Probability distributions: Assignments

EBP038A05

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general information

Here we just provide the exercises of the assignments. For information with respect to grading we refer to the course manual.

1. For each assigment you have to turn in a pdf document typeset in LaTeX. Include a title, group number, student names and ids, and date.
2. When you have to turn in a graph, provide decent labels and a legend, ensure the axes have labels too.
3. Whenever you have to program or simulate something, include your code.

1 assignment 1

1.1 *Have you read well?*

Here are a bunch of questions to help you read well. The questions are simple checks, not at exam level, but lower.

Ex 1.1. In your own words, explain what is

1. a joint PMF, PDF, CDF;
2. a conditional PMF, PDF, CDF;
3. a marginal PMF, PDF, CDF;

Ex 1.2. We have two r.v.s X and Y, both $\sim \mathrm{U}([0,1])$, and $Y > X$.

1. Why is the joint $I_{x \leq y}$?
2. Are X and Y independent?
3. Compute $F_{X,Y}(x, y)$.

1.2 *Exercises at about exam level*

Ex 1.3. Let $X \sim \mathrm{Exp}(4)$.

1. Find a simple expression for $\mathsf{P}\{1 < X \leq 4\}$ and compute the value.
2. Use simulation to check this value. Include your code and explain any difficult point (if there is any).
3. Use numerical integration to compute $\mathsf{P}\{1 < X \leq 4\}$. In python it looks like this:

```python
import numpy as np
from scipy.integrate import quad

labda = 3


def integrand(x):
    return np.exp(-labda * x)


I = quad(integrand, 1, 4)
print(I)
```

Ex 1.4. We throw an unbiased die with six sides; the result of the ith throw is X_i.

1. What is the sample space of the two throws (X_1, X_2)?
2. What is the joint CDF?
3. What is the joint PMF?
4. Marginalize out X_2 to show that $\mathsf{P}\{X_1 = 5\} = 1/6$.
5. Use the fundamental bridge and indicators to compute $\mathsf{P}\{X_1 > X_2\}$.
6. Use the fundamental bridge and indicators to compute $\mathsf{P}\{|X_1 - X_2| < 1\} = 1/6$.
7. Use the fundamental bridge and indicators to compute $\mathsf{P}\{|X_1 - X_2| \leq 1\}$.
8. Use simulation to estimate $\mathsf{P}\{|X_1 - X_2| \leq 1\}$.

Ex 1.5. We select a random married couple (a man and a woman). His height is $X \sim \mathrm{N}(1.8, 0.1)$, her height is $Y \sim \mathrm{N}(1.7, 0.08)$ in meters.

1. What is the sample space of (X, Y)?
2. What is the joint CDF?
3. What is the joint PDF?
4. Marginalize out Y to show that $X \sim \mathbb{N}1.8, 0.1$.
5. Use the fundamental bridge and indicators to write $\mathsf{P}\{X > Y\}$ as an integral.
6. Use the fundamental bridge and indicators to write $\mathsf{P}\{|X - Y| < 0.1\}$ as an integral.
7. Use a numerical integrator to compute the probability of the previous part.
8. Use simulation to estimate the probability of the previous part.

Ex 1.6. I select a random guy from the street, his height $X \sim \mathbb{N}1.8, 0.1$, and I select a random woman from the street, her height is $Y \sim \mathbb{N}1.7, 0.08$. I claim that since I selected the man and the woman independently, their heights are independent. Comment on this claim: Is it true, why, if it false, why, and what information do you have to add to the model to make the claim true?

Ex 1.7. Take $X \in \{-2, -1, 1, 2\}$ and $\eta = X^2$. What is the correlation coefficient of X and η? Comment on your answer. **Is X uniformly distributed on $\{-2, -1, 1, 2\}$? Or should an arbitrary distribution with support $\{-2, -1, 1, 2\}$ be considered?**

Ex 1.8. This is about the simplest model for an insurance company that I can think of. We start with an initial capital $I_0 = 2$. The company receives claims and contributions every period, a week say. In the ith period, we receive a contribution X_i uniform on the set $\{1, 2, \dots, 10\}$ and a claim C_i uniform on $\{0, 1, \dots 8\}$.

1. What is the meaning of $I_1 = I_0 + X_1 - C_1$?
2. What is the meaning of $I_2 = I_1 + X_2 - C_2$?
3. Provide an interpretation in terms of the inventory of rice, say, at a supermarket for I_1 and I_1'.
4. Provide also an interpretation in terms of a degradation and repair process of an item. (if you find this difficult, search a bit on the web on reliability theory.) Comment on how good you think this model is to analyze such degradation and repair processes.
5. What is the interpretation of $I_1' = \max\{\{\}I_0 - C_1, 0\} + X_1$?
6. What is the interpretation of $I_2' = \max\{\{\}I_1' - C_2, 0\} + X_2$?
7. What is $\mathsf{P}\{I_1 < 0\}$?
8. What is $\mathsf{P}\{I_1' < 0\}$?
9. What is $\mathsf{P}\{I_2 < 0\}$?
10. What is $\mathsf{P}\{I_2' < 0\}$?
11. Use simulation to estimate $\mathsf{P}\{I_{10} < 10\}$.
12. What is the interpretation of $\bar{I}_n = \min\{I_i : 0 \leq i \leq n\}$?
13. Use simulation to estimate $\mathsf{P}\{\bar{I}_{10} \geq 0\}$.
14. Similar, use simulation to estimate $\mathsf{P}\{\bar{I}_{10}' \geq 0\}$.
15. Explain the difference outcomes between the previous two parts.

