Regions in the Complex Plane

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6.9

Assume $|z| = 2$. Then consider $w = z^4 - 4z^2 + 3$. Recall that for $z_1, z_2 \in \mathbb{C}$ we know $||z_1| - |z_2|| \leq |z_1 + z_2|$.

By factoring we get $w = z^4 - 4z^2 + 3 = (z^2 - 1)(z^2 - 3)$.

Therefore $|w| = |(z^2 - 1)(z^2 - 3)| = |z^2 - 1||z^2 - 3| \geq (||z^2| - | - 1||)|z^2 - 3| = (||z|^2 - 1|)|z^2 - 3| = (|4 - 1|)|z^2 - 3| = 3|z^2 - 3| \geq 3||z^2| - | - 3|| = 3||z|^2 - 3| = 3|4 - 3| = 3$.

So we have that $|w| = |z^4 - 4z^2 + 3| \geq 3$ and hence $\frac{1}{|w|} = \frac{1}{|z^4-4z^2+3|} \leq \frac{1}{3}$.

Therefore since $|w^{-1}| = |\frac{1}{w}| = \frac{1}{|w|}$ we get:

$|\frac{1}{z^4-4z^2+3}| = \frac{1}{|z^4-4z^2+3|} \leq \frac{1}{3}$ $\square$

6.14

Let $S = \{z = x + iy : x^2 - y^2 = 1\}$. Recall that for $z_1, z_2, z_3, z_4 \in \mathbb{C}$ we know $(\frac{z_1}{z_2})(\frac{z_3}{z_4}) = \frac{z_1 z_3}{z_2 z_4}$.

Recall that for any $z \in \mathbb{C}$ we have $Re\ z = \frac{z+\overline{z}}{2}$ and $Im\ z = \frac{z-\overline{z}}{2i}$.

Therefore if $z \in S$ we have $x^2 - y^2 = (Re\ z)^2 - (Im\ z)^2 = 1$.

So for $z \in S$ we have $(Re\ z)^2 - (Im\ z)^2 = (\frac{z+\overline{z}}{2})^2 - (\frac{z-\overline{z}}{2i})^2 = (\frac{z+\overline{z}}{2})(\frac{z+\overline{z}}{2}) - (\frac{z-\overline{z}}{2i})(\frac{z-\overline{z}}{2i}) = \frac{z^2+2z\overline{z}+\overline{z}^2}{4} - \frac{z^2-2z\overline{z}+\overline{z}^2}{4i^2} = \frac{z^2+2z\overline{z}+\overline{z}^2}{4} + \frac{z^2-2z\overline{z}+\overline{z}^2}{4} = \frac{2(z^2+\overline{z}^2)}{4} = \frac{z^2+\overline{z}^2}{2} = 1$.

Therefore $z^2 + \overline{z}^2 = 2$ and without loss of generality we can say the same is true in reverse so $z \in S$ if and only if $z^2 + \overline{z}^2 = 2$. Hence $S = \{z \in \mathbb{C} : z^2 + \overline{z}^2 = 2\}$.

Since S is all the points lying on a hyperbola we have that $z^2 + \overline{z}^2 = 2$ defines the same hyperbola $\square$

9.4

Recall that for a complex number $z = re^{i\theta}$ the expression $|z-1| = |re^{i\theta}-1|$ represents the distance of z from 1.

We want to find a point $z = e^{i\theta}$ such that $|e^{i\theta}-1| = 2$ where $\theta \in [0, 2\pi)$.

Since $z = e^{i\theta}$ we get that $r = |z| = |e^{i\theta}| = |cos(\theta) + i\, sin(\theta)| = \sqrt{cos^2(\theta) + sin^2(\theta)} = 1$.

So we are looking for a complex number lying on the unit circle centered at 0 whose distance from 1 is 2.

We know that 1 is a point on the unit circle centered at 0 and that it makes an angle 0 with the real axis since $1 \in \mathbb{R}$.

