# Connected Sets and Limits of Functions

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## 3.4.8

Recall $C = \cap_{n=1}^{\infty} C_n$.
A set $E$ is totally disconnected if for all $x, y \in E$ you can find two separated sets, say $A$ and $B$ such that $x \in A$, $y \in B$ and $E = A \cup B$.

**a.** Let $x, y \in C$ where $x < y$. Let $\epsilon = y - x$.

Then $x$ and $y$ are in $C_n$ for all $n \in \mathbb{N}$. Consider the length of any interval in $C_n$.

Since the length of all intervals in $C_n$ approaches 0 we can find an $N \in \mathbb{N}$ such that the maximum length of any interval of $C_N$ is less than $\epsilon$.

Therefore since the length of any interval is less than $\epsilon$ it can not be that $x$ and $y$ are in any one interval otherwise the interval must have at least length $\epsilon$.

**b.**

For arbitrary $x, y \in C$ where $x < y$ there exists an $N \in \mathbb{N}$ such that $x$ and $y$ are in different intervals of $C_N$.

From the way the Cantor set is constructed by removing the middle third in each iteration there must exists some interval between $x$ and $y$ that is not contained in $C$.

Therefore the interval containing $x$ and the interval containing $y$ are separated.

Let $A$ be the union of the interval containing $x$ and all the intervals contained in $C$ to the left of that.

And let $B$ be the union of the interval containing $y$ and all the intervals contained in $C$ to the right of that.

Then since $x < y$ we have that $A$ and $B$ are also separated.

Furthermore $x \in A$, $y \in B$ and $A \cup B = C$ by construction.

This was for arbitrary $x, y \in C$ so for all $x, y \in C$ where $x < y$ this is the case and therefore $C$ is totally disconnected.

## 4.2.3

Recall $t(x)$ takes the value 1 if $x = 0$, the value $\frac{1}{n}$ when $x = \frac{m}{n}$ is in lowest terms, and the value 0 when $x \notin \mathbb{Q}$.

**a.** Let $(x_n) = (1 - \frac{1}{n})$, $(y_n) = (1 - \frac{1}{n^2})$, and $(z_n) = (1 - \frac{1}{n^3})$.

Then $(x_n) = (0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ...)$, $(y_n) = (0, \frac{3}{4}, \frac{8}{9}, \frac{15}{16}, ...)$, and $(z_n) = (0, \frac{7}{8}, \frac{26}{27}, \frac{63}{64})$.

Clearly all of these sequences are different.

As we have seen before $(\frac{1}{n}) \to 0$ and clearly $(0) \to 0$.

Since $0 < \frac{1}{n^3} < \frac{1}{n^2} < \frac{1}{n}$ for all $n \in \mathbb{N}$ we have by the squeeze theorem that $(\frac{1}{n^2}) \to 0$ and $(\frac{1}{n^3}) \to 0$.

Therefore by the algebraic limit theorem $(x_n) = (1 - \frac{1}{n}) \to 1$, $(y_n) = (1 - \frac{1}{n^2}) \to 1$, and $(z_n) = (1 - \frac{1}{n^3}) \to 1$.

All of these sequences do not contain the number 1 as a term so we have made three distinct sequences converging to 1 that do not contain 1.

**b.** Consider the sequences $(t(x_n))$, $(t(y_n))$, and $(t(z_n))$.

We have $(t(x_n)) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...) = (\frac{1}{n})$, $(t(y_n)) = (0, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, ...) = (\frac{1}{n^2})$, $(t(z_n)) = (1, \frac{1}{8}, \frac{1}{27}, \frac{1}{64}, ...) = (\frac{1}{n^3})$.

This comes from the definition of $t(x)$ and the fact that all terms of each sequence in part a were written in lowest terms.

As shown in part a each of these sequences converge to 0. So $lim\ t(x_n) = lim\ t(y_n) = lim\ t(z_n) = 0$.

