Compact Sets

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3.2.5

Recall that x is a limit point of A if and only if $x = lim\ a_n$ for some sequence (a_n) contained in A such that $a_n \neq x$ for all $n \in \mathbb{N}$.

- Proving if $F \subseteq \mathbb{R}$ is closed then every Cauchy sequence contained in F has its limit point in F:

  Assume that $F \subseteq \mathbb{R}$ is closed.

  Since F is closed it contains all its limit points. Let (a_n) be a Cauchy sequence contained in F.

  Then $(a_n) \to a$ for some $a \in \mathbb{R}$ since all Cauchy sequences converge.

  If $a_n \neq a$ for all $n \in \mathbb{N}$ then since (a_n) is contained in F, a is a limit point of F.

  Since F contains all its limit points $a \in F$.

  If $a_n = a$ for some $n \in \mathbb{N}$ then $a \in F$ since the sequence (a_n) is contained in F.

  Therefore if $F \subseteq \mathbb{R}$ is closed then every Cauchy sequence contained in F has its limit point in F.

- Proving if every Cauchy sequence contained in F has its limit point in F then $F \subseteq \mathbb{R}$ is closed:

  Assume that every Cauchy sequence contained in F has its limit point in F.

  Consider an arbitrary limit point a of F.

  Then for some sequence (a_n) contained in F such that $a_n \neq a$ for all $n \in \mathbb{N}$ it must be that $(a_n) \to a$.

  Since this (a_n) converges it is a Cauchy sequence, so by assumption we also have that $a \in F$.

  This was for an arbitrary limit point of F and is therefore true for all limit points of F.

  Therefore if every Cauchy sequence contained in F has its limit point in F then $F \subseteq \mathbb{R}$ is closed.

3.2.14

The interior of a set E is $E^o = \{x \in E : \exists V_\epsilon(x) \subseteq E\}$.

a. Let E be a set and L_E be the set of all the limit points of E.

Showing E is closed if and only if $\overline{E} = E$:

If E is closed then $L_E \subseteq E$ because E contains its limit points, so $\overline{E} = L_E \cup E = E$.

If $\overline{E} = L_E \cup E = E$ then $L_E \subseteq E$ so E contains its limit points and is closed.

So E is closed if and only if $\overline{E} = E$ $\square$

Showing E is open if and only if $E^o = E$:

If E is open then every $x \in E$ has some $V_\epsilon(x) \subseteq E$ and therefore $E^o = E$.

If $E^o = E$ then every $x \in E$ has some $V_\epsilon(x) \subseteq E$ and therefore E is open.

So E is closed if and only if $E^o = E$ $\square$

b. Let E be a set and L_E be the set of all the limit points of E.

Showing $\overline{E}^c = (E^c)^o$:

Let $x \in \overline{E}^c$ then $x \notin \overline{E} = E \cup L_E$. So $x \notin E$ and $x \notin L_E$, therefore $x \in E^c$ and $x \in (L_E)^c$.

So x is not a limit point of E and therefore there does not exist a $V_\epsilon(x) \subseteq E$.

Therefore, there does exist a $V_\epsilon(x) \subseteq E^c$, so $x \in (E^c)^o$.

So $\overline{E}^c \subseteq (E^c)^o$.

Let $x \in (E^c)^o$ then there exists a $V_\epsilon(x) \subseteq E^c$.

Then $x \in E^c$ and there exists a $V_\epsilon(x) \subseteq E^c$.

Therefore, there does not exist a $V_\epsilon(x) \subseteq E$.

So x is not a limit point of E, so $x \notin E$ and $x \notin L_E$.

Therefore $x \notin E \cup L_E = \overline{E}$, and $x \in \overline{E}^c$.

So $(E^c)^o \subseteq \overline{E}^c$.

Therefore $\overline{E}^c = (E^c)^o$ $\square$

Showing $(E^o)^c = \overline{E^c}$:

Let $x \in (E^o)^c$, then $x \notin E^o$. So there does not exist a $V_\epsilon(x) \subseteq E$.

Therefore there does exist a $V_\epsilon(x) \subseteq E^c$. So x is a limit point of E^c.

So $x \in L_{(E^c)}$ therefore $x \in L_{(E^c)} \cup E^c = \overline{E^c}$.

So $(E^o)^c \subseteq \overline{E^c}$.

Let $x \in \overline{E^c} = E^c \cup L_{(E^c)}$, then $x \in E^c$ or $x \in L_{(E^c)}$.

