Counting Roots and Linear Fractional Transformations

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94.2

Let f be a function that is analytic inside and on a positively oriented simple closed contour C.

Further suppose that f is never zero on C.

Let the image of C under $w = f(z)$ be the closed contour Γ as shown in the book.

If you imagine a person standing at the origin ($w = 0$), initially looking to any point $w_0 \in \Gamma$ and tracing the path of someone else walking around Γ, then that person at the origin will do 3 complete counterclockwise rotations while tracing the other's path.

Therefore we know the winding number is 3 and so $\frac{1}{2\pi}\Delta_C \, arg\, f(z) = 3$ which gives the result $\Delta_C \, arg\, f(z) = 6\pi$.

Since f is analytic inside and on C we know it has 0 poles inside C (that is $N_\infty = 0$).

Furthermore since f is nonzero on C we know that $\frac{1}{2\pi}\Delta_C \, arg\, f(z) = 3 = N_0 - N_\infty$ where N_0 is the number of zeros of f inside C and N_∞ is the number of poles of f inside C (which we know is 0).

Therefore we have that f has 3 zeros (counting multiplicity) interior to C and $\Delta_C \, arg\, f(z) = 6\pi$ $\square$

94.6

Let C be the unit circle centered at the origin $|z| = 1$, clearly C is a simple closed contour.

a.

Clearly $f(z) = -5z^4$ and $g(z) = z^6 + z^3 - 2z$ are both analytic inside and on C.

Furthermore we know that for all $z \in C$:

$$|g(z)| = |z^6 + z^3 - 2z| \leq |z^6| + |z^3| + |-2z| = |z|^6 + |z|^3 + 2|z| = 4 < 5 = 5|z|^4 = |-5z^4| = |f(z)|$$

Therefore by Rouche's Theorem we know $f(z) = -5z^4$ and $f(z) + g(z) = z^6 - 5z^4 + z^3 - 2z$ have the same number of zeros (counting multiplicity) inside C.

Clearly $f(z) = -5z^4$ has 4 zeros (counting multiplicity) inside C so we know $f(z) + g(z) = z^6 - 5z^4 + z^3 - 2z$ has 4 zeros inside C □

b.

Clearly $f(z) = 9$ and $g(z) = 2z^4 - 2z^3 + 2z^2 - 2z$ are both analytic inside and on C.

Furthermore we know that for all $z \in C$:

$$|g(z)| = |2z^4 - 2z^3 + 2z^2 - 2z| \leq |2z^4| + |-2z^3| + |2z^2| + |-2z| = 2|z|^4 + 2|z|^3 + 2|z|^2 + 2|z| = 8 < 9 = |f(z)|$$

Therefore by Rouche's Theorem we know $f(z) = 9$ and $f(z) + g(z) = 2z^4 - 2z^3 + 2z^2 - 2z + 9$ have the same number of zeros (counting multiplicity) inside C.

Clearly $f(z) = 9$ has no zeros inside C so we know $f(z) + g(z) = 2z^4 - 2z^3 + 2z^2 - 2z + 9$ has no zeros inside C □

c.

Clearly $f(z) = -4z^3$ and $g(z) = z^7 + z - 1$ are both analytic inside and on C.

Furthermore we know that for all $z \in C$:

$$|g(z)| = |z^7 + z - 1| \leq |z^7| + |z| + |-1| = |z|^7 + |z| + 1 = 3 < 4 = 4|z|^3 = |-4z^3| = |f(z)|$$

Therefore by Rouche's Theorem we know $f(z) = -4z^3$ and $f(z) + g(z) = z^7 - 4z^3 + z - 1$ have the same number of zeros (counting multiplicity) inside C.

Clearly $f(z) = -4z^3$ has 3 zeros (counting multiplicity) inside C so we know $f(z) + g(z) = z^7 - 4z^3 + z - 1$ has 3 zeros inside C □

94.9

Let $c \in \mathbb{C}$ such that $|c| > e$ then consider the functions $f(z) = cz^n$ for some $n \in \mathbb{N}$ and $g(z) = -e^z$.

Let C be the unit circle $|z| = 1$. Clearly both are analytic inside and on C.

On C we know $|f(z)| = |cz^n| = |c||z|^n = |c| > e$.

