Complex Series, Taylor Series, and Laurent Series

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61.3

Recall the triangle inequality $||z_1| - |z_2|| \leq |z_1 - z_2|$ for $z_1, z_2 \in \mathbb{C}$.

Let (z_n) be a complex sequence such that $\lim_{n\to\infty} z_n = z$.

Then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for $n \geq N$ we know $|z_n - z| < \epsilon$.

Now consider the sequence $(w_n) = \big(|z_n|\big)$, and let $w = |z|$.

Let $\epsilon > 0$, then let N be such that for $n \geq N$ we know $|z_n - z| < \epsilon$, such an N exists because $\lim_{n\to\infty} z_n = z$.

Then for $n \geq N$ we know $|w_n - w| = ||z_n| - |z|| \leq |z_n - z| < \epsilon$.

This was true for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So for all $\epsilon > 0$ we know there exists an $N \in \mathbb{N}$ such that $|w_n - w| = ||z_n| - |z|| < \epsilon$ for all $n \geq N$.

Therefore we have that $\lim_{n\to\infty} w_n = w$, or equivalently $\lim_{n\to\infty} |z_n| = |z|$ $\square$

61.6

Recall that if $z_n = x_n + iy_n$ is a sequence and $z = x + iy$ then $z_n \to z$ if and only if $x_n \to x$ and $y_n \to y$.

Let $z_n = x_n + iy_n$ be a sequence and assume:

$$\sum_{n=1}^{\infty} z_n = S = X + iY$$

Then we know that $S_N \to S$ where S_N is defined below:

$$S_N = \sum_{n=1}^{N} z_n = \sum_{n=1}^{N} x_n + iy_n = \sum_{n=1}^{N} x_n + i \sum_{n=1}^{N} y_n$$

Define the sequences X_N and Y_N as below:

$$X_N = \sum_{n=1}^{N} x_n \qquad\qquad Y_N = \sum_{n=1}^{N} y_n$$

We know that $X_N \to X$ and $Y_N \to Y$ as per the theorem before.

Now consider the sequence $w_n = \overline{z_n} = x_n - iy_n$.

Then we know:

$$T_N = \sum_{n=1}^{N} w_n = \sum_{n=1}^{N} x_n - iy_n = \sum_{n=1}^{N} x_n - i \sum_{n=1}^{N} y_n = X_N - iY_N$$

Again we know $X_N \to X$ and $Y_N \to Y$ so from the theorem above we know $T_N \to X - iY = \overline{S}$.

Since T_N, the sequence of partial sums of w_n, converges to $\overline{S}$ we know:

$$\sum_{n=1}^{\infty} \overline{z_n} = \sum_{n=1}^{\infty} w_n = \overline{S}$$

□

61.9

a. Let z_n be a sequence that converges to a complex number z.

Since $z_n \to z$ we know for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|z_n - z| < \epsilon$ for all $n \geq N$.

Let $\epsilon = 1$ then we know there exists an $N_0 \in \mathbb{N}$ such that $|z_n - z| < \epsilon = 1$ for all $n \geq N_0$.

This means for all $n \geq N_0$ we know $|z_n| = |z + (z_n - z)| \leq |z| + |z_n - z| < |z| + 1$.

Let $m = N_0 - 1$ (just for neatness), then let $M = max\{|z_1|, |z_2|, ..., |z_m|, |z| + 1\}$.

Such an $M > 0$ exists because the maximum of a finite set of real numbers always exists.

Let $n \in \mathbb{N}$ be arbitrary.

If $n \leq m = N_0 - 1$ we know that $|z_n| \leq M$ by construction of M.

If $n \geq N_0$ then we know that $|z_n| \leq |z| + 1 \leq M$ by construction.

This was true for arbitrary $n \in \mathbb{N}$ and hence is true for all $n \in \mathbb{N}$.

Therefore we have found an $M > 0$ such that $|z_n| \leq M$ for all $n \in \mathbb{N}$ $\square$

b. Let $z_n = x_n + iy_n$ be a sequence that converges to a complex number $z = x + iy$.

This means that $x_n \to x$ and $y_n \to y$ and so x_n and y_n are convergent real sequences.

