Countable Sets and Sequences

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1.5.3

a. Let A_1 and A_2 be countable sets and let $B = A_2 \backslash A_1 = A_2 \cap A_1^c$. Then $A_1 \cup A_2 = A_1 \cup B$.

Since A_1 and A_2 are countable we can represent them as $A_1 = \{a_1, a_2, a_3, ...\}$ and $A_2 = \{c_1, c_2, c_3, ...\}$.

- If B is finite we can write $A_1 \cup A_2 = A_1 \cup B$ as $\{b_1, b_2, ..., b_n, a_1, a_2, a_3, ...\}$.

  Let $f : A_1 \cup A_2 \to \mathbb{N}$ be defined by $f(b_1) = 1, f(b_2) = 2, ..., f(b_n) = n, f(a_1) = n+1, f(a_2) = n+2, ..., f(a_j) = n+j, ...$

  Clearly if $a, b \in A_1 \cup A_2$ such that $a \neq b$ then $f(a) \neq f(b)$ by the construction of f. So f is one to one.

  Let $m \in \mathbb{N}$ then if $m \leq n$, (where $n = |B|$) then $f(b) = n$ for some $b \in B$ by construction.

  If $m > n$, (where $n = |B|$) then $m = n + k$ for some $k \in \mathbb{N}$ so $f(a_k) = n + k = m$ for some $a_k \in A_1$.

  So every element in $\mathbb{N}$ is mapped to by some element of $A_1 \cup A_2$ under f. So f is also onto.

  Therefore we have found a one to one correspondence between $A_1 \cup A_2$ and $\mathbb{N}$ so $A_1 \cup A_2$ is countable.

- If B is not finite then it must be countable since $B \subseteq A_2$ where A_2 is countable.

  So let us represent B as $\{b_1, b_2, b_3, ...\}$. Now we can represent $A_1 \cup A_2 = A_1 \cup B$ as:

  $\{a_1, b_1, a_2, b_2, a_3, b_3, ...\}$ now define $f : \mathbb{N} \to A_1 \cup B$ by $f(n) = a_{\frac{n+1}{2}}$ if n is odd and $f(n) = b_{\frac{n}{2}}$ if n is odd.

  Clearly if $a, b \in \mathbb{N}$ such that $a \neq b$ then $f(a) \neq f(b)$ by the construction of f. So f is one to one.

  Furthermore if $c \in A_1 \cup B$ then $c = a_j$ for some $j \in \mathbb{N}$ or $c = b_k$ for some $k \in \mathbb{N}$.

  We also know for any $j, k \in \mathbb{N}$ that $j = \frac{n+1}{2}$ for some $n \in \mathbb{N}$ and $k = \frac{m}{2}$ for some $m \in \mathbb{N}$.

  So for any $c \in A_1 \cup B$ we have that $f(n) = c$ for some $n \in \mathbb{N}$ So f is also onto.

  Therefore we have found a one to one correspondence between $A_1 \cup B$ and $\mathbb{N}$ so $A_1 \cup B$ is countable.

  Since $A_1 \cup B = A_1 \cup A_2$ we have that $A_1 \cup A_2$ is countable.

So for any countable sets A_1 and A_2 we have shown that $A_1 \cup A_2$ is countable.

The more general statement follows from induction on this fact.

Let $S = \{n \in \mathbb{N} : A_1 \cup ... \cup A_n \textit{ is countable for arbitrary countable sets } A_1, ..., A_n\}$.

Assume $n \in S$, that is assume $A_1 \cup ... \cup A_n$ is countable for arbitrary countable sets $A_1, ..., A_n$.

Now consider an arbitrary countable set A_{n+1}. We know for any two countable sets their union is countable.

So we have that $(A_1 \cup ... \cup A_n) \cup A_{n+1} = A_1 \cup ... \cup A_n \cup A_{n+1}$ is countable and hence $n + 1 \in S$.

Clearly $1 \in S$ since for one countable set A_1 we know that A_1 is countable.

