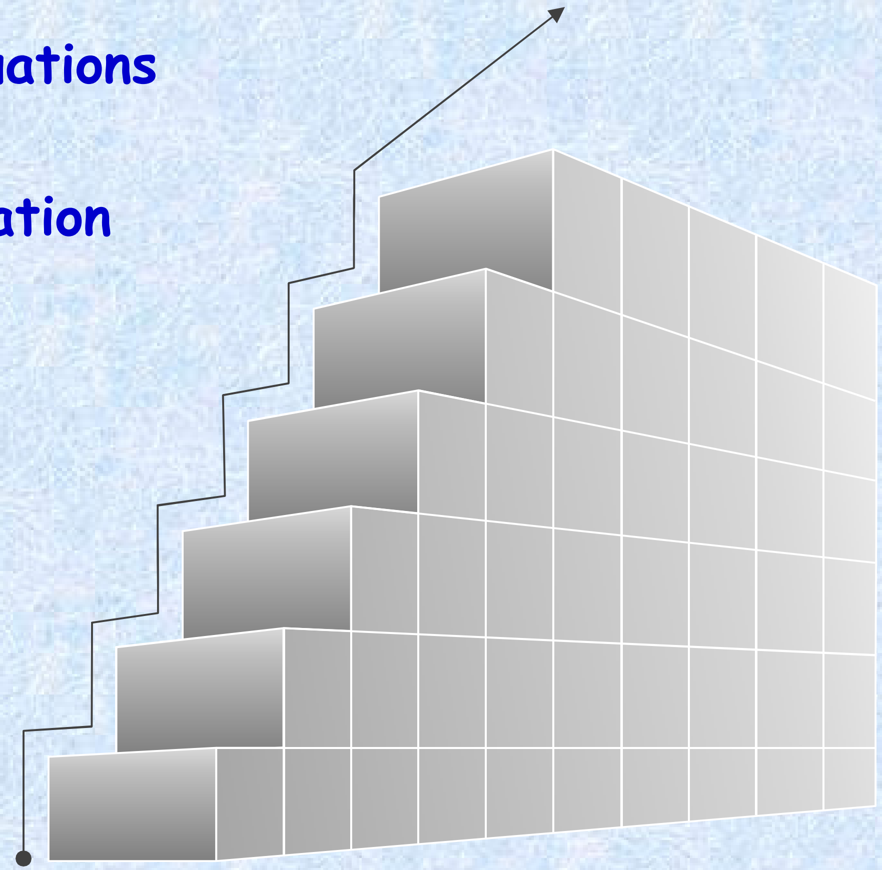
Gauss Elimination

Solving small & large number of equations

Naïve, Jordon, Seidel Gauss Elimination

Pivoting Gauss Elimination

Tridiagonal Systems



- <u>MATLAB M-files</u>
- `GaussNaive, GaussPivot, Tridiag`

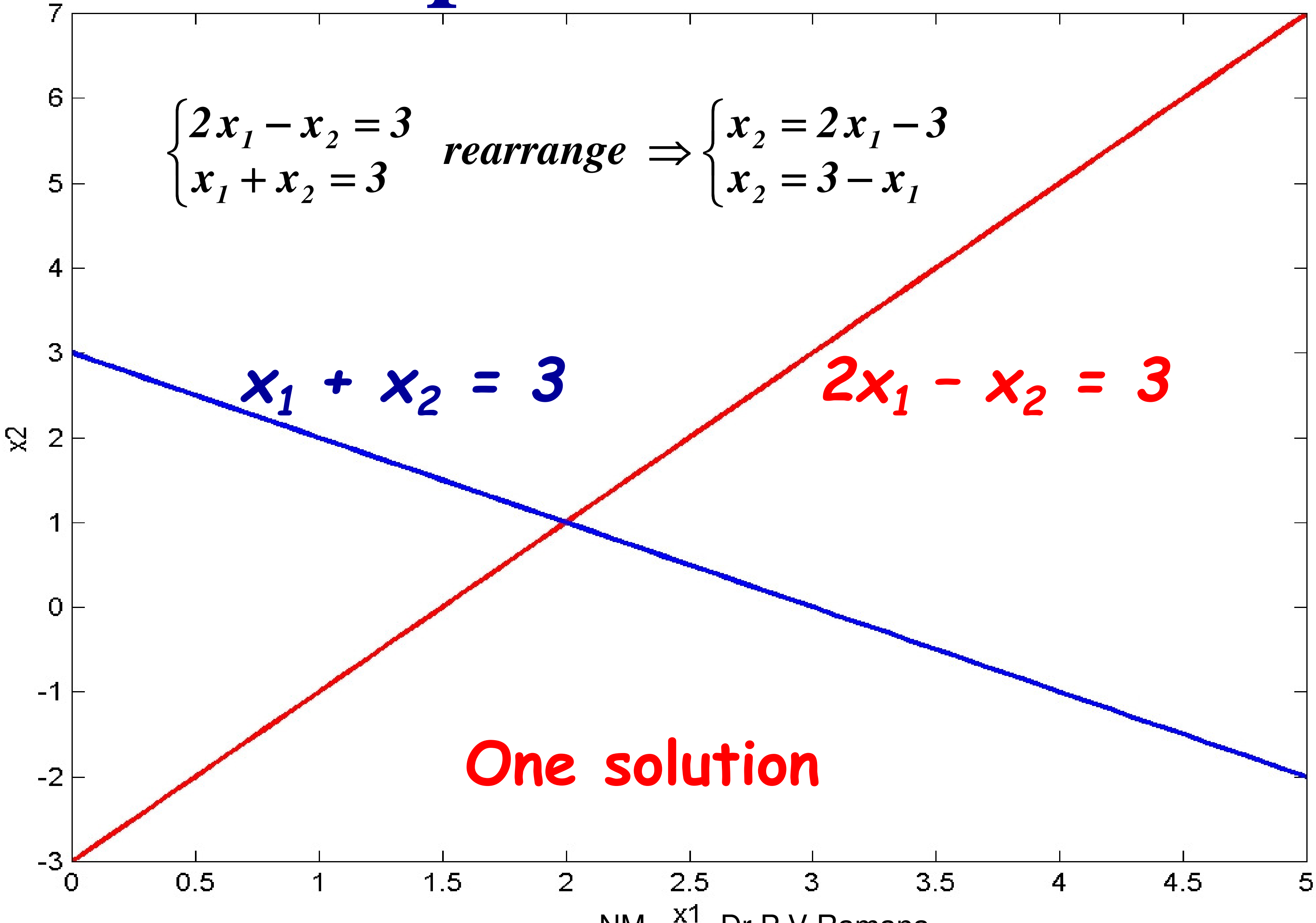
Small Matrices

$$
