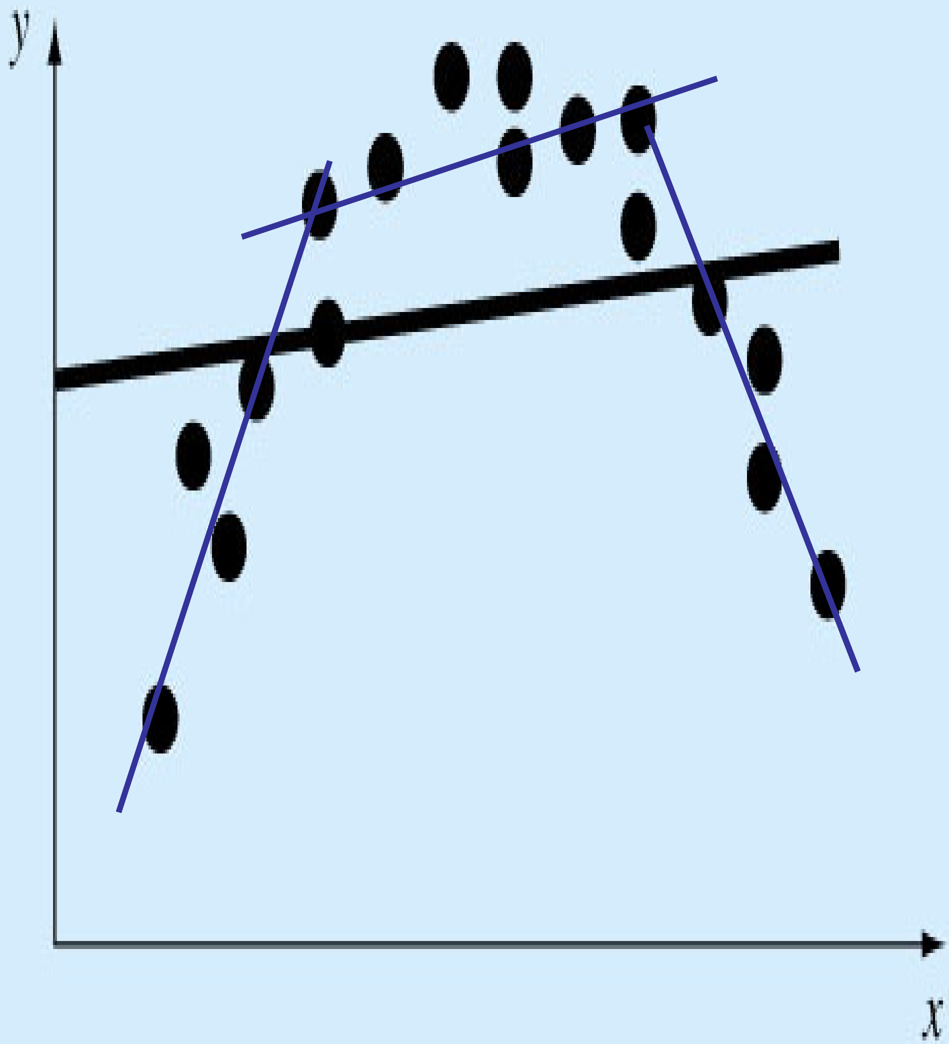
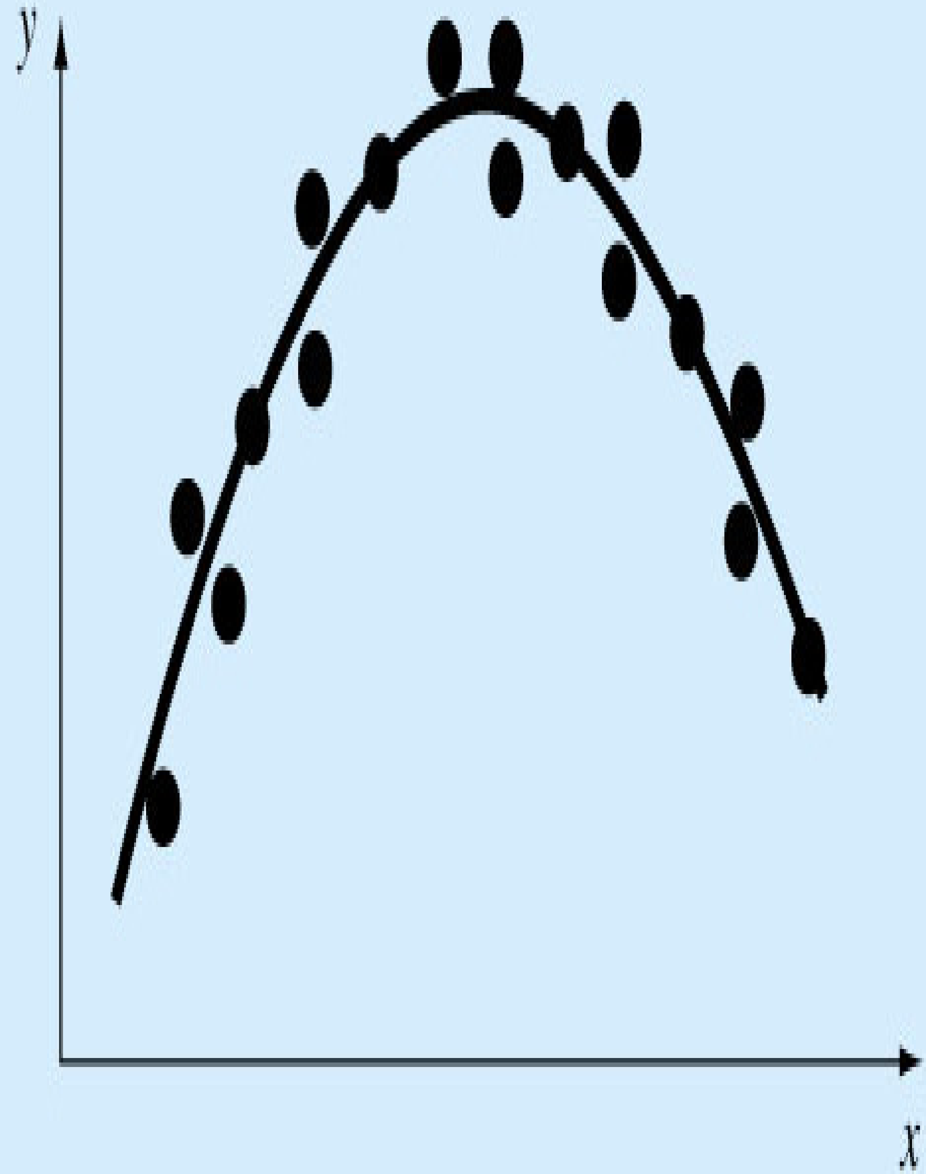
Polynomial Regression



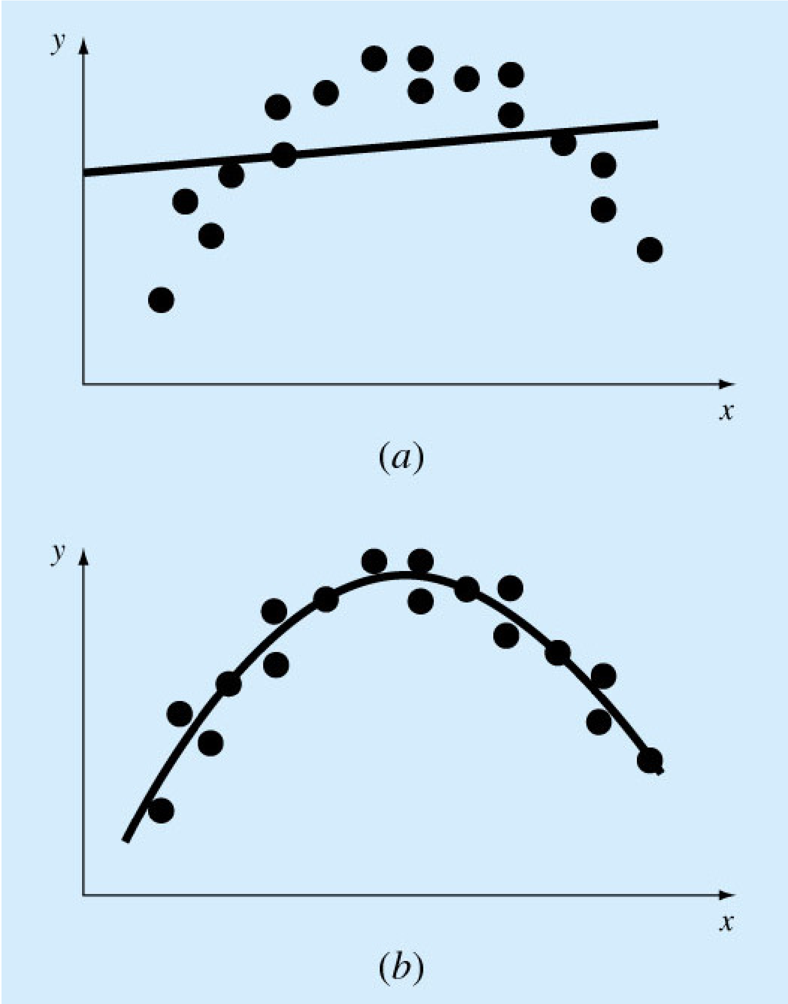


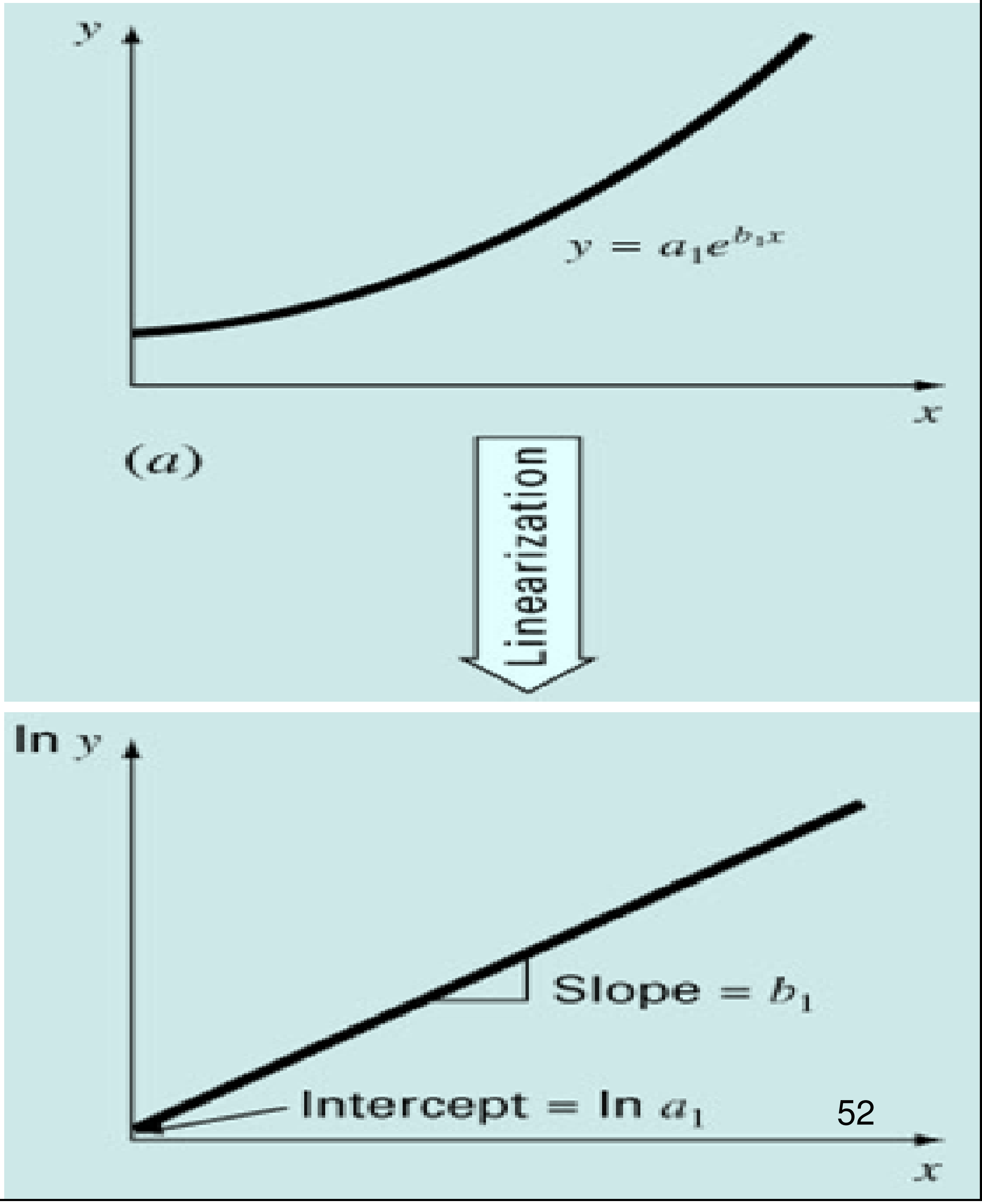
A parabola is preferable

Linearization of Nonlinear Relationships

(a) Data that is ill-suited for linear least-squares regression

(b) Indication that a parabola may be more suitable





Linearization of Nonlinear Relationships

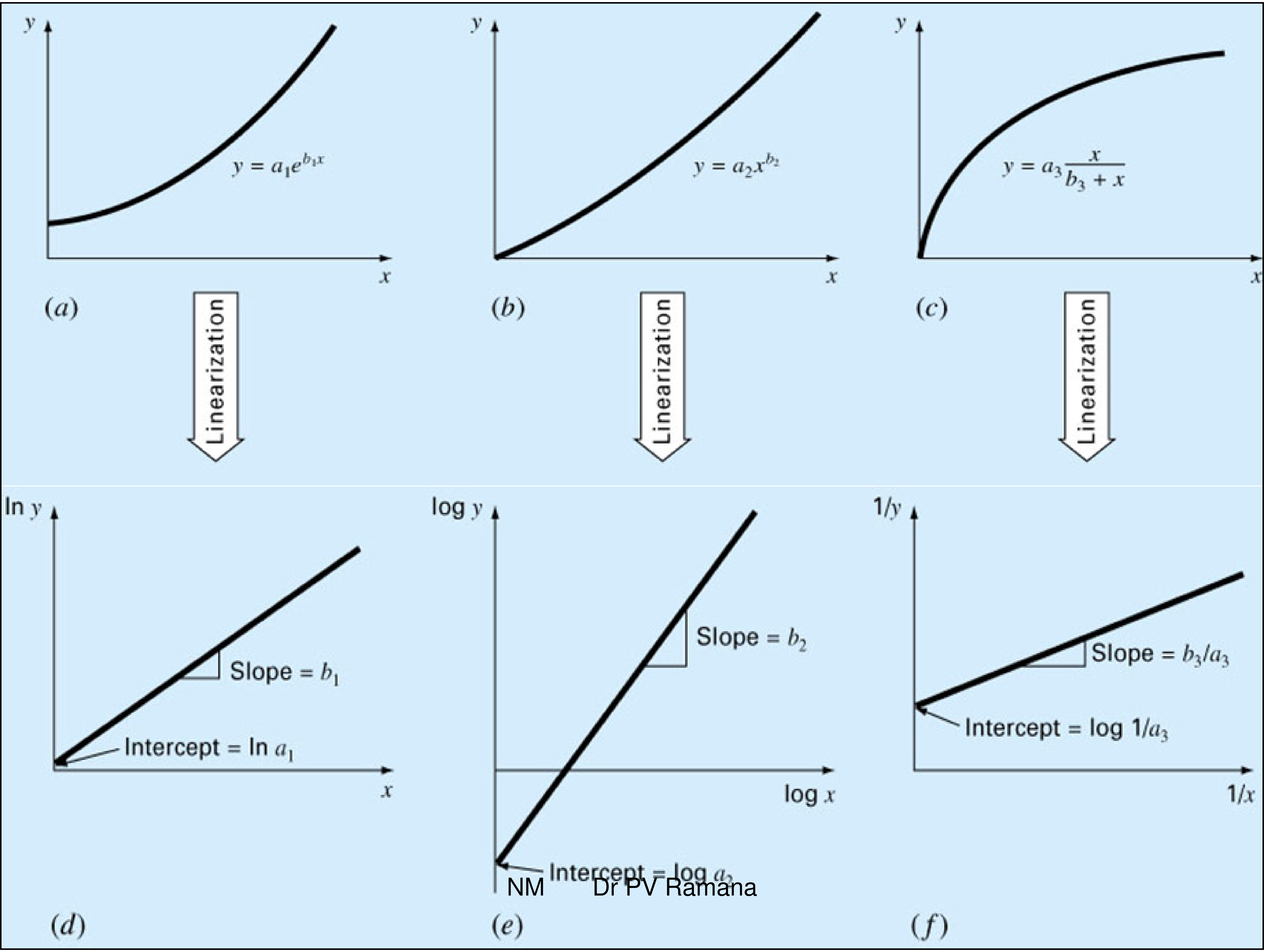
- The relationship between the dependent and independent variables is linear.
- However, a few types of nonlinear functions can be transformed into linear regression problems.

➢ The Hyperbola equation

➢ The exponential equation

➢ The power equation

➢ The saturation-growth-rate equation

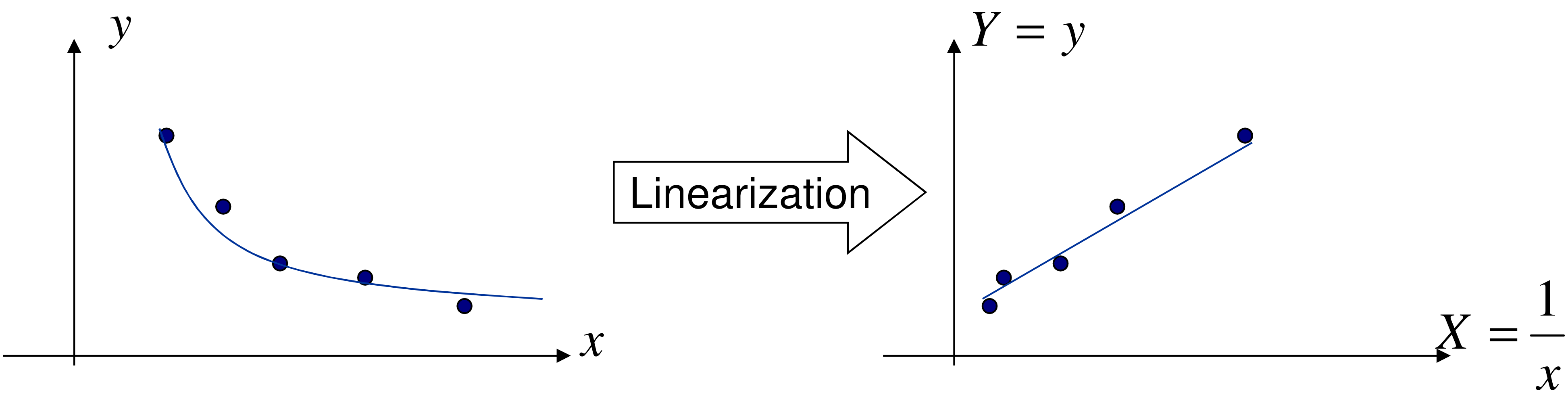


y
x
$y = a_1 e^{b_1 x}$
(a)
Linearization
y
x
$y = a_2 x^{b_2}$
(b)
Linearization
y
x
$y = a_3 \frac{x}{b_3 + x}$
(c)
Linearization
ln y
x
Slope = b_1
Intercept = ln a_1
(d)
log y
log x
Slope = b_2
Intercept = log a_2
(e)
1/y
1/x
Slope = b_3/a_3
Intercept = log $1/a_3$
(f)

