Sequences and Convergence

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2.2.8

(x_n) is "zero-heavy" if there exists an $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ where $N \leq n \leq N + M$ such that $x_n = 0$.

a. Let $(a_n) = (0, 1, 0, 1, 0, 1, 0, 1, ...)$ where for $n \in \mathbb{N}$, $a_n = 0$ if n is odd and $a_n = 1$ if n is even.

By our definition of zero-heavy this sequence a_n is zero heavy.

Let $M = 1$ then since for all $n \in \mathbb{N}$ either $a_n = 0$ or $a_{n+1} = 0$ so we have found an M such that for all $N \in \mathbb{N}$ there exists an n where $N \leq n \leq N + M$ such that $a_n = 0$ so $(a_n) = (0, 1, 0, 1, 0, 1, ...)$ is zero-heavy $\square$

b. If a sequence is zero-heavy it must contain an infinite number of zeros.

Proof:

Let (a_n) be a sequence that contains only a finite number of zeros.

Then for some $m \in \mathbb{N}$ we know that a_m is the final zero in the sequence. Consider $N \in \mathbb{N}$ where $N > m$ then $a_N \neq 0$.

So for any $M \in \mathbb{N}$ there does not exist an $n \in \mathbb{N}$ where $N \leq n \leq N + M$ and $a_n = 0$ since a_m is the final zero and $n > m$.

Consequently this (a_n) can not be zero-heavy.

This was for an arbitrary sequence with finitely many zeros so it is true for all such sequences.

So if a sequence (x_n) contains only a finite number of zeros it cannot be zero-heavy. Therefore if a sequence is zero-heavy it must contain an infinite number of zeros $\square$

c. Let (a_n) be a sequence such that the distance between zero number n and zero number $n + 1$ is strictly increasing.

Then for any $M \in \mathbb{N}$ we can choose $N \in \mathbb{N}$ large enough so there are at least $M + 1$ non-zero terms until the next zero.

So there does not exist an $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ where $N \leq n \leq N + M$ such that $x_n = 0$ and therefore (a_n) is not zero heavy yet contains an infinite number of zeros.

An example of such a sequence is $(a_n) = (0, 1, 0, 1, 2, 0, 1, 2, 3, 0, 1, 2, 3, 4, 0, ...)$.

Clearly there are n terms between zero n and $n + 1$ for all $n \in \mathbb{N}$. So for any choice of $M \in \mathbb{N}$ we can find an $N \in \mathbb{N}$ such that a_N has at least $M + 1$ non-zero terms before the next zero. Therefore (a_n) is not zero-heavy.

d. The logical negation of the definition of zero-heavy is as follows:

A sequence (x_n) is not zero-heavy if for all $M \in \mathbb{N}$ there exists an $N \in \mathbb{N}$ where for all $n \in \mathbb{N}$ where $N \leq n \leq N + M$ it is the case that $x_n \neq 0$.

2.3.1

b. Let $x_n \geq 0$ for all $n \in \mathbb{N}$ and $(x_n) \to x$. Then the sequence $(\sqrt{x_n})$ exists.

- First I would like to prove 2.3.1 part a. since it removes the issues of dealing with 0.

  Say $(x_n) \to 0$ then let $\epsilon > 0$ and $\alpha = \epsilon^2$. Then $\alpha > 0$.

  So there exists an $N \in \mathbb{N}$ such that when $n \geq N$ we have $|x_n - 0| = x_n < \alpha$.

  Now we have if $n \geq N$ then $\sqrt{x_n} < \sqrt{\alpha} = \epsilon$.

  Therefore there exists an $N \in \mathbb{N}$ such that when $n \geq N$ we have $|\sqrt{x_n} - 0| < \epsilon$.

  This was done for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

  So we have shown that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|\sqrt{x_n} - 0| < \epsilon$.

  Therefore if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $(x_n) \to 0$ then $(\sqrt{x_n}) \to 0$.

- Now let $(x_n) \to x \neq 0$. It must be that $x > 0$ by the Order Limit Theorem and our assumption that $x \neq 0$.

  We already know for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that when $n \geq N$ we have $|x_n - x| < \epsilon$.

  Furthermore $x_n - x = (\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})$ for all $n \in \mathbb{N}$.

  We want to get that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that when $n \geq N$ we have $|\sqrt{x_n} - \sqrt{x}| < \epsilon$.

