The Cauchy Integral Formula

Matthew Seguin

57.1

Recall that if a function f is analytic inside and on a simple closed contour C (taken in the positive sense) then if z_0 is any point interior to C we know:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

From which it follows:

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz = i\frac{2\pi f^{(n)}(z_0)}{n!}$$

Where $n \in \{0, 1, 2, ...\}$ and $f^{(n)}(z_0)$ is the nth derivative of f at z_0.

Let C be the positively oriented boundary of the square whose sides lie on $x = \pm 2$ and $y = \pm 2$.

Clearly C is simple and closed.

a. Let $f(z) = e^{-z}$. Since $g(z) = e^z$ and $h(z) = -z$ are entire we know f is entire and hence analytic inside and on C.

Therefore we know for any z_0 interior to C and $n \in \{0, 1, 2, ...\}$ we may use the Cauchy Integral Formula extension.

Clearly since $\frac{\pi}{2} < 2$ we have that $z_0 = \frac{\pi i}{2}$ is interior to C.

So we know:

$$\int_C \frac{e^{-z}}{(z-\frac{\pi i}{2})}dz = \int_C \frac{f(z)}{(z-z_0)^{0+1}}dz = i\frac{2\pi f^{(0)}(z_0)}{0!} = 2\pi i f(z_0) = 2\pi i e^{-\frac{\pi i}{2}} = 2\pi i(-i) = 2\pi$$

□

b. Let $f(z) = \frac{cos\ z}{z^2+8}$. Recall that $g(z) = cos\ z$ and $h(z) = z^2 + 8$ are entire.

Since $\sqrt{8} > 2$ we have that $z^2 + 8 \neq 0$ inside or on C (because $z \neq \pm i\sqrt{8}$), so f is analytic inside and on C.

Therefore we know for any z_0 interior to C and $n \in \{0, 1, 2, ...\}$ we may use the Cauchy Integral Formula extension.

Clearly $z_0 = 0$ is obviously interior to C.

So we know:

$$\int_C \frac{cos\ z}{z(z^2+8)}dz = \int_C \frac{f(z)}{(z-z_0)^{0+1}}dz = i\frac{2\pi f^{(0)}(z_0)}{0!} = 2\pi i f(z_0) = 2\pi i\frac{cos(0)}{0^2+8} = \frac{\pi i}{4}$$

□

d. Let $f(z) = cosh\ z = \frac{e^z + e^{-z}}{2}$. Then we know f is entire and hence analytic inside and on C.

Therefore we know for any z_0 interior to C and $n \in \{0, 1, 2, ...\}$ we may use the Cauchy Integral Formula extension.

Clearly $z_0 = 0$ is interior to C.

Now recall $\frac{d}{dz} cosh\ z = sinh\ z$ and $\frac{d}{dz} sinh\ z = cosh\ z$.

Then $\frac{d^3}{dz^3} cosh\ z = \frac{d^2}{dz^2}(\frac{d}{dz} cosh\ z) = \frac{d^2}{dz^2} sinh\ z = \frac{d}{dz}(\frac{d}{dz} sinh\ z) = \frac{d}{dz} cosh\ z = sinh\ z = \frac{e^z - e^{-z}}{2}$.

So we know:

$$\int_C \frac{cosh\ z}{z^4} dz = \int_C \frac{f(z)}{(z - z_0)^{3+1}} dz = i\frac{2\pi f^{(3)}(z_0)}{3!} = \frac{\pi i}{3} sinh\ z_0 = \frac{\pi i}{3}\left(\frac{e^0 - e^{-0}}{2}\right) = 0$$

□

57.4

Recall that if a function f is analytic inside and on a simple closed contour C (taken in the positive sense) then if z_0 is any point interior to C we know:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

From which it follows:

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}dz = i\frac{2\pi f^{(n)}(z_0)}{n!}$$

Where $n \in \{0, 1, 2, ...\}$ and $f^{(n)}(z_0)$ is the nth derivative of f at z_0.

Let C be any simple closed contour (taken in the positive sense) on the complex plane.

Let $f(z) = z^3 + 2z$, clearly f is entire and hence analytic inside and on C. Then let:

$$g(z) = \int_C \frac{s^3+2s}{(s-z)^3}ds = \int_C \frac{f(s)}{(s-z)^3}ds$$

- If z is interior to C:

If z is interior to C then we may use the Cauchy Integral Formula extension.

