Bode plot analysis for RC & RLC circuits as low-pass & high-pass filters

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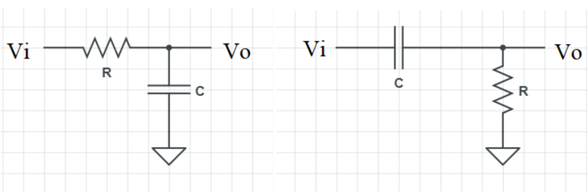
***Abstract*— The behavior of RC circuits and RLC circuits as low-pass and high-pass filters was examined by analyzing their simulated bode plots. We showed that low pass filters rejects frequencies higher than the cutoff for 1st order filters, or the natural frequency for 2nd order filters, while high-pass filters passes them through. We also showed that the gain of low-pass and high-pass 2nd order filters depends on the damping ratio at frequencies near their natural frequency.**

I. INTRODUCTION

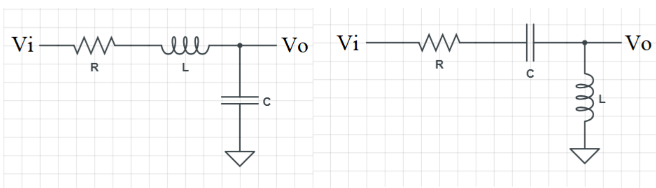
A circuit that is designed to block or to pass certain frequency ranges is called a filter. A filter modifies the relative amplitude and/or the phase of the input signal with reference to the frequency. Since filter response varies with frequency, the filter characteristics are commonly illustrated by bode plots and mathematically described by a transfer function. A bode plot shows the magnitude or the phase response of the transfer function for a range of frequencies. This transfer function or gain is the ratio of the laplace transforms of the output signal and the input signal. Getting the magnitude of the transfer function and plotting it with respect to frequency gives the the amplitude response of the filter. Similarly, the plot of the transfer function argument versus the frequency gives the phase response of the filter [1].

There are 5 basic types of a filter, namely low-pass, high-pass, band-pass, band-stop, and notch, wherein the number of active circuit components defines the filter order. The filter order is the order of filter's transfer function. It is known that a filter with a higher order attenuates more rapidly than a similar filter with a lower order, resulting to a steeper roll-off slope [1]. In this paper, our objective was to simulate the amplitude and phase responses of a 1st order and a 2nd order high-pass and low-pass filters for behavioral analysis. Figure 1 shows the 1st order high-pass and low-pass filters designed. Figure 2 shows the 2nd order high-pass and low-pass filters designed, where both circuits form a harmonic oscillator with damping oscillation.



(a) RC Low Pass Filter (b) RC High Pass Filter

Fig. 1. Circuit design of (a) RC low-pass filter, and (b) RC high-pass filter



(a) Low Pass RLC Circuit (b) High Pass RLC

Fig. 2. Circuit design of (a) RLC low-pass filter and (b) RLC high-pass filter

Let V_i and V_o be the complex amplitude of the input and output voltage, and let i be the amplitude of the current along the circuit. Considering Ohm's law and Kirchoff's law, the following equations hold for the RC low-pass filter:

$$V_o = V_C \qquad V_i = V_R + V_C$$
$$V_C = \frac{i}{j\omega C} \qquad V_R = iR$$

where R and C represent the ohmic resistance and the capacitance. The same equations hold for the RC high-pass filter except for $V_o = V_C$ which should be $V_o = V_R$.

For the RLC low-pass filter, the following equations hold:

$$V_o = V_C \qquad V_i = V_R + V_C + V_L$$
$$V_C = \frac{i}{j\omega C} \qquad V_R = iR \qquad V_L = ji\omega l$$

where L is inductance. These equations also hold for the RLC high-pass filter except for $V_o = V_C$ which should be $V_o = V_L$.

II. ALGORITHM

In sympy, variables are not defined automatically, hence, the variables in the circuit equations were first defined by symbols(). We then assigned the equations of every circuit arrangement to an equation name. Note that expressions not in an Eq() is equated to 0, thus, we equated all circuit equations to 0. Solve() was used to solve for the equations. This function returns a dictionary which consists of pairs of the variable we were solving for, the <key>, and its associated equation, the <value>. We then accessed the equation for key vo. This equation was simplified and the resulting equation was divided to vi, to get the corresponding transfer function.

