Optimization Methods

One-Dimensional Unconstrained Optimization

- **Golden-Section Search**
- **Newton's Method**
- Quadratic Interpolation

Multi-Dimensional Unconstrained Optimization

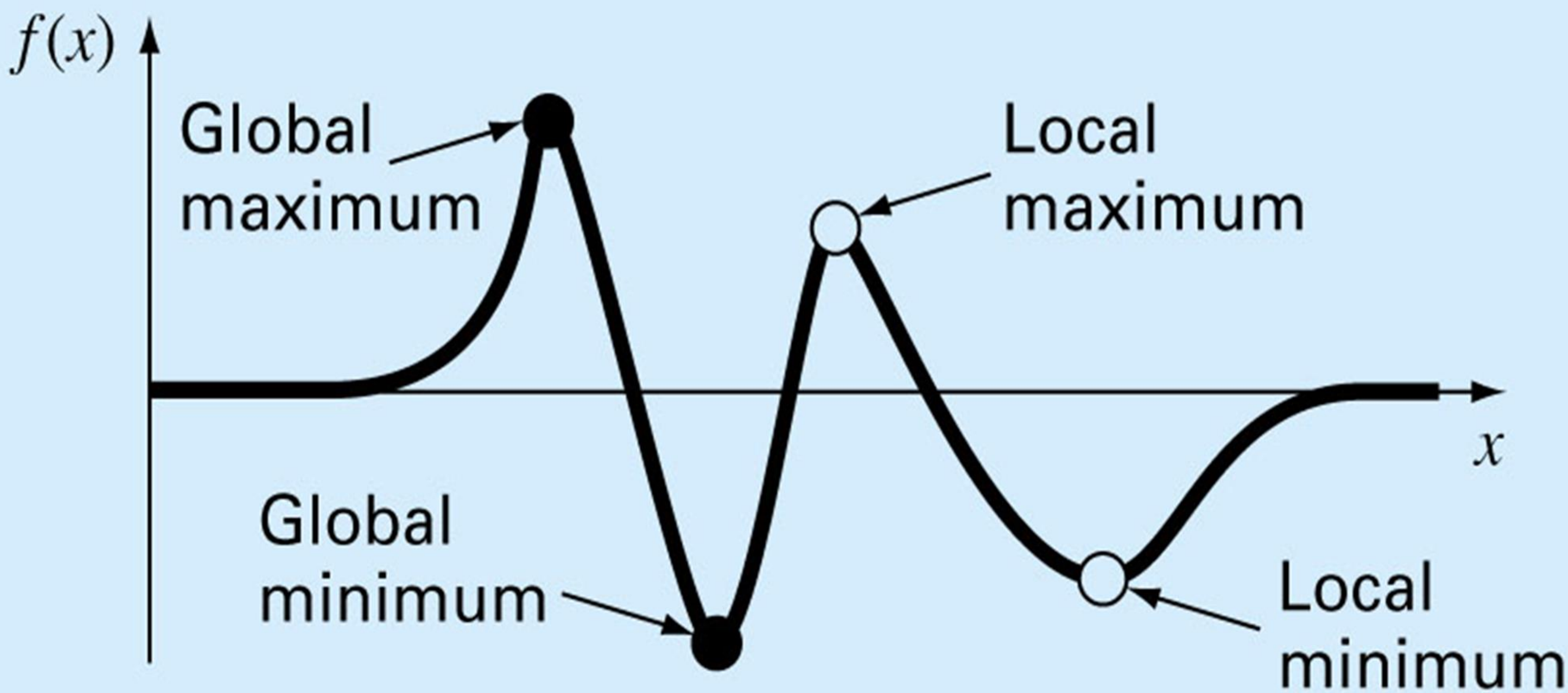
- **Non-gradient or direct methods**
- Gradient methods

Linear Programming (Constrained)

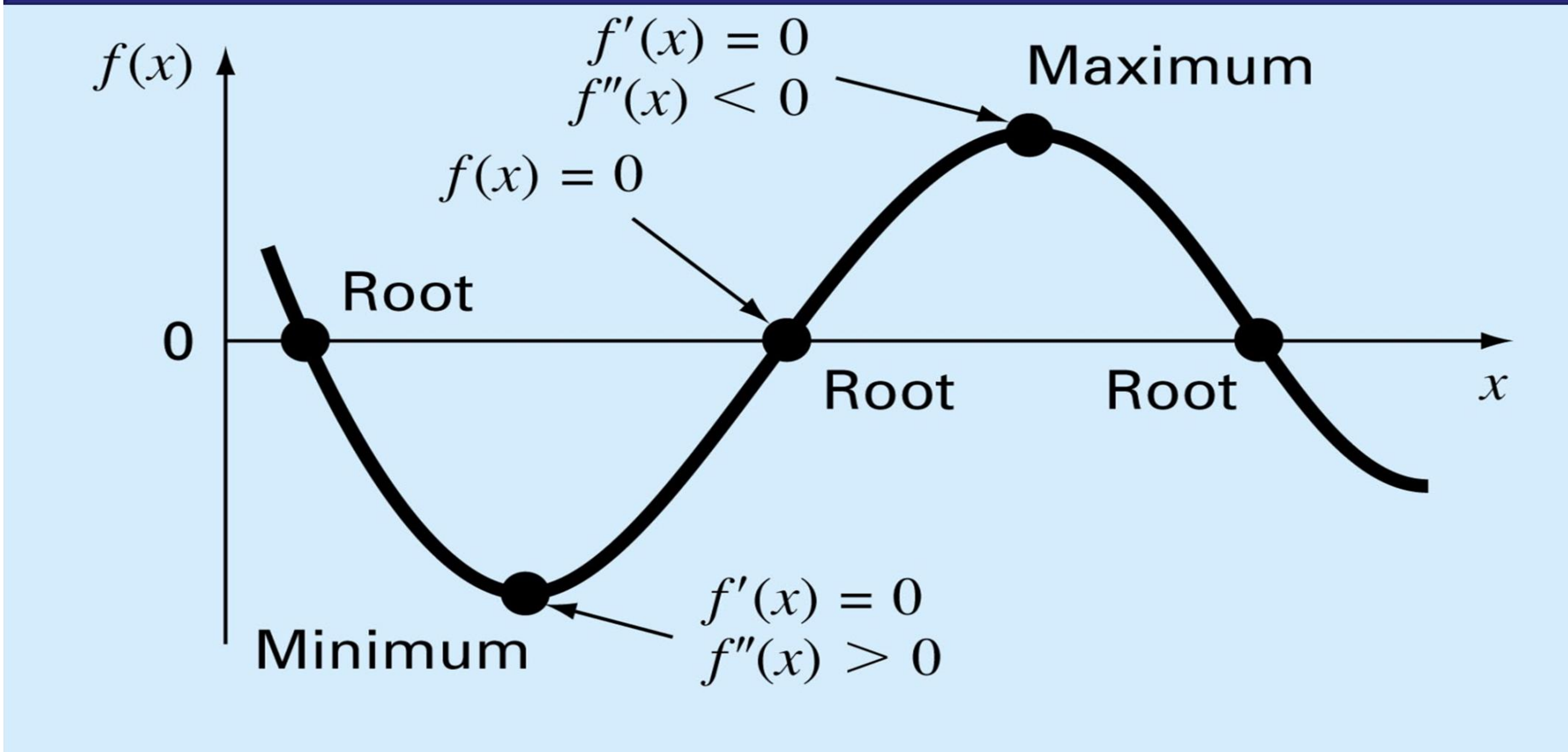
- **Graphical Solution**
- **Simplex Method**

Global and Local Optima

A function is said to be *multimodal* on a given interval if there are more than one minimum/maximum point in the interval.



Characteristics of Optima

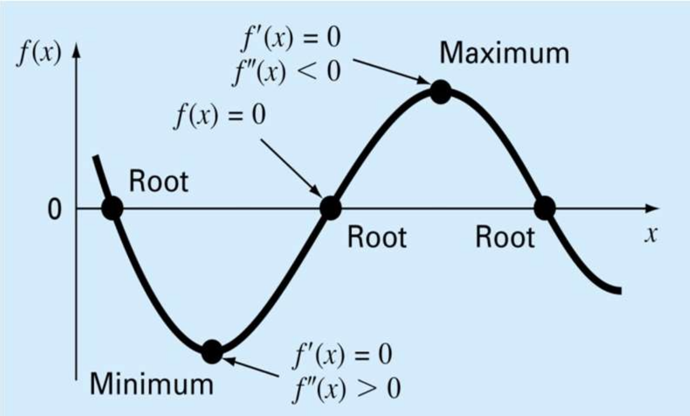


To find the optima, can find the zeroes of $f'(x)$.

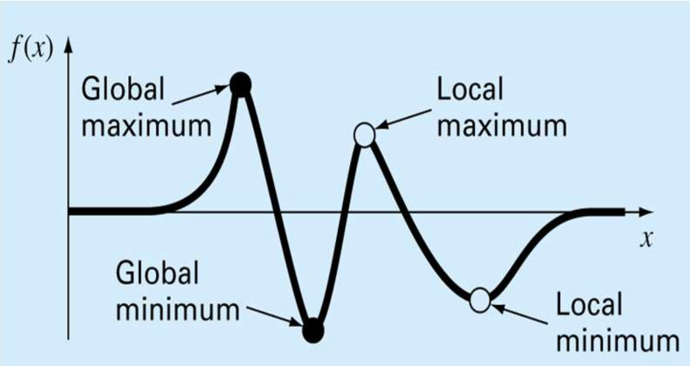
- **At a maximum point the curvature is negative, *i.e.,* f ” < 0 (2nd derivative less than zero)**
- **At a minimum point the curvature is positive, *i.e.,* f ” > 0 (2nd derivative more than zero)**

One-Dimensional Unconstrained Optimization

- **Root finding** and **optimization** are related. Both involve guessing and searching for a point on a function. Difference is:
  - Root finding is searching for zeros of a function
  - Optimization is finding the ***minimum*** or the ***maximum*** of a function of several variables.



- In *multimodal* functions, both *local* and *global* optima can occur. Mostly interested in finding the absolute highest or lowest value of a function.



How do look for the global optimum?

