SOLUTION FOR HOMEWORK ASSIGNMENT NO. 10

Nils Hoyer, Maurice Morgenthaler

Exercise 10.1

In this exercise we are asked to determine the equally tailed intervals for the $68.0\,\%$, $95.0\,\%$ and $99.7\,\%$ confidence levels for the position $y = 150\,\mu\mathrm{m}$ with a Gaussian uncertainty of $\sigma = 10\,\mu\mathrm{m}$ and a sum of two Gaussian uncertainties where $\sigma_1 = 10\,\mu\mathrm{m}$ with a total contribution of $90\,\%$ and $\sigma_2 = 200\,\mu\mathrm{m}$ and a total contribution of $90\,\%$.

To do this we will calculate

$$\alpha = \int\limits_{x_{\alpha\,,\,1}}^{x_{\alpha\,,\,2}} \mathrm{d}x \cdot f_\mu(x) \tag{1}$$

where α is the confidence level and $x_{\alpha\,,\,1/2}$ the range in x corresponding to this level.
In this exercise we use two Gaussian distributions which are symmetric with respect to their mean μ. This is why we calculate

$$\frac{\alpha}{2} = \int\limits_{\mu}^{x_\alpha} \mathrm{d}x \cdot f_\mu(x). \tag{2}$$

a) We are given a single Gaussian with $\sigma = 10\,\mu\mathrm{m}$ and $\mu = 150\,\mu\mathrm{m}$.

$$\begin{aligned}
\alpha &= 2 \cdot \int\limits_{\mu}^{x_\alpha} \mathrm{d}x \cdot \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \\
&= 2\frac{1}{\sqrt{2\pi}\sigma} \cdot \sqrt{\frac{\pi}{2}}\sigma \cdot \mathrm{Erf}\left(\frac{x_\alpha - \mu}{\sqrt{2}\sigma}\right) \\
&= \mathrm{Erf}\left(\frac{x_\alpha - \mu}{\sqrt{2}\sigma}\right) \\
\to x_\alpha &= \mu + \sqrt{2}\sigma \cdot \mathrm{Erf}^{-1}(\alpha)
\end{aligned} \tag{3}$$

Here Erf^{-1} denotes the inverse error function.

For $\alpha = 0.680$: $x_\alpha \in [140.055, 159.945]$.
For $\alpha = 0.950$: $x_\alpha \in [130.400, 169.599]$.
For $\alpha = 0.997$: $x_\alpha \in [120.322, 179.677]$.

b) As we are given a sum of two Gaussians we can reuse the calculation from a). The normed distribution is

$$f_\mu(x) = \frac{1}{\sqrt{2\pi}\,[a_1\sigma_1 + a_2\sigma_2]} \cdot \left(a_1 \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma_1}\right\} + a_2 \cdot \exp\left\{-\frac{(x-\mu)^2}{2\sigma_2}\right\}\right) \tag{4}$$

where $a_{1/2}$ indicates the strength of the contribution of that function. Following the calculation given in a) we eventually get to

$$\alpha = a_1 \cdot \mathrm{Erf}\left(\frac{x_\alpha - \mu}{\sqrt{2}\sigma_1}\right) + a_2 \cdot \mathrm{Erf}\left(\frac{x_\alpha - \mu}{\sqrt{2}\sigma_2}\right). \tag{5}$$

For $\alpha = 0.680$: $x_\alpha \in [138.485, 161.515]$.
For $\alpha = 0.950$: $x_\alpha \in [15.103, 284.897]$.
For $\alpha = 0.997$: $x_\alpha \in [-284.018$[1]$, 584.018]$.

Exercise 10.2

We were given a note on how the XENON collaboration uses a likelihood base method to set a limit on the Dark Matter - Standard Model Nuclei interaction. The following questions had to be answered.

i. Explain in a few words how the Profile Likelihood Ratio statistical approach works.

