Continuity and Cauchy Sequences

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4.3.9

Let $h : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$.

If K has no limit points then it is trivially closed.

Otherwise let x be a limit point of K. Then there exists some sequence (x_n) contained in K such that $(x_n) \to x$.

Since $x_n \in K$ for all $n \in \mathbb{N}$ it must be that $h(x_n) = 0$ for all $n \in \mathbb{N}$.

So $(h(x_n)) = (0)$ which clearly converges to 0.

Since h is continuous on $\mathbb{R}$ it must be that $lim\ h(x_n) = h(x)$. So we have that $h(x) = 0$ and therefore $x \in K$.

This was for an arbitrary limit point of K and is therefore true for all limit points of K.

So K contains all its limit points and is therefore closed $\square$

4.3.11

Let $f : \mathbb{R} \to \mathbb{R}$ such that there exists a $c \in (0, 1)$ where $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in \mathbb{R}$.

a. Let $a \in \mathbb{R}$, let $\epsilon > 0$, and let $\delta = \epsilon/c$. Then $\delta > 0$ since $c > 0$.

Then if $|x - a| < \delta$ we have that $|f(x) - f(a)| \leq c|x - a| < c\delta = \epsilon$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So for all $\epsilon > 0$ we have found a $\delta > 0$ such that when $|x - a| < \delta$ it follows that $|f(x) - f(a)| < \epsilon$.

So f is continuous at a, and this was for arbitrary $a \in \mathbb{R}$ and is therefore true for all $a \in \mathbb{R}$.

Therefore f is continuous on $\mathbb{R}$ $\square$

b. Let $y_1 \in \mathbb{R}$ and define (y_n) by $y_{n+1} = f(y_n)$.

If (y_n) is a constant sequence it is trivially convergent and therefore Cauchy.

Otherwise since $y_{n+1} = f(y_n) \in \mathbb{R}$ we have that $|y_{n+2} - y_{n+1}| = |f(y_{n+1}) - f(y_n)| \leq c|y_{n+1} - y_n|$ for all $n \in \mathbb{N}$.

Furthermore $|y_{n+2} - y_{n+1}| \leq c|y_{n+1} - y_n| = c|f(y_n) - f(y_{n-1})| \leq c^2|y_n - y_{n-1}| = ... \leq c^n|y_2 - y_1|$.

Let $m, n \in \mathbb{N}$ where $m > n$.

Then $|y_m - y_n| = |y_m - y_{m-1} + y_{m-1} - ... - y_{n+1} + y_{n+1} - y_n| \leq |y_m - y_{m-1}| + |y_{m-1} - y_{m-2}| + ... + |y_{n+1} - y_n|$.

So $|y_m - y_n| \leq |y_m - y_{m-1}| + |y_{m-1} - y_{m-2}| + ... + |y_{n+1} - y_n| \leq c^{m-2}|y_2 - y_1| + ... + c^{n-1}|y_2 - y_1|$.

Now we have $|y_m - y_n| \leq c^{m-2}|y_2 - y_1| + ... + c^{n-1}|y_2 - y_1| = c^{n-1}|y_2 - y_1|(1 + c + c^2 + ... + c^{m-n-1})$.

Since $c > 0$ we have $|y_m - y_n| \leq c^{n-1}|y_2 - y_1|(1 + c + c^2 + ... + c^{m-n-1}) < c^{n-1}|y_2 - y_1|(1 + c + c^2 + ...) = \frac{c^{n-1}|y_2 - y_1|}{1-c}$.

Let $\epsilon > 0$ then let $N > log(\frac{\epsilon(1-c)}{|y_2-y_1|})/log\ c + 1$. Such an N exists because $c, y_2, y_1,\ and\ \epsilon$ are all fixed and $c \in (0, 1)$ and $y_1 \neq y_2$ because our assumption was that (y_n) was not constant.

