The Cauchy Goursat Theorem

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91.1

Let C_R be defined by $z(\theta) = Re^{i\theta}$ in the counterclockwise direction.

Also let C_ρ be defined by $z(\theta) = \rho e^{i\theta}$ in the clockwise direction with $R > \rho$.

Now let L_1 be the line going from ρ to R and L_2 be the line going from $-R$ to $-\rho$.

Clearly $C = C_\rho + L_1 + C_R + L_2$ is a simple closed contour in the positive sense.

Now let $f(z) = \frac{e^{iaz} - e^{ibz}}{z^2}$ where $a, b \geq 0$. Clearly f has a pole of order 2 at $z = 0$.

Since $z = 0$ is not on or inside C we know that f is analytic inside and on C.

Therefore by the Cauchy Goursat theorem we know:

$$\int_C \frac{e^{iaz} - e^{ibz}}{z^2} dz = \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{z^2} dz + \int_{L_1} \frac{e^{iaz} - e^{ibz}}{z^2} dz + \int_{C_R} \frac{e^{iaz} - e^{ibz}}{z^2} dz + \int_{L_2} \frac{e^{iaz} - e^{ibz}}{z^2} dz = 0$$

We already know for all $z \in \mathbb{C}$ that:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + ...$$

Therefore taking $c \geq 0$ (it could really be any complex number though) we know for all $z \in \mathbb{C}$ that:

$$e^{icz} = \sum_{n=0}^{\infty} \frac{(icz)^n}{n!}$$

Consequently we know for all $z \in \mathbb{C}$ that:

$$e^{iaz} - e^{ibz} = \sum_{n=0}^{\infty} \frac{(iaz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(ibz)^n}{n!} = \sum_{n=0}^{\infty} \frac{(iaz)^n - (ibz)^n}{n!} = z(i(a-b)) + z^2(\frac{b^2 - a^2}{2}) + ...$$

Therefore we have the Laurent series for $|z| > 0$:

$$\frac{e^{iaz} - e^{ibz}}{z^2} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{(iaz)^n - (ibz)^n}{n!} = \frac{i(a-b)}{z} + \sum_{n=0}^{\infty} c_n z^n$$

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So $z=0$ is a simple pole of f and we know $Res_{z=0}\frac{e^{iaz}-e^{ibz}}{z^2}=i(a-b)$.

Since $z=0$ is a simple pole of f on the real axis with residue $B_0=i(a-b)$ and C_ρ denotes the upper half of the circle $|z|=\rho$ in the clockwise direction we know:

$$\lim_{\rho\to 0}\int_{C_\rho}\frac{e^{iaz}-e^{ibz}}{z^2}dz=-B_0\pi i=-i(a-b)\pi i=(a-b)\pi$$

Now notice that if $z=x+iy$ then $|e^{iz}|=|e^{i(x+iy)}|=|e^{ix-y}|=|e^{ix}||e^{-y}|=|e^{-y}|$.

Therefore if $y=Im\ z\geq 0$ we know that $|e^{iz}|\leq 1$.

So we have on C_R that $|\frac{e^{iaz}-e^{ibz}}{z^2}|=\frac{|e^{iaz}-e^{ibz}|}{|z^2|}\leq\frac{|e^{iaz}|+|e^{ibz}|}{|z|^2}=\frac{|e^{iz}|^a+|e^{iz}|^b}{R^2}\leq\frac{2}{R^2}$.

We also know that the length of C_R is πR so we have that:

$$\left|\int_{C_R}\frac{e^{iaz}-e^{ibz}}{z^2}dz\right|\leq\left(\frac{2}{R^2}\right)\pi R=\frac{2\pi}{R}$$

Clearly as $R\to\infty$ we know $\frac{2\pi}{R}\to 0$ so we have that:

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{e^{iaz}-e^{ibz}}{z^2}dz\right|=0$$

Consequently the integral itself is 0 in the limit, that is:

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{iaz}-e^{ibz}}{z^2}dz=0$$

Along L_1 we can take the parameterization $z(x)=x$ where $\rho\leq x\leq R$ to get:

$$\int_{L_1}\frac{e^{iaz}-e^{ibz}}{z^2}dz=\int_\rho^R\frac{e^{iax}-e^{ibx}}{x^2}dx=\int_\rho^R\frac{cos(ax)+i\ sin(ax)-cos(bx)-i\ sin(bx)}{x^2}dx$$

$$=\int_\rho^R\frac{cos(ax)-cos(bx)}{x^2}dx+i\int_\rho^R\frac{sin(ax)-sin(bx)}{x^2}dx$$

Along L_2 we can take the parameterization $z(x)=x$ where $-R\leq x\leq -\rho$ to get:

$$\int_{L_1}\frac{e^{iaz}-e^{ibz}}{z^2}dz=\int_{-R}^{-\rho}\frac{e^{iax}-e^{ibx}}{x^2}dx=\int_\rho^R\frac{e^{-iax}-e^{-ibx}}{x^2}dx=\int_\rho^R\frac{cos(ax)-i\ sin(ax)-cos(bx)+i\ sin(bx)}{x^2}dx$$

$$=\int_\rho^R\frac{cos(ax)-cos(bx)}{x^2}dx-i\int_\rho^R\frac{sin(ax)-sin(bx)}{x^2}dx$$