\begin{cases}
a_{11}x_1 & +a_{12}x_2 & +\Lambda & +a_{1n}x_n & =b_1 \\
a_{21}x_1 & +a_{22}x_2 & +\Lambda & +a_{2n}x_n & =b_2 \\
\mathrm{M} & & & & \mathrm{M} \\
a_{n1}x_1 & +a_{n2}x_2 & +\Lambda & +a_{nn}x_n & =b_n
\end{cases}
$$

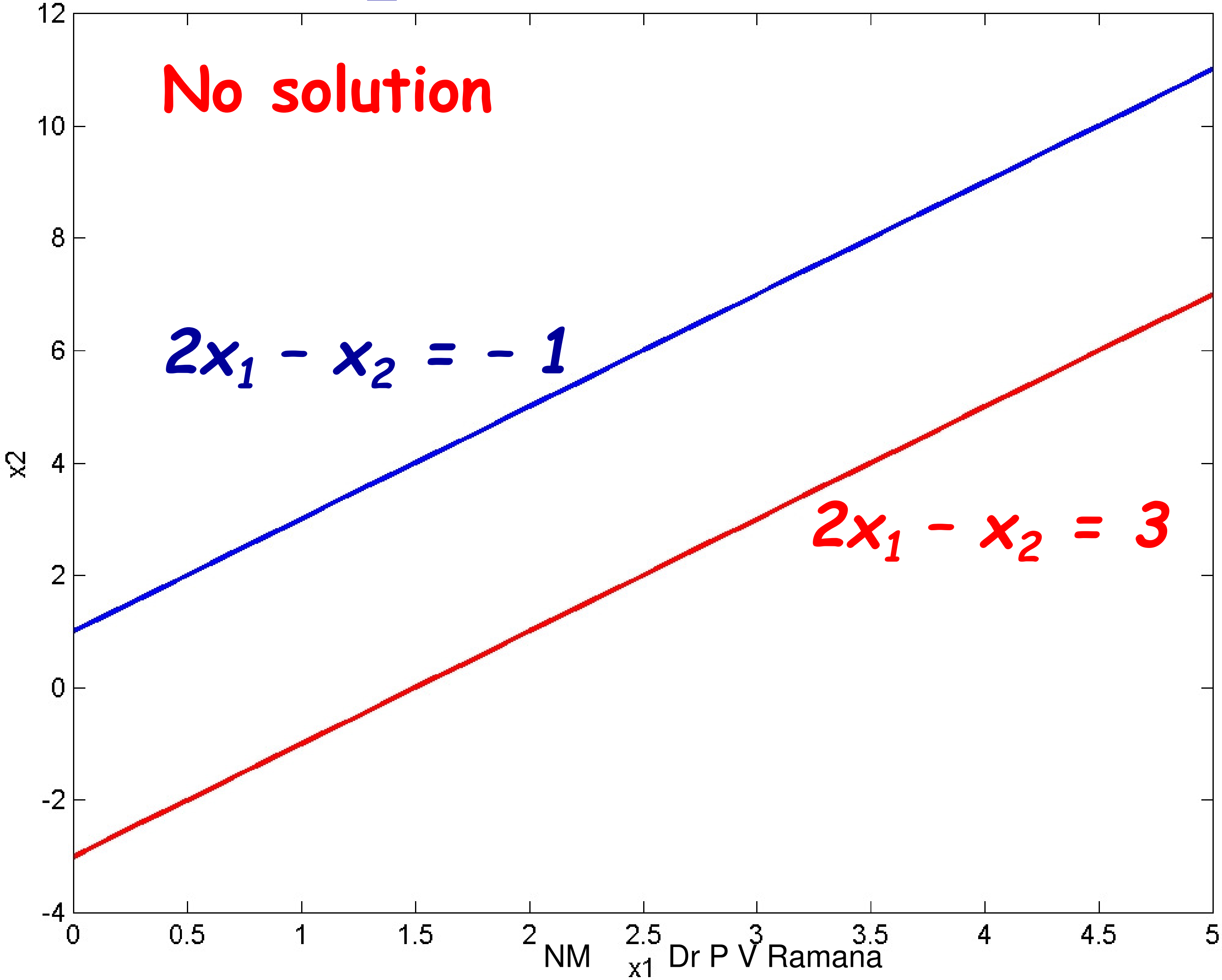
For small numbers of equations, can be solved by hand

