Sequences of Functions

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6.2.5

Let (f_n) be a sequence of functions on a common domain A.

Assume (f_n) converges uniformly on A to f.

Then for all $\delta > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \delta$ whenever $x \in A$ and $n \geq N$.

Let $\epsilon > 0$ and let $\alpha = \epsilon/2$ then let N be so that for any $k \geq N$ we have $|f_k(x) - f(x)| < \alpha$ for all $x \in A$.

So for $m, n \geq N$ we have $|f_m(x) - f_n(x)| = |f_m(x) - f(x) + f(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f_n(x) - f(x)| < 2\alpha = \epsilon$.

This was for arbitrary $x \in A$ and is therefore true for all $x \in A$.

Similarly this was for all $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that when $m, n \geq N$ and $x \in A$ we have $|f_m(x) - f_n(x)| < \epsilon$.

Now assume that for all $\delta > 0$ there exists an $N \in \mathbb{N}$ such that when $m, n \geq N$ and $x \in A$ we have $|f_m(x) - f_n(x)| < \delta$.

Note that this N does not depend on x but rather only depends on δ.

Then for each $x \in A$ we have that $(f_n(x))$ is Cauchy and therefore converges to some $y \in \mathbb{R}$.

So define the function f so that for each $x \in A$, $f(x)$ is the value that $(f_n(x))$ converges to.

Let $\epsilon > 0$ and let $\alpha = \epsilon/2$. Then let $N_1 \in \mathbb{N}$ be such that $|f_n(x) - f_m(x)| < \alpha$ whenever $m, n \geq N_1$ and $x \in A$.

For $x \in A$ let $N_2 \in \mathbb{N}$ be such that $|f_m(x) - f(x)| < \alpha$ for $m \geq N_2$.

This is possible because $(f_m(x))$ is Cauchy for all $x \in A$.

Let $x \in A$, $n \geq N_1$, and $m \geq max\{N_1, N_2\}$.

Then we have that $|f_n(x) - f(x)| = |f_n(x) - f_m(x) + f_m(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < 2\alpha = \epsilon$.

This was for arbitrary $x \in A$ and is therefore true for all $x \in A$.

Note that here we are using $n \geq N_1$ which does not depend on x.

While N_2 and therefore $max\{N_1, N_2\}$ might depend on x we are only using it when introducing a new term and not in our original expression $|f_n(x) - f(x)|$.

So for all $\epsilon > 0$ we have found an $N = N_1$ such that when $n \geq N$ and $x \in A$ it follows that $|f_n(x) - f(x)| < \epsilon$.

Therefore $(f_n) \to f$ uniformly on A.

So $(f_n) \to f$ uniformly on A if and only if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that when $m, n \geq N$ and $x \in A$ we have $|f_m(x) - f_n(x)| < \epsilon$ $\square$

6.2.7

Let f be uniformly continuous on $\mathbb{R}$ and let $f_n(x) = f(x + \frac{1}{n})$.

Then for all $\alpha > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \alpha$.

So let $\epsilon > 0$ then let δ be such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$, and let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \delta$.

Such an N exists by the Archimedean property.

Then for $n \geq N$ we have $|x + \frac{1}{n} - x| = |\frac{1}{n}| = \frac{1}{n} < \delta$ and this implies $|f(x + \frac{1}{n}) - f(x)| = |f_n(x) - f(x)| < \epsilon$.

This was for arbitrary $x \in \mathbb{R}$ and is therefore true for all $x \in \mathbb{R}$.

So for all $\epsilon > 0$ we have found an $N \in \mathbb{N}$ such that when $n \geq N$ and $x \in \mathbb{R}$ we have $|f_n(x) - f(x)| < \epsilon$.

So $(f_n) \to f$ uniformly on $\mathbb{R}$.

If we were only given that f was continuous on $\mathbb{R}$ the δ could depend on both ϵ and the point of continuity.

So we would not be able to guarantee the existence of a uniform N for $|f_n(x) - f(x)|$ that works for all $x \in \mathbb{R}$.

For an example take $f(x) = x^2$ which we have seen previously is continuous on $\mathbb{R}$.

We have $f_n(x) = f(x + \frac{1}{n}) = (x + \frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$.

