Gauss Quadrature

- Motivation
- General integration formula

Method 1 : Based on Natural Coordinates
Method 2 : Based on Polynomial functions
Method 3 : Based on Isoperimetric element

Motivation

Trapezoid Method :

$$\int_a^b f(x)\,dx \approx h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\big(f(x_0) + f(x_n)\big)\right]$$

It can be expressed as :

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} c_i\, f(x_i)$$

$$where \quad c_i = \begin{cases} h & i = 1, 2, \ldots, n-1 \\ 0.5\,h & i = 0 \ and \ n \end{cases}$$

General Integration Formula

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} c_i \; f(x_i)$$

c_i : *Weights* $\qquad$ x_i : *Nodes*

Problem :

How do select c_i *and* x_i so that th e formula gives a good approximat ion of the integral?

Lagrange Interpolation

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx$$

where $P_n(x)$ is a polynomial that interpolat es f(x) at the nodes : $x_0, x_1, \ldots, x_n$

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx = \int_a^b \left(\sum_{i=0}^{n} \lambda_i(x) f(x_i) \right) dx$$

$$\Rightarrow \int_a^b f(x)dx \approx \sum_{i=0}^{n} c_i \; f(x_i) \quad \text{where} \quad c_i = \int_a^b \lambda_i(x)\, dx$$

Example

- Determine the Gauss Quadrature Formula of

If the nodes are given as (-1, 0 , 1) $\int_{-2}^{2} f(x)dx$

- Solution: First need to find $l_0(x), l_1(x), l_2(x)$

- Then compute:

$$c_0 = \int_{-2}^{2} l_0(x)dx, \quad c_1 = \int_{-2}^{2} l_1(x)dx, \quad c_2 = \int_{-2}^{2} l_2(x)dx$$

(-1, 0 , 1)
(x0,x1,x2)

Solution

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$l_0(x) = \frac{(x - x1)(x - x2)}{(x0 - x1)(x0 - x2)} = \frac{x(x-1)}{2}$$

$$l_1(x) = \frac{(x - x0)(x - x2)}{(x1 - x0)(x1 - x2)} = -(x+1)(x-1)$$

$$l_2(x) = \frac{(x - x0)(x - x1)}{(x2 - x0)(x2 - x1)} = \frac{x(x+1)}{2}$$

$$c_0 = \int_{-2}^{2} \frac{x(x-1)}{2} dx = \frac{8}{3}, \quad c_1 = \int_{-2}^{2} -(x+1)(x-1) dx = -\frac{4}{3}, \quad c_2 = \int_{-2}^{2} \frac{x(x+1)}{2} dx = \frac{8}{3}$$

$$\text{The Gauss Quadrature Formula for } \int_{-2}^{2} f(x) dx = \frac{8}{3} f(-1) - \frac{4}{3} f(0) + \frac{8}{3} f(1)$$

Using the Gauss Quadrature Formula

Case 1 : Let $f(x) = x^2$

The exact valu e for $\int_{-2}^{2} f(x)dx = \int_{-2}^{2} x^2 dx = \frac{16}{3}$

The Gauss Quadrature Formula $= \frac{8}{3} f(-1) - \frac{4}{3} f(0) + \frac{8}{3} f(1)$

$= \frac{8}{3}(-1)^2 - \frac{4}{3}(0)^2 + \frac{8}{3}(1)^2 = \frac{16}{3}$, which is the same exact answer

Using the Gauss Quadrature Formula

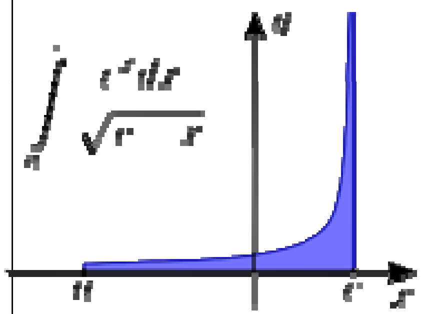
Case 2 : Let $f(x) = x^3$

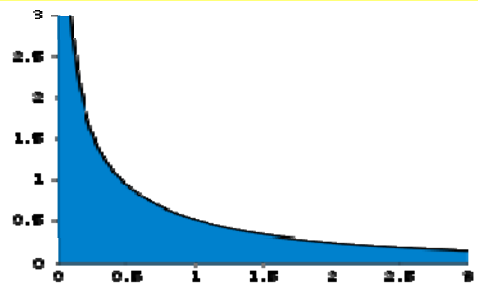
The exact value for $\int_{-2}^{2} f(x)dx = \int_{-2}^{2} x^3 dx = 0$

The Gauss Quadrature Formula $= \frac{8}{3} f(-1) - \frac{4}{3} f(0) + \frac{8}{3} f(1)$

$= \frac{8}{3}(-1)^3 - \frac{4}{3}(0)^3 + \frac{8}{3}(1)^3 = 0$, which is the same exact answer

Improper Integrals





$$\begin{aligned}\int_0^\infty \frac{dx}{(x+1)\sqrt{x}} &= \lim_{s\to 0}\int_s^1 \frac{dx}{(x+1)\sqrt{x}} + \lim_{t\to\infty}\int_1^t \frac{dx}{(x+1)\sqrt{x}} \\ &= \lim_{s\to 0}\left(\frac{\pi}{2} - 2\arctan\sqrt{s}\right) + \lim_{t\to\infty}\left(2\arctan\sqrt{t} - \frac{\pi}{2}\right) \\ &= \frac{\pi}{2} + \left(\pi - \frac{\pi}{2}\right) \\ &= \pi.\end{aligned}$$

Methods discussed earlier cannot be used directly to approximate improper integrals (one of the limits is ∞ or $-\infty$)

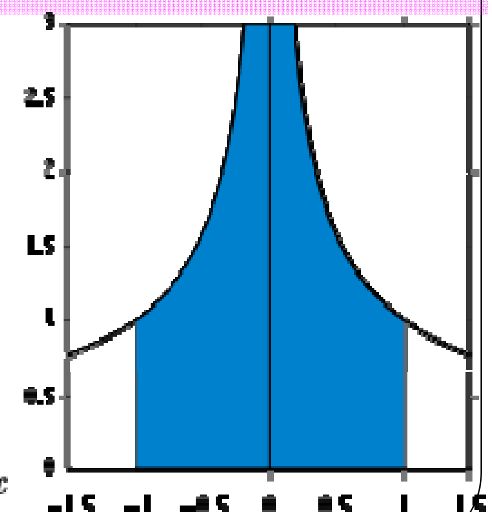
$\Rightarrow$ Use a transformation like the following

$$\int_a^b f(x)\,dx = \int_{\frac{1}{b}}^{\frac{1}{a}} \frac{1}{t^2} f\left(\frac{1}{t}\right) dt, \qquad (\text{assuming } ab > 0)$$

and apply the method on the new function.

$$Example: \quad \int_1^\infty \frac{1}{x^2}\,dx = \int_0^1 \frac{1}{t^2}\left(\frac{1}{\left(\frac{1}{t}\right)^2}\right) dt$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\lim_{b\to\infty}\int_a^b f(x)\,dx$$



Gauss Quadrature - Example

Find the integral of:

$$f(x) = 0.2 + 25\,x - 200\,x^2 + 675\,x^3 - 900\,x^4 + 400\,x^5$$

Between the limits 0 to 0.8 using:

- 2 points integration points **(ans. 1.822578)**
- 3 points integration points **(ans. 1.640533)**

Improper Integral

- Improper integrals can be evaluated by making a change of variable that transforms the infinite range to one that is finite,

$$\int_a^b f(x)dx = \int_{1/b}^{1/a} \frac{1}{t^2} f\left(\frac{1}{t}\right)dt \qquad ab > 0$$

$$\int_{-\infty}^b f(x)dx = \boxed{\int_{-\infty}^{-A} f(x)dx} + \boxed{\int_{-A}^{b} f(x)dx}$$

$$\boxed{\int_{-\infty}^{-A} f(x)dx = \int_{-1/A}^{0} \frac{1}{t^2} f\left(\frac{1}{t}\right)dt}$$

Can be evaluated by Newton-Cotes closed formula

Improper Integral - Examples

$$\int_{2}^{\infty} \frac{dx}{x(x+2)} = \int_{0}^{0.5} \frac{1}{t^2}(t)\frac{1}{1/t+2}dt = \int_{0}^{0.5} \frac{1}{1+2t}dt$$

$$\int_{0}^{\infty} e^{-y}\sin^2 y\, dy = \int_{0}^{2} e^{-y}\sin^2 y\, dy + \int_{2}^{\infty} e^{-y}\sin^2 y\, dy$$

$$\int_{2}^{\infty} e^{-y}\sin^2 y\, dy = \int_{0}^{1/2} \frac{1}{t^2} e^{-1/t}\sin^2(1/t)\, dt$$

$$\int_{-2}^{\infty} ye^{-y}\, dy = \int_{-2}^{2} ye^{-y}\, dy + \int_{2}^{\infty} ye^{-y}\, dy$$

$$\int_{2}^{\infty} ye^{-y}\, dy = \int_{0}^{1/2} \frac{1}{t^3} e^{-1/t}\, dt$$

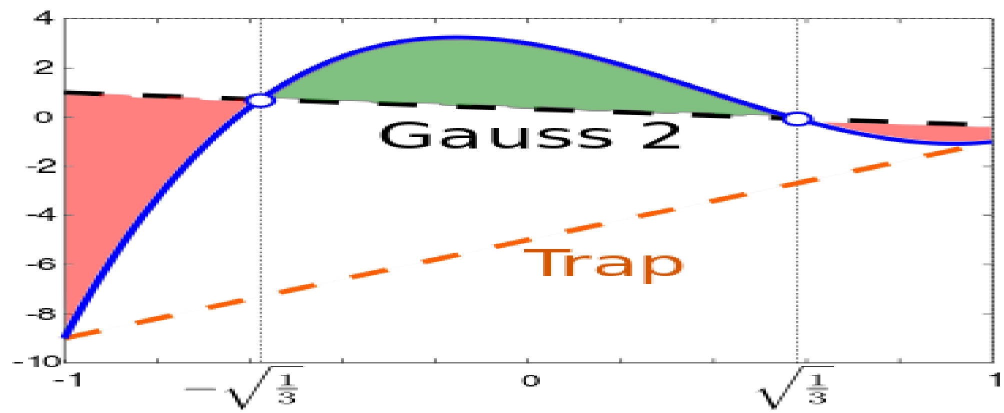
Gauss Quadrature

Method 1 : Based on Natural Coordinates

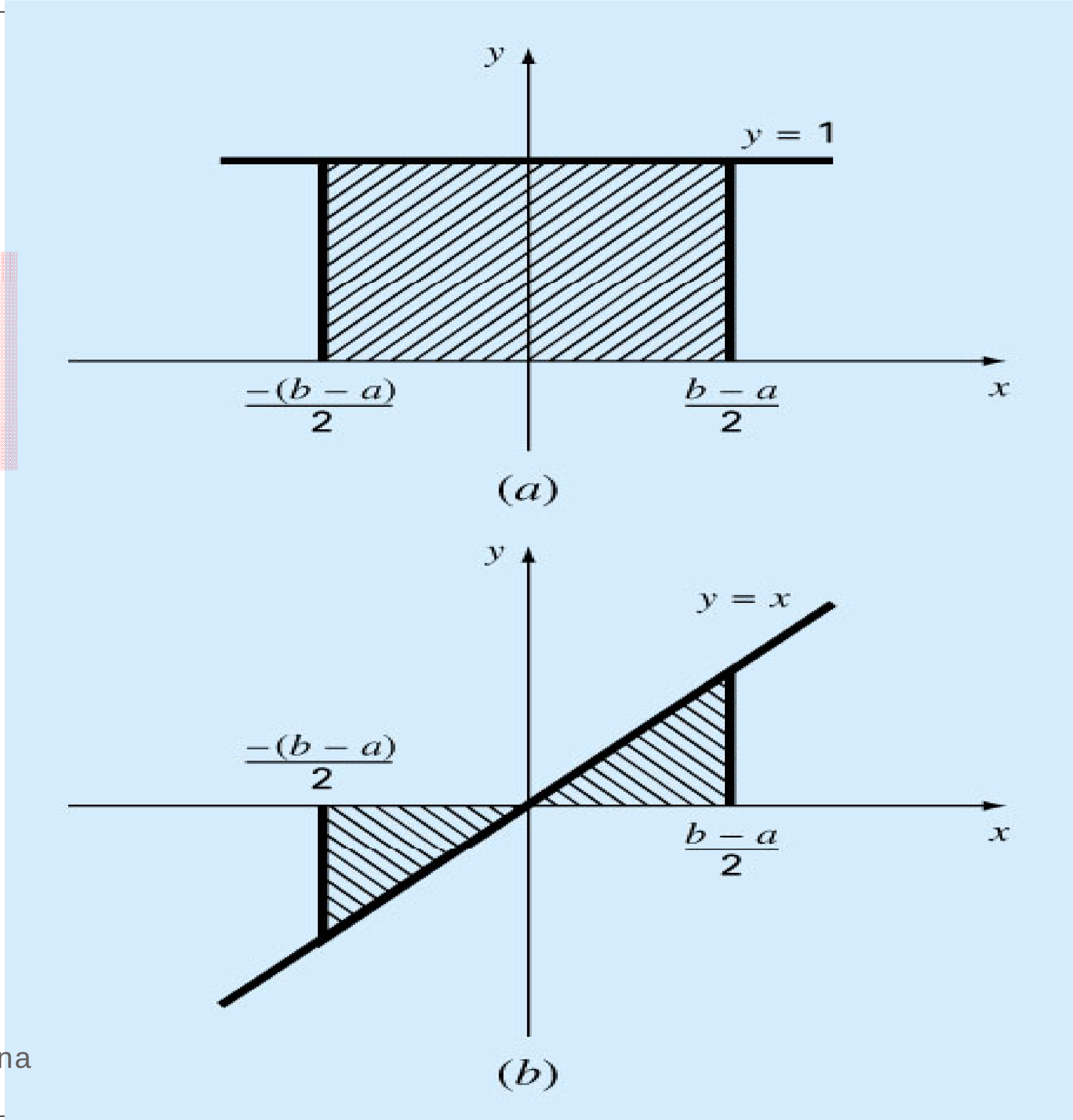
$$I = \int_a^b f(x)dx$$

- Assume $$I \cong c_0 f(a) + c_1 f(b)$$

- a and b are limits of integration
- Trapezoidal rule should give exact results for constant and linear functions



Trapezoidal rule gives exact solution for constant and linear functions



Gauss Quadrature

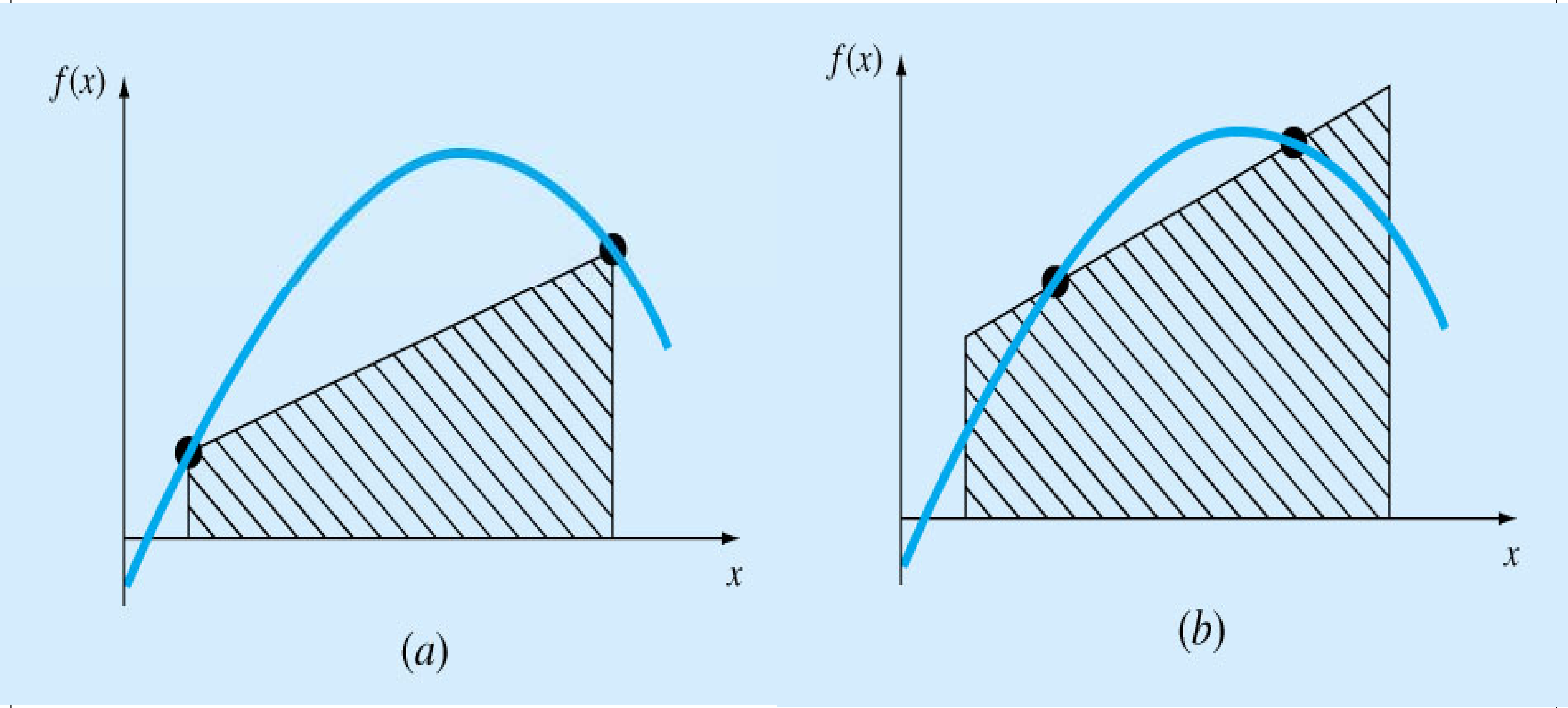
Method 1 : Based on Natural Coordinates

- Now instead of trapezoidal, which has fixed end points (*a*,*b*), let them float
- 4 unknowns - x_0, x_1, c_0, c_1

$$\boxed{I = \int_{-1}^{1} f(x)dx = c_0 f(x_0) + c_1 f(x_1)}$$

- 4 equations - constant, linear (had before in trapezoidal rule), quadratic, cubic
- Integrate from -1 to 1 to simplify math