- By graphing to gain insight into the behavior of the function.
- Using randomly generated starting guesses and picking the largest of the optima
- Perturbing the starting point to see if the routine returns a better point

Golden Ratio

A ***unimodal*** function has a **single maximum** or a **minimum** in the a given interval. For a *unimodal* function:

- First pick two points that will bracket your extremum [x_l, x_u].
- Then, pick two more points within this interval to determine whether a **maximum** has occurred within the *first three* or *last three* points

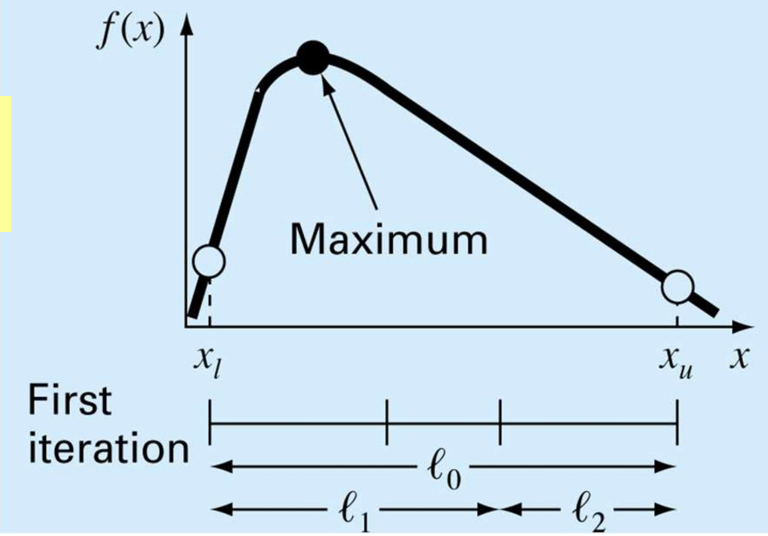
$$l_0 = l_1 + l_2 \quad \text{and} \quad \frac{l_1}{l_0} = \frac{l_2}{l_1}$$

$$\frac{l_1}{l_1 + l_2} = \frac{l_2}{l_1} \qquad R = \frac{l_2}{l_1}$$

$$1 + R = \frac{1}{R} \qquad R^2 + R - 1 = 0$$

$$R = \frac{-1 + \sqrt{1 - 4(-1)}}{2} = \frac{\sqrt{5} - 1}{2} = 0.61803$$

Golden Ratio



Golden-Section Search

- Pick two initial guesses, x_l and x_u, that bracket one local extremum of $f(x)$:

- Choose two interior points x_1 and x_2 according to the **golden ratio**

$$d = \frac{\sqrt{5}-1}{2}(x_u - x_l)$$
$$x_1 = x_l + d$$
$$x_2 = x_u - d$$

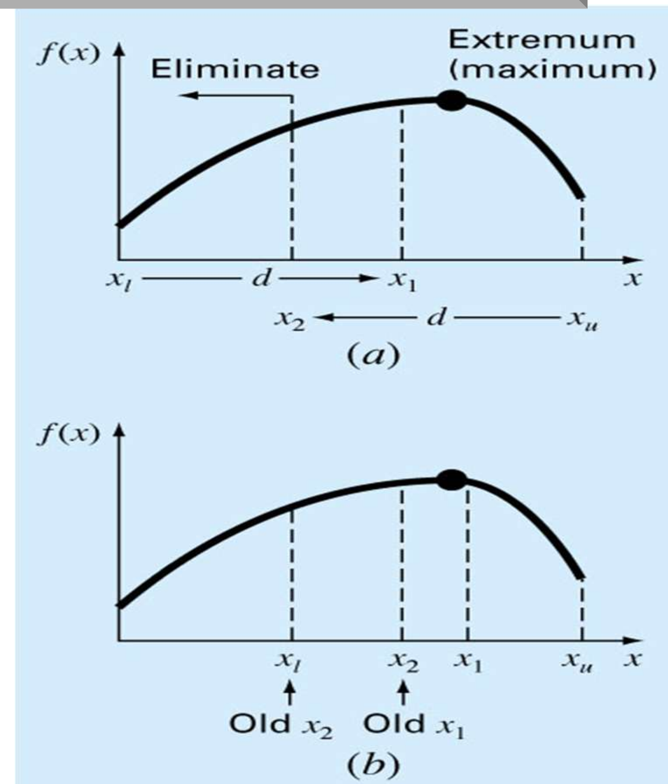
Evaluate the function at x_1 and x_2 :

- If $f(x_1) > f(x_2)$ then the domain of x to the left of x_2 (from x_l to x_2) does not contain the maximum and can be eliminated. Then, x_2 becomes the new x_l

- If $f(x_2) > f(x_1)$, then the domain of x to the right of x_1 (from x_1 to x_u) can be eliminated. In this case, x_1 becomes the new x_u .

- The benefit of using **golden ratio** is that do not need to recalculate all the function values in the next iteration.

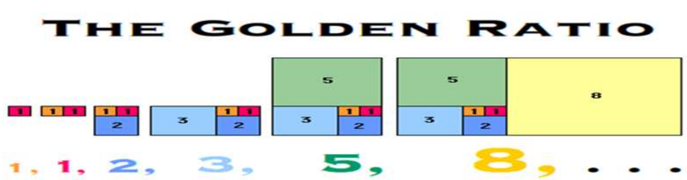
If $f(x_1) > f(x_2)$ then **New** $x_2 \leftarrow x_1$ else **New** $x_1 \leftarrow x_2$



Stopping Criteria

$| x_u - x_l | < \varepsilon$

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}$$



Use the Golden-section search to find the maximum of $f(x) = 2\sin x - x^2/10$ within the interval $x_l=0$ and $x_u=4$

$$d = \frac{\sqrt{5}-1}{2}(x_u - x_l) = 0.618(4) = 2.47$$

$$x_1 = x_l + d = 0 + 2.47 = 2.47$$

$$x_2 = x_u - d = 4 - 2.47 = 1.53$$

$$f(x_1) = (2\sin(2.47 \times \frac{\Pi}{180}) - \frac{2.47^2}{10} = -1.765$$

$$f(x_2) = (2\sin(1.53 \times \frac{\Pi}{180}) - \frac{1.53^2}{10} = -0.649$$

x_l	x_u	d	x_1	x_2	$f(x_1)$	$f(x_2)$	
0	**4**	**2.47**	**2.47**	**1.53**	**-1.765**	**-0.649**	$f(x_1)> f(x_2)$

$$d = \frac{\sqrt{5}-1}{2}(x_u - x_l) = 0.618(2.47-0) = 1.53$$

$$x_2 = x_u - d = 2.47 - 1.53 = 0.944$$

$$f(x_2) = (2\sin(0.944 \times \frac{\Pi}{180}) - \frac{0.944^2}{10} = -1.5355$$

x_l	x_u	d	x_1	x_2	$f(x_1)$	$f(x_2)$	
2.47	**0.94**	**1.53**	**1.31**	**1.52**	**-1.76**	**-1.765**	$f(x_1)< f(x_2)$; $x_1=x_U$

$$d = \frac{\sqrt{5}-1}{2}(x_u - x_l) = 0.618(2.11 - 1.889) = 0.137$$
$$x_1 = x_l + d = 1.889 + 0.137 = 2.026$$
$$x_2 = x_u - d = 2.11 - 0.137 = 1.973$$

$$f(x_1) = (2\sin(2.026 \times \frac{\Pi}{180}) - \frac{2.026^2}{10} = \quad -0.4092$$

$$f(x_2) = (2\sin(1.973 \times \frac{\Pi}{180}) - \frac{1.973^2}{10} = \quad -0.388$$

x_l	x_u	d	x_1	x_2	$f(x_1)$	$f(x_2)$	
1.889	**2.11**	**0.137**	**2.026**	**1.973**	**-0.4092**	**-0.388**	$f(x_1) < f(x_2)$; $X_1 = X_U$

$$d = \frac{\sqrt{5}-1}{2}(x_u - x_l) = 0.618(2.026 - 1.973) = 0.033$$
$$x_1 = x_l + d = 1.973 + 0.033 = 2.006$$
$$x_2 = x_u - d = 2.026 - 0.033 = 1.993$$

$$f(x_1) = (2\sin(2.006 \times \frac{\Pi}{180}) - \frac{2.006^2}{10} = \quad -0.401$$

$$f(x_2) = (2\sin(1.993 \times \frac{\Pi}{180}) - \frac{1.993^2}{10} = \quad -0.395$$

x_l	x_u	d	x_1	x_2	$f(x_1)$	$f(x_2)$	
1.973	**2.026**	**0.033**	**2.006**	**1.993**	**-0.401**	**-0.395**	$f(x_1) < f(x_2)$; $X_1 = X_U$

Golden-Section Search

Golden-Section Search

Problem Statement. Use the golden-section search to find the minimum of

$$f(x) = \frac{x^2}{10} - 2\sin x$$

within the interval from $x_l = 0$ to $x_u = 4$.

Solution. First, the golden ratio is used to create the two interior points:

$$d = 0.61803(4 - 0) = 2.4721$$
$$x_1 = 0 + 2.4721 = 2.4721$$
$$x_2 = 4 - 2.4721 = 1.5279$$

The function can be evaluated at the interior points:

$$f(x_2) = \frac{1.5279^2}{10} - 2\sin(1.5279) = -1.7647$$

$$f(x_1) = \frac{2.4721^2}{10} - 2\sin(2.4721) = -0.6300$$

Because $f(x_2) < f(x_1)$, our best estimate of the minimum at this point is that it is located at $x = 1.5279$ with a value of $f(x) = -1.7647$. In addition, we also know that the minimum is in the interval defined by x_l, x_2, and x_1. Thus, for the next iteration, the lower bound remains $x_l = 0$, and x_1 becomes the upper bound, that is, $x_u = 2.4721$. In addition, the former x_2 value becomes the new x_1, that is, $x_1 = 1.5279$. In addition, we do not have to recalculate $f(x_1)$, it was determined on the previous iteration as $f(1.5279) = -1.7647$.

All that remains is to use Eqs. (7.8) and (7.7) to compute the new value of d and x_2:

i	x_l	$f(x_l)$	x_2	$f(x_2)$	x_1	$f(x_1)$	x_u	$f(x_u)$	d
1	0	0	1.5279	−1.7647	2.4721	−0.6300	4.0000	3.1136	2.4721
2	0	0	0.9443	−1.5310	1.5279	−1.7647	2.4721	−0.6300	1.5279
3	0.9443	−1.5310	1.5279	−1.7647	1.8885	−1.5432	2.4721	−0.6300	0.9443
4	0.9443	−1.5310	1.3050	−1.7595	1.5279	−1.7647	1.8885	−1.5432	0.5836
5	1.3050	−1.7595	1.5279	−1.7647	1.6656	−1.7136	1.8885	−1.5432	0.3607
6	1.3050	−1.7595	1.4427	−1.7755	1.5279	−1.7647	1.6656	−1.7136	0.2229
7	1.3050	−1.7595	1.3901	−1.7742	1.4427	−1.7755	1.5279	−1.7647	0.1378
8	1.3901	−1.7742	1.4427	−1.7755	1.4752	−1.7732	1.5279	−1.7647	0.0851

Code for Golden-Section

```
function [x,fx,ea,iter]=goldmin(f,xl,xu,es,maxit,varargin)
% goldmin: minimization golden section search
%   [xopt,fopt,ea,iter]=goldmin(f,xl,xu,es,maxit,p1,p2,...):
%      uses golden section search to find the minimum of f
% input:
%   f = function handle
%   xl, xu = lower and upper guesses
%   es = desired relative error (default = 0.0001%)
%   maxit = maximum allowable iterations (default = 50)
%   p1,p2,... = additional parameters used by f
% output:
%   x = location of minimum
%   fx = minimum function value
%   ea = approximate relative error (%)
%   iter = number of iterations

if nargin<3,error('at least 3 input arguments required'),end
if nargin<4||isempty(es), es=0.0001;end
if nargin<5||isempty(maxit), maxit=50;end
phi=(1+sqrt(5))/2;
iter=0;
while(1)
  d = (phi-1)*(xu - xl);
  x1 = xl + d;
  x2 = xu - d;
  if f(x1,varargin{:}) < f(x2,varargin{:})
    xopt = x1;
    xl = x2;
  else
    xopt = x2;
    xu = x1;
  end
  iter=iter+1;
  if xopt~=0, ea = (2 - phi) * abs((xu - xl) / xopt) * 100;end
  if ea <= es || iter >= maxit,break,end
end
x=xopt;fx=f(xopt,varargin{:});
```

Newton's Method

Let $g(x) = f'(x)$

Thus the zeroes of $g(x)$ is the optima of $f(x)$.

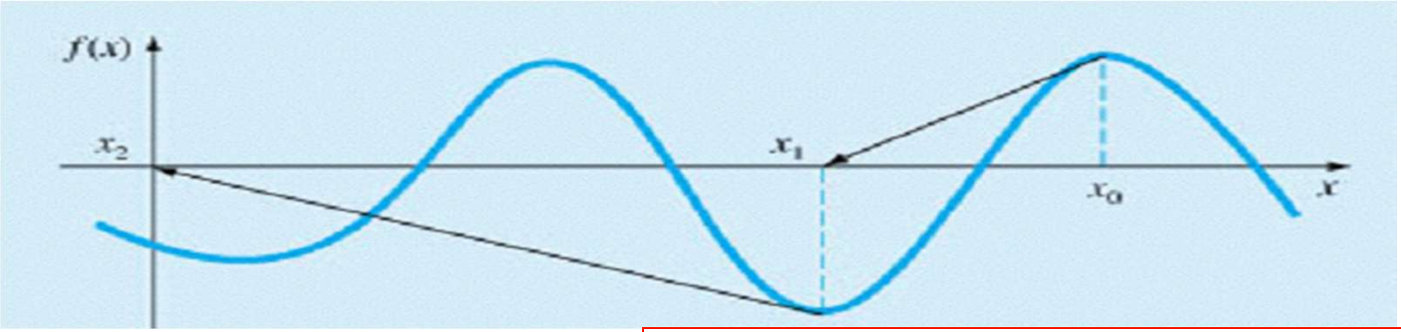
Substituting $g(x)$ into the updating formula of Newton-Rahpson method, have

$$\boxed{x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)} = x_i - \frac{f'(x_i)}{f''(x_i)}}$$

Note: Other root finding methods will also work.