**c.** I propose that $lim_{x \to 1} t(x) = 0$ since $t(x) = 0$ for all $x \notin \mathbb{Q}$ and because of the limits above.

For a specified $\epsilon > 0$ let $S = \{x \in \mathbb{R} : t(x) \geq \epsilon\}$.

Then $S \subseteq \mathbb{Q}$ since all irrational values assume the value 0 under $t(x)$ and therefore can not be in $S$.

**Proving every point in $S$ is isolated:**

Assume for the sake of contradiction that not every point in $S$ is isolated. That is say $x \in S$ is a limit point of $S$.

Then there must exist some sequence $(x_n) \subseteq S$ such that $x_n \neq x$ for all $n \in \mathbb{N}$ and $(x_n) \to x$.

Since $x \in S$ and $x_n \in S$ for all $n \in \mathbb{N}$ we have $x = \frac{p_0}{q_0}$ for some $p_0, q_0 \in \mathbb{Z}$ and $x_n = \frac{p_n}{q_n}$ for all $n \in \mathbb{N}$ and some $p_n, q_n \in \mathbb{Z}$.

We can say that all of these $p$'s and $q$'s are in lowest terms without loss of generality.

Furthermore $t(x) = \frac{1}{q_0} \geq \epsilon > 0$ and $t(x_n) = \frac{1}{q_n} \geq \epsilon > 0$ for all $n \in \mathbb{N}$. So $0 < q_0 \leq \frac{1}{\epsilon}$ and $0 < q_n \leq \frac{1}{\epsilon}$ for all $n \in \mathbb{N}$.

Since $\epsilon$ is fixed we have that $(0, \frac{1}{\epsilon}]$ must have finite length, so there are only finitely many integers in $(0, \frac{1}{\epsilon}]$.

This means that one integer in $(0, \frac{1}{\epsilon}]$ is used infinitely many times as the denominator for terms of $(x_n)$, say $q$.

Consider the subsequence $(x_{n_k})$ of $(x_n)$ where the denominator of $x_{n_k}$ is $q$ for all $k \in \mathbb{N}$.

Then $(x_{n_k}) = (\frac{p_{n_k}}{q}) \to x$ so by the algebraic limit theorem $(p_{n_k}) \to qx$.

Then $(p_{n_k})$ is a Cauchy sequence. This implies that there exists a $K \in \mathbb{N}$ such that for $k_1, k_2 \geq K$, $|p_{n_{k_1}} - p_{n_{k_2}}| < 1$.

Since $p_{n_k} \in \mathbb{Z}$ for all $k \in \mathbb{N}$ this means that there exists a $K \in \mathbb{N}$ such that for $k_1, k_2 \geq K$, $p_{n_{k_1}} = p_{n_{k_2}}$.

This means $(p_{n_k})$ contains infinitely many repeating terms, and as proved in a previous sample work these terms must be equal to $qx$ since $(p_{n_k}) \to qx$. (I will attach the proof of that below)

But this implies $(x_{n_k})$ contains infinitely many terms equal to $\frac{qx}{q} = x$, a contradiction since this implies $(x_n)$ contains $x$.

So it must be that every point of $S$ is an isolated point.

**Proving $lim_{x \to 1} t(x) = 0$:**

If $0 < \epsilon \leq 1$:

Then we know $t(1) = 1 \geq \epsilon$ so $1 \in S$, but it is also therefore an isolated point in $S$.

Therefore there must exist some $\delta > 0$ such that $V_\delta(1) \cap S = \{1\}$ by the definition of isolated points.

Therefore if $x \in V_\delta(1)$ then $x \notin S$, so $t(x) < \epsilon$. Since $t(y) \geq 0$ for all $y \in \mathbb{R}$ this implies $t(x) \in V_\epsilon(0)$.

If $\epsilon > 1$:

Simply choose any $\delta$ from the above process and you will again get that if $x \in V_\delta(1)$ then $t(x) \in V_\epsilon(0)$.