Then $(N - 1)\ log\ c < log(\frac{\epsilon(1-c)}{|y_2-y_1|})$ and $c^{N-1} < \frac{\epsilon(1-c)}{|y_2-y_1|}$ and $\frac{|y_2-y_1|c^{N-1}}{1-c} < \epsilon$.

So for $m, n \geq N$ we have $|y_m - y_n| < \frac{c^{n-1}|y_2-y_1|}{1-c} \leq \frac{c^{N-1}|y_2-y_1|}{1-c} < \epsilon$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $m, n \geq N$ it follows that $|y_m - y_n| < \epsilon$.

Therefore (y_n) is Cauchy and consequently is convergent. So we can say $lim\ y_n = y$ for some $y \in \mathbb{R}$ $\square$

c. Say $lim\ y_n = y$, then since f is continuous as per part a we know $f(y) = lim\ f(y_n) = lim\ y_{n+1} = lim\ y_n = y$.

So y is a fixed point of f.

Let x be a fixed point of f. Then $|x - y| = |f(x) - f(y)|$ and simultaneously $|f(x) - f(y)| \leq c|x - y|$.

It can not be that $|f(x) - f(y)| < c|x - y|$ since $0 < c < 1$ and $|f(x) - f(y)| = |x - y|$.

So we have that $|f(x) - f(y)| = |x - y|$ and $|f(x) - f(y)| = c|x - y|$.

Consequently, $c|x - y| = |x - y|$ and $(1 - c)|x - y| = 0$. So $|x - y| = 0$ and therefore $x = y$ so y is the only fixed point $\square$

d. Let $x \in \mathbb{R}$ then consider the sequence $(x, f(x), f(f(x)), ...)$.

Without loss of generality we can say that this sequence converges and it converges to a fixed point.

Since y is the only fixed point it must be that $(x, f(x), f(f(x)), ...) \to y$.

This was for arbitrary $x \in \mathbb{R}$ and is therefore true for all $x \in \mathbb{R}$.

So the sequence $(x, f(x), f(f(x)), ...) \to y$ for all $x \in \mathbb{R}$ $\square$

4.4.6

a. Let $f(x) = \frac{1}{x}$ for $x \in (0,1)$, and let $(x_n) = (\frac{1}{n})$.

The functions $g(x) = x$ and $h(x) = 1$ are continuous on $\mathbb{R}$ and therefore also on $(0,1)$ as shown below.

Let $c \in \mathbb{R}$ and let $\epsilon > 0$, then let $\delta = \epsilon$.

Then if $|x - c| < \delta$ we have $|g(x) - g(c)| = |x - c| < \delta = \epsilon$ and $|h(x) - h(c)| = |1 - 1| = 0 < \epsilon$.

This was for arbitrary $c \in \mathbb{R}$ and is consequently true for all $c \in \mathbb{R}$ so g and h are continuous on $\mathbb{R}$.

So by the algebraic continuity theorem $f(x) = \frac{1}{x} = \frac{h(x)}{g(x)}$ is continuous on $(0,1)$ since the quotient is defined.

Then as seen many times previously $(x_n) = (\frac{1}{n}) \to 0$ so (x_n) is Cauchy.

But $(f(x_n)) = (\frac{1}{1/n}) = (n)$ is unbounded and therefore can not be Cauchy.

This is such an example of a Cauchy sequence (x_n) and a continuous function $f : (0,1) \to \mathbb{R}$ where $f(x_n)$ is not Cauchy.

b. This is not possible. Let $f : (0,1) \to \mathbb{R}$ be uniformly continuous and let $(x_n) \subset (0,1)$ be Cauchy.

Then for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Since (x_n) is Cauchy for any $\alpha > 0$ there exists an $N \in \mathbb{N}$ such that if $m, n \geq N$ then $|x_m - x_n| < \alpha$.

So let $\epsilon > 0$, then there exists a $\delta > 0$ so that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

Furthermore there exists an $N \in \mathbb{N}$ such that if $m, n \geq N$ then $|x_m - x_n| < \delta$.