Trapezoidal vs. Gauss-Quadrature



Exact for constant and linear functions

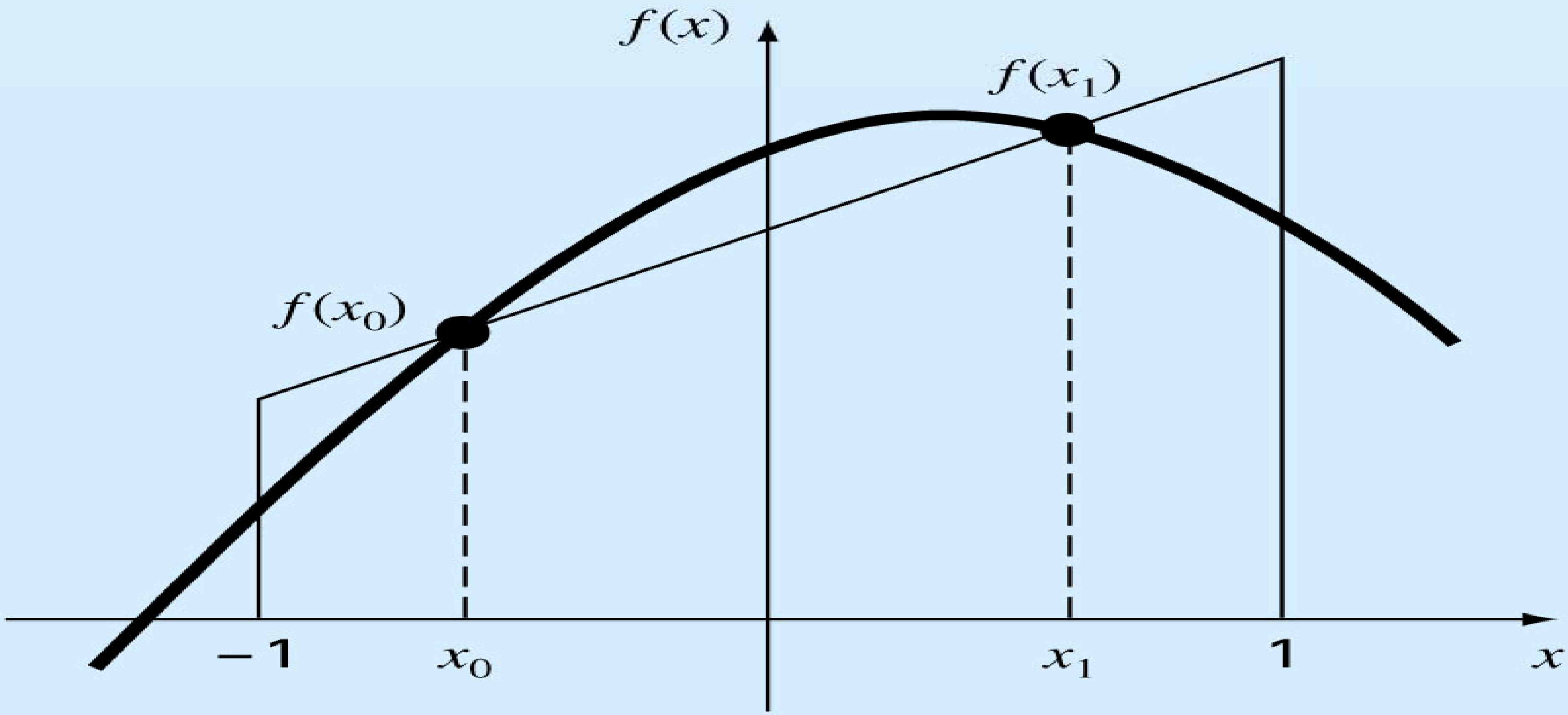
Exact for constant, linear, quadratic and cubic functions

Gauss Quadrature

Method 1 : Based on Natural Coordinates

- The idea is that if evaluate the function at certain points (non-uniformly distributed), and sum with certain weights, will get accurate integral
- Evaluation points and weights are tabulated

Gauss-Legendre Quadrature



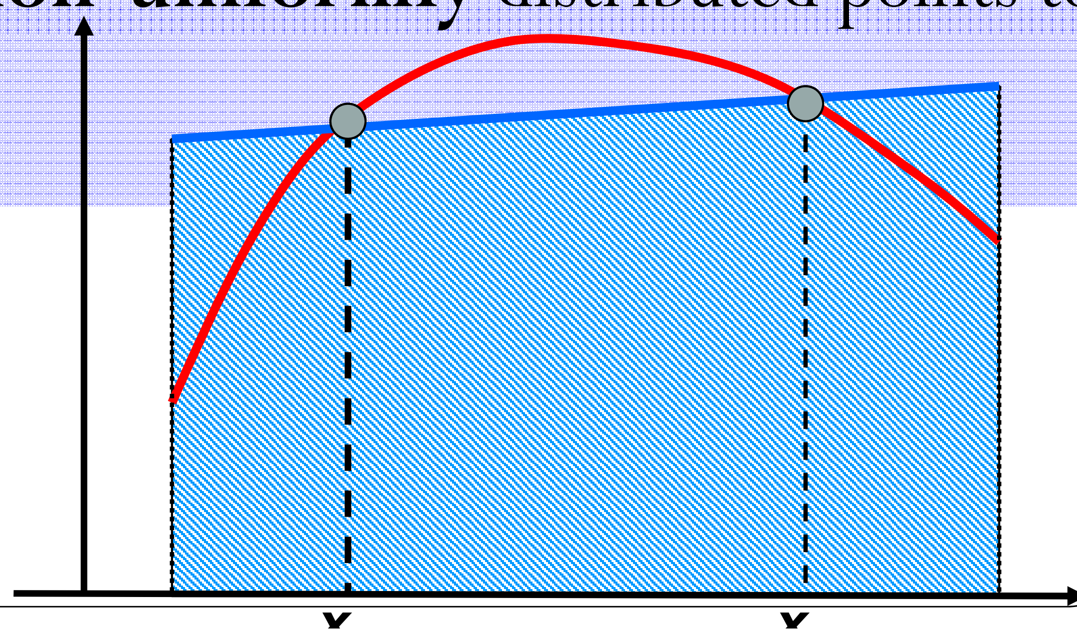
- **Choose (c_0 , c_1 , x_0 , x_1) to yield highest possible accuracy**

Gauss Quadratures

Method 1 : Based on Natural Coordinates

- **Newton-Cotes Formulas**
- use evenly-spaced functional values

- **Gauss Quadratures (Gauss-Legendre formulas)**
- change of variables so that the interval of integration is $[-1,1]$
- select functional values at **non-uniformly** distributed points to achieve higher accuracy



Gauss Quadrature on [a, b]

Method 1 : Based on Natural Coordinates

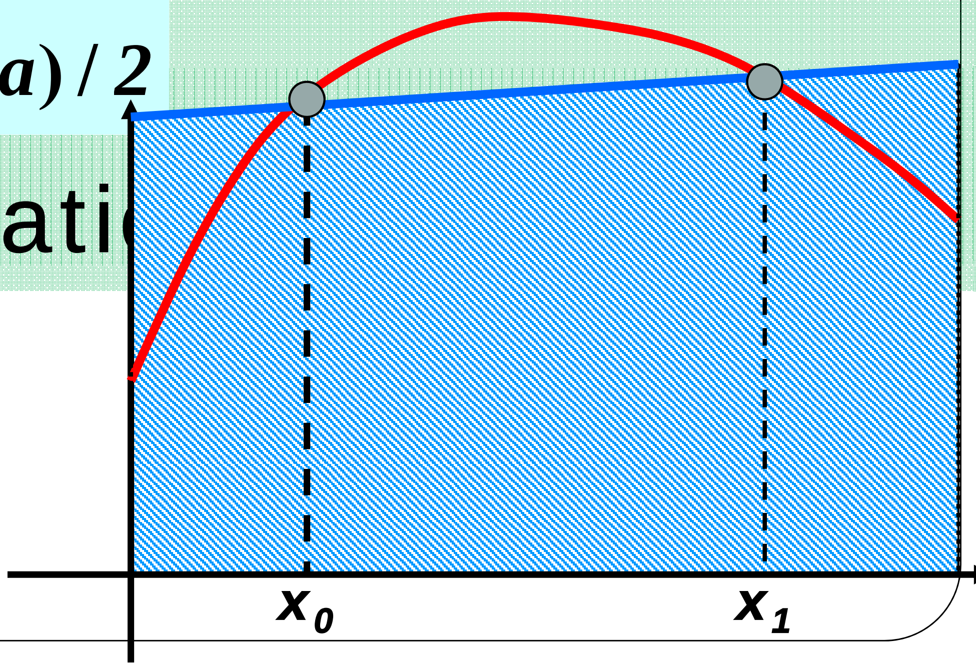
- To go to [–1,1] from other limits [*a*,*b*] - use <u>linear</u> transformation
- Change from $a \le x \le b$ to $-1 \le x_d \le 1$

$$x = a_0 + a_1 x_d$$

$$\begin{cases} a = a_0 + a_1(-1) \\ b = a_0 + a_1(1) \end{cases} \Rightarrow \begin{cases} a_0 = (a+b)/2 \\ a_1 = (b-a)/2 \end{cases}$$

- Coordinate transformatio

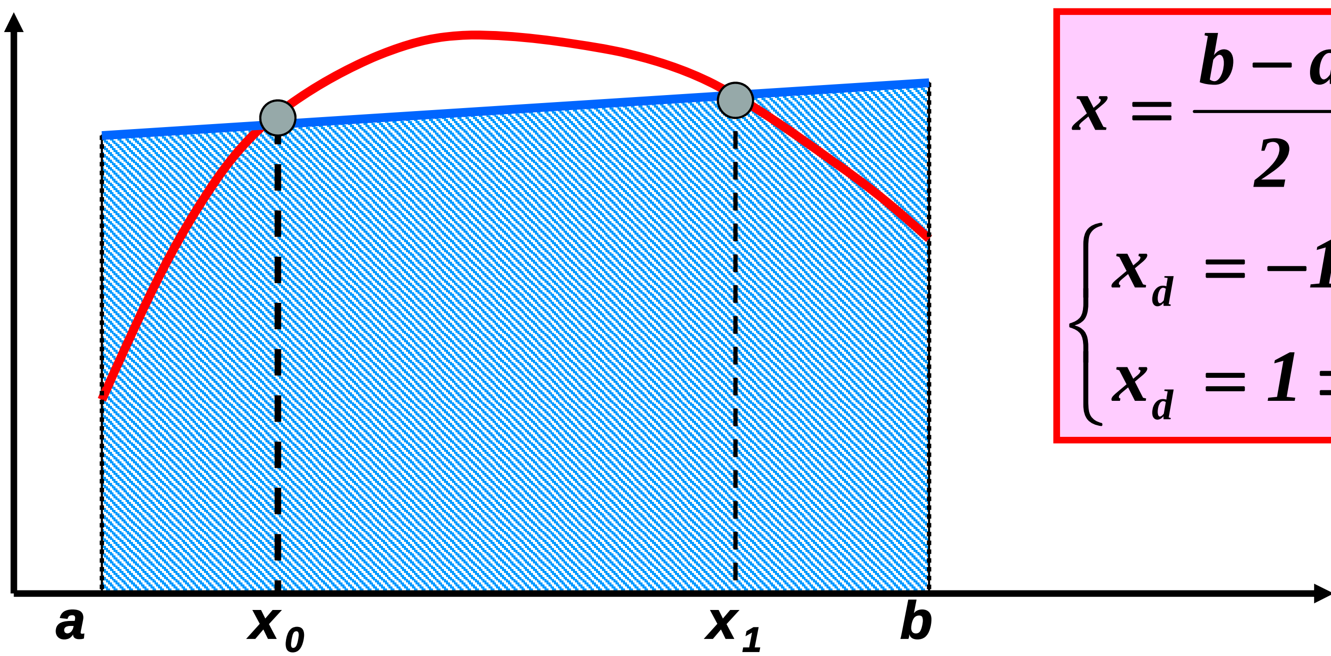
$$\boxed{x = \frac{a+b}{2} + \frac{b-a}{2} x_d}$$



Gauss Quadrature on [a, b]

Method 1 : Based on Natural Coordinates

- **Coordinate transformation from $[a,b]$ to $[-1,1]$**



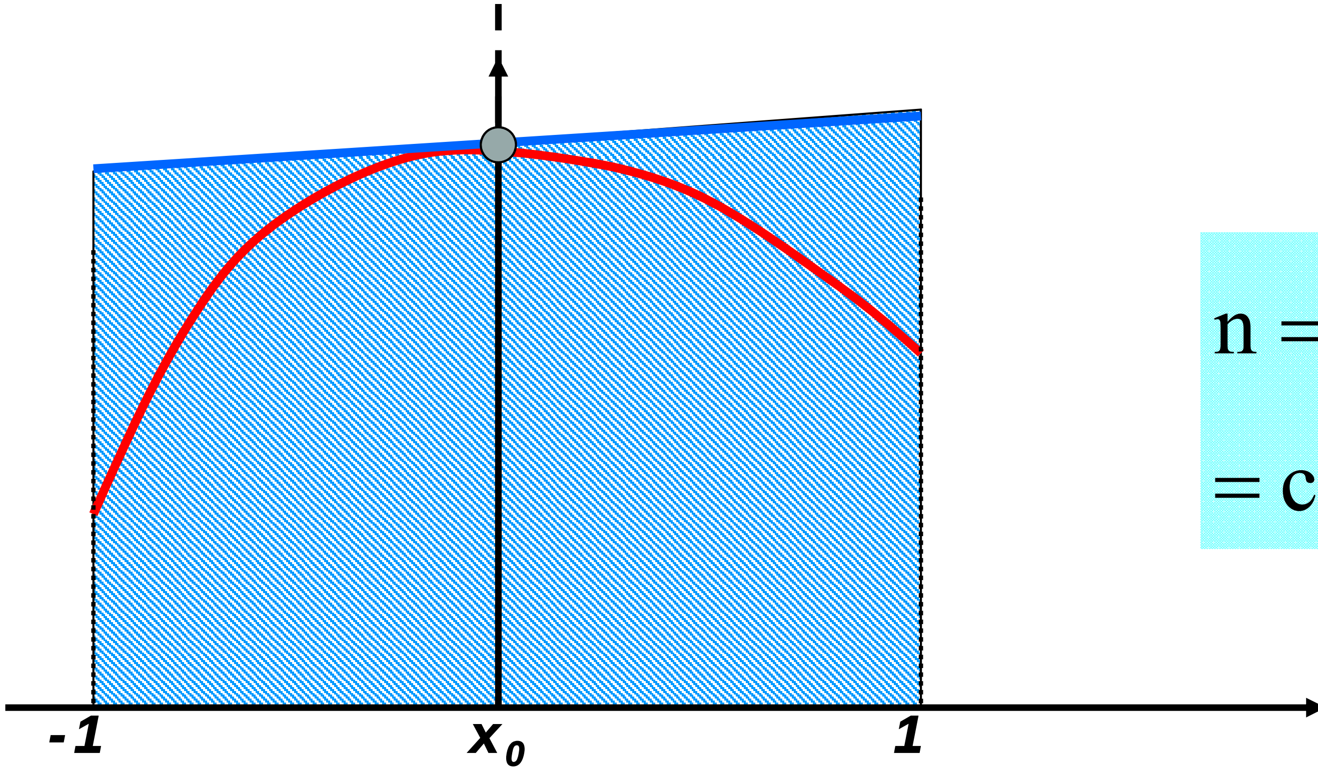
$$x = \frac{b-a}{2}x_d + \frac{a+b}{2}$$

$$\begin{cases} x_d = -1 \Rightarrow x = a \\ x_d = 1 \Rightarrow x = b \end{cases}$$

$$\int_a^b f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}x_d + \frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)dx_d = \int_{-1}^{1} g(x_d)dx_d$$

Gauss Quadrature on [–1, 1]

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + \Lambda + c_n f(x_n)$$

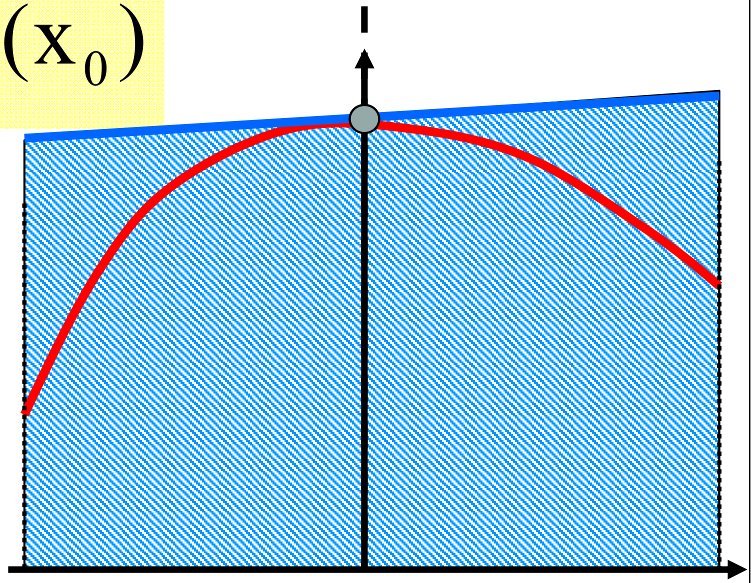


$$n = 1: \int_{-1}^{1} f(x)dx$$
$$= c_0 f(x_0)$$

- *Choose* (c_0, x_0) such that the method yields "exact integral" for $f(x) = x^0, x^1$

