Residue Theory

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77.3

Let C be the positively oriented circle $|z| = 2$ and let $f(z) = \frac{4z-5}{z(z-1)}$.

Clearly f has isolated singular points at $z = 0$ and $z = 1$ and is analytic everywhere else in the finite plane.

Since these two isolated singular points are interior to the positively oriented simple closed contour C we know:

$$\int_C \frac{4z-5}{z(z-1)} = \int_C f(z)dz = 2\pi i\, Res_{z=0}\Big(\frac{1}{z^2}f(\frac{1}{z})\Big) = 2\pi i\, Res_{z=0}\Big((\frac{1}{z^2})(\frac{\frac{4}{z}-5}{\frac{1}{z}(\frac{1}{z}-1)})\Big) = 2\pi i\, Res_{z=0}\Big(\frac{4-5z}{z(1-z)}\Big)$$

Now recall that we know for $|z| < 1$ that:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

Therefore for $|z| < 1$ we have that:

$$\frac{1}{z^2}f(\frac{1}{z}) = \frac{4-5z}{z(1-z)} = (4-5z)\Big((\frac{1}{z})(\frac{1}{1-z})\Big) = (4-5z)\left(\frac{1}{z}\sum_{n=0}^{\infty} z^n\right) = (4-5z)\left(\sum_{n=0}^{\infty} z^{n-1}\right)$$

$$= 4\left(\sum_{n=0}^{\infty} z^{n-1}\right) - 5z\left(\sum_{n=0}^{\infty} z^{n-1}\right) = \left(4\sum_{n=0}^{\infty} z^{n-1}\right) - \left(5\sum_{n=0}^{\infty} z^n\right)$$

$$= \Big(\frac{4}{z} + 4 + 4z + 4z^2 + ...\Big) - \Big(5 + 5z + 5z^2 + ...\Big) = \frac{4}{z} - \Big(1 + z + z^2 + ...\Big) = \frac{4}{z} - \sum_{n=0}^{\infty} z^n$$

Therefore we know $Res_{z=0}\Big(\frac{1}{z^2}f\big(\frac{1}{z}\big)\Big) = Res_{z=0}\Big(\frac{4-5z}{z(1-z)}\Big) = 4$.

Finally we know:

$$\int_C \frac{4z-5}{z(z-1)} = 2\pi i\, Res_{z=0}\Big(\frac{4-5z}{z(1-z)}\Big) = 8\pi i$$

$\square$

77.7

Let $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$ where $a_n \neq 0$ and $Q(z) = b_0 + b_1 z + b_2 z^2 + ... + b_m z^m$ where $b_m \neq 0$.

Also let these polynomials be such that $m \geq n+2$ which also gives $n \leq m-2$ and $m-n-2 \geq 0$.

Let C be a simple closed contour such that all of the zeros of $Q(z)$ are interior to C, such a contour exists because $Q(z)$ only has finitely many zeros.

Now let $f(z) = \frac{P(z)}{Q(z)} = \frac{a_0+a_1z+a_2z^2+...+a_nz^n}{b_0+b_1z+b_2z^2+...+b_mz^m}$.

Clearly $f(z)$ has isolated singular points at the zeros of $Q(z)$ and is analytic everywhere on and outside of C.

Therefore we know:

$$\int_C f(z)dz = \int_C \frac{P(z)}{Q(z)}dz = 2\pi i Res_{z=0}\Big(\frac{1}{z^2}f(\frac{1}{z})\Big)$$

So we need to find this residue.

We know for any $z \neq 0$ such that $Q(\frac{1}{z}) \neq 0$ that:

$$\frac{1}{z^2}f\Big(\frac{1}{z}\Big) = \Big(\frac{1}{z^2}\Big)\Big(\frac{a_0 + a_1\frac{1}{z} + a_2\frac{1}{z^2} + ... + a_n\frac{1}{z^n}}{b_0 + b_1\frac{1}{z} + b_2\frac{1}{z^2} + ... + b_m\frac{1}{z^m}}\Big) = \frac{a_0z^{m-2} + a_1z^{m-3} + a_2z^{m-4} + ... + a_nz^{m-n-2}}{b_0z^m + b_1z^{m-1} + b_2z^{m-2} + ... + b_m}$$

$$= z^{m-n-2}\frac{a_0z^n + a_1z^{n-1} + a_2z^{n-2} + ... + a_n}{b_0z^m + b_1z^{m-1} + b_2z^{m-2} + ... + b_m}$$

Note that since $b_m \neq 0$ we know that at $z = 0$ the denominator of the above expression is nonzero.

