Using Residue Theory to Solve Real Integrals

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83.5

a. Let $f(z) = tan\ z = \frac{sin\ z}{cos\ z}$. Clearly f is analytic everywhere $cos\ z \neq 0$ (i.e. for $z \neq \frac{\pi}{2} + n\pi$ with $n \in \mathbb{Z}$).

Now let C be the positively oriented circle $|z| = 2$, clearly $z = \pm\frac{\pi}{2}$ are the only isolated singular points of f interior to C.

Let us look at a derivative of $cos\ z$:

$$\frac{d}{dz}cos\ z = -sin\ z \text{ so } \frac{d}{dz}cos\ z\Big|_{-\frac{\pi}{2}} = -sin(-\frac{\pi}{2}) = 1 \text{ and } \frac{d}{dz}cos\ z\Big|_{\frac{\pi}{2}} = -sin(\frac{\pi}{2}) = -1.$$

So we have that there exists an m (namely $m = 1$) such that:

$$f(z_0) = f'(z_0) = ... = f^{(m-1)}(z_0) = 0 \text{ and } f^{(m)}(z_0) \neq 0 \text{ where } f(z) = cos\ z \text{ and } z_0 = \pm\frac{\pi}{2}.$$

Therefore $z_0 = \pm\frac{\pi}{2}$ are first order zeros of $cos\ z$ and hence are simple poles of $tan\ z$.

Recall that if $p(z)$ and $q(z)$ are analytic at z_0 with $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$ then z_0 is a simple pole of $\frac{p(z)}{q(z)}$ and $Res_{z=z_0}\frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$.

Therefore:

$$Res_{z=-\frac{\pi}{2}}tan\ z = Res_{z=-\frac{\pi}{2}}\frac{sin\ z}{cos\ z} = \frac{sin(-\frac{\pi}{2})}{-sin(-\frac{\pi}{2})} = -1 \text{ and } Res_{z=\frac{\pi}{2}}tan\ z = Res_{z=\frac{\pi}{2}}\frac{sin\ z}{cos\ z} = \frac{sin(\frac{\pi}{2})}{-sin(\frac{\pi}{2})} = -1.$$

Finally since C is positively oriented and simple closed with $z_0 = \pm\frac{\pi}{2}$ as the only isolated singular points of f interior to C:

$$\int_C tan\ z\ dz = \int_C f(z)dz = 2\pi i\big(Res_{z=-\frac{\pi}{2}}f(z) + Res_{z=\frac{\pi}{2}}f(z)\big) = 2\pi i\big(Res_{z=-\frac{\pi}{2}}tan\ z + Res_{z=\frac{\pi}{2}}tan\ z\big) = -4\pi i$$

$\square$

83.6

Let C_N be the positively oriented boundary of the square with sides on $x = \pm(N+\frac{1}{2})\pi$ and $y = \pm(N+\frac{1}{2})\pi$.

Now let $f(z) = \frac{1}{z^2 sin\ z}$. Clearly z^2 has a zero of order 2 at $z_0 = 0$.

Also $sin\ z$ has zeros only at $z_0 = \pm n\pi$ where $n \leq N$ inside C_N.

Let us look at a derivative of $sin\ z$:

$\frac{d}{dz} sin\ z = cos\ z$ so $\frac{d}{dz} sin\ z\Big|_{\pm n\pi} = cos(\pm n\pi) = \pm 1 \neq 0$.

So we have that there exists an m (namely $m = 1$) such that:

$f(z_0) = f'(z_0) = ... = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$ where $f(z) = sin\ z$ and $z_0 = \pm n\pi$.

Therefore $z_0 = \pm n\pi$ are first order zeros of $sin\ z$ and hence are simple poles of $\frac{1}{sin\ z}$.

So inside C_N we have the only isolated singular points of $f(z)$ are $z_0 = \pm n\pi$ where $n \leq N$.

All of these poles are simple except for when $n = 0$ corresponding to $z_0 = 0$ which has a pole of order 3 since z^2 has a zero of order 2 at 0 and $sin\ z$ has a zero of order 1 at 0.