So $x \notin E$ or there exists a $V_\epsilon(x) \subseteq E^c$. So $x \notin E$ or there does not exist a $V_\epsilon(x) \subseteq E$.

Therefore $x \notin E^o$, and $x \in (E^o)^c$.

So $\overline{E^c} \subseteq (E^o)^c$.

Therefore $(E^o)^c = \overline{E^c}$ $\square$

3.3.1

Let $K \subset R$ be a compact, nonempty set. Then K is closed and bounded.

Since K is bounded there exists an upper bound and a lower bound of K.

Therefore since $\mathbb{R}$ has the least upper bound property, the least upper bound and greatest lower bound of K exist in $\mathbb{R}$.

- Proving $supK \in K$:

  Say $supK = x$, then for all $\epsilon > 0$ there exists an $a \in K$ such that $x - \epsilon < a$.

  So for all $n \in \mathbb{N}$ there exists an $a_n \in K$ such that $x - \frac{1}{n} < a_n$.

  Consider the sequences $(x - a_n)$ and $(\frac{1}{n})$ then $0 < x - a_n < \frac{1}{n}$ for all $n \in \mathbb{N}$.

  As shown in previous sample works, the sequences (0) and $(\frac{1}{n})$ both converge to 0.

  Therefore by the squeeze theorem $(x - a_n) \to 0$.

  Clearly the sequence $(x) \to x$ so by the algebraic limit theorem $(a_n) \to x = supK$.

  So we have found a sequence contained in K such that its limit is $supK$, so $supK$ is a limit point of K.

  Therefore since K is closed it contains its limit points, and so $supK \in K$ $\square$

- Proving $infK \in K$:

  Say $infK = x$, then for all $\epsilon > 0$ there exists an $a \in K$ such that $x + \epsilon > a$.

  So for all $n \in \mathbb{N}$ there exists an $a_n \in K$ such that $x + \frac{1}{n} > a_n \geq x$.

  Consider the sequences $(x + \frac{1}{n})$ and (x) then $x + \frac{1}{n} > a_n \geq x$ for all $n \in \mathbb{N}$.

  As shown in previous sample works, the sequence $(\frac{1}{n})$ converges to 0. Clearly the sequence $(x) \to x$.

  By the algebraic limit theorem $(x + \frac{1}{n}) \to x$, so by the squeeze theorem $(a_n) \to x$.

  So we have found a sequence contained in K such that its limit is $infK$, so $infK$ is a limit point of K.

  Therefore since K is closed it contains its limit points, and so $infK \in K$ $\square$

3.3.9

Let K be a compact set and let this imply K is closed and bounded.

Let $\{O_\lambda : \lambda \in \Lambda\}$ be an open cover of K and assume no finite subcover exists.

Let I_0 be a closed interval containing K, such an interval exists since K is bounded.

a. Then $I_0 \cap K = K$ can not be finitely covered.

Bisect I_0, then either the left half of $I_0 \cap K$ or the right half of $I_0 \cap K$ can not be finitely covered.

Otherwise if both can be finitely covered then the union of those finite covers, which is a finite cover would cover $I_0 \cap K$ which can not happen.

Let I_1 be the half that can not be finitely covered, if both can then just pick either.

Then bisect I_1 and again we have the same process one of the two halves can not be finitely covered.

Repeat this, then $I_n \cap K$ can not be finitely covered for all $n \in \{0, 1, 2, ...\} = \{0\} \cup \mathbb{N}$.

Furthermore $I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$

And if the original length of I_0 is $|I_0| = l$ then the length of I_n is $|I_n| = \frac{l}{2^n}$.

So $(|I_n|) = (\frac{l}{2^n}) \to 0$ by the algebraic limit theorem and the fact that $(\frac{1}{2^n}) \to 0$.

b. Since K is compact so is $K \cap I_n$ for all $n \in \{0\} \cup \mathbb{N}$.

We also know $I_0 \supseteq I_1 \supseteq I_2 \supseteq ...$, so $I_0 \cap K \supseteq I_1 \cap K \supseteq I_2 \cap K \supseteq ...$

Therefore $\cap_{n=0}^{\infty} K \cap I_n \neq \phi$ since the arbitrary intersection of nested compact sets is nonempty.

So there exists an element in K that is in every I_n.

c. Since $x \in K$ there must be some open set O_{λ_0} such that $x \in O_{\lambda_0}$.

However, since O_{λ_0} is open there must exist some $\epsilon > 0$ such that $V_\epsilon(x) \subseteq O_{\lambda_0}$.