Furthermore since the denominator is a polynomial of finite degree we know it only has finitely many zeros and so there exists a neighborhood of $z_0 = 0$ where the denominator has no zeros.

Therefore there exists a neighborhood of $z_0 = 0$ where $\frac{a_0z^n+a_1z^{n-1}+a_2z^{n-2}+...+a_n}{b_0z^m+b_1z^{m-1}+b_2z^{m-2}+...+b_m}$ is analytic.

Since $m-n-2 \geq 0$ we know z^{m-n-2} is also analytic in the same neighborhood of $z_0 = 0$ (its power is non negative).

Therefore we know $\frac{1}{z^2}f\Big(\frac{1}{z}\Big) = z^{m-n-2}\frac{a_0z^n+a_1z^{n-1}+a_2z^{n-2}+...+a_n}{b_0z^m+b_1z^{m-1}+b_2z^{m-2}+...+b_m}$ is analytic in the same neighborhood which means it has a Taylor series representation about $z_0 = 0$ for that neighborhood.

That is we can say in some neighborhood of $z_0 = 0$:

$$\frac{1}{z^2}f\Big(\frac{1}{z}\Big) = z^{m-n-2}\frac{a_0z^n + a_1z^{n-1} + a_2z^{n-2} + ... + a_n}{b_0z^m + b_1z^{m-1} + b_2z^{m-2} + ... + b_m} = \sum_{k=0}^{\infty} a_k z^k$$

In the above series representation there is no $\frac{1}{z}$ term and hence the residue is 0 there.

Therefore we know:

$$\int_C f(z)dz = \int_C \frac{P(z)}{Q(z)}dz = 2\pi i Res_{z=0}\Big(\frac{1}{z^2}f(\frac{1}{z})\Big) = 0$$

□

79.3

Let $f(z)$ be analytic at z_0, then we know $f(z)$ has a Taylor expansion about z_0.

So for some neighborhood of z_0 we may write:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

a. If $f(z_0) \neq 0$ then we know the constant term in the series is $a_0 = f(z_0) \neq 0$.

Therefore we know in some deleted neighborhood of z_0 we may write:

$$g(z) = \frac{f(z)}{z - z_0} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} a_n (z - z_0)^n = \frac{a_0}{z - z_0} + a_1 + a_2(z - z_0) + a_3(z - z_0)^2 + ... = \frac{a_0}{z - z_0} + \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n$$

Since $a_0 \neq 0$ we can let $m = 1$ then we have that the coefficients in the Laurent series for $g(z)$ about z_0 are such that $b_m \neq 0$ and $b_{m+1} = b_{m+2} = ... = 0$ so z_0 is a pole of order $m = 1$ for the function $g(z)$ which means it's a simple pole.

Clearly the coefficient on the $\frac{1}{z - z_0}$ term is $a_0 = f(z_0)$ so $Res_{z=z_0} g(z) = f(z_0)$ $\square$

b. If $f(z_0) = 0$ then we know the constant term in the series is $a_0 = f(z_0) = 0$.

Therefore we know in some deleted neighborhood of z_0 we may write:

$$g(z) = \frac{f(z)}{z - z_0} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} a_n (z - z_0)^n = \frac{a_0}{z - z_0} + a_1 + a_2(z - z_0) + a_3(z - z_0)^2 + ... = \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n$$

Since $a_0 = 0$ there are no negative powers of $z - z_0$ in this power series, so the power series so the power series itself is analytic in some neighborhood of z_0 (not just a deleted neighborhood but also at z_0).