- At $z_0 = 0$:

We know for $|z| < \infty$ that:

$$sin\ z = \sum_{n=0}^{\infty} \frac{z^{2n+1}(-1)^n}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ...$$

Therefore we may find the Laurent series for $\frac{1}{sin\ z}$ about $z_0 = 0$ for $0 < |z| < \infty$ using long division:

$$\frac{1}{z} + \frac{z}{6} + ...$$

$$\frac{1}{sin\ z} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ... \ \Big) \ 1 + 0z + 0z^3 + 0z^5 + ...$$

$$-(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - ...)$$

$$\frac{z^2}{3!} - \frac{z^4}{5!} + ...$$

$$-(\frac{z^2}{6} - \frac{z^4}{36} + \frac{z^6}{720} - ...)$$

$$z^4(\frac{1}{36} - \frac{1}{5!}) + ...$$

There are many terms here but we only need the term that will give us the residue for $\frac{1}{z^2 sin\ z}$.

Therefore for $0 < |z| < \infty$ we have:

$$\frac{1}{z^2 sin\ z} = \frac{1}{z^2}\Big(\frac{1}{z} + \frac{z}{6} + ...\Big) = \frac{1}{z^3} + \Big(\frac{1}{6}\Big)\frac{1}{z} + ...$$

So $Res_{z=0} \frac{1}{z^2 sin\ z} = \frac{1}{6}$

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- At $z_0 = \pm n\pi$ for $n \leq N$ and $n \neq 0$:

Notice that $\frac{d}{dz} z^2 sin\ z = 2z\ sin\ z + z^2 cos\ z$.

Since $f(z) = \frac{1}{z^2 sin\ z}$ has a simple pole at all of these points we know:

$$Res_{z=\pm n\pi} \frac{1}{z^2 sin\ z} = \frac{1}{2z\ sin\ z + z^2 cos\ z}\bigg|_{z=\pm n\pi} = \frac{1}{0 + n^2\pi^2 cos(\pm n\pi)} = \frac{1}{n^2\pi^2 cos(n\pi)} = \frac{(-1)^n}{n^2\pi^2}$$

Since there are two residue terms for each n (because each nonzero pole occurs at $\pm n$) we know that:

$$\int_{C_N} \frac{1}{z^2 sin\ z} dz = 2\pi i \left(\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right)$$

We are given that the value of this integral tends to 0 as $N \to \infty$.

Therefore as $N \to \infty$ we have:

$$\int_{C_N} \frac{1}{z^2 sin\ z} dz = 2\pi i \left(\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \right) \to 0$$

Which implies that:

$$\frac{1}{6} + 2 \sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2} \to 0$$

Consequently:

$$\sum_{n=1}^{N} \frac{(-1)^n}{n^2} \to -\frac{\pi^2}{12}$$

Finally since we have the convergence of partial sums this gives us the result:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\left(-\frac{\pi^2}{12} \right) = \frac{\pi^2}{12}$$

$\square$

86.7

Let $f(z) = \frac{1}{z^2+2z+2} = \frac{1}{(z-(-1+i))(z-(-1-i))}$, clearly f has simple poles at $z = -1 \pm i$.

Therefore $Res_{z=-1+i} \frac{1}{z^2+2z+2} = \frac{1}{z-(-1-i)}\Big|_{z=-1+i} = \frac{1}{2i} = -\frac{i}{2}$.

Now let L_R be the line on the real axis going from $-R$ to R and let Q_R be the defined by $z(\theta) = Re^{i\theta}$ for $0 \leq \theta \leq \pi$ (i.e. the boundary of the semicircle that is the upper half of the circle $|z| = R$ excluding the real axis part).

Let $C_R = L_R + Q_R$ (i.e. the semicircle boundary that is the upper half of the circle $|z| = R$ including the real axis part).

Clearly C_R is simple, closed, positively oriented, and if $R > \sqrt{2}$ then it contains the pole $-1+i$ of f but never the other.

