Differentiation

Matthew Seguin

5.2.2

c is impossible because this would imply $(f+g)-g=f$ is differentiable at 0 by the algebraic differentiability theorem.

a. Let $f(x)=0$ and $g(x)=1$ if $x\in(-\infty,0)$ and let $f(x)=1$ and $g(x)=0$ if $x\in[0,\infty)$.

Let $\epsilon<1$ then $|f(x)-f(0)|=|0-1|=1>\epsilon$ for all $x<0$ so there does not exist a $\delta>0$ such that $|x-0|<\delta$ implies $|f(x)-f(0)|<1=\epsilon$ so f is not continuous at 0 and without loss of generality the same can be said about g.

Since f and g are both discontinuous at $x=0$ they can not be differentiable at $x=0$.

Furthermore $(fg)(x)=f(x)g(x)=0$ for all $x\in\mathbb{R}$. Which is differentiable on $\mathbb{R}$ by the following.

Let $c\in\mathbb{R}$ then consider $lim_{x\to c}\frac{(fg)(x)-(fg)(c)}{x-c}=lim_{x\to c}\frac{0-0}{x-c}=lim_{x\to c}0=0$.

So $(fg)'(c)=0$ for arbitrary $c\in\mathbb{R}$ so $(fg)'=0$ and hence fg is differentiable on all of $\mathbb{R}$.

So this is an example of functions f and g that both aren't differentiable at 0 but fg is differentiable at 0.

b. Let $f(x)=0$ if $x\in(-\infty,0)$ and $f(x)=1$ if $x\in[0,\infty)$. Let $g(x)=0$.

From the results of part a we know that f is not continuous at 0 and therefore can not be differentiable at 0.

Furthermore g is differentiable on $\mathbb{R}$ and $(fg)(x)=f(x)g(x)=0=g(x)$ for all $x\in\mathbb{R}$. So fg is differentiable on $\mathbb{R}$.

So this is an example of functions f and g where f isn't differentiable at 0 but g is and where fg is differentiable at 0.

c. I showed at the very start that this request is impossible.

d. Let $\mathbb{I}$ denote the irrationals. Then let $f(x)=0$ if $x\in\mathbb{Q}$ and let $f(x)=x$ if $x\in\mathbb{I}$.

Let $c\neq 0$, then since $\mathbb{I}$ is dense in $\mathbb{R}$ there exists some $(x_n)\subset\mathbb{I}$ such that $(x_n)\to c$.

Since $\mathbb{Q}$ is also dense in $\mathbb{R}$ there exists some $(y_n)\subset\mathbb{Q}$ such that $(y_n)\to c$.

But $(f(x_n))=(x_n)\to c\neq 0$ while $(f(y_n))=(0)\to 0$. So $lim_{x\to c}f(x)$ does not exist for $c\neq 0$.

Now at $c=0$: Let $\epsilon>0$ then let $\delta=\epsilon$.

Then if $|x-0|=|x|<\delta$ we have $|f(x)-f(0)|=|f(x)-0|=|f(x)|\leq|x|<\delta=\epsilon$ since $f(x)=0$ or $f(x)=x$.

This was for arbitrary $\epsilon>0$ and is therefore true for all $\epsilon>0$. So f is continuous at 0 and $lim_{x\to 0}f(x)=0$.

Now define the function $g(x)=xf(x)$.

We have $g'(0)=lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=lim_{x\to 0}\frac{xf(x)}{x}=lim_{x\to 0}f(x)=0$.

However at $c\neq 0$ it must be that $g'(c)$ does not exist because otherwise $lim_{x\to c}\frac{g(x)-g(c)}{x-c}$ exists.

But this would imply $lim_{x\to c}\frac{g(x)-g(c)}{x-c}(x-c)=lim_{x\to c}g(x)-g(c)$ exists by the algebraic limit theorem.

Which implies $lim_{x\to c}g(x)$ exists and hence $lim_{x\to c}\frac{g(x)}{x}=lim_{x\to c}\frac{xf(x)}{x}=lim_{x\to c}f(x)$ exists which it doesn't since $c\neq 0$.

So this is such a function g where g is differentiable only at 0.

5.2.9

a. This is true. Let f be differentiable on an interval A such that f' is non-constant. Let $\mathbb{I}$ denote the irrationals.

Then there exists some closed interval $[a, b] \subseteq A$ where f is differentiable and where $f'(a) \neq f'(b)$.