Therefore the only point on the unit circle centered at 0 that can be a distance of 2 away from 1 is the point on the opposite side of the unit circle centered at 0. This is because the diameter of the unit circle centered at 0 is 2.

So the angle between these two points will be π because this will take us half way around the circle.

Since 1 makes an angle of 0 with the real axis we get that $arg\, z = \{\pi + 2n\pi : n \in \mathbb{Z}\}$.

The only one of these angles in the interval $[0, 2\pi)$ is when $n = 0$ and hence $\theta = \pi$ $\square$

You can see this algebraically too:

We want $|z-1| = 2$ where $z = e^{i\theta}$ and hence $|z| = 1$.

So $|z-1|^2 = 4$ and $(z-1)(\overline{z-1}) = (z-1)(\overline{z}-1) = z\overline{z} - z - \overline{z} + 1 = 4$.

Therefore $|z|^2 - 2Re\, z = 1 - 2Re\, z = 3$. Giving $Re\, z = -1$.

Then if $Re\, z = -1$ we must have $(Re\, z)^2 + (Im\, z)^2 = 1 + (Im\, z)^2 = 1$. Hence $Im\, z = 0$ and $z = -1$.

This means $z = -1 = cos(\theta) + i\, sin(\theta)$ gives $cos(\theta) = -1$ and $sin(\theta) = 0$ for $\theta \in [0, 2\pi)$.

So $\theta = \pi$ as desired.

9.8

As given in the problem $(e^{i\frac{\theta_1+\theta_2}{2}})(e^{i\frac{\theta_1-\theta_2}{2}}) = e^{i\theta_1}$ and $(e^{i\frac{\theta_1+\theta_2}{2}})(\overline{e^{i\frac{\theta_1-\theta_2}{2}}}) = e^{i\theta_2}$.

Let $z_1, z_2 \in \mathbb{C}$. Recall that for $w, w_1, w_2 \in \mathbb{C}$ we know $|w_1w_2| = |w_1||w_2|$ and $|w| = |\overline{w}|$.

- Assume $r_1 = |z_1| = |z_2| = r_2 = r$:

  Represent z_1 and z_2 as $re^{i\theta_1}$ and $re^{i\theta_2}$ respectively.

  Then $z_1 = re^{i\theta_1} = r(e^{i\frac{\theta_1+\theta_2}{2}})(e^{i\frac{\theta_1-\theta_2}{2}})$ and $z_2 = re^{i\theta_2} = r(e^{i\frac{\theta_1+\theta_2}{2}})(\overline{e^{i\frac{\theta_1-\theta_2}{2}}})$.

  Let $c_1 = re^{i\frac{\theta_1+\theta_2}{2}}$ and $c_2 = e^{i\frac{\theta_1-\theta_2}{2}}$. Then we have $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$.

  So there exists $c_1, c_2 \in \mathbb{C}$ such that $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$.

- Assume there exists $c_1, c_2 \in \mathbb{C}$ such that $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$:

  Then $|z_1| = |c_1c_2| = |c_1||c_2| = |c_1||\overline{c_2}| = |c_1\overline{c_2}| = |z_2|$.

This was for arbitrary $z_1, z_2 \in \mathbb{C}$ and is therefore true for all $z_1, z_2 \in \mathbb{C}$.

So for $z_1, z_2 \in$ we have that $|z_1| = |z_2|$ if and only if $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$ for some $c_1, c_2 \in \mathbb{C}$ $\square$

9.10

a. Recall that de Moivre's formula says $(cos(\theta) + i\ sin(\theta))^n = cos(n\theta) + i\ sin(n\theta)$.

Recall the binomial theorem where for $z_1, z_2 \in \mathbb{C}$ we know

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

Consider $(cos(\theta) + i\ sin(\theta))^3$. We can apply the binomial theorem with $z_1 = cos(\theta)$ and $z_2 = i\ sin(\theta)$.