Since $(|I_n|) \to 0$ we can find an $N \in \mathbb{N}$ such that $|I_n| < \epsilon$ for all $n \geq N$.

Then O_{λ_0} contains I_n for all $n \geq N$.

But this implies that I_n can be finitely covered for all $n \geq N$, a contradiction.

Therefore our assumption that K can not be finitely covered must be false.

So for a compact set K, K can be finitely covered $\square$

3.2.12

Let A be an uncountable set and let $s \in B$ if $\{x \in \mathbb{R} : x \in A, x < s\}$ and $\{x \in \mathbb{R} : x \in A, x > s\}$ are uncountable.

For some $s \in \mathbb{R}$ let $L_s = \{x \in \mathbb{R} : x \in A, x < s\} = (-\infty, s) \cap A$ and $R_s = \{x \in \mathbb{R} : x \in A, x > s\} = (s, \infty) \cap A$.

Now let $T_1 = \{s \in \mathbb{R} : L_s \;\; is \;\; uncountable\}$ and $T_2 = \{s \in \mathbb{R} : R_s \;\; is \;\; uncountable\}$.

Recall that the countable union of countable or finite sets is countable.

Let (a_n) be a positive, monotonically decreasing sequence that converges to 0.

Then $(a_n + s) \to s$ for all $s \in \mathbb{R}$ by the algebraic limit theorem.

- Proving T_1 is nonempty and open:

  T_1 **is nonempty:**

  Assume T_1 is empty, that is $L_s = (-\infty, s) \cap A$ is countable or finite for all $s \in \mathbb{R}$.

  Then $L_n = (-\infty, n) \cap A$ is countable or finite for all $n \in \mathbb{N}$.

  This implies $\cup_{n=1}^{\infty} L_n = \cup_{n=1}^{\infty}(-\infty, n) \cap A$ is countable.

  However, $\cup_{n=1}^{\infty}(-\infty, n) \cap A = A \cap (\cup_{n=1}^{\infty}(-\infty, n)) = A \cap (-\infty, \infty) = A$ is uncountable.

  So it must be that T_1 is nonempty.

  T_1 **is open:**

  $T_1 \neq \phi$ from above. So let $s \in T_1$, then L_s is uncountable.

  Clearly for any $t > s$, $t \in T_1$ because $L_s = (-\infty, s) \cap A \subseteq (-\infty, t) \cap A = L_t$.

  And $L_s = (-\infty, s) \cap A$ is uncountable so $L_t = (-\infty, t) \cap A$ is uncountable, hence $t \in T_1$.

  Furthermore there exists some $\epsilon > 0$ such that $s - \epsilon \in T_1$.

  Otherwise $(-\infty, s - a_n) \cap A$ is countable or finite for all $n \in \mathbb{N}$.

  But this would imply $\cup_{n=1}^{\infty}(-\infty, s - a_n) \cap A$ is countable.

  However, $\cup_{n=1}^{\infty}(-\infty, s - a_n) \cap A = A \cap (\cup_{n=1}^{\infty}(-\infty, s - a_n)) = A \cap (-\infty, s) = L_s$ is uncountable since $s \in T_1$.

  Therefore for all $s \in T_1$ there exists some $\epsilon > 0$ such that $s - \epsilon \in T_1$.

  We have shown that for any $x \in T_1$ if $y > x$ then $y \in T_1$ and for all $s \in T_1$ there exists an $\epsilon > 0$ such that $s - \epsilon \in T_1$.

  Consequently we have shown that for all $s \in T_1$ there exists an $\epsilon > 0$ such that for all $t \geq s - \epsilon$, $t \in T_1$.

  So for all $s \in T_1$ there exists a $V_\epsilon(s) \subseteq T_1$.

  So T_1 is open.

- Proving T_2 is nonempty and open:

  T_2 **is nonempty:**

  Assume T_2 is empty, that is $R_s = (s, \infty) \cap A$ is countable or finite for all $s \in \mathbb{R}$.

  Then $R_{-n} = (-n, \infty) \cap A$ is countable or finite for all $n \in \mathbb{N}$.

  This implies $\cup_{n=1}^{\infty} R_{-n} = \cup_{n=1}^{\infty}(-n, \infty) \cap A$ is countable.

  However, $\cup_{n=1}^{\infty}(-n, \infty) \cap A = A \cap (\cup_{n=1}^{\infty}(-n, \infty)) = A \cap (-\infty, \infty) = A$ is uncountable.

  So it must be that T_2 is nonempty.