We also know if $z = x + iy$ then $|g(z)| = |-e^z| = |-e^{x+iy}| = |-e^x||e^{iy}| = e^x$.

On C this is clearly maximized my maximizing $x = Re\ z$ and hence is maximized at $z = 1$.

So on C we have that $|f(z)| = |c| > e = |-e^1| \geq |e^z| = |g(z)|$.

Therefore we know $f(z) = cz^n$ and $f(z)+g(z) = cz^n - e^z$ have the same number of zeros (counting multiplicity) inside C.

Clearly $f(z) = cz^n$ has n zeros (counting multiplicity) inside C (namely all at $z = 0$), so we know $f(z) + g(z) = cz^n - e^z$ has n zeros inside C.

This is equivalent to saying the equation $cz^n = e^z$ has n roots inside C $\square$

98.12

Consider the circle $|z-z_0| = R$ under the map $f(z) = \frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

Given we know that the circle is mapped to another circle we know that neither circle passes through the origin.

Recall that we may represent such a circle with the equation:

$$A(x^2+y^2) + Bx + Cy + D = 0$$

Where $A \neq 0$ because it's a circle and $D \neq 0$ because it does not pass through the origin.

This is also equivalent to:

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2+C^2-4AD}}{2A}\right)^2$$

Implying that the such a circle has center $z_0 = -\frac{B}{2A} - i\frac{C}{2A}$ and radius $R = \left(\frac{\sqrt{B^2+C^2-4AD}}{2A}\right)^2$

Further recall that this under $f(z) = \frac{1}{z} = u(x,y) + iv(x,y)$ is mapped to:

$$D(u^2+v^2) + Bu - Cv + A = 0$$

We may also write this as:

$$u^2 + \frac{B}{D}u + v^2 - \frac{C}{D}v + \frac{A}{D} = 0$$

$$u^2 + 2\frac{B}{2D}u + (\frac{B}{2D})^2 + v^2 - 2\frac{C}{2D} + (-\frac{C}{2D})^2 = -\frac{A}{D} + (\frac{B}{2D})^2 + (-\frac{C}{2D})^2$$

$$\left(u + \frac{B}{2D}\right)^2 + \left(v - \frac{C}{2D}\right)^2 = \frac{B^2+C^2-4AD}{4D^2} = \left(\frac{\sqrt{B^2+C^2-4AD}}{2D}\right)^2$$

This is clearly the equation of a circle with center $w_0 = -\frac{B}{2D} + i\frac{C}{2D}$ and radius $\rho = \left(\frac{\sqrt{B^2+C^2-4AD}}{2D}\right)^2$.

If we had $z_0 = w_0$ then it must be that $-\frac{B}{2A} - i\frac{C}{2A} = -\frac{B}{2D} + i\frac{C}{2D}$.

Giving the simultaneous equations $-\frac{B}{2A} = -\frac{B}{2D}$ (hence $A = D$) and $-\frac{C}{2A} = \frac{C}{2D}$ (hence $A = -D$).

The only way for these to be simultaneously true is if $A = D = 0$ which contradicts the initial assumption that $A \neq 0$ and $D \neq 0$.

Therefore a circle that is mapped to a circle never has it's center mapped to the new center $\square$

Note that if the center of the original circle is the origin then the center remains the origin for the new circle, however under $\frac{1}{z}$ the origin is mapped to ∞ so there is still no issue.

100.1

Recall that the implicit form of the linear fractional transformation (given below) maps z_1 to w_1, z_2 to w_2, and z_3 to w_3:

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Therefore by letting $z_1 = -1$, $z_2 = 0$, $z_3 = 1$, $w_1 = -i$, $w_2 = 1$, and $w_3 = i$ we get the following:

$$\frac{(w+i)(1-i)}{(w-i)(1+i)} = \frac{(z+1)(0-1)}{(z-1)(0+1)}$$

$$(w+i)(1-i)(z-1) = -(z+1)(1+i)(w-i)$$

$$(w-iw+i+1)(z-1) = -(z+iz+i+1)(w-i)$$

$$wz - iwz + iz + z - w + iw - i - 1 = -zw - izw - iw - w + iz - z - 1 + i$$

$$wz + z + iw - i = -wz - iw - z + i$$

$$2(wz + z + iw - i) = 0$$

$$w(z+i) + (z-i) = 0$$

$$w = -\frac{z-i}{z+i} = \frac{i-z}{i+z}$$

$\square$

Problem 2

a. Let $f(z) = \frac{1}{z+1}$ and Γ be the contour shown in the problem description.