Gauss Quadrature on [–1, 1]

$$n = 1: \quad \int_{-1}^{1} f(x)dx = c_0 f(x_0)$$



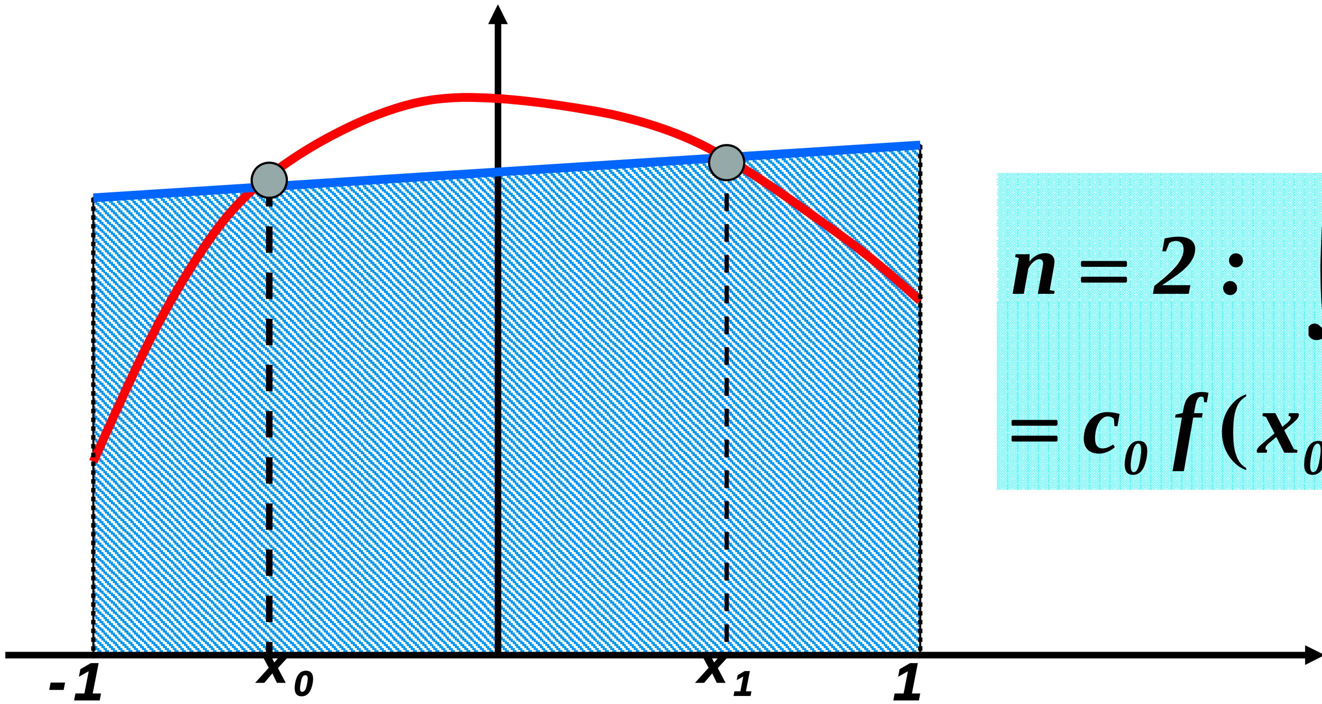
- **Exact integral for $f = x^0, x^1$**
- **Two equations for two unknowns**

$$\begin{cases} f = 1 & \Rightarrow \int_{-1}^{1} 1\,dx = 2 = c_0 \\ f = x & \Rightarrow \int_{-1}^{1} x dx = 0 = c_0 x_0 \end{cases} \Rightarrow \begin{cases} c_0 = 2 \\ x_0 = 0 \end{cases}$$

$$\boxed{I = \int_{-1}^{1} f(x)dx = 2f(0)}$$

Gauss Quadrature on [–1, 1]

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + \Lambda + c_n f(x_n)$$

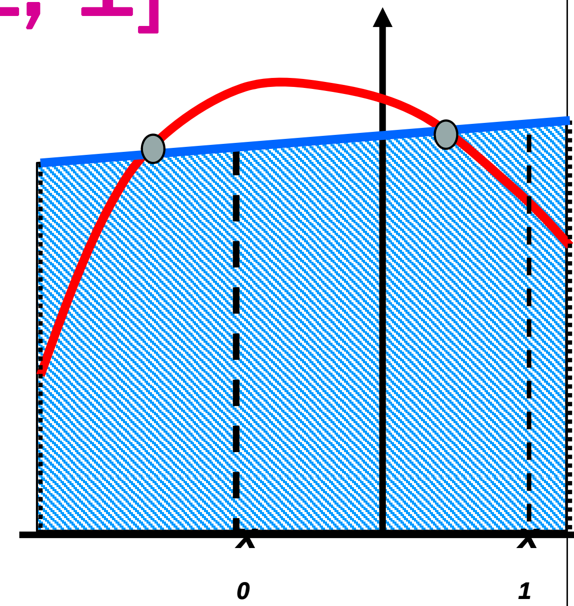


$$n = 2: \int_{-1}^{1} f(x)dx$$
$$= c_0 f(x_0) + c_1 f(x_1)$$

- *Choose* (c_0, c_1, x_0, x_1) **such that the method yields "exact integral" for** $f(x) = x^0, x^1, x^2, x^3$

Gauss Quadrature on [–1, 1]

$$n = 2: \quad \int_{-1}^{1} f(x)dx = c_0 f(x_0) + c_1 f(x_1)$$



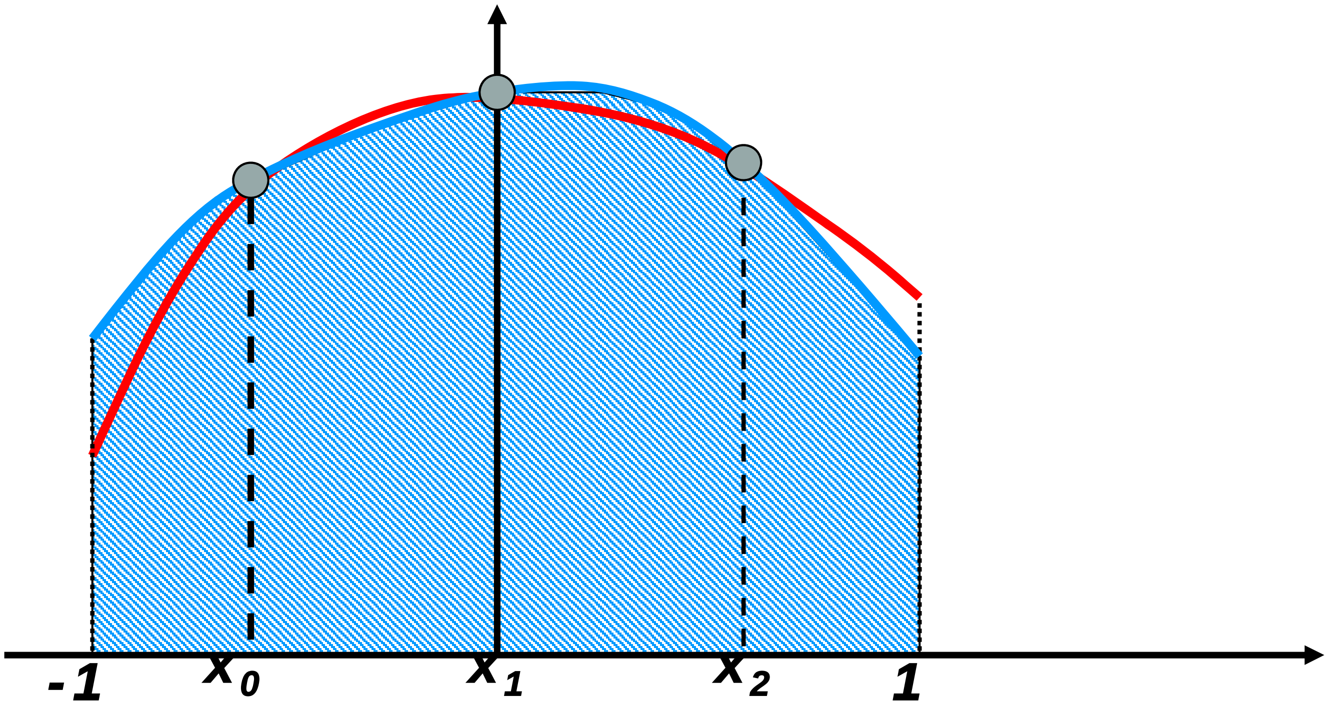
- Exact integral for $f = x^0, x^1, x^2, x^3$
- Four equations for four unknowns

$$\begin{cases} f = 1 \Rightarrow \int_{-1}^{1} 1\,dx = 2 = c_0 + c_1 \\ f = x \Rightarrow \int_{-1}^{1} x\,dx = 0 = c_0 x_0 + c_1 x_1 \\ f = x^2 \Rightarrow \int_{-1}^{1} x^2 dx = \frac{2}{3} = c_0 x_0^2 + c_1 x_1^2 \\ f = x^3 \Rightarrow \int_{-1}^{1} x^3 dx = 0 = c_0 x_0^3 + c_1 x_1^3 \end{cases} \Rightarrow \begin{cases} c_0 = 1 \\ c_1 = 1 \\ x_0 = \dfrac{-1}{\sqrt{3}} \\ x_1 = \dfrac{1}{\sqrt{3}} \end{cases}$$

$$\boxed{I = \int_{-1}^{1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)}$$

Gauss Quadrature on [–1, 1]

$$n = 3: \int_{-1}^{1} f(x)dx = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$$

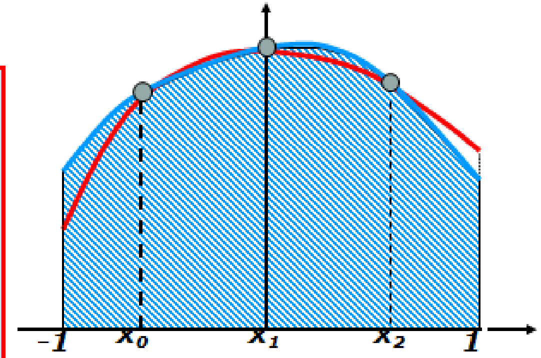


- *Choose* ($c_0, c_1, c_2, x_0, x_1, x_2$) **such that the method yields "exact integral" for** $f(x) = x^0, x^1, x^2, x^3, x^4, x^5$

Gauss Quadrature on [–1, 1]

- **Exact integral for $f = x^0, x^1, x^2, x^3, x^4, x^5$**

$$
\left\{\begin{aligned}
f &= 1 \Rightarrow \int_{-1}^{1} 1dx = 2 = c_0 + c_1 + c_2 \\
f &= x \Rightarrow \int_{-1}^{1} xdx = 0 = c_0x_0 + c_1x_1 + c_2x_2 \\
f &= x^2 \Rightarrow \int_{-1}^{1} x^2dx = \frac{2}{3} = c_0x_0^2 + c_1x_1^2 + c_2x_2^2 \\
f &= x^3 \Rightarrow \int_{-1}^{1} x^3dx = 0 = c_0x_0^3 + c_1x_1^3 + c_2x_2^3 \\
f &= x^4 \Rightarrow \int_{-1}^{1} x^4dx = \frac{2}{5} = c_0x_0^4 + c_1x_1^4 + c_2x_2^4 \\
f &= x^5 \Rightarrow \int_{-1}^{1} x^5dx = 0 = c_0x_0^5 + c_1x_1^5 + c_2x_2^5
\end{aligned}\right.
\Rightarrow
\left\{\begin{aligned}
c_0 &= 5/9 \\
c_1 &= 8/9 \\
c_2 &= 5/9 \\
x_0 &= -\sqrt{3/5} \\
x_1 &= 0 \\
x_2 &= \sqrt{3/5}
\end{aligned}\right.
$$



$$
I = \int_{-1}^{1} f(x)dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)
$$

Example: Gauss Quadrature

Method 1 : Based on Natural Coordinates

- Evaluate

$$I = \int_0^4 xe^{2x}dx = 5216.926477$$

- Coordinate transformation

$$x = \frac{b-a}{2}x_d + \frac{a+b}{2} = 2x_d + 2; \quad dx = 2dx_d$$

$$I = \int_0^4 xe^{2x}dx = \int_{-1}^{1}(4x_d + 4)e^{4x_d+4}dx_d = \int_{-1}^{1} g(x_d)dx_d$$

- Two-point formula

$$I = \int_{-1}^{1} g(x_d)dx_d = g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) = \left(4 - \frac{4}{\sqrt{3}}\right)e^{4-\frac{4}{\sqrt{3}}} + \left(4 + \frac{4}{\sqrt{3}}\right)e^{4+\frac{4}{\sqrt{3}}}$$

$$= 9.167657324 + 3468.376279 = 3477.543936 \qquad (\varepsilon = 33.34\%)$$

Points	Weighting Factors	Function Arguments
2	$c_0 = 1.0000000$	$x_0 = -0.577350269$
	$c_1 = 1.0000000$	$x_1 = 0.577350269$

Example: Gauss Quadrature

- **Three-point formula**

$$I = \int_0^4 xe^{2x}dx = \int_{-1}^{1}(4x_d + 4)e^{4x_d+4}dx_d = \int_{-1}^{1} g(x_d)dx_d$$

$$I = \int_{-1}^{1} g(x_d)dx_d = \frac{5}{9}g(-\sqrt{0.6}) + \frac{8}{9}g(0) + \frac{5}{9}g(\sqrt{0.6})$$

$$= \frac{5}{9}(4 - 4\sqrt{0.6})e^{4-\sqrt{0.6}} + \frac{8}{9}(4)e^{4} + \frac{5}{9}(4 + 4\sqrt{0.6})e^{4+\sqrt{0.6}}$$

$$= \frac{5}{9}(2.221191545) + \frac{8}{9}(218.3926001) + \frac{5}{9}(8589.142689)$$

$$= 4967.106689 \qquad (\varepsilon = 4.79\%)$$

Points	Weighting Factors	Function Arguments
3	$c_0 = 0.5555556$	$x_0 = -(0.6)^{0.5}$
	$c_1 = 0.8888889$	$x_1 = 0.000000000$
	$c_2 = 0.5555556$	$x_2 = (0.6)^{0.5}$

- **Four-point formula**

$$I = \int_{-1}^{1} g(x_d)dx_d = 0.34785\left[g(-0.861136) + g(0.861136)\right]$$

$$+ 0.652145\left[g(-0.339981) + g(0.339981)\right]$$

$$= 5197.54375 \qquad (\varepsilon = 0.37\%)$$

4	$c_0 = 0.3478548$	$x_0 = -0.861136312$
	$c_1 = 0.6521452$	$x_1 = -0.339981044$
	$c_2 = 0.6521452$	$x_2 = 0.339981044$
	$c_3 = 0.3478548$	$x_3 = 0.861136312$

Points	Weighting Factors	Function Arguments	Truncation Error
2	$c_0 = 1.0000000$	$x_0 = -0.577350269$	$\cong f^{(4)}(\xi)$
	$c_1 = 1.0000000$	$x_1 = 0.577350269$	
3	$c_0 = 0.5555556$	$x_0 = -0.774596669$	$\cong f^{(6)}(\xi)$
	$c_1 = 0.8888889$	$x_1 = 0.000000000$	
	$c_2 = 0.5555556$	$x_2 = 0.774596669$	
4	$c_0 = 0.3478548$	$x_0 = -0.861136312$	$\cong f^{(8)}(\xi)$
	$c_1 = 0.6521452$	$x_1 = -0.339981044$	
	$c_2 = 0.6521452$	$x_2 = 0.339981044$	
	$c_3 = 0.3478548$	$x_3 = 0.861136312$	
5	$c_0 = 0.2369269$	$x_0 = -0.906179846$	$\cong f^{(10)}(\xi)$
	$c_1 = 0.4786287$	$x_1 = -0.538469310$	
	$c_2 = 0.5688889$	$x_2 = 0.000000000$	
	$c_3 = 0.4786287$	$x_3 = 0.538469310$	
	$c_4 = 0.2369269$	$x_4 = 0.906179846$	
6	$c_0 = 0.1713245$	$x_0 = -0.932469514$	$\cong f^{(12)}(\xi)$
	$c_1 = 0.3607616$	$x_1 = -0.661209386$	
	$c_2 = 0.4679139$	$x_2 = -0.238619186$	
	$c_3 = 0.4679139$	$x_2 = 0.238619186$	
	$c_4 = 0.3607616$	$x_4 = 0.661209386$	
	$c_5 = 0.1713245$	$x_5 = 0.932469514$	

Gauss-Legendre Formulas

Gauss Quadrature

```
function I = Gauss_quad(f, a, b, k)
% find integral of function f on [a, b]
% using Gauss quadrature at k (k = 2, 3, 4, 5) points
t = [-0.5773502692  -0.7745966692  -0.8611363116  -0.9061798459;
      0.5773502692   0.0000000000  -0.3399810436  -0.5384693101;
      0.0            0.7745966692   0.3399810436   0.0000000000;
      0.0            0.0            0.8611363116   0.5384693101;
      0.0            0.0            0.0            0.9061798459]

c = [ 1.0            0.5555555556   0.3478548451   0.2369268850;
      1.0            0.8888888889   0.6521451549   0.4786286705;
      0.0            0.5555555556   0.6521451549   0.5688888889;
      0.0            0.0            0.3478548451   0.4786286705;
      0.0            0.0            0.0            0.2369268850]

x(1:k) = 0.5*((b-a).*t(1:k,k-1) + b + a);
y=feval(f, x);
tt = t(1 : k, k-1)
cc(1 : k) = c(1 : k, k-1);
cd = cc'
int = y*cd;
I = int*(b-a)/2;
```

Gauss Quadrature

```
» I=Gauss_quad('example1',0,pi,2);
t =
   -0.5774   -0.7746   -0.8611   -0.9062
    0.5774         0   -0.3400   -0.5385
         0    0.7746    0.3400         0
         0         0    0.8611    0.5385
         0         0         0    0.9062
c =
    1.0000    0.5556    0.3479    0.2369
    1.0000    0.8889    0.6521    0.4786
         0    0.5556    0.6521    0.5689
         0         0    0.3479    0.4786
         0         0         0    0.2369
tt =
   -0.5774
    0.5774
cd =
     1
     1
» I
I =
   -8.6878
» Q=quad8('example1',0,pi)
Q =
   -4.9348
```

$$\int_0^{\pi} x^2 \sin(2x)dx$$

k = 2

Exact Q = -4.9348

```
» I=Gauss_quad('example1',0,pi,5);
t =
   -0.5774   -0.7746   -0.8611   -0.9062
    0.5774         0   -0.3400   -0.5385
         0    0.7746    0.3400         0
         0         0    0.8611    0.5385
         0         0         0    0.9062
c =
    1.0000    0.5556    0.3479    0.2369
    1.0000    0.8889    0.6521    0.4786
         0    0.5556    0.6521    0.5689
         0         0    0.3479    0.4786
         0         0         0    0.2369
tt =
   -0.9062
   -0.5385
         0
    0.5385
    0.9062
cd =
    0.2369
    0.4786
    0.5689
    0.4786
    0.2369
» I
I =
   -4.9333
```

$$\int_0^{\pi} x^2 \sin(2x)dx$$

Gauss Quadrature

k = 5

Exact Q = -4.9348

Adaptive Quadrature

- Composite Simpson's 1/3 rule requires the use of equally spaced points
- Use adaptive refinement in regions of relatively abrupt changes
- **Estimate truncation error between two levels of refinement**
- Automatically adjust the step size so that small steps are taken in regions of sharp variations while larger steps are used elsewhere
- **MATLAB functions: `quad` and `quadl`**

MATLAB Integration Methods

- trapz(x,y)

  * Composite trapezoid rule

- q = quad('func',xmin,xmax)

  * Adaptive Simpson's rule

  more efficient for low accuracies or non- smooth functions

- q =quadl('func',xmin, xmax)