Then we know that there exists $M_1, M_2 > 0$ such that $|x_n| \leq M_1$ and $|y_n| \leq M_2$ for all $n \in \mathbb{N}$.

Which means there exists $M_1, M_2 > 0$ such that $(x_n)^2 \leq (M_1)^2$ and $(y_n)^2 \leq (M_2)^2$ for all $n \in \mathbb{N}$.

Fix such $M_1, M_2 > 0$ then let $M = \sqrt{(M_1)^2 + (M_2)^2}$.

We know $|z_n| = |x_n + iy_n| = \sqrt{(x_n)^2 + (y_n)^2} \leq \sqrt{(M_1)^2 + (M_2)^2} = M$ for all $n \in \mathbb{N}$.

Therefore we have found an $M > 0$ such that $|z_n| \leq M$ for all $n \in \mathbb{N}$ $\square$

65.5

We are given that $sinh(z+\pi i) = -sinh\, z$ and $sinh\, z$ is $2\pi i$ periodic.

Therefore we know that $sinh\, z = -sinh(z+\pi i) = -sinh(z+\pi i - 2\pi i) = -sinh(z-\pi i)$.

Recall that $sinh\, z = \frac{e^z - e^{-z}}{2}$, $\frac{d}{dz} sinh\, z = cosh\, z$, and $\frac{d}{dz} cosh\, z = sinh\, z$.

Note that $sinh\, 0 = \frac{e^0 - e^0}{2} = 0$ and $cosh\, 0 = \frac{e^0 + e^0}{2} = 1$.

This means that the Taylor series for $f(z) = sinh\, z$ about $z_0 = 0$ is given by:

$$\sum_{n=0}^{\infty} \frac{z^n f^{(n)}(0)}{n!} = \sum_{n=0}^{\infty} \frac{z^{2n} sinh\, 0}{(2n)!} + \sum_{n=0}^{\infty} \frac{z^{2n+1} cosh\, 0}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Through the use of substitution:

$$sinh(z-\pi i) = \sum_{n=0}^{\infty} \frac{(z-\pi i)^{2n+1}}{(2n+1)!}$$

Now we use $sinh\, z = -sinh(z-\pi i)$ to get the Taylor series for $sinh\, z$ about $z_0 = \pi i$ as the following:

$$sinh\, z = -sinh(z-\pi i) = -\sum_{n=0}^{\infty} \frac{(z-\pi i)^{2n+1}}{(2n+1)!}$$

65.6

Recall that $tanh\, z = \frac{sinh\, z}{cosh\, z}$ and also that $\frac{d}{dz} tanh\, z = sech^2 z = \frac{1}{cosh^2 z}$.

Note that $sinh\, z$ and $cosh\, z$ are entire, so clearly $tanh\, z$ is well defined and hence analytic wherever $cosh\, z \neq 0$.

We also know $cosh\, z = 0$ if and only if $z = (\frac{\pi}{2} + n\pi)i$ where $n \in \mathbb{Z}$.

So $tanh\, z$ is well defined and analytic when $z \neq (\frac{\pi}{2} + n\pi)i$ where $n \in \mathbb{Z}$.

The two closest zeros of $cosh\, z$ to $z_0 = 0$ are $z = \pm\frac{\pi}{2} i$ and hence $tanh\, z$ is analytic inside (but not on) the circle $|z| = \frac{\pi}{2}$.

So $tanh\, z$ is analytic throughout $|z| < R = \frac{\pi}{2}$ and hence has a power series representation (which is a Maclaurin series since they are the special case of being centered at 0) with a radius of convergence $R = \frac{\pi}{2}$.

We are now asked to find the first two nonzero terms of the Maclaurin series representation for $tanh\, z$.

So $\frac{d^2}{dz^2} tanh\, z = \frac{d}{dz}(\frac{d}{dz} tanh\, z) = \frac{d}{dz} sech^2 z = (2 sech\, z)\frac{d}{dz} sech\, z = -2 sech^2 z\, tanh\, z$.