Since $\mathbb{I}$ is dense in $\mathbb{R}$ there exists some $x \in \mathbb{I}$ such that $f'(a) < x < f'(b)$ or $f'(b) < x < f'(a)$.

Therefore by Darboux's theorem there exists a $c \in [a, b] \subseteq A$ such that $f'(c) = x \in \mathbb{I}$.

So if f' exists on an interval and if f' is non-constant then f' takes on some irrational values in that interval $\square$

b. This is false. To find a counter example consider $sin(\frac{1}{x})$ which takes negative values in every $V_\delta(0)$.

First I am going to prove a result about $g(x) = x\,cos(\frac{1}{x})$, namely $lim_{x\to 0} g(x) = 0$:

Let $\epsilon > 0$ and let $\delta = \epsilon$.

Then if $|x - 0| = |x| < \delta = \epsilon$ we have $|g(x) - 0| = |x\,cos(\frac{1}{x})| = |x||cos(\frac{1}{x})| \leq |x| < \delta = \epsilon$ since $|cos(z)| \leq 1$ for all $z \in \mathbb{R}$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$ so $lim_{x\to 0} x\,cos(\frac{1}{x}) = 0$.

Now let $f(0) = 0$ and $f(x) = cx + x^2 cos(\frac{1}{x})$ if $x \in (-1, 1)\backslash\{0\}$ and $c \in (0, 1)$.

Then when $x \neq 0$ we have $f'(x) = c + 2x\,cos(\frac{1}{x}) - sin(\frac{1}{x})$ by using properties from the algebraic differentiability theorem.

As $x \to 0$ we know by the algebraic limit theorem that $2x\,cos(\frac{1}{x}) \to 0$.

So as we get arbitrarily close to 0, $sin(\frac{1}{x})$ oscillates between -1 and 1 in every $V_\delta(0)$ and since $c < 1$ and $2x\,cos(\frac{1}{x})$ grows arbitrarily close to 0 we get that f' takes negative values in every $V_\delta(0)$.

Now let's see what happens at 0:

$$f'(0) = lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = lim_{x\to 0}\frac{cx+x^2cos(\frac{1}{x})}{x} = lim_{x\to 0} c + x\,cos(\frac{1}{x}) = c > 0$$ by the algebraic limit theorem.

So this is a function on an open interval A where $f'(a) > 0$ for some $a \in A$ but f' takes negative values in every $V_\delta(a)$.

c. This is true. Let f be differentiable on an interval containing 0 and let $lim_{x\to 0} f'(x) = L$.

Then $f'(0) = lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = lim_{x\to 0}\frac{f(x)-f(0)}{x}$ gives the 0/0 case for L'hospital's rule.

So $f'(0) = lim_{x\to 0}\frac{f(x)-f(0)}{x} = lim_{x\to 0}\frac{f'(x)-0}{1} = lim_{x\to 0} f'(x) = L$ from taking the derivative of the top and bottom and using the algebraic differentiability theorem.

So if f is differentiable on an interval containing 0 and $lim_{x\to 0} f'(x) = L$ then $f'(0) = L$ $\square$

5.3.1

a. Let $f : [a, b] \to \mathbb{R}$ be differentiable on $[a, b]$ such that f' is continuous on $[a, b]$.

Since $[a, b]$ is closed and bounded it is compact. So since f' is continuous on $[a, b]$ we know $f'([a, b])$ is compact.

So f' is bounded and there exists some $M > 0$ such that $|f'(z)| \leq M$ for all $z \in [a, b]$.

Now let $x, y \in [a, b]$ where $x \neq y$ then $x < y$ or $y < x$.

If $x < y$:

Since $[x, y] \subseteq [a, b]$ we know f is differentiable on $[x, y]$.

So by the mean value theorem there exists some $w \in [x, y]$ such that $f'(w) = \frac{f(y)-f(x)}{y-x} = \frac{f(x)-f(y)}{x-y}$.

Therefore $|\frac{f(x)-f(y)}{x-y}| = |f'(w)| \leq M$.

If $y < x$: The same process with x and y swapped leads to the same conclusion without loss of generality.

This was for arbitrary distinct $x, y \in [a, b]$ and is therefore true for all distinct $x, y \in [a, b]$.

