Integrating over Regions

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49.1

Recall that if a continuous function f has an antiderivative F throughout a domain D then for any contour C going from z_1 to z_2 contained in D we know:

$$\int_C f(z)dz = F(z)\Big|_{z_1}^{z_2}$$

Now let's look at $f(z) = z^n$ where $n \in \{0, 1, 2, ...\}$.

We have seen before that for $n \in \mathbb{Z}\backslash\{0\}$ that $\frac{d}{dz}z^n = nz^{n-1}$, therefore we can apply this to $n \in \{1, 2, 3, ...\}$.

So we have that $\frac{d}{dz}\frac{z^n}{n} = z^{n-1}$ for all $n \in \{1, 2, 3, ...\}$.

Then letting $m = n - 1$ we see that $n = m + 1$ and $\frac{d}{dz}\frac{z^{m+1}}{m+1} = z^m$ where $m \in \{0, 1, 2, ...\}$.

Therefore for all $m \in \{0, 1, 2, ...\}$ we have that the antiderivative of z^m is $\frac{z^{m+1}}{m+1}$.

Recall that z^m is entire for all $m \in \{0, 1, 2, ...\}$.

This means that for all $m \in \{0, 1, 2, ...\}$ we know that for any contour C going from z_1 to z_2:

$$\int_C z^m dz = \frac{z^{m+1}}{m+1}\Big|_{z_1}^{z_2} = \frac{1}{m+1}\left(z_2^{m+1} - z_1^{m+1}\right)$$

$\square$

49.3

Recall that if a continuous function f has an antiderivative F throughout a domain D then for any contour C going from z_1 to z_2 contained in D we know:

$$\int_C f(z)dz = F(z)\Big|_{z_1}^{z_2}$$

We have seen before that for $n \in \mathbb{Z}\backslash\{0\}$ that $\frac{d}{dz}z^n = nz^{n-1}$.

Therefore we know that $\frac{d}{dz}\frac{z^n}{n} = z^{n-1}$ for all $n \in \mathbb{Z}\backslash\{0\}$.

If a contour C_0 does not pass through z_0 then we know that for each $z \in C_0$ there must exist some neighborhood where $z_0 \notin V_{\epsilon_z}(z)$ for every point $z \in C_0$.

So there must exist some domain D_0 (which can be the union of all these neighborhoods) that contains C_0 and not z_0.

Therefore for $n \in \{\pm 1, \pm 2, \pm 3, ...\}$ we know $(z-z_0)^{n-1}$ is continuous and has an antiderivative on D_0 (negative powers aren't a problem since $z \neq z_0$).

Therefore for any contour C_0 going from z_1 to z_2 that does not pass through z_0 we may say that for $n \in \{\pm 1, \pm 2, \pm 3, ...\}$:

$$\int_{C_0} (z-z_0)^{n-1}dz = \frac{(z-z_0)^n}{n}\Big|_{z_1}^{z_2} = \frac{1}{n}\Big((z_1-z_0)^n - (z_2-z_0)^n\Big)$$

Therefore if C_0 is a closed contour (i.e. $z_1 = z_2 = z$) that does not pass through z_0 we may say that for $n \in \{\pm 1, \pm 2, \pm 3, ...\}$:

$$\int_{C_0} (z-z_0)^{n-1}dz = \frac{(z-z_0)^n}{n}\Big|_{z_1}^{z_2} = \frac{1}{n}\Big((z_1-z_0)^n - (z_2-z_0)^n\Big) = \frac{1}{n}\Big((z-z_0)^n - (z-z_0)^n\Big) = 0$$

□

49.4

Let $f_2(z)$ be the branch $f_2(z) = \sqrt{r}e^{i\frac{\theta}{2}}$ of $z^{\frac{1}{2}}$ where $\frac{\pi}{2} \leq \theta \leq \frac{5\pi}{2}$.