Therefore for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $x \in V_\delta(1)$ then $t(x) \in V_\epsilon(0)$.

So $lim_{x \to 1} t(x) = 0$ $\square$

**Used proof from previous sample work:**

Let $(b_n)$ be a convergent series that has an infinite number of terms equal to $c$ for some $c \in \mathbb{R}$.

Say $(b_n) \to b$ then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ where if $n \in \mathbb{N}$ such that $n \geq N$ then $|b_n - b| < \epsilon$.

Since $(b_n)$ contains an infinite number of terms $c$ we know for any $N$ there exists an $c$ in the sequence beyond $b_N$.

So if $b \neq c$ then for any choice of $N$ we have a term later in the sequence where $|c - b| > 0$.

So let $0 < \epsilon < |c - b|$ such an $\epsilon$ exists because of the density of $\mathbb{R}$.

Therefore if $b \neq c$ we have shown that there exists an $\epsilon > 0$ such that there does not exist an $N \in \mathbb{N}$ where if $n \geq N$ then $|b_n - b| < \epsilon$ due to the presence of infinitely many terms $c$, contradicting that $(b_n) \to b$.

Therefore a sequence that has infinitely many terms equal to $c$ can not converge to a value that is not $c$ $\square$

## 4.2.8

**a.** Let $f(x) = \frac{|x-2|}{x-2}$. Then $lim_{x \to 2} f(x)$ does not exist.

Proof:

Let $(x_n)$ be a strictly positive sequence such that $(x_n) \to 0$.

Then $(2 + x_n) \to 2$ by the algebraic limit theorem and $2 + x_n > 2$ for all $n \in \mathbb{N}$.

And $(2 - x_n) \to 2$ by the algebraic limit theorem and $2 - x_n < 2$ for all $n \in \mathbb{N}$.

Consider the sequences $(f(2 + x_n))$ and $(f(2 - x_n))$.

$f(2 + x_n) = \frac{|2+x_n-2|}{2+x_n-2} = \frac{|x_n|}{x_n} = \frac{x_n}{x_n} = 1$ since $x_n > 0$ for all $n \in \mathbb{N}$. This also exists since $x_n \neq 0$ for all $n \in \mathbb{N}$.

$f(2 - x_n) = \frac{|2-x_n-2|}{2-x_n-2} = \frac{|-x_n|}{-x_n} = \frac{x_n}{-x_n} = -1$ since $x_n > 0$ for all $n \in \mathbb{N}$. This also exists since $x_n \neq 0$ for all $n \in \mathbb{N}$.

So $lim\ f(2 + x_n) = 1$ and $lim\ f(2 - x_n) = -1$ since $(f(2 + x_n)) = (1)$ and $(f(2 - x_n)) = (-1)$.

So we have found two different sequences $(2 + x_n)$ and $(2 - x_n)$ such that 2 is not in either sequence but both converge to 2 where $lim\ f(2 + x_n) \neq lim\ f(2 - x_n)$

Therefore $lim_{x \to 2} f(x) = lim_{x \to 2} \frac{|x-2|}{x-2}$ does not exist $\square$

**b.** Let $f(x) = \frac{|x-2|}{x-2}$. Then $lim_{x \to \frac{7}{4}} f(x) = -1$.

Proof:

Let $\epsilon > 0$ and let $\delta = \frac{1}{4}$. Then if $x \in V_\delta(\frac{7}{4}) = (\frac{7}{4} - \frac{1}{4}, \frac{7}{4} + \frac{1}{4}) = (\frac{3}{2}, 2)$ we have that $x < 2$.

Therefore $x - 2 < 0$ so $|x - 2| = 2 - x = -(x - 2)$.

So $f(x) = \frac{|x-2|}{x-2} = \frac{-(x-2)}{x-2} = -1$ and this is defined since $x - 2 < 0$ so $x - 2 \neq 0$.