T_2 **is open:**

$T_2 \neq \phi$ from above. So let $s \in T_2$, then R_s is uncountable.

Clearly for any $t < s$, $t \in T_2$ because $R_s = (s, \infty) \cap A \subseteq (t, \infty) \cap A = R_t$.

And $R_s = (s, \infty) \cap A$ is uncountable so $R_t = (t, \infty) \cap A$ is uncountable, hence $t \in T_2$.

Furthermore there exists some $\epsilon > 0$ such that $s + \epsilon \in T_2$.

Otherwise $(s + a_n, \infty) \cap A$ is countable or finite for all $n \in \mathbb{N}$.

But this would imply $\cup_{n=1}^{\infty}(s + a_n, \infty) \cap A$ is countable.

However, $\cup_{n=1}^{\infty}(s + a_n, \infty) \cap A = A \cap (\cup_{n=1}^{\infty}(s + a_n, \infty)) = A \cap (s, \infty) = R_s$ is uncountable since $s \in T_2$.

Therefore for all $s \in T_2$ there exists some $\epsilon > 0$ such that $s + \epsilon \in T_2$.

We have shown that for any $x \in T_2$ if $y < x$ then $y \in T_2$ and for all $s \in T_2$ there exists an $\epsilon > 0$ such that $s + \epsilon \in T_2$.

Consequently we have shown that for all $s \in T_2$ there exists an $\epsilon > 0$ such that for all $t \leq s + \epsilon$, $t \in T_2$.

So for all $s \in T_2$ there exists a $V_\epsilon(s) \subseteq T_2$.

So T_2 is open.

- Proving $B = T_1 \cap T_2$ is nonempty and open:

  B **is nonempty:**

  We have shown that T_1 is nonempty and open and that for $s \in T_1$ if $t > s$ then it must be that $t \in T_1$.

  So T_1 is of the form $T_1 = (t_1, \infty)$ for some $t_1 \in \mathbb{R}$ or $T_1 = (-\infty, \infty) = \mathbb{R}$.

  We have shown that T_2 is nonempty and open and that for $s \in T_2$ if $t < s$ then it must be that $t \in T_2$.

  So T_2 is of the form $T_2 = (-\infty, t_2)$ for some $t_2 \in \mathbb{R}$ or $T_2 = (-\infty, \infty) = \mathbb{R}$.

  If $T_1 = \mathbb{R}$ or $T_2 = \mathbb{R}$ then $B = T_1 \cap T_2 = \mathbb{R} \cap T_2 = T_2 \neq \phi$ or $B = T_1 \cap T_2 = T_1 \cap \mathbb{R} = T_1 \neq \phi$ and we would be done.

  Otherwise we want to show $B = T_1 \cap T_2 = (t_1, \infty) \cap (-\infty, t_2) = (-\infty, t_2) \cap (t_1, \infty)$ is nonempty.

  So we want to show that $t_1 < t_2$.

  For all $x \in \mathbb{R}$ it must be that $L_x = (-\infty, x) \cap A$ or $R_x = (x, \infty) \cap A$ is uncountable.

  This is because otherwise both L_x and R_x would be countable or finite for all $x \in \mathbb{R}$.

  But this implies $L_x \cup R_x \cup (\{x\} \cap A)$ is a finite union of countable or finite sets and is therefore countable or finite.

  However, $L_x \cup R_x \cup (\{x\} \cap A) = A \cap ((-\infty, x) \cup \{x\} \cup (x, \infty)) = A \cap (-\infty, \infty) = A$ is uncountable.

  So it must be that for all $x \in \mathbb{R}$, L_x or R_x is uncountable, so $x \in T_1$ or $x \in T_2$.

  So $T_1 \cup T_2 \subseteq \mathbb{R}$ trivially and $\mathbb{R} \subseteq T_1 \cup T_2$ as we have just shown.

  Therefore $T_1 \cup T_2 = (t_1, \infty) \cup (-\infty, t_2) = (-\infty, t_2) \cup (t_1, \infty) = \mathbb{R}$.

  This can only happen if $t_1 < t_2$. So it must be that $t_1 < t_2$.

  So $B = T_1 \cap T_2 = (t_1, \infty) \cap (-\infty, t_2) = (t_1, t_2) \neq \phi$. So B is nonempty.

  B **is open:**

  $B = T_1 \cap T_2$ where T_1 and T_2 are both open.

  Since the intersection of finitely many open sets is open we have that B is open.

  So B is nonempty and open $\square$