Therefore we know that:

$$\int_{L_1}\frac{e^{iaz}-e^{ibz}}{z^2}dz+\int_{L_2}\frac{e^{iaz}-e^{ibz}}{z^2}dz=2\int_\rho^R\frac{cos(ax)-cos(bx)}{x^2}dx$$

This is true for any arbitrary $\rho < R$ and so we may simultaneously take $R \to \infty$ and $\rho \to 0$ to get:

$$2\int_0^\infty \frac{cos(ax) - cos(bx)}{x^2}dx = \lim_{R\to\infty,\rho\to 0} 2\int_\rho^R \frac{cos(ax) - cos(bx)}{x^2}dx = \lim_{R\to\infty,\rho\to 0}\left(\int_{L_1} \frac{e^{iaz} - e^{ibz}}{z^2}dz + \int_{L_2} \frac{e^{iaz} - e^{ibz}}{z^2}dz\right)$$

$$= \lim_{R\to\infty,\rho\to 0}\left(\int_C \frac{e^{iaz} - e^{ibz}}{z^2}dz - \int_{C_R} \frac{e^{iaz} - e^{ibz}}{z^2}dz - \int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{z^2}dz\right)$$

$$= \left(\lim_{R\to\infty,\rho\to 0}\int_C \frac{e^{iaz} - e^{ibz}}{z^2}dz\right) - \left(\lim_{R\to\infty,\rho\to 0}\int_{C_R} \frac{e^{iaz} - e^{ibz}}{z^2}dz\right) - \left(\lim_{R\to\infty,\rho\to 0}\int_{C_\rho} \frac{e^{iaz} - e^{ibz}}{z^2}dz\right)$$

$$= 0 - 0 - (a-b)\pi = (b-a)\pi$$

Therefore for any $a, b \geq 0$ we have that:

$$\int_0^\infty \frac{cos(ax) - cos(bx)}{x^2}dx = \frac{(b-a)\pi}{2}$$

□

We may use this fact, taking $a = 0$ and $b = 2$, as well as the identity $1 - cos(2x) = 2sin^2x$ to get:

$$\int_0^\infty \frac{cos(0) - cos(2x)}{x^2}dx = \int_0^\infty \frac{1 - cos(2x)}{x^2}dx = \int_0^\infty \frac{2sin^2x}{x^2}dx = \frac{(2-0)\pi}{2} = \pi$$

Therefore we are left with:

$$\int_0^\infty \frac{sin^2x}{x^2}dx = \frac{1}{2}\int_0^\infty \frac{2sin^2x}{x^2}dx = \frac{\pi}{2}$$

91.6

Refer to the figure in the book for the contours $\Gamma_R, \Gamma_\rho, \gamma_R, \gamma_\rho$, and L.

Let V be the line segment on the real axis going from ρ to R.

a. The contour on the left in the book is given by $C_1 = V + \Gamma_R + L + \Gamma_\rho$ and is clearly simple closed.

Let $f_1(z) = \frac{z^{-a}}{z+1}$ where the branch is $|z| > 0, -\frac{\pi}{2} < arg\, z < \frac{3\pi}{2}$.

We know f_1 is analytic inside and on C_1 except at the isolated singular point $z = -1$ because none of the branch cut for z^{-a} is inside or on C_1, so z^{-a} is hence analytic inside and on C_1.

Notice that along V we may use the parameterization $z(r) = r$ where $\rho \le r \le R$ to get:

$$\int_V f_1(z)dz = \int_\rho^R \frac{r^{-a}}{r+1}dr$$

Therefore we know by Cauchy's Residue Theorem that:

$$\int_{C_1} f_1(z)dz = \int_V f_1(z)dz + \int_{\Gamma_R} f_1(z)dz + \int_L f_1(z)dz + \int_{\Gamma_\rho} f_1(z)dz$$

$$= \int_\rho^R \frac{r^{-a}}{r+1}dr + \int_{\Gamma_R} f_1(z)dz + \int_L f_1(z)dz + \int_{\Gamma_\rho} f_1(z)dz = 2\pi i\, Res_{z=-1} f_1(z)$$

b. The contour on the left in the book is given by $C_2 = -V + \gamma_\rho - L + \gamma_R$ and is clearly simple closed.

Let $f_2(z) = \frac{z^{-a}}{z+1}$ where the branch is $|z| > 0, \frac{\pi}{2} < arg\, z < \frac{5\pi}{2}$.

Since we take this branch cut we must represent real numbers as $x = xe^{2\pi i}$ (instead of just $x = xe^{0i}$ as before) when parameterizing for our contours.

We know f_2 is analytic inside and on C_2 because none of the branch cut for z^{-a} is inside or on C_2, so z^{-a} is hence analytic inside and on C_2 and since $z = -1$ is not inside or on C_2 we thus have that f_2 is analytic inside and on C_2.

