Polar Coordinates, Cauchy Riemann Equations, and Integration

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24.1

d. Let $f(z) = e^{\overline{z}} = e^x e^{-iy} = e^x(cos(-y) + i\ sin(-y)) = e^x(cos(y) - i\ sin(y))$ when $z = x + iy$.

Recall that if a function $g(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$ then we must have:

$u_x = v_y$ and $u_y = -v_x$ at the point (x_0, y_0).

Now we know $f(z) = e^{\overline{z}} = e^x(cos(y) - i\ sin(y)) = e^x cos(y) - ie^x sin(y)$.

So we can write $f(z) = e^{\overline{z}} = u(x, y) + iv(x, y)$ where $u(x, y) = e^x cos(y)$ and $v(x, y) = -e^x sin(y)$.

From this we get $u_x = \frac{\partial}{\partial x} u = e^x cos(y)$, $u_y = \frac{\partial}{\partial y} u = -e^x sin(y)$, $v_x = \frac{\partial}{\partial x} v = -e^x sin(y)$, and $v_y = \frac{\partial}{\partial y} v = -e^x cos(y)$.

When we take the derivative with respect to only one variable (partial derivative) the other variable acts as a constant.

So we have from above that $u_x = e^x cos(y)$ while $v_y = -e^x cos(y)$.

If we want $u_x = v_y$ then $e^x cos(y) = -e^x cos(y)$. Since $e^x \neq 0$ we get $cos(y) = -cos(y)$ and $2cos(y) = 0$ i.e. $cos(y) = 0$.

So $u_x(x, y) = v_y(x, y)$ only if $y = \frac{\pi}{2} + n\pi$ for some $n \in \mathbb{Z}$.

We also have $u_y = -e^x sin(y)$ and $v_x = -e^x sin(y)$.

If we want $u_y = -v_x$ then $-e^x sin(y) = e^x sin(y)$. Since $e^x \neq 0$ we get $-sin(y) = sin(y)$ and $2sin(y) = 0$ i.e. $sin(y) = 0$.

So $u_y(x, y) = -v_x(x, y)$ only if $y = m\pi$ for some $m \in \mathbb{Z}$.

It is not possible for $y = m\pi$ and $y = \frac{\pi}{2} + n\pi$ where $m, n \in \mathbb{Z}$ simultaneously.

Proof:

Assume $y = \frac{\pi}{2} + n\pi$ and $y = m\pi$ for some $m, n \in \mathbb{Z}$. Then $\frac{\pi}{2} + n\pi = m\pi$ and $\frac{1}{2} + n = m$.

This is a contradiction because by assumption $m, n \in \mathbb{Z}$.

Therefore it is not possible for $y = m\pi$ and $y = \frac{\pi}{2} + n\pi$ where $m, n \in \mathbb{Z}$ simultaneously.

So the Cauchy Riemann equations ($u_x = v_y$ and $u_y = -v_x$) are not satisfied anywhere.

Consequently, we know that $f(z) = e^{\overline{z}}$ is not differentiable for any $z \in \mathbb{C}$ $\square$

24.4

Recall that if a function $g(z) = u(r,\theta) + iv(r,\theta)$ is defined in a neighborhood of a point $z_0 = r_0 e^{i\theta_0}$. Then if the partial derivatives of the component functions exist in that neighborhood, are continuous at z_0, and satisfy the polar Cauchy Riemann equations ($ru_r = v_\theta$ and $u_\theta = -rv_r$) at z_0 then $g(z)$ is differentiable at z_0 and $g'(z_0) = e^{-i\theta}(u_r + iv_r)\Big|_{(r_0,\theta_0)}$

a. Let $f(z) = \frac{1}{z^4}$. Then if we write $z = re^{i\theta}$ we have $f(z) = \frac{1}{(re^{i\theta})^4} = \frac{1}{r^4}e^{-4i\theta}$.