  * Labatto quadrature – more efficient for high accuracies and smooth functions

Two-Point Gaussian Quadrature Rule Method 2

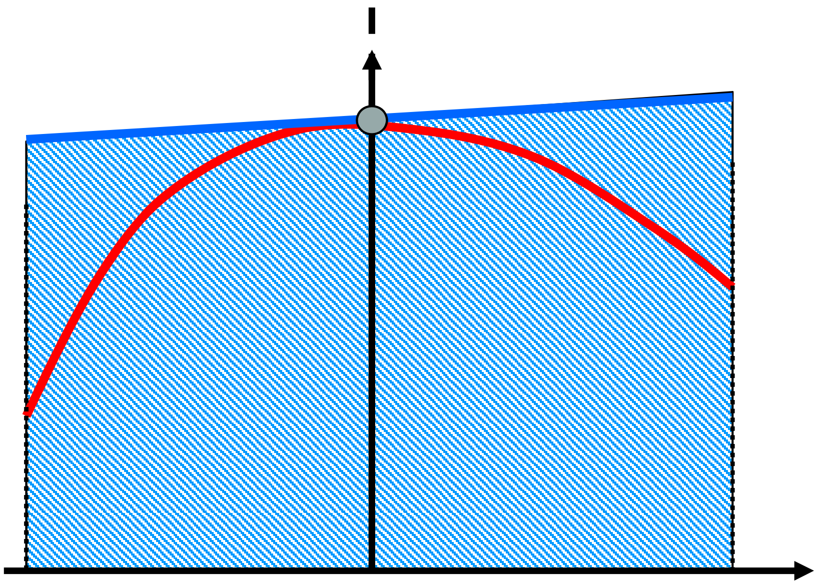
Basis of the Gaussian Quadrature Rule

Method 2 : Based on Polynomial functions

Previously, the Trapezoidal Rule was developed by the method of undetermined coefficients. The result of that development is summarized below.

$$\int_a^b f(x)dx \approx c_1 f(a)$$

$$= (b-a)\left(\frac{f(a+b)}{2}\right)$$

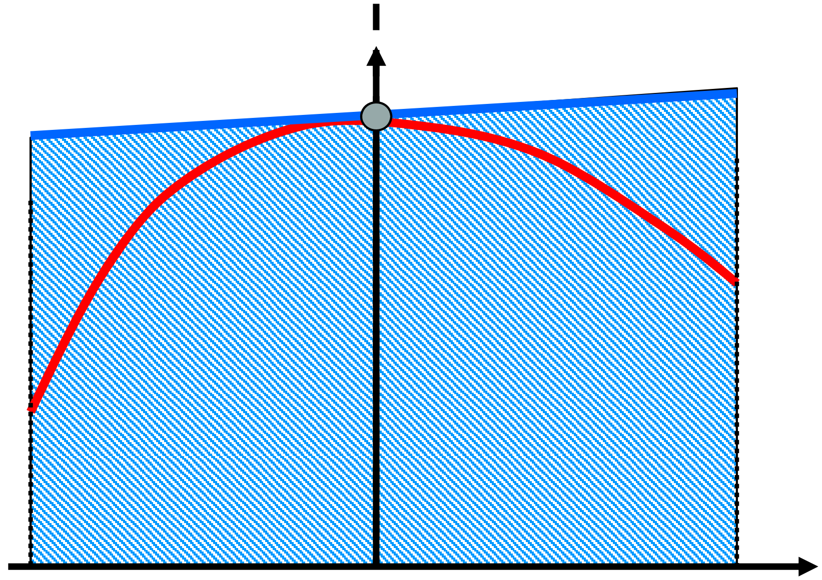


For an integral $\int_a^b f(x)dx,$ derive the one-point Gaussian Quadrature Rule.

Solution

The one-point Gaussian Quadrature Rule is

$$\int_a^b f(x)dx \approx c_1 f(x_1)$$



Solution

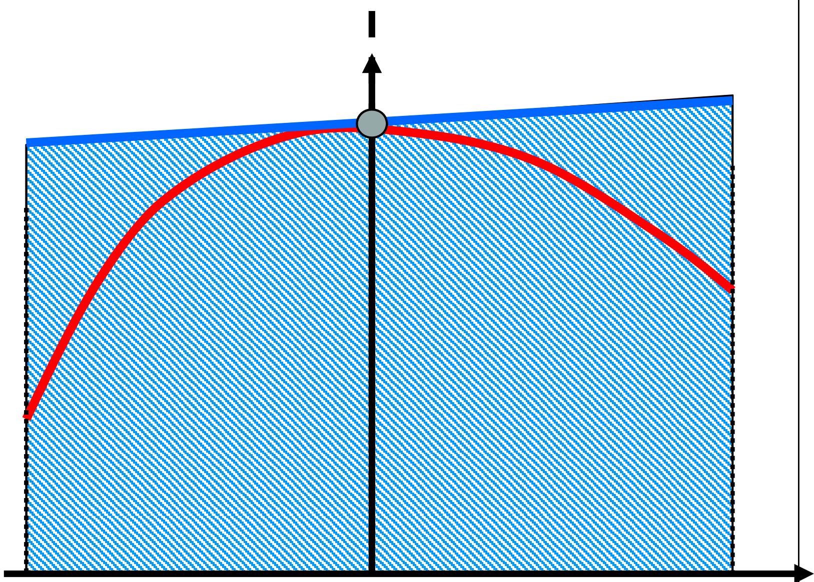
The two unknowns x_1, and c_1 are found by assuming that the formula gives exact results for integrating a general first order polynomial,

$$f(x) = a_0 + a_1 x.$$

$$\int_a^b f(x)dx = \int_a^b (a_0 + a_1 x)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} \right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right)$$



Solution

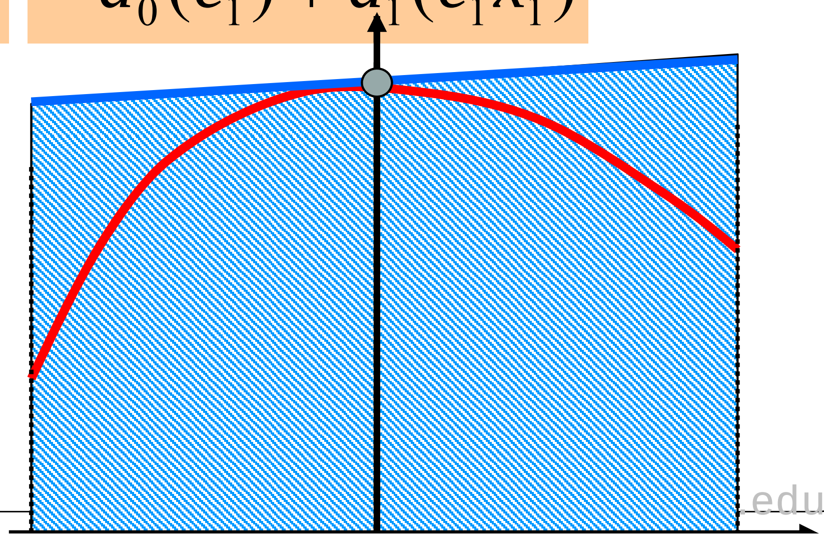
$$\int_a^b f(x)dx = \int_a^b (a_0 + a_1 x)dx = a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right)$$

It follows that

$$\int_a^b f(x)dx = c_1(a_0 + a_1 x_1)$$

Equating Equations, the two previous two expressions yield

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) = c_1(a_0 + a_1 x_1) = a_0(c_1) + a_1(c_1 x_1)$$



$$= a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right)$$

$$= a_0(c_1) + a_1(c_1 x_1)$$

Since the constants a_0, and a_1 are arbitrary

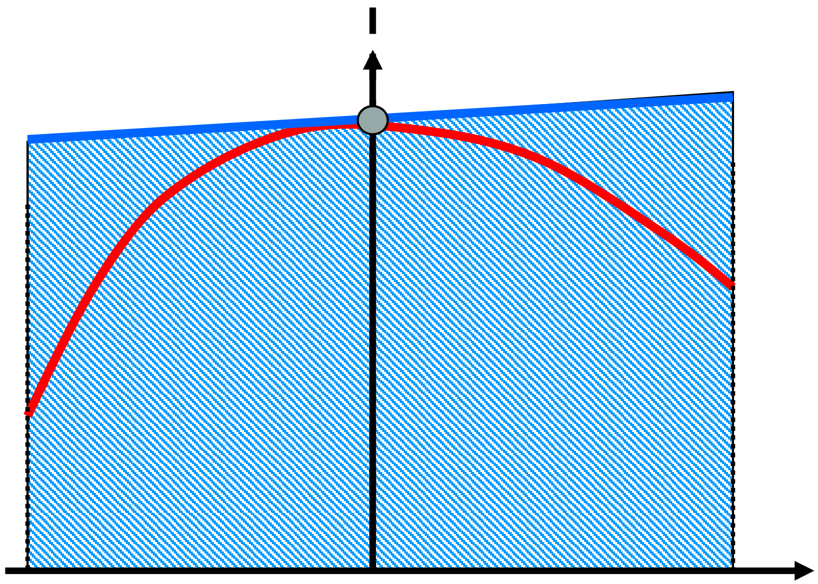
$$b - a = c_1$$

$$\frac{b^2-a^2}{2} = c_1 x_1$$

giving

$$c_1 = b - a$$

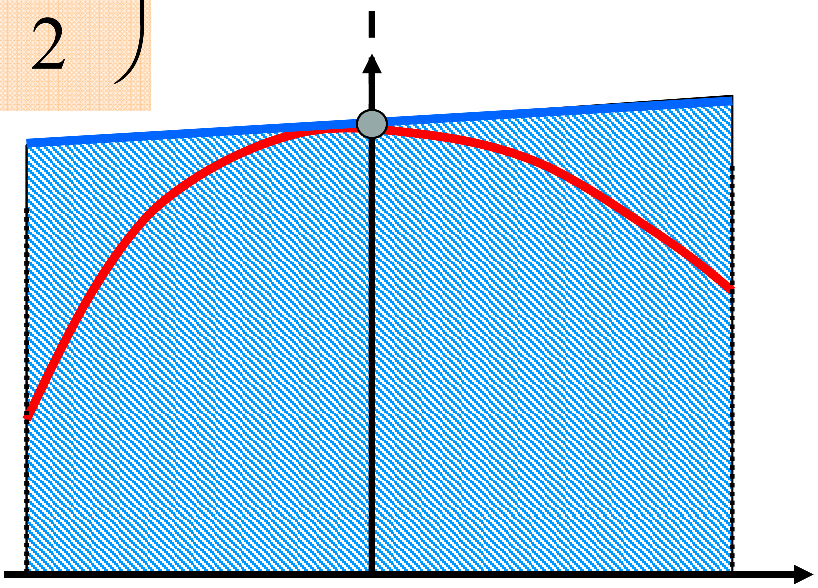
$$x_1 = \frac{b+a}{2}$$



Solution

Hence One-Point Gaussian Quadrature Rule

$$\int_a^b f(x)dx \approx c_1 f(x_1) = (b-a)\ f\left(\frac{b+a}{2}\right)$$



Two-Point Gaussian Quadrature Rule Method 2

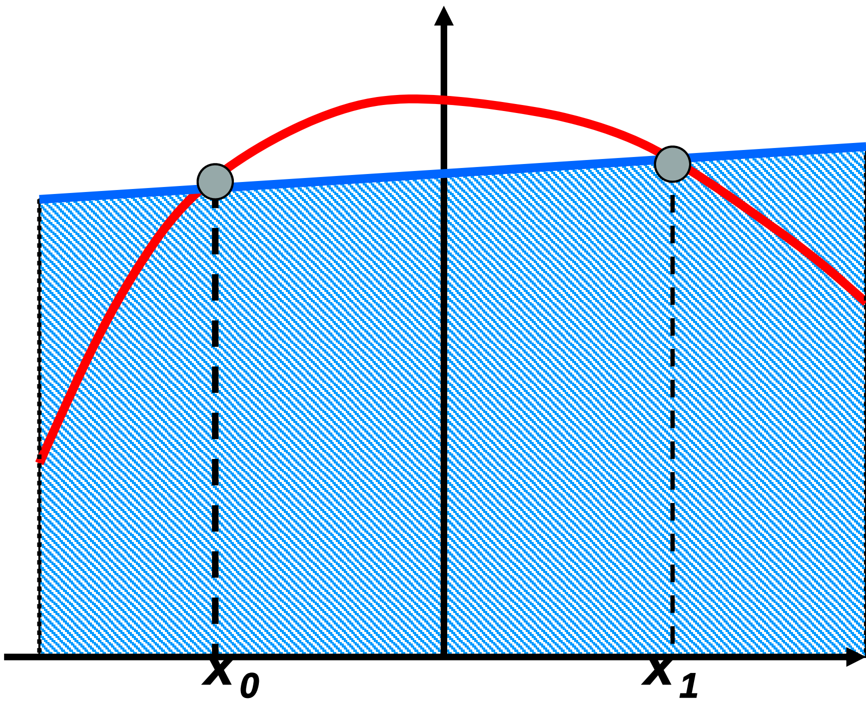
Basis of the Gaussian Quadrature Rule

Method 2 : Based on Polynomial functions

Previously, the Trapezoidal Rule was developed by the method of undetermined coefficients. The result of that development is summarized below.

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$



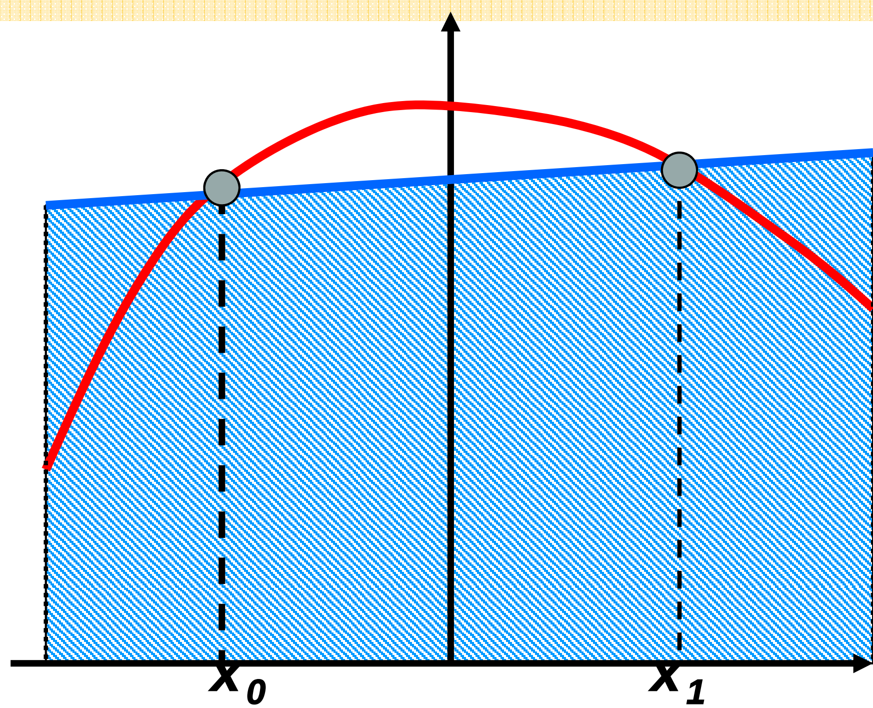
Two-Point Gaussian Quadrature Rule Method 2

Basis of the Gaussian Quadrature Rule

Method 2 : Based on Polynomial functions

The two-point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the arguments of the function are not predetermined as a and b but as unknowns x_1 and x_2. In the two-point Gauss Quadrature Rule, the integral is approximated as

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

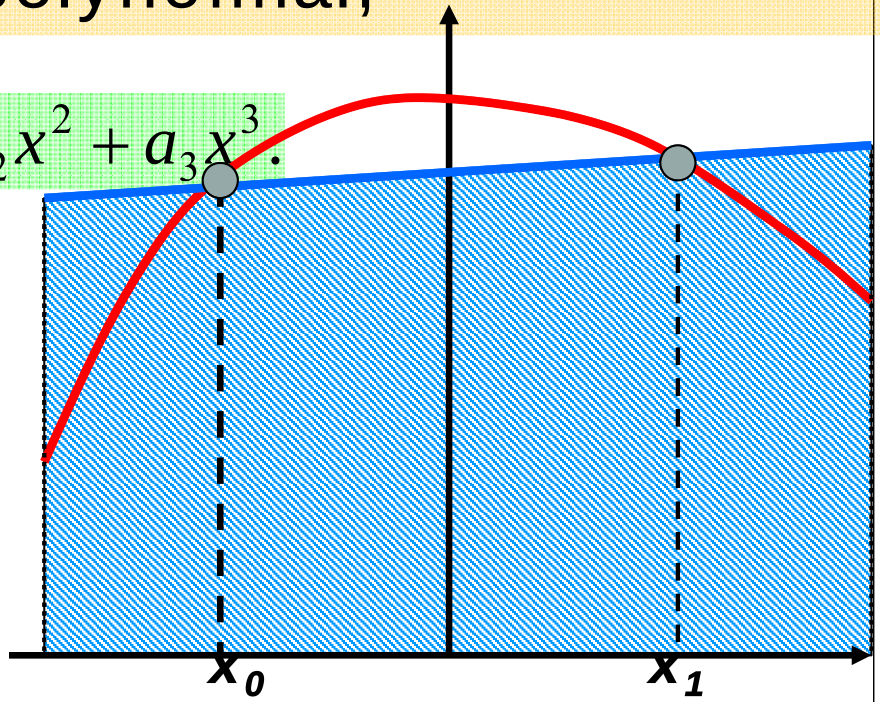


Method 2 : Based on Polynomial functions

Basis of the Gaussian Quadrature Rule

The four unknowns x_1, x_2, c_1 and c_2 are found by assuming that the formula gives exact results for integrating a general third order polynomial,

Hence $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$



$$\int_a^b f(x)dx = \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right)$$

Basis of the Gaussian Quadrature Rule

Method 2 : Based on Polynomial functions

It follows that $I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$

$$\int_a^b f(x)dx = c_1\left(a_0 + a_1x_1 + a_2x_1^{\ 2} + a_3x_1^{\ 3}\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^{\ 2} + a_3x_2^{\ 3}\right)$$

Equating Equations the two previous two expressions yield

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) + a_2\left(\frac{b^3 - a^3}{3}\right) + a_3\left(\frac{b^4 - a^4}{4}\right)$$

$$= c_1\left(a_0 + a_1x_1 + a_2x_1^{\ 2} + a_3x_1^{\ 3}\right) + c_2\left(a_0 + a_1x_2 + a_2x_2^{\ 2} + a_3x_2^{\ 3}\right)$$

$$= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2\left(c_1x_1^{\ 2} + c_2x_2^{\ 2}\right) + a_3\left(c_1x_1^{\ 3} + c_2x_2^{\ 3}\right)$$

$$a_0(b-a)+a_1\left(\frac{b^2-a^2}{2}\right)+a_2\left(\frac{b^3-a^3}{3}\right)+a_3\left(\frac{b^4-a^4}{4}\right)$$

$$=a_0(c_1+c_2)+a_1(c_1x_1+c_2x_2)+a_2\left(c_1x_1^{\,2}+c_2x_2^{\,2}\right)+a_3\left(c_1x_1^{\,3}+c_2x_2^{\,3}\right)$$