Then we get $(cos(\theta) + i\ sin(\theta))^3 = \binom{3}{0}(i\ sin(\theta))^3 + \binom{3}{1}(cos(\theta))(i\ sin(\theta))^2 + \binom{3}{2}(cos(\theta))^2(i\ sin(\theta)) + \binom{3}{3}(cos(\theta))^3 =$
$(cos^3(\theta) - 3cos(\theta)sin^2(\theta)) + i(3\ cos^2(\theta)sin(\theta) - sin^3(\theta))$.

But we also know from de Moivre's formula that $(cos(\theta) + i\ sin(\theta))^3 = cos(3\theta) + i\ sin(3\theta)$.

Therefore $(cos^3(\theta) - 3cos(\theta)sin^2(\theta)) + i(3\ cos^2(\theta)sin(\theta) - sin^3(\theta)) = cos(3\theta) + i\ sin(3\theta)$.

Hence $Re\ ((cos^3(\theta) - 3cos(\theta)sin^2(\theta)) + i(3\ cos^2(\theta)sin(\theta) - sin^3(\theta))) = Re\ (cos(3\theta) + i\ sin(3\theta))$.

So $cos^3(\theta) - 3cos(\theta)sin^2(\theta) = cos(3\theta)$ as desired $\square$

11.6

Recall that the n distinct nth roots of $z \in \mathbb{C}$ are given by $c_k = \sqrt[n]{|z|}e^{i(\frac{Arg\ z}{n} + \frac{2k\pi}{n})} = \sqrt[n]{|z|}e^{i\frac{Arg\ z}{n}}e^{i\frac{2k\pi}{n}} = c_0 w_n^k$
where c_0 is the principle root, $w_n = e^{i\frac{2\pi}{n}}$, and $k \in \{0, 1, 2, ..., n-1\}$.

We are looking for the 4 distinct solutions to $z^4 + 4 = 0$ and hence the 4 distinct 4th roots of -4.

We are given $z_0 = \sqrt{2}e^{i\frac{\pi}{4}} = 1 + i$ is the principle 4th root of -4.

Now $w_4 = e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = cos(\frac{\pi}{2}) + i\ sin(\frac{\pi}{2}) = i$.

So we have our distinct roots are:

$$z_0 = 1 + i$$

$$z_1 = z_0 w_4 = (1 + i)i = i + i^2 = -1 + i$$

$$z_2 = z_0 w_4^2 = (1 + i)i^2 = -(1 + i) = -1 - i$$

$$z_3 = z_0 w_4^3 = (1 + i)i^3 = -i(1 + i) = -i - i^2 = 1 - i$$

First notice that $z_3 = \overline{z_0}$ and $z_2 = \overline{z_1}$.

We can deconstruct $z^4 + 4$ into its roots.

$$z^4 + 4 = (z - z_0)(z - z_1)(z - z_2)(z - z_3) = (z - z_0)(z - \overline{z_0})(z - z_1)(z - \overline{z_1}) =$$
$$(z^2 - z\overline{z_0} - zz_0 + z_0\overline{z_0})(z^2 - z\overline{z_1} - zz_1 + z_1\overline{z_1}) = (z^2 - z(z_0 + \overline{z_0}) + |z_0|^2)(z^2 - z(z_1 - \overline{z_1}) + |z_1|^2) =$$
$$(z^2 - (2Re\ z_0)z + |z_0|^2)(z^2 - (2Re\ z_1)z + |z_1|^2) = (z^2 - 2z + 2)(z^2 + 2z + 2)\ \square$$

11.7

Recall that from problem 9 of section 9 for $z \neq 1$ we have $1 + z + z^2 + ... + z^n = \frac{1-z^{n+1}}{1-z}$.

Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$ such that $c \neq 1$ and $c^n = 1$. Then c is a so-called nth root of unity.

Since $c \neq 1$ we have that $1 + c + c^2 + ... + c^{n-1} = \frac{1-c^n}{1-c} = \frac{1-1}{1-c} = \frac{0}{1-c} = 0$ as desired.

This was for arbitrary $n \in \mathbb{N}$ and for arbitrary $c \neq 1$ where $c^n = 1$, so $1 + c + c^2 + ... + c^{n-1} = 0$

for all nth roots of unity $c \neq 1$ $\square$

Geometrically this makes sense as well:

The n distinct nth roots of unity are spread evenly along the unit circle.