We have that $f(z) = \frac{1}{z^4} = \frac{1}{r^4}e^{-4i\theta} = \frac{1}{r^4}(cos(-4\theta) + i\, sin(-4\theta)) = \frac{1}{r^4}(cos(4\theta) - i\, sin(4\theta)) = \frac{1}{r^4}cos(4\theta) - i\frac{1}{r^4}sin(4\theta)$.

So we can write $f(z) = u(r,\theta) + iv(r,\theta)$ where $u(r,\theta) = r^{-4}cos(4\theta)$ and $v(r,\theta) = -r^{-4}sin(4\theta)$.

Therefore $u_r = \frac{\partial}{\partial r}u = -4r^{-5}cos(4\theta)$, $u_\theta = \frac{\partial}{\partial \theta}u = -4r^{-4}sin(4\theta)$, $v_r = \frac{\partial}{\partial r}v = 4r^{-5}sin(4\theta)$, and $v_\theta = \frac{\partial}{\partial \theta}v = -4r^{-4}cos(4\theta)$.

These are all continuous if $r \neq 0$ because the product of continuous functions is continuous.

We also have that $ru_r = r(-4r^{-5}cos(4\theta)) = -4r^{-4}cos(4\theta) = v_\theta$ and $u_\theta = -4r^{-4}sin(4\theta) = -r(4r^{-5}sin(4\theta)) = -rv_r$.

So partial derivatives of the component functions exist and are continuous when $z \neq 0$ and the Cauchy Riemann equations satisfied for $z \neq 0$, therefore $f(z)$ is differentiable for $z \neq 0$ and

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(-4r^{-5}cos(4\theta) + 4ir^{-5}sin(4\theta)) = \frac{-4}{r^5}e^{-i\theta}(cos(4\theta) - i\, sin(4\theta)) =$$

$$\frac{-4}{r^5}e^{-i\theta}(cos(-4\theta) + i\, sin(-4\theta)) = \frac{-4}{r^5}e^{-i\theta}e^{-4i\theta} = \frac{-4}{r^5e^{5i\theta}} = \frac{-4}{(re^{i\theta})^5} = \frac{-4}{z^5} \ \square$$

b. Let $f(z) = e^{-\theta}cos(ln(r)) + ie^{-\theta}sin(ln(r))$ where $z = re^{i\theta}$ and $r > 0$, $\theta \in (0, 2\pi)$.

So we can write $f(z) = u(r,\theta) + iv(r,\theta)$ where $u(r,\theta) = e^{-\theta}cos(ln(r))$ and $v(r,\theta) = e^{-\theta}sin(ln(r))$.

Therefore $u_r = \frac{\partial}{\partial r}u = -e^{-\theta}sin(ln(r))\frac{1}{r}$, $u_\theta = \frac{\partial}{\partial \theta}u = -e^{-\theta}cos(ln(r))$, $v_r = \frac{\partial}{\partial r}v = e^{-\theta}cos(ln(r))\frac{1}{r}$, and $v_\theta = \frac{\partial}{\partial \theta}v = -e^{-\theta}sin(ln(r))$.

These are all continuous for $r > 0$ because the product of continuous functions is continuous.

We also have that $ru_r = r(-e^{-\theta}sin(ln(r))\frac{1}{r}) = -e^{-\theta}sin(ln(r)) = v_\theta$ and $u_\theta = -e^{-\theta}cos(ln(r)) = -r(e^{-\theta}cos(ln(r))\frac{1}{r}) = -rv_r$.

So partial derivatives of the component functions exist and are continuous when $r > 0$ and the Cauchy Riemann equations satisfied for $r > 0$, therefore $f(z)$ is differentiable for $z \neq 0$ and

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(-e^{-\theta}sin(ln(r))\tfrac{1}{r} + ie^{-\theta}cos(ln(r))\tfrac{1}{r}) = \tfrac{1}{r}e^{-\theta}e^{-i\theta}(i^2 sin(ln(r)) + i\, cos(ln(r))) =$$

$$\frac{i}{re^{i\theta}}e^{-\theta}(i\, sin(ln(r)) + cos(ln(r))) = \frac{i}{re^{i\theta}}(e^{-\theta}cos(ln(r)) + ie^{-\theta}sin(ln(r))) = \frac{if(z)}{z} \ \square$$

24.8

b. The operator $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ is given in the question. Now let $f(z) = u(x,y) + iv(x,y)$ where $z = x + iy$.