Linearization of Nonlinear Relationships

1. Hyperbola function

$$y^* = a_o + a_1 x^*$$

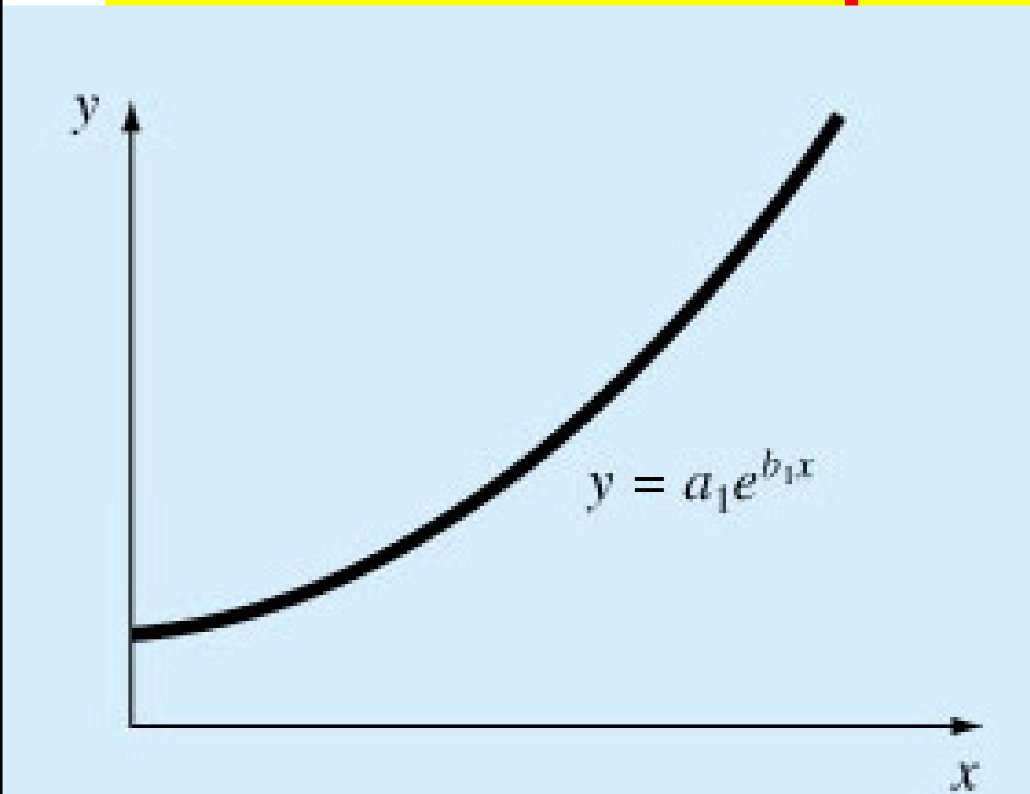


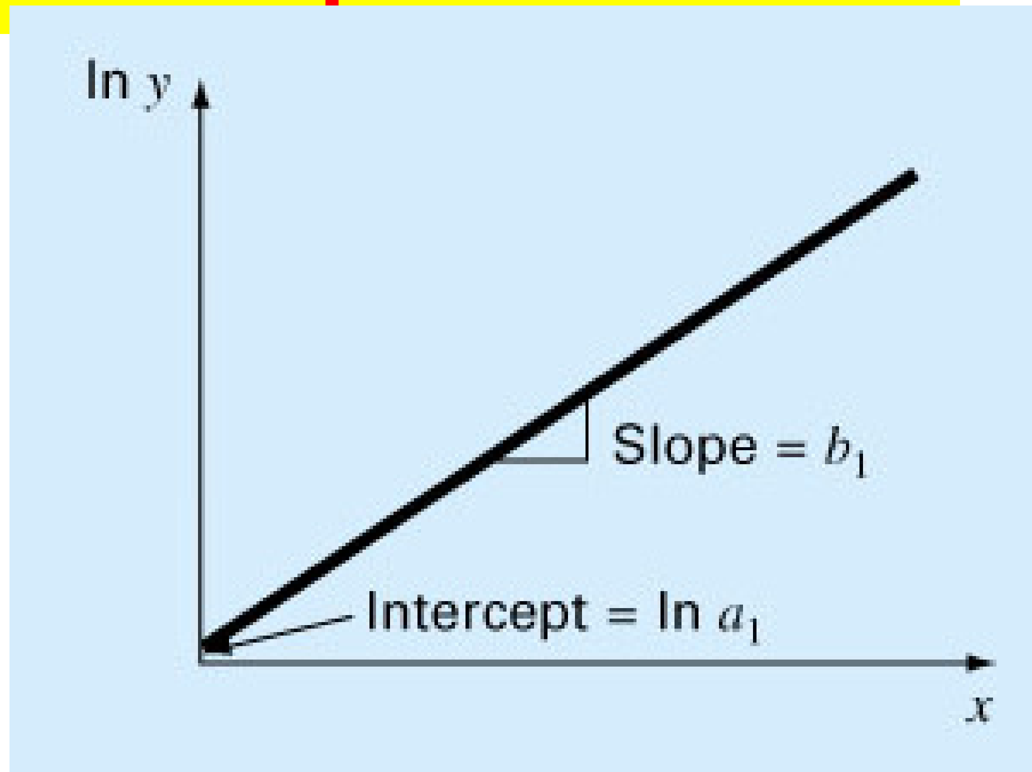
$$y = a_1 + \frac{b_1}{x}$$

$$Y = y = a_1 + b_1 \frac{1}{x} = a_0 + a_1 X$$

Linearization of Nonlinear Relationships

2. The exponential equation



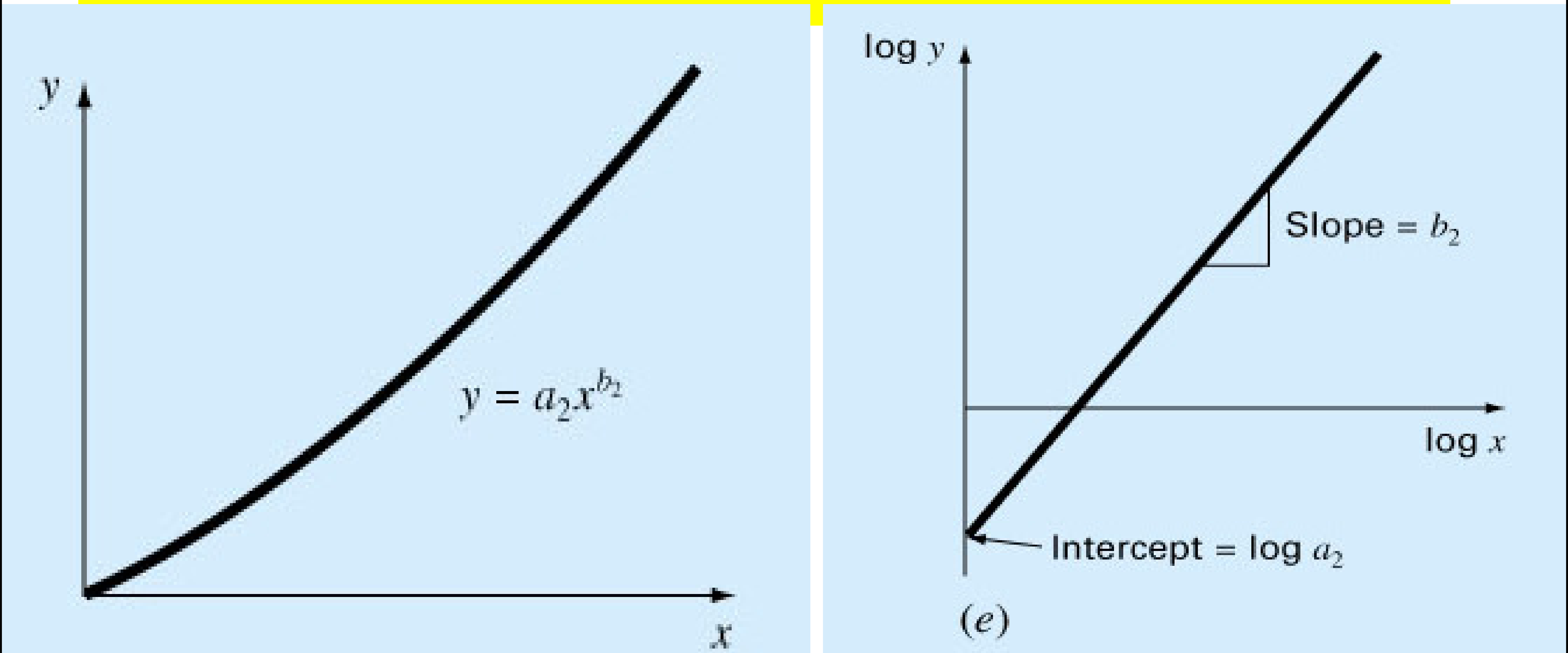


$$\ln y = \ln a_1 + b_1 x$$

$$y^* \quad = \quad a_o \; + \; a_1 \; x$$

Linearization of Nonlinear Relationships

3. The power equation

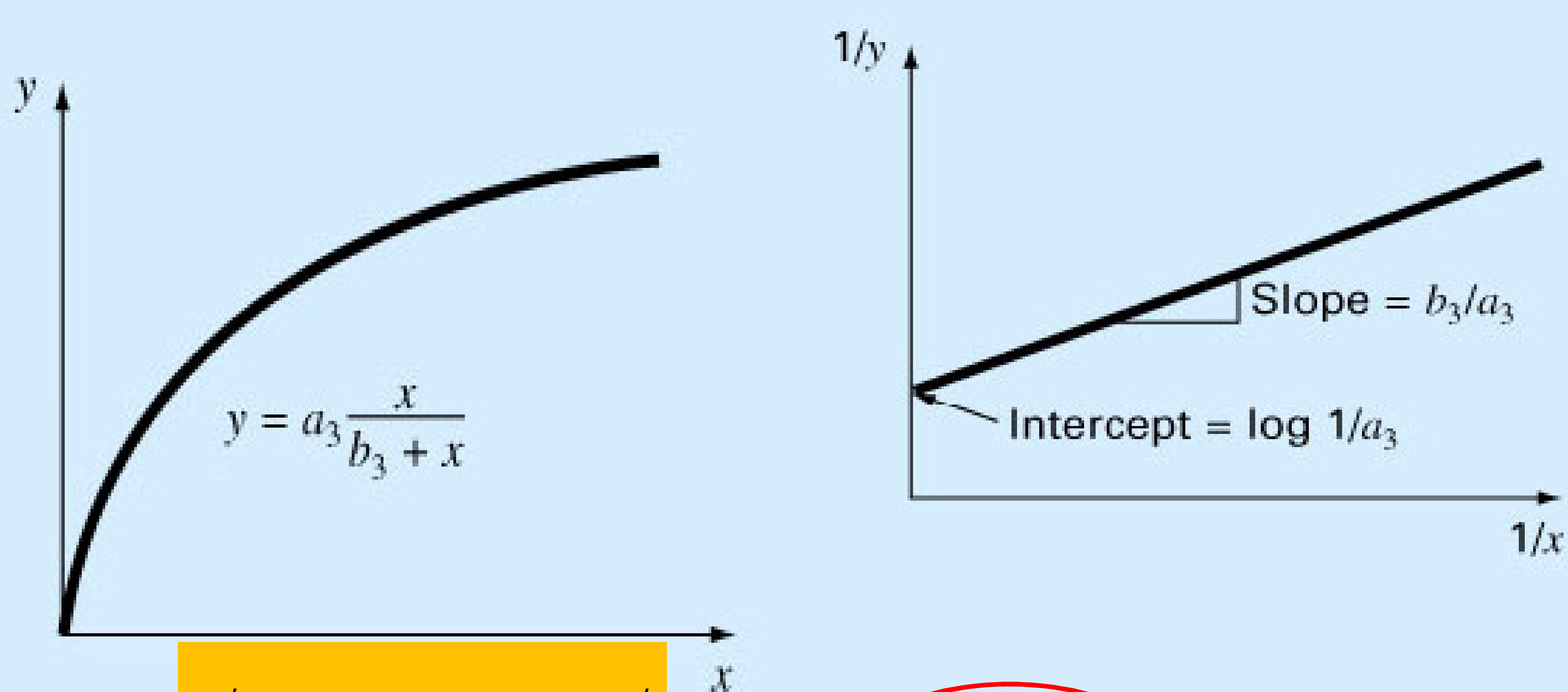


$$\log y = \log a_2 + b_2 \log x$$

$$y^* \quad = \quad a_o \; + \; a_1 \; x^*$$

Linearization of Nonlinear Relationships

4. The saturation-growth-rate equation



$$y^* = a_o + a_1 \; x^*$$

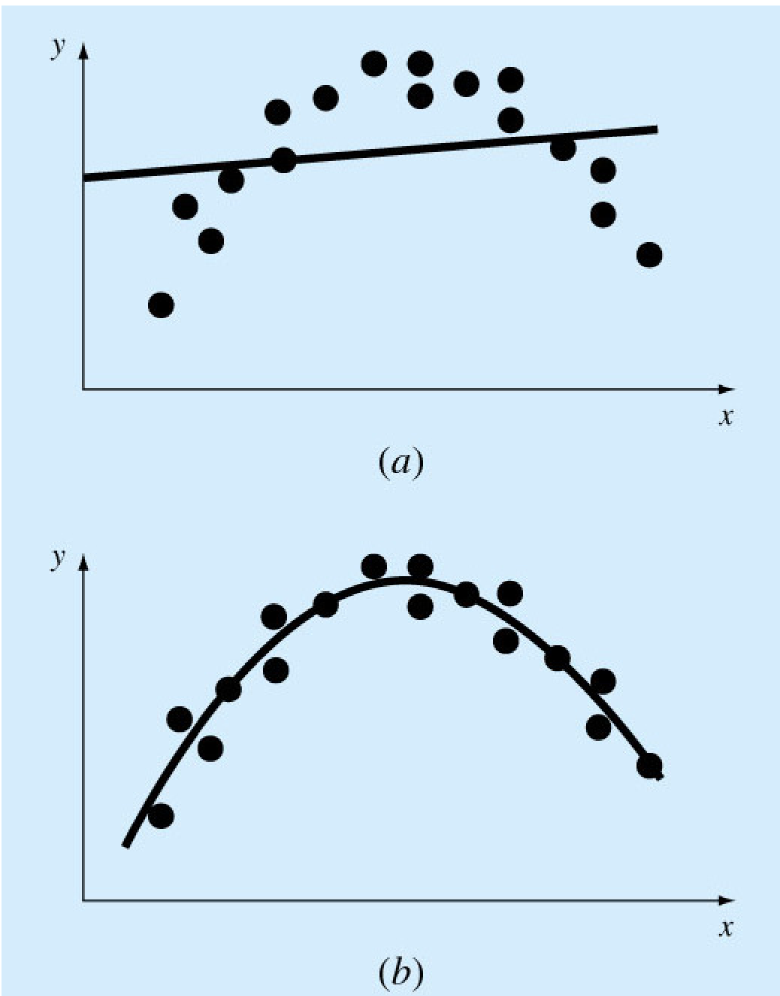
$$\frac{1}{y} = \frac{1}{a_3} + \frac{b_3}{a_3}\left(\frac{1}{x}\right)$$

$y^* = 1/y$
$a_o = 1/a_3$
$a_1 = b_3/a_3$
$x^* = 1/x$

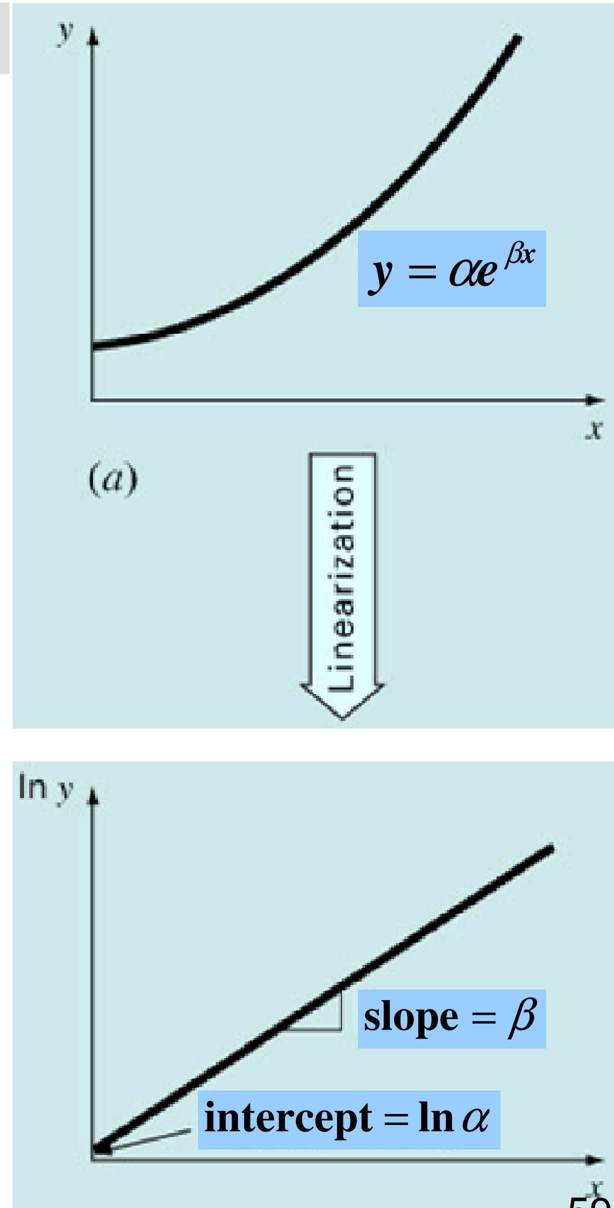
Linearization of Nonlinear Relationships

(a) Data that is ill-suited for linear least-squares regression

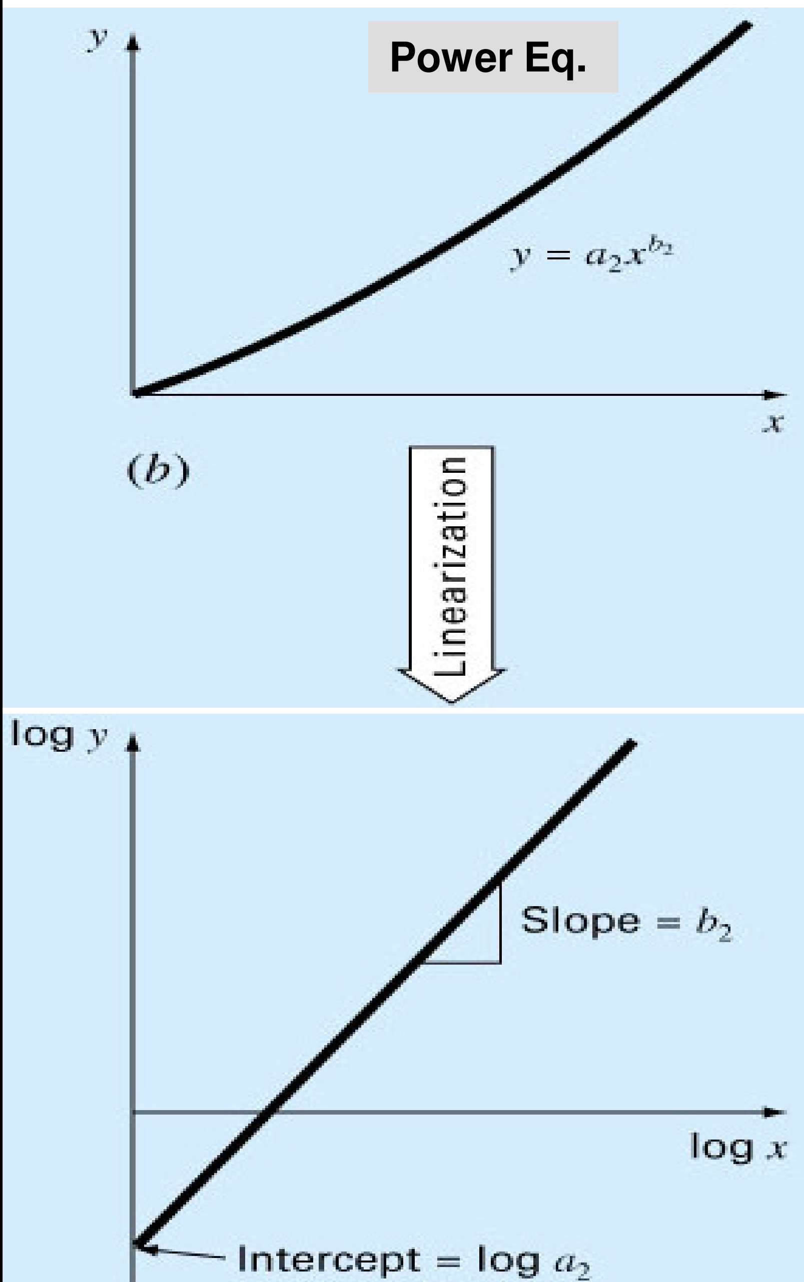
(b) Indication that a parabola may be more suitable



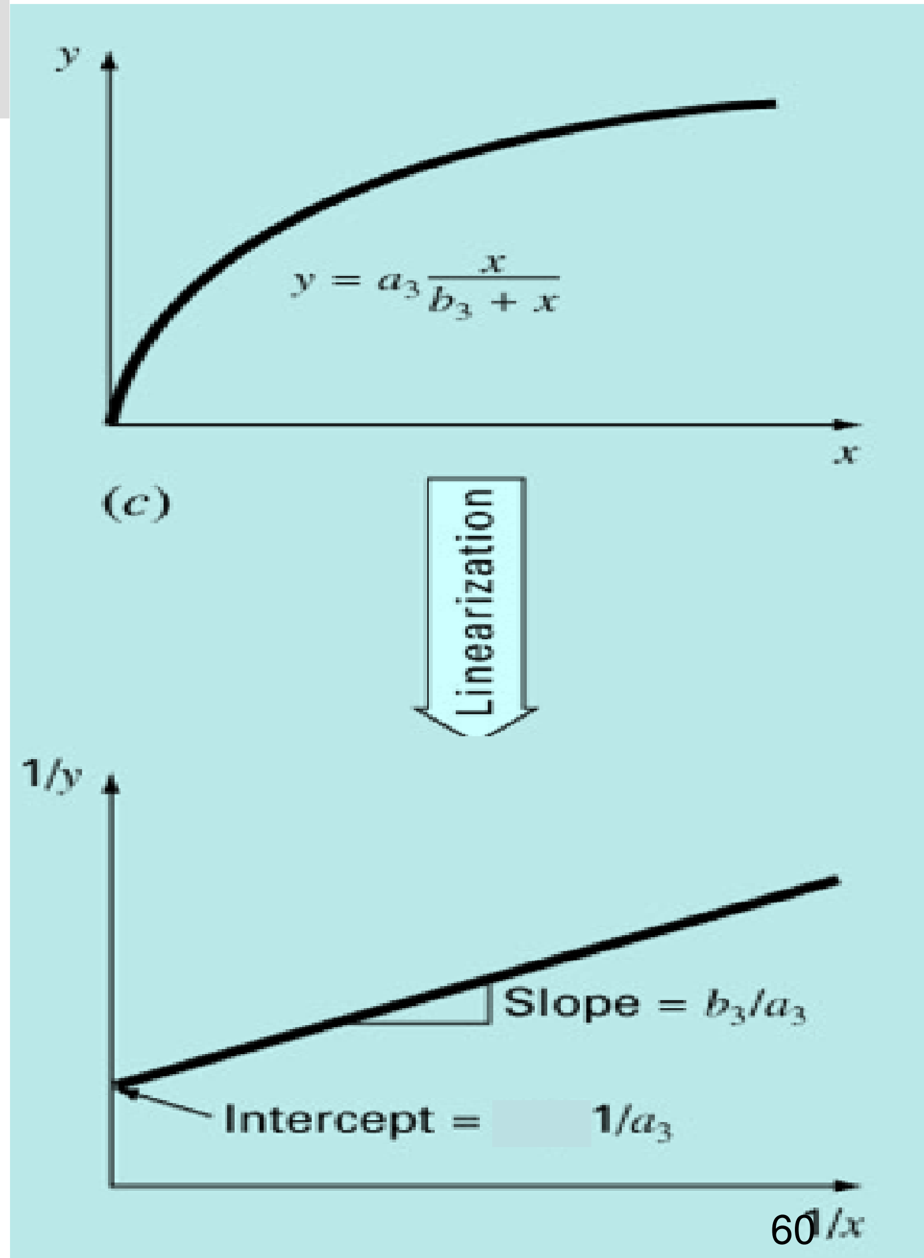
Exponential Eq.



Linearization of Nonlinear Relationships



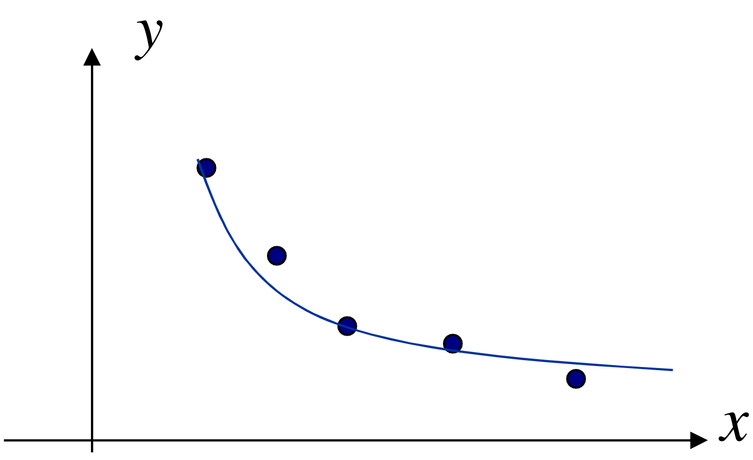
Saturation
growth-rate Eq.



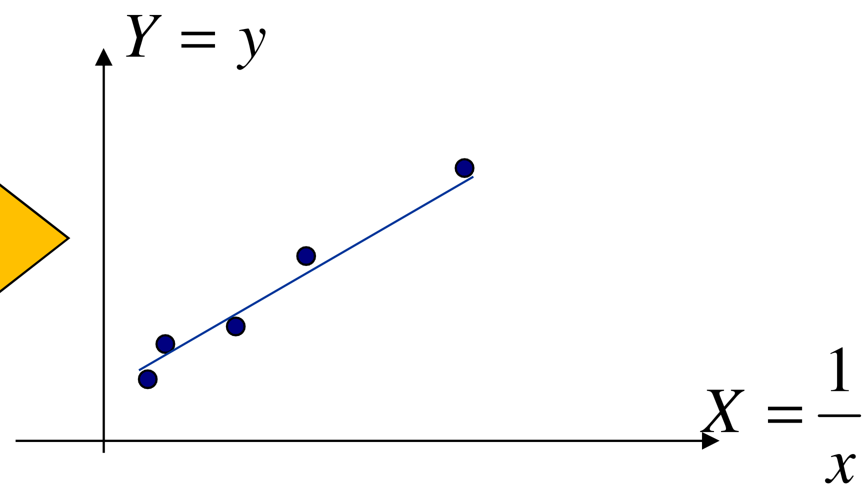
	Model	Model Transformation		Parameters Transformation
Power	$y = ax^b$	$Y = \log y$	$X = \log x$	$\alpha = \log a$ $\beta = b$
Exponential-1	$y = ae^{bx}$	$Y = \ln y$	$X = x$	$\alpha = \ln a$ $\beta = b$
Exponential-2	$y = ab^x$	$Y = \log y$	$X = x$	$\alpha = \log a$ $\beta = \log b$
Logarithmic	$y = \ln(ax^b)$	$Y = y$	$X = \ln x$	$\alpha = \ln a$ $\beta = b$
Reciprocal-1	$y = \dfrac{1}{a+bx}$	$Y = \dfrac{1}{y}$	$X = x$	$\alpha = a$ $\beta = b$
Reciprocal-2	$y = a + \dfrac{b}{1+x}$	$Y = y$	$X = \dfrac{1}{1+x}$	$\alpha = a$ $\beta = b$
Reciprocal-3	$y = \dfrac{1}{(a+bx)^2}$	$Y = \dfrac{1}{\sqrt{y}}$	$X = x$	$\alpha = a$ $\beta = b$
Square Root	$y = a + b\sqrt{x}$	$Y = y$	$X = \sqrt{x}$	$\alpha = a$ $\beta = b$

Curve Fitting

Nonlinear fit: Hyperbola function



Linearization



$$y = a_1 + \frac{b_1}{x}$$

$$Y = y = a_1 + b_1 \frac{1}{x} = a_1 + b_1 X$$

Linear Regression

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum y_i x_i \end{bmatrix}$$

Example 1

$$y = a_1 + \frac{b_1}{x}$$

Fit an **Hyperbola** function to the data below:

$$Y = y = a_0 + a_1 X = a_0 + \frac{a_1}{x}$$

x_i	1	3	4	6	9	15
y_i	**4.5**	**3.5**	**2.9**	**2.5**	**2.75**	**2.0**

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

Linearization: $X_i = 1/x$

X_i	1	1/3	1/4	1/6	1/9	1/15	∑X = 1.93
Y_i	4.5	3.5	2.9	2.5	2.75	2.0	∑Y = 18.15
$(XY)_i$	**4.5**	**1.16**	**0.725**	**0.416**	**0.305**	**0.133**	∑XY = 7.24
X^2	**1**	**1/9**	**1/16**	**1/36**	**1/81**	**1/225**	∑X^2 = 1.218

Fit the data above

$$a_1 = \frac{6(7.24) - (1.93)18.15}{6(1.218) - (1.93)^2} = 2.36$$

$$a_0 = \bar{y} - a_1\bar{x} = \frac{18.15}{6} - 2.36\left(\frac{1.93}{6}\right) = 1.8$$