Therefore since $1 \in S$ and $n \in S$ implies $n + 1 \in S$ we have that $S = \mathbb{N}$. So for all $n \in \mathbb{N}$ we have that for arbitrary countable sets $A_1, ..., A_n$ the set $A_1 \cup ... \cup A_n$ is countable.

b. Induction can not be used to prove that $\cup_{i=1}^{\infty} A_i$ is countable for arbitrary countable sets $A_1, A_2, A_3, ...$ because induction can only be used to show that a claim is true for all $n \in \mathbb{N}$ but the issue is $\infty \notin \mathbb{N}$ so we can not use induction for the countably infinite union.

c. Let $A_1, A_2, A_3, ...$ be a countably infinite collection of arbitrary disjoint countable sets. Then for each A_j we can write $A_j = \{a_{(1,1)}, a_{(1,2)}, a_{(1,3)}, ...\}$ where (m, n) denotes that $a_{(m,n)}$ is the nth element in A_m.
So we can write $A_1 \cup A_2 \cup A_3 \cup ...$ as:

$$\begin{array}{cccccc}
A_1 & A_2 & A_3 & A_4 & A_5 & \ldots \\
a_{(1,1)} & a_{(2,1)} & a_{(3,1)} & a_{(4,1)} & a_{(5,1)} & \ldots \\
a_{(1,2)} & a_{(2,2)} & a_{(3,2)} & a_{(4,2)} & \ldots & \\
a_{(1,3)} & a_{(2,3)} & a_{(3,3)} & \ldots & & \\
a_{(1,4)} & a_{(2,4)} & \ldots & & & \\
a_{(1,5)} & \ldots & & & & \\
\vdots & & & & &
\end{array}$$

So by arranging $\mathbb{N}$ as follows we can form a one to one correspondence between $\cup_{i=1}^{\infty} A_i$ and $\mathbb{N}$.

$$\begin{array}{cccccc}
B_1 & B_2 & B_3 & B_4 & B_5 & \ldots \\
1 & 3 & 6 & 10 & 15 & \ldots \\
2 & 5 & 9 & 14 & \ldots & \\
4 & 8 & 13 & \ldots & & \\
7 & 12 & \ldots & & & \\
11 & \ldots & & & & \\
\vdots & & & & &
\end{array}$$

The essence of this arrangement is that we already know we can arrange our countably infinite collection of arbitrary disjoint countable sets into such an array, so by arranging $\mathbb{N}$ into such an array we are arranging $\mathbb{N}$ into a countably infinite collection of countable subsets $B_1, B_2, B_3, ...$ of $\mathbb{N}$. Then since there is certainly a one to one correspondence, say $f_n : A_n \to B_n$, we can make a one to one correspondence between $\cup_{i=1}^{\infty} A_i$ and $\mathbb{N}$. Namely let $f : \cup_{i=1}^{\infty} A_i \to \mathbb{N}$ be defined by $f(a_{(j,k)}) = f_j(a_{(j,k)})$ then since each f_j is one to one and onto we have that every element in every column of our arrangement of $\mathbb{N}$ is uniquely mapped to:

- If $a_{(j,k)} \neq a_{(m,n)}$ then $f_j(a_{(j,k)}) = f(a_{(j,k)}) \neq f(a_{(m,n)}) = f_m(a_{(m,n)})$ since f_j and f_m have disjoint ranges by construction of $B_1, B_2, B_3, ...$ from $\mathbb{N}$.

  So f is one to one.

- If $n \in \mathbb{N}$ then $n \in B_j$ for some $j \in \mathbb{N}$ so since each f_j is onto we have $f_j(a_{(j,k)}) = f(a_{(j,k)}) = n$ for some $k \in \mathbb{N}$ so every $n \in \mathbb{N}$ is mapped to by f.

  So f is onto.

Therefore we have found a one to one correspondence $f : \cup_{i=1}^{\infty} A_i \to \mathbb{N}$. So $\cup_{i=1}^{\infty} A_i \sim \mathbb{N}$ and $\cup_{i=1}^{\infty} A_i$ is countable.