Let D be a domain that encompasses Γ but that does not contain -1, such a domain exists because -1 is exterior to Γ.

Clearly f is analytic in D since $-1 \notin D$ and -1 is clearly the only point where f is not analytic.

If you imagine a person standing outside of D, initially looking to any point $w_0 \in \Gamma$ and tracing the path of someone else walking around Γ, then that person standing outside of D won't rotate any times counterclockwise while tracing the other's path.

Therefore we know the winding number $W(\Gamma, \xi) = 0$ for $\xi \in \mathbb{C} \backslash D$.

Therefore since $0 \in D \backslash \Gamma$ and $W(\Gamma, \xi) = 0$ for $\xi \in \mathbb{C} \backslash D$ we know:

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z} dz = W(\Gamma, 0) f(0)$$

Where $W(C, z_0)$ represents the winding number for a contour C and a given point $z_0 \notin C$.

If you imagine a person standing at the origin ($z_0 = 0$), initially looking to any point $w_0 \in \Gamma$ and tracing the path of someone else walking around Γ, then that person at the origin will do 3 complete counterclockwise rotations while tracing the other's path.

Therefore we know the winding number $W(\Gamma, 0) = 3$.

So we have that:

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z} dz = W(\Gamma, 0) f(0) = 3(\frac{1}{0+1}) = 3$$

Leaving us with the final result:

$$\int_\Gamma \frac{f(z)}{z} dz = \int_\Gamma \frac{1}{z(z+1)} dz = 6\pi i$$

$\square$

Part b on next page.

b. Let $f(z) = \frac{1}{z(z^2-1)}$ and C be the contour shown in the problem description.

What we are going to do is deconstruct this into a number of simple closed contours.

Let C_1 be the simple closed contour encompassing only $z_1 = 0$, Γ_1 be the contour following directly from C_1 and ending where C intersects itself next, C_2 be the simple closed contour following directly from Γ_1 and ending where C intersects itself next, Γ_2 be the contour following directly from C_2 and ending where C intersects itself next, and finally let

$C_3 = \Gamma_1 + \Gamma_2$.

Note that C_1, C_2, and C_3 are all simple closed contours.

We know first that:

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{\Gamma_1} f(z)dz + \int_{C_2} f(z)dz + \int_{\Gamma_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz$$

- Considering C_1:

Let D_1 be a domain that encompasses C_1 but that does not contain ± 1, such a domain exists because ± 1 are exterior C_1.

Clearly $f_1(z) = \frac{1}{z^2-1}$ is analytic in D_1 since $\pm 1 \notin D_1$ and ± 1 are clearly the only points where f_1 is not analytic.

If you imagine a person standing outside of D_1, initially looking to any point $w_0 \in C_1$ and tracing the path of someone else walking around C_1, then that person standing outside of D_1 won't rotate any times counterclockwise while tracing the other's path.

Therefore we know the winding number $W(C_1, \xi) = 0$ for $\xi \in \mathbb{C} \backslash D_1$.

Therefore since $0 \in D_1 \backslash C_1$ and $W(C_1, \xi) = 0$ for $\xi \in \mathbb{C} \backslash D_1$ we know:

$$\frac{1}{2\pi i} \int_{C_1} \frac{f_1(z)}{z} dz = W(C_1, 0) f_1(0)$$

Where $W(\Gamma, z_0)$ represents the winding number for a contour Γ and a given point $z_0 \notin \Gamma$.

If you imagine a person standing at the origin ($z_0 = 0$), initially looking to any point $w_0 \in C_1$ and tracing the path of someone else walking around C_1, then that person at the origin will do 1 complete clockwise rotation while tracing the other's path.

Therefore we know the winding number $W(C_1, 0) = -1$.

