Power Series Representations

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6.4.6

Let $f(x) = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{x+n}$.

For any $x_0 > 0$ we have that $f(x_0) = \frac{1}{x_0} - \frac{1}{x_0+1} + \frac{1}{x_0+2} - \frac{1}{x_0+3} + ... = \sum_{n=0}^{\infty} \frac{(-1)^n}{x_0+n}$ is a series of real numbers.

Furthermore since $x_0 + n$ is strictly increasing as n increases so we get that $\frac{1}{x_0+n}$ is strictly decreasing as n increases.

Therefore by the alternating series test $f(x_0)$ converges for arbitrary $x_0 > 0$ and therefore for all $x_0 > 0$.

So $f(x)$ is defined pointwise for all $x > 0$.

Since $f(x)$ converges for all $x > 0$ we can regroup terms in the series.

$f(x) = \frac{1}{x} - \frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} + ... = (\frac{1}{x} - \frac{1}{x+1}) + (\frac{1}{x+2} - \frac{1}{x+3}) + ... = \frac{1}{x(x+1)} + \frac{1}{(x+2)(x+3)} + \frac{1}{(x+4)(x+5)} + ...$

Let $f_n(x) = \frac{1}{(x+n)(x+n+1)} = \frac{1}{x^2+(2n+1)x+n(n+1)}$ so that $f(x) = \sum_{n=0}^{\infty} f_n(x)$.

Since $x > 0$ we get that $|f_n(x)| = |\frac{1}{x^2+(2n+1)x+n(n+1)}| = \frac{1}{x^2+(2n+1)x+n(n+1)} < \frac{1}{n(n+1)}$ for all $n \geq 1$.

So let $M_n = \frac{1}{n(n+1)}$ for $n \geq 1$ then since $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges we get by the Weierstrass M-test that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

We have seen previously that $\frac{1}{x}$ is continuous on $(0, \infty)$ and clearly the same is true for $\frac{1}{x+n}$ as it is just a translation of $\frac{1}{x}$ that is still defined on $(0, \infty)$ for all $n \in \mathbb{N}$.

So $f_n(x)$ is continuous on $(0, \infty)$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly so $\sum_{n=1}^{\infty} f_n(x)$ is continuous on $(0, \infty)$ since uniform convergence preserves continuity.

Now we still need to take care of $n = 0$. So look at $f_0(x) = \frac{1}{x(x+1)} = (\frac{1}{x})(\frac{1}{x+1})$.

Again $\frac{1}{x}$ is continuous on $(0, \infty)$ and the same is true for $\frac{1}{x+1}$, so by the algebraic continuity theorem we have $f_0(x) = \frac{1}{x(x+1)}$ is continuous on $(0, \infty)$.

Therefore $f(x) = \sum_{n=0}^{\infty} f_n(x) = f_0(x) + \sum_{n=1}^{\infty} f_n(x)$ is continuous on $(0, \infty)$ by the algebraic continuity theorem.

Let $g_n(x) = \frac{(-1)^n}{x+n}$ for $n \geq 0$ so that $f(x) = \sum_{n=0}^{\infty} g_n(x)$.

Now for $n \in \mathbb{N}$ and $x > 0$ look at $g_n'(x) = \frac{(-1)^{n+1}}{(x+n)^2}$. Then $|g_n'(x)| = |\frac{(-1)^{n+1}}{(x+n)^2}| = \frac{1}{(x+n)^2} = \frac{1}{x^2+2nx+n^2} < \frac{1}{n^2}$ for $n \geq 1$.

So taking $M_n = \frac{1}{n^2}$ for $n \geq 1$ since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges we get by the Weierstrass M-test that $\sum_{n=1}^{\infty} g_n'(x)$ converges uniformly.

So $\sum_{n=1}^{\infty} g_n'(x)$ converges uniformly and we also know $\sum_{n=1}^{\infty} g_n(x) = f(x) - \frac{1}{x}$ converges for all $x \in (0, \infty)$ therefore $\sum_{n=1}^{\infty} g_n(x)$ is differentiable on $(0, \infty)$.

Now again we need to take care of $n = 0$, so look at $g_0(x) = \frac{1}{x}$. We have previously seen $\frac{1}{x}$ is differentiable on $(0, \infty)$.