Fix some point $z_0 = x_0 + iy_0$ and assume that $f(z)$ satisfies the Cauchy Riemann equations at z_0.

That is assume $u_x = v_y$ and $u_y = -v_x$ at (x_0, y_0).

Then we have:

$$\frac{\partial}{\partial \bar{z}} f(z) = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big) f(z) = \frac{1}{2}\Big(\frac{\partial}{\partial x} f(z) + i\frac{\partial}{\partial y} f(z)\Big) = \frac{1}{2}\Big(\frac{\partial}{\partial x}(u(x,y) + iv(x,y)) + i\frac{\partial}{\partial y}(u(x,y) + iv(x,y))\Big) =$$

$$\frac{1}{2}\Big(\frac{\partial}{\partial x} u(x,y) + i\frac{\partial}{\partial x} v(x,y) + i\frac{\partial}{\partial y} u(x,y) + i^2\frac{\partial}{\partial y} v(x,y)\Big) = \frac{1}{2}\Big((u_x(x,y) - v_y(x,y)) + i(u_y(x,y) + v_x(x,y))\Big)$$

Since we know that the Cauchy Riemann equations are satisfied at z_0 we know $u_x = v_y$ and $u_y = -v_x$ at (x_0, y_0).

Therefore $\frac{\partial f}{\partial \bar{z}}\Big|_{z_0} = \frac{1}{2}\Big((u_x(x_0,y_0) - v_y(x_0,y_0)) + i(u_y(x_0,y_0) + v_x(x_0,y_0))\Big) = \frac{1}{2}\Big(0 + 0i\Big) = 0.$

This was true for an arbitrary $z_0 \in \mathbb{C}$ and is therefore true for all $z_0 \in \mathbb{C}$.

So if the Cauchy Riemann equations are satisfied at z_0 then $\frac{\partial f}{\partial \bar{z}}\Big|_{z_0} = 0$, and clearly the reverse is true $\square$

26.2

Recall that if a function $g(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$ then we must have:

$u_x = v_y$ and $u_y = -v_x$ at the point (x_0, y_0).

Further recall that in order for a function to be analytic at $z_0 \in \mathbb{C}$ it must be differentiable in some neighborhood of z_0.

When x and y are used in this problem I am referring to the real and imaginary parts of a complex variable $z = x + iy$.

a. Let $f(z) = xy + iy$. Then $f(z) = u(x,y) + iv(x,y)$ where $u = xy$ and $v = y$.

We know that $u_x = \frac{\partial}{\partial x}u = y$, $u_y = \frac{\partial}{\partial y}u = x$, $v_x = \frac{\partial}{\partial x}v = 0$, and $v_y = \frac{\partial}{\partial y}v = 1$.

If we want $u_x = v_y$ then $y = 1$, and if we want $u_y = -v_x$ then $x = 0$.

So we have that f can not be differentiable at any point that is not $0 + 1i = i$. Note that I am not stating that f is differentiable at i, I am just saying f can not be differentiable at any other point.

Clearly if $z_0 \neq i$ then f is not differentiable at z_0 and hence not differentiable in any neighborhood of z_0.

If $z_0 = i$ then for any neighborhood of z_0 there exists some point $z \neq z_0$ in the neighborhood. Hence f is not differentiable at z and consequently not differentiable in any neighborhood of z_0.

Therefore for any $z_0 \in \mathbb{C}$ there does not exist a neighborhood of z_0 where f is differentiable.

So $f(z) = xy + iy$ where $z = x + iy$ is nowhere analytic $\square$

b. Let $f(z) = 2xy + i(x^2 - y^2)$. Then $f(z) = u(x,y) + iv(x,y)$ where $u = 2xy$ and $v = x^2 - y^2$.