Also $\frac{d^3}{dz^3} tanh\, z = \frac{d^2}{dz^2}(\frac{d}{dz} tanh\, z) = \frac{d}{dz}(\frac{d}{dz} sech^2 z) = \frac{d}{dz} - 2 sech^2 z\, tanh\, z =$

$-2(-2 sech^2 z\, tanh\, z(tanh\, z) + sech^2 z(sech^2 z)) = (2 sech\, z\, tanh\, z)^2 - 2 sech^4 z$.

Now we have $tanh\, 0 = \frac{sinh\, 0}{cosh\, 0} = 0$, also $\frac{d}{dz} tanh\, z\Big|_{z=0} = sech^2 0 = \frac{1}{cosh^2 0} = 1$, also

$\frac{d^2}{dz^2} tanh\, z\Big|_{z=0} = -2 sech^2 0\, tanh\, 0 = -2\frac{1}{cosh^2 0}\frac{sinh\, 0}{cosh\, 0} = 0$, finally

$\frac{d^3}{dz^3} tanh\, z\Big|_{z=0} = (2 sech\, 0\, tanh\, 0)^2 - 2 sech^4 0 = (2\frac{1}{cosh\, 0}\frac{sinh\, 0}{cosh\, 0})^2 - 2\frac{1}{cosh^4 0} = -2$.

So the first two nonzero terms of the Maclaurin series representation for $f(z) = tanh\, z$ are:

$f^{(1)}(0)\frac{(z-0)}{1!} = z$ and $f^{(3)}(0)\frac{(z-0)^3}{3!} = -2\frac{z^3}{6} = -\frac{1}{3}z^3$

65.10

b. Let $f(z) = \frac{sin(z^2)}{z^4}$ where $z \neq 0$.

Recall that $\frac{d}{dz} sin\, z = cos\, z$ and $\frac{d}{dz} cos\, z = -sin\, z$. Then we get the Taylor series for $g(z) = sin\, z$ about $z_0 = 0$ is given by:

$$\sum_{n=0}^{\infty} \frac{z^n g^{(n)}(0)}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} sin\, 0}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1} cos\, 0}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$$

Since $sin\ z$ is entire this expansion is valid for all $z \in \mathbb{C}$, or equivalently valid for $|z| < \infty$.

So by substituting z^2 for z we can get the Taylor series for $sin(z^2)$ about $z_0 = 0$ is:

$$sin(z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (z^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{(2n+1)!} = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + ...$$

Which again is valid for all $z \in \mathbb{C}$ since z^2 is also entire, so it's valid for $|z| < \infty$.

Then as long as $z \neq 0$ we may divide both sides by z^4 to get:

$$\frac{sin(z^2)}{z^4} = \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{z^4 (2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n-2}}{(2n+1)!} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + ...$$

Which is valid when $z \neq 0$ since before it was valid over $\mathbb{C}$ but now we are dividing by z^4, so it's valid for $0 < |z| < \infty$.

68.5

Let $f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$, we know f has the two singular points $z = 1$ and $z = 2$.

We also know f is analytic on the domains $D_1 : |z| < 1$, $D_2 : 1 < |z| < 2$, and $D_3 : 2 < |z| < \infty$.

- For D_1: Clearly D_1 is the inside of a circle and f is analytic on D_1 so f has a Taylor series representation over D_1.

We know $\frac{d}{dz}\frac{1}{z-1} = \frac{-1}{(z-1)^2}$ and $\frac{d^2}{dz^2}\frac{1}{z-1} = \frac{d}{dz}(\frac{d}{dz}\frac{1}{z-1}) = \frac{d}{dz}\frac{-1}{(z-1)^2} = \frac{2}{(z-1)^3}$. In general $\frac{d^n}{dz^n}\frac{1}{z-1} = \frac{n!(-1)^n}{(z-1)^{n+1}}$.

Similarly $\frac{d^n}{dz^n}\frac{1}{z-2} = \frac{n!(-1)^n}{(z-2)^{n+1}}$.

So we have the Taylor series representation for $g(z) = \frac{1}{z-1}$ at $z_0 = 0$ is:

$$\sum_{n=0}^{\infty} \frac{z^n g^{(n)}(0)}{n!} = \sum_{n=0}^{\infty} \frac{-z^n n!}{n!} = -\sum_{n=0}^{\infty} z^n$$

Again we are considering this over $D_1 : |z| < 1$ so we know this will converge since the series converges absolutely since in absolute value it's a geometric series with ratio less than 1.