Notice that along $-V$ we may use the parameterization $z(x) = (R - x)e^{2\pi i}$ where $0 \le x \le R - \rho$ to get:

$$\int_{-V} f_2(z)dz = \int_0^{R-\rho} \frac{((R-x)e^{2\pi i})^{-a}}{(R-x)+1}dx$$

Then using the substitution $r = R - x$ with new bounds $0 \le R - r \le R - \rho$ (meaning $\rho \le r \le R$) we get:

$$\int_{-V} f_2(z)dz = \int_0^{R-\rho} \frac{((R-x)e^{2\pi i})^{-a}}{(R-x)+1}dx = -\int_\rho^R \frac{r^{-a}e^{-2\pi a i}}{r+1}dr$$

Therefore we know by the Cauchy Goursat Theorem that:

$$\int_{C_2} f_2(z)dz = \int_{-V} f_2(z)dz + \int_{\gamma_\rho} f_2(z)dz + \int_{-L} f_2(z)dz + \int_{\gamma_R} f_2(z)dz$$

$$= -\int_\rho^R \frac{r^{-a}e^{-2\pi a i}}{r+1}dr + \int_{\gamma_\rho} f_2(z)dz - \int_L f_2(z)dz + \int_{\gamma_R} f_2(z)dz = 0$$

c. Consider the branch $f(z) = \frac{z^{-a}}{z+1}$ with $|z| > 0, 0 \leq arg\ z \leq 2\pi$.

For each of the contours $\Gamma_R, \Gamma_\rho, \gamma_R, \gamma_\rho$, and L this branch cut either doesn't intersect them or it only does so at finitely many (actually just 1) point which is on the positive side of the real axis.

Therefore f has only finitely many points where it takes a value different from f_1 and f_2 along $\Gamma_R, \Gamma_\rho, \gamma_R, \gamma_\rho$, and L and hence the integrals will be the same when swapping the branch cuts.

Since none of the branch cuts intersect $z = -1$ we will have that each of these branches have the same residue there too.

Then using the fact that $\Gamma_R + \gamma_R = C_R$ and $\Gamma_\rho + \gamma_\rho = C_\rho$ we get that:

$$\left(\int_\rho^R \frac{r^{-a}}{r+1}dr + \int_{\Gamma_R} f_1(z)dz + \int_L f_1(z)dz + \int_{\Gamma_\rho} f_1(z)dz\right) + \left(-\int_\rho^R \frac{r^{-a}e^{-2\pi ai}}{r+1}dr + \int_{\gamma_\rho} f_2(z)dz - \int_L f_2(z)dz + \int_{\gamma_R} f_2(z)dz\right)$$

$$= \int_\rho^R \frac{r^{-a}}{r+1}dr + \int_{\Gamma_R} f(z)dz + \int_L f(z)dz + \int_{\Gamma_\rho} f(z)dz - \int_\rho^R \frac{r^{-a}e^{-2\pi ai}}{r+1}dr + \int_{\gamma_\rho} f(z)dz - \int_L f(z)dz + \int_{\gamma_R} f(z)dz$$

$$= \int_\rho^R \frac{r^{-a}}{r+1}dr - \int_\rho^R \frac{r^{-a}e^{-2\pi ai}}{r+1}dr + \int_{C_\rho} f(z)dz + \int_{C_R} f(z)dz = 2\pi i\ Res_{z=-1} f(z)$$

The last inequality comes because in the first line the first term in parentheses evaluates to $2\pi i\ Res_{z=-1} f(z)$ and the second term in parentheses evaluates to 0.

So the inequality has been derived $\square$

Problem 2

Let $f(z) = \frac{z^{\frac{1}{2}}}{z^4+1}$, taking the branch $-\pi < arg\, z < \pi$. Clearly f has simple poles at all the fourth roots of -1.

If $z^4 = -1$ then $z = (-1)^{1/4} = e^{i\frac{\pi+2k\pi}{4}} = e^{i(\frac{\pi}{4}+k\frac{\pi}{2})}$ where $k \in \{0, 1, 2, 3\}$.

These roots are:

$$z_1 = e^{i\frac{\pi}{4}},\ z_2 = e^{i\frac{3\pi}{4}},\ z_3 = e^{i\frac{5\pi}{4}},\text{ and } z_4 = e^{i\frac{7\pi}{4}}$$

Since they are simple poles we know:

$$Res_{z=z_n}\frac{z^{\frac{1}{2}}}{z^4+1} = \frac{z_n^{\frac{1}{2}}}{4z_n^3} = -\frac{1}{4}z_n^{\frac{3}{2}}.$$

Let $R > 1$ and $\rho < 1$ so that the desired pole of f will lie interior to our contour.

Let γ_ρ be the clockwise part of the circle $|z| = \rho$ in the first quadrant.

Let γ_R be the counterclockwise part of the circle $|z| = R$ in the first quadrant.

Then let L_1 be the line from ρ to R on the real axis and L_2 be the line from iR to $i\rho$ on the imaginary axis.

Clearly $C = \gamma_\rho + L_1 + \gamma_R + L_2$ is a positively oriented simple closed contour.

Also f is analytic inside and on C except at $z = e^{i\frac{\pi}{4}} = \frac{1}{\sqrt{2}}(1+i)$.