Therefore in the Laurent series for $g(z)$ about z_0 we know every $b_n = 0$ so z_0 is a removable singular point of $g(z)$ and in order to remove the singular point we may redefine $g(z_0)$ to be the power series evaluated at z_0.

That is reassign $g(z_0)$ to be:

$$\left. \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n \right|_{z=z_0} = \sum_{n=0}^{\infty} a_{n+1} (z_0 - z_0)^n = \sum_{n=0}^{\infty} 0 = 0$$

So the newly defined function $g(z)$ below is analytic in some neighborhood of z_0:

$$g(z_0) = f(z_0) = 0, \quad g(z) = \frac{f(z)}{z - z_0} = \sum_{n=0}^{\infty} a_{n+1} (z - z_0)^n \quad \text{for } z \neq z_0$$

$\square$

81.2

Recall that $f(z)$ has a pole of order m at z_0 if $f(z)$ can be written as $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and nonzero at z_0, and in this case we know $Res_{z=z_0} f(z) = \frac{\phi^{(m)}(z_0)}{(m-1)!}$.

a. Let $f(z) = \frac{z^{1/4}}{z+1}$, clearly $\phi(z) = z^{1/4}$ is analytic and nonzero at $z_0 = -1$.

Therefore we know $f(z)$ may be written as $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $m = 1$, $z_0 = -1$, and $\phi(z) = z^{1/4}$.

So $f(z)$ has a simple pole ($m = 1$) at $z_0 = -1$ and $Res_{z=z_0} f(z) = \frac{\phi^{(m)}(z_0)}{(m-1)!} = \phi^{(0)}(z_0) = \phi(z_0)$.

Now we know $z^{1/4}$ is multi-valued and we can write:

$$(-1)^{1/4} = e^{\frac{1}{4}(ln|-1|+i\,arg(-1))} = e^{i\frac{arg(-1)}{4}}$$

Since we were given that we are taking $0 < arg\, z < 2\pi$ for $z^{1/4}$ we must take $arg(-1) = \pi$.

So for this branch we get $(-1)^{1/4} = e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$.

Therefore we have $Res_{z=-1} \frac{z^{1/4}}{z+1} = (-1)^{1/4} = \frac{1+i}{\sqrt{2}}$ □

b. Let $f(z) = \frac{Log\, z}{(z^2+1)^2} = \frac{Log\, z}{(z+i)^2(z-i)^2}$, clearly $\phi(z) = \frac{Log\, z}{(z+i)^2}$ is analytic and nonzero at $z_0 = i$.

Therefore we know $f(z)$ may be written as $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $m = 2$, $z_0 = i$, and $\phi(z) = \frac{Log\, z}{(z+i)^2}$.

So $f(z)$ has a pole of order 2 at $z_0 = i$ and $Res_{z=z_0} f(z) = \frac{\phi^{(m)}(z_0)}{(m-1)!} = \phi^{(1)}(z_0) = \phi'(z_0)$.

Now we know $\frac{d}{dz} Log\, z = \frac{1}{z}$, so:

$$\frac{d}{dz}\phi(z) = \frac{d}{dz}\frac{Log\, z}{(z+i)^2} = \frac{(\frac{1}{z})(z+i)^2 - 2(z+i)Log\, z}{(z+i)^4} = \frac{z+i-2zLog\, z}{z(z+i)^3}$$

Therefore $\phi'(i) = \frac{2i-2i\,Log(i)}{i(2i)^3} = \frac{2i-2i(ln|i|+i\,Arg(i))}{8} = \frac{2i+\pi}{8}$

Therefore we have $Res_{z=i} \frac{Log\, z}{(z^2+1)^2} = \phi'(i) = \frac{2i+\pi}{8}$ □

c. Let $f(z) = \frac{z^{1/2}}{(z^2+1)^2} = \frac{z^{1/2}}{(z+i)^2(z-i)^2}$, clearly $\phi(z) = \frac{z^{1/2}}{(z+i)^2}$ is analytic and nonzero at $z_0 = i$.