So there exists some $M > 0$ such that $|\frac{f(x)-f(y)}{x-y}| \leq M$ for all distinct $x, y \in [a, b]$. So f is Lipschitz $\square$

b. Yes this means that f is contractive. Add the assumption that $|f'(z)| < 1$ to part a.

If $|f'(z)| \leq 0$ then $|f'(z)| = 0$ so $f'(z) = 0$ for all $z \in [a, b]$.

This means that $f(z) = k$ for some $k \in \mathbb{R}$.

So $|f(x) - f(y)| = |k - k| = 0 \leq c|x - y|$ for all $c \in (0, 1)$ and all $x, y \in [a, b]$, hence f is contractive.

Otherwise we can say that $|f'(z)| \leq c$ for some $c \in (0, 1)$.

So for all distinct $x, y \in [a, b]$ we have there exists some $w \in [x, y]$ such that $f'(w) = \frac{f(x)-f(y)}{x-y}$ by the mean value theorem.

Therefore for all distinct $x, y \in [a, b]$ there exists a $w \in [x, y]$ such that $\frac{|f(x)-f(y)|}{|x-y|} = |\frac{f(x)-f(y)}{x-y}| = f'(w) \leq c$.

This gives us $|f(x) - f(y)| \leq c|x - y|$ for all distinct $x, y \in [a, b]$.

Now consider $x = y$, then $|f(x) - f(y)| = |f(x) - f(x)| = 0 = c|x - x| = c|x - y|$.

Therefore there exists a $c \in (0, 1)$ such that for all $x, y \in [a, b]$ we have $|f(x) - f(y)| \leq c|x - y|$, hence f is contractive.

So if we add the assumption that $|f'(z)| < 1$ then f is contractive $\square$

5.3.5

a. Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).

Let $h(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x)$.

Then h is continuous on $[a,b]$ by the algebraic continuity theorem and differentiable on (a,b) by the algebraic differentiability theorem.

Furthermore $h'(x) = [g(b) - g(a)]f'(x) - [f(b) - f(a)]g'(x)$ by the algebraic differentiability theorem.

We can see $h(a) = [g(b) - g(a)]f(a) - [f(b) - f(a)]g(a) = g(b)f(a) - g(a)f(a) - f(b)g(a) + g(a)f(a) = g(b)f(a) - f(b)g(a)$.

Also $h(b) = [g(b) - g(a)]f(b) - [f(b) - f(a)]g(b) = g(b)f(b) - g(a)f(b) - g(b)f(b) + f(a)g(b) = g(b)f(a) - f(b)g(a)$.

So $h(a) = g(b)f(a) - f(b)g(a) = h(b)$.

Then by Rolle's theorem there exists some point $c \in (a,b)$ where $h'(c) = 0$.

Therefore there exist some point $c \in (a,b)$ where $h'(c) = [g(b) - g(a)]f'(c) - [f(b) - f(a)]g'(c) = 0$.

So there exists some point $c \in (a,b)$ where $[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$ $\square$

5.3.8

Let f be continuous on an interval containing 0 and let f be differentiable for all $x \neq 0$. Assume $lim_{x\to 0} f'(x) = L$.

I believe I proved this in a previous problem but here it is again.

Let $g(x) = f(x) - f(0)$ then by the algebraic differentiability theorem g is differentiable for all $x \neq 0$.

We also know $g(x)$ is continuous on the domain of f by the algebraic differentiability theorem.

Let $h(x) = x - 0$ we have seen before that h is differentiable for all $x \in \mathbb{R}$ and is therefore also continuous for all $x \in \mathbb{R}$.

Furthermore $g(0) = f(0) - f(0) = 0 = 0 - 0 = h(0)$.

Then $f'(0) = lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = lim_{x\to 0} \frac{g(x)}{h(x)}$ satisfies the 0/0 case for L'hospital's rule.

So $f'(0) = lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = lim_{x\to 0} \frac{(f(x)-f(0))'}{(x-0)'} = lim_{x\to 0} \frac{f'(x)}{1} = lim_{x\to 0} f'(x) = L$.

From using the algebraic differentiability theorem.

So if f is continuous on an interval containing 0, differentiable for all $x \neq 0$, and $lim_{x\to 0} f'(x) = L$ then $f'(0) = L$ $\square$

5.2.7

Let $g_a(x) = 0$ if $x = 0$ and $g_a(x) = x^a sin(\frac{1}{x})$.

