Functions, Limits, Differentiation

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18.1

b. Let $f(z) = \overline{z}$. Then consider some arbitrary $z_0 \in \mathbb{C}$.

Recall that for $z, z_1, z_2 \in \mathbb{C}$ we know $|z| = |\overline{z}|$ and $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ (from which it follows $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$).

Now let $\epsilon > 0$. Then let $\delta = \epsilon$.

Then if $|z - z_0| < \delta$ we have $|f(z) - \overline{z_0}| = |\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0| < \delta = \epsilon$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

Therefore we have $lim_{z \to z_0} \overline{z} = \overline{z_0}$.

This was also for arbitrary $z_0 \in \mathbb{C}$ and is therefore true for all $z_0 \in \mathbb{C}$.

So $lim_{z \to z_0} \overline{z} = \overline{z_0}$ for all $z_0 \in \mathbb{C}$ $\square$

18.2

c. Let $f(z) = f(x + iy) = x + i(2x + y)$ where $z = x + iy$. Then we are considering the point $z_0 = 1 - i$.

Recall that for $z \in \mathbb{C}$ we know $|z| \geq |Im\ z|$ and similarly $|z| \geq |Re\ z|$.

Further recall that for $z_1, z_2, ..., z_n \in \mathbb{C}$ we know $|z_1 + z_2| \leq |z_1| + |z_2|$ and $|z_1 z_2| = |z_1||z_2|$.

Now let $\epsilon > 0$. Then let $\delta = \frac{\epsilon}{3}$.

Then if $|z - z_0| = |z - (1 - i)| = |(x - 1) + i(y + 1)| < \delta$ we have

$$|f(z) - (1 + i)| = |x + i(2x + y) - (1 + i)| = |(x - 1) + i(2x + y - 1)| = |(x - 1) + i(2x + y + 1 - 1 - 1)| =$$

$$|(x - 1) + i((2x - 2) + (y + 1))| = |(x - 1) + i(y + 1) + 2i(x - 1))| \leq |(x - 1) + i(y + 1)| + |2i(x - 1)| =$$

$$|(x - 1) + i(y + 1)| + |2i||x - 1| = |(x - 1) + i(y + 1)| + 2|x - 1| = |z - (1 - i)| + 2|Re\ (z - (1 - i))| \leq$$

$$|z - (1 - i)| + 2|z - (1 - i)| = 3|z - (1 - i)| = 3|z - z_0| < 3\delta = \epsilon$$

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So $lim_{z \to 1 - i} f(z) = 1 + i$ $\square$

18.3

Recall the limit theorems that if f, g are complex valued functions such that:

$lim_{z\to z_0} f(z) = w_0$ and $lim_{z\to z_0} g(z) = W_0$ for some $z_0 \in \mathbb{C}$.

Then it follows that:

$$lim_{z\to z_0} f(z) + g(z) = w_0 + W_0$$

$$lim_{z\to z_0} f(z)g(z) = w_0 W_0$$

And if $W_0 \neq 0$ then $lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{w_0}{W_0}$.

It was also given that for a polynomial $P(z)$, we know $lim_{z\to z_0} P(z) = P(z_0)$ for all $z_0 \in \mathbb{C}$.

a. Let $n \in \mathbb{N}$ then consider the function $f(z) = \frac{1}{z^n}$.

As seen in the book $lim_{z\to z_0} z^n = (z_0)^n$ for all $z_0 \in \mathbb{C}$.

Now let $z_0 \neq 0$, then we have that $lim_{z\to z_0} z^n = (z_0)^n \neq 0$ since $\mathbb{C}$ has no zero divisors.

Therefore $lim_{z\to z_0} f(z) = lim_{z\to z_0} \frac{1}{z^n} = \frac{1}{(z_0)^n} = (\frac{1}{z_0})^n$ since clearly $lim_{z\to z_0} 1 = 1$.

This was true for arbitrary $z_0 \neq 0$ and is therefore true for all $z_0 \neq 0$.