Newton’s Method

- **Advantages**
  - Fast convergent rate near solution
  - Hybrid approach: Use bracketing method to find an approximation near the solution, then switch to Newton's method.



- **Shortcomings**
  - Need to derive $f'(x)$ and $f''(x)$.
  - May diverge
  - May "jump" to another solution far away

$$x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)} = x_i - \frac{f'(x_i)}{f''(x_i)}$$

Newton's Method

- **A similar approach to Newton- Raphson method can be used to find an optimum of *f(x)* by finding the root of *f'(x)* (i.e. solving *f'(x)*=0):**

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}$$

- **<u>Disadvantage</u>: it may be divergent**

- **If it is difficult to find *f'(x)* and *f"(x)* analytically, then a secant-like version of Newton's technique can be developed by using finite-difference approximations for the derivatives.**

Newton's Method Procedure

1. **Assume an initial guess x_0 and calculate the first estimate x_1 from:**

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)}$$

2. Repeat step 1 several times until convergence is achieved, i.e. until $\epsilon_a < \epsilon_s$

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)}$$

3. Note that if f" = 0, the above procedure fails. However, can approximate f" by the second order centered finite difference (similar to the secant method).

$$f''(x_i) \approx \frac{(f(x_i + h) - 2f(x_i) + f(x_i - h)}{h^2}$$

Ex 1: Use Newton-Raphson method to determine the value of x that $f(x) = 2\sin x - x^2/10$ maximize the function at x = 1 of error 0.009.

Solution:

$$f'(x) = 2\cos x - \frac{x}{5}$$

$$f''(x) = -2\sin x - \frac{1}{5}$$

$$x_{i+1} = x_i - \frac{2\cos x_i - \frac{x_i}{5}}{-2\sin x_i - \frac{1}{5}}$$

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)} = x_i - \frac{2\cos x_i - x_i/5}{-2\sin x_i - 1/5}$$

$$x_1 = 1 - \frac{2\cos 1 - 1/5}{-2\sin 1 - 1/5} = 1.4676 \quad \text{and} \quad f(1.4676) = 1.774$$

Iter	X_i	$f(X_i)$	$f'(X_i)$	$f''(X_i)$	$\varepsilon_a(\%)$
1	1.	1.5829	0.8806	-1.883	31.87
2	1.4676	1.7740	-0.0002	-2.189	-2.80
3	1.4276	1.7757	0.0	-2.179	-0.006

Ex 2: Use Newton's method to find the maximum of $f(x) = 2 \sin x - x^2/10$ with an initial guess of $x_0 = 2.5$ of error 0.0009.

Solution:

$$x_{i+1} = x_i - \frac{f'(x_i)}{f''(x_i)} = x_i - \frac{2\cos x_i - x_i/5}{-2\sin x_i - 1/5}$$

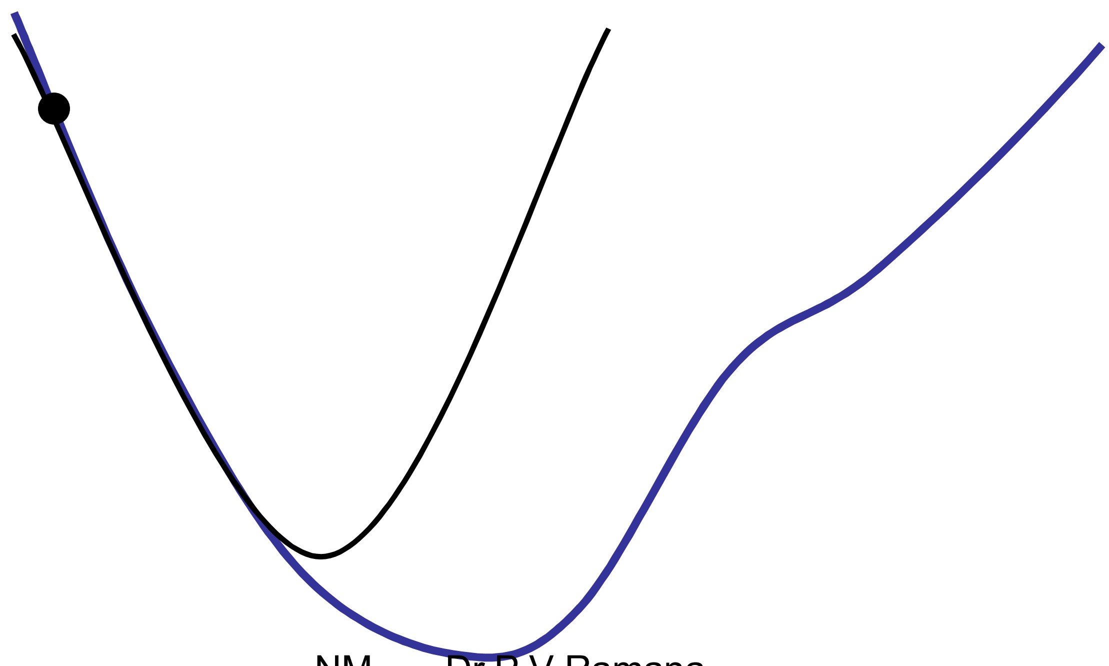
$$x_1 = 2.5 - \frac{2\cos 2.5 - 2.5/5}{-2\sin 2.5 - 1/5} = 0.995 \quad \text{and} \quad f(0.995) = 1.578$$

i	*x*	*f(x)*	*f'(x)*	*f''(x)*
0	**2.5**	**0.572**	**-2.102**	**-1.3969**
1	**0.995**	**1.578**	**0.8898**	**-1.8776**
2	**1.469**	**1.774**	**-0.0905**	**-2.1896**
3	**1.4276**	**1.77573**	**-0.0002**	**-2.17954**
4	**1.4275**	**1.77573**	**0.0000**	**-2.17952**

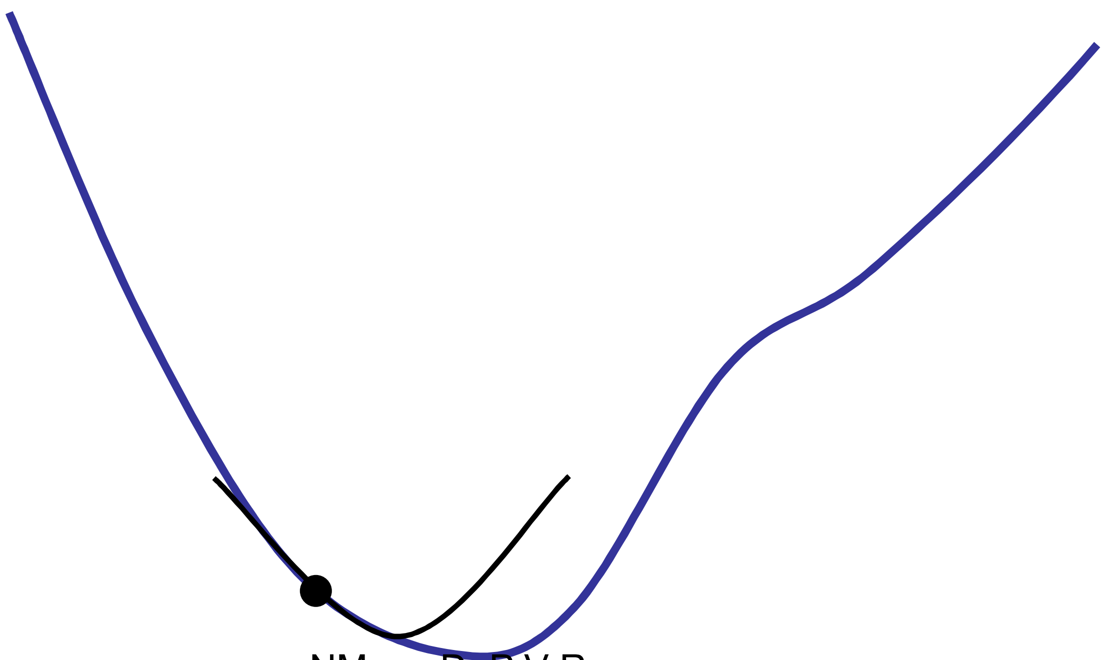
Faster 1-D Optimization

- **Trade off super-linear convergence for worse robustness**
  - **Combine with bracket method search for safety**
- **Usual bag of tricks:**
  - **Use *second* derivatives: Newton**
  - **Fit parabola through 3 points, find minimum**
  - **Compute derivatives as well as positions, fit cubic**