  Let $\epsilon > 0$ then consider $\alpha = (\sqrt{x_n} + \sqrt{x})\epsilon$. Since $\sqrt{x_m} + \sqrt{x} > 0$ for all $m \in \mathbb{N}$ we have that $\alpha = (\sqrt{x_n} + \sqrt{x})\epsilon > 0$.

  Then since $(x_n) \to x$ we know there exists an $N \in \mathbb{N}$ such that when $n \geq N$ we have $|x_n - x| < \alpha = (\sqrt{x_n} + \sqrt{x})\epsilon$.

  So we have that there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|x_n - x| = |(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})| < (\sqrt{x_n} + \sqrt{x})\epsilon$.

  Therefore if $n \geq N$ then $|(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})| = |\sqrt{x_n} - \sqrt{x}||\sqrt{x_n} + \sqrt{x}| < (\sqrt{x_n} + \sqrt{x})\epsilon$.

  Since $\sqrt{x_n} + \sqrt{x} > 0$ we have that $|\sqrt{x_n} + \sqrt{x}| = \sqrt{x_n} + \sqrt{x}$ and that we can divide by $\sqrt{x_n} + \sqrt{x}$.

  So if $n \geq N$ then $|\sqrt{x_n} - \sqrt{x}||\sqrt{x_n} + \sqrt{x}| = (\sqrt{x_n} + \sqrt{x})|\sqrt{x_n} - \sqrt{x}| < (\sqrt{x_n} + \sqrt{x})\epsilon$.

  Dividing by $\sqrt{x_n} + \sqrt{x}$ we have that there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|\sqrt{x_n} - \sqrt{x}| < \epsilon$.

  This was done for an arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

  So we have shown that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|\sqrt{x_n} - \sqrt{x}| < \epsilon$.

  Therefore if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $(x_n) \to x \neq 0$ then $(\sqrt{x_n}) \to \sqrt{x}$.

So we have shown that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $(x_n) \to 0$ then $(\sqrt{x_n}) \to 0$.

We have also shown that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $(x_n) \to x \neq 0$ then $(\sqrt{x_n}) \to \sqrt{x}$.

Since $\sqrt{0} = 0$ we have therefore shown that if $x_n \geq 0$ for all $n \in \mathbb{N}$ and $(x_n) \to x$ then $(\sqrt{x_n}) \to \sqrt{x}$ $\square$

2.3.3

Let $(x_n), (y_n), (z_n)$ be sequences such that $(x_n) \to l, (z_n) \to l$, and $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$.

We want to show for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that when $n \geq N$ then $|y_n - l| < \epsilon$.

Recall the triangle inequality $|a + b| \leq |a| + |b|$. Consider $|y_n - l|$.

Then $|y_n - l| = |(y_n - x_n) + (x_n - l)| \leq |y_n - x_n| + |x_n - l|$.

Then since $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ we have $0 \leq y_n - x_n \leq z_n - x_n$ for all $n \in \mathbb{N}$.

Since $0 \leq y_n - x_n \leq z_n - x_n$ we have that $0 \leq |y_n - x_n| \leq |z_n - x_n|$.

So we have $|y_n - l| \leq |y_n - x_n| + |x_n - l| \leq |z_n - x_n| + |x_n - l| = |(z_n - l) + (l - x_n)| + |x_n - l|$.

Again applying the triangle inequality we have $|y_n - l| \leq |z_n - l| + |l - x_n| + |x_n - l| = |z_n - l| + 2|x_n - l|$.

Let $\epsilon > 0$ and $\alpha = \epsilon/3$ then $\alpha > 0$. Since $(x_n) \to l$ and $(z_n) \to l$ we can find an $N \in \mathbb{N}$ such that when $n \geq N$ then $|x_n - l| < \alpha$ and $|z_n - l| < \alpha$ simultaneously.

So we can find an $N \in \mathbb{N}$ such that when $n \geq N$ then $|y_n - l| \leq |z_n - l| + 2|x_n - l| < 3\alpha = \epsilon$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So we have shown that for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|y_n - l| < \epsilon$.

Therefore if $(x_n) \to l, (z_n) \to l$, and $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ then $(y_n) \to l$ $\square$

2.3.7

a. Let $(x_n) = ((-1)^n) = (-1, 1, -1, 1, -1, 1, ...)$ and $(y_n) = ((-1)^{n+1}) = (1, -1, 1, -1, 1, -1, ...)$.