Since the constants a_0, a_1, a_2, a_3 are arbitrary

$$b-a=c_1+c_2$$

$$\frac{b^2-a^2}{2}=c_1x_1+c_2x_2$$

$$\frac{b^3-a^3}{3}=c_1x_1^{\,2}+c_2x_2^{\,2}$$

$$\frac{b^4-a^4}{4}=c_1x_1^{\,3}+c_2x_2^{\,3}$$

The previous four simultaneous nonlinear equations have only one acceptable solution,

$$x_1=\left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}$$

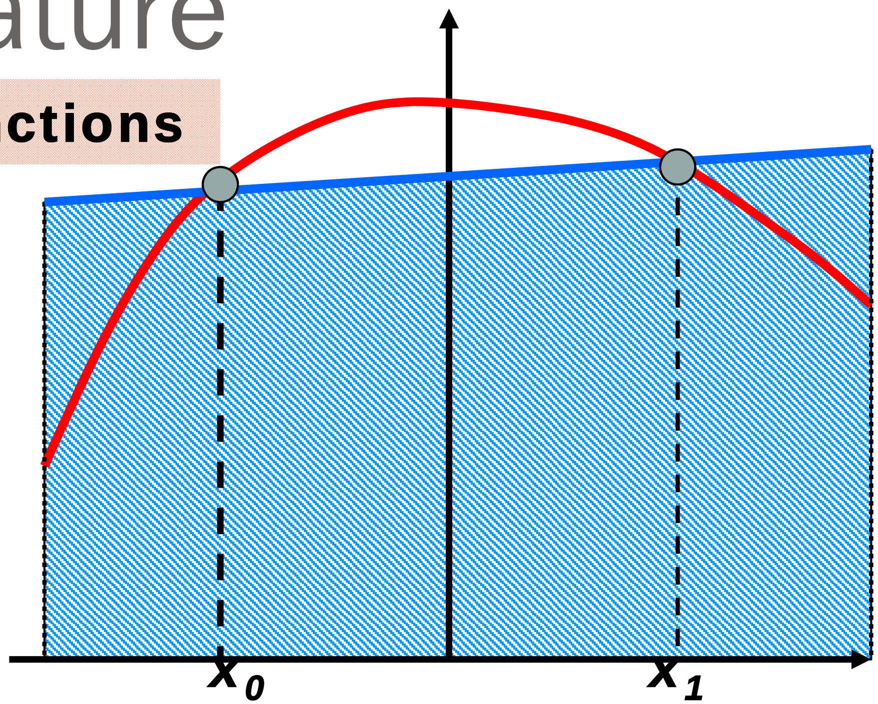
$$x_2=\left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}$$

$$c_1=\frac{b-a}{2}$$

$$c_2=\frac{b-a}{2}$$

Basis of Gauss Quadrature

Method 2 : Based on Polynomial functions



Hence Two-Point Gaussian Quadrature Rule

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{b-a}{2} f\left(\frac{b-a}{2}\left(-\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right)+\frac{b-a}{2} f\left(\frac{b-a}{2}\left(\frac{1}{\sqrt{3}}\right)+\frac{b+a}{2}\right)$$

Higher Point Gaussian Quadrature Formulas

Method 2 : Based on Polynomial functions

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

is called the three-point Gauss Quadrature Rule.

The coefficients c_1, c_2, and c_3, and the functional arguments x_1, x_2, and are calculated by assuming the formula gives exact expressions for integrating a fifth order polynomial

$$\int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right)dx$$

General n-point rules would approximate the integral

$$\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + \ldots\ldots\ldots + c_n f(x_n)$$

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

In handbooks, coefficients and arguments given for n-point

Gauss Quadrature Rule are

given for integrals

$$\int_{-1}^{1} g(x)dx \cong \sum_{i=1}^{n} c_i g(x_i)$$

as shown in Table 1.

Table 1: Weighting factors c and function arguments x used in Gauss Quadrature Formulas.

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$ $c_2 = 1.000000000$	$x_1 = -0.577350269$ $x_2 = 0.577350269$
3	$c_1 = 0.555555556$ $c_2 = 0.888888889$ $c_3 = 0.555555556$	$x_1 = -0.774596669$ $x_2 = 0.000000000$ $x_3 = 0.774596669$
4	$c_1 = 0.347854845$ $c_2 = 0.652145155$ $c_3 = 0.652145155$ $c_4 = 0.347854845$	$x_1 = -0.861136312$ $x_2 = -0.339981044$ $x_3 = 0.339981044$ $x_4 = 0.861136312$

Arguments and Weighing Factors for n-point Gauss Quadrature Formulas

Table 1 (cont.) : Weighting factors c and function arguments x used in Gauss Quadrature Formulas.

Points	Weighting Factors	Function Arguments
5	c_1 = 0.236926885 c_2 = 0.478628670 c_3 = 0.568888889 c_4 = 0.478628670 c_5 = 0.236926885	x_1 = -0.906179846 x_2 = -0.538469310 x_3 = 0.000000000 x_4 = 0.538469310 x_5 = 0.906179846
6	c_1 = 0.171324492 c_2 = 0.360761573 c_3 = 0.467913935 c_4 = 0.467913935 c_5 = 0.360761573 c_6 = 0.171324492	x_1 = -0.932469514 x_2 = -0.661209386 x_3 = -0.2386191860 x_4 = 0.2386191860 x_5 = 0.661209386 x_6 = 0.932469514

Example 1

a) Use two-point Gauss Quadrature Rule to approximate the distance covered by a rocket from t=8 to t=30 as given by

$$x = \int_{8}^{30} \left(2000\, ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

b) Find the true error, E_t

$$Exact \quad value = 11061.34m$$

c) Also, find the absolute relative true error, $\left| \in_a \right|$ for part (a).

Solution

First, change the limits of integration from [8,30] to [-1,1]

by previous relations as follows

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2}\int_{-1}^{1} f\left(\frac{30-8}{2}x+\frac{30+8}{2}\right)dx$$

$$= 11\int_{-1}^{1} f\left(11x+19\right)dx$$

Solution (cont)

$$= 11\int_{-1}^{1} f(11x+19)dx$$

Next, get weighting factors and function argument values from Table 1

for the two point rule,

$$c_1 = 1.000000000$$
$$x_1 = -0.577350269$$
$$c_2 = 1.000000000$$
$$x_2 = 0.577350269$$

Now one can use the Gauss Quadrature formula

$$11\int_{-1}^{1} f(11x+19)dx \approx 11c_1 f(11x_1+19) + 11c_2 f(11x_2+19)$$

$$= 11 f(11(-0.5773503)+19) + 11 f(11(0.5773503)+19)$$

$$= 11 f(12.64915) + 11 f(25.35085)$$

$$f(12.64915) = 2000\ln\left[\frac{140000}{140000-2100(12.64915)}\right] - 9.8(12.64915)$$

$$= 296.8317$$

$$= 11(296.8317) + 11(708.4811)$$

$$= 11058.44\ m$$

$$f(25.35085) = 2000\ln\left[\frac{140000}{140000-2100(25.35085)}\right] - 9.8(25.35085)$$

$$= 708.4811$$

$$Exact \quad value = 11061.34m$$

$$Approx = 11058.44 \ m$$

Solution (cont)

b) The true error, E_t is

$$E_t = True\ Value - Approximate\ Value$$

$$= 11061.34 - 11058.44$$

$$= 2.9000 \ m$$

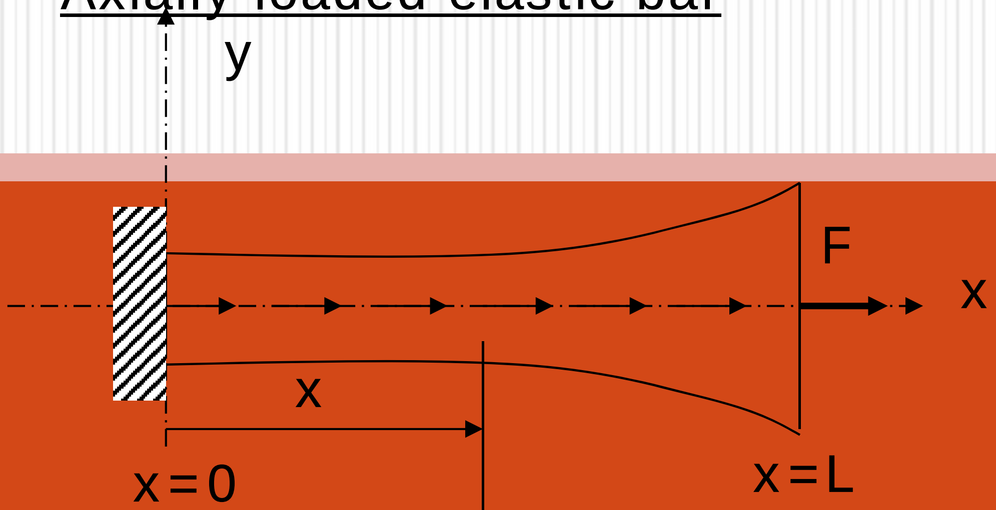
c) The absolute relative true error, $|\in_t|$ is (Exact value = 11061.34m)

$$|\in_t| = \left| \frac{11061.34 - 11058.44}{11061.34} \right| \times 100\%$$

$$= 0.0262\%$$

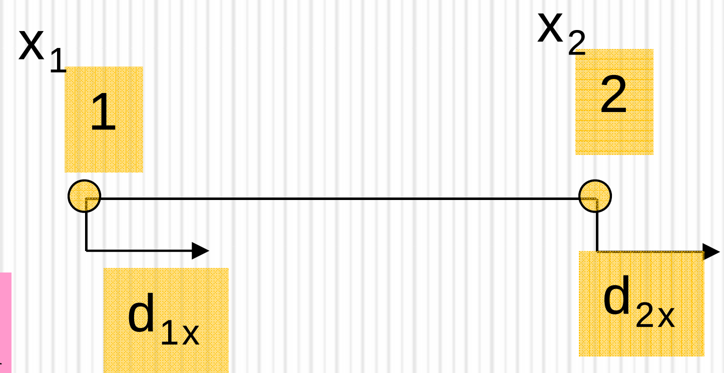
Method 3 : Based on Isoperimetric element

Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

For each element



Element stiffness matrix

$$\underline{k} = \int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx$$

$$k_{ij} = \int_{x_1}^{x_2} B_i E B_j \; A dx$$

where $$B_i = \frac{dN_i(x)}{dx}$$

Only for a linear finite element

$$\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx = \frac{1}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{x_1}^{x_2} AE dx = \left(\int_{x_1}^{x_2} AE dx \right) \frac{1}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element nodal load vector

$$\boxed{\underline{f_b} = \int_{x_1}^{x_2} \underline{N}^T \; b \, dx}$$

$$f_{bi} = \int_{x_1}^{x_2} N_i \; b \, dx$$

Question: How do compute these integrals using a computer?

Isoperimetric Procedure

Method 3 : Based on Isoperimetric element

Any integral from x_1 to x_2 can be transformed to the following integral on (-1, 1)

$$I = \int_{-1}^{1} f(\xi)\, d\xi$$

Use the following change of variables

$$x = \frac{1-\xi}{2} x_1 + \frac{1+\xi}{2} x_2$$

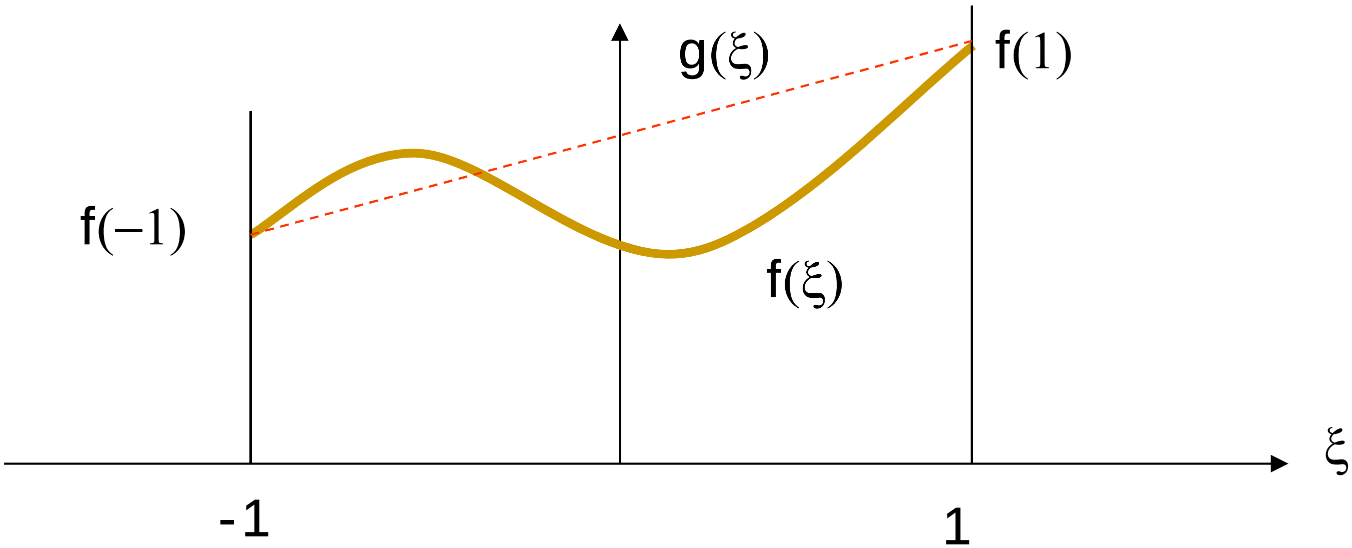
Goal: Obtain a good approximate value of this integral

1. Newton-Cotes Schemes (trapezoidal rule, Simpson’s rule, etc)
2. Gauss Integration Schemes

NOTE: Integration schemes in 1D are referred to as “quadrature rules”

Method 3 : Based on Isoperimetric element

Trapezoidal rule: Approximate the function f(ξ) by a straight line g(ξ) that passes through the end points and integrate the straight line



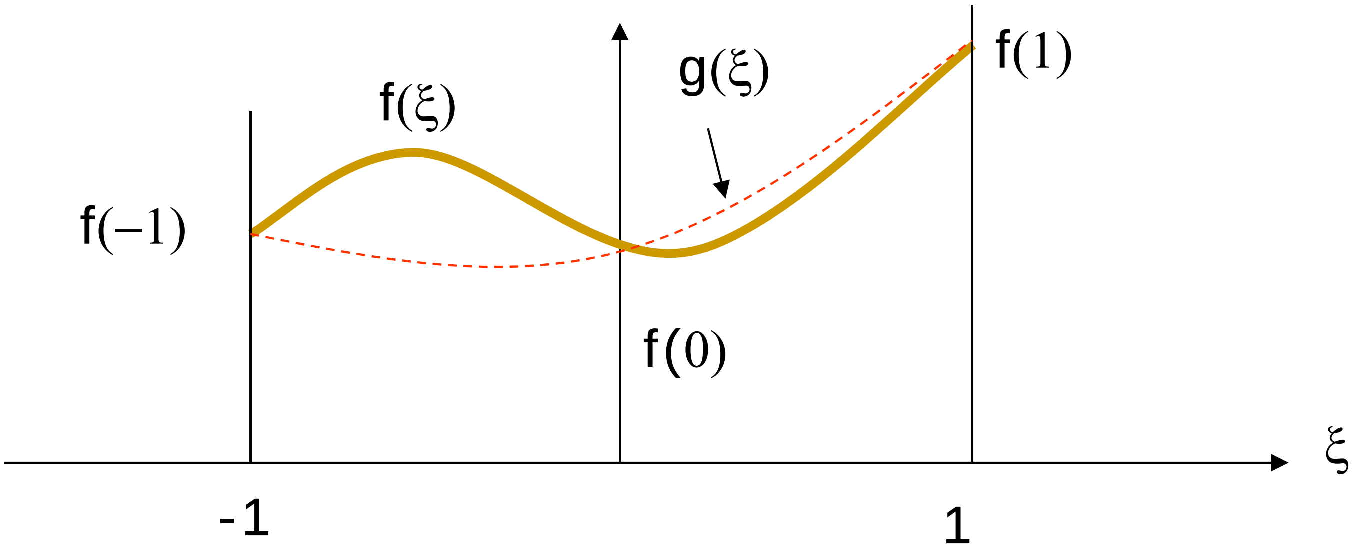
$$g(\xi) = \frac{1-\xi}{2} f(-1) + \frac{1+\xi}{2} f(1)$$

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx \int_{-1}^{1} g(\xi)\, d\xi = f(1) + f(-1)$$

- Requires the function f(x) to be evaluated at 2 points (-1, 1)
- Constants and linear functions are exactly integrated
- Not good for quadratic and higher order polynomials

How can I make this better?