So we know for $R > \sqrt{2}$:

$$\int_{C_R} \frac{1}{z^2+2z+2} dz = 2\pi i\left(-\frac{i}{2}\right) = \pi$$

We also know:

$$\int_{C_R} \frac{1}{z^2+2z+2} dz = \int_{L_R} \frac{1}{z^2+2z+2} dz + \int_{Q_R} \frac{1}{z^2+2z+2} dz = \int_{-R}^{R} \frac{1}{x^2+2x+2} dx + \int_{Q_R} \frac{1}{z^2+2z+2} dz$$

So we have that:

$$\int_{-R}^{R} \frac{1}{x^2+2x+2} dx = \pi - \int_{Q_R} \frac{1}{z^2+2z+2} dz$$

Along Q_R we know that $|z| = R$ and so $|z^2+2z+2| \geq ||z^2+2z| - |2|| = |R|z+2| - 2| \geq |R|R-2| - 2|$ which for large enough R is just $R^2 - 2R - 2$. Also clearly the length of Q_R is πR.

Then we have that $|\frac{1}{z^2+2z+2}| = \frac{1}{|z^2+2z+2|} \leq \frac{1}{R^2-2R-2}$ for large enough R.

Therefore for large enough R we know:

$$\left| \int_{Q_R} \frac{1}{z^2+2z+2} dz \right| \leq \frac{\pi R}{R^2 - 2R - 2}$$

Then we know since polynomials are continuous and the denominator of the below is nonzero:

$$\lim_{R\to\infty} \frac{\pi R}{R^2-2R-2} = \lim_{R\to 0^+} \frac{\frac{\pi}{R}}{\frac{1}{R^2} - \frac{2}{R} - 2} = \lim_{R\to 0^+} \frac{\pi R}{1 - 2R - 2R^2} = \frac{\pi(0)}{1 - 2(0) - 2(0)^2} = 0$$

Therefore by taking $R \to \infty$:

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+2x+2} dx = \pi - \lim_{R\to\infty} \int_{Q_R} \frac{1}{z^2+2z+2} dz = \pi$$

Finally we have that:

$$P.V. \int_{-\infty}^{\infty} \frac{1}{x^2+2x+2} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+2x+2} dx = \pi$$

88.6

Let $f(z) = \frac{ze^{iz}}{(z^2+1)(z^2+4)} = \frac{z(\cos z + i \sin z)}{(z+i)(z-i)(z+2i)(z-2i)}$, clearly f has simple poles at $z = \pm i, \pm 2i$.

Therefore $Res_{z=i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} = \frac{ze^{iz}}{(z+i)(z+2i)(z-2i)}\Big|_{z=i} = \frac{ie^{i^2}}{(2i)(3i)(-i)} = \frac{1}{6e}$.

Also $Res_{z=2i} \frac{ze^{iz}}{(z^2+1)(z^2+4)} = \frac{ze^{iz}}{(z+i)(z-i)(z+2i)}\Big|_{z=2i} = \frac{2ie^{2i^2}}{(3i)(i)(4i)} = -\frac{1}{6e^2}$.

Now let L_R be the line on the real axis going from $-R$ to R and let Q_R be the defined by $z(\theta) = Re^{i\theta}$ for $0 \leq \theta \leq \pi$ (i.e. the boundary of the semicircle that is the upper half of the circle $|z| = R$ excluding the real axis part).

Let $C_R = L_R + Q_R$ (i.e. the semicircle boundary that is the upper half of the circle $|z| = R$ including the real axis part).

Clearly C_R is simple, closed, positively oriented, and if $R > 2$ then it contains the poles i and $2i$ of f but never the others.

