Method of Moments and MLE

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1.

Let $X_1, X_2, ..., X_n \overset{iid}{\sim} F$ with density $f(x|\theta) = \theta x^{-2}$ for $0 < \theta \leq x < \infty$

a.

$$L(\theta) = f(X_1, X_2, ..., X_n|\theta) = \prod_{i=1}^{n} f(X_i|\theta) = \prod_{i=1}^{n} \theta X_i^{-2} = \theta^n \Big(\prod_{i=1}^{n} X_i\Big)^{-2}$$

$$l(\theta) = log\big(L(\theta)\big) = log\left(\theta^n \Big(\prod_{i=1}^{n} X_i\Big)^{-2}\right) = n\ log\ \theta - 2\sum_{i=1}^{n} log\ X_i$$

$\frac{\partial l(\theta)}{\partial \theta} = \frac{n}{\theta}$ which is monotonically decreasing in θ.

Therefore our estimate for θ should be as small as possible with respect to our data.

So the MLE is $\hat{\theta}_n = min\{X_1, X_2, ..., X_n\}$ $\square$

b.

First we will calculate $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\ f(x|\theta)\ dx = \int_{\theta}^{\infty} \theta x^{-1}\ dx = \theta\ log|x|\Big|_{\theta}^{\infty} = \theta \lim_{x\to\infty} log|x| - \theta\ log|\theta| = \infty$$

So $\mathbb{E}[X]$ does not converge and hence we can not use $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ for our MOM estimate.

Doing so would introduce the equation $\bar{X} = \mathbb{E}[X] = \infty$ which will not be true for any sample, we need the first moment to converge and be a function with θ present in order to use the first sample moment in our MOM estimate.

c.

Let $g(X) = X^{1/2}$ then let $\overline{X^{1/2}} = \sum_{i=1}^{n} X_i^{1/2}$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\ f(x|\theta)\ dx = \int_{\theta}^{\infty} \theta x^{-3/2}\ dx = -2\theta x^{-1/2}\Big|_{\theta}^{\infty} = -2\theta \lim_{x\to\infty} \frac{1}{x^{1/2}} + 2\theta\frac{1}{\theta^{1/2}} = 0 + 2\theta^{1/2} = 2\theta^{1/2}$$

Then we solve $\overline{X^{1/2}} = \mathbb{E}[g(X)] = 2\theta^{1/2}$ to get us our MOM estimate is $\tilde{\theta}_n = \frac{1}{4}\Big(\overline{X^{1/2}}\Big)^2$ $\square$

2.

Let $\theta > 0$ then let $X_1, X_2, ..., X_n \overset{iid}{\sim} N(\theta, \theta)$.

$$L(\theta) = f(X_1, X_2, ..., X_n|\theta) = \prod_{i=1}^{n} f(X_i|\theta) = \prod_{i=1}^{n} \theta X_i^{-2} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta}(X_i-\theta)^2}$$

$$l(\theta) = log\big(L(\theta)\big) = log\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{1}{2\theta}(X_i-\theta)^2}\right) = \sum_{i=1}^{n} log\big(\frac{1}{\sqrt{2\pi\theta}}\big) - \frac{1}{2\theta}(X_i-\theta)^2 = -n\, log\big(\sqrt{2\pi\theta}\big) - \sum_{i=1}^{n} \frac{1}{2\theta}(X_i-\theta)^2$$

$$= -n\, log\big(\sqrt{2\pi\theta}\big) - \sum_{i=1}^{n} \frac{1}{2\theta}(X_i^2 - 2X_i\theta + \theta^2) = -n\, log\big(\sqrt{2\pi\theta}\big) - \sum_{i=1}^{n} \frac{X_i^2}{2\theta} - X_i + \frac{\theta}{2}$$

$$= -n\, log\big(\sqrt{2\pi\theta}\big) - \frac{n\theta}{2} - \frac{1}{2\theta}\Big(\sum_{i=1}^{n} X_i^2\Big) + \Big(\sum_{i=1}^{n} X_i\Big)$$

Then:

$$\frac{\partial l(\theta)}{\partial\theta} = -n\frac{\partial}{\partial\theta} log\big(\sqrt{2\pi\theta}\big) - \frac{n}{2}\frac{\partial}{\partial\theta}\theta - \Big(\sum_{i=1}^{n} X_i^2\Big)\frac{\partial}{\partial\theta}\frac{1}{2\theta} = -n\big(\frac{1}{\sqrt{2\pi\theta}}\big)\big(\frac{\sqrt{2\pi}}{2\sqrt{\theta}}\big) - \frac{n}{2} + \frac{1}{2\theta^2}\sum_{i=1}^{n} X_i^2$$

$$= -\frac{n}{2\theta} - \frac{n}{2} + \frac{1}{2\theta^2}\sum_{i=1}^{n} X_i^2$$

Then we know the MLE will be at a critical point of $l(\theta)$ and hence we can set $\frac{\partial l(\theta)}{\partial\theta} = 0$:

$$\frac{\partial l(\theta)}{\partial\theta} = -\frac{n}{2\theta} - \frac{n}{2} + \frac{1}{2\theta^2}\sum_{i=1}^{n} X_i^2 = 0$$

Then multiplying both sides by $-\frac{2\theta^2}{n}$ implies:

$$\theta^2 + \theta - \frac{1}{n}\sum_{i=1}^{n} X_i^2 = 0$$

Which demonstrates that the MLE is a root of $\theta^2 + \theta - W$ where $W = \bar{X}_n^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ as desired.

The roots of this polynomial are:

$$\theta = \frac{-1 \pm \sqrt{1+4W}}{2}$$

To find which is the MLE we can use the fact that we know $\theta > 0$:

Since $\frac{-1-\sqrt{1+4W}}{2} < 0$ we know this can't be the MLE.

However, we know $\sqrt{1+4W} > 1$ since $W = \bar{X}_n^2 > 0$.

Therefore the MLE is $\hat{\theta}_n = \frac{-1+\sqrt{1+4W}}{2} = \frac{-1+\sqrt{1+4\bar{X}_n^2}}{2} = \frac{-1+\sqrt{1+\frac{4}{n}\sum_{i=1}^{n} X_i^2}}{2}$ $\square$