First I want to prove that $lim_{x\to 0} x^c sin(\frac{1}{x}) = lim_{x\to 0} x^c cos(\frac{1}{x}) = 0$ for $c > 0$ as long as x^c is defined for all $x \in \mathbb{R}$.

Let $\epsilon > 0$ and $\delta = \epsilon^{\frac{1}{c}}$ then if $|x - 0| = |x| < \delta$.

We have $|x^c sin(\frac{1}{x}) - 0| = |x^c sin(\frac{1}{x})| \leq |x^c| = |x|^c < \delta^c = \epsilon$ and $|x^c cos(\frac{1}{x}) - 0| = |x^c cos(\frac{1}{x})| \leq |x^c| = |x|^c < \delta^c = \epsilon$.

Such a δ exists since $c > 0$, $|x^c| = |x|^c$ since x^c is defined for all $x \in \mathbb{R}$, and $|x|^c < \delta^c$ since $c > 0$ and $|x| < \delta$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

So if $c > 0$ such that x^c is defined for all $x \in \mathbb{R}$ then $lim_{x\to 0} x^c sin(\frac{1}{x}) = lim_{x\to 0} x^c cos(\frac{1}{x}) = 0$ $\square$

Note: I require that x^c is defined for all $x \in \mathbb{R}$ because while clearly it is for $x \geq 0$ it is not always the case for $x < 0$.

Also if $c \leq 0$ this limit clearly does not converge as for $c = 0$ we have seen previously that $lim_{x\to 0} sin(\frac{1}{x})$ and $lim_{x\to 0} cos(\frac{1}{x})$ do not exist. And for $c < 0$ x^c would be unbounded when x is close to 0 and therefore can not counteract the oscillatory nature of $sin(\frac{1}{x})$ and $cos(\frac{1}{x})$.

a.

For $x = 0$ we need $lim_{x\to 0} g_a(x) = lim_{x\to 0} x^a sin(\frac{1}{x}) = g(0) = 0$ in order for g_a to be continuous at 0.

Otherwise g_a wouldn't be differentiable at 0 and therefore not differentiable on $\mathbb{R}$.

From the proof at the beginning this means we need $a > 0$ such that x^a is defined for all $x \in \mathbb{R}$.

We have $g_a'(0) = lim_{x\to 0} \frac{g_a(x) - g_a(0)}{x - 0} = lim_{x\to 0} \frac{x^a sin(\frac{1}{x})}{x} = lim_{x\to 0} x^{a-1} sin(\frac{1}{x})$ and we need this limit to exist so from the proof at the beginning we need $a - 1 > 0$ and therefore $a > 1$ giving $g_a'(0) = 0$.

For $x \neq 0$ we have $g_a'(x) = (x^a sin(\frac{1}{x}))' = (x^a)' sin(\frac{1}{x}) + x^a (sin(\frac{1}{x}))' = ax^{a-1} sin(\frac{1}{x}) - x^{a-2} cos(\frac{1}{x})$ by the algebraic differentiability theorem. We already know $a - 1 > 0$ so the first term in the derivative is bounded on $[0, 1]$ so to get unboundedness on $[0, 1]$ we need x^{a-2} to be unbounded and hence $a < 2$.

So far we have that $1 < a < 2$. Let $a = 1.2$, then x^a is defined for all $x \in \mathbb{R}$ and therefore so is g_a.

Since x^a is differentiable on $\mathbb{R}$ and $sin(\frac{1}{x})$ is differentiable when $x \neq 0$ we have that g_a is differentiable when $x \neq 0$ and from above we have g_a is differentiable at 0. So for $a = 1.2$ we have g_a is differentiable on $\mathbb{R}$.

Furthermore for $x \in (0, 1]$ we have $g_a'(x) = ax^{a-1} sin(\frac{1}{x}) - x^{a-2} cos(\frac{1}{x})$ which has a bounded first term and unbounded second term since $a = 1.2 < 2$ therefore g_a' is unbounded on $[0, 1]$.

So $a = 1.2$ is a satisfactory value for the desired qualities of g_a.

b. Again we want g_a to be differentiable on $\mathbb{R}$ so we need the condition $a > 1$ so that it is differentiable at 0.

We also want g_a' to be continuous at 0 so we want $lim_{x\to 0}g_a'(x) = g_a'(0)$ which is actually guaranteed by the previous problem as long as g_a is differentiable for $x \neq 0$ and $lim_{x\to 0}g_a'(x)$ exists.