Once you fix one root $c \neq 1$ you can find the other $n - 1$ roots are $c^2, c^3, ..., c^n$ where $c^n = 1$.

Therefore by adding all of these terms you are adding all of the evenly spaced roots on the unit circle, which will necessarily add to 0 by the nature of vectors.

Actually, for any $z \in \mathbb{C}$ and any $n \in \mathbb{N}$ where $n \geq 2$ the sum of the n distinct nth roots of z will add to 0 because they are evenly spaced along the same circle. We simply get that when $z = 1$ it is a special case where all the roots can be represented as $c, c^2, c^3, ..., c^n$ where $c \neq 1$ is one of the nth roots of unity.

Proof that $1 + z + z^2 + ... + z^n = \frac{1-z^{n+1}}{1-z}$ for $z \neq 1$ (result from P9 of Sec. 9):

Recall that the familiar distributive laws apply just the same in the complex numbers as in the real numbers.

Let $z \neq 1$ then consider the product $(1 - z)(1 + z + z^2 + ... + z^n)$.

We can write $(1 - z)(1 + z + z^2 + ... + z^n) = (1 + z + z^2 + ... + z^n) - z(1 + z + z^2 + ... + z^n) =$

$(1 + z + z^2 + ... + z^n) - (z + zz + zz^2 + ... + zz^n) = (1 + z + z^2 + ... + z^n) - (z + z^2 + z^3 + ... + z^{n+1}) =$

$1 + z + z^2 + ... + z^n - z - z^2 - z^3 + ... - z^{n+1} = 1 + (z - z) + (z^2 - z^2) + ... + (z^n - z^n) - z^{n+1} = 1 - z^{n+1}$.

Then since $z \neq 1$ we have $z - 1 \neq 0$ and we can divide both sides by $z - 1$.

So $\frac{(1-z)(1+z+z^2+...+z^n)}{1-z} = 1 + z + z^2 + ... + z^n = \frac{1-z^{n+1}}{1-z}$ as desired.

11.8

a. Let $z \in \mathbb{C}$ then consider the equation $az^2 + bz + c = 0$ where $a, b, c \in \mathbb{C}$ and $a \neq 0$.

Then we get $az^2 + bz = -c$ and $z^2 + \frac{b}{a}z = -\frac{c}{a}$ (we can divide by a since $a \neq 0$).

Then $z^2 + 2\frac{b}{2a}z = -\frac{c}{a}$ and $z^2 + 2\frac{b}{2a}z + (\frac{b}{2a})^2 = (\frac{b}{2a})^2 - \frac{c}{a}$.

So $(z + \frac{b}{2a})^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \frac{b^2-4ac}{4a^2}$.

Therefore $z + \frac{b}{2a} = ((z + \frac{b}{2a})^2)^{\frac{1}{2}} = \pm(\frac{b^2-4ac}{4a^2})^{\frac{1}{2}} = \pm\frac{(b^2-4ac)^{\frac{1}{2}}}{(4a^2)^{\frac{1}{2}}} = \pm\frac{(b^2-4ac)^{\frac{1}{2}}}{2a}$.

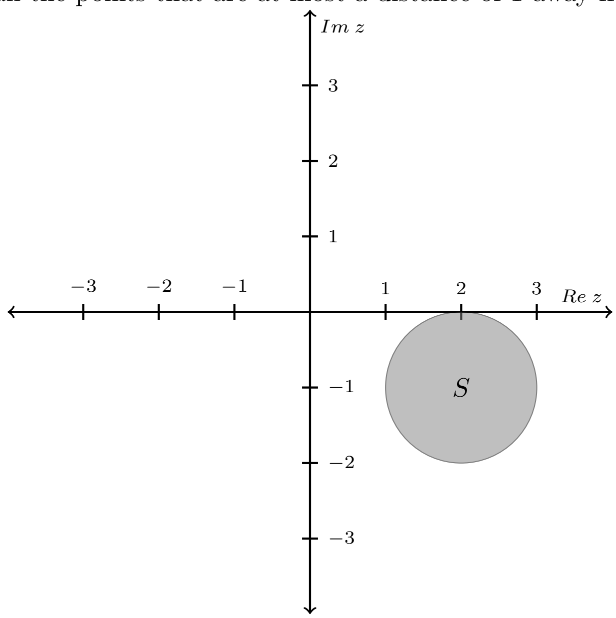
Since there are only 2 second roots for any complex number, the principal root and its rotation by $\frac{2\pi}{2} = \pi$ (its negative).