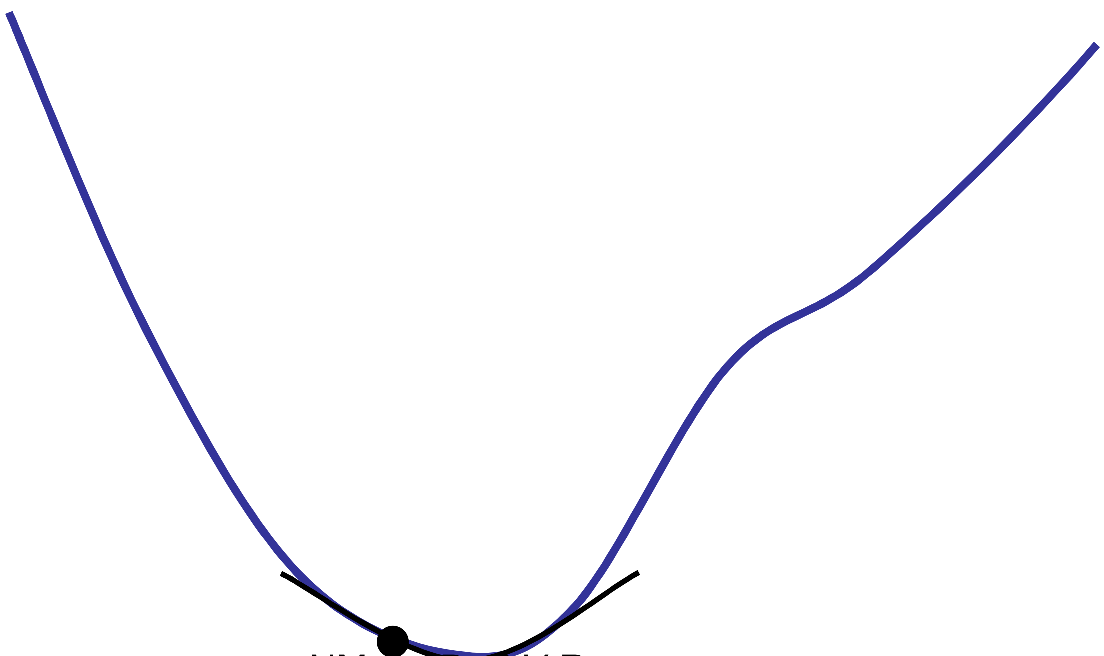
Newton's Method



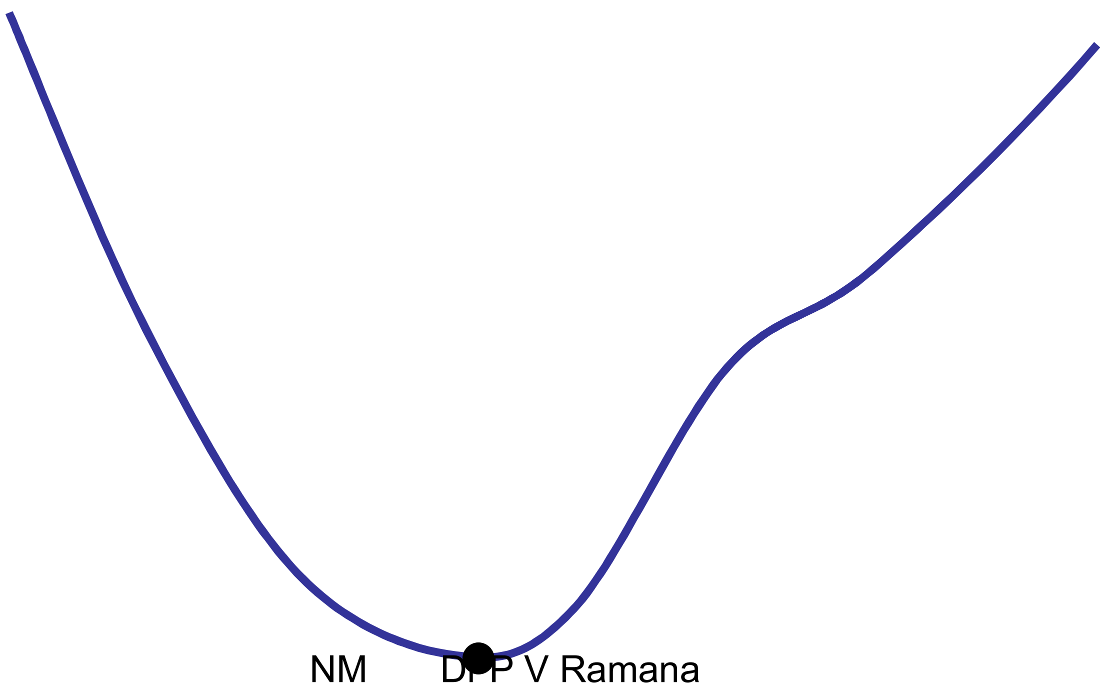
Newton's Method



Newton’s Method



Newton's Method



Newton's Method

- At each step:

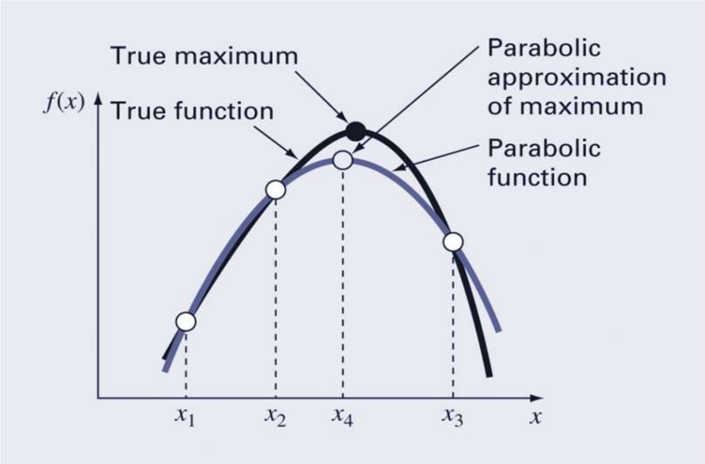
$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

- Requires 1st and 2nd derivatives
- Quadratic convergence

Parabolic Interpolation

$$f^*(x) = Ax^2 + Bx + C$$

The value which minimizes $f^*(x)$ is

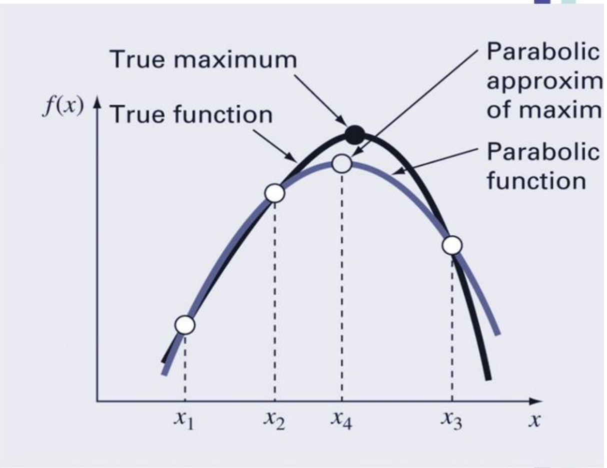


$$x_{\min} = -\frac{B}{2A}$$

Using this strategy to minimize the function is called "inverse parabolic interpolation"

Fitting a Parabola

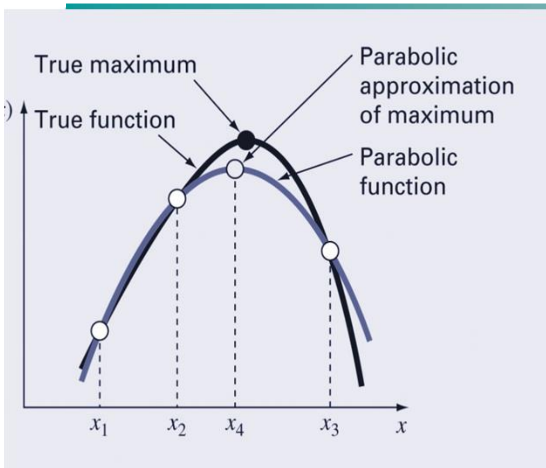
- Can be fitted with three points
  - Points must not be co-linear



$$C = f(x_1) - Ax_1^2 - Bx_1$$

$$B = \frac{A(x_2^2 - x_1^2) + (f(x_1) - f(x_2))}{x_1 - x_2}$$

$$A = \frac{f(x_3) - f(x_2)}{(x_3 - x_2)(x_3 - x_1)} - \frac{f(x_1) - f(x_2)}{(x_1 - x_2)(x_3 - x_1)}$$



$$x_{\min} = -\frac{B}{2A}$$

$$B = \frac{A(x_2^2 - x_1^2) + (f(x_1) - f(x_2))}{x_1 - x_2}$$

$$A = \frac{f(x_3) - f(x_2)}{(x_3 - x_2)(x_3 - x_1)} - \frac{f(x_1) - f(x_2)}{(x_1 - x_2)(x_3 - x_1)}$$

Minimum for a Parabola

- General expression for finding minimum of a parabola fitted through three points
  - Note repeated sub-expressions

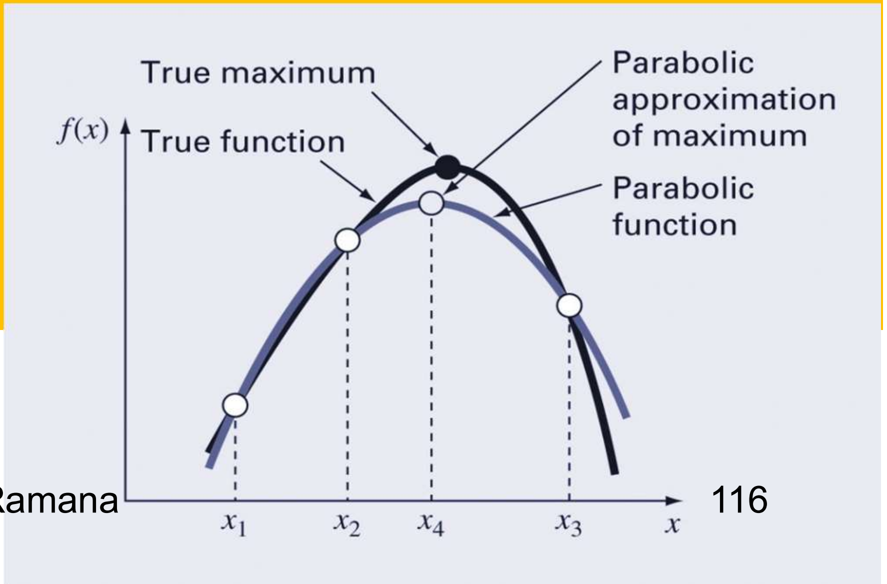
$$x_{\min} = x_2 - \frac{1}{2}\frac{(x_2 - x_1)^2(f(x_2) - f(x_3)) - (x_2 - x_3)^2(f(x_2) - f(x_1))}{(x_2 - x_1)(f(x_2) - f(x_3)) - (x_2 - x_3)(f(x_2) - f(x_1))}$$

Parabolic Interpolation

- Another algorithm uses parabolic interpolation of three points to estimate optimum location.
- The location of the maximum/minimum of a parabola defined as the interpolation of three points (x_1, x_2, and x_3) is:

$$x_4 = x_2 - \frac{1}{2}\frac{(x_2 - x_1)^2\left[f(x_2) - f(x_3)\right] - (x_2 - x_3)^2\left[f(x_2) - f(x_1)\right]}{(x_2 - x_1)\left[f(x_2) - f(x_3)\right] - (x_2 - x_3)\left[f(x_2) - f(x_1)\right]}$$

- The new point x_4 and the two surrounding it (either x_1 and x_2 or x_2 and x_3) are used for the next iteration of the algorithm.



Parabolic Interpolation

Problem Statement. Use parabolic interpolation to approximate the minimum of

$$f(x) = \frac{x^2}{10} - 2\sin x$$

with initial guesses of $x_1 = 0$, $x_2 = 1$, and $x_3 = 4$.

Solution. The function values at the three guesses can be evaluated:

$$x_1 = 0 \qquad f(x_1) = 0$$
$$x_2 = 1 \qquad f(x_2) = -1.5829$$
$$x_3 = 4 \qquad f(x_3) = 3.1136$$

$x_1 = 0;$ $f(x_1) = 0$
$x_2 = 1;$ $f(x_2) = -1.583$
$x_3 = 4;$ $f(x_3) = 3.114$

$$x_4 = x_2 - \frac{1}{2}\frac{(x_2 - x_1)^2[f(x_2) - f(x_3)] - (x_2 - x_3)^2[f(x_2) - f(x_1)]}{(x_2 - x_1)[f(x_2) - f(x_3)] - (x_2 - x_3)[f(x_2) - f(x_1)]}$$

$$x_4 = 1 - \frac{1}{2}\frac{(1-0)^2[-1.5829 - 3.1136] - (1-4)^2[-1.5829 - 0]}{(1-0)[-1.5829 - 3.1136] - (1-4)[-1.5829 - 0]} = 1.5055$$

$x_1 = 0$; $f(x_1) = 0$
$x_2 = 1$; $f(x_2) = -1.583$
$x_3 = 4$; $f(x_3) = 3.114$

which has a function value of $f(1.5055) = -1.7691$.

Next, a strategy similar to the golden-section search can be employed to determine which point should be discarded. Because the function value for the new point is lower than for the intermediate point (x_2) and the new x value is to the right of the intermediate point, the lower guess (x_1) is discarded. Therefore, for the next iteration:

$$\begin{aligned} x_1 &= 1 & f(x_1) &= -1.5829 \\ x_2 &= 1.5055 & f(x_2) &= -1.7691 \\ x_3 &= 4 & f(x_3) &= 3.1136 \end{aligned}$$

which can be substituted into Eq. (7.10) to give

$$x_4 = 1.5055 - \frac{1}{2}\frac{(1.5055-1)^2[-1.7691 - 3.1136] - (1.5055-4)^2[-1.7691 - (-1.5829)]}{(1.5055-1)[-1.7691 - 3.1136] - (1.5055-4)[-1.7691 - (-1.5829)]}$$
$$= 1.4903$$

which has a function value of $f(1.4903) = -1.7714$.