- **Graphical**
- **Cramer's rule**
- **Elimination**

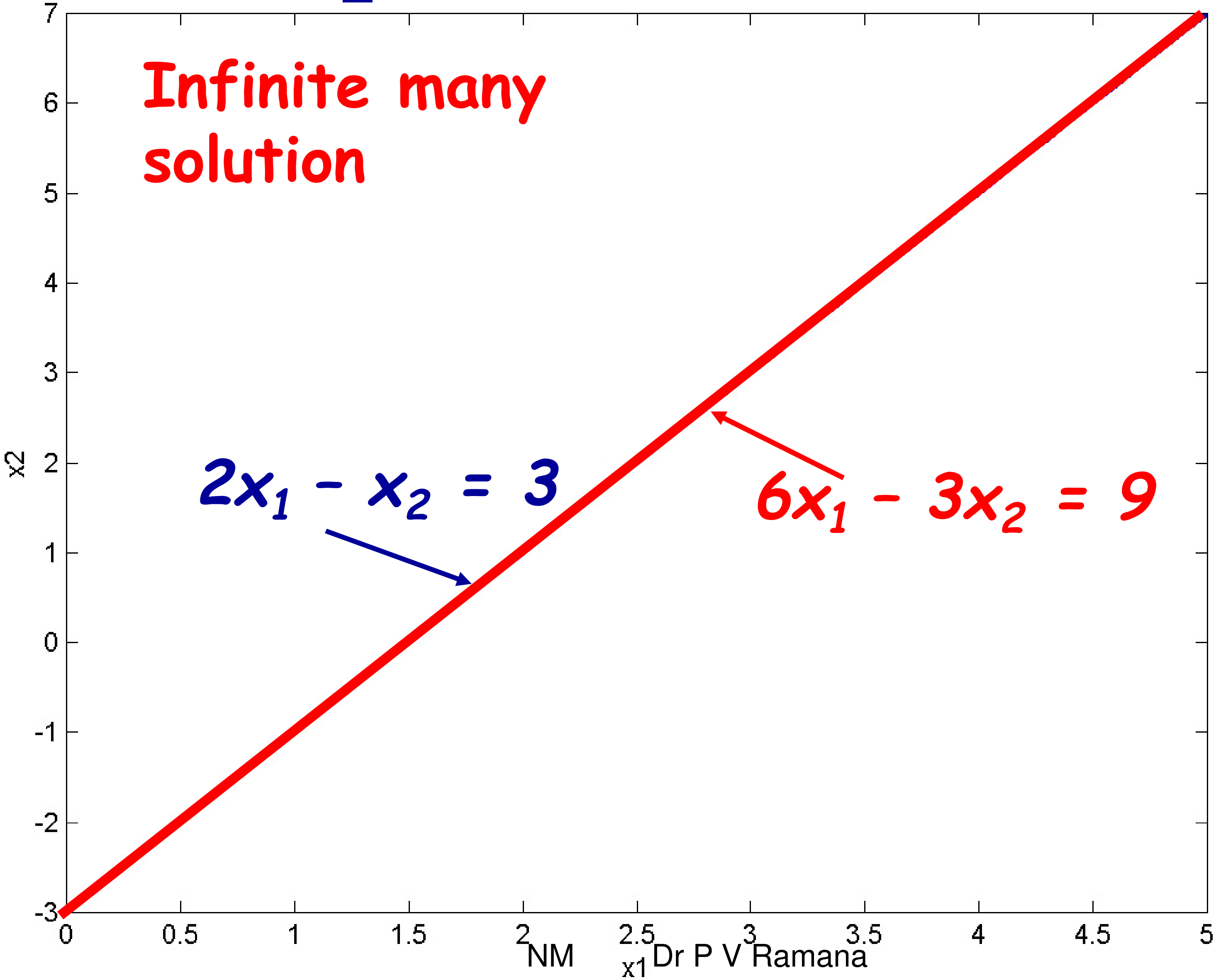
Graphical Method



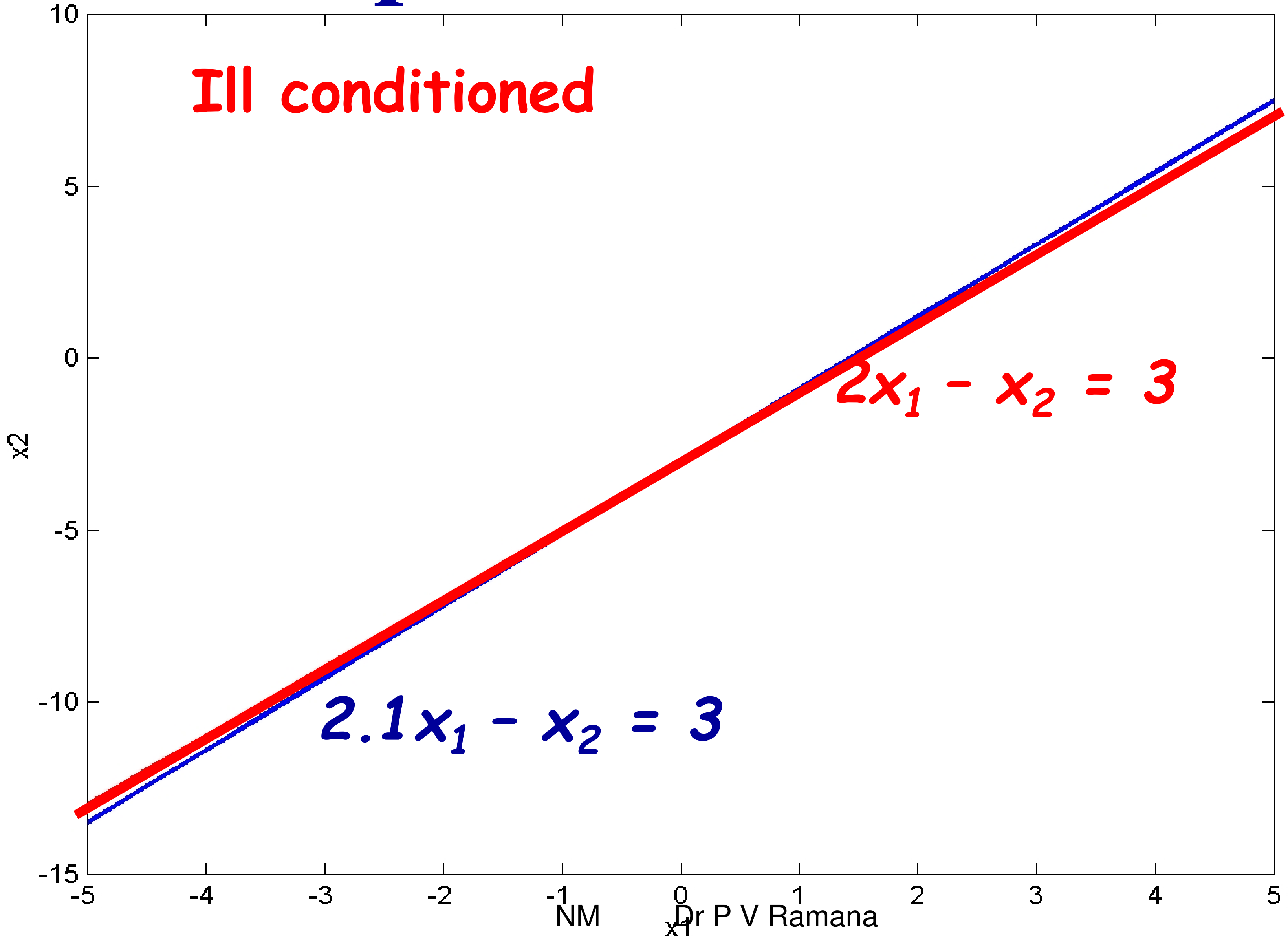
Graphical Method



Graphical Method



Graphical Method



Cramer's Rule

- Compute the **determinant D**
- 2 x 2 matrix

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- 3 x 3 matrix

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cramer's Rule

- To find x_k for the following system

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1\\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2\\
&\vdots\\
&a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n
\end{aligned}
$$