So we have that:

$$\frac{1}{2\pi i} \int_{C_1} \frac{f_1(z)}{z} dz = W(C_1, 0) f_1(0) = -(\frac{1}{0-1}) = 1$$

Leaving us with the final result:

$$\int_{C_1} \frac{f_1(z)}{z} dz = \int_{C_1} \frac{1}{z(z^2-1)} dz = 2\pi i$$

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- Considering C_3:

Let D_3 be a domain that encompasses C_3 but that does not contain 1 or 0, such a domain exists because 1 and 0 are exterior C_1.

Clearly $f_3(z) = \frac{1}{z(z-1)}$ is analytic in D_3 since $0, 1 \notin D_1$ and $0, 1$ are clearly the only points where f_3 is not analytic.

If you imagine a person standing outside of D_3, initially looking to any point $w_0 \in C_3$ and tracing the path of someone else walking around C_3, then that person standing outside of D_3 won't rotate any times counterclockwise while tracing the other's path.

Therefore we know the winding number $W(C_3, \xi) = 0$ for $\xi \in \mathbb{C} \backslash D_3$.

Therefore since $-1 \in D_3 \backslash C_3$ and $W(C_3, \xi) = 0$ for $\xi \in \mathbb{C} \backslash D_3$ we know:

$$\frac{1}{2\pi i} \int_{C_3} \frac{f_3(z)}{z+1} dz = W(C_3, -1) f_3(-1)$$

Where $W(\Gamma, z_0)$ represents the winding number for a contour Γ and a given point $z_0 \notin \Gamma$.

If you imagine a person standing at the origin ($z_0 = 0$), initially looking to any point $w_0 \in C_3$ and tracing the path of someone else walking around C_3, then that person at the origin will do 1 complete counterclockwise rotation while tracing the other's path.

Therefore we know the winding number $W(C_3, -1) = 1$.

So we have that:

$$\frac{1}{2\pi i} \int_{C_3} \frac{f_3(z)}{z+1} dz = W(C_3, -1) f_3(-1) = \frac{1}{-1(-1-1)} = \frac{1}{2}$$

Leaving us with the final result:

$$\int_{C_3} \frac{f_3(z)}{z} dz = \int_{C_3} \frac{1}{z(z^2-1)} dz = \pi i$$

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- Considering C_2:

Clearly C_2 is a simple closed contour that encompasses all of the singularities $(0, \pm 1)$ of $f(z) = \frac{1}{z(z^2-1)}$.

Furthermore we may write $f(z) = \frac{\frac{1}{z^2-1}}{z}$, $f(z) = \frac{\frac{1}{z(z+1)}}{z-1}$, and $f(z) = \frac{\frac{1}{z(z-1)}}{z+1}$.

The numerator of the first representation above is analytic at $z_1 = 0$, similarly that of the second is analytic at $z_2 = 1$, and finally that of the third is analytic at $z_3 = -1$.

Since these are simple poles we may use the fact that $Res_{z=z_0} \frac{\phi(z)}{z-z_0} = \phi(z_0)$ where $\phi(z)$ is analytic at z_0.

Therefore:

$$Res_{z=0} f(z) = Res_{z=0} \frac{1}{z(z^2-1)} = \frac{1}{z^2-1}\Big|_{z=0} = -1$$
$$Res_{z=-1} f(z) = Res_{z=-1} \frac{1}{z(z^2-1)} = \frac{1}{z(z-1)}\Big|_{z=-1} = \frac{1}{2}$$
$$Res_{z=1} f(z) = Res_{z=1} \frac{1}{z(z^2-1)} = \frac{1}{z(z+1)}\Big|_{z=1} = \frac{1}{2}$$

Since C_2 is a simple closed contour and $f(z) = \frac{1}{z(z^2-1)}$ is analytic inside and on C_2 except at isolated singularities we know:

$$\int_{C_2} f(z)dz = \int_{C_2} \frac{1}{z(z^2-1)}dz = 2\pi i\left(\sum_{k=1}^{3} Res_{z=z_k} f(z)\right) = 2\pi i\big(-1 + \frac{1}{2} + \frac{1}{2}\big) = 0$$

So finally we have that:

$$\int_C \frac{1}{z(z^2-1)}dz = \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz = 2\pi i + 0 + \pi i = 3\pi i$$