As mentioned below in equation 10 and 11 the profile likelihood method is based on the ratio between the maximized likelihood function with the relevant property σ fixed and the maximized likelihood function in case every variable is free. By definition $q_\sigma \geq 0$. It equals zero if $\hat{\sigma} = \sigma$ indicating a compatibility with a signal hypothesis. The test statistic $q_\sigma \geq 0$ is used for the p-value

$$p_s = \int_{q_\sigma^{obs}}^{\infty} f(q_\sigma | H_\sigma) dq_\sigma \tag{6}$$

which is the probability how likely it is to get less-signal like results from further experiments, in case there is indeed a signal with cross-section σ. With other words if p_s is high it is likely that the observed σ is not just a fluctuation. In case it is low there are most likely better σ to describe the data and by that the hypothesis H_σ can be rejected with $1 - p_s [\%]$ confidence. To compensate for downward fluctuations of the background p_s was further modified to

$$p'_s = \frac{p_s}{1 - p_b} \tag{7}$$

$$1 - p_b = \int_{q_\sigma^{obs}}^{\infty} f(q_\sigma | H_0) dq_\sigma \tag{8}$$

Calculating the cross-section σ for different hypothetical WIMP masses to get exactly $p'_s = 0.90$ (upper limit search) and finding the minimum of the resulting plot gave the desired limit.

ii. Why is this method advantageous compared to a “hard cut in the space parameter of interest” method? How much better is the limit with the likelihood method?

Methods which use the ”hard cut” firstly have the drawback that they are only able to give an upper limit without being able to claim a detection directly. Second, they don’t take systematic uncertainties into account. Both disadvantages are not present in the presented method. In the range of high masses where the older methods where used the likelihood ratio method gives a $20\% - 30\%$ better limit.

iii. Why is it important to model correctly the background and how is this done in XENON100?

Modelling the background precisely is of great importance as the cross-section of WIMPs is set to be lower that 10^{-43} in specific energy ranges given in figure 6 of the paper. As a result of using a wrongfully modelled background one risks of claiming the detection of WIMPs despite it only being recoils of β-radiation.

iv. Which ones are the nuisance parameters and how are they included in the full likelihood? There are five nuisance parameters which were introduced in the model.

- expected numbers of background events N_b
- the probabilities $\epsilon_s = \{\epsilon_s^j\}$ and $\epsilon_b = \left\{\epsilon_b^j\right\}$
- relative scintillation efficiency $\mathcal{L}_{eff}$
- escape velocity v_{esc}

[1] In this case a negative value is fine as we are talking about positions of particles.

The likelihood $\mathfrak{L}$ has the form

$$\mathfrak{L} = \mathfrak{L}_1\{\sigma, N_b, \epsilon_s, \epsilon_b, \mathcal{L}_{eff}, v_{esc}; m_\chi\} \times \mathfrak{L}_2\{\epsilon_s\} \times \mathfrak{L}_3\{\epsilon_b\} \times \mathfrak{L}_4\{\mathcal{L}_{eff}\} \times \mathfrak{L}_5\{v_{esc}\} \tag{9}$$

Therefore the nuisance parameters are included via the main likelihood $\mathfrak{L}_1$ as well as the subsidiary likelihoods $\mathfrak{L}_2$ to $\mathfrak{L}_5$.

v. Which are the test statistics for the exclusion and the discovery case?

In both cases the profile likelihood $\lambda(\sigma)$ is used.

$$\lambda(\sigma) = \frac{\max_{\sigma fixed} \mathfrak{L}\{\sigma; \mathcal{L}_{eff}, v_{esc}, N_b, \epsilon_s, \epsilon_b\}}{\max \mathfrak{L}\{\sigma; \mathcal{L}_{eff}, v_{esc}, N_b, \epsilon_s, \epsilon_b\}} \tag{10}$$

For the exclusion case the test statistic q_σ is defined as

$$q_\sigma = \begin{cases} -2\ln\lambda(\sigma) & \hat{\sigma} < \sigma \\ 0 & \hat{\sigma} > \sigma \end{cases} \tag{11}$$

As $0 \leq \lambda(\sigma) \leq 1$ the test statistics q_σ will be $q_\sigma \geq 0$ in which a smaller value indicates a better compatibility between data and signal hypothesis.

For the discovery case the test statistic q_0 is defined as

$$q_0 = \begin{cases} -2\ln\lambda(\sigma = 0) & \hat{\sigma} > \sigma \\ 0 & \hat{\sigma} < \sigma \end{cases} \tag{12}$$

In this method one tries to reject the background-only hypothesis.

vi. What is the difference between the exclusion sensitivity and the profiled limit? What is the observed limit in this analysis?

The exclusion sensitivity is the expected limit for given conditions concerning the experiment. It can be estimated before using actual data as the median of the test statistic distribution $f(q_\sigma|H_0)$ with background-only data simulated according to a Poisson distribution. The profiled limit on the other hand is calculated with the profile likelihood ratio method. The limit of this analysis is $\sigma^{up} < 2.4 \times 10^{-44}\mathrm{cm}^2$ for WIMP's with a mass of about $m_\chi = 50\mathrm{GeV/c}^2$