This was also true for arbitrary $n \in \mathbb{N}$ and is therefore true for all $n \in \mathbb{N}$.

So $lim_{z\to z_0} \frac{1}{z^n} = (\frac{1}{z_0})^n$ for all $n \in \mathbb{N}$ and all complex numbers $z_0 \neq 0$ □

b. Let $f(z) = \frac{iz^3 - 1}{z + i}$. Then we are considering the point $z_0 = i$.

If we let $g(z) = iz^3 - 1$ and $h(z) = z + i$ we have that $f(z) = \frac{g(z)}{h(z)}$.

We know from the previous problem that $lim_{z\to z_0} z^3 = (z_0)^3$.

Clearly $lim_{z\to z_0} i = i$ and $lim_{z\to z_0} -1 = -1$.

So by the limit theorems we know that $lim_{z\to z_0} g(z) = lim_{z\to z_0} iz^3 - 1 = i(z_0)^3 - 1 = i^4 - 1 = 0$.

Similarly we know $lim_{z\to z_0} h(z) = lim_{z\to z_0} z + i = z_0 + i = 2i$.

Since $lim_{z\to z_0} h(z) = 2i \neq 0$ we know that:

$lim_{z\to z_0} f(z) = lim_{z\to z_0} \frac{g(z)}{h(z)} = \frac{0}{2i} = 0$ (where $z_0 = i$ as stated before) □

c. Let $P(z)$ and $Q(z)$ be complex valued polynomials with complex coefficients. Then let $f(z) = \frac{P(z)}{Q(z)}$.

By the limit theorems $lim_{z\to z_0} P(z) = P(z_0)$ and $lim_{z\to z_0} Q(z) = Q(z_0)$ for all $z_0 \in \mathbb{C}$.

If $z_0 \in \mathbb{C}$ is such that $lim_{z\to z_0} Q(z) = Q(z_0) \neq 0$ then we know that:

$$lim_{z\to z_0} f(z) = lim_{z\to z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q(z_0)}.$$

This was true for arbitrary $z_0 \in \mathbb{C}$ where $Q(z_0) \neq 0$ and is therefore true for all $z_0 \in \mathbb{C}$ where $Q(z_0) \neq 0$.

So $lim_{z\to z_0} f(z) = lim_{z\to z_0} \frac{P(z)}{Q(z)} = \frac{P(z_0)}{Q(z_0)}$ for all $z_0 \in \mathbb{C}$ where $Q(z_0) \neq 0$ □

18.9

Let f, g be complex valued functions such that:

$lim_{z \to z_0} f(z) = 0$ and there exists an $M > 0$ such that $|g(z)| \leq M$ in some neighborhood of z_0.

Fix an $M > 0$ such that there exists a neighborhood of z_0 where $|g(z)| \leq M$.

Let $\alpha > 0$ be such that $|g(z)| \leq M$ for all $z \in V_\alpha(z_0)$, that is $|z - z_0| < \alpha$ implies $|g(z)| \leq M$.

Now let $\epsilon > 0$. Then let $\gamma > 0$ be such that if $|z - z_0| < \gamma$ it follows that $|f(z) - 0| = |f(z)| < \frac{\epsilon}{M}$.

Now let $\delta < min\{\alpha, \gamma\}$.

Then we have if $|z - z_0| < \delta$ it must be $|z - z_0| < \alpha$ and $|z - z_0| < \gamma$.

Therefore it follows that if $|z - z_0| < \delta$ we have $|f(z)| < \frac{\epsilon}{M}$ and $|g(z)| \leq M$.

Then if $|z - z_0| < \delta$ we have $|f(z)g(z) - 0| = |f(z)||g(z)| \leq M|g(z)| < M\frac{\epsilon}{M} = \epsilon$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So if $lim_{z \to z_0} f(z) = 0$ and $g(z)$ is bounded in some neighborhood of z_0 then we have that $lim_{z \to z_0} f(z)g(z) = 0$ $\square$

18.10

Recall that:

$lim_{z \to \infty} f(z) = w$ if $lim_{z \to 0} f(\frac{1}{z}) = w$.