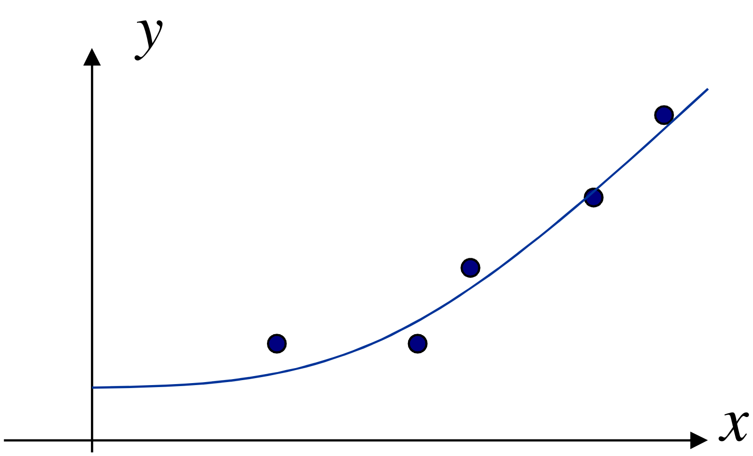
$$a_1 = 2.36, a_0 = 1.80$$

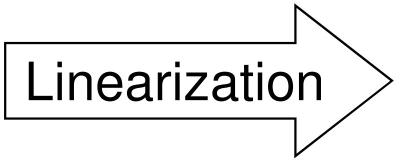
Therefore,

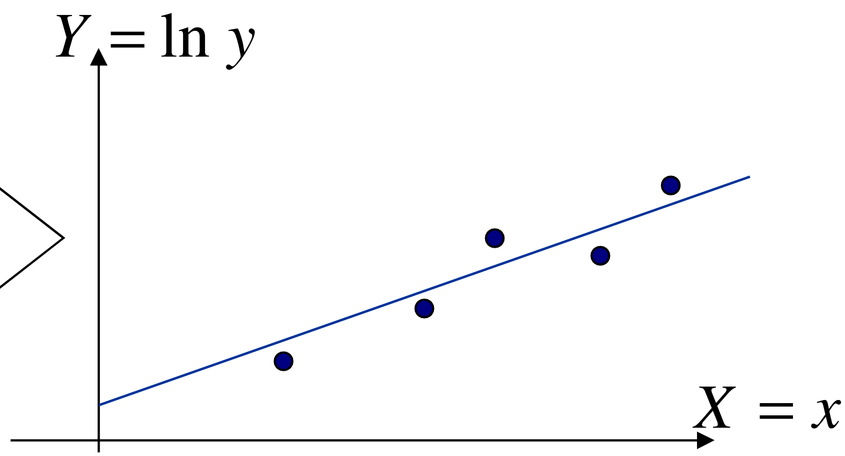
$$y = 1.8 + \frac{2.36}{x}$$

Curve Fitting

Nonlinear fit: Exponential model







$$y = a_1 e^{b_1 x}$$

$$Y = \ln y = b_1 x + \ln a_1 = b_1 X + \ln a_1$$

Linear Regression

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum y_i x_i \end{bmatrix}$$

Example 2

$$y = a_1 e^{b_1 x}$$

Fit an **exponential** function to the data below:

$$Y = \ln y = b_1 x + \ln a_1 = b_1 X + \ln a_1$$

x_i	1	3	4	6	9	15
y_i	4.5	3.5	2.9	2.5	2.75	2.0

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

Linearization: $Y_i = \ln y_i$

x_i	1	3	4	6	9	15	$\sum X = 38$
Y_i	1.5	1.25	1.06	0.92	1.01	0.69	$\sum Y = 6.43$
$(XY)_i$	1.5	3.75	4.24	5.52	9.09	10.35	$\sum XY = 34.45$
X^2	1	9	16	36	81	225	$\sum X^2 = 368$

Fit the data above

$$a_1 = \frac{6(34.45) - (38)6.43}{6(368) - (38)^2} = -0.044$$

$$a_0 = \bar{y} - a_1\bar{x} = \frac{6.43}{6} - 0.044\left(\frac{38}{6}\right) = 1.34$$

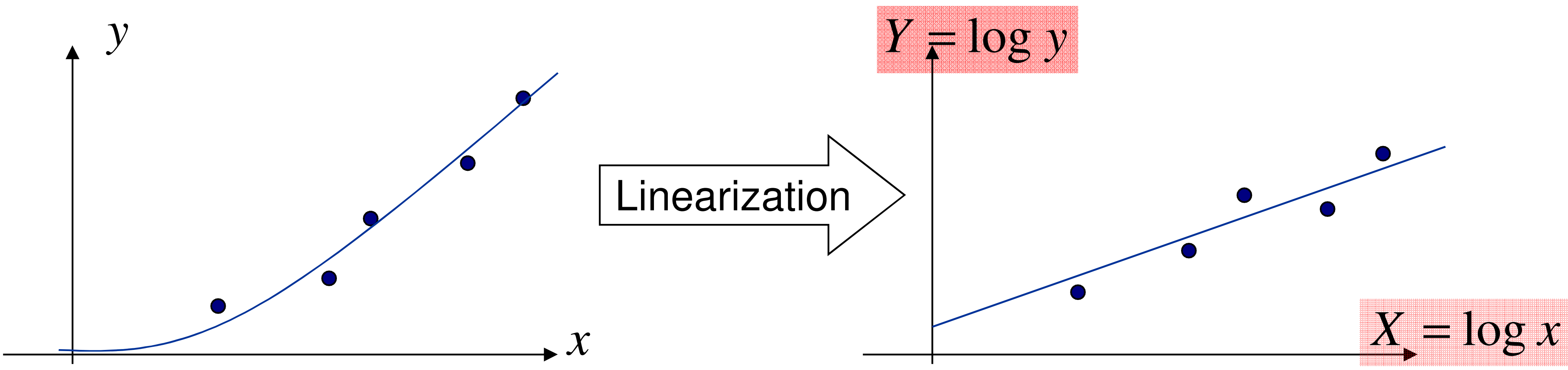
$a_0 = \ln a_1 = 1.34$ ⇒ $a_1 = e^{1.34} = 3.80$

$a_1 = b_1 = -0.044$

Therefore, $$y = 3.80 e^{-0.044x}$$

Curve Fitting

Nonlinear fit: Power equation



$$y = a_1 x^{b_1}$$

$$Y = \log y = b_1 \log x + \log a_1 = b_1 X + \log a_1$$

Linear Regression

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum y_i x_i \end{bmatrix}$$

Example 3A

$$y = a_2 x^{b_2}$$

Fit an **power** function to the data below:

$$\log y = \log a_2 + b_2 \log x$$

x_i	1	3	4	6	9	15
y_i	4.5	3.5	2.9	2.5	2.75	2.0

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

Linearization: $X_i = \ln x_i$ $\quad Y_i = \ln y_i$

X_i	0	1.1	1.386	1.79	2.197	2.7	$\sum X = 9.173$
Y_i	1.5	1.25	1.06	0.92	1.01	0.69	$\sum Y = 6.43$
$(XY)_i$	0	1.37	1.47	1.65	2.2	1.87	$\sum XY = 8.55$
X^2	0	1.027	1.922	3.21	4.83	7.34	$\sum X^2 = 18.329$

Fit the data above

$$a_1 = \frac{6(8.55) - (9.173)6.43}{6(18.329) - (9.173)^2} = -0.297$$

$$a_0 = \bar{y} - a_1\bar{x} = \frac{6.43}{6} + 0.297\left(\frac{9.173}{6}\right) = 1.526$$

$a_0 = \ln a_1 = 1.526$ ⇨ $a_1 = e^{1.526} = 4.6$

$b_2 = -0.297$

Therefore, $y = 4.6x^{-0.297}$

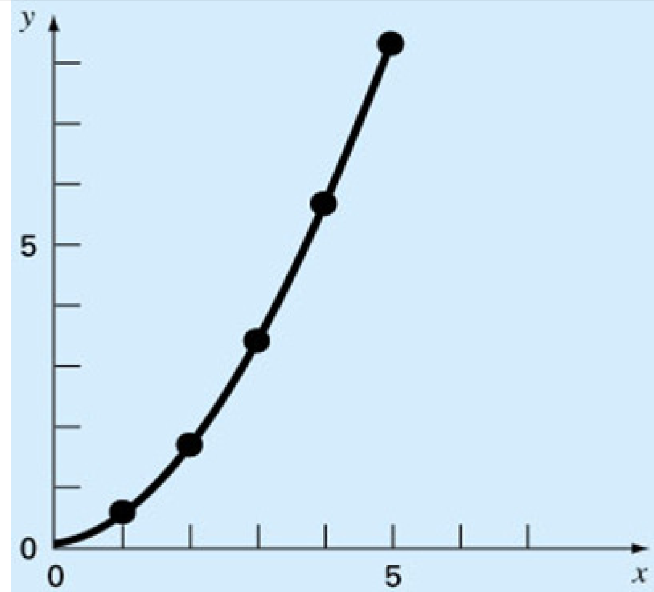
Example 3

Fit the following Equation:

$$y = a_2 x^{b_2}$$

to the data in the following table:

x_i	y_i	$X^*=\log x_i$	$Y^*=\log y_i$
1	0.5	0	-0.301
2	1.7	0.301	0.226
3	3.4	0.477	0.534
4	5.7	0.602	0.753
5	8.4	0.699	0.922
15	19.7	2.079	2.141



$$\log y = \log(a_2 x^{b_2})$$

$$\log y = \log a_2 + b_2 \log x$$

$$\text{let } Y^* = \log y,\ X^* = \log x,$$
$$a_0 = \log a_2,\ a_1 = b_2$$

$$Y^* = a_0 + a_1 X^*$$

Example 3

Xi	Yi	X^*_i=Log(X)	Y^*_i=Log(Y)	X*Y*	X*^2
1	0.5	0.0000	-0.3010	0.0000	0.0000
2	1.7	0.3010	0.2304	0.0694	0.0906
3	3.4	0.4771	0.5315	0.2536	0.2276
4	5.7	0.6021	0.7559	0.4551	0.3625
5	8.4	0.6990	0.9243	0.6460	0.4886
Sum		**2.079**	**2.141**	**1.424**	**1.169**

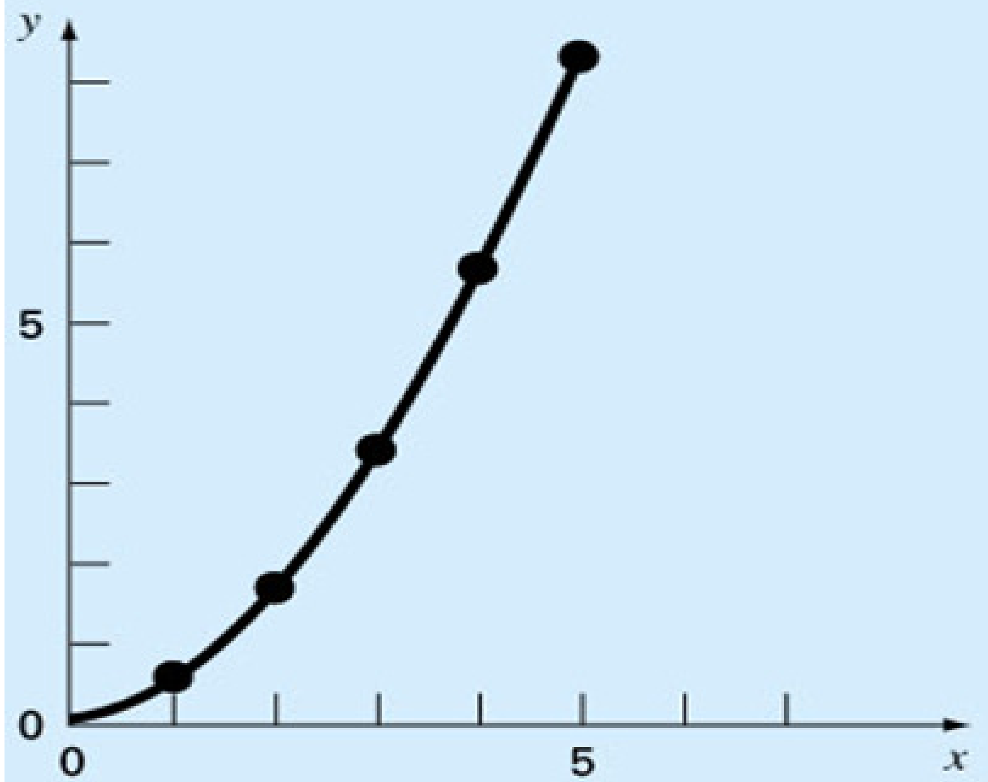
$$\begin{cases} a_1 = \dfrac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \dfrac{5\times 1.424 - 2.079\times 2.141}{5\times 1.169 - 2.079^2} = 1.75 \\ a_0 = \bar{y} - a_1\bar{x} = 0.4282 - 1.75\times 0.41584 = -0.334 \end{cases}$$

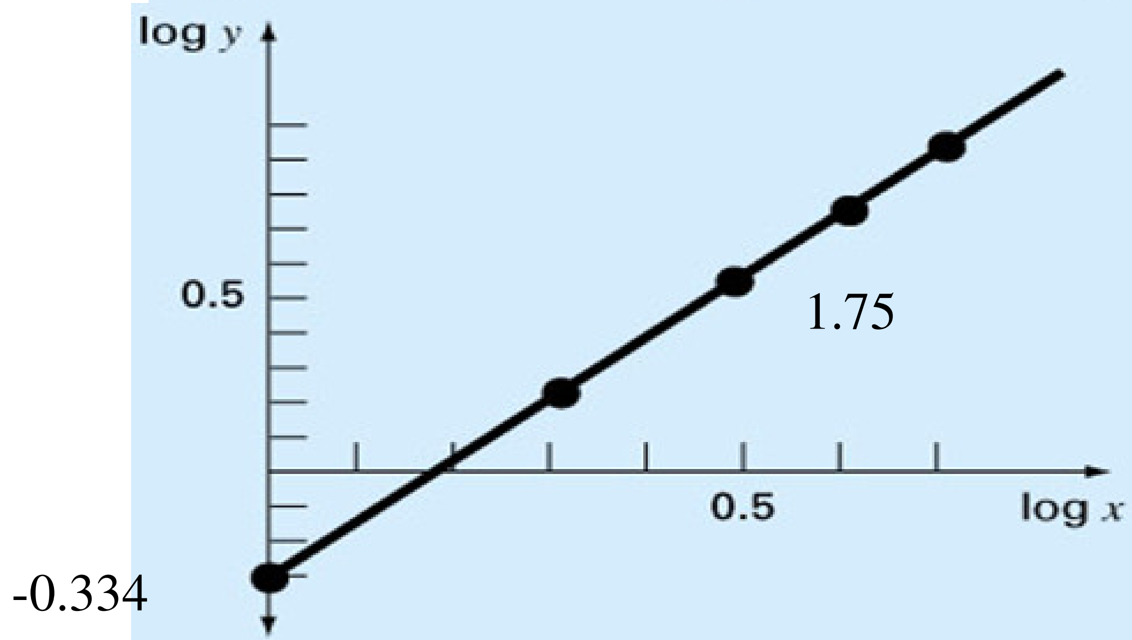
Example 3

$$\begin{cases} a_1 = \dfrac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \dfrac{5\times1.424 - 2.079\times2.141}{5\times1.169 - 2.079^2} = 1.75 \\ a_0 = \bar{y} - a_1\bar{x} = 0.4282 - 1.75\times0.41584 = -0.334 \end{cases}$$

$\log y = -0.334 + 1.75 \log x$

$$y = 0.46x^{1.75}$$





Example 3a

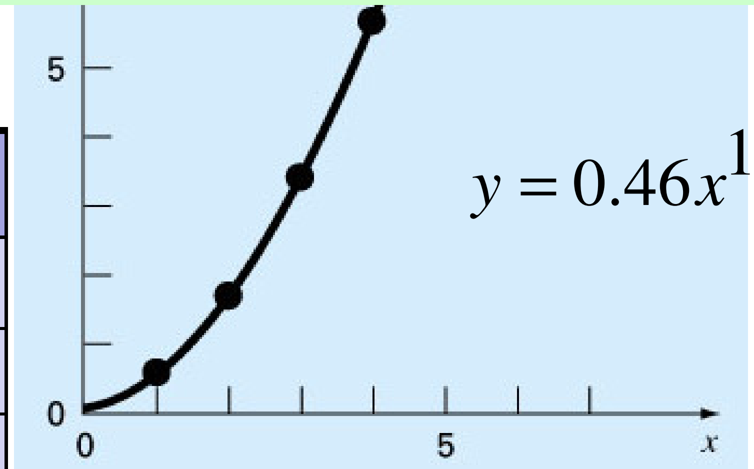
Data to be fit to the **power** equation:

$$y = \alpha_2 x^{\beta_2}$$

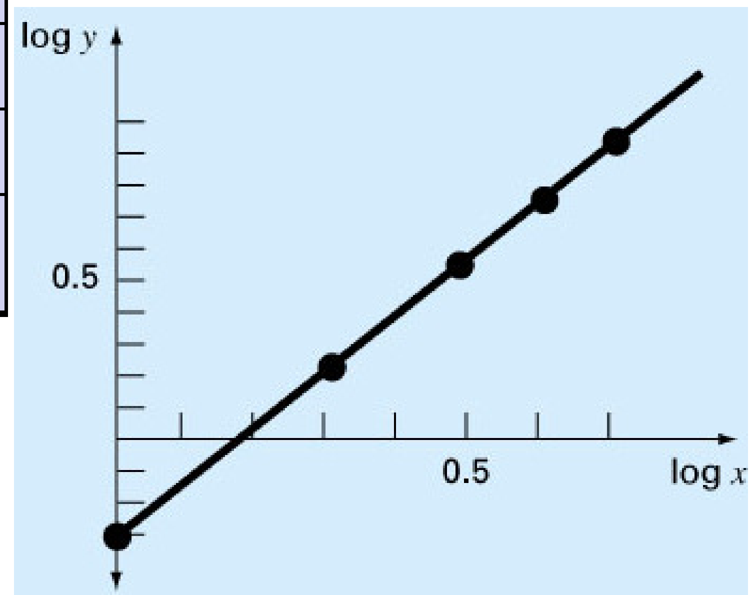
$$\Rightarrow \quad \log y = \beta_2 \log x + \log \alpha_2$$

$$\begin{cases} a_1 = \dfrac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2} = \dfrac{5\times 1.424 - 2.079\times 2.141}{5\times 1.169 - 2.079^2} = 1.75 \\ a_0 = \bar{y} - a_1\bar{x} = 0.4282 - 1.75\times 0.41584 = -0.334 \end{cases}$$

x	y	X=log x	Y=log y	XY	X²
1	0.5	0	-0.301	0	0
2	1.7	0.301	0.226	0.068	0.0906
3	3.4	0.477	0.531	0.253	0.228
4	5.7	0.602	0.756	0.455	0.362
5	8.4	0.699	0.924	0.646	0.489
		ΣX = 2.079	**ΣY = 2.136**	**ΣXY =1.422**	**ΣX² = 1.1696**



$$y = 0.46x^{1.75}$$



$$a_1 = \beta_2 = \frac{5(1.422) - (2.079)2.136}{5(1.1696) - (2.079)^2} = 1.75$$

$$a_0 = \alpha_2 = \bar{y} - a_1\bar{x} = \frac{2.136}{5} - 1.75\left(\frac{2.079}{5}\right) = -0.334$$

Linear Regression yields the result:

β_2 **= 1.75** **log** α_2 **= - 0.334** ➔ α_2 **= 0.46**

log y = 1.75*log x – 0.334

After (log y)–(log x) plot is obtained Find α_2 and β_2 using:

$$\text{Slope} = \beta_2 \qquad \text{intercept} = \log \alpha_2$$

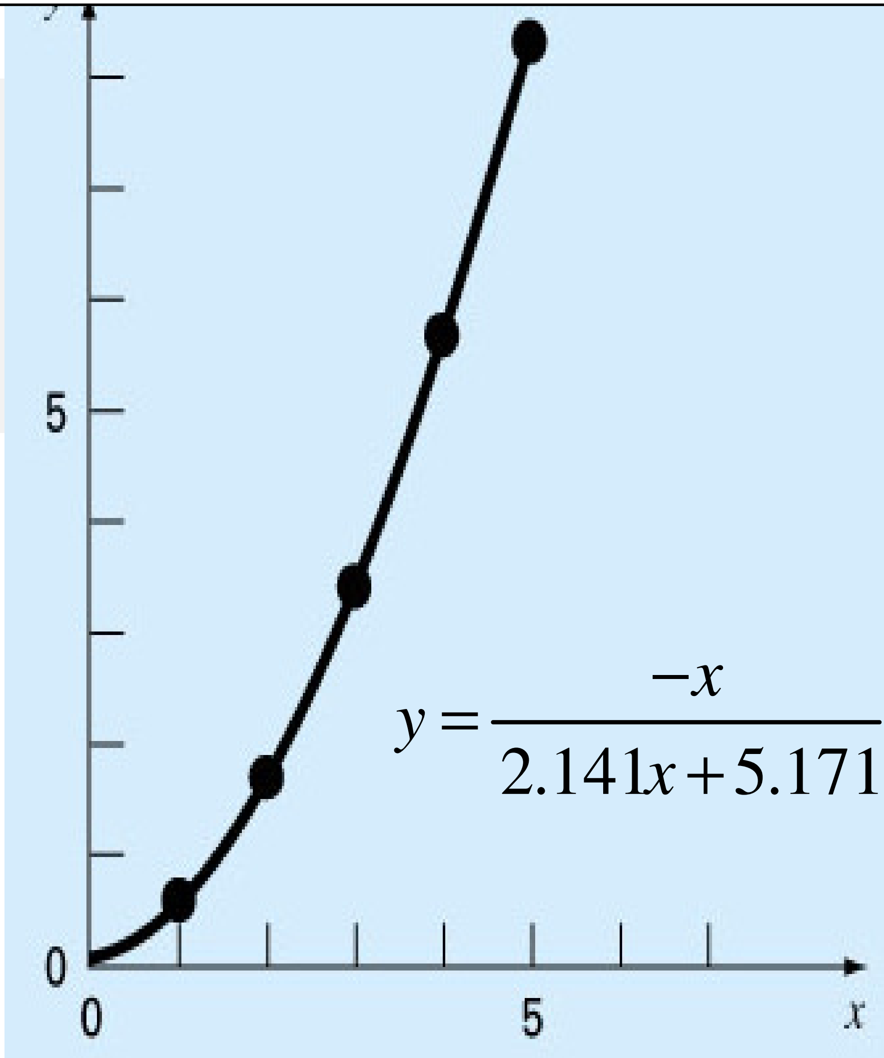
Example 4

Data to be fit to the **Saturation Growth** equation:

$$y = \frac{\alpha_o x}{a_1 + x}$$

$$\Rightarrow \frac{1}{y} = \frac{1}{a_0} + \frac{a_1}{a_0}\frac{1}{x}$$

x	y	X=1/x	Y=1/y	XY	X^2
1	0.5	1	2	2	1
2	1.7	0.5	0.588	0.294	0.25
3	3.4	0.333	0.294	0.098	0.111
4	5.7	0.25	0.176	0.044	0.062
5	8.4	0.2	0.119	0.023	0.04
		ΣX = 2.283	ΣY = 3.177	ΣXY =2.459	ΣX^2 = 1.463



Linear Regression yields the result:

$$a_1 = \frac{5(2.459) - (2.283)3.177}{5(1.46) - (2.283)^2} = 2.415$$

$$a_0 = \bar{y} - a_1\bar{x} = \frac{3.177}{5} - 2.415\left(\frac{2.283}{5}\right) = -0.467$$

$$y = \frac{-0.467x}{(2.415 + x)}$$

$$\Rightarrow \frac{1}{y} = \frac{1}{-0.467} + \frac{2.415}{-0.467}\frac{1}{x} = -2.141 - \frac{5.171}{x}$$

Example 5

The following data can be approximated by the **power** equation $y = a\,x^{b}$

Transform the relation to a linear one and determine the constants a and b.

x	y
0.5	0.012
1	0.11
1.5	0.328
2	0.81
2.5	1.543
3	2.702
3.5	4.278
4.5	9.143
5	12.48
23.5	31.4

	x	y	x'= Log (x)	y'= Log(y)	$(x')^2$	x' * y'
	0.5	0.012	-0.3	-1.94	0.09	0.584
	1	0.11	0	-0.96	0	0
	1.5	0.328	0.176	-0.48	0.031	-0.09
	2	0.81	0.301	-0.09	0.091	-0.03
	2.5	1.543	0.398	0.188	0.158	0.075
	3	2.702	0.477	0.432	0.228	0.206
	3.5	4.278	0.544	0.631	0.296	0.343
	4.5	9.143	0.653	0.961	0.427	0.628
	5	12.48	0.699	1.096	0.489	0.766
∑			2.947	-0.17	1.81	2.489

x	y(experiment)	y(estimated)
0.5	0.012	0.012265198
1	0.11	0.0989
1.5	0.328	0.335333938
2	0.81	0.797476732
2.5	1.543	1.561538994
3	2.702	2.70395362
3.5	4.278	4.301330369
4.5	9.143	9.168123515
5	12.48	12.59141571

The constants a and b can be obtained from the following table:

$$b' = \frac{n\sum x'_i\, y'_i - \sum x'_i \sum y'_i}{n\sum x'^2_i - (\sum x'_i)^2} = 3.0114$$

So that b = b' = 3.0114, a = $10^{-1.005}$ = 0.0989

$$a' = \frac{\sum y'_i}{n} - a_1 \frac{\sum x'_i}{n} = -1.005$$

Second Order Polynomial Regression

- Some engineering data is poorly represented by a straight line. A curve (polynomial) may be better suited to fit the data. The least squares method can be extended to fit the data to higher order polynomials.
- **As an example let us consider a second order polynomial to fit the data points:**

$$y = a_0 + a_1 x + a_2 x^2$$

$$\text{Minimize error:} \quad S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$$

$$\frac{\partial S_r}{\partial a_o} = -2\sum (y_i - a_o - a_1 x_i - a_2 x_i^2) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum x_i (y_i - a_o - a_1 x_i - a_2 x_i^2) = 0$$

$$\frac{\partial S_r}{\partial a_2} = -2\sum x_i^2 (y_i - a_o - a_1 x_i - a_2 x_i^2) = 0$$

$$n a_0 + \left(\sum x_i\right) a_1 + \left(\sum x_i^2\right) a_2 = \sum y_i$$

$$\left(\sum x_i\right) a_0 + \left(\sum x_i^2\right) a_1 + \left(\sum x_i^3\right) a_2 = \sum x_i y_i$$

$$\left(\sum x_i^2\right) a_0 + \left(\sum x_i^3\right) a_1 + \left(\sum x_i^4\right) a_2 = \sum x_i^2 y_i$$

Polynomial Regression

- **A 2nd order polynomial (quadratic)** is defined by:

$$y = a_o + a_1 x + a_2 x^2 + e$$

- The residuals between the model and the data:

$$e_i = y_i - a_o - a_1 x_i - a_2 x_i^2$$

- The sum of squares of the residual:

$$S_r = \sum e_i^2 = \sum \left(y_i - a_o - a_1 x_i - a_2 x_i^2 \right)^2$$

Polynomial Regression

$$\frac{\partial S_r}{\partial a_o} = -2\sum(y_i - a_o - a_1 x_i - a_2 x_i^2) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum(y_i - a_o - a_1 x_i - a_2 x_i^2)x_i = 0$$

$$\frac{\partial S_r}{\partial a_2} = -2\sum(y_i - a_o - a_1 x_i - a_2 x_i^2)x_i^2 = 0$$

$$\left.\begin{aligned}
\sum y_i &= n\cdot a_o + a_1\sum x_i + a_2\sum x_i^2 \\
\sum x_i y_i &= a_o\sum x_i + a_1\sum x_i^2 + a_2\sum x_i^3 \\
\sum x_i^2 y_i &= a_o\sum x_i^2 + a_1\sum x_i^3 + a_2\sum x_i^4
\end{aligned}\right\}$$

3 linear equations with 3 unknowns (a_o,a_1,a_2), can be solved

Polynomial Regression

- A system of 3x3 equations needs to be solved to determine the coefficients of the polynomial.