- Replace k^{th} column of a′s with b′s (i.e., $a_{ik} \leftarrow b_i$)

$$x_k = \frac{D(new\ matrix)}{D(a_{ij})}$$

Example

- 3 x 3 matrix

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$x_1 = \frac{D_1}{D} = \frac{1}{D}\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

$$x_2 = \frac{D_2}{D} = \frac{1}{D}\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$x_3 = \frac{D_3}{D} = \frac{1}{D}\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Ill-Conditioned System

- **What happen if the determinant D is very small or zero?**

$$\boxed{D = det\left[A\right] \approx 0}$$

- **Divided by zero (linearly dependent system)**
- **Divided by a small number: Round-off error**
- **Loss of significant digits**

Elimination Method

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

Eliminate $x_2 \Rightarrow \begin{cases} a_{22}a_{11}x_1 + a_{22}a_{12}x_2 = a_{22}b_1 \\ a_{12}a_{21}x_1 + a_{12}a_{22}x_2 = a_{12}b_2 \end{cases}$

Subtract to get

$$a_{22}a_{11}x_1 - a_{12}a_{21}x_1 = a_{22}b_1 - a_{12}b_2$$

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{22}a_{11} - a_{12}a_{21}} \Rightarrow x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

Not very practical for large number (> 4) of equations

MATLAB's Methods

- **Forward slash (/)**
- **Back-slash (\ )**
- **Multiplication by the inverse of the quantity under the slash**

$$Ax = b$$

$$x = A^{-1}b \Rightarrow x = A \backslash b$$

$$x = inv(A) * b$$

Gauss Elimination

- **Manipulate equations to eliminate one of the unknowns**
- **Develop algorithm to do this repeatedly**
- **The goal is to set up upper triangular matrix**

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \Lambda & a_{1n} \\ & a_{22} & a_{23} & \Lambda & a_{2n} \\ & & a_{33} & \Lambda & a_{3n} \\ & & & O & M \\ & & & & a_{nn} \end{bmatrix}$$

- **Back substitution to find solution (root)**

Basic Gauss Elimination

- Direct Method (no iteration required)
- Forward elimination
- Column-by-column elimination of the below-diagonal elements
- Reduce to upper triangular matrix
- Back-substitution

Gauss Elimination

- **Begin with**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

- **Multiply the first equation by a_{21} / a_{11} and subtract from second equation**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{n}x_n &= b_1 \\
\cancel{\left(a_{21} - \frac{a_{21}}{a_{11}}a_{11}\right)}x_1 + \left(a_{22} - \frac{a_{21}}{a_{11}}a_{12}\right)x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}}a_{1n}\right)x_n &= b_2 - \frac{a_{21}}{a_{11}}b_1 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

Gauss Elimination

- **Reduce to**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + \ldots + a'_{2n}x_n &= b'_2 \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n
\end{aligned}
$$

- **Repeat the forward elimination to get**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + \ldots + a'_{2n}x_n &= b'_2 \\
&\vdots \\
a'_{n2}x_2 + \ldots + a'_{nn}x_n &= b'_n
\end{aligned}
$$

Gauss Elimination

- First equation is **pivot equation**
- a_{11} is **pivot element**
- Now multiply second equation by a'_{32}/a'_{22} and subtract from third equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n = b'_2$$
$$\left(a'_{33} - \frac{a'_{32}}{a'_{22}}a'_{23}\right)x_3 + \ldots + \left(a'_{3n} - \frac{a'_{32}}{a'_{22}}a'_{2n}\right)x_n = \left(b'_3 - \frac{a'_{32}}{a'_{22}}b'_2\right)$$
$$\vdots$$

Gauss Elimination

- **Repeat the elimination of a'_{i2} and get**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a''_{n3}x_3 + \ldots + a''_{nn}x_n &= b''_n
\end{aligned}
$$

- **Continue and get**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

Back Substitution

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

- **Now can perform back substitution to get {*x*}**
- **By simple division**

$$
x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}
$$

- **Substitute this into (n-1)th equation**

$$
a_{n-1,n-1}^{(n-2)}x_{n-1} + a_{n-1,n}^{(n-2)}x_n = b_{n-1}^{(n-2)}
$$

- **Solve for x_{n-1}**
- **Repeat the process to solve for $x_{n-2}, x_{n-3}, \ldots, x_2, x_1$**

Back Substitution

- **Back substitution: starting with x_n**
- **Solve for $x_{n-1}, x_{n-2}, \ldots, 3, 2, 1$**