$lim_{z \to z_0} f(z) = \infty$ if $lim_{z \to z_0} \frac{1}{f(z)} = 0$.

$lim_{z \to \infty} f(z) = \infty$ if $lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$.

Also recall the limit theorems mentioned in problem 18.3 about limits of polynomials and limits of divisions of functions.

a. Let $f(z) = \frac{4z^2}{(z-1)^2}$, then for $z \neq 0$ we have $f(\frac{1}{z}) = \frac{4\frac{1}{z^2}}{(\frac{1}{z}-1)^2} = \frac{4}{z^2(\frac{1}{z}-1)^2} = \frac{4}{(z(\frac{1}{z}-1))^2} = \frac{4}{(1-z)^2}$.

Then $P(z) = (1-z)^2 = 1 - 2z + z^2$ is a complex polynomial and therefore $lim_{z \to 0} P(z) = P(0) = (1-0)^2 = 1$.

Since $lim_{z \to 0}(1-z)^2 = 1 \neq 0$ we have that $lim_{z \to 0} f(\frac{1}{z}) = lim_{z \to 0} \frac{4}{(1-z)^2} = 4$ since clearly $lim_{z \to 0} 4 = 4$.

Therefore since $lim_{z \to 0} f(\frac{1}{z}) = 4$ we have that $lim_{z \to \infty} f(z) = 4$ $\square$

b. Let $f(z) = \frac{1}{(z-1)^3}$, then we have $\frac{1}{f(z)} = \frac{1}{1/(z-1)^3} = (z-1)^3$.

Then $P(z) = (z-1)^3 = z^3 - 3z^2 + 3z - 1$ is a complex polynomial and therefore $lim_{z \to 1} P(z) = P(1) = (1-1)^3 = 0$.

So we have that $lim_{z \to 1} \frac{1}{f(z)} = lim_{z \to 1}(z-1)^3 = 0$.

Therefore since $lim_{z \to 1} \frac{1}{f(z)} = 0$ we have that $lim_{z \to 1} f(z) = \infty$ $\square$

c. Let $f(z) = \frac{z^2+1}{z-1}$, then for $z \neq 0$ we have $f(\frac{1}{z}) = \frac{(\frac{1}{z})^2+1}{\frac{1}{z}-1} = \frac{z^2(\frac{1}{z^2}+1)}{z^2(\frac{1}{z}-1)} = \frac{1+z^2}{z-z^2}$. So $\frac{1}{f(\frac{1}{z})} = \frac{1}{(1+z^2)/(z-z^2)} = \frac{z-z^2}{1+z^2}$.

Then $P(z) = z - z^2$ is a complex polynomial and therefore $lim_{z \to 0} P(z) = P(0) = 0$.

Similarly $Q(z) = 1 + z^2$ is a complex polynomial and therefore $lim_{z \to 0} Q(z) = Q(0) = 1$.

Since $lim_{z \to 0} 1 + z^2 = 1 \neq 0$ we have that $lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = lim_{z \to 0} \frac{z-z^2}{1+z^2} = \frac{0}{1} = 0$.

Therefore since $lim_{z \to 0} \frac{1}{f(\frac{1}{z})} = 0$ we have that $lim_{z \to \infty} f(z) = \infty$ $\square$

18.13

Recall that an ϵ neighborhood of the point of infinity is given by $|z| > \frac{1}{\epsilon}$.

Further recall that a set S is bounded if there exists an $R > 0$ such that every element of S is inside of the circle $|z| = R$.

Let $S \subseteq \mathbb{C}$ be an arbitrary complex set.

First, assume that S is unbounded:

Then there does not exist an $R > 0$ such that all elements of S are contained in the circle $|z| = R$.

This means that for all $R > 0$ there must be at least one element of S on the boundary of or outside of the circle $|z| = R$.