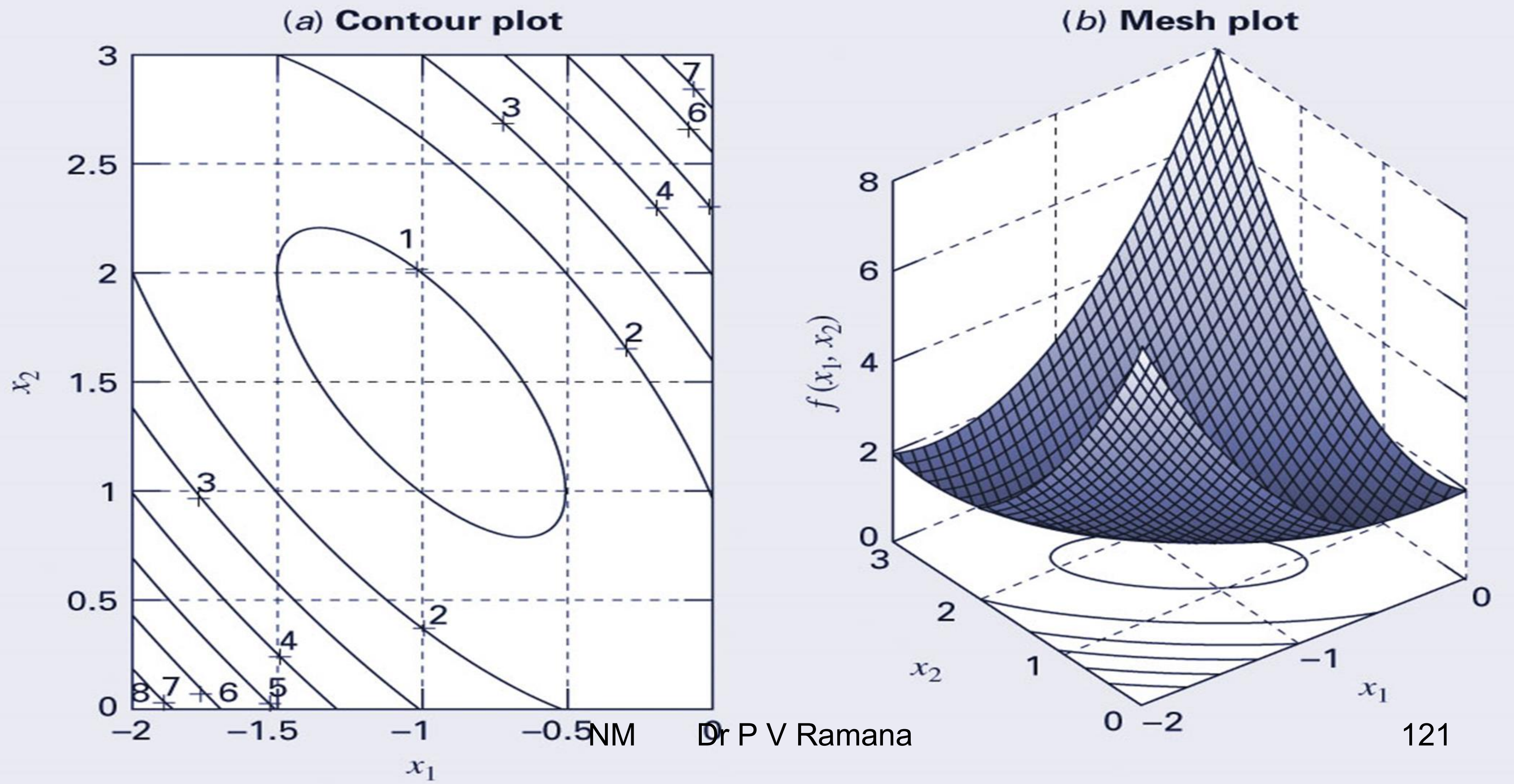
i	x_1	$f(x_1)$	x_2	$f(x_2)$	x_3	$f(x_3)$	x_4	$f(x_4)$
1	0.0000	0.0000	1.0000	−1.5829	4.0000	3.1136	1.5055	−1.7691
2	1.0000	−1.5829	1.5055	−1.7691	4.0000	3.1136	1.4903	−1.7714
3	1.0000	−1.5829	1.4903	−1.7714	1.5055	−1.7691	1.4256	−1.7757
4	1.0000	−1.5829	1.4256	−1.7757	1.4903	−1.7714	1.4266	−1.7757
5	1.4256	−1.7757	1.4266	−1.7757	1.4903	−1.7714	1.4275	−1.7757

fminbnd Function

- MATLAB has a built-in function, fminbnd, which combines the golden-section search and the parabolic interpolation.
  - [*xmin*, *fval*] = fminbnd(*function*, *x1*, *x2*)
- Options may be passed through a fourth argument using optimset, similar to fzero.

Multidimensional Visualization

- Functions of two-dimensions may be visualized using contour or surface/mesh plots.



fminsearch Function

- MATLAB has a built-in function, fminsearch, that can be used to determine the minimum of a multidimensional function.
  - [*xmin*, *fval*] = fminsearch(*function*, *x0*)
  - *xmin* in this case will be a row vector containing the location of the minimum, while *x0* is an initial guess. Note that *x0* must contain as many entries as the function expects of it.
- The function must be written in terms of a single variable, where different dimensions are represented by different indices of that variable.

fminsearch Function

- **To minimize**
  $f(x,y)=2+x-y+2x^2+2xy+y^2$
  rewrite as
  $f(x_1, x_2)=2+x_1-x_2+2(x_1)^2+2x_1x_2+(x_2)^2$
- **f=@(x) 2+x(1)-x(2)+2*x(1)^2+2*x(1)*x(2)+x(2)^2**
  [x, fval] = fminsearch(f, [-0.5, 0.5])
- **Note that *x0* has two entries - *f* is expecting it to contain two values.**
- **MATLAB reports the minimum value is 0.7500 at a location of [-1.000 1.5000]**

Multi-Dimensional Optimization

- Important in many areas
  - Fitting a model to measured data
  - Finding best design in some parameter space
- Hard in general
  - Weird shapes: multiple extrema, saddles, curved or elongated valleys, etc.
  - Can't bracket (but there are "trust region" methods)
- In general, easier than rootfinding
  - Can always walk "downhill"

Taylor expansion

A function may be approximated locally by its Taylor series expansion about a point $\mathbf{x}^*$

$$\boxed{f(\mathbf{x}^* + \mathbf{x}) \approx f(\mathbf{x}^*) + \nabla f^T \mathbf{x} + \frac{1}{2}\mathbf{x}^T \mathbf{H} \mathbf{x}}$$

where the gradient $\nabla f(\mathbf{x}^*)$ is the vector

$$\nabla f(\mathbf{x}^*) = \left[\frac{\partial f}{x_1} \dots \frac{\partial f}{x_N}\right]^T$$

and the Hessian $\mathbf{H}(\mathbf{x}^*)$ is the symmetric matrix

$$\mathbf{H}(\mathbf{x}^*) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_N} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_N \partial x_1} & \dots & \frac{\partial^2 f}{\partial x_N^2} \end{bmatrix}$$

Newton's Method in Multiple Dimensions

- Replace 1st derivative with gradient, 2nd derivative with Hessian

$$f(x, y)$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

Constrained Mathematical Background

Objective: Maximize or Minimize $f(\mathbf{x})$ subject to

$$\left.\begin{aligned} d_i(\mathbf{x}) &\le a_i \quad i = 1,2,\ldots,m^* \\ e_i(\mathbf{x}) &= b_i \quad i = 1,2,\ldots,p^* \end{aligned}\right\} \text{Constraints}$$

$\mathbf{x} = \{x_1, x_2, \ldots, x_n\}$

$f(\mathbf{x})$: *objective function*

$d_i(\mathbf{x})$: *inequality constraints*

$e_i(\mathbf{x})$: *equality constraints*

a_i and b_i are constants

$$\boxed{\begin{gathered} \textit{Maximize} \quad f(\mathbf{x}) \\ \Updownarrow \\ \textit{Minimize} \quad -f(\mathbf{x}) \end{gathered}}$$

Constrained Mathematical Background

- An ***optimization*** or ***mathematical programming*** problem is generally stated as: Find *x*, which **minimizes** or **maximizes** ***f(x)*** subject to

$$d_i(x) \le a_i \quad i = 1, 2, \ldots, m^*$$
$$e_i(x) = b_i \quad i = 1, 2, \ldots, p^*$$

Where *x* is an *n*-dimensional *design vector*, ***f(x)*** is the ***objective function***, $d_i(x)$ are ***inequality constraints***, $e_i(x)$ are ***equality constraints***, and a_i and b_i are constants

- Optimization problems can be classified on the basis of the form of *f(x):*
  - If *f(x)* and the constraints are linear, have ***linear programming***.
  - If *f(x)* is quadratic and the constraints are linear, have ***quadratic programming***.
  - If *f(x)* is not linear or quadratic and/or the constraints are nonlinear, have ***nonlinear programming***.
- When equations(*) are included, have a ***constrained optimization*** problem; otherwise, it is ***unconstrained optimization*** problem.

A Two Variable Linear Program

Objective

$$z = 3x + 5y$$

Constraints

$2x + 3y \leq 10$	(1)
$x + 2y \leq 6$	(2)
$x \leq 4$	(3)
$y \leq 3$	(4)

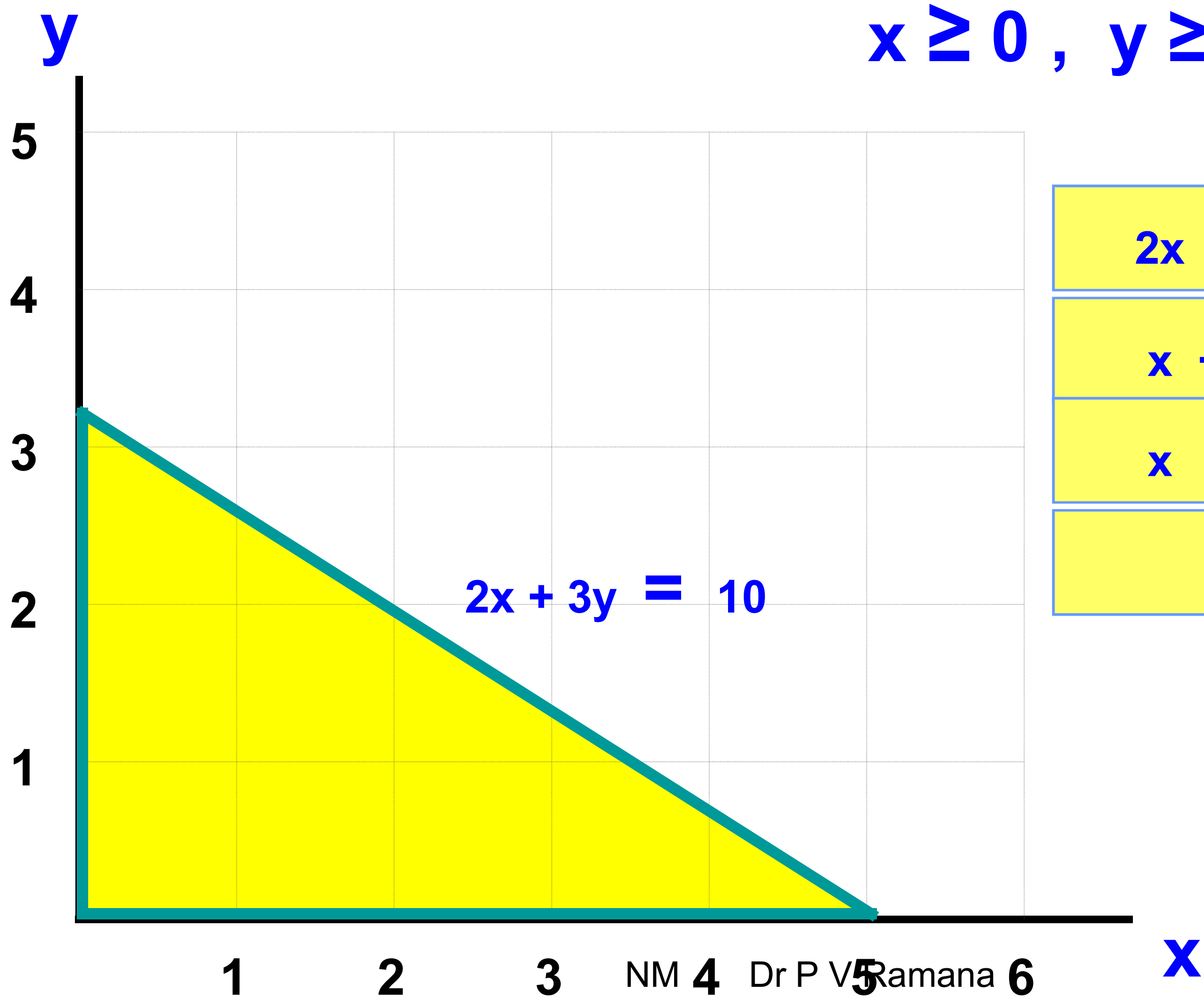
$$x, y \geq 0$$

Graphing the Feasible Region

Graph the Constraints:

$2x + 3y \leq 10 \quad (1)$

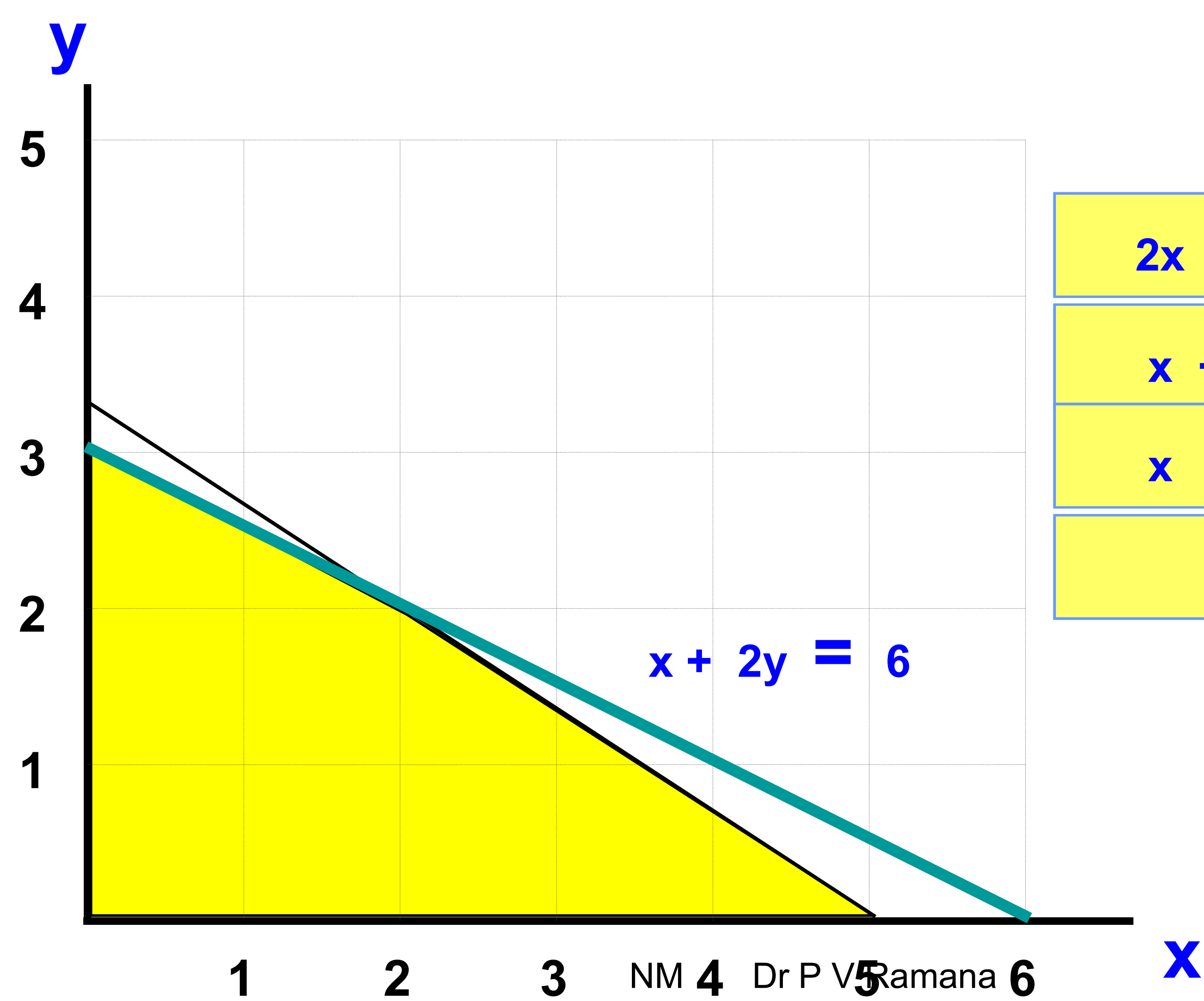
$x \geq 0, \ y \geq 0$



$2x + 3y$	$\leq$	10	1
$x + 2y$	$\leq$	6	2
x	$\leq$	4	3
y	$\leq$	3	4

Add the Constraint:

$x + 2y \le 6$ (2)

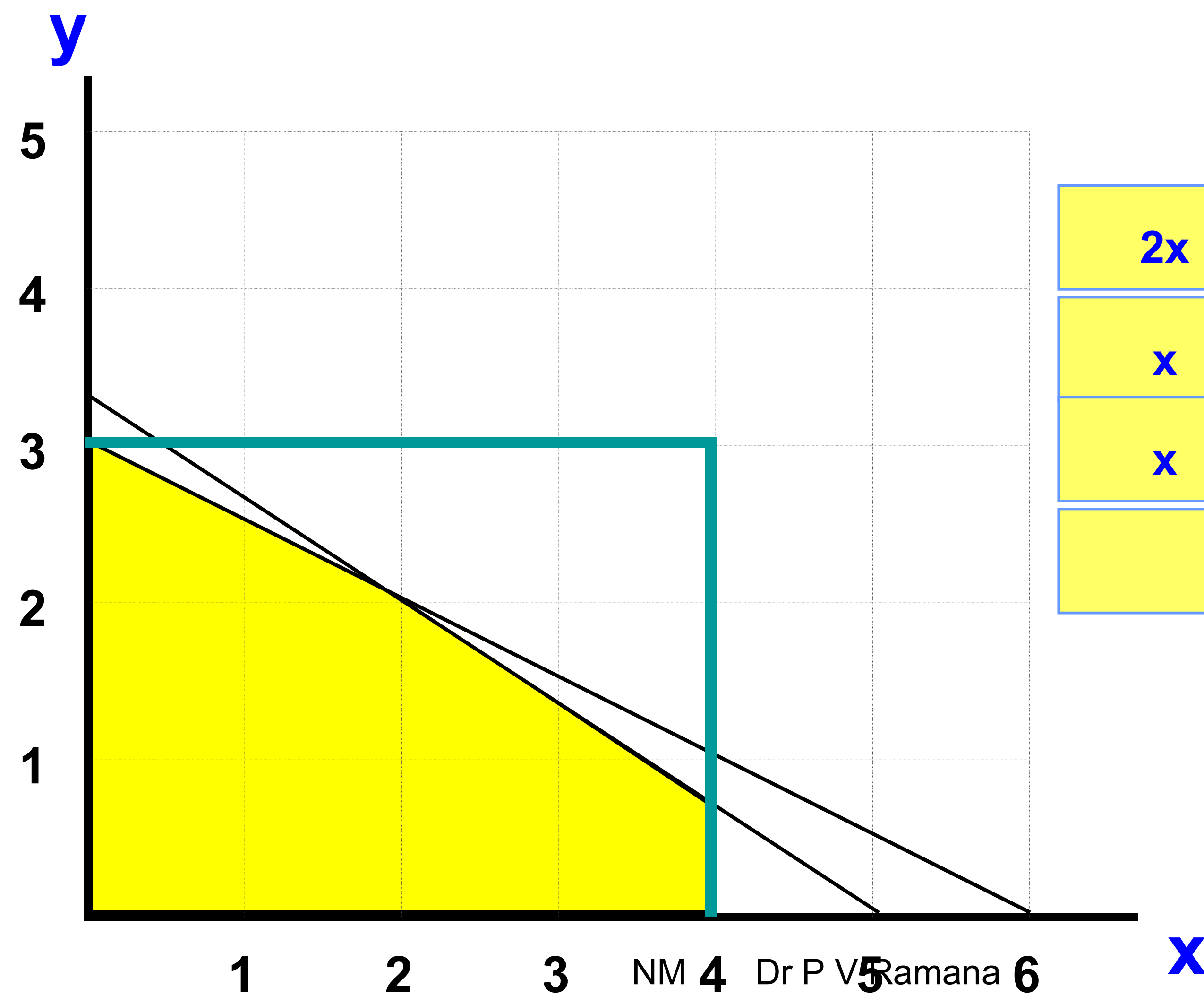


$2x + 3y$	$\le$	10	1
$x + 2y$	$\le$	6	2
x	$\le$	4	3
y	$\le$	3	4

Add the Constraints:

$x \leq 4$; $y \leq 3$

> One can have now graphed the feasible region.