Therefore $f(x) \in V_\epsilon(-1)$ since $f(x) = -1$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So for all $\epsilon > 0$ we have found a $\delta > 0$ such that if $x \in V_\delta(\frac{7}{4})$ then $f(x) \in V_\epsilon(-1)$.

So $lim_{x \to \frac{7}{4}} f(x) = lim_{x \to \frac{7}{4}} \frac{|x-2|}{x-2} = -1$ $\square$

**c.** Let $f(x) = (-1)^{\frac{1}{x}}$ then $lim_{x \to 0} f(x)$ does not exist.

Proof:

Let $(x_n) = (1, \frac{1}{3}, \frac{1}{5}, ...) = (\frac{1}{2n-1})$. Let $(y_n) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ...) = (\frac{1}{2n})$.

Clearly $(x_n) \to 0$ and $(y_n) \to 0$ and 0 is not in either sequence.

Consider the sequences $(f(x_n))$ and $(f(y_n))$.

$f(x_n) = (-1)^{\frac{1}{1/2n-1}} = (-1)^{2n-1} = (-1)^{2n}(-1)^{-1} = -1$ for all $n \in \mathbb{N}$. This also exists since $\frac{1}{2n-1} \neq 0$ for all $n \in \mathbb{N}$.

$f(y_n) = (-1)^{\frac{1}{1/2n}} = (-1)^{2n} = ((-1)^2)^n = (1)^n = 1$ for all $m \in \mathbb{N}$. This also exists since $\frac{1}{2n} \neq 0$ for all $n \in \mathbb{N}$.

So $lim\ f(x_n) = -1$ and $lim\ f(y_n) = 1$ since $(f(x_n)) = (-1)$ and $(f(y_n)) = (1)$.

So we have found two different sequences $(x_n)$ and $(y_n)$ such that 0 is not in either sequence but both converge to 0 where $lim\ f(x_n) \neq lim\ f(y_n)$

Therefore $lim_{x \to 0} f(x) = lim_{x \to 0} (-1)^{\frac{1}{x}}$ does not exist $\square$

**d.** Let $f(x) = \sqrt[3]{x}(-1)^{\frac{1}{x}}$ then $lim_{x \to 0} f(x) = 0$.

Proof:

Let $\epsilon > 0$ then let $\delta = \epsilon^3$.

If $|x - 0| = |x| < \delta = \epsilon^3$ then $|f(x) - 0| = |\sqrt[3]{x}(-1)^{\frac{1}{x}} - 0| = |\sqrt[3]{x}(-1)^{\frac{1}{x}}| = |\sqrt[3]{x}| = |x^{\frac{1}{3}}| = |x|^{\frac{1}{3}} < \sqrt[3]{\delta} = \epsilon$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

Therefore $lim_{x \to 0} f(x) = lim_{x \to 0} \sqrt[3]{x}(-1)^{\frac{1}{x}} = 0$ $\square$

Note however that this function is not continuous in the slightest. When I say if $|x| < \delta$ I mean those parts of the $\delta$ neighborhood where $f(x)$ is defined.

## 4.2.10

**a.** Let $f : A \to \mathbb{R}$ be a function and let $a$ be a limit point of $A$.

Starting with the left hand limit $lim_{x \to a^-} f(x)$:

We say $lim_{x \to a^-} f(x) = L$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < a - x < \delta$ then $|f(x) - L| < \epsilon$.

Now for the right hand limit $lim_{x \to a^+} f(x)$:

We say $lim_{x \to a^+} f(x) = L$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < x - a < \delta$ then $|f(x) - L| < \epsilon$.

**b.** Let $f : A \to \mathbb{R}$ be as before and $a$ be a limit point of $A$. Let the left and right hand limits be defined as before.