Then let C_2 be the contour as shown in the example (although the exact shape doesn't matter) which goes from -3 to 3.

Now consider the function $F_2(z) = \frac{2}{3}z^{\frac{3}{2}} = \frac{2}{3}\sqrt{r^3}e^{i\frac{3\theta}{2}} = \frac{2}{3}r^{\frac{3}{2}}(cos(\frac{3\theta}{2}) + i\,sin(\frac{3\theta}{2}))$ with the same bounds on θ.

Then we can write $F_2(z) = u(r,\theta) + iv(r,\theta)$ where $u(r,\theta) = \frac{2}{3}r^{\frac{3}{2}}cos(\frac{3\theta}{2})$ and $v(r,\theta) = \frac{2}{3}r^{\frac{3}{2}}sin(\frac{3\theta}{2})$.

Looking at the partial derivatives:

$u_r = r^{\frac{1}{2}}cos(\frac{3\theta}{2})$, $u_\theta = -r^{\frac{3}{2}}sin(\frac{3\theta}{2})$, $v_r = r^{\frac{1}{2}}sin(\frac{3\theta}{2})$, $v_\theta = r^{\frac{3}{2}}cos(\frac{3\theta}{2})$

Clearly $ru_r = r^{\frac{3}{2}}cos(\frac{3\theta}{2}) = v_\theta$ and $u_\theta = -r^{\frac{3}{2}}sin(\frac{3\theta}{2}) = -r(r^{\frac{1}{2}}sin(\frac{3\theta}{2})) = -rv_r$.

So the polar Cauchy Riemann equations are satisfied.

Furthermore the partial derivatives are continuous so we know that $F_2(z)$ is differentiable and $F_2'(z) = e^{-i\theta}(u_r + iv_r)$.

So $F_2'(z) = e^{-i\theta}(r^{\frac{1}{2}}(cos(\frac{3\theta}{2}) + ir^{\frac{1}{2}}sin(\frac{3\theta}{2})) = r^{\frac{1}{2}}e^{-i\theta}e^{i\frac{3\theta}{2}} = \sqrt{r}e^{i\frac{\theta}{2}} = f_2(z)$ since they have the same θ bounds.

So $F_2(z) = \frac{2}{3}z^{\frac{3}{2}} = \frac{2}{3}\sqrt{r^3}e^{i\frac{3\theta}{2}}$ is an antiderivative for $f_2(z)$.

Therefore we know:

$$\int_{C_2} f_2(z)dz = F_2(z)\Big|_{-3}^{3} = \frac{2}{3}\sqrt{27}(e^{i\frac{3(2\pi)}{2}} - e^{i\frac{3(\pi)}{2}}) = 2\sqrt{3}(e^{3i\pi} - e^{i\frac{3\pi}{2}}) = 2\sqrt{3}(-1+i)$$

$\square$

Note then that:

$$\int_{C_2 - C_1} z^{\frac{1}{2}}dz = \int_{C_2} f_2(z)dz - \int_{C_1} f_1(z)dz = 2\sqrt{3}(-1+i) - 2\sqrt{3}(1+i) = -4\sqrt{3}$$

53.1

c. Let $f(z) = \frac{1}{z^2+2z+2}$ and let C be the unit circle $|z| = 1$.

We know we can solve for the roots of $z^2 + 2z + 2$ with the quadratic equation, as shown in a previous sample work.

So $z = \frac{-2\pm\sqrt{2^2-4(1)(2)}}{2} = \frac{-2\pm\sqrt{-4}}{2} = \frac{-2\pm 2i}{2} = -1 \pm i$.

Clearly neither of these roots are interior to or on the contour C, this can be easily shown by noticing

$$|-1 \pm i| = \sqrt{(-1)^2 + (\pm 1)^2} = \sqrt{2} > 1.$$

Recall both polynomials and constant functions are entire, and therefore analytic at all points interior to and on C.