So for all $m, n \geq N$ we have $|f(x_m) - f(x_n)| < \epsilon$.

Therefore for all $\epsilon > 0$ we have found an $N \in \mathbb{N}$ such that if $m, n \geq N$ then $|f(x_m) - f(x_n)| < \epsilon$.

So $f(x_n)$ is Cauchy $\square$

c. This is not possible. Let $f : [0, \infty) \to \mathbb{R}$ be continuous and let $(x_n) \subset [0, \infty)$ be Cauchy.

Then $(x_n) \to x$ for some $x \in \mathbb{R}$ and x is therefore a limit point of $[0, \infty)$.

Since $[0, \infty)$ is closed we have that $x \in [0, \infty)$.

So $f(x)$ is defined and since f is continuous we have $(f(x_n)) \to f(x)$.

Therefore $(f(x_n))$ is Cauchy $\square$

4.4.9

a. Assume $f : A \to \mathbb{R}$ is Lipschitz, then there exists an $M > 0$ such that $|\frac{f(x)-f(y)}{x-y}| \leq M$ for all $x, y \in A$ where $x \neq y$.

Let $\epsilon > 0$ then let $\delta = \epsilon/M$. Let $x, y \in A$.

If $x = y$ then $|x - y| = 0 < \delta$ and $|f(x) - f(y)| = 0 < \epsilon$.

Otherwise if $|x - y| < \delta = \epsilon/M$ we have $|f(x) - f(y)| \leq M|x - y| < M\delta = \epsilon$.

This was for arbitrary $x, y \in A$ and is therefore true for all $x, y \in A$.

So for all $x, y \in A$ and for all $\epsilon > 0$ we have found a $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

So f is uniformly continuous $\square$

b. No, the converse is not true. Let $f(x) = \sqrt{x}$ which is uniformly continuous on $[0, \infty)$ as seen in class.

However, f is not Lipschitz:

Let $y = 0$ and $x \neq 0$, then $|\frac{f(x)-f(y)}{x-y}| = |\frac{\sqrt{x}-0}{x-0}| = |\frac{\sqrt{x}}{x}| = \frac{1}{\sqrt{x}}$.

Also $lim_{x\to 0^+}|\frac{f(x)-f(y)}{x-y}| = lim_{x\to 0^+}\frac{1}{\sqrt{x}} = \infty$.

Let $M > 0$ then let $\delta = \frac{1}{M^2}$. Then if $0 < x < \delta = \frac{1}{M^2}$ we have $\sqrt{x} < \sqrt{\frac{1}{M^2}} = \frac{1}{M}$ and $\frac{1}{\sqrt{x}} > M$.

This was for arbitrary $M > 0$ and is therefore true for all $M > 0$ so $lim_{x\to 0^+}\frac{1}{\sqrt{x}} = \infty$.

So $|\frac{f(x)-f(y)}{x-y}|$ is unbounded for $y = 0$ and $x \to 0$.

Therefore $f(x) = \sqrt{x}$ is not Lipschitz but is uniformly continuous.

4.5.2

a. Let $f(x) = |x|$ if $x \in [-1, 1]$ and $f(x) = 1$ if $x \in (-2, -1) \cup (1, 2)$.

Since $|x|$ is continuous on $\mathbb{R}$ we have f is continuous on $(-1, 1)$.

Since constant functions are continuous on $\mathbb{R}$ we have f is continuous on $(-2, -1) \cup (1, 2)$.

Since $lim_{x \to -1^+} f(x) = lim_{x \to -1^+} |x| = 1 = lim_{x \to -1^-} 1 = lim_{x \to -1^-} f(x) = f(-1)$ we have that f is continuous at -1.

Since $lim_{x \to 1^-} f(x) = lim_{x \to 1^-} |x| = 1 = lim_{x \to 1^+} 1 = lim_{x \to 1^+} f(x) = f(1)$ we have that f is continuous at 1.

Let $I = (-2, 2)$ then $f : I \to \mathbb{R}$ is continuous, and clearly $f(I) = [0, 1]$.