We know that $u_x = \frac{\partial}{\partial x}u = 2y$, $u_y = \frac{\partial}{\partial y}u = 2x$, $v_x = \frac{\partial}{\partial x}v = 2x$, and $v_y = \frac{\partial}{\partial y}v = -2y$.

If we want $u_x = v_y$ then $2y = -2y$ and hence $y = 0$, and if we want $u_y = -v_x$ then $2x = -2y$ and hence $x = -y$.

If both of these are simultaneously true then we must have $y = 0$ and $x = -y = 0$.

So we have that f can not be differentiable at any point that is not $0 + 0i = 0$. Note that I am not stating that f is differentiable at 0, I am just saying f can not be differentiable at any other point.

Clearly if $z_0 \neq 0$ then f is not differentiable at z_0 and hence not differentiable in any neighborhood of z_0.

If $z_0 = 0$ then for any neighborhood of z_0 there exists some point $z \neq z_0$ in the neighborhood. Hence f is not differentiable at z and consequently not differentiable in any neighborhood of z_0.

Therefore for any $z_0 \in \mathbb{C}$ there does not exist a neighborhood of z_0 where f is differentiable.

So $f(z) = 2xy + i(x^2 - y^2)$ where $z = x + iy$ is nowhere analytic $\square$

b. Let $f(z) = e^y e^{ix} = e^y(cos(x) + i\,sin(x))$. Then $f(z) = u(x,y) + iv(x,y)$ where $u = e^y cos(x)$ and $v = e^y sin(x)$.

We know that $u_x = \frac{\partial}{\partial x}u = -e^y sin(x)$, $u_y = \frac{\partial}{\partial y}u = e^y cos(x)$, $v_x = \frac{\partial}{\partial x}v = e^y cos(x)$, and $v_y = \frac{\partial}{\partial y}v = e^y sin(x)$.

If we want $u_x = v_y$ then $-e^y sin(x) = e^y sin(x)$ and hence $sin(x) = 0$, and if we want $u_y = -v_x$ then $e^y cos(x) = -e^y cos(x)$ and hence $cos(x) = 0$.

I showed earlier in this sample work that these equations can not be simultaneously true.

So we have that f can not be differentiable at any point.

Clearly if $z_0 \in \mathbb{C}$ then f is not differentiable at z_0 and hence not differentiable in any neighborhood of z_0.

Therefore for any $z_0 \in \mathbb{C}$ there does not exist a neighborhood of z_0 where f is differentiable.

So $f(z) = e^y e^{ix}$ where $z = x + iy$ is nowhere analytic $\square$

26.4

Recall that a point z_0 is a singular point of f if f fails to be analytic at z_0 but is analytic at some point for every neighborhood of z_0.

c. Let $f(z) = \frac{z^2+1}{(z+2)(z^2+2z+2)}$.

The roots of a complex polynomial $P(z)$ can be found with the quadratic equation as shown in a previous sample work.
So $z = \frac{-2\pm\sqrt{2^2-4(1)(2)}}{2} = \frac{-2\pm\sqrt{-4}}{2} = -1 \pm i$ are the roots of z^2+2z+2
Therefore $z^2+2z+2 = (z-(-1-i))(z-(-1+i))$ and consequently $f(z) = \frac{z^2+1}{(z+2)(z-(-1-i))(z-(-1+i))}$.
Clearly if $z_0 = -2$ or $z_0 = -1-i$ or $z_0 = -1+i$ then $f(z_0)$ does not exist and hence f is not continuous at z_0 and can not be differentiable at z_0. Consequently f is not differentiable in any neighborhood of -2, $-1-i$, and $-1+i$.
So f is not analytic at -2, $-1-i$, and $-1+i$.