Both (x_n) and (y_n) diverge as shown many times in previous samples.

However, $(z_n) = (x_n + y_n) = ((-1)^n + (-1)^{n+1}) = (-1 + 1, 1 - 1, -1 + 1, 1 - 1, ...) = (0, 0, 0, 0, ...)$ clearly converges.

Proof: Let $\epsilon > 0$ then let $N = 1$ now if $n \geq N$ then $|z_n - 0| = |0 - 0| = 0 < \epsilon$ so $(z_n) \to 0$.

So this is such an example of two divergent sequences whose sum converges.

b. This is not possible. Let $(x_n) \to x$ for some $x \in \mathbb{R}$ and (y_n) diverge.

By the algebraic limit theorem. $(-x_n) \to -x$.

Assume for the sake of contradiction that $(x_n + y_n) \to z$ for some $z \in \mathbb{R}$.

Then by the algebraic limit theorem $(x_n + y_n - x_n) = (y_n) \to z - x$.

But $z - x \in \mathbb{R}$ therefore since (y_n) diverges it can not be that $(y_n) \to z - x$ so we have a contradiction.

Therefore if (x_n) converges and (y_n) diverges it can not be that $(x_n + y_n)$ converges $\square$

c. Let $(b_n) = (\frac{1}{n}) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$. Then $b_n \neq 0$ for all $n \in \mathbb{N}$. Furthermore $(b_n) \to 0$.

- Proving $(b_n) \to 0$:

  Let $\epsilon > 0$ then by the Archimedean property there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.

  Then $|b_N - 0| = b_N = \frac{1}{N} < \epsilon$. Since (b_n) is strictly decreasing if $b_N < \epsilon$ then $b_n < \epsilon$ for $n \geq N$.

  Therefore for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|b_n - 0| < \epsilon$. So $(b_n) \to 0$.

- Showing $(\frac{1}{b_n})$ diverges:

  The sequence $(\frac{1}{b_n}) = (\frac{1}{1/n}) = (n)$ is unbounded since $\mathbb{N}$ is unbounded.

  Since every convergent sequence is bounded it can not be that $(\frac{1}{b_n}) = (n)$ converges.

So this is such an example of a sequence (b_n) where $b_n \neq 0$ for all $n \in \mathbb{N}$ such that $(\frac{1}{b_n})$ diverges.

d. This is not possible. Let (a_n) be unbounded and $(b_n) \to b$.

Since (a_n) is unbounded there does not exist an $M \in \mathbb{N}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Since (b_n) converges it is bounded so there exists an $N \in \mathbb{N}$ such that $|b_n| \leq N$ for all $n \in \mathbb{N}$.

Therefore there does not exist an $M \in \mathbb{N}$ such that $|a_n| + N \leq M + N$ for all $n \in \mathbb{N}$.

We know $|b_n| \leq N$ for all $n \in \mathbb{N}$ and from the triangle inequality that $|a_n - b_n| \leq |a_n| + |b_n|$.

So there does not exist an $M \in \mathbb{N}$ such that $|a_n - b_n| \leq |a_n| + |b_n| \leq |a_n| + N \leq M + N$ for all $n \in \mathbb{N}$.

Let $L = M + N$. Then we have that there does not exist an $L \in \mathbb{N}$ such that $|a_n - b_n| \leq L$ for all $n \in \mathbb{N}$.

So if (a_n) is unbounded and (b_n) converges then $(a_n - b_n)$ is unbounded $\square$

e. Let $(a_n) = (\frac{1}{n}) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$ and $(b_n) = (n) = (1, 2, 3, 4, ...)$.

As shown in 2.3.7.c the sequence $(a_n) \to 0$. Now consider $(a_n b_n) = (\frac{1}{n} n) = (1)$.

Let $\epsilon > 0$ then let $N = 1$ now if $n \geq N$ then $|a_n b_n - 1| = |1 - 1| = 0 < \epsilon$ so $(a_n b_n) \to 1$.

$(b_n) = (n)$ is unbounded since $\mathbb{N}$ is unbounded. Since all convergent sequences are bounded, (b_n) can not converge.

Therefore this is such an example of sequences (a_n) and (b_n) such that (a_n) and $(a_n b_n)$ converge but (b_n) diverges.

2.3.11

a. Let $(x_n) \to x$ and $(y_n) = (\frac{\sum_{i=1}^{n} x_n}{n}) = (x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, ...)$.