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{Bmatrix}$$

- The standard error & the coefficient of determination

$$s_{y/x} = \sqrt{\frac{S_r}{n-3}}$$

$$r^2 = \frac{S_t - S_r}{S_t}$$

Polynomial Regression - Ex

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{Bmatrix}$$

Example 1

Fit a second order polynomial to six data points:

x_i	y_i	x_i^2	x_i^3	x_i^4	$x_i y_i$	$x_i^2 y_i$
0	2.1	0	0	0	0	0
1	7.7	1	1	1	7.7	7.7
2	13.6	4	8	16	27.2	54.4
3	27.2	9	27	81	81.6	244.8
4	40.9	16	64	256	163.6	654.4
5	61.1	25	125	625	305.5	1527.5
15	152.6	55	225	979	585.6	2489

$$\sum x_i = 15$$

$$\sum y_i = 152.6$$

$$\sum x_i^2 = 55$$

$$\sum x_i^3 = 225$$

$$\sum x_i^4 = 979$$

$$\sum x_i y_i = 585.6$$

$$\sum x_i^2 y_i = 2488.8$$

$$\bar{x} = \frac{15}{6} = 2.5, \quad \bar{y} = \frac{152.6}{6} = 25.433$$

Polynomial Regression - Ex

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{Bmatrix}$$

- The system of simultaneous linear equations:

$$\begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 152.6 \\ 585.6 \\ 2488.8 \end{Bmatrix}$$

$$a_0 = 2.47857,\ a_1 = 2.35929,\ a_2 = 1.86071$$

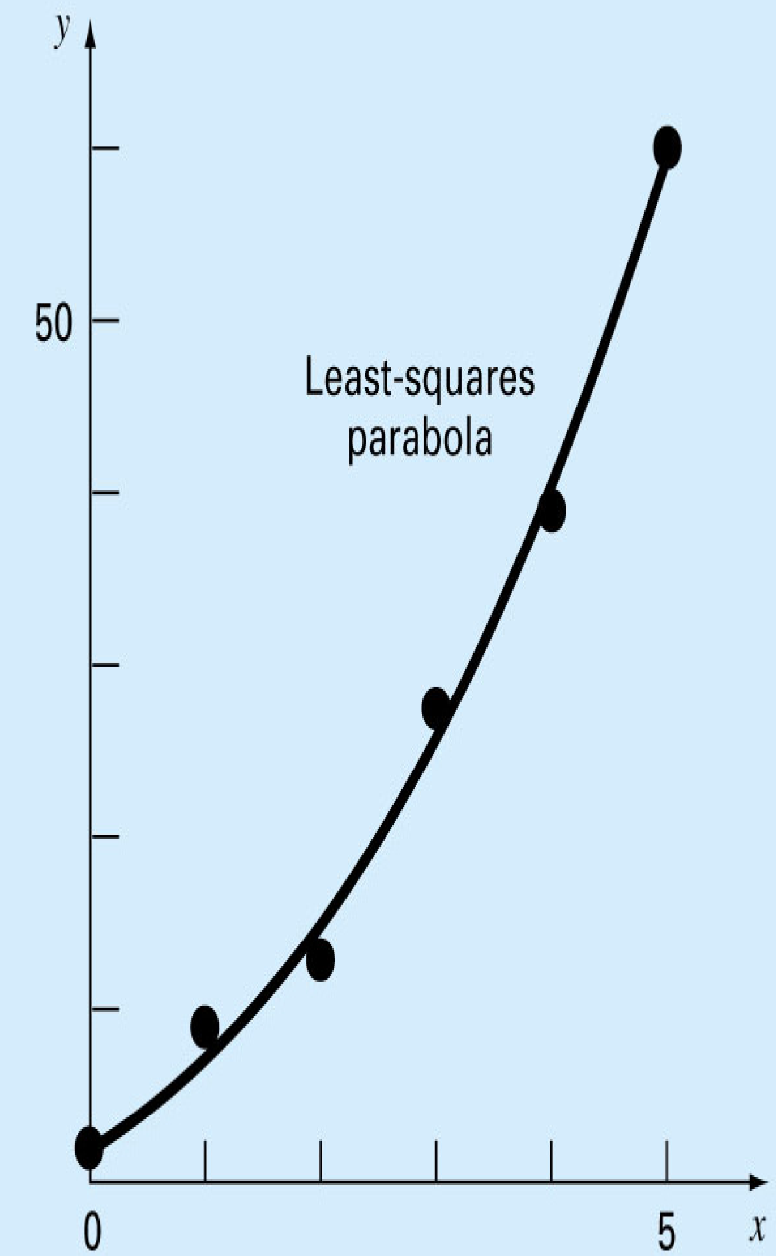
$$y = 2.47857 + 2.35929\,x + 1.86071\,x^2$$

$$S_t = \sum (y_i - \bar{y})^2 = 2513.39$$

$$S_r = \sum e_i^{\ 2} = 3.74657$$

Polynomial Regression - Ex

x_i	y_i	y_{model}	e_i^2	$(y_i-\bar{y})^2$
0	2.1	2.4786	0.14332	544.42889
1	7.7	6.6986	1.00286	314.45929
2	13.6	14.64	1.08158	140.01989
3	27.2	26.303	0.80491	3.12229
4	40.9	41.687	0.61951	239.22809
5	61.1	60.793	0.09439	1272.13489
15	**152.6**		**3.74657**	**2513.39333**



- The standard error of estimate:

$$s_{y/x} = \sqrt{\frac{3.74657}{6-3}} = 1.12$$

- **The coefficient of determination:**

$$r^2 = \frac{2513.39 - 3.74657}{2513.39} = 0.99851, \quad r = \sqrt{r^2} = 0.99925$$

1st Order Polynomial Regression

$y = a_0 + a_1 x$

2 X 2

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum y_i x_i \end{bmatrix}$$

$$a_1 = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - \left(\sum x_i \right)^2}$$

using $\bar{y} = a_0 + a_1 \bar{x}$, a_0 can be expressed as $a_0 = \bar{y} - a_1 \bar{x}$

2nd Order Polynomial Regression

3 X 3

$y = a_0 + a_1 x + a_2 x^2$

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{Bmatrix}$$

3^{rd} Order Polynomial Regression

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

4 X 4

$$\text{Minimize error :} \quad S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2 - a_3 x_i^3)^2$$

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \sum x_i^5 \\ \sum x_i^3 & \sum x_i^4 & \sum x_i^5 & \sum x_i^6 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \\ \sum x_i^3 y_i \end{Bmatrix}$$

m^{th} order Polynomial Regression

- To fit the data to an m^{th} order polynomial, need to solve the following system of linear equations ((m+1) equations with (m+1) unknowns)

$$\begin{bmatrix} n & \sum x_i & \cdots & \sum x_i^m \\ \sum x_i & \sum x_i^2 & \cdots & \sum x_i^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum x_i^m & \sum x_i^{m+1} & \cdots & \sum x_i^{m+m} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \\ \vdots \\ \sum x_i^m y_i \end{bmatrix}$$

Matrix Form

Polynomial Regression

General:

The mth-order polynomial:

$$y = a_o + a_1 x + a_2 x^2 + \ldots.. + a_m x^m + e$$

- A system of (m+1)x(m+1) linear equations must be solved for determining the coefficients of the mth-order polynomial.
- The standard error:

$$s_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$$

- The coefficient of determination:

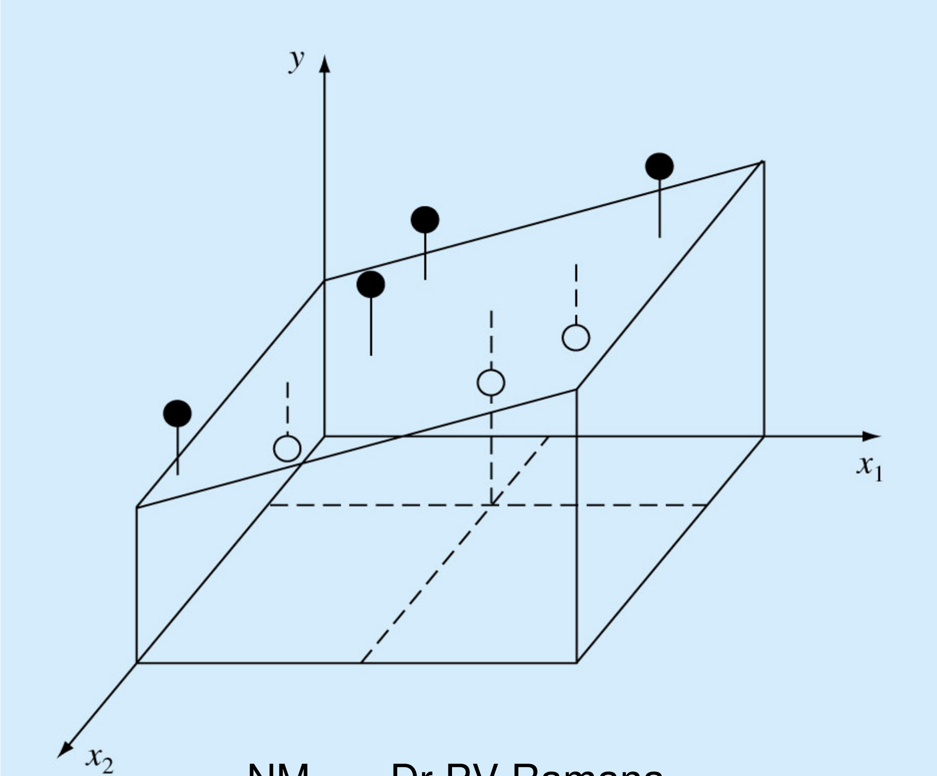
$$r^2 = \frac{S_t - S_r}{S_t}$$

Multiple Linear Regression

- A useful extension of linear regression is the case where *y* is a linear function of two or more independent variables. For example:

$$y = a_o + a_1x_1 + a_2x_2 + e$$

- For this 2-dimensional case, the regression line becomes a plane as shown in the figure below.



Multiple Linear Regression

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{Bmatrix}$$

Example (2 - vars) : Minimize error : $S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_{1i} - a_2 x_{2i})^2$

$$\frac{\partial S_r}{\partial a_o} = -2\sum (y_i - a_o - a_1 x_{1i} - a_2 x_{2i}) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum x_{1i}(y_i - a_o - a_1 x_{1i} - a_2 x_{2i}) = 0$$

$$\frac{\partial S_r}{\partial a_2} = -2\sum x_{2i}(y_i - a_o - a_1 x_{1i} - a_2 x_{2i}) = 0$$

$$n a_0 + \left(\sum x_{1i}\right) a_1 + \left(\sum x_{2i}\right) a_2 = \sum y_i$$

$$\left(\sum x_{1i}\right) a_0 + \left(\sum x_{1i}^2\right) a_1 + \left(\sum x_{1i} x_{2i}\right) a_2 = \sum x_{1i} y_i$$

$$\left(\sum x_{2i}\right) a_0 + \left(\sum x_{1i} x_{2i}\right) a_1 + \left(\sum x_{2i}^2\right) a_2 = \sum x_{2i} y_i$$

$$\begin{bmatrix} n & \sum x_{1i} & \sum x_{2i} \\ \sum x_{1i} & \sum x_{1i}^2 & \sum x_{1i} x_{2i} \\ \sum x_{2i} & \sum x_{1i} x_{2i} & \sum x_{2i}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_{1i} y_i \\ \sum x_{2i} y_i \end{bmatrix}$$

Which method would one can use to solve this Linear System of Equations?

Multiple Linear Regression

$$\begin{bmatrix} n & \sum x_{1i} & \sum x_{2i} \\ \sum x_{1i} & \sum x_{1i}^2 & \sum x_{1i}x_{2i} \\ \sum x_{2i} & \sum x_{1i}x_{2i} & \sum x_{2i}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_{1i} y_i \\ \sum x_{2i} y_i \end{bmatrix}$$

Example 1

The following data is calculated from the equation

x_1	x_2	y	$x^2{}_1$	$x^2{}_2$	x_1x_2
0	0	5	0	0	0
2	1	10	4	1	2
2.5	2	9	6.25	4	5
1	3	0	1	9	3
4	6	3	16	36	24
7	2	27	49	4	14
16.5	**14**	**54**	**76.25**	**54**	**48**

Use multiple linear regression to fit this data.

Solution:

$y = 5 + 4x_1 - 3x_2$

this system can be solved using Gauss Elimination

The result is: $a_0=5$ $a_1=4$ and $a_2= -3$

$y = 5 + 4x_1 -3x_2$

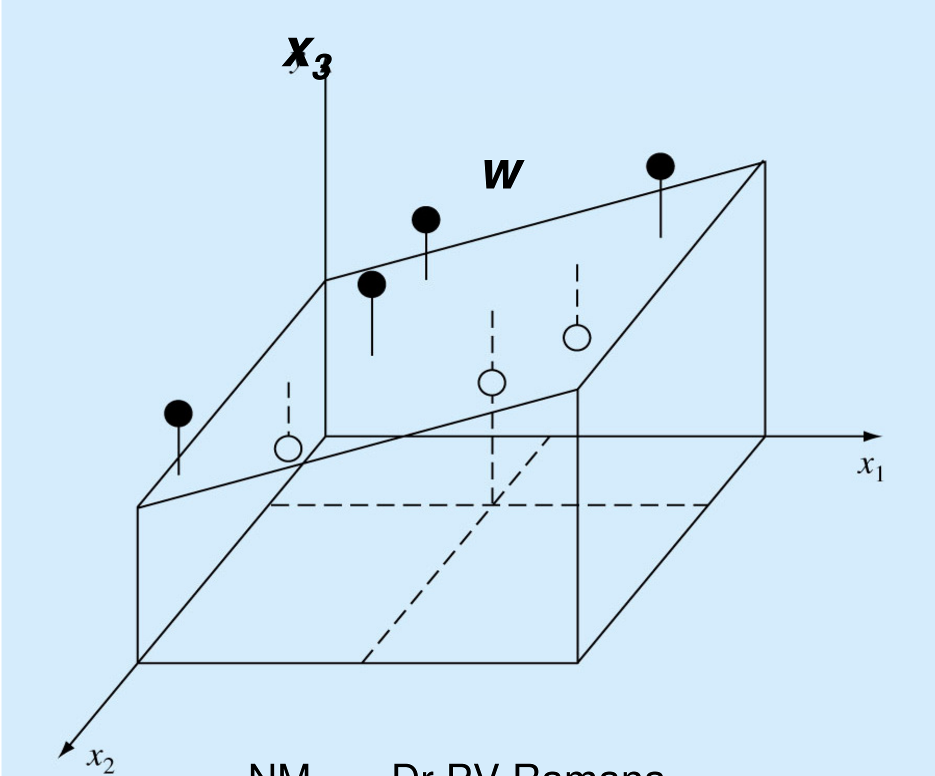
$$\begin{bmatrix} 6 & 16.5 & 14 \\ 16.5 & 76.25 & 48 \\ 14 & 48 & 54 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 54 \\ 243.5 \\ 100 \end{bmatrix}$$

3D Multiple Linear Regression

- A useful extension of linear regression is the case where *w* is a linear function of three or more independent variables. For example:

$$w = a_o + a_1x_1 + a_2x_2 + a_3x_3 + e$$

- For this three - dimensional case, the regression line becomes a plane as shown in the figure below.



Multiple Linear Regression

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \sum x_i^5 \\ \sum x_i^3 & \sum x_i^4 & \sum x_i^5 & \sum x_i^6 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \\ \sum x_i^3 y_i \end{Bmatrix}$$

Example (3 - variables): Minimize error: $S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_{1i} - a_2 x_{2i} - a_3 x_{3i})^2$

$$\frac{\partial S_r}{\partial a_o} = -2\sum (y_i - a_o - a_1 x_{1i} - a_2 x_{2i} - a_3 x_{3i}) = 0 \qquad n a_0 + \left(\sum x_{1i}\right) a_1 + \left(\sum x_{2i}\right) a_2 + \left(\sum x_{3i}\right) a_3 = \sum y_i$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum x_{1i}(y_i - a_o - a_1 x_{1i} - a_2 x_{2i} - a_3 x_{3i}) = 0 \qquad \left(\sum x_{1i}\right) a_0 + \left(\sum x_{1i}^2\right) a_1 + \left(\sum x_{1i} x_{2i}\right) a_2 + \left(\sum x_{1i} x_{3i}\right) a_3 = \sum x_{1i} y_i$$

$$\frac{\partial S_r}{\partial a_2} = -2\sum x_{2i}(y_i - a_o - a_1 x_{1i} - a_2 x_{2i} - a_3 x_{3i}) = 0 \qquad \left(\sum x_{2i}\right) a_0 + \left(\sum x_{1i} x_{2i}\right) a_1 + \left(\sum x_{2i}^2\right) a_2 + \left(\sum x_{2i} x_{3i}\right) a_3 = \sum x_{2i} y_i$$

$$\frac{\partial S_r}{\partial a_3} = -2\sum x_{3i}(y_i - a_o - a_1 x_{1i} - a_2 x_{2i} - a_3 x_{3i}) = 0 \qquad \left(\sum x_{3i}\right) a_0 + \left(\sum x_{1i} x_{3i}\right) a_1 + \left(\sum x_{2i} x_{3i}\right) a_2 + \left(\sum x_{3i}^2\right) a_3 = \sum x_{3i} y_i$$

$$\begin{bmatrix} n & \sum x_{1i} & \sum x_{2i} & \sum x_{3i} \\ \sum x_{1i} & \sum x_{1i}^2 & \sum x_{1i} x_{2i} & \sum x_{1i} x_{3i} \\ \sum x_{2i} & \sum x_{1i} x_{2i} & \sum x_{2i}^2 & \sum x_{2i} x_{3i} \\ \sum x_{3i} & \sum x_{1i} x_{3i} & \sum x_{2i} x_{3i} & \sum x_{3i}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_{1i} y_i \\ \sum x_{2i} y_i \\ \sum x_{3i} y_i \end{bmatrix}$$

Interpolation

- General formula for an *n-th* order polynomial
  - $y = a_0 + a_1x + a_2x^2 + \ldots. a_mx^m$
- For *m+1* data points, there is one, and only one polynomial of order *m* or less that passes through all points

Example: $y = a_0 + a_1x$
fits between 2 points
1^{st} order

Example: $y = a_0 + a_1x + a_2x^2$
fits between 3 points
2^{nd} order

Interpolation

- One can explore two mathematical methods well suited for computer implementation

- **Lagrange Interpolating Polynomial**

- **Newton's Divided Difference Interpolating Polynomials**

Given the data

- Employ 1st and 2nd order Lagrange polynomial to find f(1.4)

x	1	2	3	4	5	6	7
f(x)	1.01	0.49	0.34	0.24	0.21	0.16	0.14

x	1	2
f(x)	1.01	0.49

Solution:

1st order Lagrange polynomial f(1.4) = 0.698

x	1	2	3
f(x)	1.01	0.49	0.34

2nd order Lagrange polynomial f(1.4) = 0.7576

Solution:
Method2

x	x_0	x_1	x_2	x_3	x_4	x_5	x_6
f(x)	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$	$f(x_6)$

Lagrange Interpolating Polynomials

1st order Lagrange interpolating polynomial

$$f_1(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)$$

x	f (x)
x_0	$f(x_0)$
x_1	$f(x_1)$

2nd order Lagrange interpolating polynomial

The above polynomial is known as the 1st order Lagrange interpolating polynomial. The same procedure can be followed for the 2nd order Lagrange interpolating polynomial:

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

x	f (x)
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$

3^{rd} order Lagrange interpolating polynomial

It can be shown that for the 3^{rd} order polynomials have:

$$f_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

x	f (x)
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$

Given the data:

Calculate f(4) using 1st, 2nd, and 3rd order Lagrange interpolating polynomials

x	1	2	3	5	6
f(x)	4.75	4	5.25	19.75	36

- 1st order Lagrange interpolating polynomial:

3	5
5.25	19.75

$$f_1(x)=\frac{(x-x_1)}{(x_0-x_1)}f(x_0)+\frac{(x-x_0)}{(x_1-x_0)}f(x_1)$$

$$f_1(x)=\frac{(x-5)}{(3-5)}(5.25)+\frac{(x-3)}{(5-3)}(19.75)$$
$$=7.25x-16.5$$

$$f(4)=12.5$$

$$f_1(x)=\frac{(x-2)}{(1-2)}(4.75)+\frac{(x-1)}{(2-1)}(4)$$
$$=5.75-0.75x$$

$$f(4)=5.75-0.75x=5.5-0.75(4)=2.5$$

1	2	3
4.75	4	5.25

- **2nd order Lagrange interpolating polynomial:**

$$f_2(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0)+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1)+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

$$f_2(x)=\frac{(x-2)(x-3)}{(1-2)(1-3)}(4.75)+\frac{(x-1)(x-3)}{(2-1)(2-3)}(4)+\frac{(x-1)(x-2)}{(3-1)(3-2)}(5.25)$$

$f_2(x) = x^2 - (15x)/4 + 15/2$

$$f(4)=\frac{15}{2}+4^2-\frac{15}{4}4=8.5$$

2^{nd} order Lagrange interpolating polynomial:

x	1	2	3	5	6
f(x)	4.75	4	5.25	19.75	36

$$f_2(x) = \frac{(x-3)(x-5)}{(2-3)(2-5)}(4) + \frac{(x-2)(x-5)}{(3-2)(3-5)}(5.25) + \frac{(x-3)(x-2)}{(5-3)(5-2)}(19.75)$$

2	3	5
4	5.25	19.75

$f_2(x) = 2x^2 - 8.715x + 13.44$

$f_2(x) = 2(4)^2 - 8.715(4) + 13.44 = 10.58$

3	5	6
5.25	19.75	36

3^{rd} order Lagrange interpolating polynomial:

$$f_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$f_3(x) = \frac{(x-2)(x-3)(x-5)}{(1-2)(1-3)(1-5)}(4.75) + \frac{(x-1)(x-3)(x-5)}{(2-1)(2-3)(2-5)}(4) +$$

$$\frac{(x-1)(x-2)(x-5)}{(3-1)(3-2)(3-5)}(5.25) + \frac{(x-1)(x-2)(x-3)}{(5-1)(5-2)(5-3)}(19.75)$$

$f_3(x) = x^3/4 - x^2/2 - x + 6$

1	2	3	5
4.75	4	5.25	19.75

2	3	5	6
4	5.25	19.75	36

$$f(4) = \frac{4^3}{4} - \frac{4^2}{2} - 4 + 6 = 10$$

Newton's Divided Difference & Basis Spline Interpolating Polynomials

- Linear Interpolation
- Quadratic Interpolation
- General Form

B-Splines

- Linear B-Spline
- Quadratic B-Spline
- Cubic B-Spline

Linear Interpolation

Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2

How would approach estimating the density at 17^0 C?

x	x_0	x_1	x_2	x_3	x_4
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$

Interpolantion

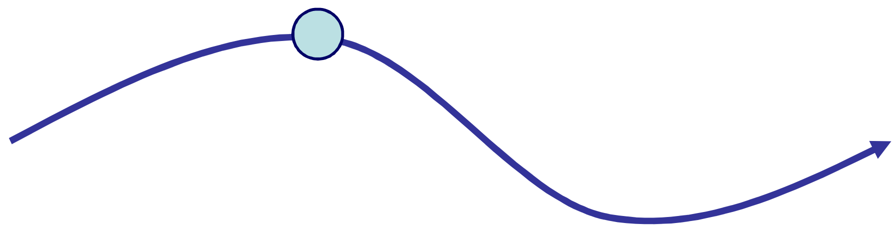
Polynomials are the most common choice of interpolation because they are easy to:

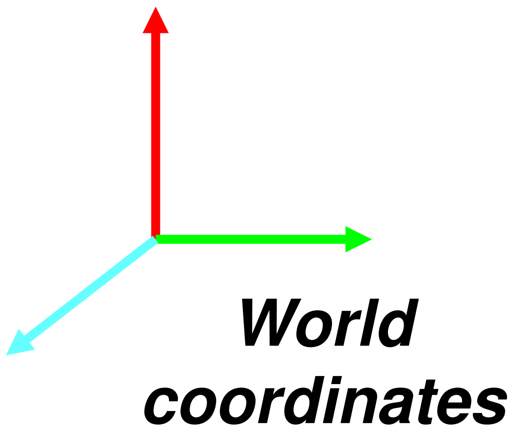
- Evaluate
- Differentiate, and
- Integrate.