However, if $z_0 \notin \{-2, -1-i, -1+i\}$ then there exists some neighborhood of z_0 that contains none of these points. Simply let $\epsilon < min\{|z_0-(-2)|, |z_0-(-1-i)|, |z_0-(-1+i)|\}$, that is let ϵ be less than the minimum distance to any of the points -2, $-1-i$, and $-1+i$.
Then the neighborhood $\{z \in \mathbb{C} : |z_0 - z| < \epsilon\}$ of z_0 won't contain any of the points -2, $-1-i$, and $-1+i$ since they are more than a distance of ϵ away from z_0.
The numerator of $f(z)$ is a complex polynomial and hence is differentiable at all $z_0 \in \mathbb{C}$.
Similarly the denominator of $f(z)$ is a complex polynomial and hence is differentiable at all $z_0 \in \mathbb{C}$.
Therefore for any $z_0 \notin \{-2, -1-i, -1+i\}$ we have that the denominator of f evaluated at z_0 is not 0.
Then since the quotient of differentiable functions is differentiable when the denominator is not 0, we have that $f(z)$ is differentiable at z_0 when $z_0 \notin \{-2, -1-i, -1+i\}$.

Now we know for any neighborhood of any of any the points -2, $-1-i$, and $-1+i$ we can find a $z_0 \notin \{-2, -1-i, -1+i\}$ in that neighborhood.
For such a z_0 we know that we can find a neighborhood of z_0 that does not contain any of the points -2, $-1-i$, and $-1+i$. Hence we can find a neighborhood of z_0 where f is differentiable since its denominator is nonzero.

Similarly if we just start with a $z \notin \{-2, -1-i, -1+i\}$ then we know there exists a neighborhood of z that does not contain any of the points -2, $-1-i$, and $-1+i$. Hence there exists a neighborhood of z where f is differentiable since its denominator is nonzero.

Therefore f fails to be analytic at each of the points -2, $-1-i$, and $-1+i$, but is analytic at some point in every neighborhood of each of these points. Also, f is analytic at every $z \notin \{-2, -1-i, -1+i\}$.
Therefore -2, $-1-i$, and $-1+i$ are the singular points of f and f is analytic everywhere else. $\square$

27.2

Recall that two lines are perpendicular in $\mathbb{R}^2$ if their slopes m_1, m_2 satisfy $m_1 = -\frac{1}{m_2}$.

Proof:

Let L_1 and L_2 be two lines in $\mathbb{R}^2$ with slopes $m_1 \neq 0$ and $m_2 \neq 0$ respectively.

Then the coordinate rates of change are given by $(1, m_1)$ and $(1, m_2)$ for lines one and two respectively.

Taking the dot product we get $1 + m_1 m_2$, if we want this to be equal to 0 (meaning the lines are perpendicular we get):

$1 + m_1 m_2 = 0$ and $1 = -m_1 m_2$ and finally $m_1 = -\frac{1}{m_2}$.

Let $f(z) = u(x, y) + iv(x, y)$ be analytic over some domain D. Let $c_1, c_2 \in \mathbb{R}$ be arbitrary.

Consider the level curves $u(x, y) = c_1$ and $v(x, y) = c_2$.

Fix some $z_0 = x_0 + iy_0$ that is common to both curves.

Further assume that $f'(z_0) \neq 0$.

Then by differentiating the equations $u(x, y) = c_1$ and $v(x, y) = c_2$ with respect to x we get:

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = \frac{d}{dx}c_1 = 0$ and $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = \frac{d}{dx}c_2 = 0$.

We also know that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ since f is analytic over D and hence the Cauchy Riemann equations must be satisfied over D.

Furthermore we know that $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \neq 0$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \neq 0$ at (x_0, y_0) by our assumption that $f'(z_0) \neq 0$.

Therefore for the first curve, $u(x, y) = c_1$, we have:

$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\frac{dy}{dx} = 0$ and $\frac{\partial v}{\partial x}\frac{dy}{dx} = \frac{\partial u}{\partial x}$.

Finally since $v_x = \frac{\partial v}{\partial x} \neq 0$ at (x_0, y_0) we have $\frac{dy}{dx} = \frac{u_x}{v_x}$ at (x_0, y_0).