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \qquad \text{for } i = n-1, n-2, \ldots, 1$$

$$a_{ii}^{(i-1)} \neq 0$$

Pivot, Naïve, Jordon, Seidel Gauss Elimination

Elimination of first column

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \vdots & b_1 \\ a_{21} & a_{22} & a_{23} & a_{24} & \vdots & b_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & \vdots & b_3 \\ a_{41} & a_{42} & a_{43} & a_{44} & \vdots & b_4 \end{bmatrix}$$

$$f_{21} = a_{21} / a_{11}$$
$$f_{31} = a_{31} / a_{11}$$
$$f_{41} = a_{41} / a_{11}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \vdots & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & \vdots & b'_2 \\ 0 & a'_{32} & a'_{33} & a'_{34} & \vdots & b'_3 \\ 0 & a'_{42} & a'_{43} & a'_{44} & \vdots & b'_4 \end{bmatrix}$$

$$(2) - f_{21} \times (1)$$
$$(3) - f_{31} \times (1)$$
$$(4) - f_{41} \times (1)$$

Elimination of second column

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & a'_{32} & a'_{33} & a'_{34} & b'_3 \\ 0 & a'_{42} & a'_{43} & a'_{44} & b'_4 \end{array}\right]$$

$$f_{32} = a'_{32} / a'_{22}$$

$$f_{42} = a'_{42} / a'_{22}$$

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & 0 & a''_{43} & a''_{44} & b''_4 \end{array}\right]$$

$$(3) - f_{32} \times (2)$$

$$(4) - f_{42} \times (2)$$

Elimination of third column

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & a''_3 \\ 0 & 0 & a''_{43} & a''_{44} & a''_4 \end{array}\right]$$

$$f_{43} = a''_{43} / a''_{33}$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & 0 & 0 & a'''_{44} & b'''_4 \end{array}\right]$$

Upper triangular matrix

$$(4) - f_{43} \times (3)$$

Back-Substitution

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & a_{14} & b_1 \\ 0 & a'_{22} & a'_{23} & a'_{24} & b'_2 \\ 0 & 0 & a''_{33} & a''_{34} & b''_3 \\ 0 & 0 & 0 & a'''_{44} & b'''_4 \end{array}\right]$$

Upper triangular matrix

$$\boxed{a_{11}, a'_{22}, a''_{33}, a'''_{44} \neq 0}$$

$$x_4 = b'''_4 / a'''_{44}$$

$$x_3 = (b''_3 - a''_{34} x_4) / a''_{33}$$

$$x_2 = (b'_2 - a'_{23} x_3 - a'_{24} x_4) / a'_{22}$$

$$x_1 = (b_1 - a_{12} x_2 - a_{13} x_3 - a_{14} x_4) / a_{11}$$

Gaussian elimination Problem 1

Consider the following system:

$$x - 2y - 2z = -3$$
$$2x + y - z = 7$$
$$3x - 2y + 5z = 10$$

First convert the system above into a matrix:

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 2 & 1 & -1 & 7 \\ 3 & -2 & 5 & 10 \end{array}\right]$$

By Gaussian elimination, end up with a matrix that looks like this:

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & 5 & 3 & 13 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Instead of having zeroes above and below the diagonal

Just produce zeroes below the main diagonal.

Consider the following system:

$$x - 2y - 2z = -3$$
$$2x + y - z = 7$$
$$3x - 2y + 5z = 10$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 2 & 1 & -1 & 7 \\ 3 & -2 & 5 & 10 \end{array}\right]$$

Now let's learn about back substitution.

First put the variables back and create the equations again.

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & -3 \\ 0 & 5 & 3 & 13 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

First, notice that our first equation is still the same. Only equations 2 and 3 changed.

$$X - 2y - 2y = -3$$
$$5y + 3z = 13$$
$$z = 1$$

Now look at equation 3, z = 1.

That's our first solution. By backing up to equation 2 and using the information about z can find y.

$5y + 3(1) = 13$

$5y = 10$

$y = 2$ Now know y's value. Back up to equation 1 to find x's value.

$X - 2(2) - 2(1) = -3$

$x - 4 - 2 = -3$

$x - 6 = -3$

$x = 3$

X = 3, y = 2, and z = 1 or (3,2,1)

Example 1b

Now try to find the solution by back substitution:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -5 \\ 0 & 1 & -4 & 9 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

One can put the variables back as find solutions.
For example the correct first step would be to write:

$2z = 8$

or

$Y - 4z = 9$

Now try to find the solution by back substitution:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -5 \\ 0 & 1 & -4 & 9 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

One can put the variables back as find solutions.
For example the correct first step would be to write:

$2z = 8$

That's correct, and thus $z = 4$.

or

$Y - 4z = 9$

Now try to find the solution by back substitution:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -5 \\ 0 & 1 & -4 & 9 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

Put the variables back earlier found solutions.
For example the correct first step would be to write:

$2z = 8$ and thus $z = 4$.