Therefore for all $R > 0$ there exists some $z \in S$ where $|z| \geq R$.

Let $\epsilon > 0$, then $\frac{1}{\epsilon}$ is well defined. Now let $R \in \mathbb{R}$ be such that $R > \frac{1}{\epsilon}$.

Such an R exists due to the unboundedness of $\mathbb{R}$.

Then we know $R > 0$ and therefore there exists some $z_0 \in S$ such that $|z_0| \geq R > \frac{1}{\epsilon}$.

So we have found a $z \in S$ where z is in an ϵ neighborhood of the point at infinity.

This was true for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

Therefore if S is unbounded then there exists some element of S in every neighborhood of the point at infinity.

Now, assume that every neighborhood of the point at infinity contains some element of S:

Then for all $\epsilon > 0$ there exists some $z \in S$ such that $|z| > \frac{1}{\epsilon}$.

Let $R > 0$, then let $\epsilon > 0$ be such that $R \leq \frac{1}{\epsilon}$.

Such an ϵ exists due to the unboundedness of $\mathbb{R}$.

So we know there exists some $z_0 \in S$ such that $|z_0| > \frac{1}{\epsilon} \geq R$.

So there exists some element of S that lies outside of the circle $|z| = R$.

This was true for arbitrary $R > 0$ and is therefore true for all $R > 0$.

So there does not exist an $R > 0$ such that every element of S is contained in the circle $|z| = R$.

This means that S is not bounded, and hence is unbounded.

Therefore if every neighborhood of the point at infinity contains some element of S then S is unbounded.

So $S \subseteq \mathbb{C}$ is unbounded if and only if every neighborhood of the point at infinity contains some element of S $\square$

20.1

Recall the limit theorems mentioned in problem 18.3 about limits of polynomials.

Let $f(z) = w = z^2$.

Then $\Delta w = f(z + \Delta z) - f(z) = (z + \Delta z)^2 - z^2 = z^2 + 2z\Delta z + (\Delta z)^2 - z^2 = (\Delta z)^2 + 2z\Delta z$.

Therefore when $\Delta z \neq 0$ we get $\frac{\Delta w}{\Delta z} = \frac{(\Delta z)^2 + 2z\Delta z}{\Delta z} = \frac{\Delta z(\Delta z + 2z)}{\Delta z} = \Delta z + 2z$.

Since z does not depend on Δz we can write $P(\Delta z) = \Delta z + 2z$ as a complex polynomial in Δz where z acts as a constant.

Hence $lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = lim_{\Delta z \to 0} \Delta z + 2z = lim_{\Delta z \to 0} P(\Delta z) = P(0) = 2z$.

Therefore since $lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = 2z$ we have $\frac{dw}{dz} = 2z$ $\square$

20.2

Recall the following:

$$\frac{d}{dz}(f(z)+g(z)) = \frac{d}{dz}f(z) + \frac{d}{dz}g(z) \text{ for any two differentiable functions } f, g.$$

$$\frac{d}{dz}cf(z) = c\frac{d}{dz}f(z) \text{ and } \frac{d}{dz}c = 0 \text{ for all } c \in \mathbb{C}.$$

$$\frac{d}{dz}z^n = nz^{n-1} \text{ for all } n \in \mathbb{N}$$

$$\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{g(z)f'(z)-f(z)g'(z)}{(g(z))^2} \text{ when } g(z) \neq 0 \text{ for any two differentiable functions } f, g.$$

$$\frac{d}{dz}f(g(z)) = f'(g(z))g'(z) \text{ for any two differentiable functions } f, g.$$

a. Let $f(z) = 3z^2 - 2z + 4$.

We know from the theorems above that $\frac{d}{dz}3z^2 = 3\frac{d}{dz}z^2 = 6z$, $\frac{d}{dz}(-2z) = -2\frac{d}{dz}z = -2$, and $\frac{d}{dz}4 = 0$.