This was for a countably infinite collection of arbitrary disjoint countable sets but we can generalize this to any countably infinite collection of arbitrary sets. Say $C_1, C_2, C_3, ...$ are arbitrary countable sets (not necessarily disjoint). Then let $A_1 = C_1, A_2 = C_2 \backslash A_1, A_3 = C_3 \backslash A_2, ...$ then $A_1, A_2, A_3, ...$ are all disjoint and therefore we know $\cup_{i=1}^{\infty} A_i$ is countable. Since $\cup_{i=1}^{\infty} A_i = \cup_{i=1}^{\infty} C_i$ we have that $\cup_{i=1}^{\infty} C_i$ is countable for arbitrary countable sets $C_1, C_2, C_3, ...$ $\square$

2.2.1

- This vercongent definition does not work for convergence. This definition says that a sequence (a_n) verconges to a if for a single choice of $\epsilon > 0$ all values of the sequence (a_n) lie within an epsilon neighborhood of a. This is because by the definition if (a_n) verconges to a then there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ when $n \geq N$ then $|a_n - a| < \epsilon$ so since $1 \in \mathbb{N}$ we have $|a_n - a| < \epsilon$ for all $n \in \mathbb{N}$ such that $n \geq 1$. This is essentially saying that (a_n) is bounded because $|a_n - a| < \epsilon$ implies $-\epsilon < a_n - a < \epsilon$ implies $a - \epsilon < a_n < a + \epsilon$ for all $n \in \mathbb{N}$ and some finite $a, \epsilon \in \mathbb{R}$.

- The sequence $(a_n) = ((-1)^n) = (-1, 1, -1, 1, -1, ...)$ for $n \in \mathbb{N}$ is vercongent by this definition. Let $a = 0$ and $\epsilon = 2$ then $|a_n - a| = |(-1)^n - 0| = |(-1)^n| = 1 < \epsilon = 2$ for all $n \in \mathbb{N}$.

  Therefore $(a_n) = ((-1)^n)$ verconges to $a = 0$.

- However, this sequence is known to diverge due to its oscillating and non-absolutely-decreasing nature. So this is an example of a divergent series that verconges according to this definition.

  Proof:

  Let $(a_n) = ((-1)^n)$ and $a \in \mathbb{R}$ then if n is odd $|a_n - a| = |-1-a| = |a+1|$ and if n is even $|a_n - a| = |1-a| = |a-1|$. Let $0 < \epsilon < min(|a+1|, |a-1|)$, such an ϵ exists because of the density of $\mathbb{R}$, then clearly $|a_n - a| \not< \epsilon$ for any $n \in \mathbb{N}$. We have found an $\epsilon > 0$ where there does not exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - a| < \epsilon$ for any $a \in \mathbb{R}$. Therefore by the definition of convergence $(a_n) = ((-1)^n)$ does not converge, so (a_n) diverges.

- Furthermore, a sequence can verconge to more than one value.

  Let (a_n) be as before and $a = 1$ then if n is odd $|a_n - a| = |(-1)^n - 1| = |-1-1| = 2$ or if n is even $|a_n - a| = |(-1)^n - 1| = |1 - 1| = 0$ so choosing $\epsilon = 3$ we see that $|a_n - a| < \epsilon$ for all $n \in \mathbb{N}$.

  Therefore (a_n) verconges to $a = 1$ as well.

2.2.2

a. Let $(a_n) = (\frac{2n+1}{5n+4})$ for $n \in \mathbb{N}$ and let $\epsilon > 0$. The proposed limit is $\frac{2}{5}$.

We know $|a_n - \frac{2}{5}| = |\frac{2n+1}{5n+4} - \frac{2}{5}| = |\frac{5(2n+1)-2(5n+4)}{5(5n+4)}| = |\frac{-3}{25n+20}| = \frac{3}{25n+20}$ since $n > 0$.

This shows that as $n \in \mathbb{N}$ increases $|a_n - \frac{2}{5}|$ decreases.