So we have $|f_n(x) - f(x)| = |\frac{2x}{n} + \frac{1}{n^2}| = \frac{1}{n}|2x + \frac{1}{n}|$.

Since f has its domain as all of $\mathbb{R}$ the $2x$ term is unbounded so for $\epsilon > 0$ you can not find an $N \in \mathbb{N}$ such that when $n \geq N$ and $x \in \mathbb{R}$ we have $|f_n(x) - f(x)| = \frac{1}{n}|2x + \frac{1}{n}| < \epsilon$ simply because you can always choose x large enough so that this inequality doesn't hold.

6.3.2

Let $h_n(x) = \sqrt{x^2 + \frac{1}{n}}$.

a.

Simply taking the limit we get $lim_{n\to\infty} h_n(x) = lim_{n\to\infty}\sqrt{x^2 + \frac{1}{n}} = \sqrt{x^2} = |x|$.

Now proving this is the limit and that convergence is uniform:

We have seen previously that $\sqrt{y}$ is uniformly continuous on $[0, \infty)$.

So for all $\alpha > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ implies $|\sqrt{x} - \sqrt{y}| < \alpha$ when $x, y \in [0, \infty)$.

Let $\epsilon > 0$ and let δ be such that $|x - y| < \delta$ implies $|\sqrt{x} - \sqrt{y}| < \epsilon$. Then let $N \in \mathbb{N}$ be so that $\frac{1}{N} < \delta$.

Such an N exists by the Archimedean property.

Note that $x^2 \geq 0$ for all $x \in \mathbb{R}$ and $\frac{1}{m} > 0$ for all $m \in \mathbb{N}$, so $x^2 + \frac{1}{n} \in (0, \infty) \subset [0, \infty)$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$.

Also $x^2 \in [0, \infty)$ for all $x \in \mathbb{R}$.

Then for $n \geq N$ we have $|x^2 + \frac{1}{n} - x^2| = |\frac{1}{n}| = \frac{1}{n} \leq \frac{1}{N} < \delta$.

This implies $|h_n(x) - h(x)| = |\sqrt{x^2 + \frac{1}{n}} - \sqrt{x^2}| < \epsilon$ whenever $n \geq N$ and $x \in \mathbb{R}$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So $(h_n(x)) \to |x|$ uniformly $\square$

b.

We know $h'_n(x) = (\frac{1}{2\sqrt{x^2+\frac{1}{n}}})(2x) = \frac{x}{\sqrt{x^2+\frac{1}{n}}}$ by using the algebraic differentiability theorem.

Fix some $x \in \mathbb{R}$ then $(h'_n(x))$ is a sequence of real numbers.

If $x = 0$ then $(h'_n(x)) = (\frac{0}{\sqrt{0+\frac{1}{n}}}) = (0) \to 0$ clearly.

Otherwise $(x) \to x \neq 0$ and we have seen from part a that $(\sqrt{x^2 + \frac{1}{n}}) \to |x| \neq 0$.

Since each of these is a sequence of real numbers we have by the algebraic limit theorem that $(h'_n(x)) \to \frac{x}{|x|}$ for $x \neq 0$.

So for $x \in \mathbb{R}$ if $x < 0$ then $(h'_n(x)) \to -1$, if $x = 0$ then $(h'_n(x)) \to 0$, and if $x > 0$ then $(h'_n(x)) \to 1$.

So $(h'_n(x))$ converges for all $x \in \mathbb{R}$ and therefore $(h'_n(x)) \to g(x)$ pointwise, where $g(0) = 0$, $g(x) = -1$ for $x < 0$, and $g(x) = 1$ for $x > 0$.

However this convergence can not be uniform for any neighborhood of 0 because if it was this would imply that the limit of (h_n) is differentiable at 0, but we know that the limit is $|x|$ which is not differentiable at 0 so the convergence of (h'_n) can not be uniform.

6.3.5

Let $g_n(x) = \frac{nx+x^2}{2n} = \frac{x}{2} + \frac{x^2}{2n}$ for $n \in \mathbb{N}$.

a. Simply taking the limit we get $lim_{n\to\infty} g_n(x) = lim_{n\to\infty} \frac{x}{2} + \frac{x^2}{2n} = \frac{x}{2}$.