Substitute $z = 4$ into this equation to find y.

$Y - 4(4) = 9$ or $y = 25$

Now try to find the solution by back substitution:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -5 \\ 0 & 1 & -4 & 9 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

Now z = 4 and y = 25.
How do find x?

$2x + 3(25) - 1(4) = -5$

or

$2x + 3 - 1 = -5$

Now try to find the solution by back substitution:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -5 \\ 0 & 1 & -4 & 9 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

We know z = 4 and y = 25.
How do find x?

2x +3(25) -1(4) = -5

That's correct, and thus x = -38.

or

2x +3 - 1 = -5

Now that one can back substitute, need to learn how to get the matrix in the form one can call Gaussian (zeroes below the main diagonal).

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & -5 \\ 0 & 1 & -4 & 9 \\ 0 & 0 & 2 & 8 \end{array}\right]$$

Zeroes needed here only.

Gaussian elimination Problem 2

Gaussian Elimination in 3D

$$\begin{aligned}
2x &+ 4y - 2z = 2 \\
4x &+ 9y - 3z = 8 \\
-2x &- 3y + 7z = 10
\end{aligned}$$

- Using the first equation to eliminate *x* from the next two equations

Gaussian Elimination in 3D

$$\begin{array}{rcrcrcr} 2x & + & 4y & - & 2z & = & 2 \\ & & y & + & z & = & 4 \\ & & y & + & 5z & = & 12 \end{array}$$

- Using the second equation to eliminate *y* from the third equation

Gaussian Elimination in 3D

$$
\begin{aligned}
2x + 4y - 2z &= 2 \\
y + z &= 4 \\
4z &= 8
\end{aligned}
$$

- Using the second equation to eliminate *y* from the third equation

Solving Triangular Systems

- Now have a triangular system which is easily solved using a technique called ***Backward-Substitution***.

$$
\begin{aligned}
2x + 4y - 2z &= 2 \\
y + z &= 4 \\
4z &= 8
\end{aligned}
$$

Backward Substitution

- From the previous work, have

$$
\begin{aligned}
2x + 4y - 2z &= 2 \\
y + z &= 4 \\
z &= 2
\end{aligned}
$$

- And substitute z in the first two equations

Backward Substitution

$$\begin{aligned} 2x + 4y - 4 &= 2 \\ y + 2 &= 4 \\ z &= 2 \end{aligned}$$

- One can solve *y*

Backward Substitution

$$\begin{aligned} 2x + 4y - 4 &= 2 \\ y &= 2 \\ z &= 2 \end{aligned}$$

- Substitute to the first equation

Backward Substitution

$$
\begin{aligned}
2x + 8 - 4 &= 2 \\
y &= 2 \\
z &= 2
\end{aligned}
$$

- One can solve the first equation

Backward Substitution

$$\begin{aligned} x \quad\quad\quad\quad &= -1 \\ y \quad\quad &= 2 \\ z &= 2 \end{aligned}$$

Gaussian elimination Problem 3

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ -1 & 2 & 2 & -3 & -1 \\ 0 & 1 & 1 & 4 & 2 \\ 6 & 2 & 2 & 4 & 1 \end{array}\right] \quad \begin{array}{l} \\ f_{21} = -1 \\ f_{31} = 0 \\ f_{41} = 6 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & 2 & -10 & -14 & -5 \end{array}\right] \quad \begin{array}{l} \\ (2)-(1)\times f_{21} \\ (3)-(1)\times f_{31} \\ (4)-(1)\times f_{41} \end{array}$$

Forward Elimination

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & 2 & -10 & -14 & -5 \end{array}\right] \qquad \begin{array}{l} \\ \\ f_{32} = 1/2 \\ f_{42} = 1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & -14 & -14 & -5 \end{array}\right] \qquad \begin{array}{l} \\ \\ (3) - (2) \times f_{32} \\ (4) - (2) \times f_{42} \end{array}$$

Upper Triangular Matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 1 & -1 & 4 & 2 \\ 0 & 2 & -14 & -14 & -5 \end{array}\right] \qquad f_{43} = 14$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & 0 & -70 & -33 \end{array}\right] \qquad (4)-(3)\times f_{43}$$