Particle Motion

- A curve in 3-dimensional space

$$\mathbf{p}(t) = \big(x(t), y(t), z(t)\big)$$

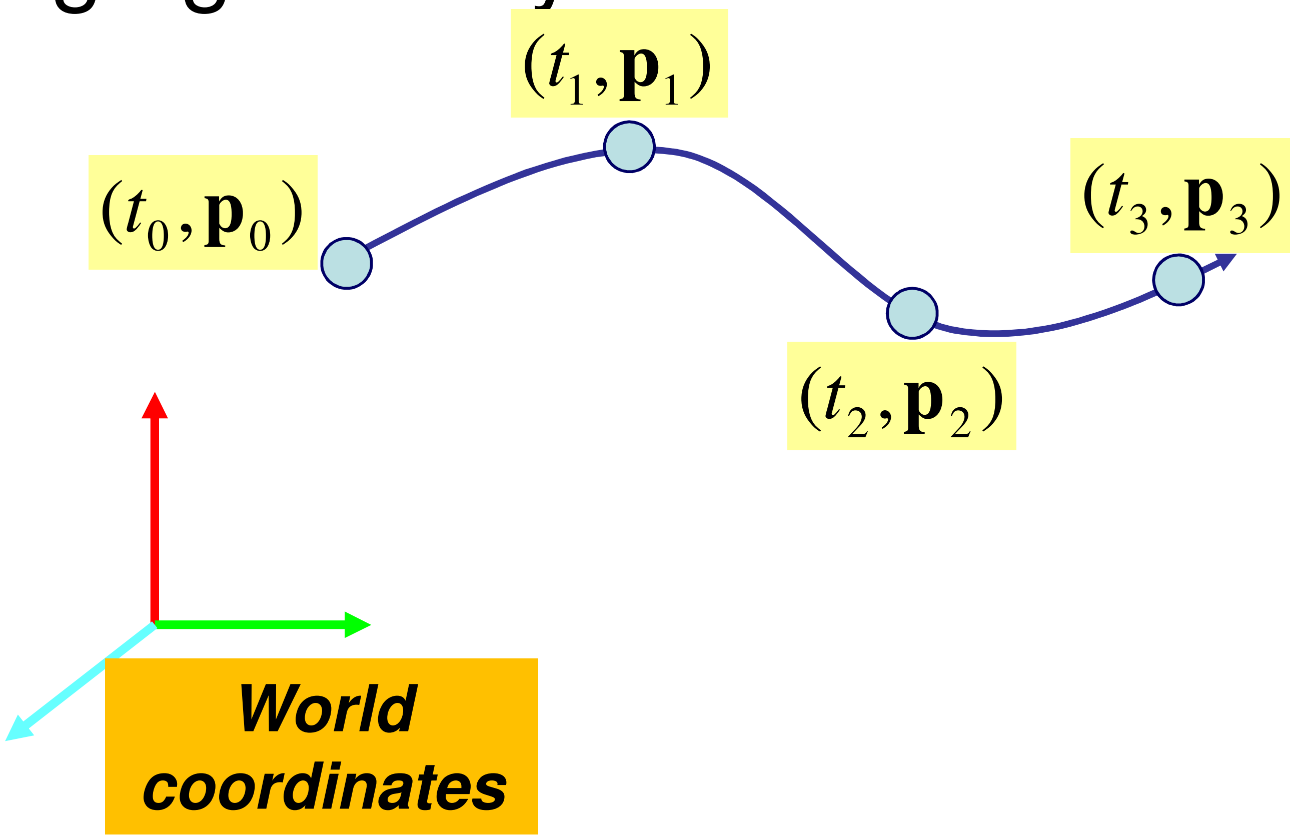




Keyframing Particle Motion

$(t_i, \mathbf{p}_i), 0 \le i \le n.$

- Find a smooth function $\mathbf{p}(t)$ that passes through given keyframes



Polynomial Curve

- Mathematical function vs. discrete samples
  - Compact
  - Resolution independence

- Why polynomials ?
  - Simple
  - Efficient
  - Easy to manipulate
  - Historical reasons

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

or

$$\mathbf{p}(t) = \mathbf{a} t^3 + \mathbf{b} t^2 + \mathbf{c} t + \mathbf{d}$$

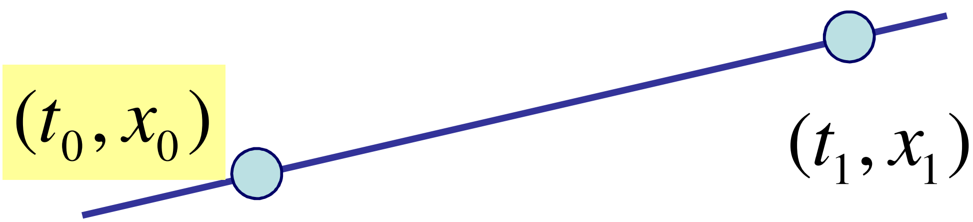
Degree and Order

- Polynomial
  - Order *n+1* (= number of coefficients)
  - Degree *n*

$$x(t) = a_n t^n + a_{n-1} t^{n-1} + \Lambda + a_1 t + a_0$$

Polynomial Interpolation

- Linear interpolation with a polynomial of degree one
  - **Input: two nodes**
  - **Output: Linear polynomial**



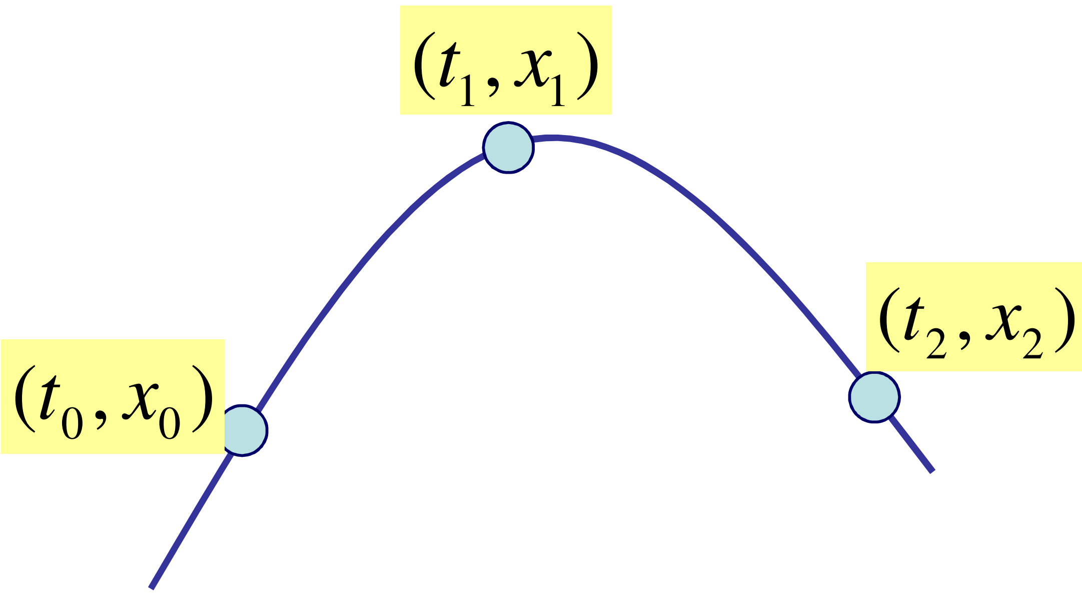
$$a_1 t_0 + a_0 = x_0$$
$$a_1 t_1 + a_0 = x_1$$

$$x(t) = a_1 t + a_0$$

$$\begin{pmatrix} 1 & t_0 \\ 1 & t_1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

Polynomial Interpolation

- Quadratic interpolation with a polynomial of degree two



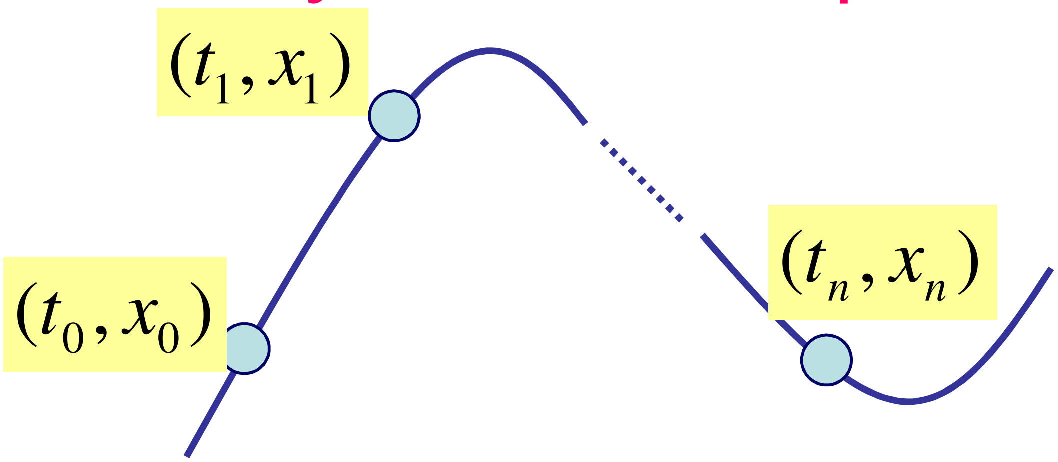
$$x(t) = a_2 t^2 + a_1 t + a_0$$

$$\begin{aligned} a_2 t_0^2 + a_1 t_0 + a_0 &= x_0 \\ a_2 t_1^2 + a_1 t_1 + a_0 &= x_1 \\ a_2 t_2^2 + a_1 t_2 + a_0 &= x_2 \end{aligned}$$

$$\begin{pmatrix} 1 & t_0 & t_0^2 \\ 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

Polynomial Interpolation

- Polynomial interpolation of degree n



$$x(t) = a_n t^n + \cdots + a_1 t + a_0$$

$$\begin{pmatrix} 1 & \cdots & t_0^{n-1} & t_0^n \\ 1 & \cdots & t_1^{n-1} & t_1^n \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & t_n^{n-1} & t_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Lagrange Polynomial

- Weighted sum of data points and cardinal functions

$$x(t) = L_0(t)x_0 + L_1(t)x_1 + \Lambda + L_n(t)x_n$$

$$L_k(t) = \frac{(t-t_0)\Lambda\,(t-t_{k-1})(t-t_{k+1})\Lambda\,(t-t_n)}{(t_k-t_0)\Lambda\,(t_k-t_{k-1})(t_k-t_{k+1})\Lambda\,(t_k-t_n)}$$

- Cardinal polynomial functions

$$L_k(t_i) = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

Limitation of Polynomial Interpolation

- **Oscillations at the ends**
  - **Nobody uses higher-order polynomial interpolation now**

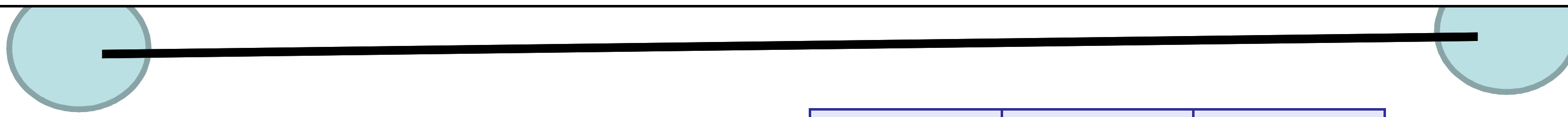
Lagrange Interpolating Polynomial

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

where Π designates the "product of"

The linear version of this expression is at n=1



$X_I=0$ $X_J=L$

x	x_0	x_1
$f(x)$	$f(x_0)$	$f(x_1)$

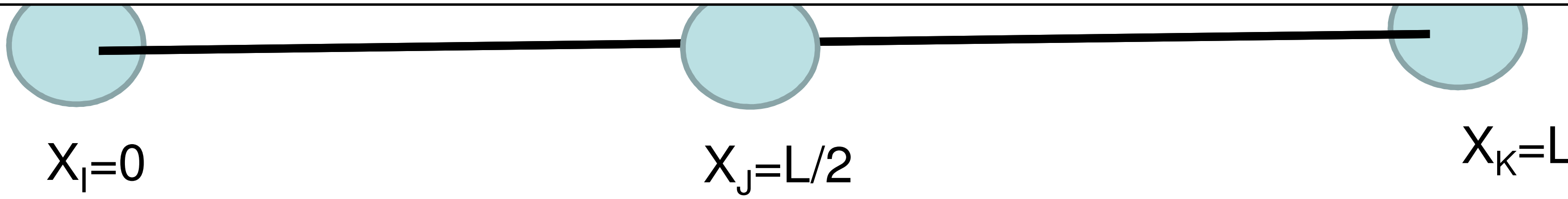
Linear version: n=1

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$f_1 = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

Next, how to do n=2 (second order).
What would third order be?



x	x_0	x_1	x_2
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$f_2 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

x	x_0	x_1	x_2	x_3
f(x)	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$f_3 = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

....

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$f_n = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

....

Note:
x_1 is
not being subtracted
from the constant
term x
or $x_i = x_1$ in
the numerator
or the denominator
$j= 1$

x	x_0	x_1	x_2	x_3	x_4	x_5	x_6	…	$X_{(n-1)}$
f(x)	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$	$f(x_6)$	…	$f(x_{(n-1)})$

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Note:
x_2 is
not being subtracted
from the constant
term x or
$x_i = x_2$ in
the numerator
or the denominator
$j = 2$

$$f_3 = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0)$$

$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2)$$

$$\ldots\ldots$$

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

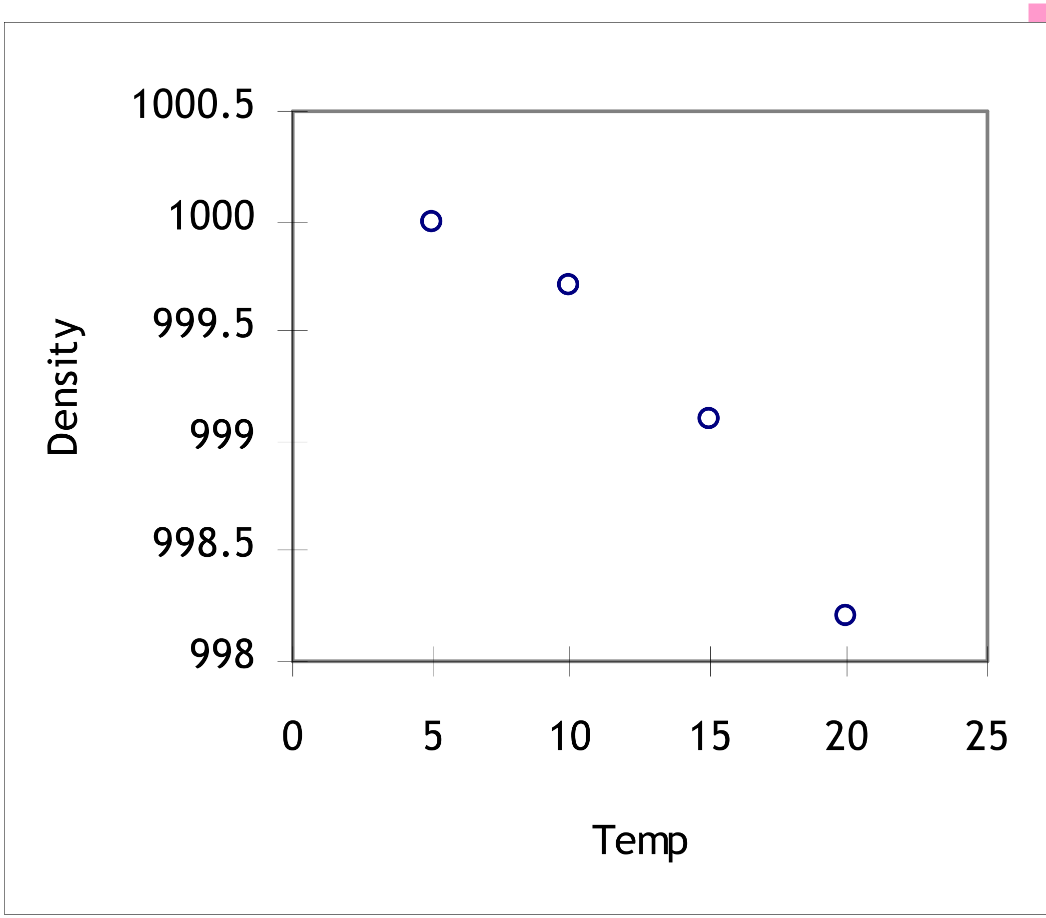
$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$f_3 = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0)$$
$$+ \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} f(x_2)$$
$$+ \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

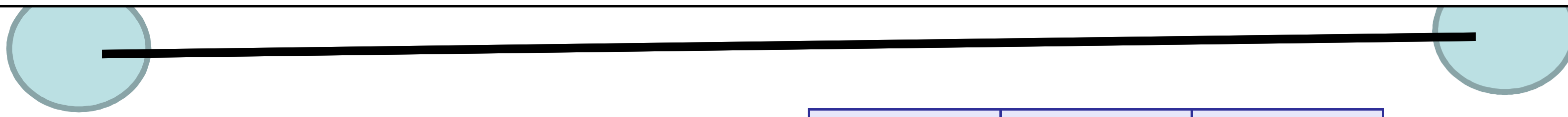
Note:
x_3 is
not being subtracted
from the constant
term *x* or
$x_i = x_3$ in
the numerator
or the denominator
j= 3

Example

Determine the density at 17 degrees.



Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2



$X_I=0$ $X_J=L$

x	x_0	x_1
f(x)	$f(x_0)$	$f(x_1)$

Linear version: n=1

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2

$$f_1 = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

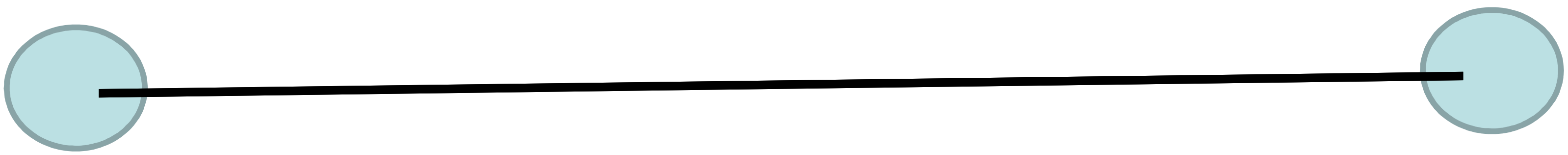
$$f_1 = \frac{x-5}{0-5}(999.9) + \frac{x-0}{5-0}(1000)$$
$$= 0.02x + 999.95$$

$$f(17) = 0.02(17) + 999.95 = 1000.29$$

x	x_0	x_1
f(x)	$f(x_0)$	$f(x_1)$

Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2

$$f(17) = 0.02(17) + 999.9 = 1000.29$$

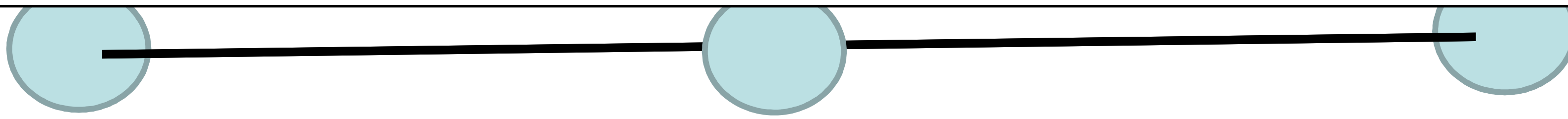


$X_I=15$ $X_J=20$

$$f_1 = \frac{x-20}{15-20}(999.1) + \frac{x-15}{20-15}(998.2)$$

$$= -0.18x + 1001.8$$

$$f(17) = -0.18(17) + 1001.8 = 998.74$$



$X_I=0$ $X_J=L/2$ $X_K=L$

x	x_0	x_1	x_2
f(x)	$f(x_0)$	$f(x_1)$	$f(x_2)$

Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$f_2 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

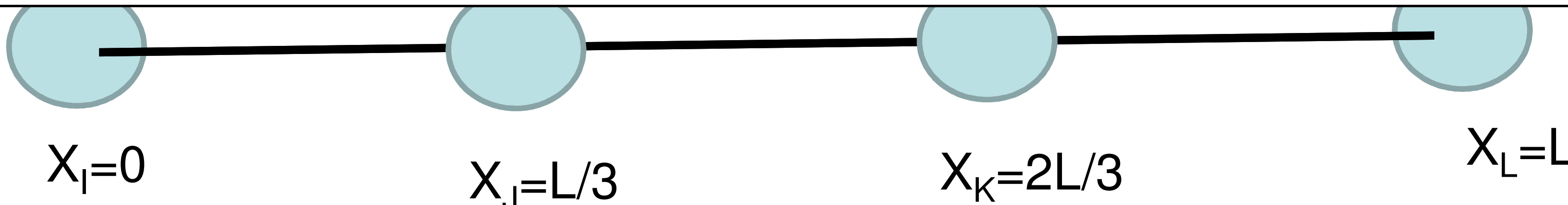
$$f_2 = \frac{(x-5)(x-10)}{(0-5)(0-10)}(999.9) + \frac{(x-0)(x-10)}{(5-0)(5-10)}1000 + \frac{(x-0)(x-5)}{(10-0)(10-5)}(999.7)$$

$$f_2 = \frac{-x^2}{125} + \frac{3x}{50} + 999.9$$

$$f_2(17) = \frac{-(17)^2}{125} + \frac{3(17)}{50} + 999.9 = 998.608$$

$$f_2 = \frac{(x-15)(x-20)}{(10-15)(10-20)}(999.7) + \frac{(x-10)(x-20)}{(15-10)(15-20)}(999.1) + \frac{(x-10)(x-15)}{(20-10)(20-15)}(998.2)$$

$$f_2 = \frac{-3x^2}{500} + \frac{3x}{100} + 1000 = 998.776$$



x	x_0	x_1	x_2	x_3
f(x)	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$f_3 = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

....

$$f_3 = \frac{(x-10)(x-15)(x-20)}{(5-10)(5-15)(5-20)}(1000) + \frac{(x-5)(x-15)(x-20)}{(10-5)(10-15)(10-20)}(999.7) +$$

$$\frac{(x-10)(x-5)(x-20)}{(15-10)(15-5)(15-20)}(999.1) + \frac{(x-5)(x-10)(x-15)}{(20-5)(20-10)(20-15)}(998.2)$$

$f_3 =$ (3 * x)/100 - (3 * x^2)/500 + 1000 = 998.776

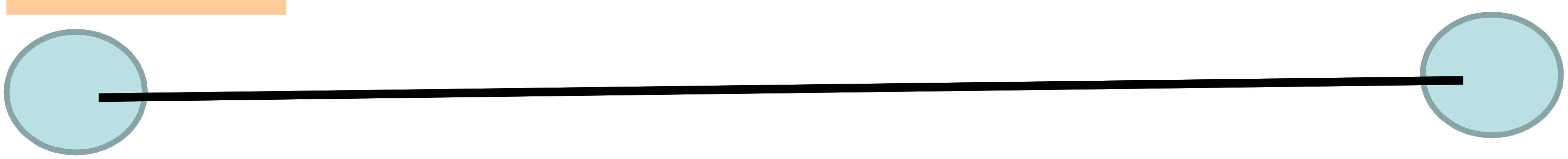
Lagrange Interpolating Polynomials - Example

Use a Lagrange interpolating polynomial of the first and second order to evaluate ln(2) on the basis of the data:

$$x_0 = 1 \qquad f(x_0) = \ln(1) = 0$$

$$x_1 = 4 \qquad f(x_1) = \ln(4) = 1.386294$$

$$x_2 = 6 \qquad f(x_2) = \ln(6) = 1.791760$$



$X_I=0$

$X_J=L$

Lagrange Interpolating Polynomials – Example

- **First order polynomial:**

$x_0 = 1$ $\quad f(x_0) = \ln(1) = 0$

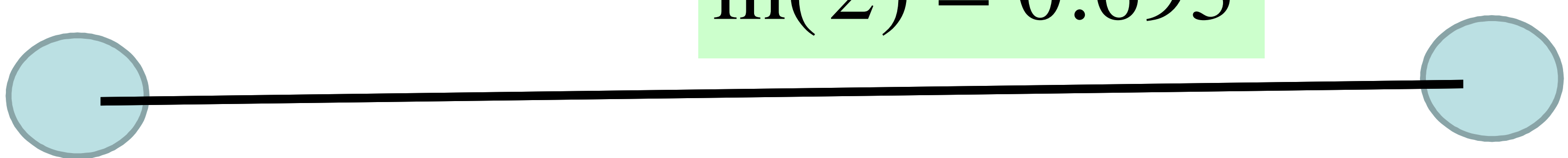
$x_1 = 4$ $\quad f(x_1) = \ln(4) = 1.386294$