Clearly $(-2, 2)$ is an open interval and $[0, 1]$ is a closed interval.

So this is such an example of a continuous function whose domain is an open interval and whose range is a closed interval.

b. This is not possible. Let $f : [a, b] \to \mathbb{R}$ be continuous.

Then since $[a, b]$ is closed and bounded it is compact.

So since f is continuous the range of f is compact and is therefore closed and bounded.

Therefore the range of f can not be open because the only sets that are open and closed are ϕ and $\mathbb{R}$.

But the range of f is not empty since its domain is nonempty and the range of f is not $\mathbb{R}$ since it is bounded.

Therefore since the range of f is already closed and it is neither ϕ or $\mathbb{R}$, the range of f is not open $\square$

c. Let $f(x) = \frac{1}{x}$ for $x \in (0, 1]$ and $f(x) = 1$ for $x \in (1, 2)$.

Since $\frac{1}{x}$ is continuous on $(0, \infty)$ we have f is continuous on $(0, 1)$.

Since constant functions are continuous on $\mathbb{R}$ we have f is continuous on $(1, 2)$.

Since $lim_{x \to 1^-} f(x) = lim_{x \to 1^-} \frac{1}{x} = 1 = lim_{x \to 1^+} 1 = lim_{x \to 1^+} f(x) = f(1)$ we have that f is continuous at 1.

Let $I = (0, 2)$ then $f : I \to \mathbb{R}$ is continuous and clearly $f(I) = [1, \infty)$.

Clearly $(0, 2)$ is an open interval and $[1, \infty)$ is an unbounded, closed set.

So this is such an example of a continuous function whose domain is an open interval and whose range is an unbounded, closed set that is not $\mathbb{R}$.

d. This is not possible. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Let $\mathbb{I}$ denote the set of irrational numbers.

If f is constant we are done since then $f(\mathbb{R}) = \{c\} \neq \mathbb{Q}$.

Otherwise consider f over an interval $[a, b]$ where $f(a) \neq f(b)$, this can be done because f is non-constant here.

Then there must exist some $y \in \mathbb{I}$ such that $f(a) < y < f(b)$ since $\mathbb{I}$ is dense in $\mathbb{R}$.

Since f is continuous on $[a, b]$ by the intermediate value theorem there exists some $x \in (a, b)$ such that $f(x) = y \notin \mathbb{Q}$.

Therefore $f(\mathbb{R}) \neq \mathbb{Q}$ $\square$

4.5.8

Let $f : [a,b] \to \mathbb{R}$ be continuous and one to one. Then $f^{-1} : f([a,b]) \to [a,b]$ exists.

Assume $f^{-1} : f([a,b]) \to [a,b]$ is not continuous for the sake of contradiction.

Then there exists some $(x_n) \subset f([a,b])$ such that $(x_n) \to x$ but $(f^{-1}(x_n)) \not\to f^{-1}(x)$.

Note that $[a,b]$ is compact and since f is continuous $f([a,b])$ is compact and is consequently closed so $x \in f([a,b])$.

Therefore $f(y) = x$ for some $y \in [a,b]$. So our above statement amounts to $(x_n) \to x$ but $(f^{-1}(x_n)) \not\to y$.

Let $(y_n) = (f^{-1}(x_n))$. Then $(y_n) \not\to y$ by assumption.

However since $f^{-1} : f([a,b]) \to [a,b]$ we have that $(y_n) \subset [a,b]$ and therefore (y_n) is bounded.

Therefore there exists a subsequence (y_{n_k}) that converges to some $y_0 \neq y$.

Since f is continuous it must be that $(f(y_{n_k})) \to f(y_0)$ but $(f(y_{n_k})) = (f(f^{-1}(x_{n_k}))) = (x_{n_k}) \to x$

This implies $f(y_0) = x$ and $f(y) = x$ but $y_0 \neq y$, a contradiction since f is one to one.

Therefore it must be that f^{-1} is continuous $\square$