Therefore we know:

$$\int_C f(z)dz = \int_{\gamma_\rho} f(z)dz + \int_{L_1} f(z)dz + \int_{\gamma_R} f(z)dz + \int_{L_2} f(z)dz = 2\pi i Res_{z=\frac{1}{\sqrt{2}}(1+i)} f(z) = -\frac{\pi i}{2}e^{i\frac{3\pi}{8}}$$

We know that $\left|\frac{z^{\frac{1}{2}}}{z^4+1}\right| = \frac{|z|^{\frac{1}{2}}}{|z^4+1|} \le \frac{\sqrt{|z|}}{||z^4|-|1||} = \frac{\sqrt{|z|}}{||z|^4-1|}$

Along γ_R we know $|z| = R$ and clearly γ_R has length $\frac{\pi R}{2}$.

Since $R > 1$ we know $\left|\frac{z^{\frac{1}{2}}}{z^4+1}\right| \le \frac{\sqrt{|z|}}{||z|^4-1|} = \frac{\sqrt{R}}{|R^4-1|} = \frac{\sqrt{R}}{R^4-1}$ on γ_R.

Since clearly $\frac{\pi R^{\frac{3}{2}}}{2(R^4-1)} \to 0$ as $R \to \infty$ we know that:

$$\left|\int_{\gamma_R} f(z)dz\right| \le \frac{\pi R^{\frac{3}{2}}}{2(R^4-1)} \quad \text{and} \quad \lim_{R\to\infty}\int_{\gamma_R} f(z)dz = 0$$

Along γ_ρ we know $|z| = \rho$ and clearly γ_ρ has length $\frac{\pi\rho}{2}$.

Since $\rho < 1$ we know $\left|\frac{z^{\frac{1}{2}}}{z^4+1}\right| \le \frac{\sqrt{|z|}}{||z|^4-1|} = \frac{\sqrt{\rho}}{|\rho^4-1|} = \frac{\sqrt{\rho}}{1-\rho^4}$ on γ_ρ.

Since clearly $\frac{\pi\rho^{\frac{3}{2}}}{2(1-\rho^4)} \to 0$ as $\rho \to 0$ we know that:

$$\left|\int_{\gamma_\rho} f(z)dz\right| \le \frac{\pi\rho^{\frac{3}{2}}}{2(1-\rho^4)} \quad \text{and} \quad \lim_{\rho\to 0}\int_{\gamma_R} f(z)dz = 0$$

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Along L_1 we have the parameterization $z(r) = re^{0i}$ where $\rho \leq r \leq R$, so:

$$\int_{L_1} f(z)dz = \int_{\rho}^{R} \frac{(re^{0i})^{\frac{1}{2}}}{r^4+1} dr = \int_{\rho}^{R} \frac{\sqrt{r}}{r^4+1} dr$$

Along L_2 we have the parameterization $z(x) = (R-x)e^{i\frac{\pi}{2}}$ where $0 \leq x \leq R-\rho$, so:

$$\int_{L_2} f(z)dz = \int_{0}^{R-\rho} \frac{((R-x)e^{i\frac{\pi}{2}})^{\frac{1}{2}}}{((R-x)e^{i\frac{\pi}{2}})^4+1} dx$$

Then using the substitution $r = R-x$ with new bounds $0 \leq R-r \leq R-\rho$ (meaning $\rho \leq r \leq R$) we get:

$$\int_{L_2} f(z)dz = \int_{0}^{R-\rho} \frac{((R-x)e^{i\frac{\pi}{2}})^{\frac{1}{2}}}{((R-x)e^{i\frac{\pi}{2}})^4+1} dx = -\int_{\rho}^{R} \frac{(re^{i\frac{\pi}{2}})^{\frac{1}{2}}}{(re^{i\frac{\pi}{2}})^4+1} dr = -\int_{\rho}^{R} \frac{e^{i\frac{\pi}{4}}\sqrt{r}}{e^{2\pi i}r^4+1} dr = -e^{i\frac{\pi}{4}} \int_{\rho}^{R} \frac{\sqrt{r}}{r^4+1} dr$$

Therefore we have:

$$\lim_{R\to\infty,\rho\to 0} \int_C f(z)dz = \lim_{R\to\infty,\rho\to 0} \left(\int_{\gamma_\rho} f(z)dz + \int_{L_1} f(z)dz + \int_{\gamma_R} f(z)dz + \int_{L_2} f(z)dz \right)$$

$$= \lim_{R\to\infty,\rho\to 0} \int_{\gamma_\rho} f(z)dz + \lim_{R\to\infty,\rho\to 0} \int_{\gamma_R} f(z)dz + \lim_{R\to\infty,\rho\to 0} (1-e^{i\frac{\pi}{4}}) \int_{\rho}^{R} \frac{\sqrt{r}}{r^4+1} dr$$

$$= (1-e^{i\frac{\pi}{4}}) \int_{0}^{\infty} \frac{\sqrt{r}}{r^4+1} dr = 2\pi i Res_{z=\frac{1}{\sqrt{2}}(1+i)} f(z) = -\frac{\pi i}{2} e^{i\frac{3\pi}{8}}$$

Then by taking the imaginary part of both sides:

$$-sin(\frac{\pi}{4}) \int_{0}^{\infty} \frac{\sqrt{r}}{r^4+1} dr = -\frac{\pi}{2} cos(\frac{3\pi}{8})$$

So finally we have the result:

$$\int_{0}^{\infty} \frac{\sqrt{r}}{r^4+1} dr = \frac{\pi cos(\frac{3\pi}{8})}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}} sin(\frac{\pi}{8})$$

Problem 3

We are considering the integral:

$$\int_0^{2\pi} \frac{1}{2+cos\theta} d\theta$$

Using the contour C as the unit circle centered at the origin with the parameterization $z(\theta) = e^{i\theta}$ we get that

$cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$, and $\frac{dz}{d\theta} = ie^{i\theta} = iz$.