Similarly for the second curve, $v(x, y) = c_2$, we have:

$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{dy}{dx} = 0$ and $\frac{\partial u}{\partial x}\frac{dy}{dx} = -\frac{\partial v}{\partial x}$.

Finally since $u_x = \frac{\partial u}{\partial x} \neq 0$ at (x_0, y_0) we have $\frac{dy}{dx} = -\frac{v_x}{u_x}$ at (x_0, y_0).

Let m_1 be the rate of change of the line tangent to the first curve $(u(x, y) = c_1)$ at (x_0, y_0), then $m_1 = \frac{u_x}{v_x}\Big|_{(x_0, y_0)}$

Let m_2 be the rate of change of the line tangent to the second curve $(v(x, y) = c_2)$ at (x_0, y_0), then $m_2 = -\frac{v_x}{u_x}\Big|_{(x_0, y_0)}$

Therefore we have that:

$$m_1 = \frac{u_x}{v_x} = -(-\frac{u_x}{v_x}) = -(\frac{1}{-\frac{v_x}{u_x}}) = -\frac{1}{m_2}$$

By the proof at the start of this problem we have shown that $u(x, y) = c_1$ is perpendicular to $v(x, y) = c_2$ at (x_0, y_0) $\square$

Problem 2

Let $f(z) = z^2$ if $z \in \mathbb{R}$ and $f(z) = z^3$ otherwise.

- Showing f is differentiable at $z_0 = 0$:

Let $\epsilon > 0$ then let $\delta < min\{\epsilon, 1\}$. Then $\delta < \epsilon$.

We know that $|\frac{f(z)-f(0)}{z-0}| = |\frac{f(z)}{z}|$ for $z \neq 0$.

Therefore $|\frac{f(z)-f(0)}{z-0} - 0| = |\frac{f(z)}{z}| = |\frac{z^2}{z}| = |z|$ for all nonzero $z \in \mathbb{R}$.

Similarly $|\frac{f(z)-f(0)}{z-0} - 0| = |\frac{f(z)}{z}| = |\frac{z^3}{z}| = |z^2|$ for all nonzero $z \in \mathbb{C} \cap \mathbb{R}^c$.

Now if $|z - 0| = |z| < \delta$ we have that $|z| < \delta < 1$ and therefore $|z^2| = |z|^2 < |z|$.

Therefore we have that if $|z - 0| < \delta$ then $|\frac{f(z)-f(0)}{z-0} - 0| = |\frac{f(z)}{z}| < |z| < \delta < \epsilon$.

This was true for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

Therefore $f'(0) = lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = 0$, so f is differentiable at 0.

- Showing f is not analytic at 0:

When $z \notin \mathbb{R}$ clearly $f(z) = z^3$ is differentiable because it is a complex polynomial, but that is not what we are looking at.

We will now look at $lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ from the vertical direction when $z_0 \in \mathbb{R}\backslash\{0, 1\}$.

By taking the vertical approach we have $\Delta z = \Delta x + i\Delta y = i\Delta y$ since we take $\Delta x = 0$.

So $z_0 + \Delta z = z_0 + i\Delta y \notin \mathbb{R}$ since $z_0 \in \mathbb{R}$ and $i\Delta y \notin \mathbb{R}$, and $f(z_0 + \Delta z) = f(z_0 + i\Delta y) = (z_0 + i\Delta y)^3$.

I claim that for $z_0 \in \mathbb{R}\backslash\{0, 1\}$ from the vertical approach $lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = lim_{\Delta y\to 0}\frac{f(z_0+i\Delta y)-f(z_0)}{i\Delta y} = \infty$.