Therefore $\frac{d}{dz}f(z) = \frac{d}{dz}(3z^2 - 2z + 4) = \frac{d}{dz}3z^2 + \frac{d}{dz}(-2z) + \frac{d}{dz}4 = 6z - 2$ □

b. Let $f(z) = (2z^2 + i)^5$.

Then we can write $f(z) = g(h(z))$ where $g(z) = z^5$ and $h(z) = 2z^2 + i$.

We know from the theorems above that $\frac{d}{dz}g(z) = \frac{d}{dz}z^5 = 5z^4$ and $\frac{d}{dz}h(z) = \frac{d}{dz}(2z^2 + i) = \frac{d}{dz}(2z^2) + \frac{d}{dz}i = 2\frac{d}{dz}z^2 = 4z$.

We also know that $\frac{d}{dz}f(z) = \frac{d}{dz}g(h(z)) = g'(h(z))h'(z)$.

So we have that $\frac{d}{dz}f(z) = 5(2z^2 + i)^4(4z) = 20z(2z^2 + i)^4$ □

c. Let $f(z) = \frac{z-1}{2z+1}$ where $z \neq -\frac{1}{2}$.

Then we can write $f(z) = \frac{g(z)}{h(z)}$ where $g(z) = z - 1$ and $h(z) = 2z + 1$.

We know from the theorems above that:

$$\frac{d}{dz}g(z) = \frac{d}{dz}(z-1) = \frac{d}{dz}z + \frac{d}{dz}(-1) = 1 \text{ and } \frac{d}{dz}h(z) = \frac{d}{dz}(2z+1) = \frac{d}{dz}(2z) + \frac{d}{dz}1 = 2\frac{d}{dz}z = 2.$$

We also know that $\frac{d}{dz}f(z) = \frac{d}{dz}\frac{g(z)}{h(z)} = \frac{h(z)g'(z)-g(z)h'(z)}{(h(z))^2}$ when $z \neq -\frac{1}{2}$.

So we have that $\frac{d}{dz}f(z) = \frac{(2z+1)(1)-(z-1)(2)}{(2z+1)^2} = \frac{2z+1-2z+2}{(2z+1)^2} = \frac{3}{(2z+1)^2}$ when $z \neq -\frac{1}{2}$ □

d. Let $f(z) = \frac{(1+z^2)^4}{z^2}$ where $z \neq 0$.

Then we can write $f(z) = \frac{l(z)}{k(z)} = \frac{g(h(z))}{k(z)}$ where $g(z) = z^4$, $h(z) = 1 + z^2$, and $k(z) = z^2$, and $l(z) = g(h(z))$.

We know from the theorems above that:

$$\frac{d}{dz}g(z) = \frac{d}{dz}(z^4) = 4z^3, \ \frac{d}{dz}h(z) = \frac{d}{dz}(1+z^2) = \frac{d}{dz}1 + \frac{d}{dz}z^2 = 2z, \text{ and } \frac{d}{dz}k(z) = \frac{d}{dz}z^2 = 2z.$$

We also know that $\frac{d}{dz}l(z) = \frac{d}{dz}g(h(z)) = g'(h(z))h'(z)$.

Furthermore we know $\frac{d}{dz}f(z) = \frac{d}{dz}\frac{l(z)}{k(z)} = \frac{k(z)l'(z)-l(z)k'(z)}{(k(z))^2}$.

So we have that

$$\frac{d}{dz}f(z) = \frac{(z^2)(4(1+z^2)^3(2z)) - ((1+z^2)^4)(2z)}{(z^2)^2} = \frac{8z^3(1+z^2)^3 - 2z(1+z^2)^4}{z^4} =$$

$$\frac{2z(1+z^2)^3(4z^2 - (1+z^2))}{z^4} = \frac{2(1+z^2)^3(3z^2-1)}{z^3}$$

for $z \neq 0$ □

20.7

Recall the following:

$\frac{d}{dz}c = 0$ for all $c \in \mathbb{C}$.