So we know for $R > 2$:

$$\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz = 2\pi i \Big(\frac{1}{6e} - \frac{1}{6e^2}\Big) = i\frac{\pi(e-1)}{3e^2}$$

We also know:

$$\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz = \int_{L_R} \frac{ze^{ix}}{(z^2+1)(z^2+4)} dz + \int_{Q_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz$$

$$= \int_{-R}^{R} \frac{x \cos x}{(x^2+1)(x^2+4)} dx + i \int_{-R}^{R} \frac{x \sin x}{(x^2+1)(x^2+4)} dx + \int_{Q_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)} dz$$

So we have that:

$$\int_{-R}^{R} \frac{x \cos x}{(x^2+1)(x^2+4)} dx + i \int_{-R}^{R} \frac{x \sin x}{(x^2+1)(x^2+4)} dx = i\frac{\pi(e-1)}{3e^2} - \int_{Q_R} \frac{z \sin z}{(z^2+1)(z^2+4)} dz$$

We know $g(z) = \frac{z}{(z^2+1)(z^2+4)}$ is analytic in the upper half of the complex plane ($Im\ z \geq 0$) exterior to the circle $|z| = 2$.

Clearly the length of Q_R is πR. Now along Q_R we know that $|z| = R$ and so:

$|z^2 + 1| \geq ||z^2| - |1|| = |R^2 - 1|$ which for large enough R is just $R^2 - 1$.

$|z^2 + 4| \geq ||z^2| - |4|| = |R^2 - 4|$ which for large enough R is just $R^2 - 4$.

Then we have that $|\frac{z}{(z^2+1)(z^2+4)}| = \frac{|z|}{|z^2+1||z^2+4|} \leq \frac{R}{(R^2-1)(R^2-4)}$ for large enough R.

Since polynomials are continuous and the denominator of the below is nonzero:

$$\lim_{R\to\infty} \frac{R}{(R^2-1)(R^2-4)} = \lim_{R\to 0^+} \frac{\frac{1}{R}}{(\frac{1}{R^2}-1)(\frac{1}{R^2}-4)} = \lim_{R\to 0^+} \frac{R}{(1-R^2)(1-4R^2)} = \frac{0}{(1-(0)^2)(1-4(0)^2)} = 0$$

Therefore by Jordan's Lemma we know for all $a > 0$ (and hence for $a = 1$) that:

$$\lim_{R\to\infty} \int_{Q_R} \frac{ze^{iaz}}{(z^2+1)(z^2+4)} e^{iaz} dz = 0$$

Therefore we have that:

$$\lim_{R\to\infty}\int_{Q_R}\frac{ze^{iz}}{(z^2+1)(z^2+4)}e^{iaz}dz = 0$$

Which means:

$$\lim_{R\to\infty}\left(\int_{-R}^{R}\frac{x\ cos\ x}{(x^2+1)(x^2+4)}dx + i\int_{-R}^{R}\frac{x\ sin\ x}{(x^2+1)(x^2+4)}dx\right)$$

$$= \lim_{R\to\infty}\int_{-R}^{R}\frac{x\ cos\ x}{(x^2+1)(x^2+4)}dx + i\lim_{R\to\infty}\int_{-R}^{R}\frac{x\ sin\ x}{(x^2+1)(x^2+4)}dx$$

$$= i\frac{\pi(e-1)}{3e^2} - \lim_{R\to\infty}\int_{Q_R}\frac{z\ sin\ z}{(z^2+1)(z^2+4)}dz = i\frac{\pi(e-1)}{3e^2}$$

Which after taking the imaginary part of both sides we get:

$$Im\left(\lim_{R\to\infty}\int_{-R}^{R}\frac{x\ cos\ x}{(x^2+1)(x^2+4)}dx + i\lim_{R\to\infty}\int_{-R}^{R}\frac{x\ sin\ x}{(x^2+1)(x^2+4)}dx\right) = \lim_{R\to\infty}\int_{-R}^{R}\frac{x\ sin\ x}{(x^2+1)(x^2+4)}dx$$

$$= Im\left(i\frac{\pi(e-1)}{3e^2}\right) = \frac{\pi(e-1)}{3e^2}$$

Therefore we have:

$$P.V.\int_{-\infty}^{\infty}\frac{x\ sin\ x}{(x^2+1)(x^2+4)}dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{x\ sin\ x}{(x^2+1)(x^2+4)}dx = \frac{\pi(e-1)}{3e^2}$$