Sequences and Series

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2.5.1

a. This is not possible. Let (a_n) be a sequence that contains a bounded subsequence, say (b_n).

Then by the Bolzano-Weierstrass Theorem (b_n) has some subsequence that is convergent.

Therefore (a_n) has a subsequence that is convergent.

b. Let (a_n) be defined by $a_n = \frac{1}{n+1}$ if n is odd and $a_n = 1 - \frac{1}{n+1}$ if n is even.

Then $(a_n) = (\frac{1}{2}, 1 - \frac{1}{3}, \frac{1}{4}, 1 - \frac{1}{5}, ...) = (\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{4}{5}, ...)$. Clearly $0 < a_n < 1$ for all $n \in \mathbb{N}$. So $0, 1 \notin (a_n)$.

Consider $(a_{2n-1}) = (a_1, a_3, a_5, ...) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ...) = (\frac{1}{2n})$. Then $a_{2n-1} = (\frac{1}{2})\frac{1}{n}$.

Since as shown in previous sample works $(\frac{1}{n}) \to 0$ we have by the Algebraic Limit Theorem that $(a_{2n-1}) \to (\frac{1}{2})0 = 0$.

So the subsequence (a_{2n-1}) of a_n converges to 0.

Consider $(a_{2n}) = (a_2, a_4, a_6, ...) = (1 - \frac{1}{3}, 1 - \frac{1}{5}, 1 - \frac{1}{7}, ...) = (1 - \frac{1}{2n+1})$.

Then $1 - \frac{1}{2n} < 1 - \frac{1}{2n+1} = a_{2n} < 1$ for all $n \in \mathbb{N}$.

By the Algebraic Limit Theorem, $(1 - \frac{1}{2n}) \to 1 - lim_{n\to\infty} \frac{1}{2n} = 1 - 0 = 1$. Clearly $(1) \to 1$.

So by the squeeze theorem, proved in a previous sample work, we have that $(a_{2n}) \to 1$.

So the subsequence (a_{2n}) of a_n converges to 1.

So this is an example of such a sequence that does not contain 0 or 1 but has subsequences that converge to 0 and 1.

Note: If you start with a convergent sequence this is not possible since its subsequences must converge to the same value.

c. Let $(a_n) = (1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$. Then let $x \in \{\frac{1}{n} : n \in \mathbb{N}\}$.

Then by construction of (a_n) there exists a point in the sequence after which there are infinitely many terms equal to x.

Therefore for any $x \in \{\frac{1}{n} : n \in \mathbb{N}\}$ you can make a subsequence $(x, x, x, x, ...)$ of (a_n) which clearly converges to x.

So this is such a sequence that for every element of $\{\frac{1}{n} : n \in \mathbb{N}\}$ there is a subsequence converging to that element.

d. This is not possible. Let (a_n) be a sequence that for every $x \in \{\frac{1}{n} : n \in \mathbb{N}\}$ there is a subsequence converging to x.

Let $\epsilon_n = \frac{1}{n}$. Then choose n_1 such that $|a_{n_1} - 1| < \epsilon_1$, $n_2 > n_1$ such that $|a_{n_2} - \frac{1}{2}| < \epsilon_2$, and so on.

Then $n_1 < n_2 < n_3 < ... < n_m < ...$ and n_m is such that $|a_{n_m} - \frac{1}{m}| < \epsilon_m = \frac{1}{m}$.

Each n_m exists since for each $x \in \{\frac{1}{n} : n \in \mathbb{N}\}$ there is a subsequence of (a_n) converging to x.

Then $-\frac{1}{m} < a_{n_m} - \frac{1}{m} < \frac{1}{m}$ and $0 < a_{n_m} < \frac{2}{m}$ for all $m \in \mathbb{N}$.

By the Algebraic Limit Theorem $(\frac{2}{m}) \to 2\ lim_{m\to\infty} \frac{1}{m} = 0$. Clearly $(0) \to 0$.

So by the squeeze theorem, the subsequence $(a_{n_1}, a_{n_2}, a_{n_3}, ..., a_{n_m}, ...) \to 0$.