$\frac{d}{dz}z^n = nz^{n-1}$ for all $n \in \mathbb{N}$.

$\frac{d}{dz}\frac{f(z)}{g(z)} = \frac{g(z)f'(z)-f(z)g'(z)}{(g(z))^2}$ when $g(z) \neq 0$ for any two differentiable functions f, g.

Let $n \in \mathbb{N}$ be arbitrary. Then let $m = -n$ and $z \neq 0$.

Then we have that $z^m = z^{-n} = \frac{1}{z^n}$.

We know from the theorems above that:

$$\frac{d}{dz}z^m = \frac{d}{dz}\frac{1}{z^n} = \frac{z^n(\frac{d}{dz}1) - 1(\frac{d}{dz}z^n)}{(z^n)^2} = \frac{-nz^{n-1}}{z^{2n}} = \frac{-n}{z^{2n-(n-1)}} = \frac{-n}{z^{n+1}} = -nz^{-(n+1)} = -nz^{-n-1} = mz^{m-1}$$

This was for arbitrary $n \in \mathbb{N}$ and is therefore true for all $n \in \mathbb{N}$.

So $\frac{d}{dz}z^{-n} = -nz^{-n-1}$ for all $n \in \mathbb{N}$ when $z \neq 0$.

Rewriting with $m = -n$ we get $\frac{d}{dz}z^m = mz^{m-1}$ for all $m \in \{-1, -2, -3, ...\} = \mathbb{Z}^-$ when $z \neq 0$ $\square$

20.8

Recall that for a complex valued function $f(z)$ and $z_0 \in \mathbb{C}$ in order for $lim_{z \to z_0} f(z)$ to exist the limit must be the same from all directions you can approach z_0 due to the uniqueness of limits.

a. Let $f(z) = Re\, z$. Then for $z = x + iy$ we can write $f(z) = f(x + iy) = x$.

Let us begin by considering approaching along the real axis.

That is we take $\Delta y = 0$ giving $\Delta z = \Delta x + i\Delta y = \Delta x$.

Then using this approach we get:

$$f(z + \Delta z) - f(z) = Re\,(z + \Delta z) - Re\, z = Re\,(x + iy + \Delta x) - Re\,(x + iy) = x + \Delta x - x = \Delta x.$$

Therefore $\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\Delta x}{\Delta x} = 1$ where $z = x + iy$.

This gives $lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = lim_{\Delta z \to 0} 1 = 1$.

Now consider approaching along the imaginary axis.

That is we take $\Delta x = 0$ giving $\Delta z = \Delta x + i\Delta y = i\Delta y$.

Then using this approach we get:

$$f(z + \Delta z) - f(z) = Re\,(z + \Delta z) - Re\, z = Re\,(x + iy + i\Delta y) - Re\,(x + iy) = x - x = 0.$$

Therefore $\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{0}{\Delta x} = 0$ where $z = x + iy$.

This gives $lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = lim_{\Delta z \to 0} 0 = 0$.

Since $0 \neq 1$ we have that $lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$ does not exist. Therefore $\frac{d}{dz} Re\, z$ does not exist anywhere $\square$

b. Let $f(z) = Im\, z$. Then for $z = x + iy$ we can write $f(z) = f(x + iy) = y$.

Let us begin by considering approaching along the real axis.

That is we take $\Delta y = 0$ giving $\Delta z = \Delta x + i\Delta y = \Delta x$.

Then using this approach we get:

$$f(z + \Delta z) - f(z) = Im\,(z + \Delta z) - Im\, z = Im\,(x + iy + \Delta x) - Im\,(x + iy) = y - y = 0.$$

Therefore $\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{0}{\Delta y} = 0$ where $z = x + iy$.

This gives $lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = lim_{\Delta z \to 0} 0 = 0$.

Now consider approaching along the imaginary axis.

That is we take $\Delta x = 0$ giving $\Delta z = \Delta x + i\Delta y = i\Delta y$.