Simpson's rule: Approximate the function f(ξ) by a parabola g(ξ) that passes through the end points and through f(0) and integrate the parabola



$$g(\xi)=\frac{\xi(\xi-1)}{2}f(-1)+(1-\xi)(1+\xi)f(0)+\frac{\xi(1+\xi)}{2}f(1)$$

$$I=\int_{-1}^{1}f(\xi)\,d\xi\approx\int_{-1}^{1}g(\xi)\,d\xi=\frac{1}{3}f(1)+\frac{4}{3}f(0)+\frac{1}{3}f(-1)$$

- Requires the function f(x) to be evaluated at 3 points (-1,0, 1)
- Constants, linear functions and parabolas are exactly integrated
- Not good for cubic and higher order polynomials

Notice that both the integration formulas had the general form

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{i=1}^{M} W_i \ f(\xi_i)$$

Weight Integration point

Trapezoidal rule:
M=2

$$W_1 = 1 \qquad \xi_1 = -1$$
$$W_2 = 1 \qquad \xi_2 = 1$$

Accurate for polynomial of degree at most 1 (=M-1)

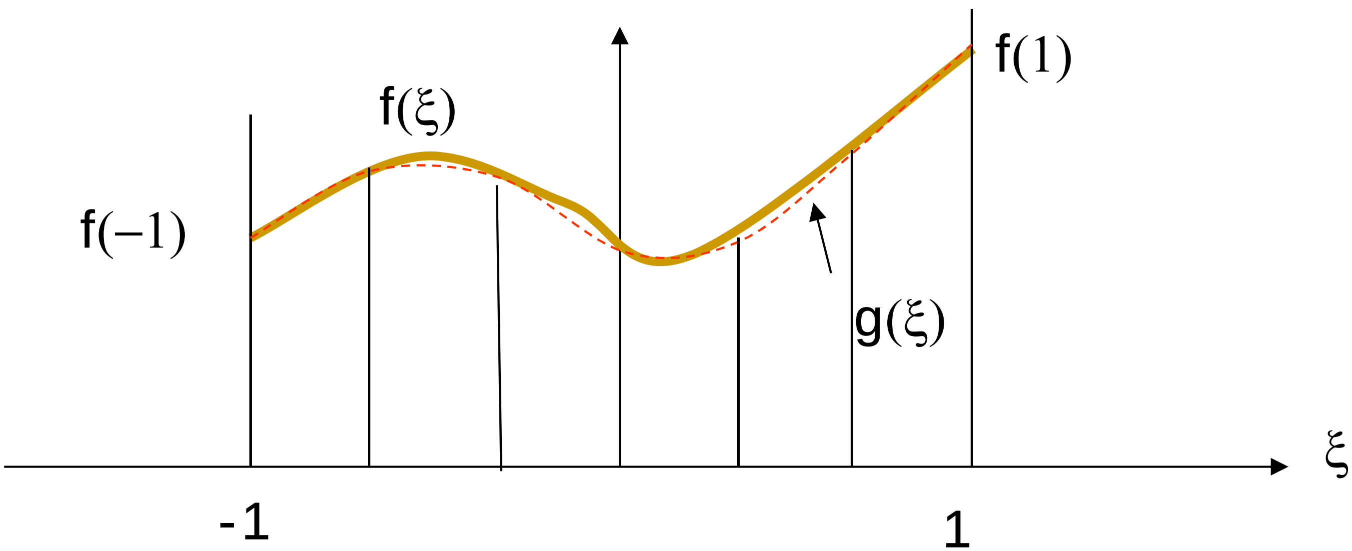
Simpson's rule:
M=3

$$W_1 = 1/3 \qquad \xi_1 = -1$$
$$W_2 = 4/3 \qquad \xi_2 = 0$$
$$W_2 = 1/3 \qquad \xi_2 = 1$$

Accurate for polynomial of degree at most 2 (=M-1)

Generalization of these two integration rules: **Newton-Cotes**

- Divide the interval (-1,1) into M-1 **equal** intervals using M points
- Pass a polynomial of degree M-1 through these M points (the value of this polynomial will be equal to the value of the function at these M-1 points)
- Integrate this polynomial to obtain an approximate value of the integral



Gauss quadrature

$$I=\int_{-1}^{1} f(\xi)\,d\xi \approx \sum_{i=1}^{M} W_i \; f(\xi_i)$$

Weight Integration point

How can choose the integration points **and** weights to **exactly integrate a polynomial of degree 2M-1**?

Remember that now do not know, a priori, the location of the integration points.

Method 3 : Based on Isoperimetric element

Example: M=1 (Midpoint qudrature)

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx W_1 \; f(\xi_1)$$

How can choose W_1 and x_1 so that may integrate a (2M-1=1) linear polynomial exactly?

$$f(\xi) = a_0 + a_1 \xi$$

$$\int_{-1}^{1} f(\xi)\, d\xi = 2a_0$$

But want

$$\int_{-1}^{1} f(\xi)\, d\xi = W_1 f(\xi_1) = a_0 W_1 + a_0 W \xi_1$$

Hence, obtain the identity

$$2a_0 = a_0 W_1 + a_1 W_1 \xi_1$$

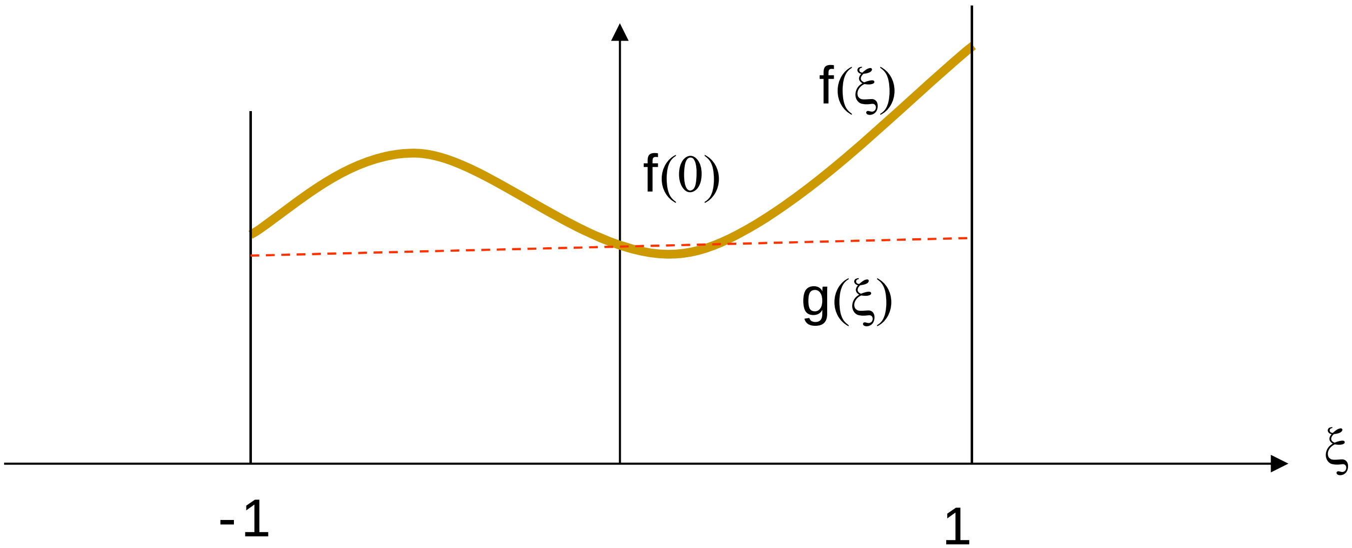
For this to hold for arbitrary a_0 and a_1 need to satisfy 2 conditions

$$Condition \quad 1: \ W_1 = 2$$
$$Condition \quad 2: \ W_1 \xi_1 = 0$$

i.e., $W_1 = 2; \xi_1 = 0$

For M=1

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx 2\, f(0)$$



Midpoint quadrature rule:

- Only one evaluation of f(ξ) is required at the midpoint of the interval.
- Scheme is accurate for constants and linear polynomials (compare with Trapezoidal rule)

Method 3 : Based on Isoperimetric element

Example: M=2

$$I=\int_{-1}^{1} f(\xi)\,d\xi \approx W_1\ f(\xi_1)+W_2\ f(\xi_2)$$

How can choose W_1 ,W_2 ξ_1 and ξ_2 so that may integrate a **polynomial** of degree (2M-1=4-1=3) exactly?

$$f(\xi)=a_0+a_1\xi+a_2\xi^2+a_3\xi^3$$

$$\int_{-1}^{1} f(\xi)\,d\xi=2a_0+\frac{2}{3}a_2$$

But want

$$\int_{-1}^{1} f(\xi)\,d\xi=W_1 f(\xi_1)+W_2\ f(\xi_2)$$
$$=a_0(W_1+W_2)+a_1(W_1\xi_1+W_2\xi_2)+a_2\left(W_1\xi_1{}^2+W_2\xi_2{}^2\right)+a_3\left(W_1\xi_1{}^3+W_2\xi_2{}^3\right)$$

$$\int_{-1}^{1} f(\xi)\, d\xi = W_1 f(\xi_1) + W_2\ f(\xi_2)$$

$$= a_0\left(W_1 + W_2\right) + a_1\left(W_1\xi_1 + W_2\xi_2\right) + a_2\left(W_1\xi_1^{\ 2} + W_2\xi_2^{\ 2}\right) + a_3\left(W_1\xi_1^{\ 3} + W_2\xi_2^{\ 3}\right)$$

Hence, obtain 4 conditions to determine the 4 unknowns (W_1 ,W_2 ξ_1 and ξ_2)

$$Condition\quad 1:\ W_1 + W_2 = 2$$

$$Condition\quad 2:\ W_1\xi_1 + W_2\xi_2 = 0$$

$$Condition\quad 3:\ W_1\xi_1^{\ 2} + W_2\xi_2^{\ 2} = \frac{2}{3}$$

$$Condition\quad 4:\ W_1\xi_1^{\ 3} + W_2\xi_2^{\ 3} = 0$$

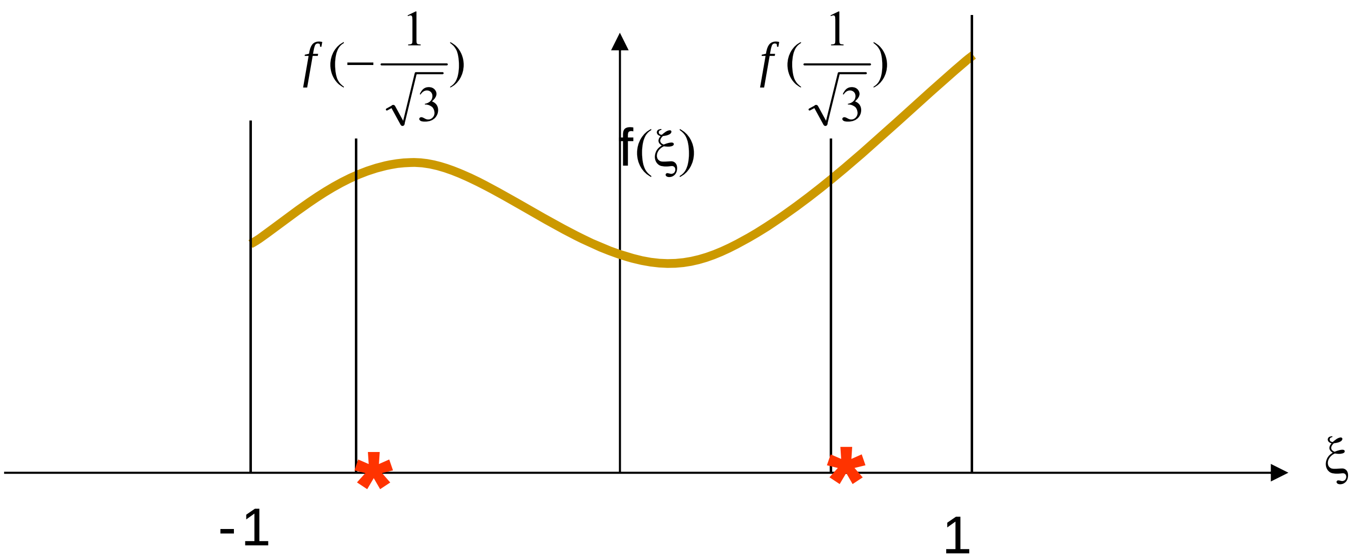
Check that the following is the solution

$$W_1 = W_2 = 1$$

$$\xi_1 = -\frac{1}{\sqrt{3}};\ \xi_2 = \frac{1}{\sqrt{3}}$$

For M=2

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$$



- Only two evaluations of f(ξ) is required.
- Scheme is accurate for polynomials of degree at most 3 (compare with Simpson's rule)

Exercise: Derive the 6 conditions required to find the integration points and weights for a 3-point Gauss quadrature rule

Newton-Cotes	Gauss quadrature
1. 'M' integration points are necessary to exactly integrate a polynomial of degree 'M-1' 2. More expensive	1. 'M' integration points are necessary to exactly integrate a polynomial of degree '2M-1' 2. Less expensive 3. Exponential convergence, error proportional to $\left(\frac{1}{2M}\right)^{2M}$

Gauss quadrature:

Example

$$I = \int_{-1}^{1} \mathrm{f}(\xi)\mathrm{d}\xi \quad where \ \mathrm{f}(\xi) = \xi^3 + \xi^2$$

Exact integration

$$I = \frac{2}{3}$$

Integrate and check!

Newton-Cotes

To exactly integrate this need a 4-point Newton-Cotes formula. Why?

Gauss

To exactly integrate this I need a 2-point Gauss formula. Why?

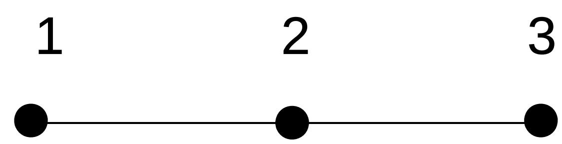
$$I = \int_{-1}^{1} \mathrm{f}(\xi)\mathrm{d}\xi$$

$$= \mathrm{f}(-\frac{1}{\sqrt{3}}) + \mathrm{f}(\frac{1}{\sqrt{3}})$$

$$= \frac{2}{3}$$

Exact answer!

Quadratic Element



$\xi = -1$ $\quad$ $\xi = 0$ $\quad$ $\xi = 1$

Nodal shape functions

$$N_1(\xi) = \frac{\xi}{2}(\xi - 1)$$

$$N_2(\xi) = \left(1 - \xi^2\right)$$

$$N_3(\xi) = \frac{\xi}{2}(\xi + 1)$$

Stiffness matrix

$$\underline{k} = \int_{-1}^{1} \underline{\mathrm{B}}^{\mathrm{T}} \mathrm{E} \underline{\mathrm{B}} \;\; A d\xi = AE \int_{-1}^{1} \underline{\mathrm{B}}^{\mathrm{T}} \underline{\mathrm{B}} \;\; \mathrm{d}\xi$$

Assuming E and A are constants

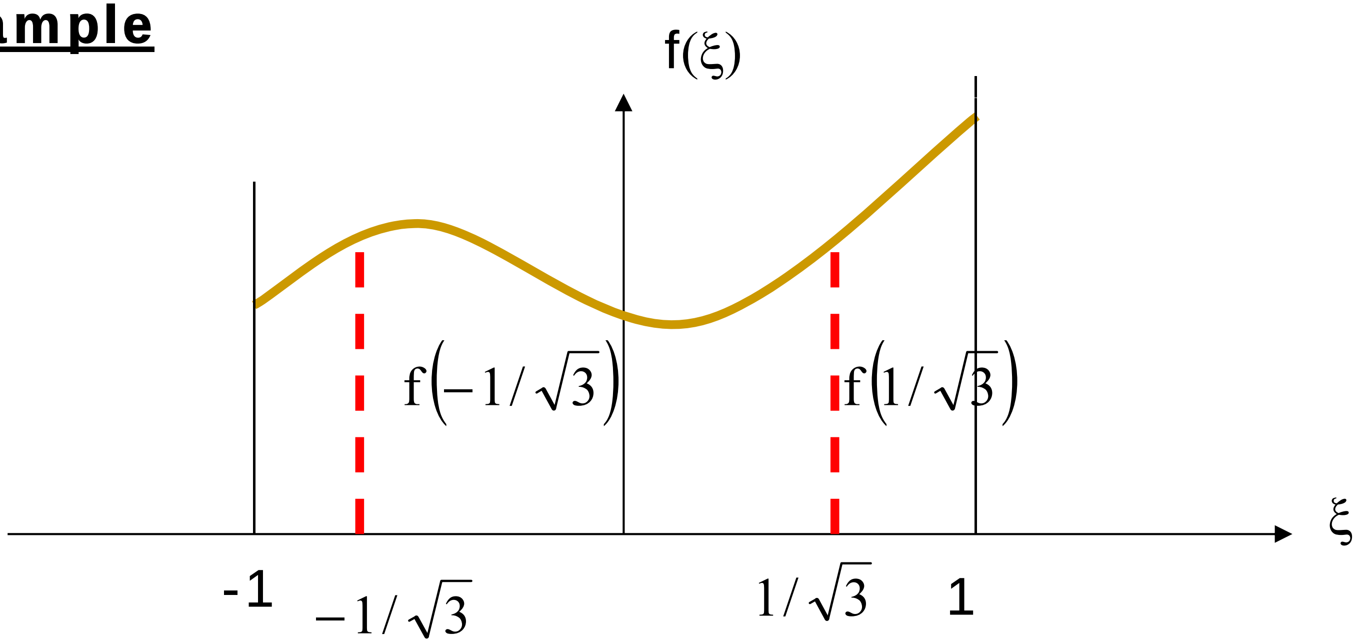
$$\underline{\mathrm{B}} = \frac{d\,\underline{N}}{d\xi} = \left[\frac{dN_1}{d\xi} \;\; \frac{dN_2}{d\xi} \;\; \frac{dN_3}{d\xi} \right] = \left[\frac{1}{2}(2\xi - 1) \quad -2\xi \quad \frac{1}{2}(2\xi + 1) \right]$$

$$\underline{k} = \int_{-1}^{1} \underline{B}^{T} E \underline{B} \; A d\xi = AE \int_{-1}^{1} \underline{B}^{T} \underline{B} \; d\xi$$

$$= AE \int_{-1}^{1} \begin{bmatrix} (\xi - 1/2)^2 & -2\xi(\xi - 1/2) & (\xi^2 - 1/4) \\ -2\xi(\xi - 1/2) & 4\xi^2 & -2\xi(\xi + 1/2) \\ (\xi^2 - 1/4) & -2\xi(\xi + 1/2) & (\xi + 1/2)^2 \end{bmatrix} d\xi$$

Need to exactly integrate **quadratic** terms.
Hence need a **2-point Gauss** quadrature scheme..Why?