And finally $z = -\frac{b}{2a} \pm \frac{(b^2-4ac)^{\frac{1}{2}}}{2a} = \frac{-b\pm(b^2-4ac)^{\frac{1}{2}}}{2a}$ as desired $\square$

12.1

a. Let $S = \{z \in \mathbb{C} : |z - 2 + i| \leq 1\} = \{z \in \mathbb{C} : |z - (2 - i)| \leq 1\}$.

Then S consists of all the points that are at most a distance of 1 away from the point $2 - i$.



Clearly the boundary of S is given by $\{z \in \mathbb{C} : |z - (2 - i)| = 1\} \subseteq S$.

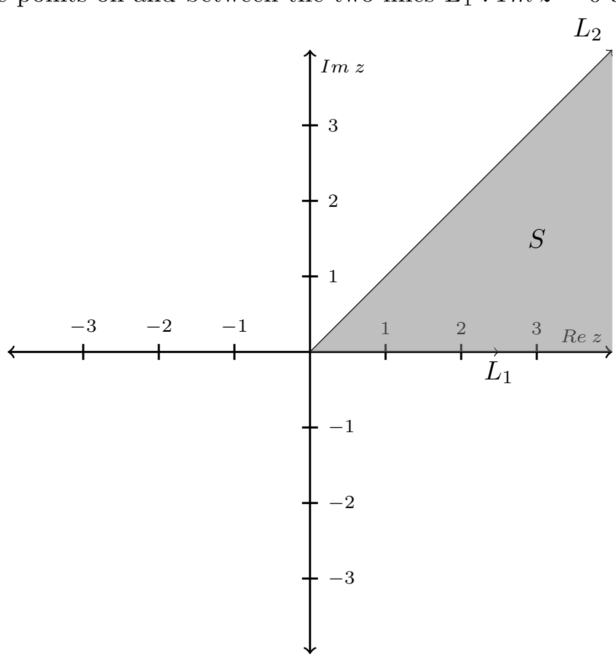
Therefore S contains all its boundary points and hence is not open. So S is not a domain.

e. Let $S = \{z \in \mathbb{C} : 0 \leq arg\ z \leq \frac{\pi}{4}\}$.

I am assuming this is referring to z's principle argument otherwise the inequalities don't make sense.

Then S consists of all the points whose principle angle from the real axis is equal to or between 0 and $\frac{\pi}{4}$.

So S consists of all the points on and between the two lines $L_1 : Im\ z = 0$ and $L_2 : Im\ z = Re\ z$.



Clearly the boundary of S is given by $\{z \in \mathbb{C} : arg\ z \in \{0, \frac{\pi}{4}\}\} \subseteq S$.

Therefore S contains all its boundary points and hence is not open. So S is not a domain.

f. Let $S = \{z \in \mathbb{C} : |z-4| \geq |z|\} = \{z \in \mathbb{C} : |z-4| \geq |z-0|\}$.

Then S consists of all the points whose distance from 4 is at least their distance from 0.

To put it in terms easier to visualize:

$z \in S$ if and only if the following holds.

$\sqrt{(z-4)(\overline{z-4})} \geq \sqrt{z\overline{z}}$. So $(z-4)(\overline{z}-4) \geq z\overline{z}$.