So if $|a_n - \frac{2}{5}| < c$ then when $m \in \mathbb{N}$ such that $m \geq n$ we have $|a_m - \frac{2}{5}| \leq |a_n - \frac{2}{5}| < c$ for $c \in \mathbb{R}$.

So we want to find an $N \in \mathbb{N}$ such that $|a_N - \frac{2}{5}| < \epsilon$ and it will follow that if $n \geq N$ then $|a_n - \frac{2}{5}| < \epsilon$.

Let $|a_N - \frac{2}{5}| = \frac{3}{25N+20} < \epsilon$ then $3 < \epsilon(25N+20) = 25N\epsilon + 20\epsilon$ then $3 - 20\epsilon < 25N\epsilon$.

So for $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $N > \frac{3-20\epsilon}{25\epsilon}$. Such an N exists because $\mathbb{N}$ is unbounded.

Then we will have that $|a_N - \frac{2}{5}| < \epsilon$ and if $n \in \mathbb{N}$ such that $n \geq N$ then $|a_n - \frac{2}{5}| < \epsilon$.

So we have shown that for all $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - \frac{2}{5}| < \epsilon$.

Therefore $(a_n) = (\frac{2n+1}{5n+4})$ converges to $\frac{2}{5}$ $\square$

b. Let $(a_n) = (\frac{2n^2}{n^3+3})$ for $n \in \mathbb{N}$ and let $\epsilon > 0$. The proposed limit is 0.

We know $|a_n - 0| = |\frac{2n^2}{n^3+3} - 0| = |\frac{2n^2}{n^3+3}| = \frac{2n^2}{n^3+3}$ since $n > 0$.

This shows that as $n \in \mathbb{N}$ increases $|a_n - 0|$ decreases because $n^3 + 3$ grows faster than $2n^2$.

So if $|a_n - 0| < c$ then when $m \in \mathbb{N}$ such that $m \geq n$ we have $|a_m - 0| \leq |a_n - 0| < c$ for $c \in \mathbb{R}$.

So we want to find an $N \in \mathbb{N}$ such that $|a_N - 0| < \epsilon$ and it will follow that if $n \geq N$ then $|a_n - 0| < \epsilon$.

Let $\frac{2N^2}{N^3} < \epsilon$ then $|a_N - 0| = \frac{2N^2}{N^3+3} < \frac{2N^2}{N^3} = \frac{2}{N} < \epsilon$.

So for $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Such an N exists because $\mathbb{N}$ is unbounded.

Then we will have that $|a_N - 0| < \epsilon$ and if $n \in \mathbb{N}$ such that $n \geq N$ then $|a_n - 0| < \epsilon$.

So we have shown that for all $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - 0| < \epsilon$.

Therefore $(a_n) = (\frac{2n^2}{n^3+3})$ converges to 0 $\square$

c. Let $(a_n) = (\frac{sin(n^2)}{\sqrt[3]{n}})$ for $n \in \mathbb{N}$ and let $\epsilon > 0$. The proposed limit is 0.

We know $|a_n - 0| = |\frac{sin(n^2)}{\sqrt[3]{n}} - 0| = |\frac{sin(n^2)}{\sqrt[3]{n}}| \leq \frac{1}{\sqrt[3]{n}}$ since $-1 \leq sin(n^2) \leq 1$.

This shows that as $n \in \mathbb{N}$ increases $|a_n - 0|$ decreases.

So if $|a_n - 0| < c$ then when $m \in \mathbb{N}$ such that $m \geq n$ we have $|a_m - 0| \leq |a_n - 0| < c$ for $c \in \mathbb{R}$.

So we want to find an $N \in \mathbb{N}$ such that $|a_N - 0| < \epsilon$ and it will follow that if $n \geq N$ then $|a_n - 0| < \epsilon$.

Let $\frac{1}{\sqrt[3]{N}} < \epsilon$ then $|a_N - 0| = |\frac{sin(N^2)}{\sqrt[3]{N}}| \leq \frac{1}{\sqrt[3]{N}} < \epsilon$.

