More on Sets and Irrationality

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1.3.2

a. Let $S = \mathbb{R}$ and $B = \{b\}$ for any $b \in \mathbb{R}$ then $B \subset S$.

Clearly here $infB = b$ and $supB = b$. So this is an example of a set B where $infB \geq supB$.

Note: if the question asked for a set B where $infB > supB$ this would not be possible as by definition $infB$ is less than or equal to any element of B and $supB$ is greater than or equal to any element of B. So we can only possibly get $infB \geq supB$ and not $infB > supB$.

b. It is not possible to have a finite set that does not contain its own supremum.

Proof:

Let $A = \{a_1, a_2, ..., a_n\}$ be a finite set and $S \supseteq A$ be the superset of A for finding the infimum and supremum.

We can assume that A is ordered so that if $j, k \in [1, n] \subset \mathbb{Z}$ with $j < k$ then $a_j < a_k$ (A is in increasing order) because if it isn't we can simply rearrange A so that it is.

Then since every element of A is in S we have $a_1 \in S$ and $a_n \in S$. Clearly for all $x \in A$, $a_1 \leq x$ and $a_n \geq x$.

So a_1 is a lower bound of A and a_n is an upper bound of A.

Say $x \in S$ such that x is a lower bound of A then for all $y \in A$, $x \leq y$ and since $a_1 \in A$ we have that $x \leq a_1$.

So $infA = a_1$

Say $x \in S$ such that x is an upper bound of A then for all $y \in A$, $x \geq y$ and since $a_n \in A$ we have that $x \geq a_n$.

So $supA = a_n$

Therefore since A and S were arbitrary choices of a finite set and any superset, any finite set contains its supremum and its infimum. $\square$

c. Let $S = \mathbb{R}$ and $A = \{x \in \mathbb{Q} : a_1 < x \leq a_2\}$. For some $a_1, a_2 \in \mathbb{Q}$. Clearly A is a bounded subset of $\mathbb{Q}$.

We know $a_2 \in S$, $x \leq a_2$ for all $x \in A$. Therefore a_2 is an upper bound of A. If $y \in S$ is an upper bound of A then for all $x \in A$ we know $y \geq x$ so since $a_2 \in A$ we have that $y \geq a_2$.

Therefore $supA = a_2$.

We also know $a_1 \in S$, $x > a_1$ for all $x \in A$. Therefore a_1 is a lower bound of A. If $y \in S$ such that $y > a_1$ then since $\mathbb{Q}$ is dense in $\mathbb{R}$ there exists an $r \in \mathbb{Q}$ such that $a_1 < r < y$. So either $a_1 < r \leq a_2$ so that $r \in A$ and therefore $y > z$ for some $z \in A$ and is not a lower bound of A or $a_2 < r < y$ and therefore y is not a lower bound of A.

Therefore $infA = a_1$.

Since $supA = a_2 \in A$ and $infA = a_1 \notin A$ this is an example of a bounded subset of $\mathbb{Q}$ that contains its supremum but not its infimum.

1.3.11

a. This is true. Let A and B both be nonempty sets so $A \subseteq B$.

$$\text{Let } x = supB. \text{ Then } x \geq b \text{ for all } b \in B. \text{ Since } A \subseteq B \text{ if } a \in A \text{ then } a \in B.$$

So we also know $x \geq a$ for all $a \in A$ and is an upper bound of A, meaning it must be greater than or equal to $supA$.

$$\text{Therefore } x = supB \geq supA \; \square$$

b. This is true. Let A and B be sets such that $supA < infB$.

$$\text{Let } x = supA \text{ and } y = infB, \text{ according to our assumptions then } x < y.$$

$$\text{Consider } z = \frac{x+y}{2} \text{ then since } x < y \text{ by adding } x \text{ to both sides and dividing by 2 we get } x < \frac{x+y}{2} = z.$$

$$\text{Similarly since } x < y \text{ by adding } y \text{ to both sides and dividing by 2 we get } z = \frac{x+y}{2} < y.$$

$$\text{So } x < z < y, \text{ and since } x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ we know } \frac{x+y}{2} = z \in \mathbb{R}$$

$$\text{Since } x = supA \text{ and } y = infB \text{ we know } x \geq a \text{ for all } a \in A \text{ and } y \leq b \text{ for all } b \in B.$$

$$\text{So we have that } a \leq x < z < y \leq b \text{ for all } a \in A \text{ and all } b \in B.$$

$$\text{So } z \in \mathbb{R} \text{ is such an example where } a < z < b \text{ for all } a \in A \text{ and all } b \in B.$$

$$\text{Therefore if } supA < infB \text{ there does exist a } c \in \mathbb{R} \text{ such that } a < c < b \text{ for all } a \in A \text{ and all } b \in B \; \square$$

c. This is false. Let $A = (-\infty, t)$ and $B = (t, \infty)$ for some $t \in \mathbb{R}$.