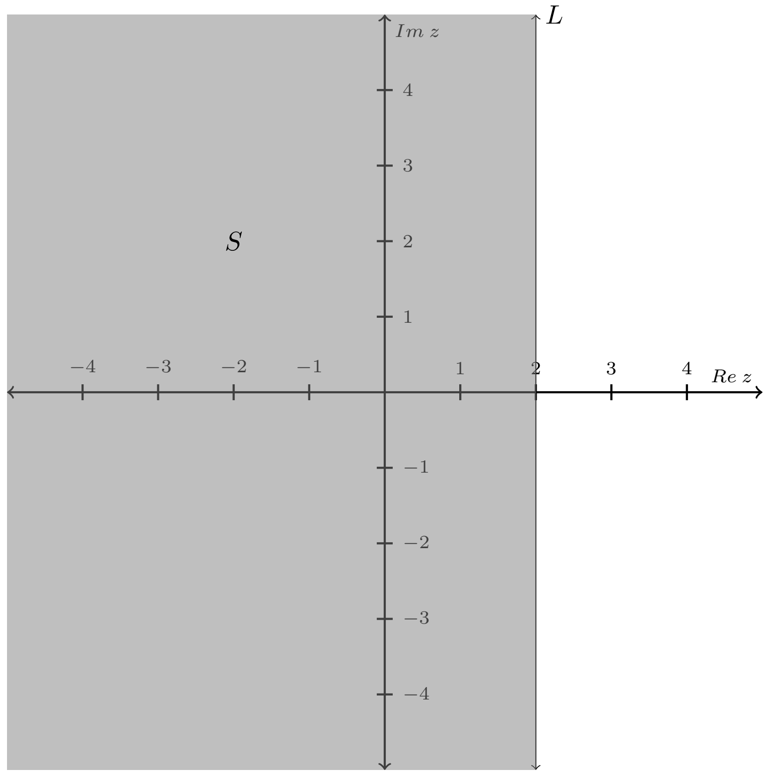
Therefore $z\overline{z} - 4z - 4\overline{z} + 16 \geq z\overline{z}$ and $16 \geq 4(z+\overline{z})$.

Finally we get $4 \geq 2Re\, z$ and $Re\, z \leq 2$.

So $S = \{z \in \mathbb{C} : Re\, z \leq 2\}$.

This makes sense geometrically as if $Re\, z > 2$ then z will be closer to 4 than to 0. This is due to both 0 and 4 being on the real line making $Im\, z$ irrelevant as it will have the same effect on the distance from both 0 and 4.

So S consists of all the points on and to the left of the line $L : Re\, z = 2$



Clearly the boundary of S is given by $\{z \in \mathbb{C} : Re\, z = 2\} \subseteq S$.

Therefore S contains all its boundary points and hence is not open. So S is not a domain.

12.6

Let $S \subseteq \mathbb{C}$ be an arbitrary set of complex numbers.

We know that z is a boundary point of S if z is not an interior or exterior point of S.

In order for z to be an interior point of S there must exist some $\epsilon > 0$ such that $V_\epsilon(z) \subseteq S$.

In order for z to be an exterior point of S there must exist some $\epsilon > 0$ such that $V_\epsilon(z) \subseteq S^c$.

Therefore if z is not a boundary point of S then there must exist some $\epsilon > 0$ where either $V_\epsilon(z) \subseteq S$ or $V_\epsilon(z) \subseteq S^c$.

- Assume that S is open:

Then S does not contain any of its boundary points.

Consider some arbitrary point $z \in S$. We know z can not be a boundary point of S.

So there exists an $\epsilon > 0$ where we have either $V_\epsilon(z) \subseteq S$ or $V_\epsilon(z) \subseteq S^c$.

Take any $\epsilon > 0$ then we know $z \in V_\epsilon(z)$, so $V_\epsilon(z) \nsubseteq S^c$ since $z \in S$.

So there does not exist an $\epsilon > 0$ where $V_\epsilon(z) \subseteq S^c$.

Therefore we must have that there exists an $\epsilon > 0$ where $V_\epsilon(z) \subseteq S$.

This means that z is an interior point of S.

This was true for arbitrary $z \in S$ and is therefore true for all $z \in S$.

