_1 z^2 + a_2 z^4 + a_3 z^6 + ...) = z + \sum_{n=1}^{\infty} a_n z^{2n}$$

$$z(a_1 - 1) + z^2(a_2 - a_1) + z^3(a_3 - 0) + z^4(a_4 - a_2) + z^5(a_5 - 0) + z^6(a_6 - a_3) + ... = 0$$

From which we know:

$a_1 = 1, a_{2k+1} = 0$ for any $k \in \mathbb{N}$ (i.e. odd coefficients are 0), and $a_{2n} = a_n$ for any $n \in \mathbb{N}$ divisible by 2.

So if n is divisible by any odd natural number greater than 1, then $a_{2n} = a_n = ... = a_{2k+1} = 0$ for some $k \in \mathbb{N}$.

Which means that $a_n = 1$ if $n = 2^k$ for some $k \in \{0, 1, 2, ...\}$, $a_0 = 1$, and otherwise $a_n = 0$.

Therefore we have for $|z| < 1$ that:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = 1 + z + z^2 + z^4 + z^8 + z^{16} + ... = 1 + \sum_{n=0}^{\infty} z^{(2^n)}$$

$\square$