$0 \notin \{\frac{1}{n} : n \in \mathbb{N}\}$ so any sequence that has a subsequence converging to x for all $x \in \{\frac{1}{n} : n \in \mathbb{N}\}$ also has a subsequence converging to some value not in $\{\frac{1}{n} : n \in \mathbb{N}\}$ because it must contain a subsequence converging to 0 $\square$

2.5.5

Let (a_n) be a bounded sequence such that every convergent subsequence converges to the same $a \in \mathbb{R}$.

Assume for the sake of contradiction that (a_n) does not converge to a.

That is there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists an $n \geq N$ such that $|a_n - a| \geq \epsilon$.

Let $\epsilon^{'}$ be such an ϵ and let n_1 be such that $|a_{n_1} - a| \geq \epsilon^{'}$, $n_2 > n_2$ be such that $|a_{n_2} - a| \geq \epsilon^{'}$, and so on.

Then $n_1 < n_2 < n_3 < ... < n_m < ...$ and $|a_{n_m} - a| \geq \epsilon^{'}$ for all $m \in \mathbb{N}$.

This set of n_m exists because of our assumption that for all $N \in \mathbb{N}$ there exists an $n \geq N$ such that $|a_n - a| \geq \epsilon^{'}$.

Then since (a_n) is bounded we have that (a_{n_m}) is also bounded.

Then by the Bolzano-Weierstrass Theorem (a_{n_m}) has some subsequence that is convergent.

Since this subsequence of (a_{n_m}) is also a subsequence of (a_n) it must converge to a by our assumption that every convergent subsequence converges to the same $a \in \mathbb{R}$.

But we have constructed (a_{n_m}) so that $|a_{n_m} - a| \geq \epsilon^{'}$ for all $m \in \mathbb{N}$.

Therefore a subsequence of (a_{n_m}) can not converge to a and we have a contradiction.

So it must be that our assumption was wrong and therefore $(a_n) \to a$ $\square$

2.6.2

a. Let $(a_n) = (\frac{(-1)^n}{n})$. This series converges to 0.

Proof:

Let $\epsilon > 0$ then $|a_N - 0| = |\frac{(-1)^N}{N}| = \frac{1}{N}$.

By the Archimedean Property there exists an $N \in \mathbb{N}$ such that $|a_N - 0| = \frac{1}{N} < \epsilon$.

When $n \geq N$ we now have $|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$.

So for arbitrary $\epsilon > 0$ we have shown there exists an $N \in \mathbb{N}$ such that for $n \geq N$, $|a_n - 0| < \epsilon$.

Therefore $(a_n) = (\frac{(-1)^n}{n}) \to 0$.

Since all convergent sequences are Cauchy we have that (a_n) is a Cauchy sequence.

Furthermore (a_n) is not monotone as its terms oscillate from positive to negative.

So this is an example of such a sequence that is Cauchy but not monotone.

b. This is not possible. Let (a_n) be a sequence that has an unbounded subsequence.

Then (a_n) must be unbounded itself. Since all Cauchy sequences are bounded (a_n) can not be a Cauchy sequence $\square$

c. This is not possible. Let (a_n) be a divergent, monotone sequence.

Then (a_n) is unbounded because if it were bounded then it would converge by the Monotone Convergence Theorem.

Let (a_{n_m}) be a subsequence of (a_n) then (a_{n_m}) is monotone as well.

If (a_{n_m}) were Cauchy then it would be bounded, but this would imply that (a_n) is bounded as well.

Because if $(a_{n_m}) = (a_{n_1}, a_{n_2}, a_{n_3}, ...)$ is bounded then we have $|a_{n_m}| \leq M$ for all $m \in \mathbb{N}$ and some $M \in \mathbb{N}$.

Since (a_{n_m}) is monotonically increasing and n_m is unbounded we would have that

$$|a_1| \leq |a_2| \leq ... \leq |a_{n_1}| \leq ... \leq |a_{n_m}| \leq ... \leq M.$$

This would imply $|a_n| \leq M$ for all $n \in \mathbb{N}$ and (a_n) is bounded. But (a_n) is unbounded.

Therefore if (a_{n_m}) is a subsequence of (a_n) then (a_{n_m}) can not be Cauchy $\square$

d. Let (a_n) be defined by $a_n = n$ if n is odd and $a_n = \frac{1}{n}$ if n is even.