Therefore $f(x) = \sum_{n=0}^{\infty} g_n(x) = g_0(x) + \sum_{n=1}^{\infty} g_n(x)$ is differentiable on $(0, \infty)$ by the algebraic differentiability theorem.

So we have shown $f(x)$ is defined on $(0, \infty)$, and is continuous and differentiable on $(0, \infty)$ $\square$

6.5.5

a. Let $s \in (0,1)$ then let $(x_n) = (ns^{n-1})$.

Since $s > 0$ clearly $x_n > 0$ for all $n \in \mathbb{N}$.

So all we need to show is that eventually (x_n) is decreasing because this would mean all but finitely many points are decreasing and so we can bound (x_n) by the maximum of those finitely many points.

Let $(y_n) = (x_{n+1} - x_n) = ((n+1)s^n - ns^{n-1}) = (s^{n-1}(n(s-1)+s))$.

Let $N > \frac{s}{1-s}$, then for $n \geq N$ we have $n \geq N > \frac{s}{1-s}$ so $n(1-s) > s$ and $0 > s - n(1-s) = s + n(s-1)$ and so $0 > s^{n-1}(s + n(s-1))$ for all $n \geq N$ since $s > 0$.

So we have that $y_n = x_{n+1} - x_n < 0$ and hence $x_{n+1} < x_n$ so (x_n) is decreasing after x_N.

Again let $M = max\{x_1, x_2, ..., x_N\}$ then we have that $M \geq x_n$ for all $n \in \mathbb{N}$ and since $0 < x_n$ for all $n \in \mathbb{N}$ we have that $|x_n| \leq M$ for all $n \in \mathbb{N}$ hence the sequence $(x_n) = (ns^{n-1})$ is bounded $\square$

b. Assume that $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$. Let $x \in (-R, R)$ then let $|x| < t < R$.

Then if $\sum_{n=0}^{\infty} |na_n x^{n-1}|$ converges we know that $\sum_{n=0}^{\infty} na_n x^{n-1}$ converges.

So $\sum_{n=0}^{\infty} |na_n x^{n-1}| = \sum_{n=0}^{\infty} n|a_n||x|^{n-1}|\frac{t^n}{t^n}| = \sum_{n=0}^{\infty} \frac{n}{t}|a_n t^n||\frac{x}{t}|^{n-1} = \sum_{n=0}^{\infty} \frac{1}{t}|a_n t^n|(n|\frac{x}{t}|^{n-1})$.

Since $|x| < t$ we have that $|\frac{x}{t}| = \frac{|x|}{t} < 1$ and hence by part a we can let $M > 0$ be such that $n|\frac{x}{t}|^{n-1} \leq M$ for all $n \in \mathbb{N}$.

So we have that $\sum_{n=0}^{\infty} |na_n x^{n-1}| = \sum_{n=0}^{\infty} \frac{1}{t}|a_n t^n|(n|\frac{x}{t}|^{n-1}) \leq \sum_{n=0}^{\infty} \frac{M}{t}|a_n t^n| = \frac{M}{t}\sum_{n=0}^{\infty} |a_n t^n|$.

Since $t \in (|x|, R)$ there exists some $r \in (-R, R)$ satisfying $t < r < R$.

Since $r \in (-R, R)$ we know $\sum_{n=0}^{\infty} a_n r^n$ converges and hence $\sum_{n=0}^{\infty} a_n t^n$ converges absolutely since $|t| < |r|$.

Therefore $\sum_{n=0}^{\infty} |a_n t^n|$ converges and so $\frac{M}{t}\sum_{n=0}^{\infty} |a_n t^n|$ converges so $\sum_{n=0}^{\infty} |na_n x^{n-1}|$ converges.

Since $\sum_{n=0}^{\infty} |na_n x^{n-1}|$ converges this means $\sum_{n=0}^{\infty} na_n x^{n-1}$ converges.

This was for arbitrary $x \in (-R, R)$ and is therefore true for all $x \in (-R, R)$.

So if $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$ then so does $\sum_{n=0}^{\infty} na_n x^{n-1}$ $\square$

6.5.7

Let $\sum a_n x^n$ be a power series where $a_n \neq 0$ and assume $L = lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|$ exists.

a. Assume that $L \neq 0$, then let $x \in (-\frac{1}{L}, \frac{1}{L})$. Then let $y_n = a_n x^n$.

If $x = 0$ then $y_n = 0$ for all $n \in \mathbb{N}$ and clearly then $\sum y_n = \sum a_n x^n$ converges.