$x_2 = 6$ $\quad f(x_2) = \ln(6) = 1.791760$

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$f_1(2) = \frac{2-4}{1-4} \cdot 0 + \frac{2-1}{4-1} \cdot 1.386294 = 0.4620981$$

$$\ln(2) = 0.693$$



Lagrange Interpolating Polynomials – Example

- **First order polynomial:**

$x_0 = 1$ $\quad f(x_0) = \ln(1) = 0$

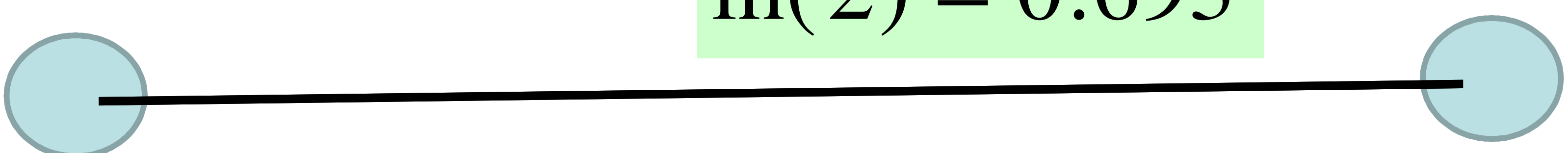
$x_1 = 4$ $\quad f(x_1) = \ln(4) = 1.386294$

$x_2 = 6$ $\quad f(x_2) = \ln(6) = 1.791760$

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$f_1(2) = \frac{2-6}{4-6}(1.386) + \frac{2-4}{6-4} \cdot 1.791 = 0.9808$$

$$\ln(2) = 0.693$$



$X_I=0$ $\quad X_J=L$

$x_0 = 1$ $\quad f(x_0) = \ln(1) = 0$

$x_1 = 4$ $\quad f(x_1) = \ln(4) = 1.386294$

$x_2 = 6$ $\quad f(x_2) = \ln(6) = 1.791760$

- **Second order polynomial:**

$$L_o(x) = \frac{x - x_1}{x_o - x_1} \cdot \frac{x - x_2}{x_o - x_2} = \frac{x-4}{1-4} \cdot \frac{x-6}{1-6}$$

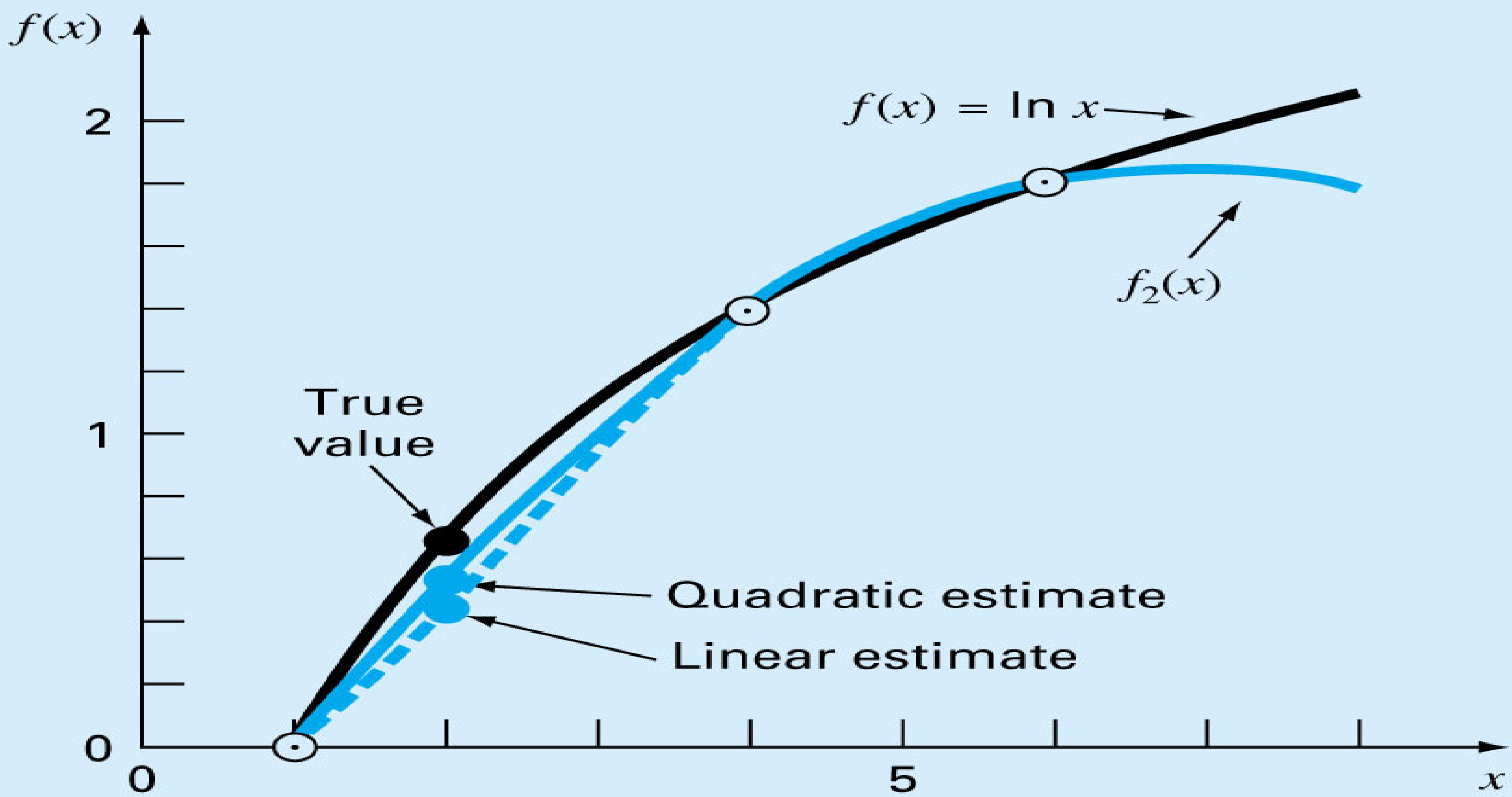
$$L_1(x) = \frac{x - x_o}{x_1 - x_o} \cdot \frac{x - x_2}{x_1 - x_2} = \frac{x-1}{4-1} \cdot \frac{x-6}{4-6}$$

$$L_2(x) = \frac{x - x_o}{x_2 - x_o} \cdot \frac{x - x_1}{x_2 - x_1} = \frac{x-1}{6-1} \cdot \frac{x-4}{6-4}$$

$$\ln(2) = 0.693$$

$$f_1(2) = \frac{1}{3}(0) + L_1(1.386) + L_2(1.791) = 0.789$$

Lagrange Interpolating Polynomials –



FD Linear Interpolation

$$f_1(x) = f(x_o) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

Alternate interpretation

$$f_1(x) = a_0 + a_1(x - x_0)$$

intercept is $f(x_0)$

slope is a finite difference approx. of dy/dx

Divided differences and the coefficients

The divided difference of a function, f

with respect to x_i is denoted as $f[x_i]$

It is called as *zeroth divided difference* and is simply the value of the function, f at x_i

$$f[x_i] = f(x_i)$$

x	x_0	x_1	x_2	x_3		x_i
f(x)	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$		$f(x_i)$

The divided difference of a function, f with respect to x_i and x_{i+1} called as the *first divided difference*, is denoted

$$f\left[x_i, x_{i+1}\right]$$

$$f\left[x_i, x_{i+1}\right] = \frac{f\left[x_{i+1}\right] - f\left[x_i\right]}{x_{i+1} - x_i}$$

The divided difference of a function, f with respect to x_i , x_{i+1} and x_{i+2} called as the *second divided difference*, is denoted as

$$f\left[x_i, x_{i+1}, x_{i+2}\right]$$

$$f\left[x_i, x_{i+1}, x_{i+2}\right] = \frac{f\left[x_{i+1}, x_{i+2}\right] - f\left[x_i, x_{i+1}\right]}{x_{i+2} - x_i}$$

$$f\left[x_{i+1}, x_{i+2}\right] = \frac{f\left[x_{i+2}\right] - f\left[x_{i+1}\right]}{x_{i+2} - x_{i+1}}$$

$$f\left[x_i, x_{i+1}\right] = \frac{f\left[x_{i+1}\right] - f\left[x_i\right]}{x_{i+1} - x_i}$$

The *third divided difference* with respect to x_i **,** x_{i+1} **,** x_{i+2} **and** x_{i+3}

$$f\left[x_i, x_{i+1}, x_{i+2}, x_{i+3}\right]$$

$$= \frac{f\left[x_{i+1}, x_{i+2}, x_{i+3}\right] - f\left[x_i, x_{i+1}, x_{i+2}\right]}{x_{i+3} - x_i}$$

The coefficients of Newton's interpolating polynomial are:

$$a_0 = f[x_0]$$

$$a_1 = f[x_0, x_1]$$

$$a_2 = f[x_0, x_1, x_2]$$

$$a_3 = f[x_0, x_1, x_2, x_3]$$

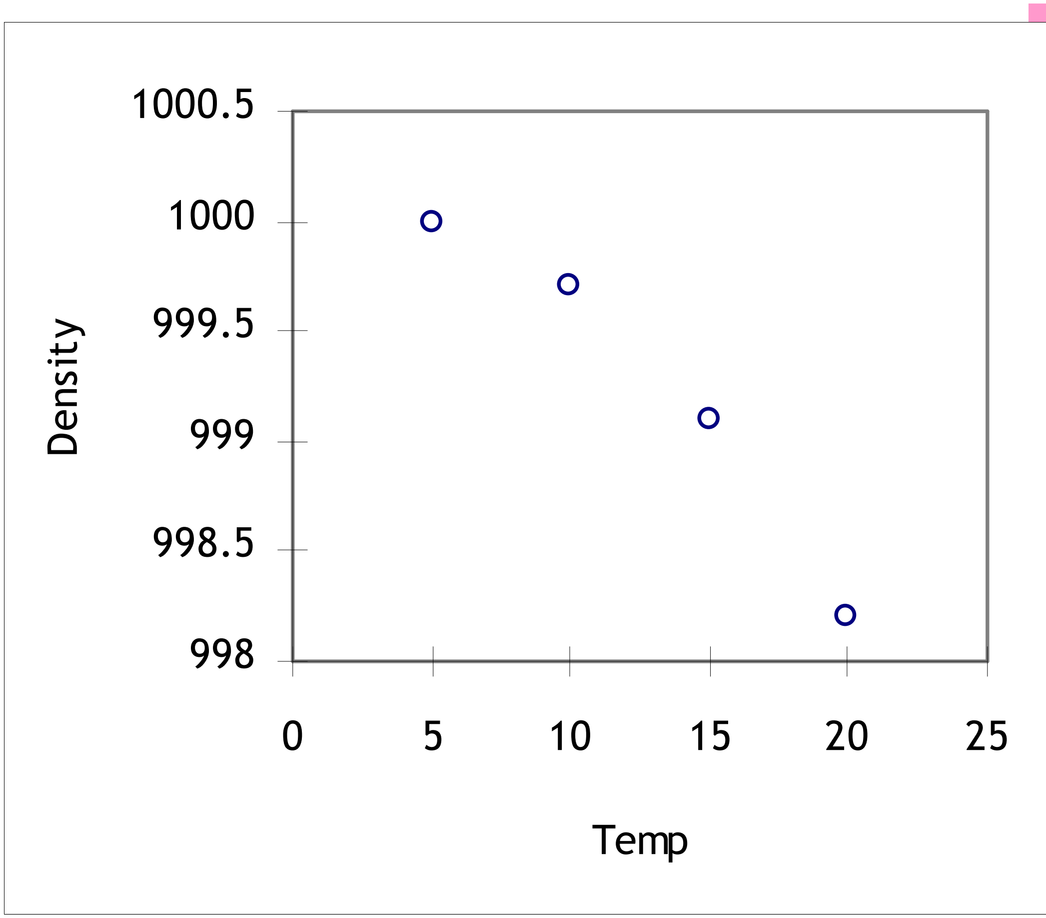
$$a_4 = f[x_0, x_1, x_2, x_3, x_4]$$

and so on.

	$f(x)$	First divided differences	Second divided differences	Third divided differences
x_0	$f[x_0]$			
		$f[x_0,x_1]=\dfrac{f[x_1]-f[x_0]}{x_1-x_0}$		
x_1	$f[x_1]$		$f[x_0,x_1,x_2]=\dfrac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$	
		$f[x_1,x_2]=\dfrac{f[x_2]-f[x_1]}{x_2-x_1}$		$f[x_0,x_1,x_2,x_3]=\dfrac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$
x_2	$f[x_2]$		$f[x_1,x_2,x_3]=\dfrac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$	
		$f[x_2,x_3]=\dfrac{f[x_3]-f[x_2]}{x_3-x_2}$		$f[x_1,x_2,x_3,x_4]=\dfrac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$
x_3	$f[x_3]$		$f[x_2,x_3,x_4]=\dfrac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$	
		$f[x_3,x_4]=\dfrac{f[x_4]-f[x_3]}{x_4-x_3}$		$f[x_2,x_3,x_4,x_5]=\dfrac{f[x_3,x_4,x_5]-f[x_2,x_3,x_4]}{x_5-x_2}$
x_4	$f[x_4]$		$f[x_3,x_4,x_5]=\dfrac{f[x_4,x_5]-f[x_3,x_4]}{x_5-x_3}$	
		$f[x_4,x_5]=\dfrac{f[x_5]-f[x_4]}{x_5-x_4}$		
x_5	$f[x_5]$			

Example

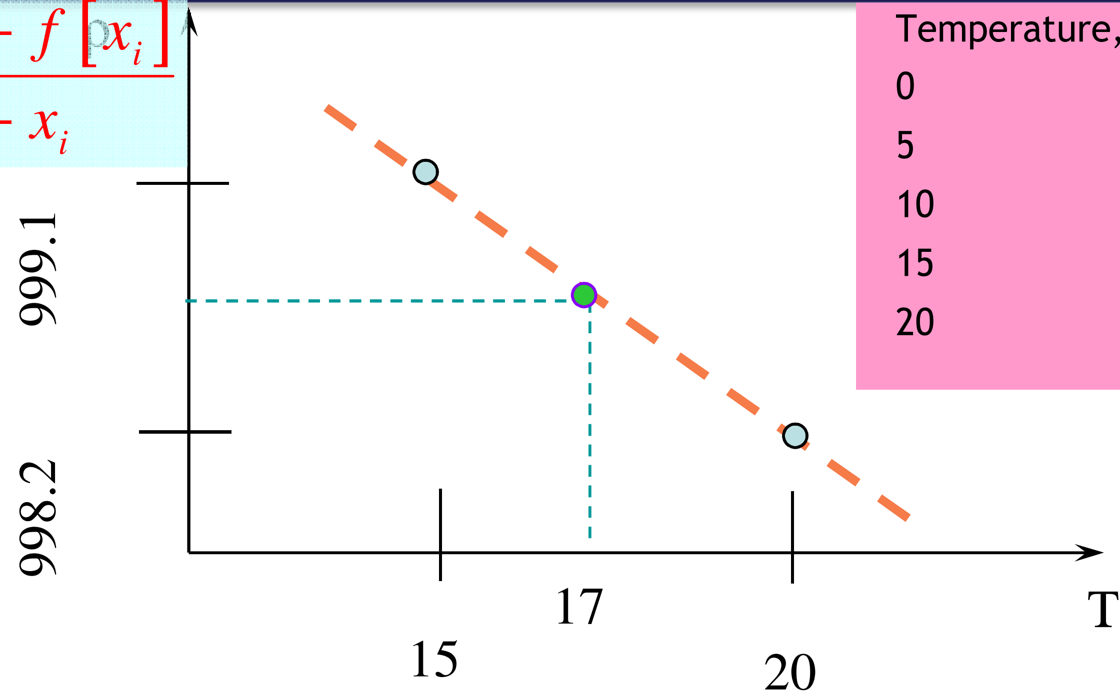
Determine the density at 17 degrees.



Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2

Linear Interpolation

$$f\left[x_i, x_{i+1}\right] = \frac{f\left[x_{i+1}\right] - f\left[x_i\right]}{x_{i+1} - x_i}$$



Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2

$$\frac{998.2 - 999.1}{20 - 15} = \frac{\rho - 999.1}{17 - 15} = \frac{\Delta\rho}{\Delta T}$$

Solve for ρ

Therefore, the slope of one interval will equal the slope of the other interval.

$$\rho = 998.74$$

Linear Interpolation

$$\frac{998.2 - 999.1}{20 - 15} = \frac{998.2 - \rho}{20 - 17}$$

$$f_1(x) = f(x_o) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2

Alternate interpretation

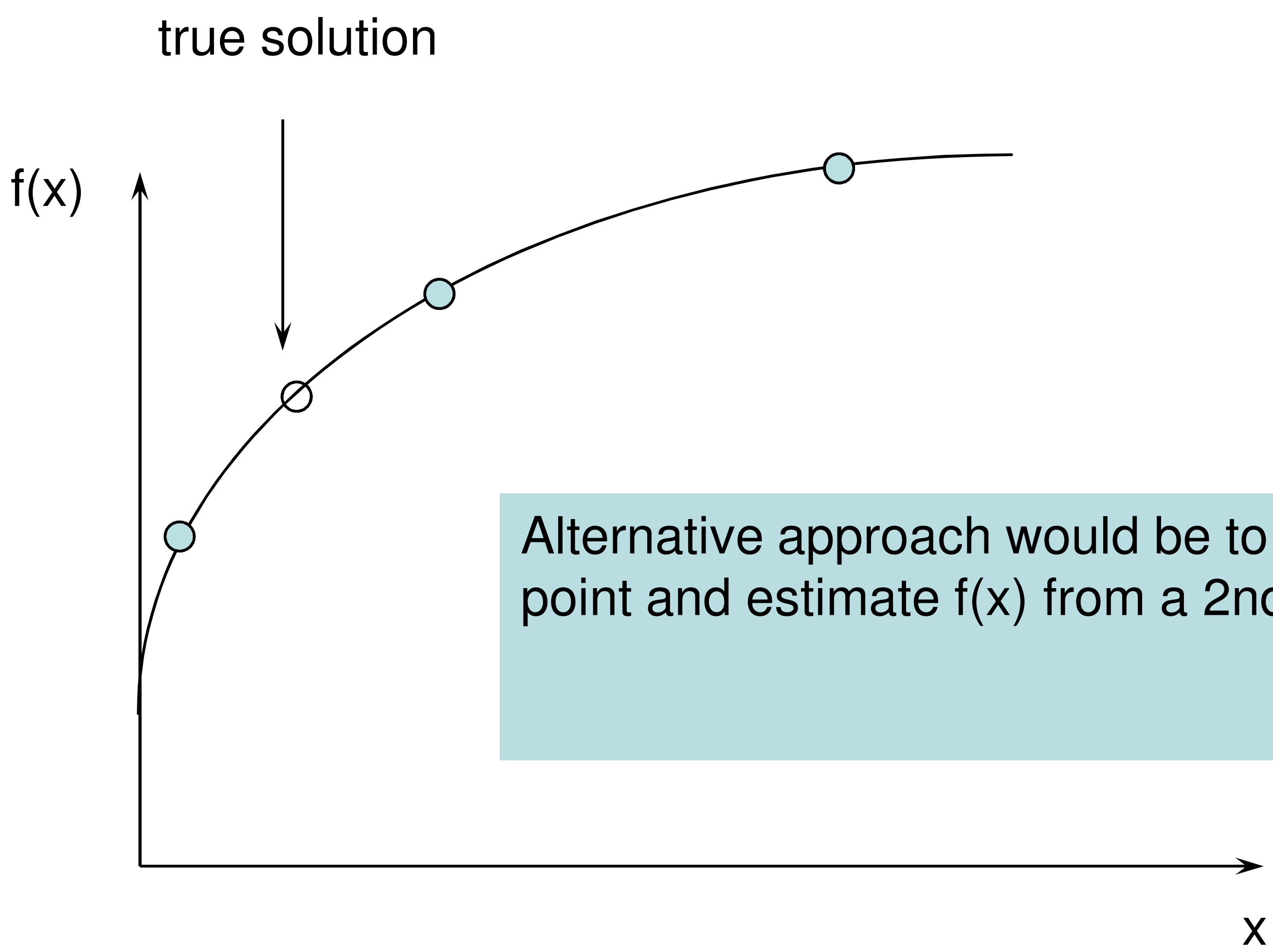
$$f_1(x) = a_0 + a_1(x - x_0)$$

$$f_1(17) = 999.1 + \frac{998.2 - 999.1}{20 - 15}(17 - 15) = 998.74$$

the intercept is $f(x_0)$

the slope is a finite difference approx. of dy/dx

Linear Interpolation



Alternative approach would be to include a third point and estimate f(x) from a 2nd order polynomial.

$$f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2$$
$$f(x_1) = a_0 + a_1 x_1 + a_2 x_1^2$$
$$f(x_2) = a_0 + a_1 x_2 + a_2 x_3^2$$

This is a 2nd order polynomial.

Need three data points.

Plug the value of x_i and $f(x_i)$ directly into equations.

This gives three simultaneous equations to solve for a_0, a_1, and a_2.

Coefficients of an Interpolating Polynomial

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots + b_n(x - x_0)(x - x_1) \text{☹} (x - x_{n-1})$$

$$y = a_0 + a_1 x + a_2 x^2 + \ldots a_m x^m$$

HOW CAN BE MORE STRAIGHT FORWARD IN GETTING VALUES?

Quadratic Interpolation

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Prove that this a 2nd order polynomial of the form:

$$f(x) = a_0 + a_1 x + a_2 x^2$$

Quadratic Interpolation

First, multiply the terms

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

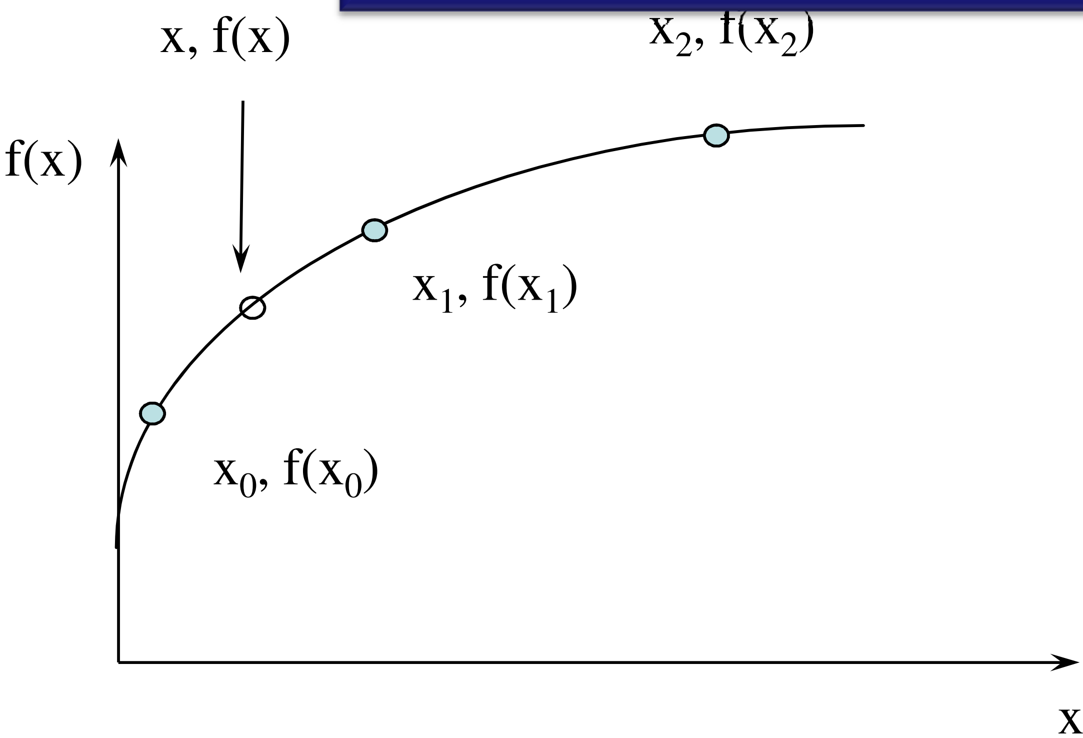
$$\begin{aligned} f_2(x) &= b_0 + b_1 x - b_1 x_0 + b_2 x^2 + b_2 x_0 x_1 - b_2 x x_0 - b_2 x x_1 \\ &= b_0 - b_1 x_0 + b_2 x_0 x_1 + b_1 x - b_2 x x_0 - b_2 x x_1 + b_2 x^2 \\ &= (b_0 - b_1 x_0 + b_2 x_0 x_1) + (b_1 - b_2 x_0 - b_2 x_1)x + b_2 x^2 \end{aligned}$$

Collect terms and recognize that:

$$\begin{aligned} a_0 &= b_0 - b_1 x_0 + b_2 x_0 x_1 \\ a_1 &= b_1 - b_2 x_0 - b_2 x_1 \\ a_2 &= b_2 \end{aligned}$$

$$f(x) = a_0 + a_1 x + a_2 x^2$$

Quadratic Interpolation



Procedure for Quadratic Interpolation

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Quadratic Interpolation

Procedure for Quadratic Interpolation

$$b_0 = f(x_0)$$

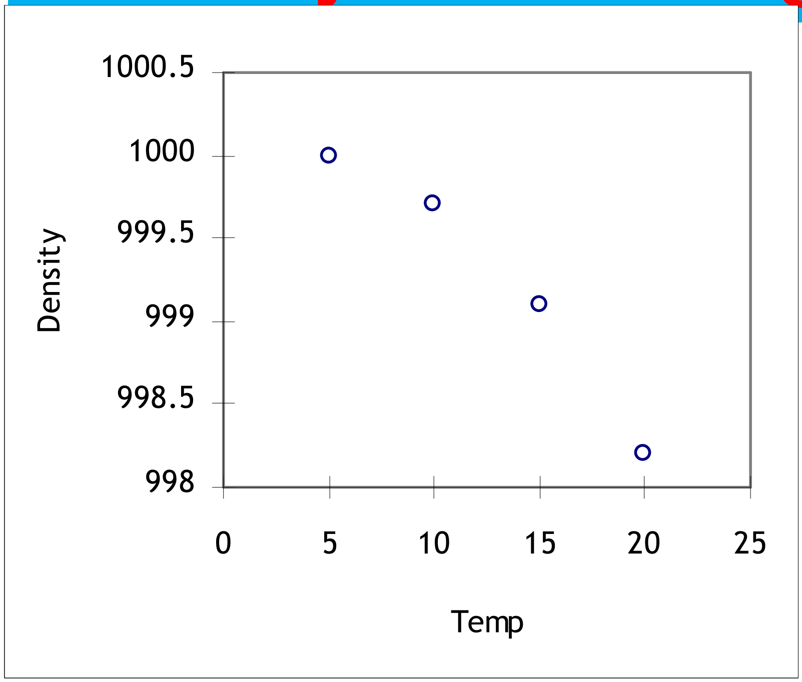
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Quadratic Interpolation- Ex

Include 10 degrees, calculation of the density at 17 degrees.



Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2

Quadratic Interpolation- Ex

Include 10 degrees, calculation of the density at 17 degrees.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Temperature, C°	Density, kg/m^3
0	999.9
5	1000.0
10	999.7
15	999.1
20	998.2

Solution

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$b_0 = f(x_0) = 999.7$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{999.1 - 999.7}{15 - 10} = -0.12$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{\dfrac{998.2 - 999.1}{20 - 15} - (-0.12)}{20 - 10} = -0.006$$

Temperature, C°	Density, kg/m^3
10	999.7
15	999.1
20	998.2

Solution

$$f_2(17) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$
$$= b_0 + b_1(17 - 10) + b_2(17 - 10)(17 - 15)$$

$$f_2(17) = 998.776$$
$$f_1(17) = 998.74$$

Example 2

Find a polynomial to interpolate:

x	y
0	1
1	3
2	2
3	5
4	4

Both Newton's interpolation method and Lagrange interpolation method must give the same answer.

$f(x)$	First divided differences	Second divided differences	Third divided differences
$f[x_0]$			
	$f[x_0, x_1] = \dfrac{f[x_1] - f[x_0]}{x_1 - x_0}$		
$f[x_1]$		$f[x_0, x_1, x_2] = \dfrac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
	$f[x_1, x_2] = \dfrac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2, x_3] = \dfrac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$
$f[x_2]$		$f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
	$f[x_2, x_3] = \dfrac{f[x_3] - f[x_2]}{x_3 - x_2}$		$f[x_1, x_2, x_3, x_4] = \dfrac{f[x_2, x_3, x_4] - f[x_1, x_2, x_3]}{x_4 - x_1}$
$f[x_3]$		$f[x_2, x_3, x_4] = \dfrac{f[x_3, x_4] - f[x_2, x_3]}{x_4 - x_2}$	
	$f[x_3, x_4] = \dfrac{f[x_4] - f[x_3]}{x_4 - x_3}$		$f[x_2, x_3, x_4, x_5] = \dfrac{f[x_3, x_4, x_5] - f[x_2, x_3, x_4]}{x_5 - x_2}$
$f[x_4]$		$f[x_3, x_4, x_5] = \dfrac{f[x_4, x_5] - f[x_3, x_4]}{x_5 - x_3}$	
	$f[x_4, x_5] = \dfrac{f[x_5] - f[x_4]}{x_5 - x_4}$		
$f[x_5]$			

0	1	2	-3/2	7/6	-5/8
1	3	-1	2	-4/3	
2	2	3	-2		
3	5	-1			
4	4				

Newton's FD

Interpolating Polynomial

0	1	2	-3/2	7/6	-5/8
1	3	-1	2	-4/3	
2	2	3	-2		
3	5	-1			
4	4				

$$f_4(x) = 1 + 2(x) - \frac{3}{2}x(x-1) + \frac{7}{6}x(x-1)(x-2)$$

$$-\frac{5}{8}x(x-1)(x-2)(x-3)$$

$$f_4(x) = 1 + \frac{115}{12}x - \frac{95}{8}x^2 + \frac{59}{12}x^3 - \frac{5}{8}x^4$$

Interpolating Polynomial Using Lagrange Interpolation Method

x	y
0	1
1	3
2	2
3	5
4	4

$$f_4(x) = \sum_{i=0}^{4} f(x_i)\,\lambda_i = \lambda_0 + 3\lambda_1 + 2\lambda_2 + 5\lambda_3 + 4\lambda_4$$

$$\lambda_0 = \frac{(x-1)}{(0-1)}\,\frac{(x-2)}{(0-2)}\,\frac{(x-3)}{(0-3)}\,\frac{(x-4)}{(0-4)} = \frac{(x-1)(x-2)(x-3)(x-4)}{24}$$

$$\lambda_1 = \frac{(x-0)}{(1-0)}\,\frac{(x-2)}{(1-2)}\,\frac{(x-3)}{(1-3)}\,\frac{(x-4)}{(1-4)} = \frac{x(x-2)(x-3)(x-4)}{24}$$

$$\lambda_2 = \frac{(x-0)}{(2-0)}\,\frac{(x-1)}{(2-1)}\,\frac{(x-3)}{(2-3)}\,\frac{(x-4)}{(2-4)} = \frac{x(x-1)(x-3)(x-4)}{24}$$

$$\lambda_3 = \frac{(x-0)}{(3-0)}\,\frac{(x-1)}{(3-1)}\,\frac{(x-2)}{(3-2)}\,\frac{(x-4)}{(3-4)} = \frac{x(x-1)(x-2)(x-4)}{24}$$

$$\lambda_4 = \frac{(x-0)}{(4-0)}\,\frac{(x-1)}{(4-1)}\,\frac{(x-2)}{(4-2)}\,\frac{(x-3)}{(4-3)} = \frac{x(x-1)(x-2)(x-3)}{24}$$

Interpolating Polynomial Using Lagrange Interpolation Method

x	y
0	1
1	3
2	2
3	5
4	4

$$f_4(x) = \sum_{i=0}^{4} f(x_i)\ \lambda_i = \lambda_0 + 3\lambda_1 + 2\lambda_2 + 5\lambda_3 + 4\lambda_4$$

$$\lambda_0 = \frac{(x-1)}{(0-1)}\frac{(x-2)}{(0-2)}\frac{(x-3)}{(0-3)}\frac{(x-4)}{(0-4)} = \frac{(x-1)(x-2)(x-3)(x-4)}{24}$$

$$\lambda_1 = \frac{(x-0)}{(1-0)}\frac{(x-2)}{(1-2)}\frac{(x-3)}{(1-3)}\frac{(x-4)}{(1-4)} = \frac{x(x-2)(x-3)(x-4)}{24}$$

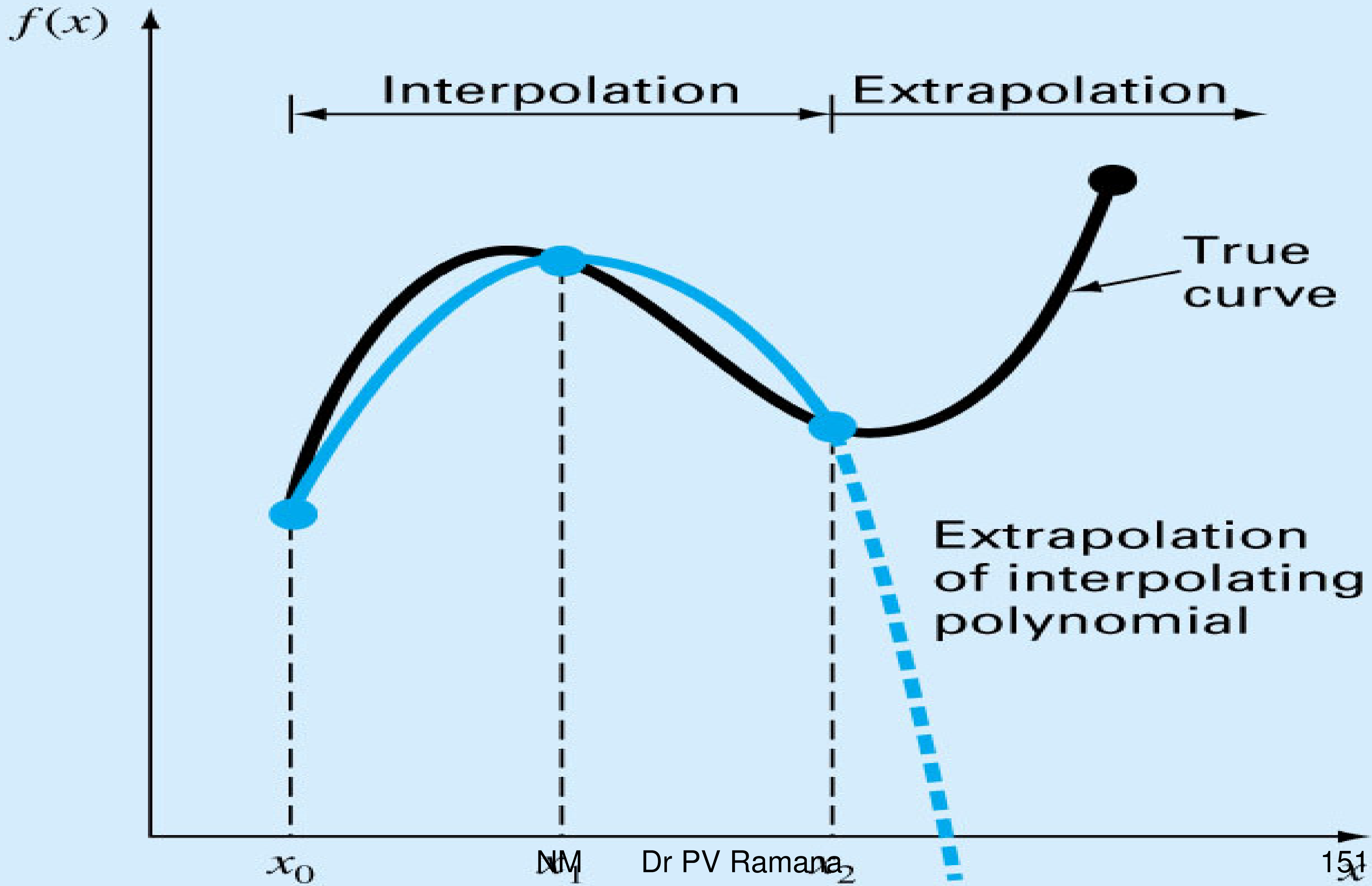
$$\lambda_2 = \frac{(x-0)}{(2-0)}\frac{(x-1)}{(2-1)}\frac{(x-3)}{(2-3)}\frac{(x-4)}{(2-4)} = \frac{x(x-1)(x-3)(x-4)}{24}$$

$$\lambda_3 = \frac{(x-0)}{(3-0)}\frac{(x-1)}{(3-1)}\frac{(x-2)}{(3-2)}\frac{(x-4)}{(3-4)} = \frac{x(x-1)(x-2)(x-4)}{24}$$

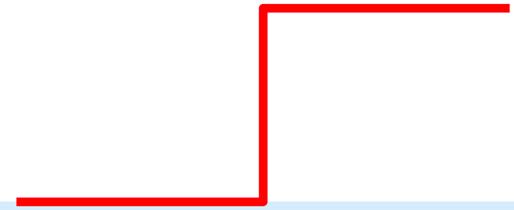
$$\lambda_4 = \frac{(x-0)}{(4-0)}\frac{(x-1)}{(4-1)}\frac{(x-2)}{(4-2)}\frac{(x-3)}{(4-3)} = \frac{x(x-1)(x-2)(x-3)}{24}$$

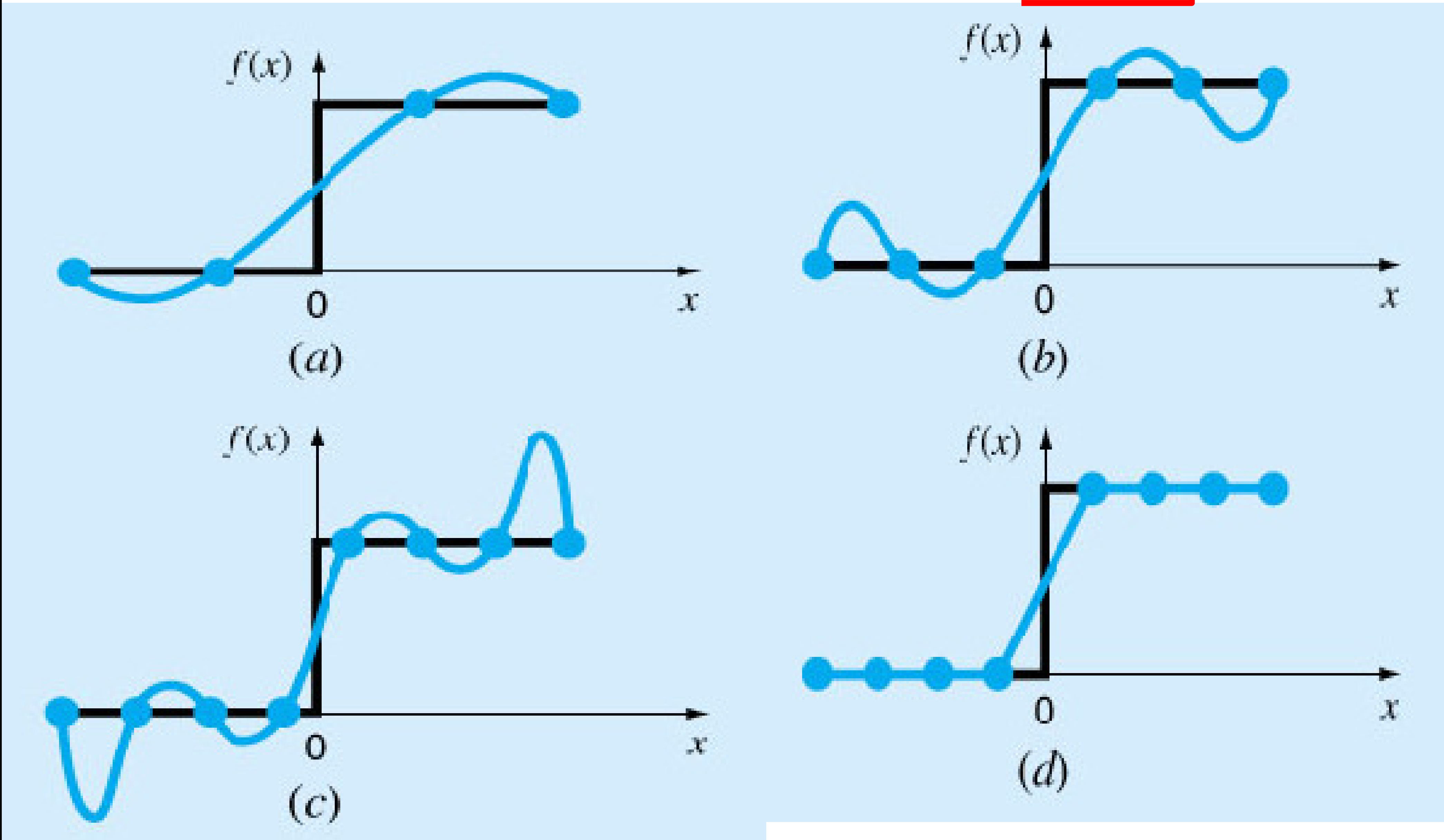
$$f_4(x) = 1 + \frac{115}{12}x - \frac{95}{8}x^2 + \frac{59}{12}x^3 - \frac{5}{8}x^4$$

Possible divergence of an extrapolated production



Why Spline Interpolation?





Apply lower-order polynomials to subsets of data points. Spline provides a superio approximation of the behavior of functions that have local, abrupt changes.

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

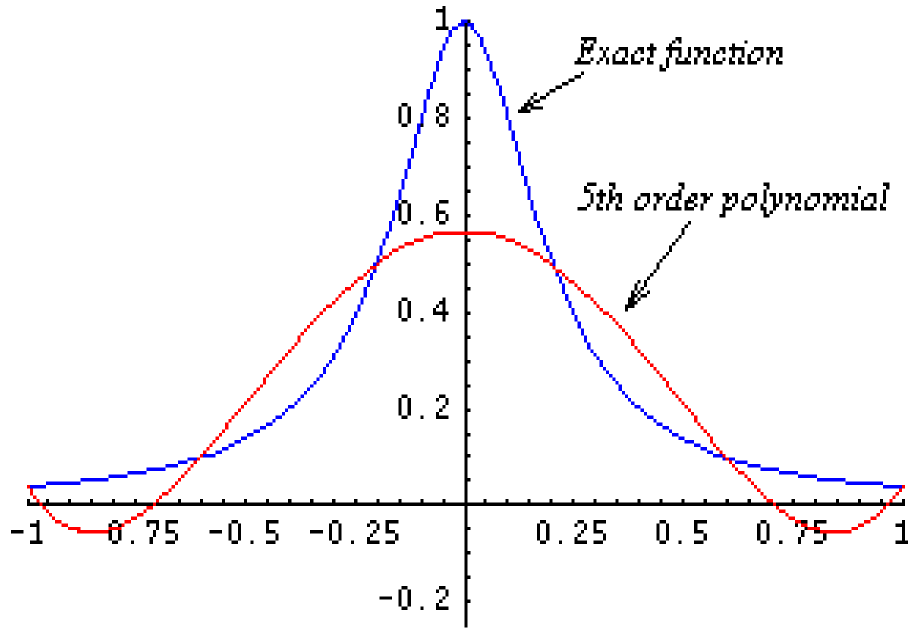
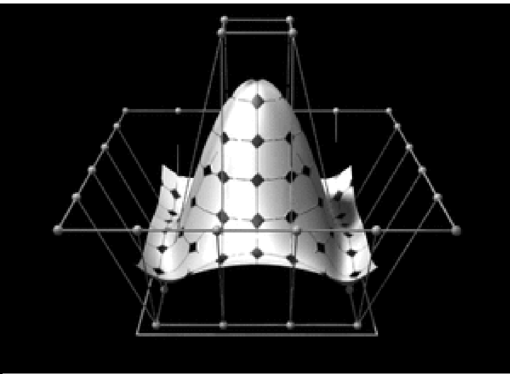


Figure : 5th order polynomial vs. exact function



Why Splines ?

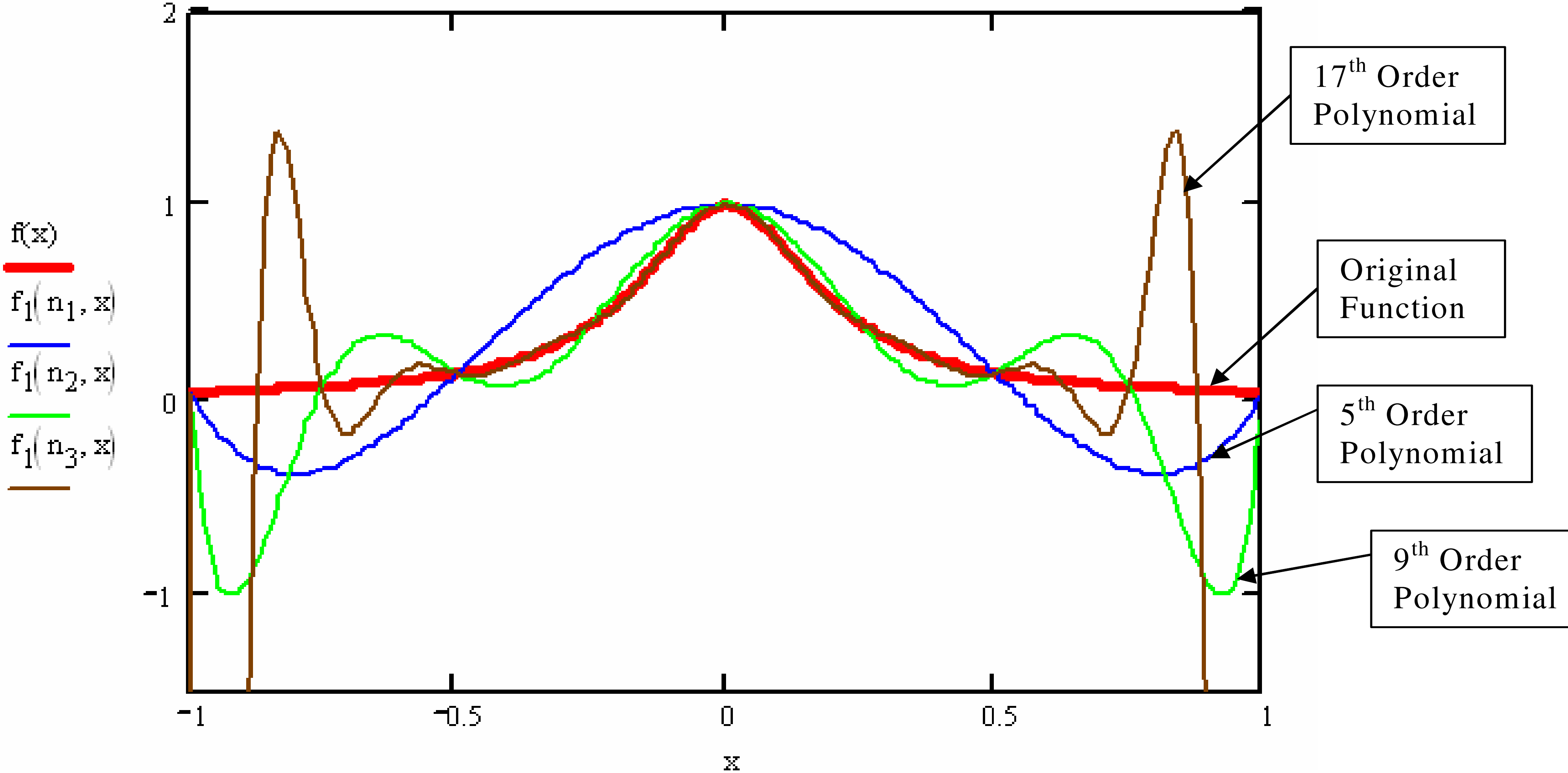
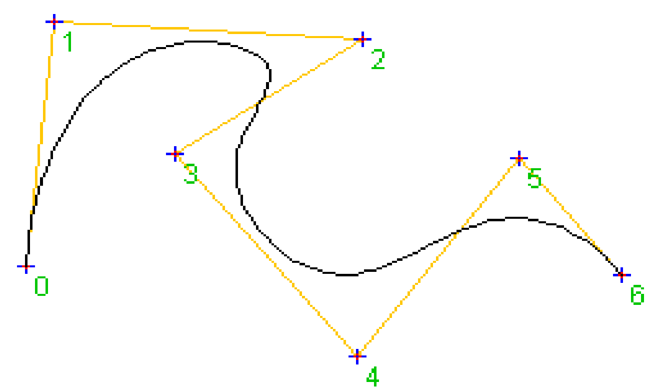


Figure : Higher order polynomial interpolation is a bad idea

Spline Interpolation

- Our previous approach was to derive an *nth* order polynomial for *n+1* data points.
- An alternative approach is to apply lower-order polynomials to subset of data points.
- Such connecting polynomials are called spline functions.
- Adaptation of drafting techniques



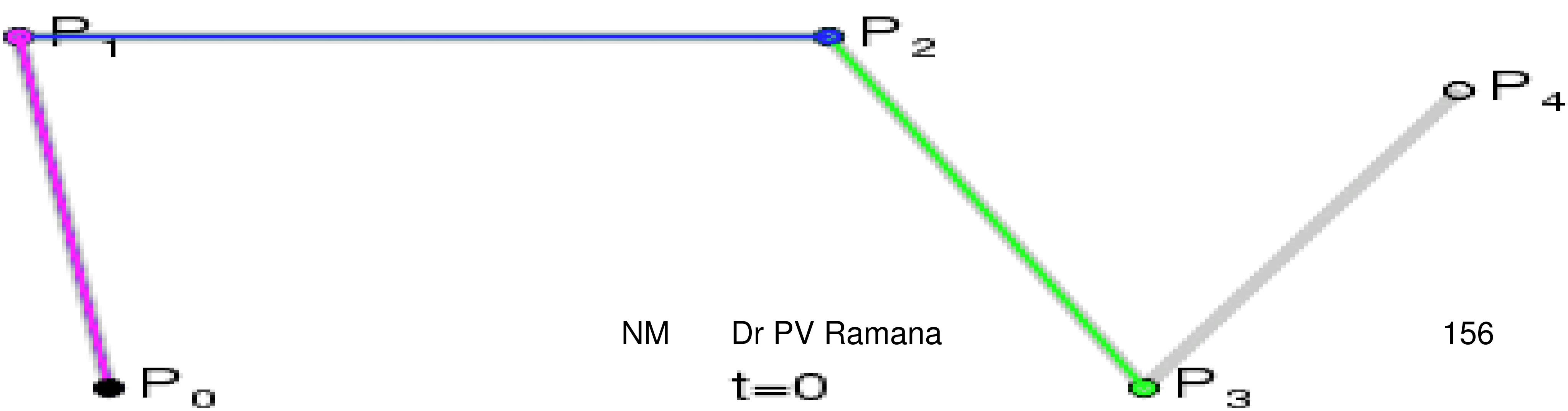
Piecewise Polynomials and Splines

- Spline:

➢ In Mathematics, a spline is a special function defined piecewise by polynomials;

➢ In Computer Science, the term spline more frequently refers to a piecewise polynomial (parametric) curve.