So every point in S is an interior point.

- Assume that every point in S is an interior point:

Then if $z \in S$ we have that z can not be a boundary point because z is already an interior point.

Consider some arbitrary boundary point w of S. Then it must be $w \notin S$.

Otherwise w would be an interior point of S and hence not a boundary point of S.

This was true for an arbitrary boundary point w of S and is therefore true for every boundary point of S.

Therefore S does not contain any of its boundary points.

So S is open.

So we have that S is open if and only if every point in S is an interior point of S $\square$

14.2

Recall that for a complex number $z = x + iy$ we know $\overline{z} = \overline{x+iy} = x - iy$, also recall that $i^2 = -1$.

Further recall the binomial theorem where for $z_1, z_2 \in \mathbb{C}$ we know

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

a. Let $f(z) = z^3 + z + 1$.

Then if $z = x + iy$ we have:

$$f(z) = f(x+iy) = (x+iy)^3 + (x+iy) + 1 = \left(\sum_{k=0}^{3} \binom{3}{k} x^k (iy)^{3-k}\right) + (x+iy) + 1 =$$

$$(x^3 + 3ix^2y - 3xy^2 - iy^3) + (x+iy) + 1 = (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)$$

So $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where:

$$u(x,y) = x^3 - 3xy^2 + x + 1$$

and

$$v(x,y) = 3x^2y - y^3 + y$$

b. Let $f(z) = \frac{\overline{z}^2}{z} = \frac{\overline{z}^2\overline{z}}{z\overline{z}} = \frac{\overline{z}^3}{z\overline{z}}$.

Then if $z = x + iy$ we have:

$$f(z) = f(x+iy) = \frac{(\overline{x+iy})^3}{(x+iy)(\overline{x+iy})} = \frac{(x-iy)^3}{(x+iy)(x-iy)} = \frac{1}{x^2 - ixy + ixy - i^2y}(x-iy)^3 =$$

$$\frac{1}{x^2+y^2}\left(\sum_{k=0}^{3} \binom{3}{k} x^k (-iy)^{3-k}\right) = \frac{1}{x^2+y^2}(x^3 - 3ix^2y - 3xy^2 + iy^3) = \frac{x^3 - 3xy^2}{x^2+y^2} + i\frac{y^3 - 3x^2y}{x^2+y^2}$$

So $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where:

$$u(x,y) = \frac{x^3 - 3xy^2}{x^2+y^2}$$

and

$$v(x,y) = \frac{y^3 - 3x^2y}{x^2+y^2}$$

14.3

Recall that for a complex number $z = x + iy$ we know $x = Re\, z = \frac{z+\overline{z}}{2}$ and $y = Im\, z = \frac{z-\overline{z}}{2i}$.

Now let $f(z) = f(x+iy) = (x^2 - y^2 - 2y) + i(2x - 2xy)$.

We can substitute with the equations given before to get:

$$f(z) = \Big((\frac{z+\overline{z}}{2})^2 - (\frac{z-\overline{z}}{2i})^2 - 2\frac{z-\overline{z}}{2i} \Big) + i\Big(2\frac{z+\overline{z}}{2} - 2(\frac{z+\overline{z}}{2})(\frac{z-\overline{z}}{2i}) \Big) =$$

$$\Big(\frac{(z+\overline{z})^2}{2^2} - \frac{(z-\overline{z})^2}{(2i)^2} - \frac{i(z-\overline{z})}{i^2} \Big) + i\Big(z + \overline{z} - 2\frac{i(z+\overline{z})(z-\overline{z})}{4i^2} \Big) =$$

$$\frac{z^2 + 2z\overline{z} + \overline{z}^2}{4} + \frac{z^2 - 2z\overline{z} + \overline{z}^2}{4} + i(z - \overline{z}) + iz + i\overline{z} + \frac{i^2(z+\overline{z})(z-\overline{z})}{2} =$$

$$\frac{z^2 + \overline{z}^2}{2} + 2iz - \frac{z^2 - z\overline{z} + z\overline{z} - \overline{z}^2}{2} = \frac{z^2 + \overline{z}^2}{2} + 2iz - \frac{z^2 - \overline{z}^2}{2} =$$

$$\overline{z}^2 + 2iz \ \square$$

14.6

Let $f(z) = z^2$. Then for $z = x + iy$ we have $f(x+iy) = (x+iy)^2 = x^2 + 2ixy + (iy)^2 = (x^2 - y^2) + i(2xy)$.