Note that $\frac{d^2}{dz^2}f(z) = \frac{d^2}{dz^2}(z^3+2z) = \frac{d}{dz}(\frac{d}{dz}(z^3+2z)) = \frac{d}{dz}(3z^2+2) = 6z$.

So we know:

$$g(z) = \int_C \frac{s^3+2s}{(s-z)^3}ds = \int_C \frac{f(s)}{(s-z)^3}ds = \int_C \frac{f(s)}{(s-z)^{2+1}}ds = i\frac{2\pi f^{(2)}(z)}{2!} = 6\pi i z$$

$\square$

- If z is exterior to C:

If z is exterior to C then we know $s - z \neq 0$ inside or on C and hence $(s-z)^3 \neq 0$ inside or on C.

Note that $h(s) = (s-z)^3$ is entire and hence analytic inside and on C.

Since we already know $f(s) = s^3 + 2s$ is analytic inside and on C, so is $h(s) = (s-z)^3$, and $h(s) \neq 0$ inside or on C we know that $\frac{s^3+2s}{(s-z)^3} = \frac{f(s)}{h(s)}$ is analytic inside and on C.

Then since C is a simple closed contour we know via the Cauchy-Goursat Theorem:

$$g(z) = \int_C \frac{s^3+2s}{(s-z)^3}ds = 0$$

$\square$

57.5

Recall the Cauchy Integral Formula's extension which has been stated in the previous two problems.

Let C be a simple closed contour, f be a function that is analytic inside and on C, and z_0 be a point not on C.

Then since f is analytic inside and on C so is f', the derivative of f.

- If z_0 is interior to C:

First take C in the positive sense (call it C^+), then since z_0 is interior to C then we may use the Cauchy Integral Formula extension:

$$\int_{C^+} \frac{f'(z)}{(z-z_0)}dz = \int_{C^+} \frac{f'(z)}{(z-z_0)^{0+1}}dz = i\frac{2\pi f'^{(0)}(z_0)}{0!} = 2\pi i f'(z_0)$$

$$\int_{C^+} \frac{f(z)}{(z-z_0)^2}dz = \int_{C^+} \frac{f'(z)}{(z-z_0)^{1+1}}dz = i\frac{2\pi f^{(1)}(z_0)}{1!} = 2\pi i f'(z_0)$$

Now take C to be in the negative sense (call it C^-):

$$\int_{C^-} \frac{f'(z)}{(z-z_0)}dz = \int_{-C^+} \frac{f'(z)}{(z-z_0)}dz = -\int_{C^+} \frac{f'(z)}{(z-z_0)}dz = -2\pi i f'(z_0)$$

$$\int_{C^-} \frac{f(z)}{(z-z_0)^2}dz = \int_{-C^+} \frac{f(z)}{(z-z_0)^2}dz = -\int_{C^+} \frac{f(z)}{(z-z_0)^2}dz = -2\pi i f'(z_0)$$

Therefore when z_0 is interior to C, we have that in either orientation of C:

$$\int_C \frac{f'(z)}{(z-z_0)}dz = \int_C \frac{f(z)}{(z-z_0)^2}dz$$

- If z is exterior to C:

If z_0 is exterior to C then $z - z_0 \neq 0$ inside or on C.

Since we already know $f(z)$ and $f'(z)$ are analytic inside and on C, and $z - z_0$ and $(z-z_0)^2$ are analytic and nonzero inside and on C, both $\frac{f'(z)}{z-z_0}$ and $\frac{f(z)}{(z-z_0)^2}$ are analytic inside and on C.

Then since C is a simple closed contour we know via the Cauchy-Goursat Theorem:

$$\int_C \frac{f'(z)}{(z-z_0)}dz = 0 = \int_C \frac{f(z)}{(z-z_0)^2}dz$$

Therefore if C is a simple closed contour, f is analytic inside and on C, and z_0 is not on C (meaning it must be interior or exterior to C) then:

$$\int_C \frac{f'(z)}{(z-z_0)}dz = \int_C \frac{f(z)}{(z-z_0)^2}dz$$

$\square$

59.7

Recall that if a function f is continuous on a closed bounded region R and it is also analytic and non-constant in the interior of R then the maximum value of $|f(z)|$ (which will exist since f is continuous over R which is closed and bounded) occurs somewhere on the boundary of R and never in the interior.