- Showing if $lim_{x \to a^-} f(x) = L$ and $lim_{x \to a^+} f(x) = L$ then $lim_{x \to a} f(x) = L$:

  Assume $lim_{x \to a^-} f(x) = L$ and $lim_{x \to a^+} f(x) = L$.

  Then for all $\epsilon > 0$ there exists a $\delta_1$ such that if $0 < a - x < \delta_1$ then $|f(x) - L| < \epsilon$, and there exists a $\delta_2$ such that if $0 < x - a < \delta_2$ then $|f(x) - L| < \epsilon$.

  For each $\epsilon > 0$ let $\delta = min\{\delta_1, \delta_2\}$. Then $0 < \delta \leq \delta_1$ and $0 < \delta \leq \delta_2$.

  So if $0 < a - x < \delta$ it follows $|f(x) - L| < \epsilon$ and if $0 < x - a < \delta$ it follows that $|f(x) - L| < \epsilon$.

  So if $|x - a| < \delta$ then $|f(x) - L| < \epsilon$. Such a $\delta$ was found for all $\epsilon > 0$.

  Therefore for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$ then it follows that $|f(x) - L| < \epsilon$.

  So $lim_{x \to a} f(x) = L$.

- Showing if $lim_{x \to a} f(x) = L$ then $lim_{x \to a^-} f(x) = L$ and $lim_{x \to a^+} f(x) = L$:

  Assume $lim_{x \to a} f(x) = L$. And let $\epsilon > 0$.

  Then there exists a $\delta > 0$ such that if $|x - a| < \delta$ it follows that $|f(x) - L| < \epsilon$.

  So if $0 < a - x < \delta$ then $|x - a| < \delta$ and therefore it follows that $|f(x) - L| < \epsilon$.

  This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

  So for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < a - x < \delta$ it follows that $|f(x) - L| < \epsilon$.

  Therefore $lim_{x \to a^-} f(x) = L$.

  Similarly if $0 < x - a < \delta$ then $|x - a| < \delta$ and therefore it follows that $|f(x) - L| < \epsilon$.

  This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

  So for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < x - a < \delta$ it follows that $|f(x) - L| < \epsilon$.

  Therefore $lim_{x \to a^+} f(x) = L$.

Therefore $lim_{x \to a} f(x) = L$ if and only if $lim_{x \to a^-} f(x) = L$ and $lim_{x \to a^+} f(x) = L$ $\square$

## 4.2.11

Let $f, g, h$ be functions with a common domain $A$ such that $f(x) \leq g(x) \leq h(x)$ for all $x \in A$.

Let $c$ be a limit point of $A$ and assume that $lim_{x \to c} f(x) = L$ and $lim_{x \to c} h(x) = L$.

Let $\epsilon > 0$ and let $\alpha = \epsilon/3$ there exists a $\delta_1 > 0$ such that if $|x - c| < \delta_1$ it follows that $|f(x) - L| < \alpha$.

And there exists a $\delta_2 > 0$ such that if $|x - c| < \delta_2$ it follows that $|h(x) - L| < \alpha$.

Let $\delta = min\{\delta_1, \delta_2\}$. Then if $|x - c| < \delta$ it follows that $|f(x) - L| < \alpha$ and $|h(x) - L| < \alpha$.

Note that $f(x) - h(x) \leq g(x) - h(x) \leq 0$ so $|g(x) - h(x)| \leq |f(x) - h(x)|$.

So if $|x - c| < \delta$ then $|g(x) - L| = |g(x) - h(x) + h(x) - L| \leq |g(x) - h(x)| + |h(x) - L| \leq |f(x) - h(x)| + |f(x) - L| =$

$|f(x) - L + L - h(x)| + |f(x) - L| \leq |f(x) - L| + |L - h(x)| + |f(x) - L| < 3\alpha = \epsilon$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - c| < \delta$ it follows that $|g(x) - L| < \epsilon$.

Therefore $lim_{x \to c} g(x) = L$ $\square$