So $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ where $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$.

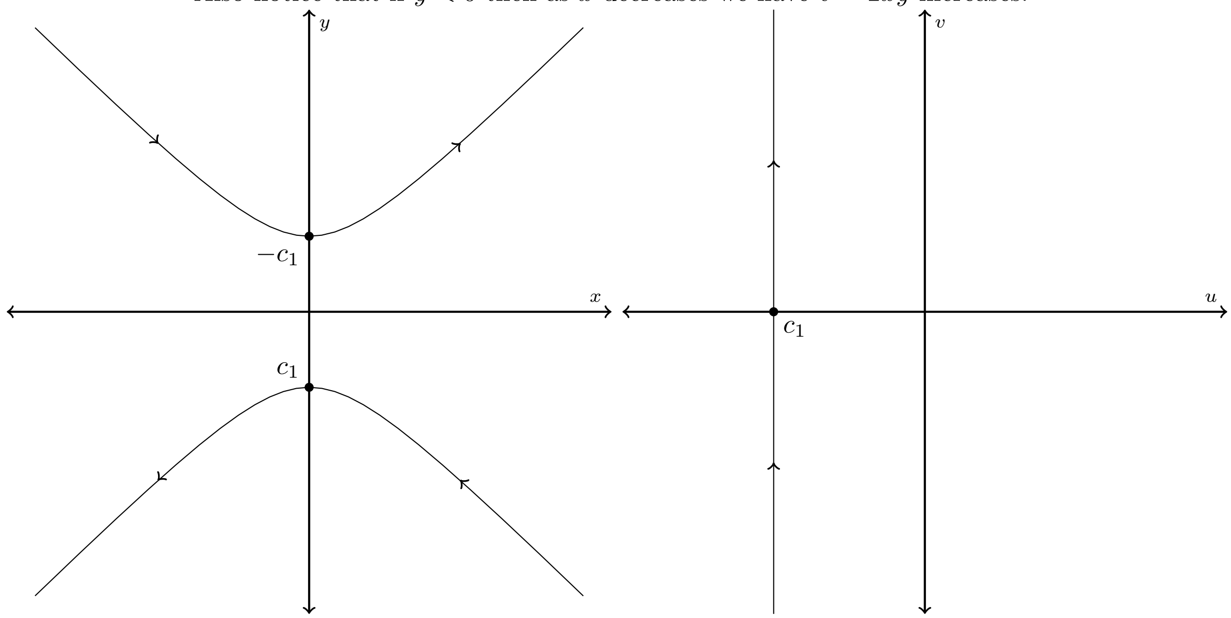
Therefore if $z \in \mathbb{C}$ where $z = x + iy$ such that $x^2 - y^2 = c_1$ for some $c_1 < 0$ then $u(x,y) = Re\, f(z) = c_1$.

Furthermore if $z \in \mathbb{C}$ where $z = x + iy$ such that $2xy = c_2$ for some $c_2 < 0$ then $v(x,y) = Im\, f(z) = c_2$.

Let us first look at the case where $z = x + iy$ and $x^2 - y^2 = c_1$:

Notice that if $y > 0$ then as x increases we have $v = 2xy$ increases.

Also notice that if $y < 0$ then as x decreases we have $v = 2xy$ increases.



Now let us look at the case where $z = x + iy$ and $2xy = c_2$:

Notice that as $|x|$ increases it must be that $|y|$ decreases in order for $2xy = c_2$ to stay constant.

This means that when $|x|$ increases (causing x^2 to increase and y^2 to decrease) we have $u = x^2 - y^2$ increases.