To simulate the bode plots for the 1st order filters, the symbols c and r were first set to the desired C and R values using a dictionary. They were substituted to the transfer function using eq.subs(), where L is 0 in the algorithm. Then, we defined an array of angular frequencies and we solved for the cutoff frequency given by:

$$\omega_c = \frac{1}{RC}$$

At this frequency, the gain falls by 3dB. To simulate a Bode plot that applies to any R and C values, the ratio of the angular frequencies and the cutoff angular frequency were taken, and then were converted to log scale for comparison over a large range of values along the vertical axis. Here, we randomly set the resistance to $0.01\text{x}10^{-6}$ ohms and the capacitance to $15\text{x}10^{3}$ farads. These values do not affect the shape of the simulated bode plot. The angular frequencies we defined were substituted to the transfer function using the subs method. This results to an array of complex values. We took the phase of each complex value for the phase response plot and the modulus converted to decibels for the amplitude response plot.

Almost the same process was done for the simulation of the bode plots of 2nd order RLC filters, except that for these filters, L was not set to 0 and the substituted R was varied. It was varied to analyze the effect of different damping ratios and quality factors on the output. Damping ratio describes the decay of a system after a disturbance while quality factor describes how underdamped a system is. Both are measures of the "sharpness" of the amplitude response at the natural frequency [1,2]:

$$\omega_o = \frac{1}{\sqrt{LC}}$$

This is also known as the undamped resonant frequency, where we set the inductance to 1x10^{-3} henry and the capacitance to 1x10^{-9} farad. The relation of damping ratio and quality factor to R, L and C is given by the following equation:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{2d}$$

In the code, damping ratio values were chosen. Since L and C were also set, the R values corresponding to the chosen damping ratio values were calculated. The R values and an array of angular frequencies were then substituted to the transfer equation. From that, we obtained their amplitudes and converted them to dB for the amplitude response plot, and their phases for the phase response plot.

The code can be accessed using the link: https://github.com/MsTimeTraveler/156_manuscript/blob/master/thisisit.ipynb

III. Results

Figure 3 shows the phase and amplitude response of an RC low-pass filter. The algorithm solved that the transfer function for this filter is given by:

$$H(j\omega) = \frac{-j}{c\omega r - j}$$

It is emphasized that it has a polar representation of $|H(j\omega)|exp(j\phi)$. Thus, the equations for its magnitude and for its phase are:

$$|H(j\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}} \qquad \phi = -\tan^{-1}(\omega RC)$$

Considering that $\omega_c = 1/(rc)$, then these equations show that:

(1) When $\omega << \omega_c$, $|H(j\omega)| \approx 1 \approx 0$ dB and phase shift $\phi = 0$. Hence, the output is approximately the same as the input.

(2) When $\omega = \omega_c$, $|H(j\omega)| = 0.707 = -3$ dB and phase shift $\phi = -\pi/4$. Near the cutoff, the behavior of the output changes.

(3) When $\omega >> \omega_c$, $|H(j\omega)| \approx \omega_c/\omega$ and phase shift $\phi = -\pi/2$. Hence, the output is much smaller than the input.

These cases are illustrated by Figure 3. Note that negative log in the vertical axis of the bode plot stands for $\omega/\omega_c < 1$. Thus, large negative $\log(\omega/\omega_c)$ values displays case 1, $\log(\omega/\omega_c) = 0$ displays case 2, and large positive $\log(\omega/\omega_c)$ values displays case 3. It was observed that for a low angular frequency, the gain is nearly 1, until it reaches the cutoff. Then for angular frequencies greater than the cutoff, the gain approaches 0 with a slope of -20. This shows that a low-pass filter allows frequencies below its cutoff and impedes frequencies above its cutoff [3].

Figure 4 shows the phase and amplitude response of RC high-pass filter. The transfer function of this filter was solved by the algorithm to be:

$$H(j\omega) = \frac{c\omega r}{c\omega r - j}$$

The equations for its amplitude and its phase are:

$$|H(j\omega)| = \frac{\omega rc}{\sqrt{1+(\omega RC)^2}} \qquad \phi = \frac{\pi}{2} - \tan^{-1}(\omega RC)$$