This is the pointwise limit and works because of the following:

Consider any point $x \in \mathbb{R}$ then $(g_n(x)) = (\frac{x}{2} + \frac{x^2}{2n})$. Clearly $(\frac{x}{2}) \to \frac{x}{2}$ since x is a fixed constant. We also know $(\frac{1}{n}) \to 0$ so by the algebraic limit theorem $(\frac{x^2}{2n}) \to 0$. So by the algebraic limit theorem $(g_n(x)) \to \frac{x}{2}$.

We have seen that x is differentiable on $\mathbb{R}$ so $lim_{n\to\infty} g_n(x) = g(x) = \frac{x}{2}$ is differentiable on $\mathbb{R}$ by the algebraic differentiability theorem and $g'(x) = \frac{1}{2}$.

b. For each $n \in \mathbb{N}$ we get $g_n'(x) = \frac{1}{2} + \frac{2x}{2n} = \frac{1}{2} + \frac{x}{n}$.

Let $M > 0$, $\epsilon > 0$, and $N > \frac{M}{\epsilon}$, such an $N \in \mathbb{N}$ exists since $\epsilon \neq 0$ and $\mathbb{N}$ is unbounded.

Then for $n \geq N$ and $x \in [-M, M]$ we have $|g_n(x) - \frac{1}{2}| = |\frac{1}{2} + \frac{x}{n} - \frac{1}{2}| = |\frac{x}{n}| = \frac{|x|}{n} \leq \frac{M}{n} \leq \frac{M}{N} < \frac{M}{M/\epsilon} = \epsilon$.

This was for arbitrary $x \in [-M, M]$ and is therefore true for all $x \in [-M, M]$. So $(g_n') \to \frac{1}{2}$ uniformly on $[-M, M]$.

This was also for arbitrary $M > 0$ and is therefore true for all $M > 0$ so $(g_n') \to \frac{1}{2}$ uniformly on every interval $[-M, M]$.

We have already seen in part a that $(g_n(x))$ converges for every $x \in \mathbb{R}$ so there exists an $x_0 \in \mathbb{R}$ such that $(g_n(x_0))$ converges and therefore $(g_n(x)) \to g(x) = \frac{x}{2}$ uniformly on every interval $[-M, M]$ and therefore on $\mathbb{R}$. Furthermore $(g_n') \to g'$, all of this is by theorem 6.3.3 in the book that is mentioned in the problem.

c. Let $f_n(x) = \frac{nx^2+1}{2n+x} = \frac{x^2+\frac{1}{n}}{2+\frac{x}{n}}$ for $n \in \mathbb{N}$.

First we want to compute the pointwise limit of (f_n) so let $x \in \mathbb{R}$.

Then by the algebraic limit theorem we know $(\frac{x}{n}) \to 0$, and so $(2 + \frac{x}{n}) \to 2$.

We also know by the algebraic limit theorem that $(x^2 + \frac{1}{n}) \to x^2$, and so $(f_n(x)) = (\frac{x^2+\frac{1}{n}}{2+\frac{x}{n}}) \to \frac{x^2}{2} = f(x)$.

So $f'(x) = \frac{2x}{2} = x$

For each $n \in \mathbb{N}$ we have $f_n'(x) = \frac{(2nx)(2n+x)-(1)(nx^2+1)}{(2n+x)^2} = \frac{4n^2x+2nx^2-nx^2-1}{4n^2+4nx+x^2} = \frac{nx^2+4n^2x-1}{x^2+4nx+4n^2}$.

Let $M > 0$ and $\epsilon > 0$. Then for $x \in [-M, M]$ we have the following:

$x^2 + 4nx + 4n^2 = (x + 2n)^2 \geq 0$ so $|x^2 + 4nx + 4n^2| = x^2 + 4nx + 4n^2$.

Also note that $x + M \geq 0$ so $4n(x + M) \geq 0$ and $x^2 + 4n(x + M) \geq 0$ since $x^2 \geq 0$.

So $x^2 + 4n(x + M) + 4n^2 \geq 4n^2$ and therefore $x^2 + 4nx + 4n^2 \geq 4n^2 - 4nM$.

Then by letting $n > M$ we have $x^2 + 4nx + 4n^2 \geq 4n^2 - 4nM > 0$. So $0 < \frac{1}{x^2+4nx+4n^2} \leq \frac{1}{4n^2-4nM}$.