We want to show for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|y_n - x| < \epsilon$.

Since (x_n) converges it is bounded so there exists an $L \in \mathbb{N}$ such that $|x_n| \leq L$ for all $n \in \mathbb{N}$.

So $|x_n| + |x| \leq L + |x|$, and $|x_n - x| \leq |x_n| + |x| \leq L + |x|$ for all $n \in \mathbb{N}$.

Let $\beta > L + |x|$ then we have that $|x_n - x| \leq L + |x| < \beta$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$ and $\alpha = \epsilon/2$. Then we know there exists an $N \in \mathbb{N}$ such that for $n \geq N$, $|x_n - x| < \alpha$.

Let $K = N - 1$ then we have for $n > K$, $|x_n - x| < \alpha$.

So $|y_n - x| = |\frac{x_1+...+x_n}{n} - \frac{nx}{n}| = |\frac{(x_1-x)+...+(x_K-x)+...+(x_n-x)}{n}| \leq |\frac{(x_1-x)+...+(x_K-x)}{n}| + |\frac{(x_{K+1}-x)+...+(x_n-x)}{n}|$

And $|y_n - x| \leq |\frac{(x_1-x)+...+(x_K-x)}{n}| + |\frac{(x_{K+1}-x)+...+(x_n-x)}{n}| \leq |\frac{|x_1-x|+...+|x_K-x|}{n}| + |\frac{|x_{K+1}-x|+...+|x_n-x|}{n}|$.

So for $n > K$: $|y_n - x| \leq |\frac{|x_1-x|+...+|x_K-x|}{n}| + |\frac{|x_{K+1}-x|+...+|x_n-x|}{n}| < |\frac{K\beta}{n}| + |\frac{(n-K)\alpha}{n}| < \frac{K\beta}{n} + \alpha$.

Since K and β are fixed we can choose an M large enough such that $\frac{K\beta}{n} < \alpha$ for $n > M$.

So when both $n > N$ and $n > M$: $|y_n - x| < \frac{K\beta}{n} + \alpha < 2\alpha = \epsilon$.

Let $J = 1 + max\{N, M\}$. Then for $n \geq J$ we have $|y_n - x| < \epsilon$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So we have shown that for all $\epsilon > 0$ there exists a $J \in \mathbb{N}$ such that $|y_n - x| < \epsilon$ for $n \geq J$.

Therefore if $(x_n) \to x$ then $(y_n) = (\frac{\sum_{i=1}^{n} x_n}{n}) = (x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, ...) \to x$ $\square$

b. Let $(x_n) = ((-1)^n)$. We have shown many times (x_n) does not converge.

Consider $(y_n) = (\frac{\sum_{i=1}^{n} x_n}{n}) = (x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, ...) = (-1, \frac{-1+1}{2}, \frac{-1+1-1}{3}, ...) = (-1, 0, -\frac{1}{3}, 0, -\frac{1}{5}, ...)$

If n is even then $y_n = 0$ and if n is odd then $y_n = -\frac{1}{n}$. Here $(y_n) \to 0$.

Proof:

Let $\epsilon > 0$ then by the Archimedean property there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$.

So for $n \geq N$ if n is even then $|y_n - 0| = |0 - 0| = 0 < \epsilon$ and if n is odd then $|y_n - 0| = |-\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|y_n - 0| < \epsilon$.

Therefore for $(x_n) = ((-1)^n)$ we have that $(y_n) \to 0$.

So this is such an example of a sequence (x_n) that does not converge where (y_n) does.

2.4.3

a. Let $(a_n) = (\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, ...)$. Then $a_{n+1} = \sqrt{2+a_n}$ for all $n \in \mathbb{N}$.

- Proving (a_n) is monotonically increasing:

  Let $S = \{n \in \mathbb{N} : a_n < a_{n+1}\}$. We know $a_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = a_2$. So $1 \in S$.

  Now assume that $n \in S$, that is assume $a_n < a_{n+1}$. Then $2 + a_n < 2 + a_{n+1}$ and $\sqrt{2+a_n} < \sqrt{2+a_{n+1}}$.

  Since $a_{m+1} = \sqrt{2+a_m}$ we have that $a_{n+1} < a_{n+2}$. So $n+1 \in S$.

  Therefore since $1 \in S$ and if $n \in S$ then $n+1 \in S$ we have that $S = \mathbb{N}$.

  So $a_n < a_{n+1}$ for all $n \in \mathbb{N}$ and therefore (a_n) is monotonically increasing.