$2x + 3y$	$\leq$	10	1
$x + 2y$	$\leq$	6	2
x	$\leq$	4	3
y	$\leq$	3	4

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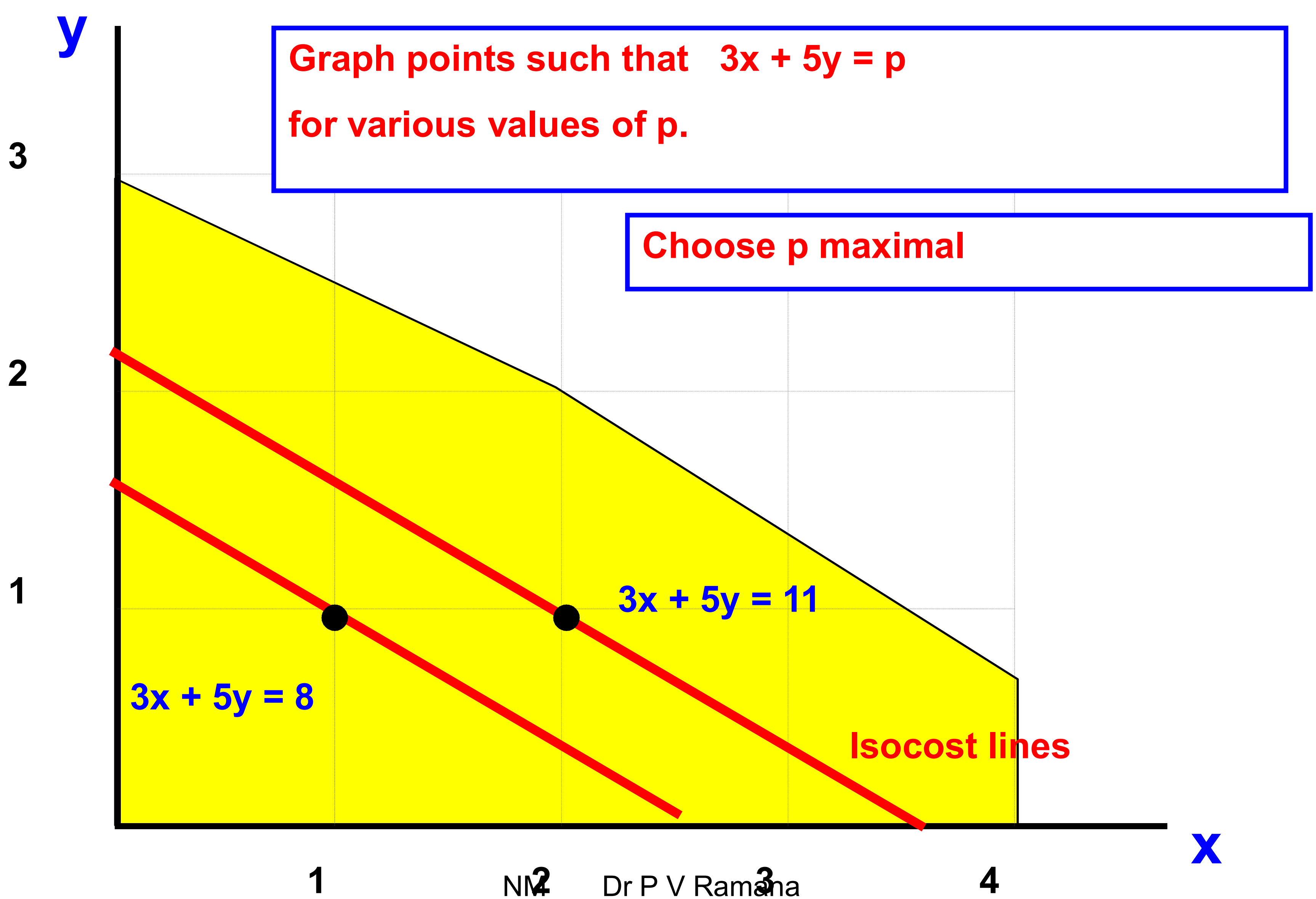
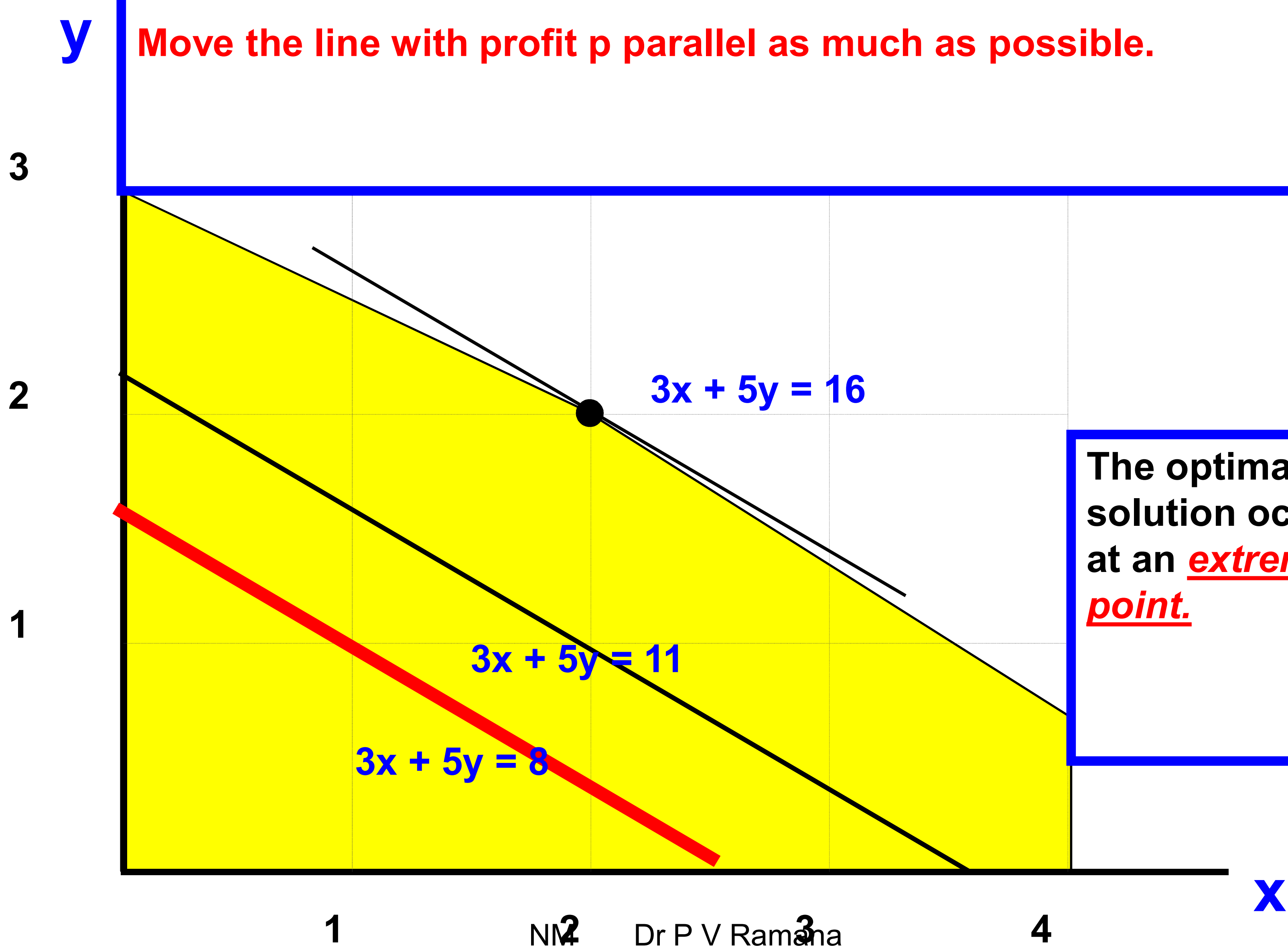
The geometrical method for optimizing 3x + 5y



Find the maximum value p such that there is a feasible solution with 3x + 5y = p.

Move the line with profit p parallel as much as possible.

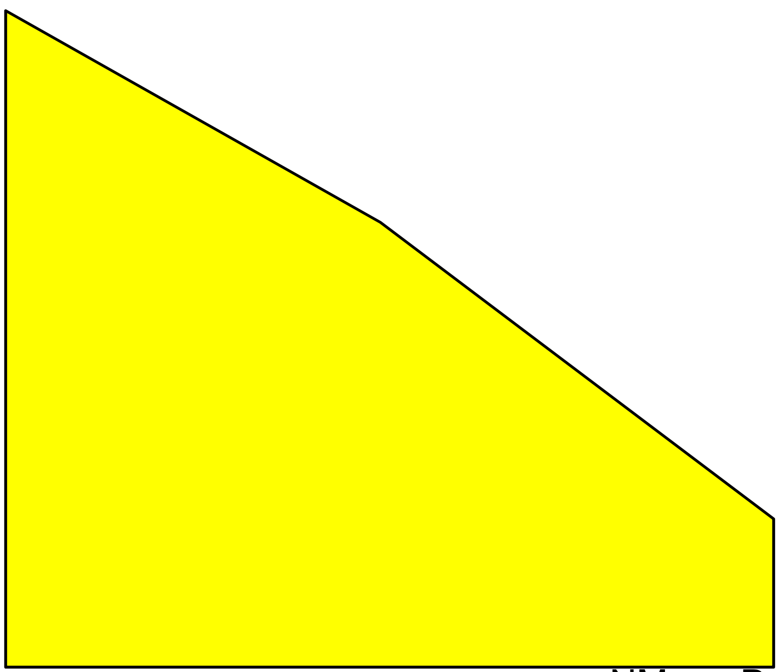


The optimal solution occurs at an ***extreme point.***

Extreme Points (Corner Points)

An ***extreme point*** (also called n ***corner point***) of the feasible region is a point that is not the midpoint of two other points of the feasible region.

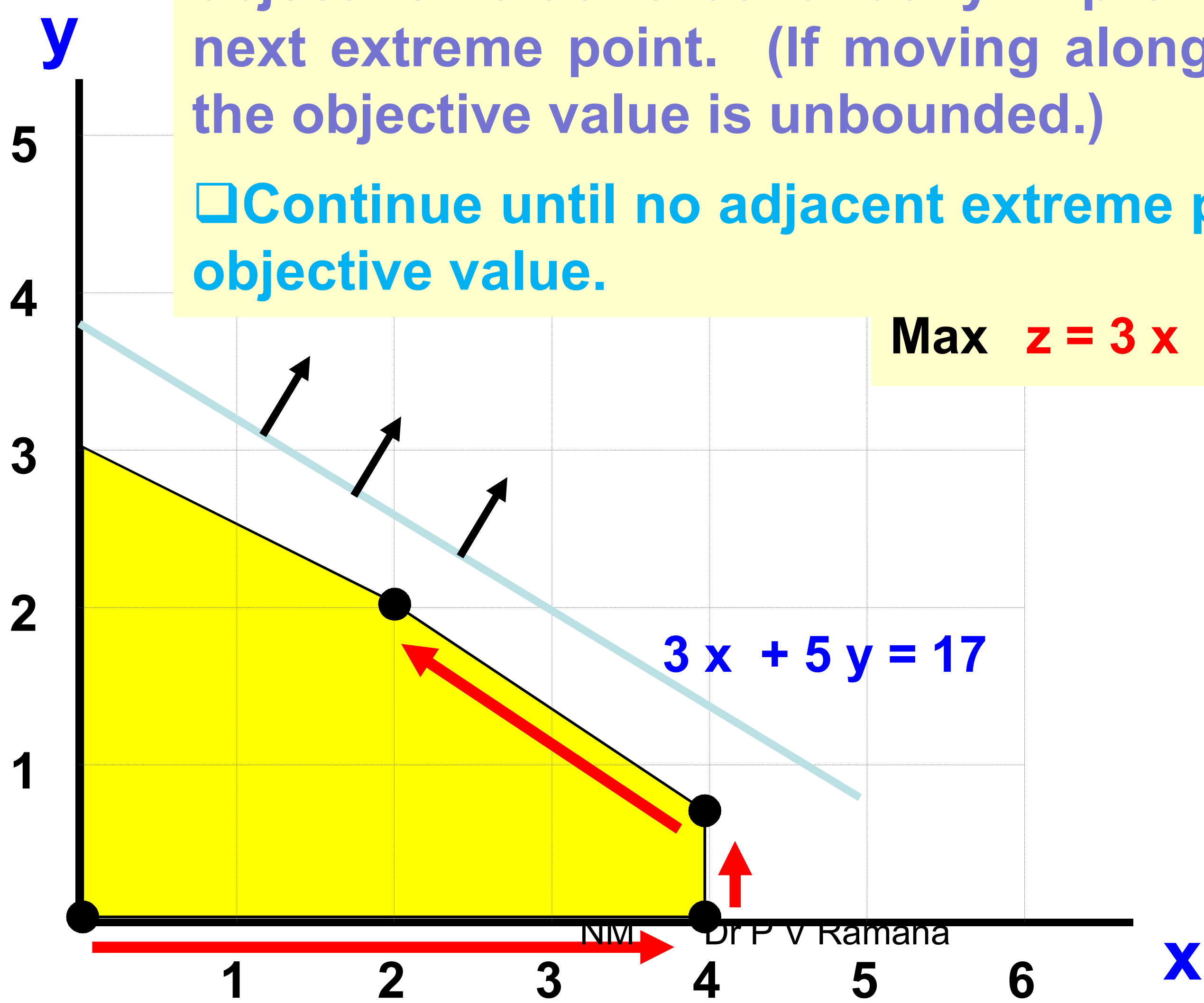
Objective function z = 3x + 5y



Where are the extreme points of this feasible region?

The Simplex Method

- Start at any feasible extreme point.
- Move along an edge (or extreme ray) in which the objective value is continually improving. Stop at the next extreme point. (If moving along an extreme ray, the objective value is unbounded.)
- Continue until no adjacent extreme point has a better objective value.