I am excluding the point 1 for reasons that will be obvious when taking the limit, but since we are only removing a finite number of points we will still get the desired result.

$$lim_{\Delta y\to 0}\frac{1}{\frac{f(z_0+i\Delta y)-f(z_0)}{i\Delta y}} = lim_{\Delta y\to 0}\frac{i\Delta y}{f(z_0 + i\Delta y) - f(z_0)} = lim_{\Delta y\to 0}\frac{i\Delta y}{(z_0 + i\Delta y)^3 - z_0^2} =$$

$$lim_{\Delta y\to 0}\frac{i\Delta y}{z_0^3 + 3iz_0^2\Delta y - 3z_0(\Delta y)^2 - i(\Delta y)^3 - z_0^2}$$

We know that clearly $lim_{\Delta y\to 0}(z_0^3 + 3iz_0^2\Delta y - 3z_0(\Delta y)^2 - i(\Delta y)^3 - z_0^2) = z_0^3 - z_0^2 = z_0^2(z_0 - 1)$ and $lim_{\Delta y\to 0}i\Delta y = 0$.

Assume $z_0 \notin \{0, 1\}$. Then $lim_{\Delta y\to 0}(z_0^3 + 3iz_0^2\Delta y - 3z_0(\Delta y)^2 - i(\Delta y)^3 - z_0^2) = z_0^3 - z_0^2 = z_0^2(z_0 - 1) \neq 0$.

Therefore if $z_0 \in \mathbb{R}\backslash\{0, 1\}$ we have $lim_{\Delta y\to 0}\frac{i\Delta y}{f(z_0+i\Delta y)-f(z_0)} = 0$ and consequently,

$lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = lim_{\Delta y\to 0}\frac{f(z_0+i\Delta y)-f(z_0)}{i\Delta y} = \infty$ from the vertical approach.

So for $z_0 \in \mathbb{R}\backslash\{0, 1\}$ the derivative of f does not exist since the limit approaching vertically is not finite.

Now consider an arbitrary neighborhood of 0, then you will always be able to find a point $z_0 \in \mathbb{R}\backslash\{0, 1\}$ and hence in any neighborhood of 0 you will always be able to find a point where f is not differentiable.

Therefore f can not be differentiable in any neighborhood of 0 and so f is not analytic at 0 $\square$

It is actually the case that f is not analytic at any $z_0 \in \mathbb{R}$ by similar reasoning.

Problem 3

Let $h(z)$ be a function such that both $h(z)$ and $zh(z)$ solve the Laplace equation over a domain D.

Then if we write $h(z) = u(x,y) + iv(x,y)$ we know that

$$zh(z) = (x+iy)(u(x,y)+iv(x,y)) = xu(x,y) - yv(x,y) + i(yu(x,y) + xv(x,y)).$$

So let $zh(z) = s(x,y) + it(x,y)$ where $s(x,y) = xu(x,y) - yv(x,y)$ and $t(x,y) = yu(x,y) + xv(x,y)$.

We also know $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$ over D. Similarly we know $s_{xx} + s_{yy} = 0$ and $t_{xx} + t_{yy} = 0$ over D.

Let's actually find s_{xx}, s_{yy}, t_{xx}, and t_{yy} in terms of u and v:

For $s(x,y)$:

To start $\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}(xu - yv) = u + xu_x - yv_x$.

So $s_{xx} = \frac{\partial^2 s}{\partial x^2} = \frac{\partial}{\partial x}(u + xu_x - yv_x) = u_x + u_x + xu_{xx} - yv_{xx} = 2u_x + xu_{xx} - yv_{xx}$.

Similarly $\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}(xu - yv) = xu_y - v - yv_y$.

So $s_{yy} = \frac{\partial^2 s}{\partial y^2} = \frac{\partial}{\partial y}(xu_y - v - yv_y) = xu_{yy} - v_y - v_y - yv_{yy} = xu_{yy} - 2v_y - yv_{yy}$.

For $t(x,y)$:

To start $\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}(yu + xv) = yu_x + v + xv_x$.

So $t_{xx} = \frac{\partial^2 t}{\partial x^2} = \frac{\partial}{\partial x}(yu_x + v + xv_x) = yu_{xx} + v_x + v_x + xv_{xx} = yu_{xx} + 2v_x + xv_{xx}$.