We have $lim_{x\to 0}g_a'(x) = lim_{x\to 0}ax^{a-1}sin(\frac{1}{x}) - x^{a-2}cos(\frac{1}{x})$.

By letting $a > 2$ we get from the proof at the beginning and by the algebraic limit theorem $lim_{x\to 0}g_a'(x) = 0$ for $a > 2$ and hence $g_a'(0) = 0$ so g_a' is continuous at 0 for $a > 2$.

We also want g_a' to not be differentiable at 0 so we want $lim_{x\to 0}\frac{g_a'(x)-g_a'(0)}{x-0} = lim_{x\to 0}\frac{g_a'(x)}{x}$ to not exist.

We have $lim_{x\to 0}\frac{g_a'(x)}{x} = lim_{x\to 0}\frac{ax^{a-1}sin(\frac{1}{x})-x^{a-2}cos(\frac{1}{x})}{x} = lim_{x\to 0}ax^{a-2}sin(\frac{1}{x}) - x^{a-3}cos(\frac{1}{x})$.

Since $a > 2$ we have seen that the first term will converge to 0.

By letting $a < 3$ this limit does not exist since x^{a-3} is unbounded and won't counteract the oscillatory nature of $cos(\frac{1}{x})$.

So let $a = 2.2$ then x^{a-2} and g_a are defined for all $x \in \mathbb{R}$. Also g_a is differentiable on all of $\mathbb{R}$ and g_a' is continuous but not differentiable at 0.

So $a = 2.2$ is a satisfactory value for the desired qualities of g_a.

c. Again we want g_a to be differentiable on $\mathbb{R}$ so we need the condition $a > 1$ so that it is differentiable at 0.

Since we want g_a' to be continuous at 0 we get from the previous part that $a > 2$ and $g_a'(0) = 0$.

We have $g_a'(x) = ax^{a-1}sin(\frac{1}{x}) - x^{a-2}cos(\frac{1}{x})$ for $x \neq 0$ which is differentiable since $a > 2$ and $x \neq 0$.

So $g_a''(x) = a(a-1)x^{a-2}sin(\frac{1}{x}) - ax^{a-3}cos(\frac{1}{x}) - (a-2)x^{a-3}cos(\frac{1}{x}) - x^{a-4}sin(\frac{1}{x}) =$

$a(a-1)x^{a-2}sin(\frac{1}{x}) - 2(a-1)x^{a-3}cos(\frac{1}{x}) - x^{a-4}sin(\frac{1}{x})$ for $x \neq 0$ by using the algebraic differentiability theorem.

We want $g_a''(0) = lim_{x\to 0}\frac{g_a'(x)-g_a'(0)}{x-0} = lim_{x\to 0}\frac{g_a'(x)}{x} = lim_{x\to 0}\frac{ax^{a-1}sin(\frac{1}{x})-x^{a-2}cos(\frac{1}{x})}{x} = lim_{x\to 0}ax^{a-2}sin(\frac{1}{x}) - x^{a-3}cos(\frac{1}{x})$ to exist. By letting $a > 3$ we know from the proof at the beginning and the algebraic limit theorem that $g_a''(0) = 0$.

So provided $a > 3$ and x^{a-3} is defined on $\mathbb{R}$ we get that g_a' is differentiable on $\mathbb{R}$.

We also want g_a'' to be discontinuous at 0.

So we want $lim_{x\to 0}g_a''(x) \neq g_a''(0) = 0$.

We have $lim_{x\to 0}g_a''(x) = lim_{x\to 0}a(a-1)x^{a-2}sin(\frac{1}{x}) - 2(a-1)x^{a-3}cos(\frac{1}{x}) - x^{a-4}sin(\frac{1}{x})$.

We have $a > 3$ so the first two terms converge to 0 by the proof at the beginning and the algebraic limit theorem.

So to have this limit not be 0 we need $a \leq 4$ and then $a - 4 \leq 0$ which causes the limit of the last term to not exist.

Then we would have g_a'' is discontinuous at 0 since its limit at 0 doesn't exist.

So let $a = 3.2$ then x^{a-2} and g_a are defined for all $x \in \mathbb{R}$. Also g_a and g_a' are differentiable on all of $\mathbb{R}$ but g_a'' is not continuous at 0.

So $a = 3.2$ is a satisfactory value for the desired qualities of g_a.