Then using this approach we get:

$$f(z + \Delta z) - f(z) = Im\,(z + \Delta z) - Im\, z = Im\,(x + iy + i\Delta y) - Im\,(x + iy) = y + \Delta y - y = \Delta y.$$

Therefore $\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\Delta y}{\Delta y} = 1$ where $z = x + iy$.

This gives $lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = lim_{\Delta z \to 0} 1 = 1$.

Since $0 \neq 1$ we have that $lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z}$ does not exist. Therefore $\frac{d}{dz} Im\, z$ does not exist anywhere $\square$

Extra Problem

Let the Riemann sphere be the unit sphere centered at the origin.

We are asked to find the point on the Riemann sphere corresponding to $z_0 = 2 + 3i$.

I will actually provide a method for finding the corresponding point P on the Riemann sphere for any complex number.

Let $z \in \mathbb{C}$ then we can write $z = x + iy$, or in three dimensions $z = (x, y, 0)$.

Then since the Riemann sphere is the unit sphere centered at the origin we know $N = (0, 0, 1)$ is its north pole.

We can construct a parametric equation of a line segment connecting z and N.

This will be given by: $tz + (1-t)N = t(x, y, 0) + (1-t)(0, 0, 1) = (tx, ty, 0) + (0, 0, 1-t) = (tx, ty, 1-t)$ for $t \in [0, 1]$.

Notice at $t = 0$ we get $0z + (1-0)N = N$ and at $t = 1$ we get $z + (1-1)N = z$.

What we are doing here is taking the vector going from N to z (which is $z - N = (x, y, 0) - (0, 0, 1) = (x, y, -1)$), scaling its size with a parameter t. We do this because the direction of our parameterized vector stays the same, we are just varying the distance in order to be able to "travel" to any point connecting N and z.

Then $t(z - N) = t(x, y, -1) = (tx, ty, -t)$, but this vector does not start at N as we want (it starts at the origin) so we add N to recenter it.

This finally gives $t(z - N) + N = (tx, ty, -t) + (0, 0, 1) = (tx, ty, 1-t)$, the equation of our parametric line.

Now that we have an equation representing the line segment connecting N and z we can find P on the sphere corresponding to z.

Since this point P lies on the sphere (which is a unit sphere) we know its distance from the origin must be 1.

The distance from the origin of any point on our parametric line is $\sqrt{(tx)^2 + (ty)^2 + (1-t)^2}$, using the normal Euclidean norm.

Therefore we must have $||P|| = \sqrt{(tx)^2 + (ty)^2 + (1-t)^2} = 1$ and hence $(tx)^2 + (ty)^2 + (1-t)^2 = 1$.

So $x^2t^2 + y^2t^2 + 1 - 2t + t^2 = t^2(x^2 + y^2 + 1) - 2t + 1 = 1$ and $t^2(x^2 + y^2 + 1) = 2t$.

We know $t \neq 0$ because at $t = 0$ we are at N, the point at infinity but we are dealing with a finite $z \in \mathbb{C}$.

Therefore we can divide both sides by t and get $t(x^2 + y^2 + 1) = 2$ and get $t = \frac{2}{x^2+y^2+1} = \frac{2}{|z|^2+1}$.

Plugging this t back into our parametric equation we get:

$$P = (tx, ty, 1-t) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, 1 - \frac{2}{x^2+y^2+1}\right) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right)$$

Finally:

$$P = \left(\frac{2Re\ z}{|z|^2+1}, \frac{2Im\ z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$

For our particular desired value, $z_0 = 2 + 3i$ we know $|z_0|^2 = 2^2 + 3^2 = 13$, $Re\ z_0 = 2$, and $Im\ z_0 = 3$.

Therefore $P_{z_0} = \left(\frac{4}{13+1}, \frac{6}{13+1}, \frac{13-1}{13+1}\right) = \left(\frac{4}{14}, \frac{6}{14}, \frac{12}{14}\right) = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right)$ $\square$