Therefore we know:

$$\int_0^{2\pi} \frac{1}{2+cos\theta} d\theta = \int_C \frac{1}{2+\frac{z+z^{-1}}{2}} \frac{dz}{iz} = -2i \int_C \frac{1}{4z+z^2+1} dz$$

Clearly $\frac{1}{z^2+4z+1}$ will have simple poles at the zeros of z^2+4z+1.

From a previous sample work we know $z = \frac{-4\pm\sqrt{4^2-4}}{2} = -2\pm\sqrt{3}$ are the zeros of z^2+4z+1.

Clearly the only one of these that will be interior to C is $-2+\sqrt{3}$.

Since this is a simple pole we know:

$Res_{z=-2+\sqrt{3}}\left(\frac{1}{z^2+4z+1}\right) = Res_{z=-2+\sqrt{3}}\left(\frac{1/(z-(-2-\sqrt{3}))}{z-(-2+\sqrt{3})}\right) = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$.

Since C is simple closed, and positively oriented we know:

$$\int_C \frac{1}{4z+z^2+1} dz = 2\pi i Res_{z=-2+\sqrt{3}}\left(\frac{1}{4z+z^2+1}\right) = 2\pi i\left(\frac{1}{2\sqrt{3}}\right) = \frac{\pi i}{\sqrt{3}}$$

Therefore we have:

$$\int_0^{2\pi} \frac{1}{2+cos\theta} d\theta = -2i \int_C \frac{1}{4z+z^2+1} dz = -2i\left(\frac{\pi i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$

□

Problem 4

Let $f(z) = \frac{1}{z^{\frac{1}{2}}(z^3+1)} = \frac{z^{\frac{1}{2}}}{z(z^3+1)}$, taking the branch $0 < arg\, z < 2\pi$. Clearly f has simple poles at all the third roots of -1.

If $z^3 = -1$ then $z = (-1)^{1/3} = e^{i\frac{\pi+2k\pi}{3}} = e^{i(\frac{\pi}{3}+k\frac{2\pi}{3})}$ where $k \in \{0,1,2\}$.

These roots are:

$z_1 = e^{i\frac{\pi}{3}}$, $z_2 = e^{i\pi}$, and $z_3 = e^{i\frac{5\pi}{3}}$

Since they are simple poles we know:

$Res_{z=z_n} \frac{1}{z^{\frac{1}{2}}(z^3+1)} = Res_{z=z_n} \frac{z^{\frac{1}{2}}}{z(z^3+1)} = \frac{z_n^{\frac{1}{2}}}{4z_n^3+1} = -\frac{z_n^{\frac{1}{2}}}{3}$.

Let $R > 1$ and $\rho < 1$ so that the poles of f will lie interior to our contour.

Let Γ_ρ be the upper half of the clockwise circle $|z| = \rho$.

Let Γ_R be the upper half of the counterclockwise circle $|z| = R$.

Let γ_ρ be the lower half of the clockwise circle $|z| = \rho$.

Let γ_R be the lower half of the counterclockwise circle $|z| = R$.

Let L_1 be the line from ρ to R on the real axis approached from above.

Let L_2 be the line from ρ to R on the real axis approached from below.

Clearly $C = \Gamma_\rho + L_1 + \Gamma_R + \gamma_R + L_2 + \gamma_\rho$ is a positively oriented simple closed contour.

Also f is analytic inside and on C except at the simple poles that are the three third roots of -1.

Therefore we know:

$$\int_C f(z)dz = \int_{\Gamma_\rho} f(z)dz + \int_{L_1} f(z)dz + \int_{\Gamma_R} f(z)dz + \int_{\gamma_R} f(z)dz + \int_{L_2} f(z)dz + \int_{\gamma_\rho} f(z)dz$$

$$= 2\pi i\Big(\sum_{n=1}^{3} Res_{z=z_n} \frac{1}{z^{\frac{1}{2}}(z^3+1)}\Big) = -\frac{2\pi i}{3}\Big(\sum_{n=1}^{3} z_n^{\frac{1}{2}}\Big) = -\frac{2\pi i}{3}(e^{i\frac{\pi}{6}} + e^{i\frac{\pi}{2}} + e^{i\frac{5\pi}{6}})$$

$$= \frac{2\pi}{3}\Big(sin(\frac{\pi}{6}) + sin(\frac{\pi}{2}) + sin(\frac{5\pi}{6})\Big) - \frac{2\pi i}{3}\Big(cos(\frac{\pi}{6}) + cos(\frac{\pi}{2}) + cos(\frac{\pi}{6})\Big) = \frac{4\pi}{3}$$

We know that $\left|\frac{1}{z^{\frac{1}{2}}(z^3+1)}\right| = \frac{1}{|z^{\frac{1}{2}}||z^3+1|} \leq \frac{1}{||z^3|-|1||\sqrt{|z|}} = \frac{1}{||z|^3-1|\sqrt{|z|}}$

Along Γ_R and γ_R we know $|z| = R$ and clearly they both have length πR.

