

## SOLUTION FOR HOMEWORK ASSIGNMENT NO. 02

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### Exercise 2.1

We are given a function  $f$  which is defined as

$$f(x) = \begin{cases} 0 & \text{for } a \leq x \leq b \\ \frac{2(x-a)}{(b-a)(c-a)} & \text{for } a \leq x < c \\ \frac{2(b-x)}{(b-a)(b-c)} & \text{for } c \leq x < b \end{cases} \quad (1)$$

and are asked to calculate its mean, mode, median and variance. We will consider the case where  $c = 0$ .

$$\text{Mean } \mu: \mu = E[x] = \int_{-\infty}^{\infty} dx \cdot x f(x)$$

$$\text{Median } M: \frac{1}{2} = \int_{-\infty}^M dx \cdot f(x)$$

$$\text{Mode } m: \frac{\partial f}{\partial x}(x)|_{x=m} = 0$$

$$\text{Varianz } \sigma^2: \sigma^2 = E[(x - E[x])^2] = \int_{-\infty}^{\infty} dx \cdot (x - \mu)^2 f(x)$$

a) For  $a = -b$  the function  $f(x)$  is defined as

$$f(x) = \begin{cases} 0 & \text{for } x \leq -b \text{ and } x \geq b \\ \frac{x+b}{b^2} & \text{for } -b \leq x \leq 0 \\ \frac{b-x}{b^2} & \text{for } 0 \leq x \leq b \end{cases}$$

**Mean:** Since  $f(x)$  and  $x$  are odd functions their product is an even function. Since we integrate over a symmetric interval the mean is zero.

**Median:** Because of symmetry we know that  $M = 0$ .

**Mode:** Again, due to symmetry and because of the fact that the function is rising for  $x \leq 0$  and falling for  $x \geq 0$ , the mode is zero.

**Varianz:**

$$\begin{aligned}\sigma^2 &= \int_{-\infty}^{\infty} dx \cdot (x - \mu)^2 f(x) = \int_{-b}^0 dx \cdot x^2 \frac{x+b}{b^2} + \int_0^b dx \cdot x^2 \frac{b-x}{b^2} \\ &= \frac{1}{b^2} \left[ -\frac{b^4}{4} + \frac{b^4}{3} + \frac{b^4}{3} - \frac{b^4}{4} \right] = \frac{b^2}{6}\end{aligned}$$

**b)** For  $a = -2b$  the function  $f(x)$  is defined as

$$f(x) = \begin{cases} 0 & \text{for } x \leq -2b \text{ and } x \geq b \\ \frac{x+b}{3b^2} & \text{for } -2b \leq x \leq 0 \\ \frac{2}{3} \frac{b-x}{b^2} & \text{for } 0 \leq x \leq b \end{cases}$$

**Mean:**

$$\begin{aligned}\mu &= \int_{-\infty}^{\infty} dx \cdot x f(x) = \int_{-2b}^0 dx \cdot x \frac{x+b}{3b^2} + \int_0^b dx \cdot x \frac{2}{3} \frac{b-x}{b^2} \\ &= \frac{1}{3b^2} \left[ \frac{6b^3}{3} - 4b^3 + \frac{b^3}{2} - \frac{b^3}{3} \right] = -\frac{11b}{18}\end{aligned}$$

**Median:** I suspect that the median lies between  $x = -2b$  and  $x = 0$ .

$$\begin{aligned}\frac{1}{2} &= \int_{-\infty}^M dx \cdot f(x) = \int_{-2b}^M dx \cdot \frac{x+b}{3b^2} \\ &= \frac{1}{3b^2} \left[ \frac{M^2}{2} + Mb \right] \\ \Rightarrow M_{1,2} &= -3b, b\end{aligned}$$

**Mode:** Same argument as before: Since  $f_1$  (function defined for  $x \leq 0$ ) is rising and  $f_2$  is falling the mode lies at zero.