Similarly the Taylor series representation for $h(z) = \frac{1}{z-2}$ at $z_0 = 0$ is:

$$\sum_{n=0}^{\infty} \frac{z^n h^{(n)}(0)}{n!} = \sum_{n=0}^{\infty} \frac{-z^n n!}{n! 2^{n+1}} = -\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

Again we are considering this over $D_1 : |z| < 1$ so we know this will converge since the series converges absolutely since in absolute value it's a geometric series with ratio less than 1.

So for $f(z)$ over $D_1 : |z| < 1$ we get the Taylor series representation at $z_0 = 0$ is:

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = -\sum_{n=0}^{\infty} z^n - \left(-\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \right) = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^n \Big(\frac{1}{2^{n+1}} - 1 \Big)$$

The cases for D_2 and D_3 are continued on the next pages.

- For D_2: Clearly D_2 is an annular domain and f is analytic on D_2 so f has a Laurent series representation over D_2.

We already know that $\frac{1}{w-1}$ has a Taylor series representation when $|w| < 1$ (which was found in part a).

If $|z| > 1$ then we know $|\frac{1}{z}| = \frac{1}{|z|} < 1$.

Therefore substituting $\frac{1}{z}$ for w we get the Taylor series representation for $\frac{1}{\frac{1}{z}-1}$ when $|z| > 1$ is:

$$\frac{1}{\frac{1}{z}-1} = -\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{z^n}$$

Since D_2 is given by $1 < |z| < 2$ we know over D_2 that:

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} = -\frac{1}{z(\frac{1}{z}-1)} = -\frac{1}{z}\left(\frac{1}{\frac{1}{z}-1}\right) = -\frac{1}{z}\left(-\sum_{n=0}^{\infty}\frac{1}{z^n}\right) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

Similarly we know that $\frac{1}{z-2}$ has a Taylor series representation when $|z| < 2$ (which was found in part a).

Since D_2 is given by $1 < |z| < 2$ we know over D_2 that:

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

So for $f(z)$ over $D_2 : 1 < |z| < 2$ we get the series representation at $z_0 = 0$ is:

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \left(-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

The case of D_3 is continued on the next page.

- For D_3: Clearly D_3 is an annular domain and f is analytic on D_3 so f has a Laurent series representation over D_3.

We saw in the previous part that the Taylor series representation for $\frac{1}{\frac{1}{z}-1}$ when $|z| > 1$ is:

$$\frac{1}{\frac{1}{z}-1} = -\sum_{n=0}^{\infty} \frac{1}{z^n}$$

Since D_3 is given by $|z| > 2$ we know over D_3 that:

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} = -\frac{1}{z(\frac{1}{z}-1)} = -\frac{1}{z}\left(\frac{1}{\frac{1}{z}-1}\right) = -\frac{1}{z}\left(-\sum_{n=0}^{\infty}\frac{1}{z^n}\right) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

We already know that $\frac{1}{w-1}$ has a Taylor series representation when $|w| < 1$ (which was found in part a).

If $|z| > 2$ then we know $|\frac{2}{z}| = \frac{2}{|z|} < 1$.

Therefore substituting $\frac{2}{z}$ for w we get the Taylor series representation for $\frac{1}{\frac{2}{z}-1}$ when $|z| > 2$ is:

$$\frac{1}{\frac{2}{z}-1} = -\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{2^n}{z^n}$$

Since D_3 is given by $|z| > 2$ we know over D_3 that:

$$\frac{1}{z-2} = \frac{1}{z(1-\frac{2}{z})} = -\frac{1}{z(\frac{2}{z}-1)} = -\frac{1}{z}\left(\frac{1}{\frac{2}{z}-1}\right) = -\frac{1}{z}\left(-\sum_{n=0}^{\infty}\frac{2^n}{z^n}\right) = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

So for $f(z)$ over $D_3 : |z| > 2$ we get the series representation at $z_0 = 0$ is:

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}}$$