Since $R > 1$ we know $\left|\frac{1}{z^{\frac{1}{2}}(z^3+1)}\right| \leq \frac{1}{||z|^3-1|\sqrt{|z|}} = \frac{1}{|R^3-1|\sqrt{R}} = \frac{1}{(R^3-1)\sqrt{R}}$ on Γ_R and γ_R.

Therefore we know:

$$\left|\int_{\Gamma_R} f(z)dz\right| \leq \frac{\pi\sqrt{R}}{R^3-1} \quad \text{and} \quad \left|\int_{\gamma_R} f(z)dz\right| \leq \frac{\pi\sqrt{R}}{R^3-1}$$

Since clearly $\frac{\pi\sqrt{R}}{R^3-1} \to 0$ as $R \to \infty$ we know that:

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0 \quad \text{and} \quad \lim_{R\to\infty} \int_{\gamma_R} f(z)dz = 0$$

Along Γ_ρ and γ_ρ we know $|z| = \rho$ and clearly they both have length $\pi\rho$.

Since $\rho < 1$ we know $\left|\frac{1}{z^{\frac{1}{2}}(z^3+1)}\right| \leq \frac{1}{||z|^3-1|\sqrt{|z|}} = \frac{1}{|\rho^3-1|\sqrt{\rho}} = \frac{1}{(1-\rho^3)\sqrt{\rho}}$ on Γ_ρ and γ_ρ.

Therefore we know:

$$\left|\int_{\Gamma_\rho} f(z)dz\right| \leq \frac{\pi\sqrt{\rho}}{1-\rho^3} \quad \text{and} \quad \left|\int_{\gamma_\rho} f(z)dz\right| \leq \frac{\pi\sqrt{\rho}}{1-\rho^3}$$

Since clearly $\frac{\pi\sqrt{\rho}}{1-\rho^3} \to 0$ as $\rho \to 0$ we know that:

$$\lim_{\rho\to 0}\int_{\Gamma_\rho} f(z)dz = 0 \quad \text{and} \quad \lim_{\rho\to 0}\int_{\gamma_\rho} f(z)dz = 0$$

Along L_1 we have the parameterization $z(r) = re^{0i}$ where $\rho \leq r \leq R$, so:

$$\int_{L_1} f(z)dz = \int_\rho^R \frac{1}{(re^{0i})^{\frac{1}{2}}((re^{0i})^3+1)}dr = \int_\rho^R \frac{1}{\sqrt{r}(r^3+1)}dr$$

Along L_2 we have the parameterization $z(x) = (R-x)e^{2\pi i}$ where $0 \leq x \leq R-\rho$, so:

$$\int_{L_2} f(z)dz = \int_0^{R-\rho} \frac{1}{((R-x)e^{2\pi i})^{\frac{1}{2}}(((R-x)e^{2\pi i})^3+1)}dx$$

Then using the substitution $r = R-x$ with new bounds $0 \leq R-r \leq R-\rho$ (meaning $\rho \leq r \leq R$) we get:

$$\int_{L_2} f(z)dz = \int_0^{R-\rho} \frac{1}{((R-x)e^{2\pi i})^{\frac{1}{2}}(((R-x)e^{2\pi i})^3+1)}dx = -\int_\rho^R \frac{1}{(re^{2\pi i})^{\frac{1}{2}}((re^{2\pi i})^3+1)}dr$$

$$= -\int_\rho^R \frac{1}{e^{\pi i}\sqrt{r}(e^{6\pi i}r^3+1)}dr = \int_\rho^R \frac{1}{\sqrt{r}(r^3+1)}dr$$

Therefore we have:

$$\lim_{R\to\infty,\rho\to 0}\int_C f(z)dz = \lim_{R\to\infty,\rho\to 0}\left(\int_{\Gamma_\rho} f(z)dz + \int_{L_1} f(z)dz + \int_{\Gamma_R} f(z)dz + \int_{\gamma_R} f(z)dz + \int_{L_2} f(z)dz + \int_{\gamma_\rho} f(z)dz\right)$$

$$= \lim_{R\to\infty,\rho\to 0}\int_{\Gamma_\rho} f(z)dz + \lim_{R\to\infty,\rho\to 0}\int_{\Gamma_R} f(z)dz + \lim_{R\to\infty,\rho\to 0}\int_{\gamma_R} f(z)dz + \lim_{R\to\infty,\rho\to 0}\int_{\gamma_\rho} f(z)dz + \lim_{R\to\infty,\rho\to 0} 2\int_\rho^R \frac{1}{\sqrt{r}(r^3+1)}dr$$

$$= 2\int_0^\infty \frac{1}{\sqrt{r}(r^3+1)}dr = 2\pi i\Big(\sum_{n=1}^{3} Res_{z=z_n}\frac{1}{z^{\frac{1}{2}}(z^3+1)}\Big) = \frac{4\pi}{3}$$

So finally we have the result:

$$\int_0^\infty \frac{1}{\sqrt{r}(r^3+1)}dr = \frac{2\pi}{3}$$

In order to do this formally we would need to use the same process as in problem 91.6 where we switch the branch cut, but doing that is essentially the same process it just takes longer.