Example



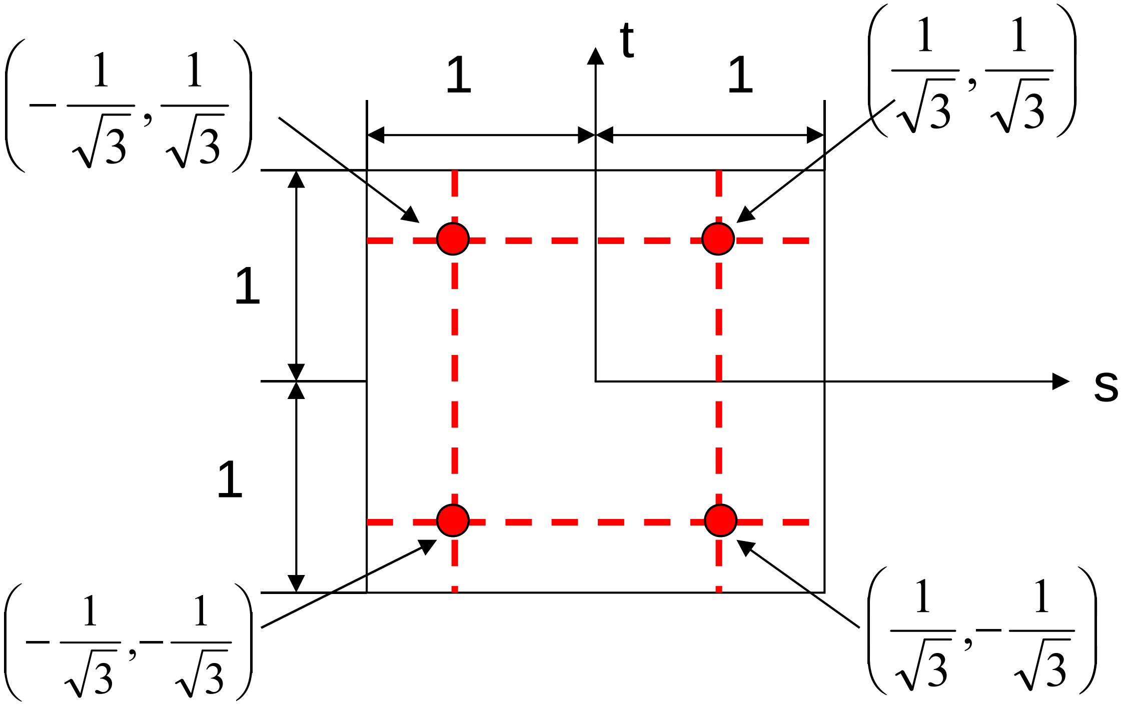
A 2-point Gauss quadrature rule

$$\int_{-1}^{1} f(\xi)\, d\xi \approx f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

is **exact** for a polynomial of degree 3

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$	$x_1 = -0.577350269$
	$c_2 = 1.000000000$	$x_2 = 0.577350269$
3	$c_1 = 0.555555556$	$x_1 = -0.774596669$
	$c_2 = 0.888888889$	$x_2 = 0.000000000$
	$c_3 = 0.555555556$	$x_3 = 0.774596669$
4	$c_1 = 0.347854845$	$x_1 = -0.861136312$
	$c_2 = 0.652145155$	$x_2 = -0.339981044$
	$c_3 = 0.652145155$	$x_3 = 0.339981044$
	$c_4 = 0.347854845$	$x_4 = 0.861136312$

2D square domain



$$I=\int_{-1}^{1}\int_{-1}^{1} f(s,t)\,dsdt$$

$$
\begin{aligned}
I &= \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,dsdt \\
&\approx \int_{-1}^{1}\left(\sum_{j=1}^{M} W_j\, f(s,t_j)\right) ds \\
&\approx \sum_{i=1}^{M}\sum_{j=1}^{M} W_i W_j\, f(s_i,t_j) \\
&= \sum_{i=1}^{M}\sum_{j=1}^{M} W_{ij}\, f(s_i,t_j)
\end{aligned}
$$

Using 1D Gauss rule to integrate along ‘t’

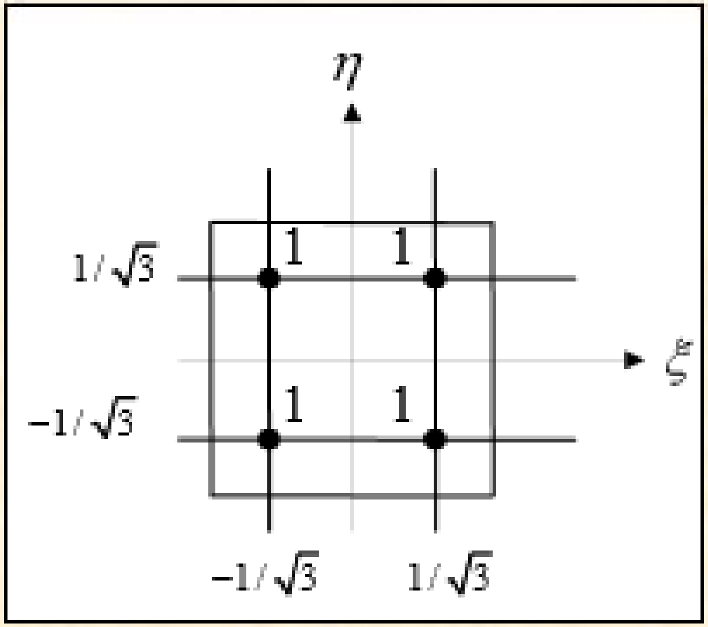
Using 1D Gauss rule to integrate along ‘s’

Where $W_{ij} = W_i\, W_j$

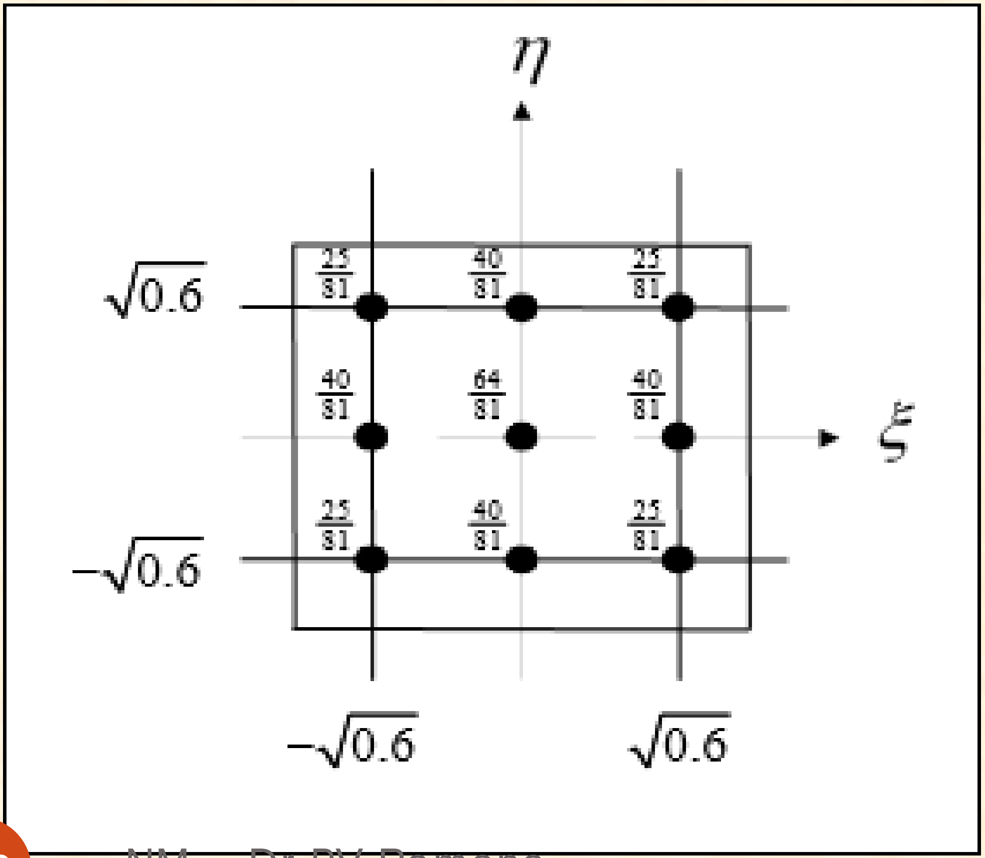
Gauss quadrature in two dimensions

$$I = \int_{-1}^{1}\int_{-1}^{1} \phi(\xi,\eta)\,d\xi\,d\eta \approx \int_{-1}^{1}\left[\sum_i W_i \phi(\xi_i,\eta)\right]d\eta$$

$$= \sum_j W_j \left[\sum_i W_i \phi(\xi_i,\eta_j)\right] = \sum_j \sum_i W_j W_i \phi(\xi_i,\eta_j)$$



2 point rule



m x n rule possible but not recommended

3 point rule

For M=2

$$I \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_{ij}\, f(s_i, t_j) \qquad W_{ij} = W_i\, W_j = 1$$

$$= f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

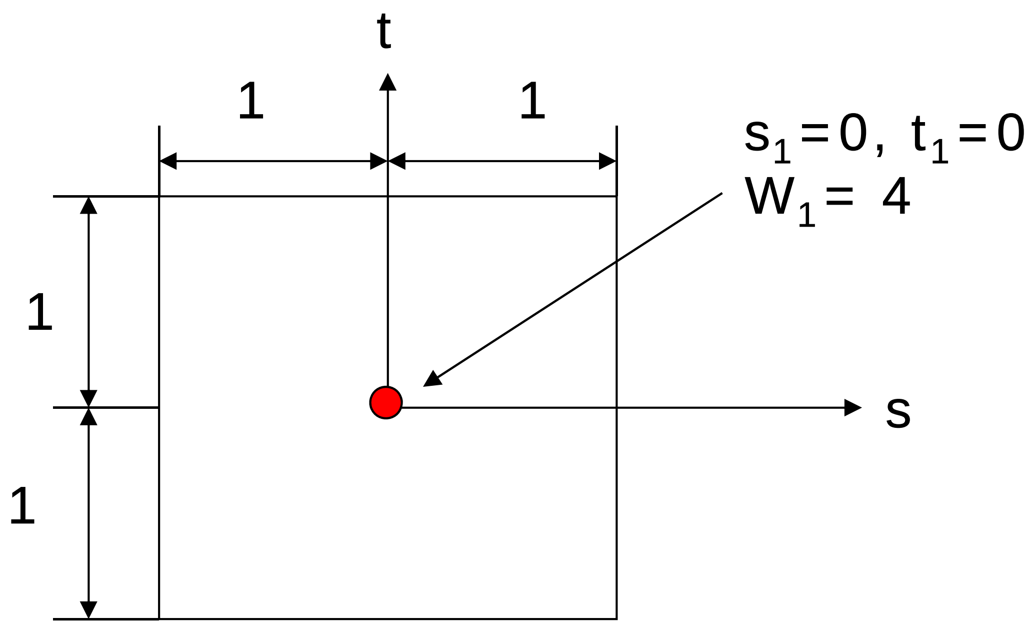
Number the Gauss points IP=1,2,3,4

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, ds dt \approx \sum_{IP=1}^{4} W_{IP}\, f_{IP}$$

Points	Weighting Factors	Function Arguments
2	c_1 = 1.000000000	x_1 = -0.577350269
	c_2 = 1.000000000	x_2 = 0.577350269
3	c_1 = 0.555555556	x_1 = -0.774596669
	c_2 = 0.888888889	x_2 = 0.000000000
	c_3 = 0.555555556	x_3 = 0.774596669
4	c_1 = 0.347854845	x_1 = -0.861136312
	c_2 = 0.652145155	x_2 = -0.339981044
	c_3 = 0.652145155	x_3 = 0.339981044
	c_4 = 0.347854845	x_4 = 0.861136312

CASE I: M=1 (One-point GQ rule)

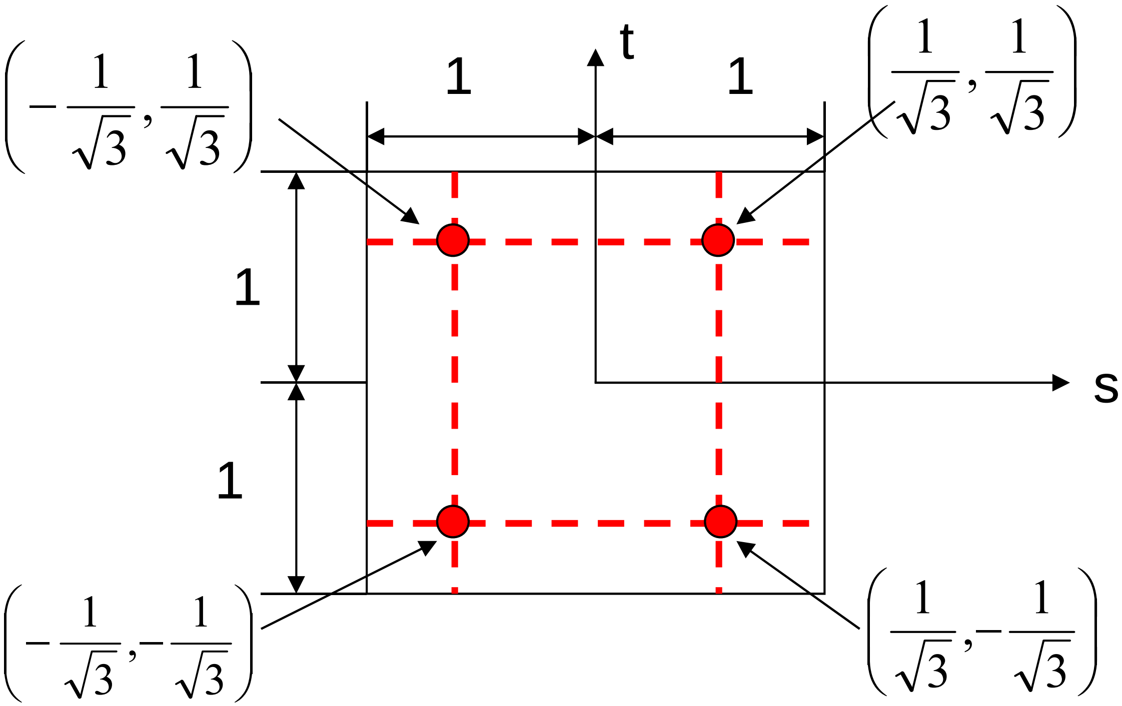
$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt \approx 4 f(0,0)$$



is exact for a product of two linear polynomials

<u>CASE II:</u> M=2 (2x2 GQ rule)

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$	$x_1 = -0.577350269$
	$c_2 = 1.000000000$	$x_2 = 0.577350269$
3	$c_1 = 0.555555556$	$x_1 = -0.774596669$
	$c_2 = 0.888888889$	$x_2 = 0.000000000$
	$c_3 = 0.555555556$	$x_3 = 0.774596669$
4	$c_1 = 0.347854845$	$x_1 = -0.861136312$
	$c_2 = 0.652145155$	$x_2 = -0.339981044$
	$c_3 = 0.652145155$	$x_3 = 0.339981044$
	$c_4 = 0.347854845$	$x_4 = 0.861136312$



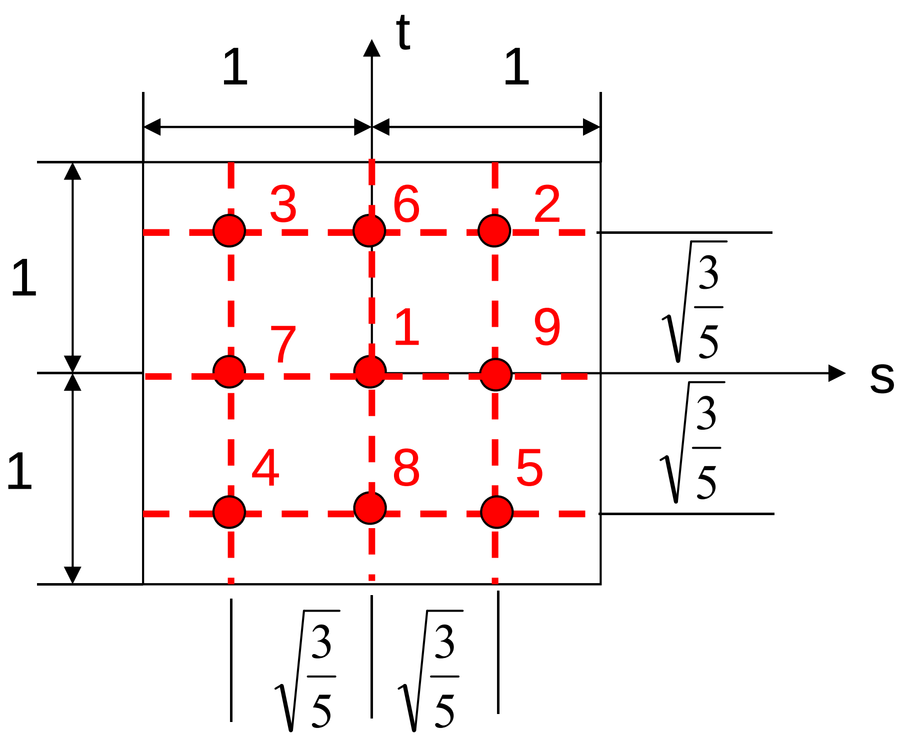
$$I \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_{ij}\, f(s_i, t_j)$$

$$= f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$$

is exact for a product of two cubic polynomials

CASE III: M=3 (3x3 GQ rule)

Points	Weighting Factors	Function Arguments
2	$c_1 = 1.000000000$	$x_1 = -0.577350269$
	$c_2 = 1.000000000$	$x_2 = 0.577350269$
3	$c_1 = 0.555555556$	$x_1 = -0.77459666$
	$c_2 = 0.888888889$	$x_2 = 0.00000000$
	$c_3 = 0.555555556$	$x_3 = 0.77459666$
4	$c_1 = 0.347854845$	$x_1 = -0.86113631$
	$c_2 = 0.652145155$	$x_2 = -0.33998104$
	$c_3 = 0.652145155$	$x_3 = 0.33998104$
	$c_4 = 0.347854845$	$x_4 = 0.86113631$



$$W_1 = \frac{64}{81},$$

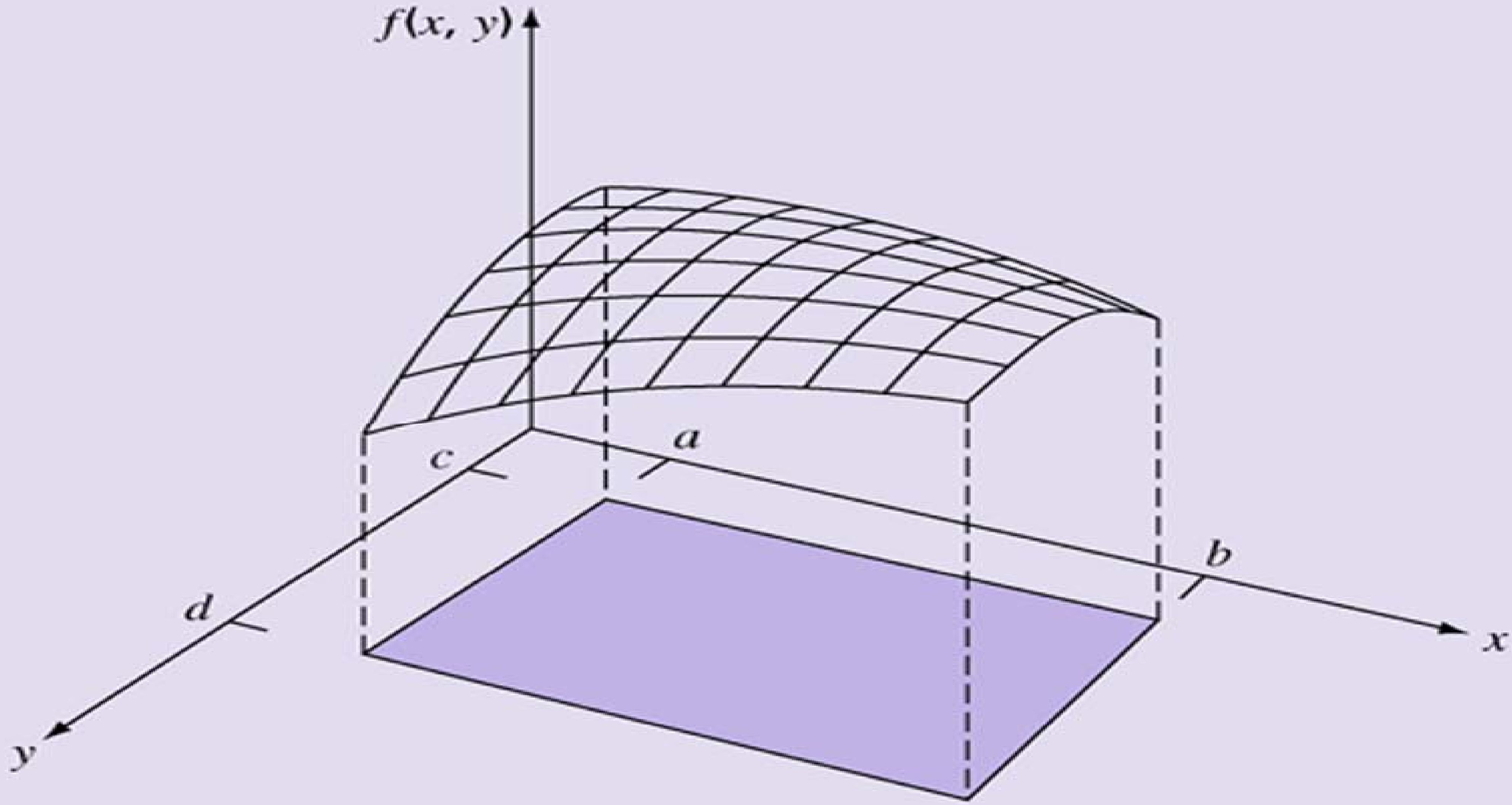
$$W_2 = W_3 = W_4 = W_5 = \frac{25}{81}$$