Then (a_n) is unbounded since the set of all odd natural numbers is unbounded.

The subsequence $(a_{2n}) = (a_2, a_4, a_6, ...) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, ...) = \frac{1}{2}(\frac{1}{n})$ clearly converges.

Since $(\frac{1}{n}) \to 0$ we have $(a_{2n}) \to \frac{1}{2}(0) = 0$ by the algebraic limit theorem.

Since (a_{2n}) converges it is Cauchy.

Therefore this is an example of such a sequence that is unbounded and has a Cauchy subsequence.

Note: The key is in not letting the sequence be monotone as then it is not possible.

2.7.1

c. Let (a_n) be a sequence where $a_1 \geq a_2 \geq a_3 \geq ...$ and $(a_n) \to 0$.

Then it must be that $a_n \geq 0$ for all $n \in \mathbb{N}$, otherwise (a_n) is getting further away from 0 and can not converge to 0.

Consider the alternating sequence of partial sums $(s_n) = (a_1 - a_2 + a_3 - a_4 + ... \pm a_n)$.

The subsequence $(s_{2n}) = (a_1 - a_2, a_1 - a_2 + a_3 - a_4, ...)$ is clearly bounded above by a_1 and is monotonically increasing.

This is because $a_n \geq a_{n+1}$ and $a_n - a_{n+1} \geq 0$ for all $n \in \mathbb{N}$ so $s_{2(n+1)} = s_{2n} + (a_{2n+1} - a_{2n+2}) \geq s_{2n}$.

Also $s_2 = a_1 - a_2 \leq a_1$. So (s_{2n}) is bounded and monotone and converges by the monotone convergence theorem.

Similarly $(s_{2n+1}) = (a_1 - a_2 + a_3, a_1 - a_2 + a_3 - a_4 + a_5, ...) = (s_{2n} + a_{2n+1})$.

So by the algebraic limit theorem we have $(s_{2n+1}) = (s_{2n} + a_{2n+1}) \to lim\ s_{2n} + lim\ a_{2n+1} = lim\ s_{2n}$.

Since $(s_n) = (s_1, s_{2(1)}, s_{2(1)+1}, s_{2(2)}, s_{2(2)+1}, ...)$ we have that (s_n) alternates between s_{2n} and s_{2n+1}.

Let $\epsilon > 0$ then $(s_{2n}) \to lim\ s_{2n}$, $(s_{2n+1}) \to lim\ s_{2n+1}$, and $lim\ s_{2n} = lim_{2n+1} = x$.

So we can find an $N \in \mathbb{N}$ where $|s_n - x| < \epsilon$ for $n \geq N$ because either $n = 2k+1$ or $n = 2k$ for some $k \in \mathbb{N}$.

So $|s_n - x| = |s_{2n} - x| < \epsilon$ or $|s_n - x| = |s_{2n+1} - x| < \epsilon$ since $lim\ s_{2n} = lim\ s_{2n+1} = x$.

This was for arbitrary $\epsilon > 0$ and is therefore true for all $\epsilon > 0$.

Therefore (s_n) converges as well $\square$

2.7.2

a. The series $\sum_{n=1}^{\infty}\frac{1}{2^n+n}$ converges.

Consider the sequence $(\frac{1}{2^n})$. Then $0 < \frac{1}{2^n+n} < \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

The geometric series $\sum_{n=0}^{\infty}\frac{1}{2^n} = \sum_{n=0}^{\infty}(\frac{1}{2})^n$ converges to $\frac{1}{1-\frac{1}{2}} = 2$

So we have that $\sum_{n=1}^{\infty}\frac{1}{2^n} = (\sum_{n=0}^{\infty}\frac{1}{2^n}) - \frac{1}{2^0}$ converges to $2-1=1$.

Since $0 < \frac{1}{2^n+n} < \frac{1}{2^n}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges, by the comparison test for series $\sum_{n=1}^{\infty}\frac{1}{2^n+1}$ converges $\square$

b. The series $\sum_{n=1}^{\infty}\frac{sin(n)}{n^2}$ converges.

