SOLUTION FOR HOMEWORK ASSIGNMENT NO. 02

Nils Hoyer

Exercise 2.1

We are given a function f which is defined as

$$f(x) = \begin{cases} 0 & \text{for } a \leq x \leq b \\ \frac{2(x-a)}{(b-a)(c-a)} & \text{for } a \leq x < c \\ \frac{2(b-x)}{(b-a)(b-c)} & \text{for } c \leq x < b \end{cases} \tag{1}$$

and are asked to calculate its mean, mode, median and variance. We will consider the case where $c = 0$.

Mean μ: $\mu = E[x] = \int\limits_{-\infty}^{\infty} dx \cdot xf(x)$
Median M: $\frac{1}{2} = \int\limits_{-\infty}^{M} dx \cdot f(x)$
Mode m: $\frac{\partial f}{\partial x}(x)|_{x=m} = 0$
Varianz σ^2: $\sigma^2 = E[(x - E[x])^2] = \int\limits_{-\infty}^{\infty} dx \cdot (x-\mu)^2 f(x)$

a) For $a = -b$ the function $f(x)$ is defined as

$$f(x) = \begin{cases} 0 & \text{for } x \leq -b \text{ and } x \geq b \\ \frac{x+b}{b^2} & \text{for } -b \leq x \leq 0 \\ \frac{b-x}{b^2} & \text{for } 0 \leq x \leq b \end{cases}$$

Mean: Since $f(x)$ and x are odd functions their product is an even function. Since we integrate over a symmetric interval the mean is zero.

Median: Because of symmerty we know that $M = 0$.

Mode: Again, due to symmetry and because of the fact that the function is rising for $x \leq 0$ and falling for $x \geq 0$, the mode is zero.

Varianz:

$$\sigma^2 = \int\limits_{-\infty}^{\infty} dx \cdot (x-\mu)^2 f(x) = \int\limits_{-b}^{0} dx \cdot x^2 \frac{x+b}{b^2} + \int\limits_{0}^{b} dx \cdot x^2 \frac{b-x}{b^2}$$
$$= \frac{1}{b^2}\left[-\frac{b^4}{4} + \frac{b^4}{3} + \frac{b^4}{3} - \frac{b^4}{4}\right] = \frac{b^2}{6}$$

b) For $a = -2b$ the function $f(x)$ is defined as

$$f(x) = \begin{cases} 0 & \text{for } x \leq -2b \text{ and } x \geq b \\ \frac{x+b}{3b^2} & \text{for } -2b \leq x \leq 0 \\ \frac{2}{3}\frac{b-x}{b^2} & \text{for } 0 \leq x \leq b \end{cases}$$

Mean:

$$\mu = \int\limits_{-\infty}^{\infty} dx \cdot x f(x) = \int\limits_{-2b}^{0} dx \cdot x \frac{x+2b}{3b^2} + \int\limits_{0}^{b} dx \cdot x \frac{2}{3}\frac{b-x}{b^2}$$
$$= \frac{1}{3b^2}\left[\frac{6b^3}{3} - 4b^3 + \frac{b^3}{2} - \frac{b^3}{3}\right] = -\frac{17}{18b}$$

Median: I suspect that the median lies between $x = -2b$ and $x = 0$.

$$\frac{1}{2} = \int\limits_{-\infty}^{M} dx \cdot f(x) = \int\limits_{-2b}^{M} dx \cdot \frac{x+b}{3b^2}$$
$$= \frac{1}{3b^2}\left[\frac{M^2}{2} + Mb\right]$$
$$\Rightarrow M_{1,2} = -3b, b$$

Mode: Same argument as before: Since f_1 (function defined for $x \leq 0$) is rising and f_2 is falling the mode lies at zero.

Varianz:

$$\sigma^2 = \int\limits_{-\infty}^{\infty} dx \cdot (x-\mu)^2 f(x) = \int\limits_{-2b}^{0} dx \cdot (x-\mu)^2 \frac{x+b}{3b^2} + \int\limits_{0}^{b} dx \cdot (x-\mu)^2 \frac{2(b-x)}{3b^2}$$
$$= \dots$$
$$= \frac{1}{3b^2}\left[-\frac{5b^4}{6} + \frac{154b^3}{27} - \frac{238b^2}{27} + \left(\frac{34}{18}\right)^2\right]$$