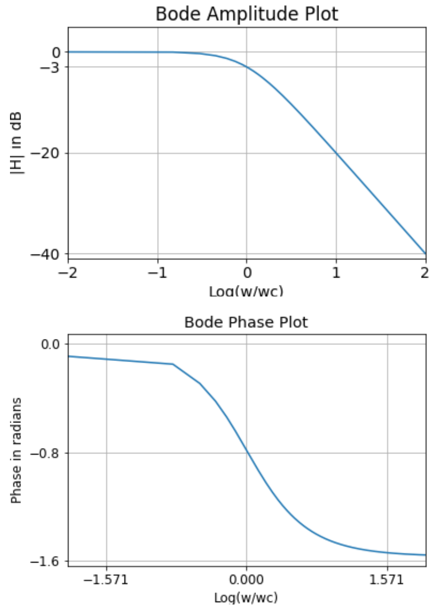


Fig. 3. Bode plots for RC low-pass filter

where it has the following cases:

(1) When $\omega << \omega_c$, $|H(j\omega)| \approx \omega/\omega_c$ and phase shift $\phi = \pi/2$. Thus, the output approaches 0.

(2) When $\omega = \omega_c$, $|H(j\omega)| = 0.707 = -3$ dB and phase shift $\phi = \pi/4$. Near the cutoff, the amplitude response slowly changes.

(3) When $\omega >> \omega_c$, $|H(j\omega)| \approx 1 \approx 0$ dB and phase shift $\phi = 0$. Thus, the output is approximately equal to the input.

These are illustrated by Figure 4. Plot analysis tells us that at angular frequencies lower than the cutoff, the output approaches 0 and for angular frequencies larger than the cutoff, the output approaches the input with as slope of 20, hence, the circuit filters low frequencies [3].

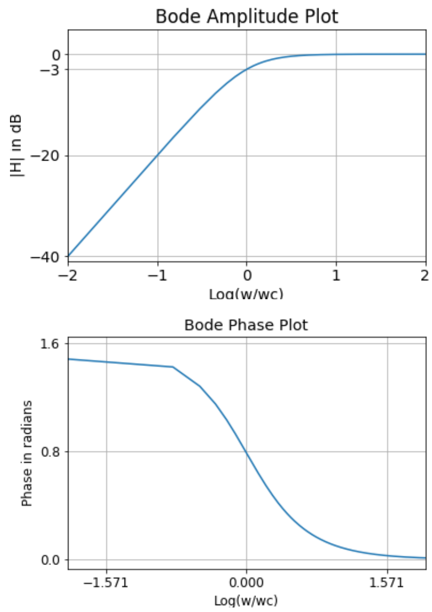


Fig. 4. Bode plots for RC high-pass filter

Figure 5 shows the phase and amplitude response of RLC low-pass filter for damping ratios $d = 0.4, 0.6, 0.7$ and 1. Its transfer function was solved by the algorithm to be:

$$H(j\omega) = \frac{-j}{jcl\omega^2 r + c\omega r - j}$$

Considering its polar representation in terms of its natural angular frequency, the equations of its amplitude and its phase are:

$$|H(j\omega)| = \frac{\omega_o^2}{\sqrt{(\omega_o^2-\omega^2)^2+\omega^2(\frac{r}{l})^2}} \qquad \phi = \tan(\frac{\frac{\omega r}{l}}{\omega_o^2-\omega^2})$$

From these equations, we can deduce the following cases :

(1) When $\omega << \omega_c$, $|H(j\omega)| \approx 1 \approx 0$ dB and phase shift $\phi = 0$. In this case, the output is approximately the same as the input.

(2) When $\omega = \omega_c$, $|H(j\omega)| = Q$, where Q is the quality factor and phase shift $\phi = -\pi/2$. Hence, the amplitude depends on R, C and L values.

(3) When $\omega >> \omega_c$, $|H(j\omega)| \approx (\omega_c/\omega)^2$ and phase shift $\phi = -\pi$, showing that the output is very small compared to the input.

These cases are illustrated by Figure 5. When angular frequency is equal to the natural angular frequency, gain is dependent on the damping ratio, thus, it is possible for the output to be larger than the input at certain R, L and C values. For this angular frequency, we expected that (1) when d = 0.707, gain is always smaller than 1 or 0 dB, (2) when d < 0.707, gain can be greater than 1 or 0 dB, and (3) when d = 1, circuit is critically damped. These claims were shown by Figure 3. It was also observed that angular frequencies lower than the natural angular frequency has a gain of 1, while the gain of angular frequencies above the natural angular frequency approaches 0 with a slope of -40, even when the damping ratio varies. This shows that the low pass filter still allows frequencies lower than the cutoff and rejects frequencies higher than the cutoff [3].