Extra Problem

We are considering the integral below:

$$\int_0^1 \frac{1}{x^{\frac{2}{3}}(1-x)^{\frac{1}{3}}}dx$$

We can not easily use results from complex analysis yet so first I will use some substitutions.

First let $u = \frac{1}{x}$, then $du = -\frac{dx}{x^2}$ and our new bounds are ∞ and 1 since we approach 0 from the positive side:

$$\int_0^1 \frac{1}{x^{\frac{2}{3}}(1-x)^{\frac{1}{3}}}dx = \int_0^1 \frac{-x^2}{-x^2x^{\frac{2}{3}}(1-x)^{\frac{1}{3}}}dx = \int_0^1 \frac{-x^{\frac{4}{3}}}{-x^2(1-x)^{\frac{1}{3}}}dx = \int_\infty^1 \frac{-1}{u^{\frac{4}{3}}(1-\frac{1}{u})^{\frac{1}{3}}}du$$

Then after taking out the negative to switch the bounds of integration and simplifying:

$$\int_0^1 \frac{1}{x^{\frac{2}{3}}(1-x)^{\frac{1}{3}}}dx = \int_\infty^1 \frac{-1}{u^{\frac{4}{3}}(1-\frac{1}{u})^{\frac{1}{3}}}du = \int_1^\infty \frac{1}{(u^4(1-\frac{1}{u}))^{\frac{1}{3}}}du = \int_1^\infty \frac{1}{u(u-1)^{\frac{1}{3}}}du$$

Finally using the substitution $v = u - 1$ where $dv = du$ and our new bounds are 0 and ∞ we have:

$$\int_0^1 \frac{1}{x^{\frac{2}{3}}(1-x)^{\frac{1}{3}}}dx = \int_1^\infty \frac{1}{u(u-1)^{\frac{1}{3}}}du = \int_0^\infty \frac{1}{(v+1)v^{\frac{1}{3}}}dv$$

Now the process is similar to the previous problem.

Let $f(z) = \frac{1}{z^{\frac{1}{3}}(z+1)}$, taking the branch $0 < arg\, z < 2\pi$. Clearly f has a simple pole at $z = -1$.

Since it is a simple pole we know:

$$Res_{z=-1}\frac{1}{z^{\frac{1}{3}}(z+1)} = \frac{1}{(-1)^{\frac{1}{3}}} = \frac{1}{(e^{i\pi})^{\frac{1}{3}}} = \frac{1}{e^{i\frac{\pi}{3}}} = e^{-i\frac{\pi}{3}} = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Let $R > 1$ and $\rho < 1$ so that the pole of f will lie interior to our contour.

Let Γ_ρ be the upper half of the clockwise circle $|z| = \rho$.

Let Γ_R be the upper half of the counterclockwise circle $|z| = R$.

Let γ_ρ be the lower half of the clockwise circle $|z| = \rho$.

Let γ_R be the lower half of the counterclockwise circle $|z| = R$.

Let L_1 be the line from ρ to R on the real axis approached from above.

Let L_2 be the line from ρ to R on the real axis approached from below.

Clearly $C = \Gamma_\rho + L_1 + \Gamma_R + \gamma_R + L_2 + \gamma_\rho$ is a positively oriented simple closed contour.

Also f is analytic inside and on C except at the simple pole -1 of f.

Therefore we know:

$$\int_C f(z)dz = \int_{\Gamma_\rho} f(z)dz + \int_{L_1} f(z)dz + \int_{\Gamma_R} f(z)dz + \int_{\gamma_R} f(z)dz + \int_{L_2} f(z)dz + \int_{\gamma_\rho} f(z)dz$$

$$= 2\pi i Res_{z=-1}\frac{1}{z^{\frac{1}{3}}(z+1)} = 2\pi i(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = \pi(\sqrt{3} + i)$$

We know that $\left|\frac{1}{z^{\frac{1}{3}}(z+1)}\right| = \frac{1}{|z^{\frac{1}{3}}||z+1|} \leq \frac{1}{|z|^{\frac{1}{3}}||z|-|1||} = \frac{1}{|z|^{\frac{1}{3}}||z|-1|}$

Along Γ_R and γ_R we know $|z| = R$ and clearly they both have length πR.

Since $R > 1$ we know $\left|\frac{1}{z^{\frac{1}{3}}(z+1)}\right| \leq \frac{1}{|z|^{\frac{1}{3}}||z|-1|} = \frac{1}{R^{\frac{1}{3}}|R-1|} = \frac{1}{R^{\frac{1}{3}}(R-1)}$ on Γ_R and γ_R.