Therefore we know $f(z)$ may be written as $f(z) = \frac{\phi(z)}{(z-z_0)^m}$ where $m = 2$, $z_0 = i$, and $\phi(z) = \frac{z^{1/2}}{(z+i)^2}$.

So $f(z)$ has a pole of order 2 at $z_0 = i$ and $Res_{z=z_0} f(z) = \frac{\phi^{(m)}(z_0)}{(m-1)!} = \phi^{(1)}(z_0) = \phi'(z_0)$.

Now we know $\frac{d}{dz} z^{1/2} = \frac{1}{2z^{1/2}}$, so:

$$\frac{d}{dz}\phi(z) = \frac{d}{dz}\frac{z^{1/2}}{(z+i)^2} = \frac{(\frac{1}{2z^{1/2}})(z+i)^2 - 2(z+i)z^{1/2}}{(z+i)^4} = \frac{z+i-4z}{2z^{1/2}(z+i)^3} = \frac{i-3z}{2z^{1/2}(z+i)^3}$$

We were given $0 < arg\, z < 2\pi$ so for $z^{1/2}$ we must take $arg(i) = \frac{\pi}{2}$, then $(i)^{1/2} = e^{\frac{1}{2}(ln|i|+i\,arg(i))} = e^{i\frac{arg(i)}{2}} = e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}}$.

So for this branch we get:

$$\phi'(i) = \frac{-2i}{2(i)^{1/2}(2i)^3} = \frac{-1}{-8(\frac{1+i}{\sqrt{2}})} = \frac{1}{8}\Big(\frac{\frac{1-i}{\sqrt{2}}}{(\frac{1-i}{\sqrt{2}})(\frac{1+i}{\sqrt{2}})}\Big) = \frac{1-i}{8\sqrt{2}}$$

Therefore we have $Res_{z=i} \frac{z^{1/2}}{(z^2+1)^2} = \phi'(i) = \frac{1-i}{8\sqrt{2}}$ □

81.4

b. Let $f(z) = \frac{3z^3+2}{(z-1)(z^2+9)}$, clearly $f(z)$ has isolated singular points at $z_1 = 1$, $z_2 = 3i$, and $z_3 = -3i$.

Furthermore we know each of these isolated singular points is a simple pole ($m = 1$) since we can write $f(z) = \frac{3z^3+2}{(z-1)(z-3i)(z+3i)}$, so if we consider each point (z_1, z_2, z_3) one at a time we know we can write:

$$f(z) = \frac{\phi_k(z)}{z - z_k} \;\text{ where }\; k \in \{1, 2, 3\} \;\text{ and }\; \phi_k(z) \;\text{ is analytic and nonzero at }\; z_k$$

So at each of these simple poles ($m = 1$) we get $Res_{z=z_k} f(z) = \frac{\phi_k^{(m-1)}(z_k)}{(m-1)!} = \phi_k(z_k)$.

Let's actually find these residues:

For $z_1 = 1$ we know $\phi_1(z) = \frac{3z^3+2}{z^2+9}$ so $Res_{z=1} f(z) = Res_{z=z_1} f(z) = \phi_1(z_1) = \phi_1(1) = \frac{3+2}{10} = \frac{1}{2}$.

For $z_2 = 3i$ we know $\phi_2(z) = \frac{3z^3+2}{(z-1)(z+3i)}$ so

$Res_{z=3i} f(z) = Res_{z=z_2} f(z) = \phi_2(z_2) = \phi_2(3i) = \frac{3(3i)^3+2}{6i(3i-1)} = \frac{81i-2}{18+6i} = \frac{(81i-2)(18-6i)}{(18-6i)(18+6i)} = \frac{30(15+49i)}{360} = \frac{1}{12}(15 + 49i)$.

For $z_3 = -3i$ we know $\phi_3(z) = \frac{3z^3+2}{(z-1)(z-3i)}$ so

$Res_{z=-3i} f(z) = Res_{z=z_3} f(z) = \phi_3(z_3) = \phi_3(-3i) = \frac{3(-3i)^3+2}{-6i(-3i-1)} = \frac{81i+2}{6i-18} = \frac{(81i+2)(-18-6i)}{(-18-6i)(-18+6i)} = \frac{30(15-49i)}{360} = \frac{1}{12}(15 - 49i)$.