Ex 1.9. We have a machine that consists of two components. The machine works as long as not both components have failed. Let X_i be the lifetime of component i.

1. What is the interpretation of $\min\{X_1, X_2\}$?
2. What is the interpretation of $\max\{X_1, X_2\}$?
3. If X_1, X_2 iid $\sim \text{Exp}(10)$ (in hours), what is the expected time until the machine fails?
4. If X_1, X_2 iid $\sim \text{Exp}(10)$ (in hours), what is the probability that the machine is still 'up' (i.e., not failed) at time $T = 50$?
5. If $X_1 \sim \text{Exp}(10)$, $X_2 \sim \text{N}(20, 3)$, use simulation to estimate the expected time until the machine fails.
6. If $X_1 \sim \text{Exp}(10)$, $X_2 \sim \text{N}(20, 3)$, use simulation to determine the probability that the machine is still 'up' (i.e., not failed) at time $T = 50$.

Ex 1.10. Let X be the result of the throw of a coin. It is given that $\mathsf{P}\{X = H\} = p = 1 - \mathsf{P}\{X = T\}$. When $X = H$, we choose a fair die with 4 sides with values $1, 2, 3, 4$, when $X = T$ we choose a fair die with 6 sides with values $1, \dots, 6$. Let Y_i be the value of the ith throw with the die.

1. What is the PMF of X and Y_1?

2. Marginalize the answer of part a to show that $\mathsf{P}\{X = H\} = p$.
3. What is $\mathsf{P}\{Y_1 = 1\}$?
4. What is $\mathsf{P}\{X = 1 \,|\, Y_1 = 1\}$?
5. What is $\mathsf{P}\{X = 1 \,|\, Y_1 = 1, Y_2 = 2\}$?
6. What is $\mathsf{P}\{X = 1 \,|\, Y_1 = Y_2 = \cdots = Y_n = 1\}$?

Ex 1.11. Derive the results of BH 7.3.6 without smart tricks. Thus, you have to use the fundamental bridge to show that

$$\mathsf{E}[ML] = \mathsf{E}[X]\,\mathsf{E}[Y] = 1, \qquad \mathsf{E}[M] = 3/2, \qquad \mathsf{E}[L] = 1/2,$$

$$\mathsf{E}\left[L^2\right] = 1/4, \qquad \mathsf{E}\left[M^2\right] = 2\,\mathsf{E}\left[X^2\right] - \mathsf{E}\left[L^2\right] = 7/2$$

$$\mathsf{V}[M] = \mathsf{E}\left[M^2\right] - (\mathsf{E}[M])^2, \quad \mathsf{V}[L] = \mathsf{E}\left[L^2\right] - (\mathsf{E}[L])^2.$$

Ex 1.12. Check first BH 7.2.3. If X, Y iid $\sim \mathrm{P}(\lambda)$, then prove first that $X + Y \sim \mathrm{P}(2\lambda)$, but $X - Y$ is not $\sim \mathrm{P}(0)$. But when X, Y iid $\sim \mathrm{N}(0, 1)$, then $X - Y \sim \mathrm{N}(0, 2)$. Explain this difference between the Poisson and normal distribution.

Ex 1.13. Use simulation to estimate the answer of exercise 7.1 of BW. Here is a way to do it in python. Port this to R and explain how your code works. Then compare the value of the simulation to the exact value. Include the integrals in your answer to show how you found the exact answer.

```
import numpy as np

np.random.seed(3)

num = 1000

a = np.random.uniform(size=num)
b = np.random.uniform(size=num)
success = np.abs(a - b) ¡ 0.25
print(success.mean(), success.var())
```

If you like, you can include a plot of the region (in time) in which Alice and Bob meet, and put marks on the points of the simulation that were 'successful'.

1.3 *Exam level, TBD*

Ex 1.14. Assume that X has the Cauchy distribution.

1. Does $\mathsf{E}\left[\frac{X}{X^2+1}\right]$ exist? If so, find its value.
2. Does $\mathsf{E}\left[\frac{|X|}{X^2+1}\right]$ exist? If so, find its value.

1.4 *Above exam level, TBD*

Ex 1.15. Consider the again the chicken-egg story (BH 7.1.9): A chicken lays a random number of eggs N an each egg independently hatches with probability p and fails to hatch with probability $q = 1 - p$. Let X be the number of eggs that hatch and let Y be the number of eggs that do not hatch, so $X + Y = N$. For $N \sim \mathrm{P}(\lambda)$ it is shown in BH 7.1.9 that X and Y are independent. This exercise asks for the converse. Assume that X and Y are independent. Prove that there exists a $\lambda > 0$ such that $N \sim \mathrm{P}(\lambda)$.