Therefore we know:

$$\left|\int_{\Gamma_R} f(z)dz\right| \leq \frac{\pi R^{\frac{2}{3}}}{R-1} \quad \text{and} \quad \left|\int_{\gamma_R} f(z)dz\right| \leq \frac{\pi R^{\frac{2}{3}}}{R-1}$$

Since clearly $\frac{\pi R^{\frac{2}{3}}}{R-1} \to 0$ as $R \to \infty$ we know that:

$$\lim_{R\to\infty}\int_{\Gamma_R} f(z)dz = 0 \quad \text{and} \quad \lim_{R\to\infty}\int_{\gamma_R} f(z)dz = 0$$

Along Γ_ρ and γ_ρ we know $|z| = \rho$ and clearly they both have length $\pi\rho$.

Since $\rho < 1$ we know $\left|\frac{1}{z^{\frac{1}{3}}(z+1)}\right| \leq \frac{1}{|z|^{\frac{1}{3}}||z|-1|} = \frac{1}{\rho^{\frac{1}{3}}|\rho-1|} = \frac{1}{\rho^{\frac{1}{3}}(1-\rho)}$ on Γ_ρ and γ_ρ.

Therefore we know:

$$\left|\int_{\Gamma_\rho} f(z)dz\right| \leq \frac{\pi \rho^{\frac{2}{3}}}{1-\rho} \quad \text{and} \quad \left|\int_{\gamma_\rho} f(z)dz\right| \leq \frac{\pi \rho^{\frac{2}{3}}}{1-\rho}$$

Since clearly $\frac{\pi\rho^{\frac{2}{3}}}{1-\rho} \to 0$ as $\rho \to 0$ we know that:

$$\lim_{\rho\to 0}\int_{\Gamma_\rho} f(z)dz = 0 \quad \text{and} \quad \lim_{\rho\to 0}\int_{\gamma_\rho} f(z)dz = 0$$

Along L_1 we have the parameterization $z(r) = re^{0i}$ where $\rho \leq r \leq R$, so:

$$\int_{L_1} f(z)dz = \int_\rho^R \frac{1}{(re^{0i})^{\frac{1}{3}}(re^{0i}+1)}dr = \int_\rho^R \frac{1}{r^{\frac{1}{3}}(r+1)}dr$$

Along L_2 we have the parameterization $z(x) = (R-x)e^{2\pi i}$ where $0 \leq x \leq R-\rho$, so:

$$\int_{L_2} f(z)dz = \int_0^{R-\rho} \frac{1}{((R-x)e^{2\pi i})^{\frac{1}{3}}((R-x)e^{2\pi i}+1)}dx$$

Then using the substitution $r = R-x$ with new bounds $0 \leq R-r \leq R-\rho$ (meaning $\rho \leq r \leq R$) we get:

$$\int_{L_2} f(z)dz = \int_0^{R-\rho} \frac{1}{((R-x)e^{2\pi i})^{\frac{1}{3}}((R-x)e^{2\pi i}+1)}dx = -\int_\rho^R \frac{1}{(re^{2\pi i})^{\frac{1}{3}}(re^{2\pi i}+1)}dr$$

$$= -\int_\rho^R \frac{1}{e^{i\frac{2\pi}{3}}r^{\frac{1}{3}}(r+1)}dr = -e^{-i\frac{2\pi}{3}}\int_\rho^R \frac{1}{r^{\frac{1}{3}}(r+1)}dr = (\frac{1}{2}+i\frac{\sqrt{3}}{2})\int_\rho^R \frac{1}{r^{\frac{1}{3}}(r+1)}dr$$

Therefore we have:

$$\lim_{R\to\infty,\rho\to 0}\int_C f(z)dz = \lim_{R\to\infty,\rho\to 0}\left(\int_{\Gamma_\rho} f(z)dz + \int_{L_1} f(z)dz + \int_{\Gamma_R} f(z)dz + \int_{\gamma_R} f(z)dz + \int_{L_2} f(z)dz + \int_{\gamma_\rho} f(z)dz\right)$$

$$= \lim_{R\to\infty,\rho\to 0}\int_{\Gamma_\rho} f(z)dz + \lim_{R\to\infty,\rho\to 0}\int_{\Gamma_R} f(z)dz + \lim_{R\to\infty,\rho\to 0}\int_{\gamma_R} f(z)dz + \lim_{R\to\infty,\rho\to 0}\int_{\gamma_\rho} f(z)dz + \lim_{R\to\infty,\rho\to 0}(\frac{3}{2}+i\frac{\sqrt{3}}{2})\int_\rho^R \frac{1}{r^{\frac{1}{3}}(r+1)}dr$$

$$= (\frac{3}{2}+i\frac{\sqrt{3}}{2})\int_0^\infty \frac{1}{r^{\frac{1}{3}}(r+1)}dr = 2\pi i Res_{z=-1}\frac{1}{z^{\frac{1}{3}}(z+1)} = \pi(\sqrt{3}+i)$$

Then taking the imaginary part of both sides we have:

$$\frac{\sqrt{3}}{2}\int_0^\infty \frac{1}{r^{\frac{1}{3}}(r+1)}dr = \pi$$

So finally we have the result:

$$\int_0^1 \frac{1}{x^{\frac{2}{3}}(1-x)^{\frac{1}{3}}}dx = \int_0^\infty \frac{1}{r^{\frac{1}{3}}(r+1)}dr = \frac{2\pi}{\sqrt{3}}$$