Now let C be the counterclockwise oriented circle $|z| = 4$, clearly $f(z)$ is analytic inside and on C except at each of these singular points which are interior to C.

Therefore we know:

$$\int_C f(z)dz = \int_C \frac{3z^3+2}{(z-1)(z^2+9)} = 2\pi i \sum_{k=1}^{3} Res_{z=z_k} f(z) = 2\pi i\Big(\frac{1}{2} + \frac{1}{12}(15+49i) + \frac{1}{12}(15-49i)\Big)$$

$$= 2\pi i\Big(\frac{1}{2} + \frac{30}{12}\Big) = \pi i\Big(1 + 5\Big) = 6\pi i$$

$\square$

Problem 2

Let f, g be analytic functions on a bounded domain D where $f = g$ at infinitely many points in a closed set $S \subset D$.

Now define the function $h(z) = f(z) - g(z)$.

We know h is analytic in D and that $h(z) = 0$ at infinitely many points in the closed set $S \subset D$.

From these infinitely many points where $h(z) = 0$ in S define a sequence (z_n), this need not contain every such zero of h and in fact can't if there are unaccountably many but such a sequence does exist.

Clearly the sequence is bounded since it is contained within S which is a subset of D which is bounded.

Therefore the real sequence $(Re\ z_n)$ is bounded.

Then by the Bolzano Weierstrass theorem (in the real case) there exists a convergent subsequence $(Re\ z_{n_k})$ of $(Re\ z_n)$.

Now consider the real sequence $(Im\ z_{n_k})$ where all these n_k are the same as in the above sequence $(Re\ z_{n_k})$.

Again clearly $(Im\ z_{n_k})$ is bounded.

So by the Bolzano Weierstrass theorem (in the real case) there exists a convergent subsequence $(Im\ z_{n_p})$ of $(Im\ z_{n_k})$.

Now we consider the real sequence $(Re\ z_{n_p})$ where all these n_p are the same as in the above sequence $(Im\ z_{n_p})$.

Since $(Re\ z_{n_p})$ is a subsequence of the convergent sequence $(Re\ z_{n_k})$ we know that $(Re\ z_{n_p})$ converges.

Therefore the sequence $(z_{n_p}) = (Re\ z_{n_k} + i\ Im\ z_{n_p})$ converges to say z_0.

Since S is closed it contains all its limit points and we know $z_0 \in S$.

Consider the sequence $(h(z_{n_p})) = (f(z_{n_p}) - g(z_{n_p}))$. By construction $(h(z_{n_p})) = (0, 0, 0, ...)$ which clearly converges to 0.

Since $h(z) = f(z) - g(z)$ is analytic and hence continuous in D and consequently also in S we know:

$$\text{The limit } \lim_{z \to z_0} h(z) \text{ exists and it must be that } \lim_{z \to z_0} h(z) = h(z_0)$$

Since we already know the sequences $(z_{n_p}) \to z_0$ and $(h(z_{n_p})) = (0, 0, 0, ...) \to 0$ it must be that:

$$\lim_{z \to z_0} h(z) = 0 = h(z_0)$$

Since there are points of the sequence (z_{n_p}) in every neighborhood of z_0 (by the definition of convergence) we know that there are zeros of h in every neighborhood of z_0.

Therefore z_0 is a point such that $h(z_0) = 0$, h is analytic at z_0, and there does not exist a deleted neighborhood of z_0 where $h(z) \neq 0$, so there must exist a neighborhood of z_0 where $h(z) = 0$ identically (otherwise there would exist a deleted neighborhood of z_0 where $h(z) \neq 0$).

Then we know this neighborhood is a domain containing z_0, $h(z_0) = 0$, and h is analytic in a neighborhood of z_0.

Therefore we know that $h(z) = f(z) - g(z) = 0$ identically throughout D, and hence $f(z) = g(z)$ throughout D $\square$