So for $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$ choose $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^3}$. Such an N exists because $\mathbb{N}$ is unbounded.

Then we will have that $|a_N - 0| < \epsilon$ and if $n \in \mathbb{N}$ such that $n \geq N$ then $|a_n - 0| < \epsilon$.

So we have shown that for all $\epsilon \in \mathbb{R}$ such that $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - 0| < \epsilon$.

Therefore $(a_n) = (\frac{sin(n^2)}{\sqrt[3]{n}})$ converges to 0 $\square$

2.2.4

a. Let $(a_n) = ((-1)^n) = (-1, 1, -1, 1, -1, 1, ...)$ for $n \in \mathbb{N}$.

Then (a_n) has an infinite number of ones.

Proof:

Say (a_n) has a finite number of ones. That is say for some $k \in \mathbb{N}$ that a_k is the last one in the sequence.

But consider a_{k+2}. Since $a_k = (-1)^k = 1$ we have that $a_{k+2} = (-1)^{k+2} = (-1)^k(-1)^2 = (-1)^k(1) = (-1)^k = 1$.

So we have a contradiction and there can not be a finite number of ones in the sequence.

So there are an infinite number of ones in the sequence $(a_n) = ((-1)^n)$.

However, as proved in problem 2.2.1 this sequence diverges.

So this is such a sequence that contains an infinite number of ones that does not converge to one.

b. This is not possible. Let (a_n) be a convergent series that has an infinite number of ones.

Say $(a_n) \to a$ then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ where if $n \in \mathbb{N}$ such that $n \geq N$ then $|a_n - a| < \epsilon$.

Since (a_n) contains an infinite number of ones we know that for any N there exists a one in the sequence beyond a_N.

So if $a \neq 1$ then for any choice of N we have a point later in the sequence where $|1 - a| > 0$.

So let $0 < \epsilon < |1 - a|$ such an ϵ exists because of the density of $\mathbb{R}$.

Therefore if $a \neq 1$ we have shown that there exists an $\epsilon > 0$ such that there does not exist an $N \in \mathbb{N}$ where if $n \geq N$ then $|a_n - a| < \epsilon$ due to the presence of infinitely many ones.

Therefore a sequence that has infinitely many ones can not converge to a value that is not one $\square$

c. Let $(a_n) = (1, a, 1, 1, a, 1, 1, 1, a, 1, 1, 1, 1, a, 1, 1, 1, 1, 1, a, ...)$ where $a \in \mathbb{R}$ and $a \neq 1$.

By construction for any $n \in \mathbb{N}$ you can find n consecutive ones in the sequence. This sequence also diverges.

Proof:

- To show this I will prove a more generalized form of the previous part of the problem:

  Let (b_n) be a convergent series that has an infinite number of terms equal to c for some $c \in \mathbb{R}$.

  Say $(b_n) \to b$ then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ where if $n \in \mathbb{N}$ such that $n \geq N$ then $|b_n - b| < \epsilon$.

  Since (b_n) contains an infinite number of terms c we know for any N there exists an c in the sequence beyond b_N.

  So if $b \neq c$ then for any choice of N we have a term later in the sequence where $|c - b| > 0$.

  So let $0 < \epsilon < |c - b|$ such an ϵ exists because of the density of $\mathbb{R}$.

  Therefore if $b \neq c$ we have shown that there exists an $\epsilon > 0$ such that there does not exist an $N \in \mathbb{N}$ where if $n \geq N$ then $|b_n - b| < \epsilon$ due to the presence of infinitely many terms c.

  Therefore a sequence that has infinitely many terms equal to c can not converge to a value that is not c $\square$

- Since this was for arbitrary $c \in \mathbb{R}$ it applies to our construction for both a and one.

  So if our constructed sequence converges then it must converge to a and to one. So we would need $a = 1$.

  However by our construction $a \neq 1$ so it can not be that our sequence converges.

Therefore this is such a divergent sequence where for any $n \in \mathbb{N}$ you can find n consecutive ones in the sequence.