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Simplex Method

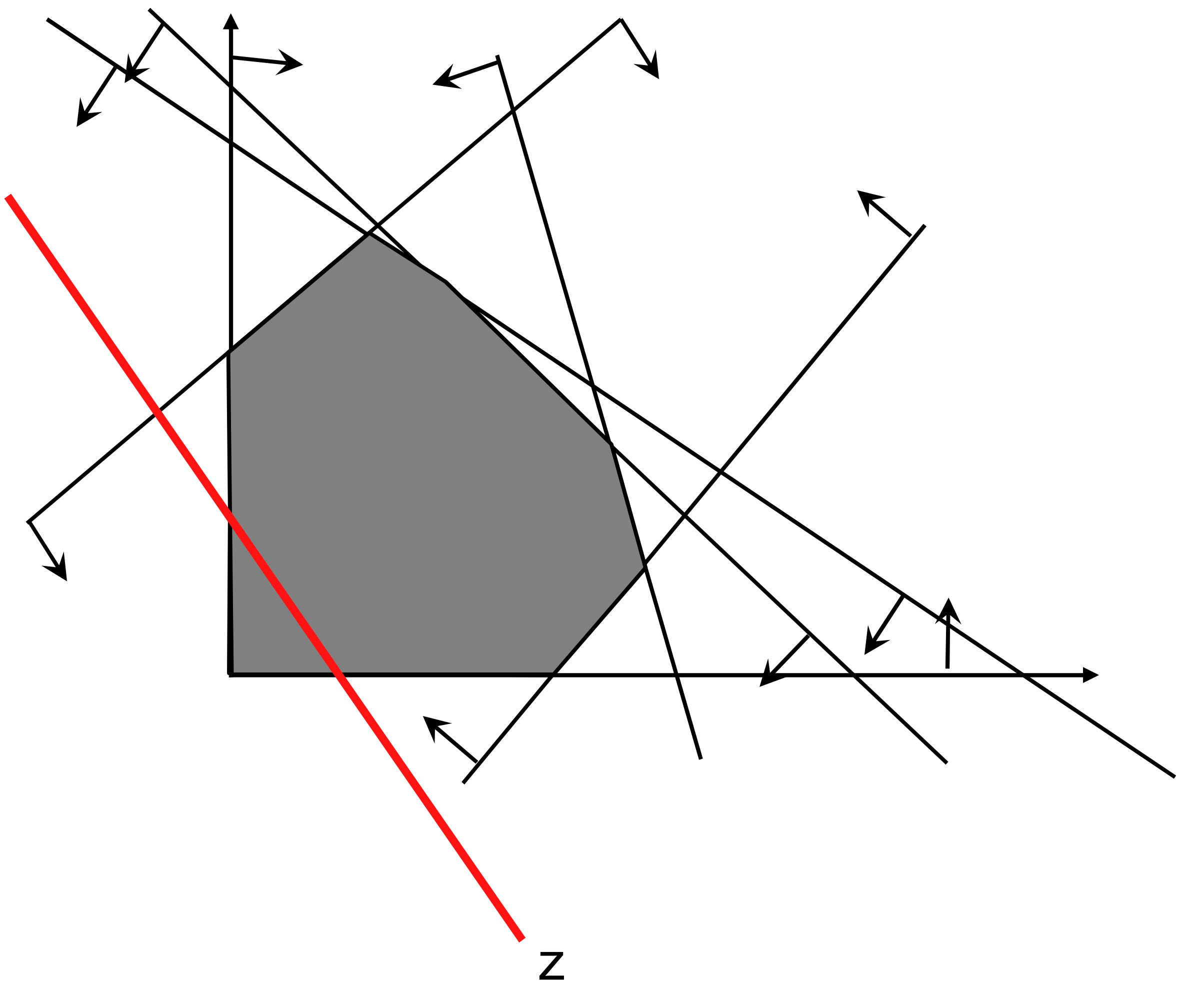
Step 1: Find an initial BFS of the LP.

Step 2: Determine if the current BFS is an optimal to the LP. If it is not, find an adjacent BFS to the LP that has a larger objective function value.

Step 3. Return to Step 2 using the new BFS as the current BFS.

This method performs at most ^nC_m iterations.

Simplex Method



Simplex Method

- The process of moving from one basic feasible solution to another basic feasible solution is called a pivot iteration. (or a pivot step)

- If the basic feasible solution is non-degenerate, then the new basic feasible solution has a lower cost.

- In the absence of degeneracy, the simplex method terminate in a finite number of iterations.

Simplex Method

Standard Form- A linear program in which all the constraints are written as equalities.

Slack Variable- A variable added to the LHS of *"Less than or equal to"* constraint to convert it into an equality.

Surplus Variable- A variable subtracted from the LHS of *"More than or equal to"* constraint to convert it into an equality.

Basic Solution- For a system of m linear equations in n variables ($n>m$),a solution obtained by setting (n-m) variables equal to ***zero*** and solving the system of equations for remaining m variables.

Basic Feasible Solution(BFS)- If all the variables in basic solution are *more than or equal* to zero.

Simplex Method

Optimum Solution- Any BFS which optimizes(maximizes or minimizes) the objective function.

Tableau Form- When a LPP is written in a tabular form prior to setting up the Initial Simplex Tableau.

Simplex Tableau- A table which is used to keep track of the calculations made at each iteration when the simplex method is employed.

Pivotal Column- The column having largest positive(or negative) value in the Net Evaluation Row for a maximization(or minimization) problem.

Pivotal Row- The row corresponding to variable that will leave the table in order to make room for another variable.

Pivotal Element- Element at the intersection of *pivotal row* and *pivotal column*.

Comments about the simplex algorithm

- Each step is called a pivot.
- Pivots are carried out using linear algebra.
- Pivots for network flow problems can be carried out directly by changing flows in arcs.
- Typically, the simplex method finds the optimal solution after a "small" number of pivots (but can be exponential in the worst case).
- The simplex algorithm is VERY efficient in practice.

Take an example containing two decision variables and two constraints

Example:

Maximize $Z = 7X_1+5X_2$, subject to the constraints,

$X_1+2X_2 \leq 6$

$4X_1+3X_2 \leq 12$ **and X_1 & X_2 are non-negative.**

Step1: Convert the LP problem into a system of linear equations.

One can do this by rewriting the constraint inequalities as equations by adding new "slack variables" and assigning them ***zero*** coefficients(profits) in the objective function as shown below:

$$X_1+2X_2+S_1 = 6$$
$$4X_1+3X_2+S_2 = 12$$

And the Objective Function would be:

$Z = 7X_1+5X_2+0.S_1+0.S_2$

Step 2: Obtain a Basic Solution to the problem.

One can do this by putting the decision variables $X_1=X_2=0$, so that $S_1= 6$ and $S_2=12$.

These are the initial values of ***slack variables***.

$Z = 7X_1+5X_2+0.S_1+0.S_2$

$X_1+2X_2 +s_1=6$

$4X_1+3X_2 +s_2= 12$

Step 3: Form the Initial Tableau as shown.

Initial Tableau

		C_j	7	5	0	0	Min.Ratio (X_B/Pivotal Col.)
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	
0	S_1	6	1	2	1	0	6/1=6
0	S_2	12	4☆	3	0	1	12/4=3 →
		Z_j	0	0	0	0	
		(Net Evaluation)C_j - Z_j	7 ↑	5	0	0	

Step 4: Find (C_j-Z_j) having highest positive value. The column corresponding to this value is called the ***Pivotal Column*** and enters the table. In the previous table, column corresponding to variable X_1 is the pivotal column.

Step 5: Find the Minimum Positive Ratio. Divide X_B values by the corresponding values of Pivotal Column. The row corresponding to the minimum positive value is the ***Pivotal Row*** and leaves the table. In below table, row corresponding to the slack variable S_2 is the pivotal row.

Initial Tableau

		C_j	7	5	0	0	
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	Min.Ratio (X_B/Pivotal Col.)
0	S_1	6	1	2	1	0	6/1=6
0	S_2	12	4☆	3	0	1	12/4=3 →
		Z_j	0	0	0	0	
		(Net Evaluation)C_j - Z_j	7 ↑	5	0	0	

Step 6: First Iteration or First Simplex Tableau.

In the new table, shall place the incoming variable(X_1) instead of the outgoing variable(S_2). Accordingly, new values of this row have to be obtained in the following way :

R_2(New)=R_2(Old)/Pivotal Element = R_2(Old)/4

R_1(New)=R_1(Old) - (Intersecting value of R_1(Old) & Pivotal Col)*R_2(New)

=R_1(Old) - 1*R_2(New)

		C_j	7	5	0	0	
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	Min.Ratio (X_B/Pivotal Col.)
0	S_1	3	0	5/4	1	- 1/4	
7	X_1	3	1	3/4	0	1/4	
		Z_j	7	21/4	0	7/4	
		(Net Evaluation)C_j - Z_j	0	- 1/4	0	-7/4	

Step 7: If all the (C_j-Z_j) values are zero or negative,an optimum point is reached *otherwise* repeat the process as given in Step 4,5 & 6.

$Z = 7X_1+5X_2+0.S_1+0.S_2$
$X_1+2X_2+s_1=6$
$4X_1+3X_2+s_2=12$

Since all the (Cj-Zj) values are either negative or zero, hence an optimum solution has been achieved. The optimum values are: $X_1=3$, $X_2=0$ and, Max Z=21.

		C_j	7	5	0	0	
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	Min.Ratio (X_B/Pivotal Col.)
0	S_1	3	0	5/4	1	- 1/4	
7	X_1	3	1	3/4	0	1/4	
		Z_j	7	21/4	0	7/4	
(Net Evaluation)Cj - Zj			0	- 1/4	0	-7/4	

Example: Minimize $Z= 600X_1+500X_2$ subject to constraints,

$2X_1+ X_2 \geq 80$

$X_1+2X_2 \geq 60$ and $X_1, X_2 \geq 0$

Step1: Convert the LP problem into a system of linear equations.

One can do this by rewriting the constraint inequalities as equations by subtracting new "surplus & artificial variables" and assigning them *zero & +M* coefficients respectively in the objective function as shown below.

So the Objective Function would be:

$Z=600X_1+500X_2+0.S_1+0.S_2+MA_1+MA_2$ subject to constraints,

$$2X_1+ X_2-S_1+A_1 = 80$$

$$X_1+2X_2-S_2+A_2 = 60$$

$$X_1,X_2,S_1,S_2,A_1,A_2 \geq 0$$

Step 2: Obtain a Basic Solution to the problem.

One can do this by putting the decision variables $X_1=X_2=S_1=S_2=0$, so that $A_1= 80$ and $A_2=60$.

These are the initial values of ***artificial variables*****.**

Step 3: Form the Initial Tableau as shown.

		C_j	600	500	0	0	M	M	
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	A_1	A_2	Min.Ratio (X_B/Pivotal Col.)
M	A1	80	2	1	-1	0	1	0	80
M	A2	60	1	2 ☆	0	-1	0	1	60 →
		Z_j	3M	3M	−M	−M	M	M	
		$C_j - Z_j$	600-3M	500-3M ↑	M	M	0	0	

It is clear from the tableau that X_2 will enter and A_2 will leave the basis. Hence 2 is the key element in pivotal column. Now, the new row operations are as follows:

$R_2(New) = R_2(Old)/2$

$R_1(New) = R_1(Old) - 1*R_2(New)$

		C_j	600	500	0	0	M	
C_B	Basic Variable (B)	Basic Soln(X_B)	X_1	X_2	S_1	S_2	A_1	**Min.Ratio (X_B/Pivotal Col.)**
M	A_1	50	3/2☆	0	-1	1/2	1	100/3 →
500	X_2	30	1/2	1	0	- 1/2	0	60
		Z_j	3M/2+250	500	− M	M/2-250	M	
		$C_j - Z_j$	**350-3M/2** ↑	0	M	250-M/2	0	

It is clear from the tableau that X_1 will enter and A_1 will leave the basis. Hence 2 is the key element in pivotal column. Now, the new row operations are as follows:

$R_1(New) = R_1(Old)*2/3$

$R_2(New) = R_2(Old) – (1/2)*R_1(New)$

		Cj - Zj	**0**	0	700/3	400/3	

Since all the values of (C_j-Z_j) are either zero or positive and also both the artificial variables have been removed, an optimum solution has been arrived at with X_1=100/3 , X_2=40/3 and Z=80,000/3.