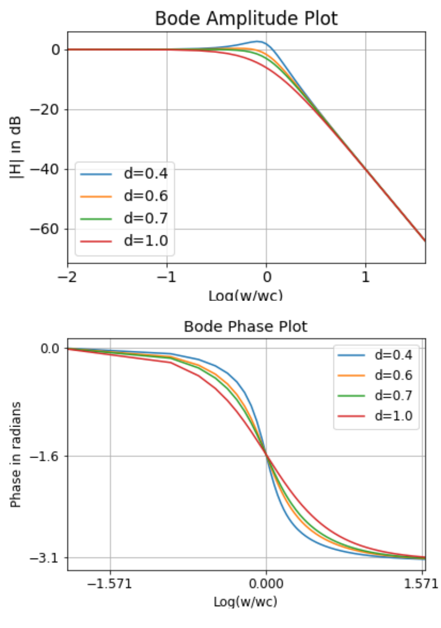


Fig. 5. Bode plots for RLC low-pass filter for d = 0.4, 0.6, 0.7 and 1.0

Figure 6 shows the phase and amplitude response of RLC high-pass filter for damping ratios $d = 0.4, 0.6, 0.7$ and 1. It has a transfer function of

$$H(j\omega) = \frac{-j}{jcl\omega^2 r + c\omega r - j}$$

The equations for its amplitude and its phase are:

$$|H(j\omega)| = \frac{\omega^2}{\sqrt{(\omega_o^2-\omega^2)^2+\omega^2(\frac{r}{l})^2}} \qquad \phi = \pi - \tan(\frac{\frac{\omega r}{l}}{\omega_o^2-\omega^2})$$

where these equations apply to the following cases:

(1) When $\omega << \omega_c$, $|H(j\omega)| \approx (\omega_c/\omega)^2$ and phase shift $\phi = \pi$, resulting to a very small output amplitude

(2) When $\omega = \omega_c$, $|H(j\omega)| = Q$, where Q is the quality factor and phase shift $\phi = \pi/2$. Since Q depends on R, C, and L values, then the output amplitude also depends on them.

(3) When $\omega >> \omega_c$, $|H(j\omega)| \approx 1 \approx 0$ dB and phase shift $\phi = 0$, showing that the output approaching the input.

These cases are shown by Figure 6. As observed, when the angular frequency is smaller than the natural angular frequency, it has a gain approaching 0 with a slope of 40, hence, it is rejected by the filter. When the angular frequency is larger than the natural angular frequency, it has a gain near 1, hence, it is allowed by the filter to pass through. For angular frequencies equal to the natural angular frequency, gain is dependent on the damping ratio and its behavior is similar to the RLC high pass filter [3].

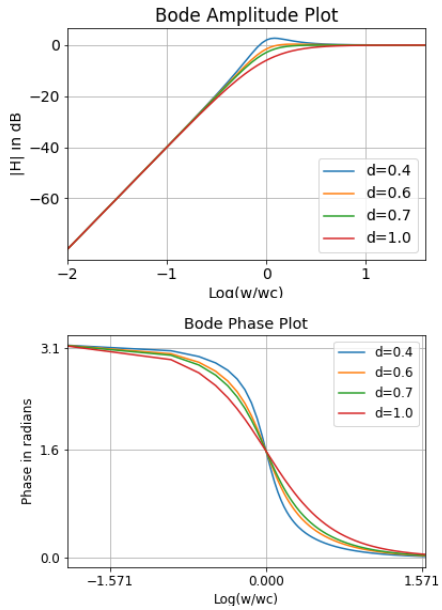


Fig. 6. Bode plots for RLC high-pass filter for d = 0.4, 0.6, 0.7 and 1.0

Aside from the number of active components, one obvious difference between the 1st order filter and the 2nd order filter is their roll-off slope. Table 1 shows the roll-off slope of each filter obtained from its bode amplitude plot.

filter type	roll-off slope
RC low-pass	-20
RC high-pass	20
RLC low-pass	-40
RLC high-pass	40

IV. Conclusion

Simulating the bode plots of high pass and low pass RC and RLC filters gives an accurate graphical representation of their behavior. In this study, we analyzed the simulated bode plots and in turn, an thzese balyehavior of low pass and high pass filters.

References

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