Since $|sin(x)| \leq 1$ for all $x \in \mathbb{R}$ we have that $0 \leq |\frac{sin(n)}{n^2}| = \frac{|sin(n)|}{n^2} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

We have seen in class that the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges and $0 \leq |\frac{sin(n)}{n^2}| \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

So by the comparison test, the series $\sum_{n=1}^{\infty}|\frac{sin(n)}{n^2}|$ converges.

Therefore $\sum_{n=1}^{\infty}\frac{sin(n)}{n^2}$ converges by the absolute convergence test $\square$

c. The series $1 - \frac{3}{4} + \frac{4}{6} - \frac{5}{8} + \frac{6}{10} - \frac{7}{12} + ... = \sum_{n=1}^{\infty}\frac{n+1}{2n}(-1)^{n+1}$ diverges.

Clearly $0 < \frac{n+1}{2n+2} = \frac{n+1}{2(n+1)} = \frac{1}{2} < \frac{n+1}{2n}$ for all $n \in \mathbb{N}$.

If $\sum_{n=1}^{\infty}\frac{n+1}{2n}(-1)^{n+1}$ were to converge then the sequence $(\frac{n+1}{2n}(-1)^{n+1}) \to 0$.

However as above $\frac{1}{2} < \frac{n+1}{2n}$ for all $n \in \mathbb{N}$. So $|\frac{n+1}{2n}(-1)^{n+1} - 0| = \frac{n+1}{2n} > \frac{1}{2}$ and it can not be that $(\frac{n+1}{2n}(-1)^{n+1}) \to 0$.

Therefore $\sum_{n=1}^{\infty}\frac{n+1}{2n}(-1)^{n+1}$ must diverge $\square$

d. The series $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$ diverges.

We can group the series as $1 + (\frac{1}{2} - \frac{1}{3}) + \frac{1}{4} + (\frac{1}{5} - \frac{1}{6}) + ... + \frac{1}{n} + (\frac{1}{n+1} - \frac{1}{n+2}) + ...$

Since $\frac{1}{n+1} > \frac{1}{n+2}$ we have $\frac{1}{n} + (\frac{1}{n+1} - \frac{1}{n+2}) > \frac{1}{n} > 0$ for all $n \in \mathbb{N}$.

Therefore $s_{3n} = 1 + (\frac{1}{2} - \frac{1}{3}) + \frac{1}{4} + (\frac{1}{5} - \frac{1}{6}) + ... + \frac{1}{3n-2} + (\frac{1}{3n-1} - \frac{1}{3n}) > 1 + \frac{1}{4} + \frac{1}{7} + ... + \frac{1}{3n-2} > 0$ for all $n \in \mathbb{N}$.

Clearly $\frac{1}{3n-2} > \frac{1}{4n} > 0$ for all $n \in \mathbb{N}$. The series $\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n} = \sum_{n=1}^{\infty}\frac{1}{4n}$ diverges since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

So by the comparison test we have that $\sum_{n=1}^{\infty}\frac{1}{3n-2}$ diverges.

Since the series $\sum_{n=1}^{\infty}\frac{1}{3n-2}$ diverges, we have by the comparison test that $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + ...$ diverges $\square$

e. The series $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + ...$ diverges.

This series is equal to $(\sum_{n=1}^{\infty}\frac{1}{2n-1}) - (\sum_{n=1}^{\infty}\frac{1}{(2n)^2})$.

Since $\sum_{n=1}^{\infty}\frac{1}{(n)^2}$ converges and $\frac{1}{(2n)^2} < \frac{1}{n^2}$ for all $n \in \mathbb{N}$ we have that $\sum_{n=1}^{\infty}\frac{1}{(2n)^2}$ converges by the comparison test.

Clearly $\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n} = \sum_{n=1}^{\infty}\frac{1}{3n}$ diverges since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, and $0 < \frac{1}{3n} < \frac{1}{2n-1}$ for all $n \in \mathbb{N}$.

So we have that $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges by the comparison test.

Therefore $1 - \frac{1}{2^2} + \frac{1}{3} - \frac{1}{4^2} + ... = (\sum_{n=1}^{\infty}\frac{1}{2n-1}) - (\sum_{n=1}^{\infty}\frac{1}{(2n)^2})$ diverges since you are taking a divergent series and subtracting a convergent one $\square$