- Proving (a_n) is bounded:

  $a_n > 0$ for all $n \in \mathbb{N}$ since $a_1 = \sqrt{2} > 0$ and (a_n) is monotonically increasing. So $|a_n| = a_n$ for all $n \in \mathbb{N}$.

  Let $S = \{n \in \mathbb{N} : a_n < 2\}$. We know $a_1 = \sqrt{2} < 2$. So $1 \in S$.

  Now assume that $n \in S$, that is assume $a_n < 2$. Then $2 + a_n < 2 + 2 = 4$ and $\sqrt{2+a_n} < \sqrt{4} = 2$.

  Since $a_{m+1} = \sqrt{2+a_m}$ we have that $a_{n+1} < 2$. So $n+1 \in S$.

  Therefore since $1 \in S$ and if $n \in S$ then $n+1 \in S$ we have that $S = \mathbb{N}$.

  So $|a_n| = a_n < 2$ for all $n \in \mathbb{N}$ and therefore (a_n) is bounded.

- The limit of (a_n):

  Since (a_n) is monotonically increasing and bounded, by the monotone convergence theorem (a_n) converges.

  The limit of (a_n) should satisfy the equation $a = \sqrt{2+a}$. So $a^2 = a + 2$ and $a^2 - a - 2 = (a-2)(a+1) = 0$.

  The limit can not be 1 since $a_1 = \sqrt{2} > 1$ and (a_n) is monotonically increasing so $|a_n - 1|$ is increasing.

  So $(a_n) \to 2$.

2.4.5

a. Define (x_n) by $x_1 = 2$ and $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$ for $n \in \mathbb{N}$.

- Showing $x_n^2 \geq 2$ for all $n \in \mathbb{N}$:

  We know $x_n^2 = (\frac{1}{2}(x_{n-1} + \frac{2}{x_{n-1}}))^2 = \frac{1}{4}(x_{n-1}^2 + 4 + \frac{4}{x_{n-1}^2}) = \frac{x_{n-1}^2}{4} + 1 + \frac{1}{x_{n-1}^2}$ for $n \in \mathbb{N}$.

  $x_n > 0$ for all $n \in \mathbb{N}$ because $x_1 = 2 > 0$, and if $x_n > 0$ then $\frac{2}{x_n} > 0$ and so $\frac{1}{2}(x_n + \frac{2}{x_n}) = x_{n+1} > 0$.

  So $x_n > 0$ for all $n \in \mathbb{N}$ and therefore $\frac{1}{x_n}$ exists for all $n \in \mathbb{N}$.

  Now consider $0 \leq (\frac{x_{n-1}}{2} - \frac{1}{x_{n-1}})^2 = \frac{x_{n-1}^2}{4} - 1 + \frac{1}{x_{n-1}^2} = \frac{x_{n-1}^2}{4} + 1 + \frac{1}{x_{n-1}^2} - 2 = x_n^2 - 2$.

  So $x_n^2 - 2 \geq 0$ is always true and therefore $x_n^2 \geq 2$ for all $n \in \mathbb{N}$.

- Showing $x_n - x_{n+1} \geq 0$ for all $n \in \mathbb{N}$:

  Consider $x_n - x_{n+1} = x_n - \frac{1}{2}(x_n + \frac{2}{x_n}) = \frac{x_n}{2} - \frac{1}{x_n} = \frac{x_n^2-2}{2x_n}$.

  From the results above we know $x_n^2 - 2 \geq 0$ and $x_n > 0$ for all $n \in \mathbb{N}$.

  Therefore $x_n - x_{n+1} = \frac{x_n^2-2}{2x_n} \geq 0$ for all $n \in \mathbb{N}$.

- This shows (x_n) is bounded and monotonically decreasing so (x_n) converges by the monotone convergence theorem.

  The limit of (x_n) should satisfy the equation $x = \frac{1}{2}(x + \frac{2}{x})$. So $2x = x + \frac{2}{x}$, and $x = \frac{2}{x}$.

  So $x^2 = 2$ and therefore $x = \sqrt{2}$. So $(x_n) \to \sqrt{2}$.

b. We want to modify (x_n) so that $(x_n) \to \sqrt{c}$.

We should focus on the recursive definition to set the limit, $x = \frac{1}{2}(x + \frac{2}{x})$.