$$W_6 = W_7 = W_8 = W_9 = \frac{40}{81}$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, ds dt \approx \sum_{i=1}^{3}\sum_{j=1}^{3} W_{ij}\, f(s_i, t_j)$$

is exact for a product of two 1D polynomials of degree 5

Double Integral



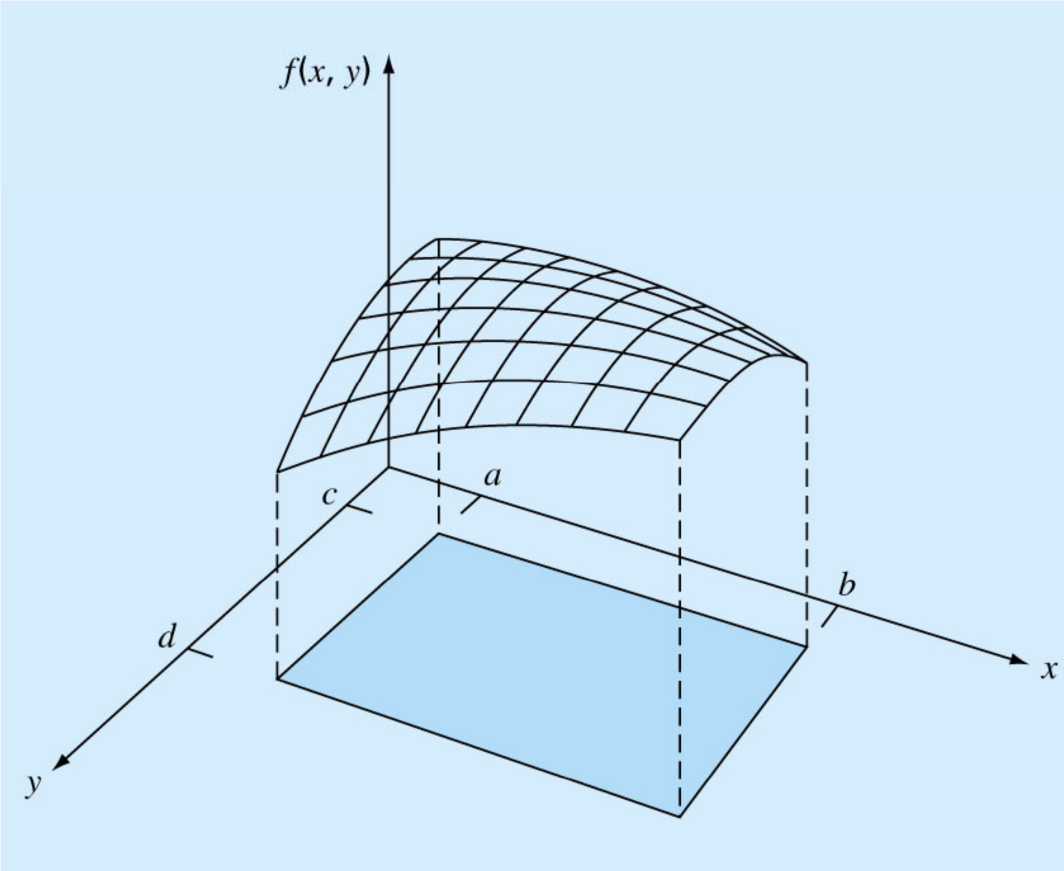
- Area under the function surface

$$\int_c^d \left(\int_a^b f(x,y)dx \right) dy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \int_a^b f(x,y)dxdy$$

Multiple Integration

- Double integral:

$$I = \int_{y=c}^{d} \int_{x=a}^{b} f(x, y)\,dx\,dy$$



Multiple Integration using Gauss Quadrature Technique

- $$x = b \to \xi = 1 \qquad \& \qquad x = a \to \xi = -1$$

$$x = \frac{b+a}{2} + \frac{b-a}{2}\xi \qquad \& \qquad dx = \frac{b-a}{2} d\xi$$

- $$y = d \to \eta = 1 \qquad \& \qquad x = c \to \eta = -1$$

$$y = \frac{d+c}{2} + \frac{d-c}{2}\eta \qquad \& \qquad dy = \frac{d-c}{2} d\eta$$

- $$I = \int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\,dx\,dy = \frac{d-c}{2}\frac{b-a}{2}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\,d\xi\,d\eta$$

Multiple Integration using Gauss Quadrature Technique

Now we can use the Gauss Quadrature technique:

$$I \cong \frac{d-c}{2}\frac{b-a}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} c_j c_i f(\xi_i,\eta_j)$$

If we use two points Gauss Formula:

$$I = \frac{d-c}{2}\frac{b-a}{2}\sum_{j=1}^{2}\sum_{i=1}^{2} c_j c_j f(\xi_i,\eta_j)$$

$$I = \frac{d-c}{2}\frac{b-a}{2}\sum_{j=1}^{2} c_j[c_1 f(-\frac{1}{\sqrt{3}},\eta_j)+c_2 f(\frac{1}{\sqrt{3}},\eta_j)]$$

$$I = \frac{d-c}{2}\frac{b-a}{2}[c_1\{c_1 f(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})+c_2 f(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})\}+ c_2\{c_1 f(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})+c_2 f(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})\}]$$

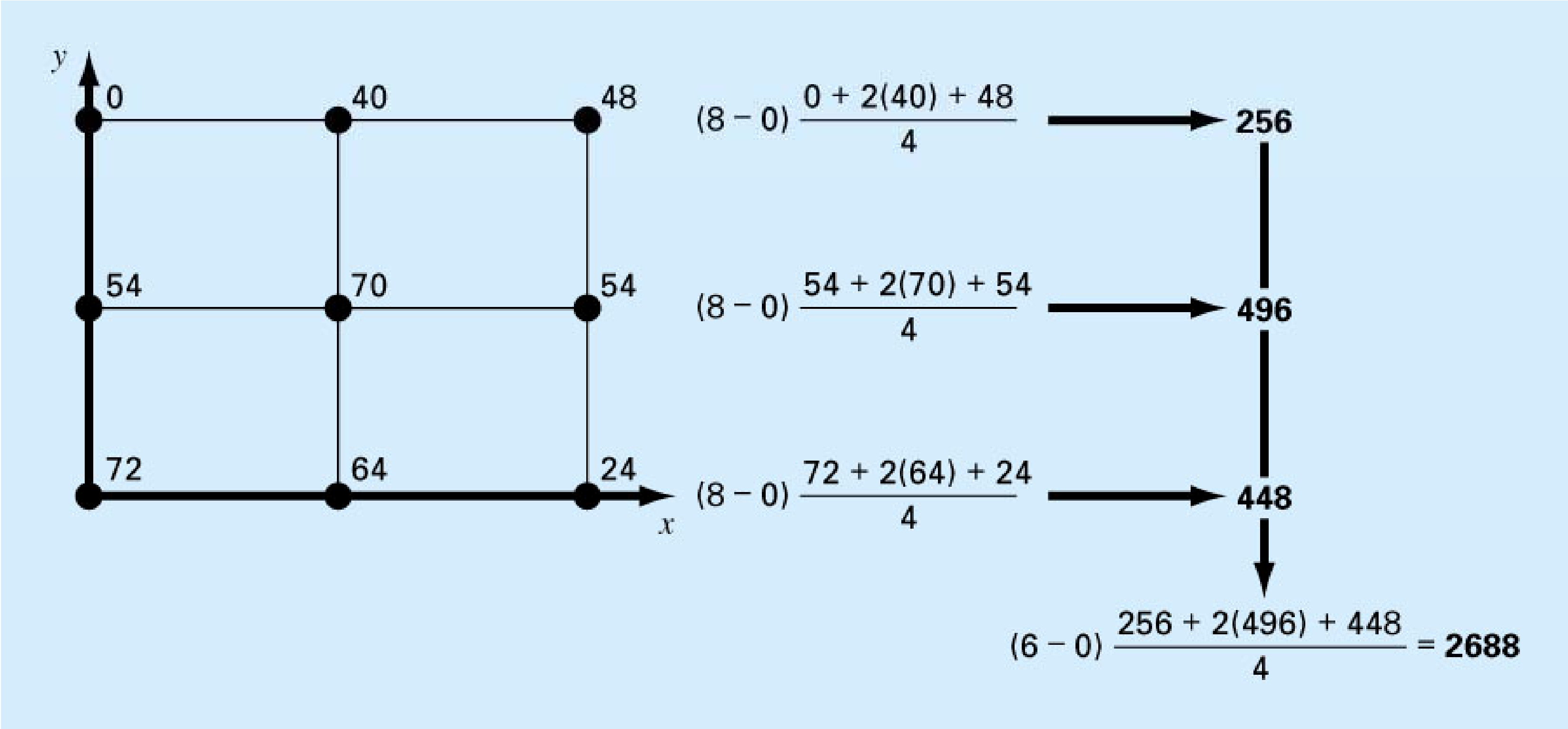
Double integral - Example

- Compute the average temperature of a rectangular heated plate which is 8m long in the x direction and 6 m wide in the y direction. The temperature is given as:

$$T(x,y) = 2xy + 2x - x^2 - 2y^2 + 72$$

- (Use 2 segment applications of the trapezoidal rule in each dimension)

Double integral - Example



$$I = \int_0^6 \int_0^8 (2xy + 2x - x^2 - 2y^2 + 72)dxdy$$

$$Multiple\ Trapezoidal\ rule\ (n = 2) \rightarrow I = 2688, T_{avg} = 2688/(6 \times 8) = 56$$

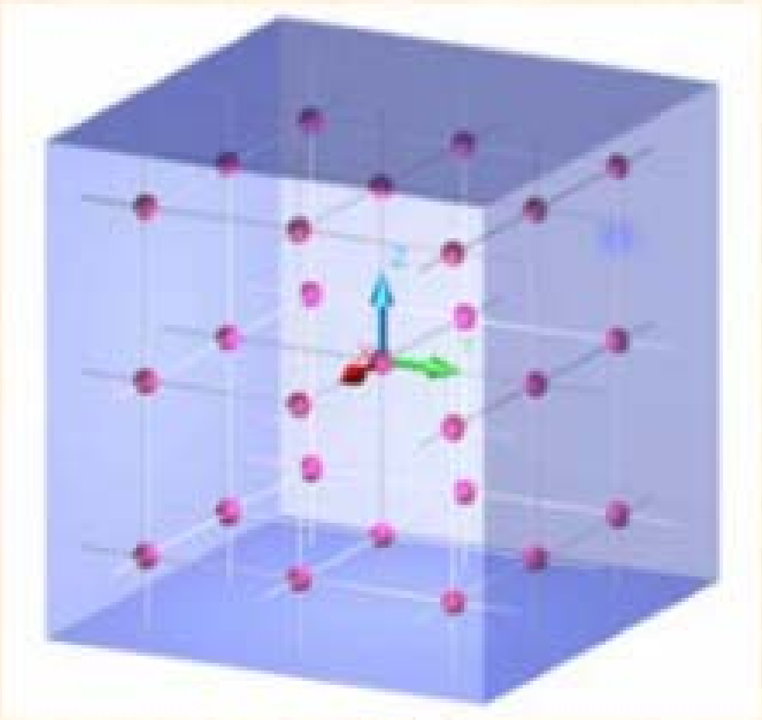
$$Simpson\ 1/3\ rule \rightarrow I = 2816, T_{avg} = 2816/(6 \times 8) = 58.6667$$

HW: Use two points Gauss formula to solve the problem

Three dimensions

$$I = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \phi(\xi,\eta,\varsigma)\,d\xi\,d\eta\,d\varsigma$$

$$\approx \sum_i \sum_j \sum_k W_i W_j W_k \phi(\xi_i,\eta_j,\varsigma_k)$$



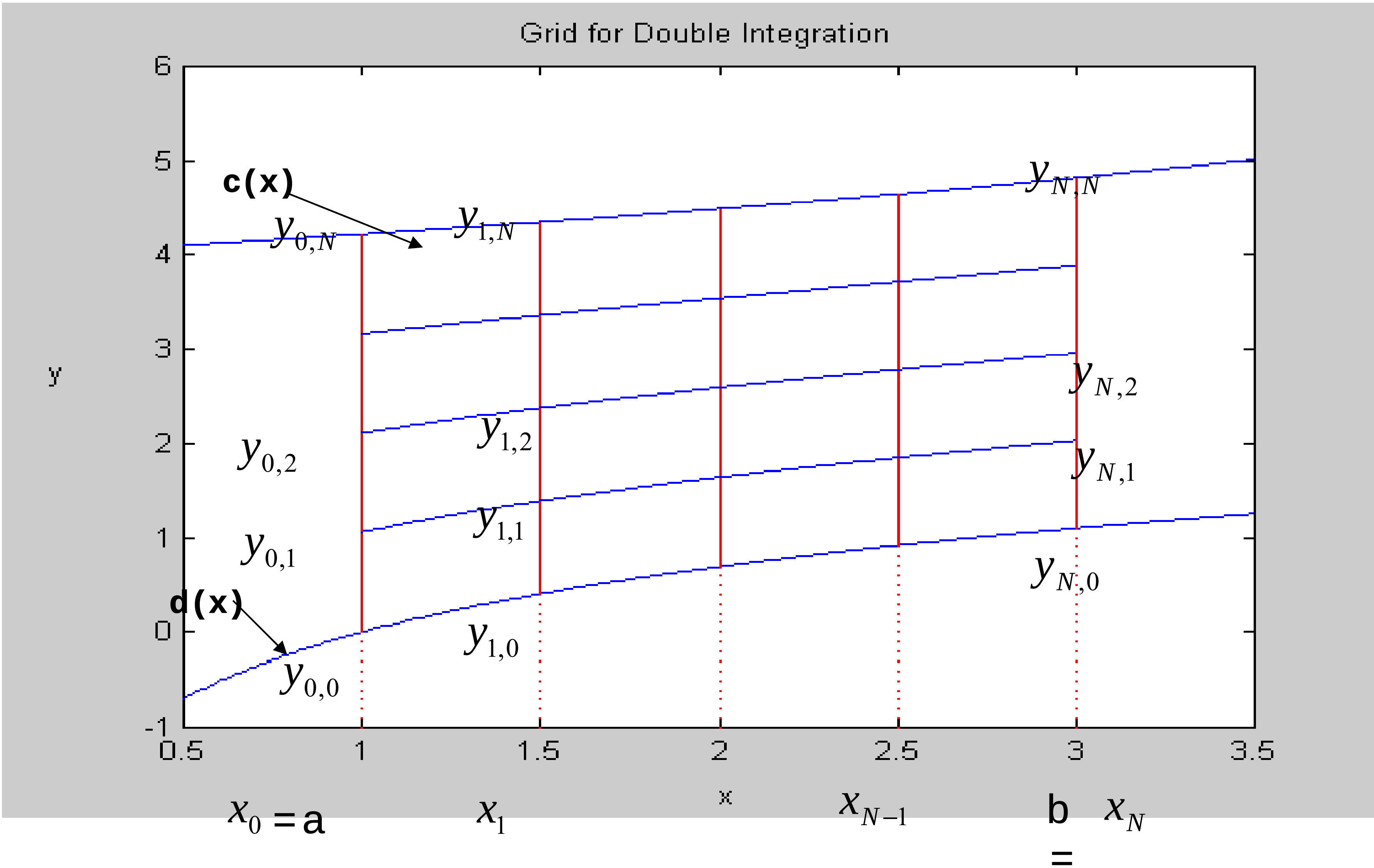
3x3x3 Gauss points in 3D

Computational cost of 3x3x3 Gauss quadrature for the brick element:

27 (Gauss points) x 24x24 = 15,552 (function evaluations per element)


Thank You!

Numerical Integration in a Two Dimensional Domain



- A double integration in the domain is written as

$$I = \int_a^b \left[\int_{c(x)}^{d(x)} f(x, y) dy \right] dx$$

- The numerical integration of above equation is to reduce to a combination of one-dimensional problems

Procedure:

- **Step 1:** Define $$G(x)=\int_{c(x)}^{d(x)} f(x,y)dy$$

So, the solution is $$I=\int_a^b G(x)dx$$

- **Step 2:** Divided the range of integration [a,b] into N equispaced intervals with the interval size

$$h_x=\frac{b-a}{N}$$

So, the grid points will be denoted by $x_0, x_1, \ldots, x_N$ and then we have

$$G(x_i)=\int_{c(x_i)}^{d(x_i)} f(x_i,y)dy,$$

- **Step 3:** Divided the domain of integration $[d(x_i), c(x_i)]$ into N equispaced intervals with the interval size

$$h_y = \frac{[d(x_i) - c(x_i)]}{N}$$

So, the grid points denoted by $y_{i,0}, y_{i,1}, \ldots, y_{i,N}$

- **Step 4:** By Applying numerical integration for one-dimensional (for example the trapezoidal rule) we have

$$G(x_i) = \frac{h_y}{2}\left\{ f(x_i, y_{i,0}) + 2\sum_{j=1}^{N-1} f(x_i, y_{i,j}) + f(x_i, y_{i,N}) \right\}$$

for $i = 0, 1, 2, \ldots, N$

- **Step 5:** By applying numerical integration (for example trapezoidal rule) in one-dimensional domain we have the solution of double integration is

$$I = \frac{h_x}{2}\left\{G(x_0) + 2\sum_{i=1}^{N-1} G(x_i) + G(x_N)\right\}$$