So since $P(z) = z^2 + 2z + 2$ and $g(z) = 1$ are analytic at all points interior to and on C, and since $P(z) \neq 0$ for any point interior to or on C we know that $f(z) = \frac{1}{z^2+2z+2} = \frac{g(z)}{P(z)}$ is analytic at all points interior to and on C.

Therefore since C is a simple closed contour (being just the unit circle $|z| = 1$) we may use the Cauchy-Goursat Theorem.

So we have that:

$$\int_C f(z)dz = \int_C \frac{1}{z^2 + 2z + 2} dz = 0$$

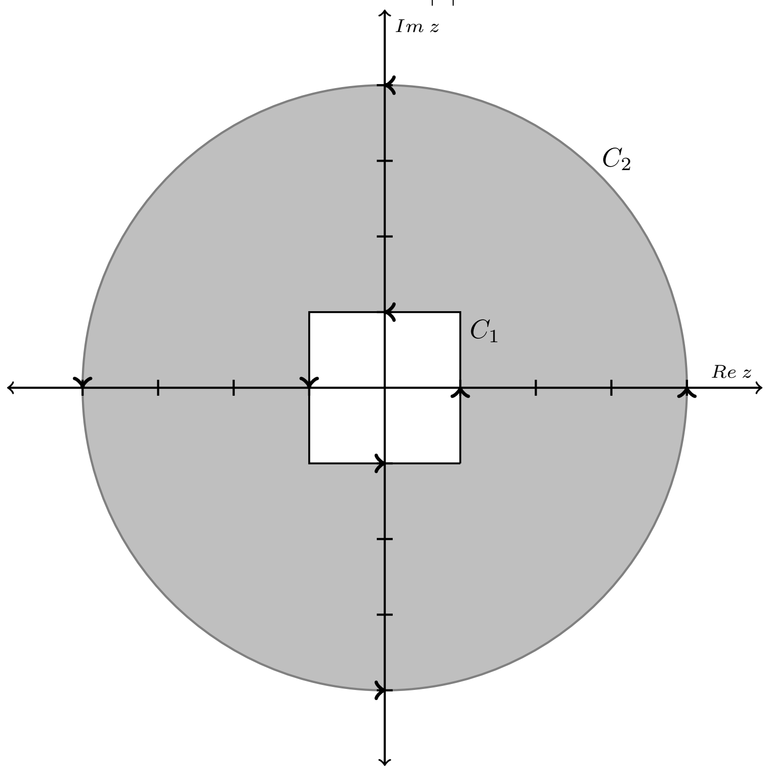
Notice that the direction of C was unspecified and the integral still evaluates to 0, this is because no matter the direction it is still a simple closed contour.

□

53.2

b. Let $f(z) = \frac{z+2}{sin(\frac{z}{2})}$.

Then let C_1 be the positively oriented unit square whose sides lie on $x = \pm 1$ and $y = \pm 1$ and C_2 be the positively oriented circle $|z| = 4$.



Clearly C_1 is interior to C_2.

Furthermore since $sin(\frac{z}{2}) = 0$ if and only if $\frac{z}{2} = n\pi$ and hence only if $z = 2n\pi$ for some $n \in \mathbb{Z}$ we know that $sin(\frac{z}{2}) \neq 0$ for any point lying on or in between the contours C_1 and C_2.

This is because for $n \in \mathbb{Z}\backslash\{0\}$ we know $|\pm 2n\pi| = 2n\pi \geq 2\pi = |\pm 2\pi|$ and clearly $2\pi > 4$ so all of these points lie outside of the region between C_1 and C_2. Also, as clearly seen above $z = 0$ is not on or lying between the contours C_1 and C_2.

Recall that polynomials and $sin(w)$ are entire, therefore we have that $g(z) = sin(\frac{z}{2})$ and $P(z) = z + 2$ are analytic on and between the contours C_1 and C_2.