And so $|f_n'(x) - x| = |\frac{nx^2+4n^2x-1}{x^2+4nx+4n^2} - x| = |\frac{nx^2+4n^2x-1-x(x^2+4nx+4n^2)}{x^2+4nx+4n^2}| = |\frac{nx^2+4n^2x-1-x^3-4nx^2-4n^2x}{x^2+4nx+4n^2}| = |\frac{-1(x^3+3nx^2+1)}{x^2+4nx+4n^2}| = |\frac{x^3+3nx^2+1}{x^2+4nx+4n^2}| \leq |\frac{x^3}{x^2+4nx+4n^2}| + |\frac{3nx^2}{x^2+4nx+4n^2}| + |\frac{1}{x^2+4nx+4n^2}| = \frac{|x|^3}{x^2+4nx+4n^2} + \frac{3n|x|^2}{x^2+4nx+4n^2} + \frac{1}{x^2+4nx+4n^2} \leq \frac{M^3}{x^2+4nx+4n^2} + \frac{3nM^2}{x^2+4nx+4n^2} + \frac{1}{x^2+4nx+4n^2} = \frac{M^3+3nM^2+1}{x^2+4nx+4n^2} \leq \frac{M^3+3nM^2+1}{4n^2-4nM} = \frac{\frac{M^3}{n}+3M^2+\frac{1}{n}}{4n-4M}$ which clearly approaches 0 as $n \to \infty$ because the denominator gets arbitrarily large while the numerator gets arbitrarily close to $3M^2$, a constant.

Since this doesn't depend on x we can choose an N large enough so that for $n \geq N$ and $x \in [-M, M]$ we have $|f_n'(x) - x| < \epsilon$ so $(f_n'(x)) \to x$ uniformly on every interval $[-M, M]$ and therefore on $\mathbb{R}$.

So again by theorem 6.3.3 $(f_n(x)) \to f(x) = \frac{x^2}{2}$ uniformly and $(f_n'(x)) \to f'(x) = x$.

6.3.6

a.

From section 5.4 let $h(x) = |x|$ for $x \in [-1,1]$ and $h(x+2) = h(x)$ extending h to all of $\mathbb{R}$ and take $g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} h(2^n x)$, this section showed that this function is nowhere differentiable. Furthermore this function is bounded uniformly for all $x \in \mathbb{R}$ because $g(x+2) = g(x)$ by the nature of $h(x)$ and g is continuous on $[-1,1]$ and therefore the range of g on $[-1,1]$ is also compact and hence bounded, so g is bounded on uniformly on all of $\mathbb{R}$.

So define $(f_n(x)) = (\frac{g(x)}{n})$ then for each $n \in \mathbb{N}$ we have f_n is nowhere differentiable, however since g is bounded we get $(f_n(x)) \to 0$ which is clearly everywhere differentiable.

So this is such an example of a sequence of nowhere differentiable functions that converge to an everywhere differentiable function.

b. For $n \in \mathbb{N}$ let $f_n(x) = n$.

Clearly $f_n(x)$ does not converge for any $x \in \mathbb{R}$ since the sequence (n) is unbounded and therefore can not converge.

However for any fixed $n \in \mathbb{N}$ we have $f_n'(x) = 0$.

Letting $\epsilon > 0$ and $N = 1$ we get for all $n \geq 1$ and therefore for all $n \in \mathbb{N}$ when $x \in \mathbb{R}$ we have $|f_n'(x) - 0| = |0 - 0| = 0 < \epsilon$.

So $(f_n'(x)) \to 0$ uniformly but $f_n(x)$ does not converge for any $x \in \mathbb{R}$.

c. This is not possible, let (f_n) be a sequence of functions defined on A such that (f_n) and (f_n') converge uniformly.

Then we have that $(f_n') \to g$ uniformly for some function g.

Furthermore there exists an $x_0 \in A$ such that $(f_n(x_0))$ converges since (f_n) converges uniformly and therefore converges pointwise for all $x \in A$.

This implies that $(f_n) \to f$ uniformly for some function f (this we already knew) but this also implies that f is differentiable on A and $f' = g$ via the theorems in section 6.3.