Let $f(z) = u(x, y) + iv(x, y)$ be a continuous function over a closed and bounded region R, and suppose f is analytic and non-constant on the interior of R.

- Showing v attains a maximum on the boundary of R:

$$\text{Let } g(z) = e^{-i\,f(z)} = e^{-i(u(x,y)+iv(x,y))} = e^{v(x,y)-iu(x,y)} = e^{v(x,y)}e^{-iu(x,y)}, \text{ then}$$
$$|g(z)| = |e^{v(x,y)}e^{-iu(x,y)}| = |e^{v(x,y)}||e^{-iu(x,y)}| = e^{v(x,y)}.$$

Since e^w is entire and f is analytic and non-constant on the interior of R we know g is analytic and non-constant on the interior of R. Since e^w is continuous everywhere and f is continuous over R we know that g is continuous over R. The conditions for the above theorem are satisfied and we may use it on g.

So we know that $|g(z)| = e^{v(x,y)}$ attains its maximum on the boundary of R and never in the interior of R. Therefore since the real function e^t is strictly increasing we know that this must mean $v(x, y)$ attains its maximum on the boundary of R and never in the interior of R.

- Showing v attains a minimum on the boundary of R:

$$\text{Let } h(z) = \frac{1}{g(z)} = \frac{1}{e^{-i\,f(z)}} = \frac{1}{e^{v(x,y)}e^{-iu(x,y)}}, \text{ then } |h(z)| = |\frac{1}{g(z)}| = \frac{1}{|g(z)|} = \frac{1}{e^{v(x,y)}}.$$
$$\text{Note that } g(z) = e^{-if(z)} \neq 0 \text{ since } e^w \neq 0 \text{ for all } w \in \mathbb{C}.$$

We know from before that $g(z)$ is analytic and non-constant on the interior of R, and also that it is continuous over R. Since $g(z) \neq 0$ anywhere over R we know h is analytic and non-constant on the interior of R. Similarly since $g(z) \neq 0$ anywhere over R we know h is continuous over R.

The conditions for the above theorem are satisfied and we may use it on h.

So we know that $|h(z)| = \frac{1}{e^{v(x,y)}}$ attains its maximum on the boundary of R and never in the interior of R, which means that $e^{v(x,y)}$ attains its minimum on the boundary of R and never in the interior of R.

Therefore since the real function e^t is strictly increasing we know that this must mean $v(x, y)$ attains its minimum on the boundary of R and never in the interior of R.

So if $f(z) = u(x, y) + iv(x, y)$ is a continuous function over a closed and bounded region R where f is analytic and non-constant on the interior of R then the component function $v(x, y)$ attains both a maximum and minimum value on the boundary of R and never in the interior of R $\square$

Problem 2

Recall that if a function is entire and bounded (in modulus) on all of $\mathbb{C}$ then it is constant.

Assume that f is entire and that there exists an $M > 0$ such that $|f(z)| > M$ for all $z \in \mathbb{C}$.

Then since $|f(z)| > M > 0$ for all $z \in \mathbb{C}$ we know that $|f(z)| > 0$ and hence $f(z) \neq 0$ for all $z \in \mathbb{C}$.

Therefore the function $g(z) = \frac{1}{f(z)}$ is well defined and is also entire.

Furthermore we know $|g(z)| = |\frac{1}{f(z)}| = \frac{1}{|f(z)|} < \frac{1}{M}$ for all $z \in \mathbb{C}$ (since $|f(z)| > M$).

Since $g(z)$ is entire and bounded (in modulus) on all of $\mathbb{C}$ we know it must be constant.

That is $g(z) = \frac{1}{f(z)} = c$ for some $c \in \mathbb{C}$ (note that $|g(z)| = \frac{1}{|f(z)|} > 0$ and hence $c \neq 0$).

Which implies $f(z) = \frac{1}{c}$ for some $c \in \mathbb{C}$ (which is well defined because $c \neq 0$).

Therefore if f is entire and there exists an $M > 0$ such that $|f(z)| > M$ for all $z \in \mathbb{C}$, then it follows that f is constant $\square$