Otherwise consider $|\frac{y_{n+1}}{y_n}| = |\frac{a_{n+1}x^{n+1}}{a_n x^n}| = |\frac{a_{n+1}x}{a_n}| = |x||\frac{a_{n+1}}{a_n}|$. Note that here $|x|$ is a fixed constant.

So by the algebraic limit theorem $lim_{n\to\infty}|\frac{y_{n+1}}{y_n}| = lim_{n\to\infty}|x||\frac{a_{n+1}}{a_n}| = |x|\ lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = |x|L < \frac{1}{L}L = 1$.

So by the ratio test if $L \neq 0$ and $x \in (-\frac{1}{L}, \frac{1}{L})$ then $\sum a_n x^n$ converges $\square$

b. Assume that $L = 0$, then let $x \in \mathbb{R}$. Then let $y_n = a_n x^n$.

If $x = 0$ then $y_n = 0$ for all $n \in \mathbb{N}$ and clearly then $\sum y_n = \sum a_n x^n$ converges.

Otherwise consider $|\frac{y_{n+1}}{y_n}| = |\frac{a_{n+1}x^{n+1}}{a_n x^n}| = |\frac{a_{n+1}x}{a_n}| = |x||\frac{a_{n+1}}{a_n}|$. Note that here $|x|$ is a fixed constant.

So by the algebraic limit theorem $lim_{n\to\infty}|\frac{y_{n+1}}{y_n}| = lim_{n\to\infty}|x||\frac{a_{n+1}}{a_n}| = |x|\ lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = 0 < 1$.

So by the ratio test if $L = 0$ and $x \in \mathbb{R}$ then $\sum a_n x^n$ converges $\square$

c. Now let $L' = lim_{n\to\infty}s_n$ where $s_n = sup\{|\frac{a_{k+1}}{a_k}| : k \geq n\}$.

First I will prove that for a sequence (b_n) if $lim_{n\to\infty}t_n = M < 1$ then $\sum b_n$ converges where $t_n = sup\{|\frac{b_{k+1}}{b_k}| : k \geq n\}$.

Choose $y \in (M, 1)$ such a y exists because $M < 1$.

Clearly (t_n) is decreasing as if n increases then the supremum is of a subset of the original one and is therefore less than or equal to the supremum before.

Since $(t_n) \to M < y$ and (t_n) is decreasing there must exist some $N \in \mathbb{N}$ such that $|\frac{b_{n+1}}{b_n}| < y$ for all $n \geq N$.

So $|b_{n+1}| < y|b_n|$ for all $n \geq N$. I will show by induction that $|b_n| < y^{n-N}|b_N|$ for all $n > N$.

Let $S = \{n \in \mathbb{N} : |b_n| < y^{n-N}|b_N|\}$.

For our base case we know that for $n = N+1$ we have $|b_n| = |b_{N+1}| < y|b_N| = y^{n-N}|b_N|$. So $N+1 \in S$.

Assume that $n \in S$, then $|b_n| < y^{n-N}|b_N|$ so $|b_{n+1}| < y|b_n| < y(y^{n-N}|b_N|) = y^{(n+1)-N}|b_N|$ and $n+1 \in S$.

Therefore $n \in S$ for all $n > N$ by induction and hence $|b_n| < y^{n-N}|b_N|$ for all $n > N$.

So $\sum_{n=N+1}^{\infty}|b_n| < \sum_{n=N+1}^{\infty} y^{n-N}|b_N| = |b_N|\sum_{k=1}^{\infty} y^k$ which converges since $y < 1$ and this is a geometric series.

By the comparison test $\sum_{n=N+1}^{\infty}|b_n|$ converges and therefore $\sum|b_n| = \sum_{n\leq N}|b_n| + \sum_{n=N+1}^{\infty}|b_n|$ converges since the first sum is finite. So $\sum b_n$ converges and we are done with this proof.

Now for our example let $t_n = sup\{|\frac{a_{k+1}x^{k+1}}{a_k x^k}| : k \geq n\}$. Assume that $L' \neq 0$ and that $x \in (-\frac{1}{L'}, \frac{1}{L'})$. Then let $y_n = a_n x^n$

Again if $x = 0$ then clearly $\sum a_n x^n$ converges.