Similarly $\frac{\partial t}{\partial y} = \frac{\partial}{\partial y}(yu + xv) = u + yu_y + xv_y$.

So $t_{yy} = \frac{\partial^2 t}{\partial y^2} = \frac{\partial}{\partial y}(u + yu_y + xv_y) = u_y + u_y + yu_{yy} + xv_{yy} = 2u_y + yu_{yy} + xv_{yy}$.

Now lets plug these into the equations $s_{xx} + s_{yy} = 0$ and $t_{xx} + t_{yy} = 0$:

Keep in mind that $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$.

For $s(x,y)$:

$$s_{xx} + s_{yy} = 2u_x + xu_{xx} - yv_{xx} + xu_{yy} - 2v_y - yv_{yy} = x(u_{xx} + u_{yy}) - y(v_{xx} + v_{yy}) + 2(u_x - v_y) = 2(u_x - v_y) = 0.$$

Therefore we have that $u_x = v_y$ over D.

For $t(x,y)$:

$$t_{xx} + t_{yy} = yu_{xx} + 2v_x + xv_{xx} + 2u_y + yu_{yy} + xv_{yy} = y(u_{xx} + u_{yy}) + x(v_{xx} + v_{yy}) + 2(u_y + v_x) = 2(u_y + v_x) = 0.$$

Therefore we have that $u_y = -v_x$ over D.

The professor said that we may assume u and v are twice differentiable with continuous derivatives therefore we know that the first order partial derivatives of u and v are continuous.

Therefore Cauchy Riemann equations are satisfied over D and the first order partial derivatives of the component functions are continuous over D.

So $h(z)$ is analytic over D $\square$

Bonus

Let $h(t)$ be a complex valued function continuous on $[0,1]$.

Then define a new function for $z \in \mathbb{C}\backslash[0,1]$:

$$f(z) = \int_0^1 \frac{h(t)}{z-t} dt$$

Then we want to find $f'(z)$:

Let $z_0 \in \mathbb{C}\backslash[0,1]$ be arbitrary, then for $z \neq z_0$:

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{1}{z-z_0}\int_0^1 \frac{h(t)}{z-t}dt - \frac{1}{z-z_0}\int_0^1 \frac{h(t)}{z_0-t}dt = \frac{1}{z-z_0}\int_0^1 \frac{h(t)}{z-t} - \frac{h(t)}{z_0-t}dt =$$

$$\frac{1}{z-z_0}\int_0^1 \frac{h(t)(z_0-t)-h(t)(z-t)}{(z-t)(z_0-t)}dt = \frac{1}{z-z_0}\int_0^1 \frac{z_0h(t)-zh(t)}{(z-t)(z_0-t)}dt = \frac{1}{z-z_0}\int_0^1 \frac{(z_0-z)h(t)}{(z-t)(z_0-t)}dt =$$

$$\int_0^1 \frac{h(t)}{(z-t)(z_0-t)}dt$$

Therefore:

$$lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = lim_{z\to z_0}\int_0^1 \frac{h(t)}{(z-t)(z_0-t)}dt = \int_0^1 lim_{z\to z_0}\frac{h(t)}{(z-t)(z_0-t)}dt = \int_0^1 \frac{h(t)}{(z_0-t)^2}dt$$

Since we know $z_0 \notin [0,1]$ we know that the limit of the denominator inside the integral is not 0 for any $t \in [0,1]$ and hence the limit is straightforward since it is the limit of a constant function (with respect to z) divided by a polynomial.

This was true for arbitrary $z_0 \in \mathbb{C}\backslash[0,1]$ and therefore true for all $z_0 \in \mathbb{C}\backslash[0,1]$.

So f is analytic since it is differentiable at every $z_0 \in \mathbb{C}\backslash[0,1]$, and:

$$f'(z_0) = lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = \int_0^1 \frac{h(t)}{(z_0-t)^2}dt$$

$\square$