$$\text{Then } a < t < b \text{ for all } a \in A \text{ and all } b \in B. \text{ So } t \text{ is both an upper bound for } A \text{ and a lower bound for } B.$$

- Showing $t = supA$:

$$\text{If } z \in \mathbb{R} \text{ such that } z < t \text{ then adding } t \text{ to both sides and dividing by 2 we get } \frac{z+t}{2} < t.$$

$$\text{Similarly since } z < t \text{ by adding } z \text{ to both sides and dividing by 2 we get } z < \frac{z+t}{2}.$$

$$\text{So } r = \frac{z+t}{2} \in A \text{ and therefore } z < w \text{ for some } w \in A \text{ and can't be an upper bound of } A.$$

$$\text{Therefore if } z \in \mathbb{R} \text{ is an upper bound of } A \text{ then } z \geq t, \text{ so } supA = t.$$

- Showing $t = infB$:

$$\text{If } z \in \mathbb{R} \text{ such that } z > t \text{ then adding } t \text{ to both sides and dividing by 2 we get } \frac{z+t}{2} > t.$$

$$\text{Similarly since } z > t \text{ by adding } z \text{ to both sides and dividing by 2 we get } z > \frac{z+t}{2}.$$

$$\text{So } r = \frac{z+t}{2} \in B \text{ and therefore } z > w \text{ for some } w \in B \text{ and can't be a lower bound of } B.$$

$$\text{Therefore if } z \in \mathbb{R} \text{ is a lower bound of } B \text{ then } z \leq t, \text{ so } infB = t.$$

$$\text{Since } supA = t = infB, \text{ we have that } supA \not< infB.$$

So this is such an example where there exists a $c \in \mathbb{R}$ such that $a < c < b$ for all $a \in A$ and all $b \in B$ but $supA \not< infB$.

Therefore the existence of a $c \in \mathbb{R}$ such that $a < c < b$ for all $a \in A$ and all $b \in B$ does not imply that $supA < infB$ $\square$

1.4.5

From problem 1.4.1 we have: If $a \in \mathbb{Q}$ and $t \in \mathbb{I}$ then $a + t \in \mathbb{I}$ and if $t \neq 0$ then $at \in \mathbb{I}$.

Let $a, b \in \mathbb{R}$ such that $a < b$, then $a - \sqrt{2}, b - \sqrt{2} \in \mathbb{R}$ and $a - \sqrt{2} < b - \sqrt{2}$.

Furthermore, as proved in a previous Sample Work, $\sqrt{2} \in \mathbb{I}$.

So for all $a, b \in \mathbb{R}$ such that $a < b$, since $\mathbb{Q}$ is dense in $\mathbb{R}$ there exists a $q \in \mathbb{Q}$ such that $a - \sqrt{2} < q < b - \sqrt{2}$.

By adding $\sqrt{2}$ to each side we have that for all $a, b \in \mathbb{R}$ such that $a < b$, there exists a $q \in \mathbb{Q}$ such that $a < q + \sqrt{2} < b$.

Let $t = q + \sqrt{2}$ for this $q \in \mathbb{Q}$. By the result of problem 1.4.1 we have that $t = q + \sqrt{2} \in \mathbb{I}$

Therefore for all $a, b \in \mathbb{R}$ there exists a $t \in \mathbb{I}$ such that $a < t < b$ $\square$

1.4.8

a. Let $A = \{x \in \mathbb{Q} : x < t\}$ and $B = \{x \in \mathbb{I} : x < t\}$ for some $t \in \mathbb{R}$.

Clearly since if $x \in \mathbb{Q}$ then $x \notin \mathbb{I}$ and vice versa we know that $A \cap B = \phi$.

- Showing $t = supA$:

  Since $t > x$ for all $x \in A$, t is an upper bound of A.

  If $y \in \mathbb{R}$ such that $y < t$ then since $\mathbb{Q}$ is dense in $\mathbb{R}$ there exists an $x \in \mathbb{Q}$ such that $y < x < t$.

  Since for this $x \in \mathbb{Q}$ we know $x < t$, $x \in A$ so $y < z$ for some $z \in A$ and therefore can't be an upper bound of A.

  So if $w \in \mathbb{R}$ is an upper bound of A then $w \geq t$.

  Therefore $supA = t$.

- Showing $t = supB$:

  Since $t > x$ for all $x \in B$, t is an upper bound of B.

  If $y \in \mathbb{R}$ such that $y < t$ then since $\mathbb{I}$ is dense in $\mathbb{R}$ there exists an $x \in \mathbb{I}$ such that $y < x < t$.