Otherwise consider $|\frac{y_{n+1}}{y_n}| = |\frac{a_{n+1}x^{n+1}}{a_n x^n}| = |x||\frac{a_{n+1}}{a_n}|$, then clearly $lim_{n\to\infty}t_n = |x|lim_{n\to\infty}s_n = |x|L' < \frac{1}{L'}L' = 1$.

So by the proof before we know that $\sum a_n x^n$ converges.

Now assume that $L' = 0$ and let $x \in \mathbb{R}$. Then let $y_n = a_n x^n$.

Again if $x = 0$ then clearly $\sum a_n x^n$ converges.

Otherwise consider $|\frac{y_{n+1}}{y_n}| = |\frac{a_{n+1}x^{n+1}}{a_n x^n}| = |x||\frac{a_{n+1}}{a_n}|$, then clearly $lim_{n\to\infty}t_n = |x|lim_{n\to\infty}s_n = 0 < 1$.

So by the proof before we know that $\sum a_n x^n$ converges.

Therefore the result still holds if L is replaced by $L' = lim_{n\to\infty}s_n$ $\square$

6.6.6

Let $g(0)=0$ and $g(x)=e^{-\frac{1}{x^2}}$ for $x\neq 0$.

a. We are given that $g'(0)=0$.

Let $c\neq 0$ then $g'(c)=(e^{-\frac{1}{c^2}})(\frac{2}{c^3})$ by using the chain rule and the fact that e^y is its own derivative.

Now let's find $g''(0)=lim_{x\to 0}\frac{g'(x)-g'(0)}{x-0}=lim_{x\to 0}\frac{\frac{2}{x^3}e^{-\frac{1}{x^2}}}{x}=lim_{x\to 0}\frac{2e^{-\frac{1}{x^2}}}{x^4}$ which satisfies the 0/0 case for L'hospital's rule because $\frac{1}{x^2}$ grows arbitrarily large and hence $e^{-\frac{1}{x^2}}$ grows arbitrarily close to 0.

So $g''(0)=lim_{x\to 0}\frac{2e^{-\frac{1}{x^2}}}{x^4}=lim_{x\to 0}\frac{2}{x^4e^{\frac{1}{x^2}}}=lim_{x\to 0}\frac{0}{4x^3e^{\frac{1}{x^2}}+x^4e^{\frac{1}{x^2}}(-\frac{2}{x^3})}=0$.

b. We have from before that $g'(x)=\frac{2}{x^3}e^{-\frac{1}{x^2}}$ for $x\neq 0$.

So for $x\neq 0$ we have $g''(x)=-\frac{6}{x^4}e^{-\frac{1}{x^2}}+(\frac{2}{x^3})(\frac{2}{x^3})e^{-\frac{1}{x^2}}=(\frac{4}{x^6}-\frac{6}{x^4})e^{-\frac{1}{x^2}}$.

And thus for $x\neq 0$ we have $g'''(x)=(-\frac{24}{x^7}+\frac{24}{x^5})e^{-\frac{1}{x^2}}+(\frac{4}{x^6}-\frac{6}{x^4})(\frac{2}{x^3})e^{-\frac{1}{x^2}}=(\frac{8}{x^9}-\frac{36}{x^7}+\frac{24}{x^5})e^{-\frac{1}{x^2}}$.

I claim that $f^{(n)}(x)$ is of the form $(\sum_{k=1}^n a_kx^{-(n+2k)})e^{-\frac{1}{x^2}}$ for all $n\in\mathbb{N}$ when $x\neq 0$ and I will show this by induction.

Let $S=\{n\in\mathbb{N}: f^{(n)}(x)=(\sum_{k=1}^n a_kx^{-(n+2k)})e^{-\frac{1}{x^2}}\}$.

For our base case we know $g'(x)=g^{(1)}(x)=(2x^{-3})e^{-\frac{1}{x^2}}=(a_1x^{-(1+2(1))})e^{-\frac{1}{x^2}}$ so $1\in S$.

Now assume $n\in S$, that is assume $f^{(n)}(x)=(\sum_{k=1}^n a_kx^{-(n+2k)})e^{-\frac{1}{x^2}}$.