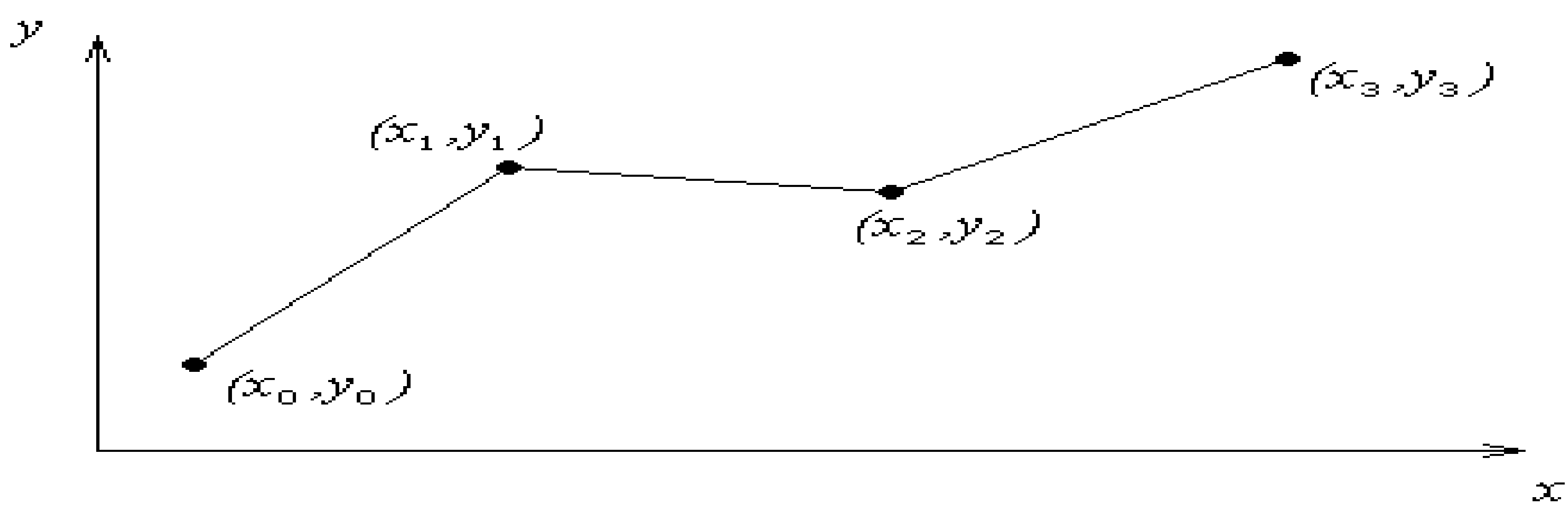
- Simple construction, ease and accuracy of evaluation, capacity to approximate complex shapes through curve fitting and interactive curve design.



Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $\left(y_i = f(x_i)\right)$

Figure : Linear splines



Linear Interpolation (contd)

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

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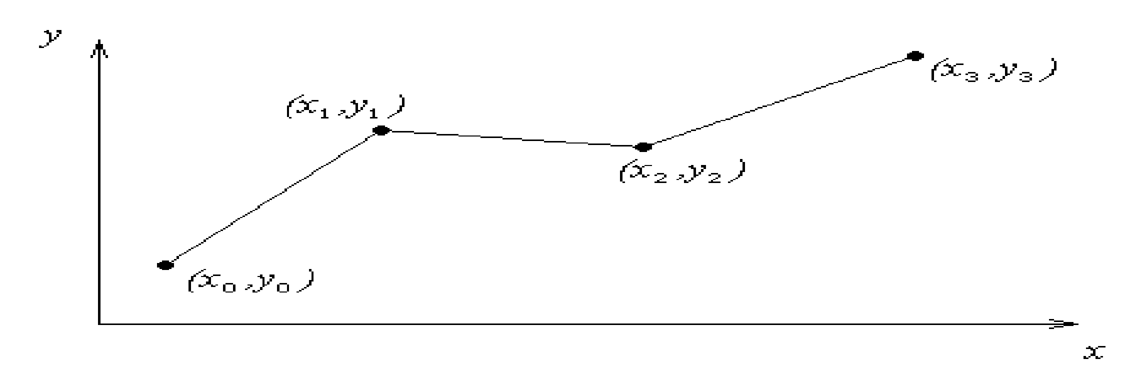


$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

in the above function are simply slopes between x_{i-1} and x_i.

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using linear splines.

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

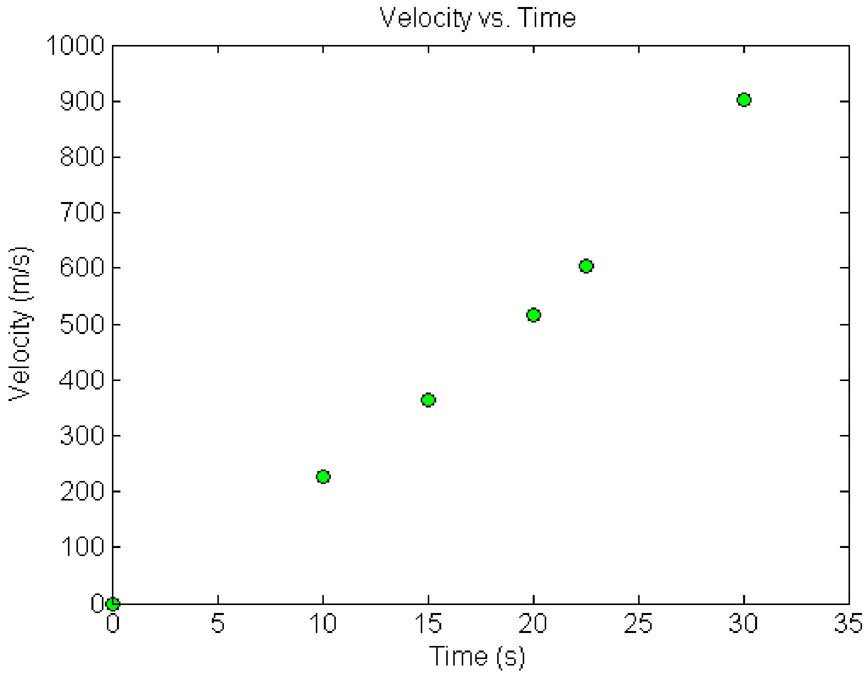
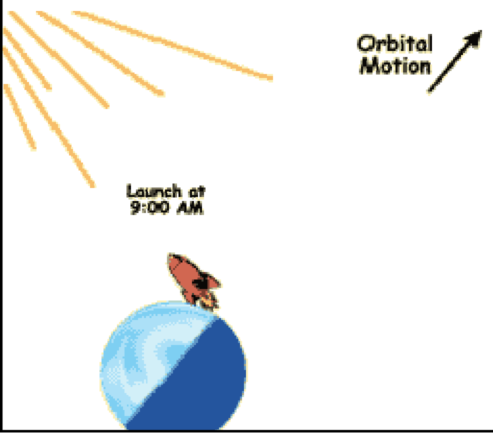
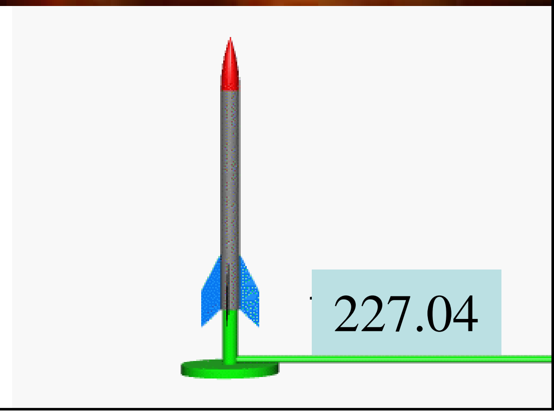


Figure. Velocity vs. time data for the rocket example





Linear Interpolation

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$t_0 = 15, \qquad v(t_0) = 362.78$

$t_1 = 20, \qquad v(t_1) = 517.35$

$$v(t) = v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0}(t - t_0)$$

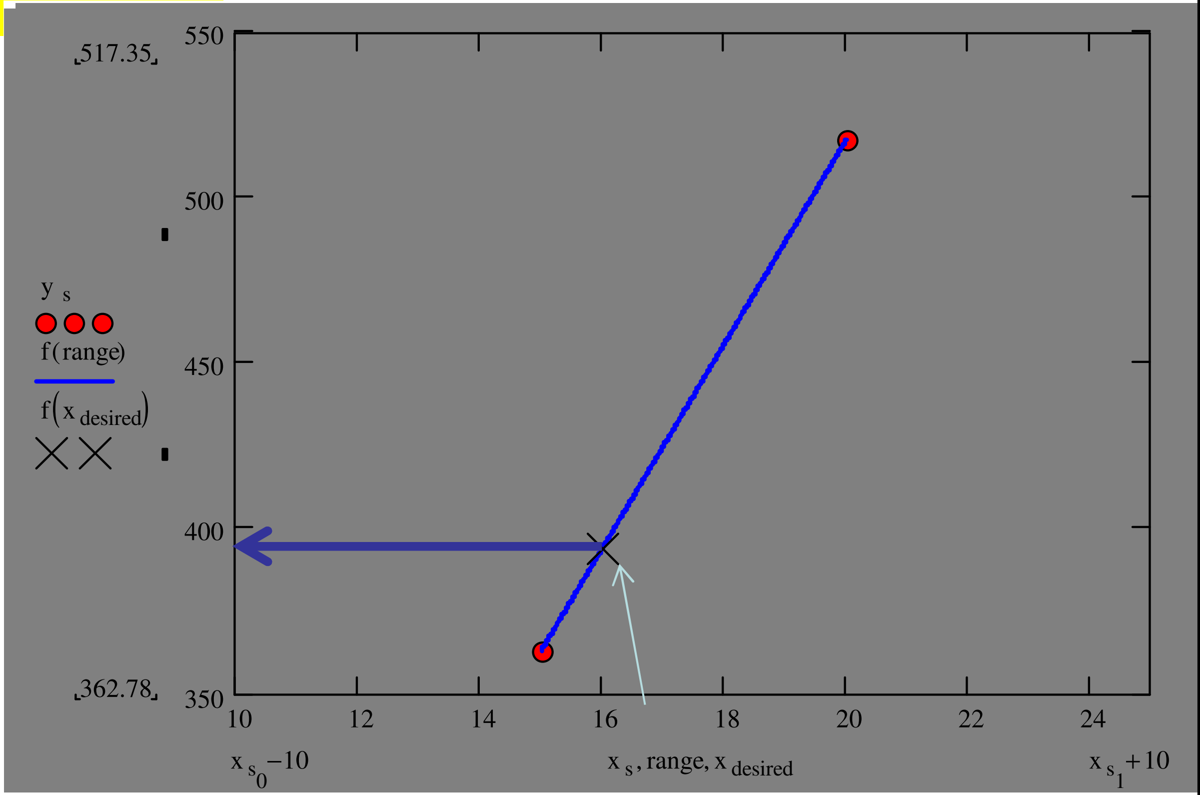
$$= 362.78 + \frac{517.35 - 362.78}{20 - 15}(t - 15)$$

$$v(t) = 362.78 + 30.913(t - 15)$$

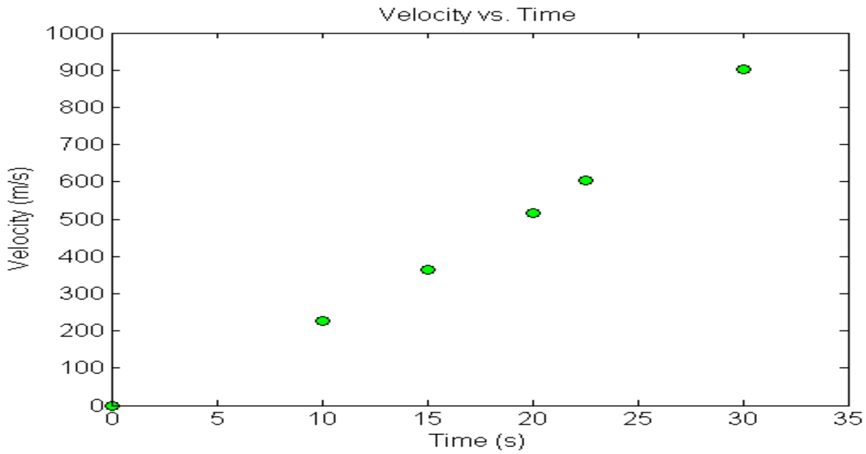
At $t = 16$,

$$v(16) = 362.78 + 30.913(16 - 15)$$

$$= 393.7 \text{ m/s}$$



t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



$$f_1(x) = 0 + 22.704(x - 0)$$

$$f_2(x) = 0 + 22.704(x - 0) + 0.296(x)(x - 10)$$

$$f_3(x) = 0 + 22.704(x - 0) + 0.296(x)(x - 10) + 4.05\left(10^{-3}\right)(x)(x - 10)(x - 15)$$

x (s)	$v(x)$ (m/s)					
0	0	22.704	0.296	**4.05×10^{-3}**	5.82×10^{-5}	1.79×10^{-6}
10	227.04	27.148	0.377	**5.36×10^{-3}**	11.2×10^{-5}	
15	362.78	30.914	0.444	**7.60×10^{-3}**		
20	517.35	34.248	0.558			
22.5	602.97	39.827				
30	901.67					

X = 16 s

1st = 363.264

2nd = 391.680

3rd = 392.069

4th = 392.046

5th = 392.051

$$f_4(x) = 0 + 22.704(x - 0) + 0.296(x)(x - 10) + 4.05\left(10^{-3}\right)(x)(x - 10)(x - 15) + 5.82\left(10^{-5}\right)(x)(x - 10)(x - 15)(x - 20)$$

$$f_5(x) = 0 + 22.704(x - 0) + 0.296(x)(x - 10) + 4.05\left(10^{-3}\right)(x)(x - 10)(x - 15) + 5.82\left(10^{-5}\right)(x)(x - 10)(x - 15)(x - 20) + 1.79\left(10^{-6}\right)(x)(x - 10)(x - 15)(x - 20)(x - 22.5)$$

Linear Splines

- Connect each two points with straight line functions connecting each pair of points

$$s_1(x) = a_1 + b_1(x - x_1)$$
$$s_2(x) = a_2 + b_2(x - x_2)$$
$$\mathrm{M}$$
$$s_i(x) = a_i + b_i(x - x_i)$$
$$\mathrm{M}$$
$$s_{n-1}(x) = a_{n-1} + b_{n-1}(x - x_{n-1})$$

b is the <u>slope</u> between points

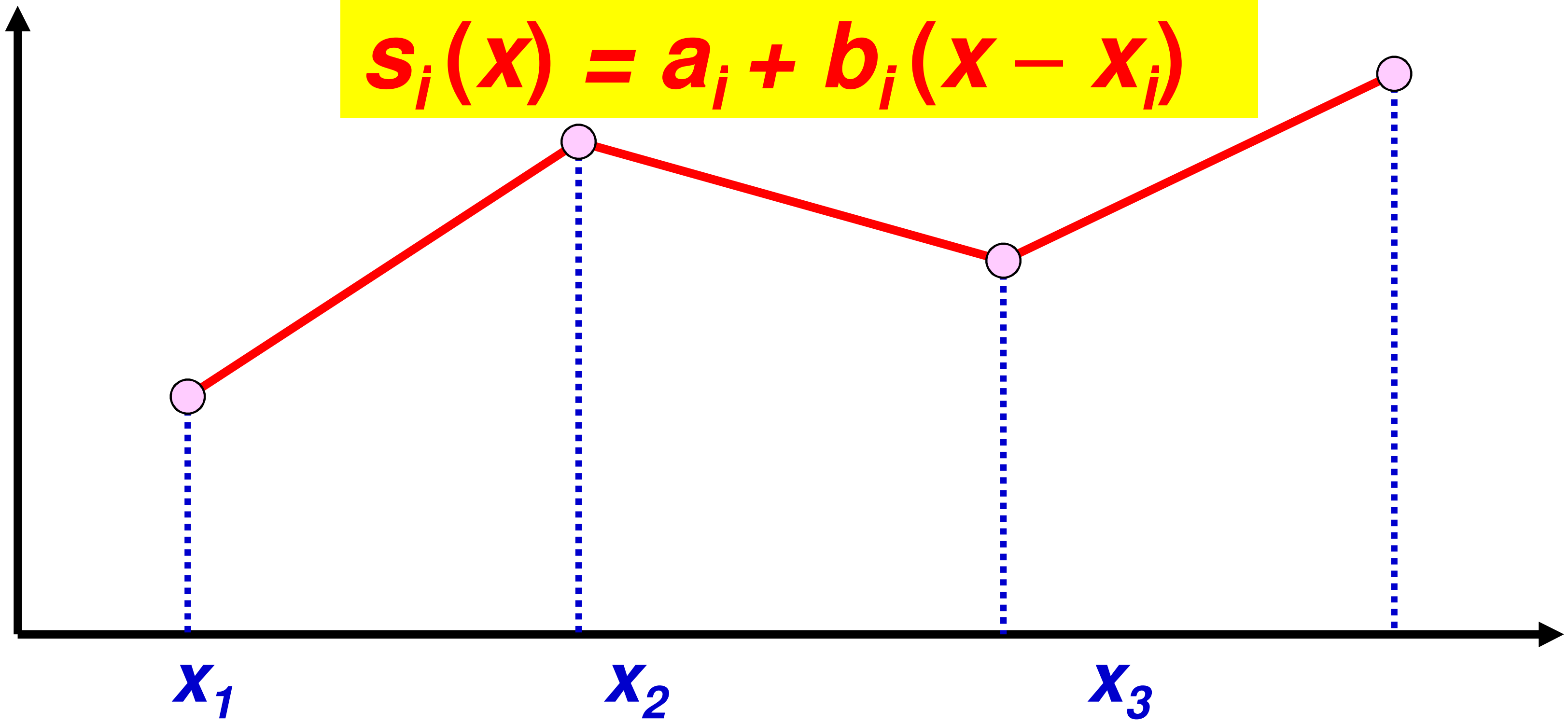
$$a_i = f_i$$
$$b_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

Linear Splines

$$data\ points: (x_{1,} y_1), (x_{2,} y_2), (x_{3,} y_3), \Lambda\ , (x_{n,} y_n)$$

$$interval: \quad I_1 = [x_1, x_2], I_2 = [x_2, x_3], \Lambda\ , I_{n-1} = [x_{n-1}, x_n]$$

$$s_i(x) = a_i + b_i(x - x_i)$$



Linear Splines

$$\textit{data points}: (x_{1,} y_1), (x_{2,} y_2), (x_{3,} y_3), \Lambda\ , (x_{n,} y_n)$$

$$\textit{interval}: \quad I_1 = [x_1, x_2], I_2 = [x_2, x_3], \Lambda\ , I_{n-1} = [x_{n-1}, x_n]$$

$$s_i(x) = \begin{cases} \left(\dfrac{x - x_2}{x_1 - x_2}\right) f(x_1) + \left(\dfrac{x - x_1}{x_2 - x_1}\right) f(x_2), & x_1 \le x \le x_2 \\ \left(\dfrac{x - x_3}{x_2 - x_3}\right) f(x_2) + \left(\dfrac{x - x_2}{x_3 - x_2}\right) f(x_3), & x_2 \le x \le x_3 \\ \quad \mathrm{M} & \\ \left(\dfrac{x - x_n}{x_{n-1} - x_n}\right) f(x_{n-1}) + \left(\dfrac{x - x_{n-1}}{x_n - x_{n-1}}\right) f(x_n), & x_{n-1} \le x \le x_n \end{cases}$$

Identical to Lagrange interpolating polynomials

Linear splines

- Connect each two points with straight line
- Functions connecting each pair of points are

$$
\begin{aligned}
&s_1(x) = a_1 + b_1(x - x_1) \;; && x_1 \le x \le x_2 \\
&s_2(x) = a_2 + b_2(x - x_2) \;; && x_2 \le x \le x_3 \\
&\quad\vdots \\
&s_i(x) = a_i + b_i(x - x_i) \;; && x_i \le x \le x_{i+1} \\
&\quad\vdots \\
&s_{n-1}(x) = a_{n-1} + b_{n-1}(x - x_{n-1}) \;; && x_{n-1} \le x \le x_n
\end{aligned}
$$

- **slope** $b_i = \dfrac{f_{i+1} - f_i}{x_{i+1} - x_i}$

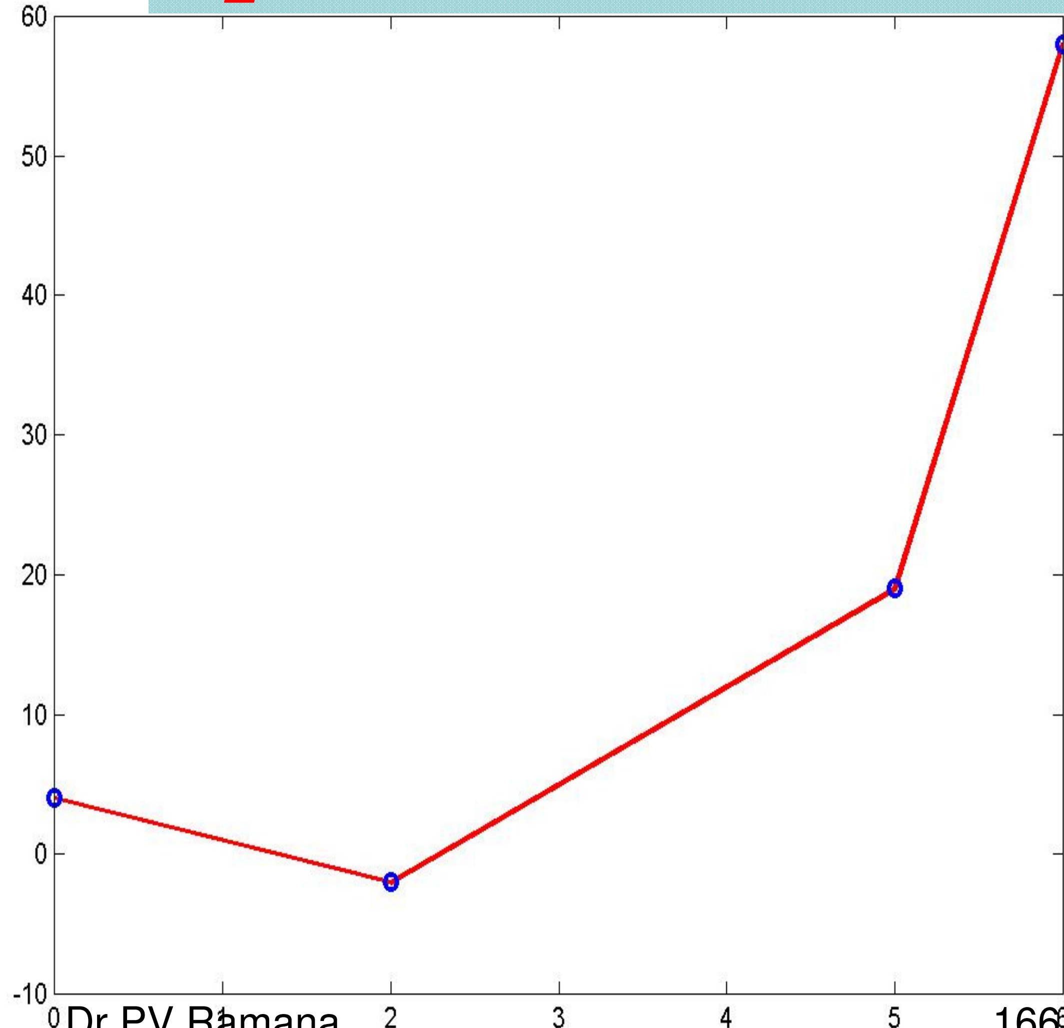
Linear splines are exactly the same as linear interpolation!

$s_1(x) = 4-3(x)$

$s_2(x) = 4-3(x)+2(x)(x-2)$

Example:

x	$f(x)$	b_i		
0	4			
		-3		
2	-2		2	
		7		1
5	19		8	
		39		
6	58			



$s_3(x) = 4-3(x)+2(x)(x-2) +1(x)(x-2)(x-5)$