  Since for this $x \in \mathbb{I}$ we know $x < t$, $x \in B$ so $y < z$ for some $z \in B$ and therefore can't be an upper bound of B.

  So if $w \in \mathbb{R}$ is an upper bound of B then $w \geq t$.

  Therefore $supB = t$.

So $supA = t = supB$, and $t \notin A$, $t \notin B$

So this is an example of sets A and B where $A \cap B = \phi$, $supA \notin A$, $supB \notin B$, and $supA = supB$.

b. For $n \in \mathbb{N}$, let $J_n = (-\frac{1}{n}, \frac{1}{n})$. Notice that $0 \in J_n$ for all $n \in \mathbb{N}$.

Let $x \in \mathbb{R}$ be such that $x > 0$. By the Archimedean property there exists an $m \in \mathbb{N}$ such that $\frac{1}{m} < x$.

Since $\frac{1}{m} < x$ we have that $x \notin J_m$, and therefore $x \notin \cap_{i=1}^{\infty} J_i$.

Let $x \in \mathbb{R}$ be such that $x < 0$. Then $-x > 0$. By the Archimedean property there exists an $m \in \mathbb{N}$ such that $\frac{1}{m} < -x$.

Since $\frac{1}{m} < -x$ we have that $x < -\frac{1}{m}$ and consequently $x \notin J_m$, and therefore $x \notin \cap_{i=1}^{\infty} J_i$.

So if $x \neq 0$ then $x \notin J_n$ for some $n \in \mathbb{N}$ therefore $\cap_{i=1}^{\infty} J_i = \{0\}$.

So this is an example of a sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq ...$ such that $\cap_{i=1}^{\infty} J_i \neq \phi$ and $\cap_{i=1}^{\infty} J_i$ only has a finite number of elements.

c. For $n \in \mathbb{N}$ let $L_n = [n, \infty)$. Then $L_1 \supseteq L_2 \supseteq L_3 \supseteq ...$

Let $x \in \mathbb{R}$ then by the Archimedean property there exists an $n \in \mathbb{N}$ such that $n > x$.

Since $x < n$ we have that $x \notin L_n$ and therefore $x \notin \cap_{i=1}^{\infty} L_i$.

Therefore $\cap_{i=1}^{\infty} L_i = \phi$.

So this is an example of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq ...$ such that $\cap_{i=1}^{\infty} L_i = \phi$.

d. Let $I_1, I_2, I_3, ...$ be closed bounded intervals such that $\cap_{i=1}^{n} I_i \neq \phi$ for all $n \in \mathbb{N}$.

Then $\cap_{i=1}^{n} I_i$ is a closed bounded interval itself for all $n \in \mathbb{N}$. That is $\cap_{i=1}^{n} I_i = [a_n, b_n]$ for some $a_n, b_n \in \mathbb{R}$.

Furthermore $\cap_{i=1}^{n+1} I_i = (\cap_{i=1}^{n} I_i) \cap I_{n+1} \subseteq \cap_{i=1}^{n} I_i$ because if $x \in \cap_{i=1}^{n+1} I_i$ then $x \in \cap_{i=1}^{n} I_i$.

Denote $\cap_{i=1}^{n} I_i$ as $[a_n, b_n]$ since each $\cap_{i=1}^{n} I_i$ is a closed bounded interval.

So we have that $[a_1, b_1] \supseteq [a_2, b_2] \supseteq [a_3, b_3] \supseteq ...$ as a sequence of closed bounded nested intervals.

The nested interval property tells us that $\cap_{n=1}^{\infty} [a_n, b_n] \neq \phi$.

Therefore $\cap_{n=1}^{\infty} [a_n, b_n] = \cap_{n=1}^{\infty} \cap_{i=1}^{n} I_i = I_1 \cap (I_1 \cap I_2) \cap (I_1 \cap I_2 \cap I_3) \cap ... = I_1 \cap I_2 \cap I_3 \cap ... = \cap_{n=1}^{\infty} I_n \neq \phi$.

So if you have closed bounded intervals $I_1, I_2, I_3, ...$ (not necessarily nested) such that $\cap_{i=1}^{n} I_i \neq \phi$ for all $n \in \mathbb{N}$ then it can not be that $\cap_{n=1}^{\infty} I_n = \phi$ $\square$

External Sources

I believe that Abbott's book had an analogous example to my solution for 1.4.8.c. where they took $A_1 = \mathbb{N} = \{1, 2, 3, ...\}$, $A_2 = \{2, 3, 4, ...\}$, $A_3 = \{3, 4, 5, ...\}$, ... then proceeded to show that $\cap_{i=1}^{\infty} A_i = \phi$.

This idea from the book also contributed to my solution for 1.4.8.b.