Then $f^{(n+1)}(x)=(\sum_{k=1}^n(a_k)(-(n+2k))x^{-(n+2k+1)})e^{-\frac{1}{x^2}}+(\sum_{k=1}^n a_kx^{-(n+2k)})(\frac{2}{x^3})e^{-\frac{1}{x^2}}=$ $((\sum_{k=1}^n b_kx^{-((n+1)+2k)})+(\sum_{k=1}^n c_kx^{-((n+1)+2(k+1))}))e^{-\frac{1}{x^2}}=(\sum_{k=1}^{n+1}d_kx^{-((n+1)+2k)})e^{-\frac{1}{x^2}}$. Therefore $n+1\in S$.

So by induction $S=\mathbb{N}$ and hence $f^{(n)}(x)=(\sum_{k=1}^n a_kx^{-(n+2k)})e^{-\frac{1}{x^2}}$ for all $n\in\mathbb{N}$ when $x\neq 0$.

c. Let $S=\{n\in\mathbb{N}: g^{(n)}(0)=0\}$. We already know $g'(0)=g^{(1)}(0)=0$, so $1\in S$.

Now assume $n\in S$, that is $g^{(n)}(0)=0$.

Then $g^{(n+1)}(0)=lim_{x\to 0}\frac{g^{(n)}(x)-g^{(n)}(0)}{x-0}=lim_{x\to 0}\frac{(\sum_{k=1}^n a_kx^{-(n+2k)})e^{-\frac{1}{x^2}}}{x}=lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{(\sum_{k=1}^n a_kx^{(n+2k)})x}$ satisfies the 0/0 case for L'hospital's rule.

So $g^{(n+1)}(0)=lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{(\sum_{k=1}^n a_kx^{(n+2k)})x}=lim_{x\to 0}\frac{1}{(\sum_{k=1}^n a_kx^{(n+2k+1)})e^{\frac{1}{x^2}}}=$
$lim_{x\to 0}\frac{0}{(\sum_{k=1}^n b_kx^{(n+2k+2)})e^{\frac{1}{x^2}}+(\sum_{k=1}^n b_kx^{(n+2k+1)})(\frac{2}{x^3})e^{\frac{1}{x^2}}}=0$

Therefore $n+1\in S$. So by induction $S=\mathbb{N}$ and hence $g^{(n)}(0)=0$ for all $n\in\mathbb{N}$ $\square$

6.3.7

Let (f_n) be a sequence of differentiable functions on $[a,b]$ such that (f_n') converges uniformly and at some point $x_0 \in [a,b]$, $(f_n(x_0))$ is convergent.

Let $\epsilon > 0$ and let $\alpha = \epsilon/2$ and let $\beta = \frac{\epsilon}{2(b-a)}$.

Then since $(f_n(x_0))$ is convergent it is also Cauchy so let $N_1 \in \mathbb{N}$ be such that for all $m,n \geq N_1$ we have $|f_n(x_0) - f_m(x_0)| < \alpha$.

Furthermore we know each f_n is differentiable so for all $m,n \in \mathbb{N}$ we have $(f_n - f_m)' = f_n' - f_m'$.

Since $(f_n') \to g$ uniformly for some g we have that (f_n') satisfies the Cauchy criterion for uniform convergence.

So let N_2 be such that $|f_n'(x) - f_m'(x)| < \beta$ when $m,n \geq N_2$ and $x \in [a,b]$.

Then for any $x \in [a,b]$ we can apply the mean value theorem to $f_n - f_m$ on $[x, x_o]$ to conclude there exists a $c \in [x, x_0]$ such that $|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| = |x - x_0||f_n'(c) - f_m'(c)|$.

Therefore for $m,n \geq N_2$ we have

$$|(f_n(x)-f_m(x))-(f_n(x_0)-f_m(x_0))| = |x-x_0||f_n'(c)-f_m'(c)| \leq (b-a)|f_n'(c)-f_m'(c)| < (b-a)\beta = (b-a)\frac{\epsilon}{2(b-a)} = \epsilon/2 = \alpha.$$

Let $N = max\{N_1, N_2\}$ this N exists since we are looking at a finite set.

Then for $m,n \geq N$ we have that $|f_n(x_0) - f_m(x_0)| < \alpha$ and $|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| < \alpha$.

Therefore for $m,n \geq N$ we have $|f_n(x) - f_m(x)| = |f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0)) + (f_n(x_0) - f_m(x_0))| \leq$ $|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)| < 2\alpha = \epsilon$.

So for all $\epsilon > 0$ we have found an $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ when $m,n \geq N$ and $x \in [a,b]$.

Therefore (f_n) converges uniformly on $[a,b]$ $\square$