If we change the recursive definition to $x_{n+1} = \frac{1}{2}(x_n + \frac{c}{x_n})$ we get the limit condition $x = \frac{1}{2}(x + \frac{c}{x})$. So $2x = x + \frac{c}{x}$.

We have $x = \frac{c}{x}$ and so $x^2 = c$. Therefore $x = \sqrt{c}$ so by this definition $(x_n) \to \sqrt{c}$.

To make it look nicer you can also let $x_1 = c$ and the limit will not change.

2.4.7

Let (a_n) be a bounded sequence. Then there exists an $M \in \mathbb{N}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

a. Let $(y_n) = (sup\{a_k : k \geq n\})$. Then I will show (y_n) is monotonically decreasing.

- Showing (y_n) is monotonically decreasing:

  Let $A_n = \{a_k : k \geq n\}$. Then $A_1 \supseteq A_2 \supseteq ... \supseteq A_n \supseteq ...$ and as proved in Sample Work 3, $supA_1 \geq supA_2 \geq ...supA_n \geq ...$

  Since $y_n = supA_n$ we have that $y_1 \geq y_2 \geq ... \geq y_n \geq ...$ so (y_n) is monotonically decreasing.

- Showing (y_n) is bounded:

  Since (a_n) is bounded we have that $|a_n| \leq M$ for some $M \in \mathbb{N}$ and all $n \in \mathbb{N}$. So $-M \leq a_n \leq M$ for all $n \in \mathbb{N}$.

  Since $a_n \leq M$ for all $n \in \mathbb{N}$ we have that M is an upper bound of each A_n. So $supA_n \leq M$ for all $n \in \mathbb{N}$.

  Furthermore for any A_n we have that for all $a_k \in A_n$, $-M \leq a_k \leq supA_n \leq M$.

  So $|y_n| = |supA_n| \leq M$ for all $n \in \mathbb{N}$ and (y_n) is bounded.

Therefore since (y_n) is monotonically decreasing and bounded (y_n) converges by the monotone convergence theorem $\square$

b. Let $(x_n) = (inf\{a_k : k \geq n\})$. Since (a_n) is bounded the infimum exists for each of these sets.

- Proving $infA \leq infB$ for two sets A, B such that $A \supseteq B$:

  Let A and B both be nonempty sets so $A \supseteq B$.

  Let $x = infA$. Then $x \leq a$ for all $a \in A$. Since $A \supseteq B$ if $b \in B$ then $b \in A$.

  So we also know $x \leq b$ for all $b \in B$ and is a lower bound of B, meaning it must be less than or equal to $infB$.

  Therefore $x = infA \leq infB$

- Showing (x_n) is monotonically increasing:

  Let $A_n = \{a_k : k \geq n\}$. Then $A_1 \supseteq A_2 \supseteq ... \supseteq A_n \supseteq ...$ and from above $infA_1 \leq infA_2 \leq ... \leq A_n \leq ...$

  Since $x_n = infA_n$ we have that $x_1 \leq x_2 \leq ... \leq x_n \leq ...$ so (x_n) is monotonically increasing.

- Showing (x_n) is bounded:

  Since (a_n) is bounded we have that $|a_n| \leq M$ for some $M \in \mathbb{N}$ and all $n \in \mathbb{N}$. So $-M \leq a_n \leq M$ for all $n \in \mathbb{N}$.

  Since $-M \leq a_n$ for all $n \in \mathbb{N}$ we have that $-M$ is a lower bound of each A_n. So $infA_n \geq -M$ for all $n \in \mathbb{N}$.

  Furthermore for any A_n we have that for all $a_k \in A_n$, $-M \leq infA_n \leq a_k \leq M$.

  So $|x_n| = |infA_n| \leq M$ for all $n \in \mathbb{N}$ and (x_n) is bounded.

Therefore since (x_n) is monotonically increasing and bounded (x_n) converges by the monotone convergence theorem $\square$

c. For any A_n we have shown in the previous two parts that $-M \leq infA_n \leq a_k \leq supA_n \leq M$.

So we have that $x_n = infA_n \leq supA_n = y_n$ for all $n \in \mathbb{N}$.

Therefore since (x_n) and (y_n) converge, say to x and y respectively, we may use the order limit theorem to say $x \leq y$.

Since $lim\ inf\ a_n = x$ and $lim\ sup\ a_n = y$ we have shown that $lim\ inf\ a_n \leq lim\ sup\ a_n$ for any bounded sequence (a_n).

d.