Since $sin(\frac{z}{2}) \neq 0$ on or between the contours C_1 and C_2 we know $f(z) = \frac{z+2}{sin(\frac{z}{2})} = \frac{P(z)}{g(z)}$ is analytic on and between the contours C_1 and C_2.

Therefore since C_1 and C_2 are positively oriented simple closed contours where C_1 is interior to C_2 and we know $f(z) = \frac{z+2}{sin(\frac{z}{2})}$ is analytic on and between the contours C_1 and C_2 we may use the corollary in the book to say:

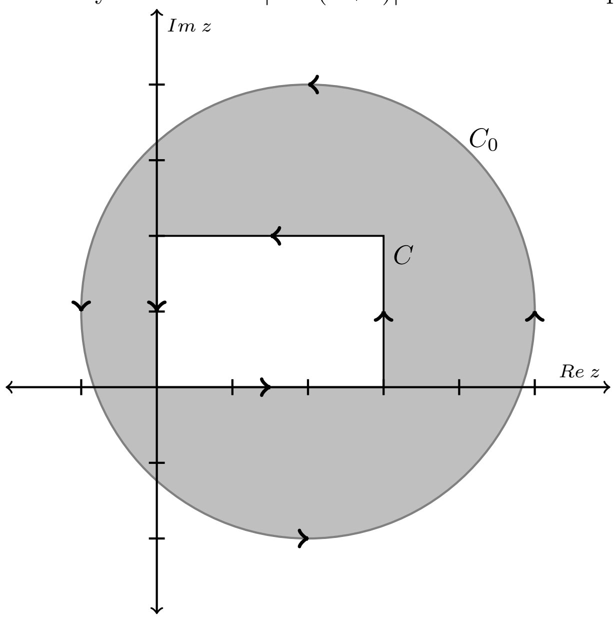
$$\int_{C_1} f(z)dz = \int_{C_1} \frac{z+2}{sin(\frac{z}{2})}dz = \int_{C_2} \frac{z+2}{sin(\frac{z}{2})}dz = \int_{C_2} f(z)dz$$

$\square$

53.3

Let C be the boundary of the rectangle $0 \leq x \leq 3$ and $0 \leq y \leq 2$ taken in the positive orientation.

Now let C_0 be the boundary of the circle $|z - (2+i)| = 3$ taken in the positive orientation.



Clearly C is interior to C_0.

Since polynomials are entire we have that $z - (2+i)$ is analytic on and between the contours C and C_0.

Furthermore we know that $z - (2+i) \neq 0$ for any z lying on or between the contours C and C_0.

Consequently $(z - (2+i))^{n-1}$ is analytic on and between the contours C and C_0 for any $n \in \mathbb{Z}$ (since $z - (2+i) \neq 0$ for any z on or between the contours C and C_0 negative powers are not a problem).

Therefore since C and C_0 are positively oriented simple closed contours where C is interior to C_0 and we know $(z - (2+i))^{n-1}$ is analytic on and between the contours C and C_0 we may use the corollary in the book to say:

$$\int_C (z - (2+i))^{n-1} dz = \int_{C_0} (z - (2+i))^{n-1} dz$$

Recall from problem in a previous sample work that when C_0 denotes the positively oriented circle of radius R centered at z_0:

If $n = 0$:

$$\int_{C_0} (z - z_0)^{n-1} dz = 2\pi i$$

And if $n \in \{\pm 1, \pm 2, \pm 3, ...\}$:

$$\int_{C_0} (z - z_0)^{n-1} dz = 0$$

Therefore we have for our contour C_0 centered at $z_0 = 2+i$ with $R = 3$:

If $n = 0$:

$$\int_C (z-(2+i))^{n-1} dz = \int_{C_0} (z-(2+i))^{n-1} dz = 2\pi i$$

And if $n \in \{\pm 1, \pm 2, \pm 3, ...\}$:

$$\int_C (z-(2+i)^{n-1} dz = \int_{C_0} (z-(2+i)^{n-1} dz = 0$$

□

53.7

Even if we have an function $f = u + iv$ that is nowhere analytic we can still write the following (for a simple closed contour C) so long as the partial derivatives of u and v are well defined and integrable:

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_a^b \Big(u\big(x(t),y(t)\big) + iv\big(x(t),y(t)\big)\Big)\Big(x'(t) + iy'(t)\Big)dt =$$

$$\int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')dt = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy$$

Now recalling Green's Theorem (R is the region bounded by C):

$$\int_C P\,dx + Q\,dy = \int\int_R (Q_x - P_y)dA$$

We now have that:

$$\int_C f(z)dz = \int_C u\,dx - v\,dy + i\int_C v\,dx + u\,dy = \int\int_R (-v_x - u_y)dA + i\int\int_R (u_x - v_y)dA$$

Let $f(z) = \overline{z} = x - iy$ then $f(z) = u + iv$ where $u = Re\,z = x$ and $v = -Im\,z = -y$.

Then we have that $u_x = 1$, $u_y = 0$, $v_x = 0$, and $v_y = -1$, all of which are clearly well defined and integrable.

Now let C be any positively oriented simple closed contour.

Then we have the following where R is the region bounded by C):

$$\int_C \overline{z}dz = \int\int_R (-0-0)dA + i\int\int_R (1-(-1))dA = i\int\int_R 2dA = 2i\int\int_R dA = 2Ai$$

Where A is the area of the region bounded by C.

Therefore we have that:

$$\frac{1}{2i}\int_C \overline{z}dz = \frac{1}{2i}(2Ai) = A$$

So if C is any simple closed contour taken in the positive sense then the area enclosed by C can be written as:

$$A = \frac{1}{2i}\int_C \overline{z}dz$$

$\square$

Problem 2

Suppose that f is analytic on and inside a simple closed contour C except at one point z_0 in the interior of C.

Further assume that f is bounded in some neighborhood of z_0.

Then we know there exists some $\alpha > 0$ such that if $|z - z_0| < \alpha$ then $|f(z)| \leq M$ for some $M > 0$.

Let $\epsilon > 0$ be arbitrary. Then let C_ϵ be a simple closed contour in the positive sense such that it is contained in both the above α neighborhood of z_0, $V_\alpha(z_0)$, and the interior of C.

Furthermore let C_ϵ be such that z_0 is in the interior of C_ϵ, and the length of C_ϵ is less than $\frac{\epsilon}{M}$ (that is $L < \frac{\epsilon}{M}$).

Such a contour exists because $\alpha > 0$, so for any given direction there is always a point between z_0 and the boundary of $V_\alpha(z_0)$. We already know that since z_0 is interior to C there must exist a point for any given direction between z_0 and C. Also the distance of points from z_0 can be made arbitrarily small and hence the length of the contour can be made arbitrarily small.

Take C in the positive sense and call it C_+ (I will show later that the same holds true for negative orientation).

Then since z_0 is in the interior of C_ϵ we know that f is analytic everywhere on and between the simple closed contours C_+ and C_ϵ and so we may use the deformation of paths theorem.

So we have that:

$$\int_{C_+} f(z)dz = \int_{C_\epsilon} f(z)dz$$

Therefore since C_ϵ is contained inside of $V_\alpha(z_0)$, f is bounded (by M) on C_ϵ so:

$$\left| \int_{C_+} f(z)dz \right| = \left| \int_{C_\epsilon} f(z)dz \right| \leq ML < M\frac{\epsilon}{M} = \epsilon$$

This was true for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

Therefore when C is taken in the positive sense:

$$\int_C f(z)dz = 0$$

By which it follows immediately that if C is taken in the negative sense:

$$\int_C f(z)dz = \int_{-C_+} f(z)dz = -\int_{C_+} f(z)dz = 0$$

So when C is taken in either the positive or negative sense we get:

$$\int_C f(z)dz = 0$$

$\square$