- Showing if $lim\ \ inf\ a_n = lim\ \ sup\ a_n = a$ then $(a_n) \to a$:

  Assume $lim\ x_n = lim\ \ inf\ a_n = lim\ \ sup\ a_n = lim\ y_n = a$. Let $\epsilon > 0$ and $\alpha = \epsilon/2$.

  Then there exists an $N \in \mathbb{N}$ such that $|x_n - a| < \alpha$ and $|y_n - a| < \alpha$ for $n \geq N$.

  Since $x_n \leq a_n \leq y_n$ we have $0 \leq a_n - x_n \leq y_n - x_n$.

  So $|a_n - x_n| \leq |y_n - x_n| = |(y_n - a) + (a - x_n)| \leq |y_n - a| + |x_n - a| < 2\alpha = \epsilon$ for $n \geq N$.

  Therefore if $lim\ \ inf\ a_n = lim\ \ sup\ a_n = a$ then $(a_n) \to a$.

- Showing if $(a_n) \to a$ then $lim\ \ inf\ a_n = lim\ \ sup\ a_n = a$:

  Assume $(a_n) \to a$. Let $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for $n \geq N$.

  So $-\epsilon < a_n - a < \epsilon$ and $a - \epsilon < a_n < a + \epsilon$ for $n \geq N$.

  Therefore $a - \epsilon$ is a lower bound for A_N and $a + \epsilon$ is an upper bound for A_N.

  Since for $n > N$, $A_n \subseteq A_N$ we have that if $b \in A_n$ then $b \in A_N$ so $a - \epsilon < b < a + \epsilon$.

  So $a - \epsilon$ is a lower bound for A_n and $a + \epsilon$ is an upper bound for A_n when $n \geq N$.

  So we have $a - \epsilon \leq inf\ a_n \leq sup\ a_n \leq a + \epsilon$ for $n \geq N$.

  Therefore $|inf\ a_n - a| < \epsilon$ and $|sup\ a_n - a| < \epsilon$ for $n \geq N$.

  So if $(a_n) \to a$ then $lim\ \ inf\ a_n = lim\ \ sup\ a_n = a$.

Therefore $(a_n) \to a$ and $lim\ a_n$ exists if and only if $lim\ \ inf\ a_n = lim\ \ sup\ a_n = a$ $\square$

Extra Problem

Let $(x_n) = (\frac{n^2+1}{2n+1})$ for $n \in \mathbb{N}$.

$$\text{Then } \frac{n^2+1}{2n+1} > \frac{n^2+1-2}{2n+1} = \frac{n^2-1}{2n+1} = \frac{(n+1)(n-1)}{2n+1} > \frac{(n+1)(n-1)}{2n+1+1} = \frac{(n+1)(n-1)}{2n+2} = \frac{(n+1)(n-1)}{2(n+1)} = \frac{n-1}{2}$$

Clearly $(y_n) = (\frac{n-1}{2})$ diverges to ∞ as $\mathbb{N}$ is unbounded.

Proof:

(y_n) is monotonically increasing since $y_{n+1} - y_n = \frac{(n+1)-1}{2} - \frac{n-1}{2} = \frac{n-(n-1)}{2} = \frac{1}{2} > 0$.

So $y_{n+1} - y_n > 0$ for all $n \in \mathbb{N}$ and therefore $y_{n+1} > y_n$ for all $n \in \mathbb{N}$.

Since $y_1 = 0$ and (y_n) is monotonically increasing we have that $y_n \geq 0$ for all $n \in \mathbb{N}$.

Assume for the sake of contradiction that (y_n) is bounded, that is assume $|y_n| \leq M$ for some $M \in \mathbb{N}$.

Then $|y_n| = y_n = \frac{n-1}{2} \leq M$ for all $n \in \mathbb{N}$.

But consider $y_{2M+2} = \frac{(2M+2)-1}{2} = \frac{2M+1}{2} = M + \frac{1}{2} > M$

So we have a contradiction and therefore (y_n) can not be bounded.

Since (y_n) is unbounded it can not converge since all convergent series are bounded.

So (y_n) diverges, is monotonically increasing, non-negative, and unbounded, and consequently y_n diverges to ∞.

Therefore since $x_n = \frac{n^2+1}{2n+1} > \frac{n-1}{2} = y_n$ and (y_n) diverges to ∞ we have that (x_n) diverges to ∞ $\square$