Back-Substitution

$$\left[\begin{array}{cccc:c} 1 & 0 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & -1 & 4 & 2 \\ 0 & 0 & 0 & -70 & -33 \end{array}\right] \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$x_4 = -33/-70 = 33/70$$

$$x_3 = 4x_4 - 2 = -4/35$$

$$x_2 = -2x_3 = 8/35$$

$$x_1 = 1 - 2x_3 - 3x_4 = -13/70$$

$$\vec{x} = \begin{bmatrix} -13/70 \\ 8/35 \\ -4/35 \\ 33/70 \end{bmatrix}$$

MATLAB Script File: GaussNaive

```
function x = GaussNaive(A,b)

% GaussNaive(A,b)  :
% Solve Ax =b using Gaussian elimination without pivoting
% Input:
%     A = coefficient matrix
%     b = right-hand-side matrix
%
% Output:
%     x = solution matrix

% compute the matrix sizes
[m, n] = size(A);
if m ~= n, error('Matrix A must be square'); end
nb = n + 1;
Aug = [A b];

% forward elimination
for k = 1 : n-1
  for i = k+1 : n
     factor = Aug(i,k) / Aug(k,k);
     Aug(i,k:nb) = Aug(i,k:nb) - factor*Aug(k,k:nb);
  end;
end

% back-substitution
x = zeros(n,1);
x(n) = Aug(n,nb) / Aug(n,n);
for i = n-1 : -1 : 1
     x(i) = (Aug(i,nb) - Aug(i,i+1:n)*x(i+1:n)) / Aug(i,i);
end
```

```
>> format short
>> x = GaussNaive(A,b)
m =
     4
n =
     4
Aug =
     1     0     2     3     1
    -1     2     2    -3    -1
     0     1     1     4     2
     6     2     2     4     1
factor =
    -1
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     1     1     4     2
     6     2     2     4     1
factor =
     0
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     1     1     4     2
     6     2     2     4     1
factor =
     6
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     1     1     4     2
     0     2   -10   -14    -5
```

Aug = [A, b]

Eliminate first column

```
factor =
    0.5000
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     0    -1     4     2
     0     2   -10   -14    -5
factor =
     1
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     0    -1     4     2
     0     0   -14   -14    -5
```

Eliminate second column

```
factor =
    14
Aug =
     1     0     2     3     1
     0     2     4     0     0
     0     0    -1     4     2
     0     0     0   -70   -33
```

Eliminate third column

```
x =
         0
         0
         0
    0.4714
x =
         0
         0
   -0.1143
    0.4714
x =
         0
    0.2286
   -0.1143
    0.4714
x =
   -0.1857
    0.2286
   -0.1143
    0.4714
```

x_4

x_3

x_2

x_1

Back-substitution

Print all `factor` and `Aug`

(do not suppress output)

Gaussian elimination Problem 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

Part 1 : Forward Elimination

Step1 : Eliminate x_1 from equations 2, 3, 4

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -27 \\ -18 \end{bmatrix}$$

Example

Forward Elimination

$$\text{Step2: Eliminate } x_2 \text{ from equations } 3, 4$$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 4 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -21 \end{bmatrix}$$

$$\text{Step3: Eliminate } x_3 \text{ from equation } 4$$

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Example

Forward Elimination

Summary of the Forward Elimination :

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Example

Backward Substitution

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -6 \\ -9 \\ -3 \end{bmatrix}$$

Solve for x_4, then solve for $x_3, \ldots$ solve for x_1

$$x_4 = \frac{-3}{-3} = 1, \qquad x_3 = \frac{-9+5}{2} = -2$$

$$x_2 = \frac{-6-2(-2)-2(1)}{-4} = 1, \quad x_1 = \frac{16+2(1)-2(-2)-4(1)}{6} = 3$$

Forward Elimination

$$\text{To eliminate } x_1 \quad \left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i1}}{a_{11}}\right) a_{1j} \quad (1 \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{i1}}{a_{11}}\right) b_1 \end{aligned}\right\} 2 \le i \le n$$

$$\text{To eliminate } x_2 \quad \left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{i2}}{a_{22}}\right) a_{2j} \quad (2 \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{i2}}{a_{22}}\right) b_2 \end{aligned}\right\} 3 \le i \le n$$

Forward Elimination

To eliminate x_k

$$\left.\begin{aligned} a_{ij} &\leftarrow a_{ij} - \left(\frac{a_{ik}}{a_{kk}}\right) a_{kj} \quad (k \le j \le n) \\ b_i &\leftarrow b_i - \left(\frac{a_{ik}}{a_{kk}}\right) b_k \end{aligned}\right\} k+1 \le i \le n$$

Continue until x_{n-1} is eliminated.

Backward Substitution

$$x_n = \frac{b_n}{a_{n,n}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n}x_n - a_{n-2,n-1}x_{n-1}}{a_{n-2,n-2}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{i,j}x_j}{a_{i,i}}$$

Algorithm for Gauss elimination

1. Forward elimination

- for each equation j, $j = 1$ to $n\text{-}1$

  for all equations k greater than j

  (a) multiply equation j by a_{kj}/a_{jj}

  (b) subtract the result from equation k

- This leads to an upper triangular matrix

2. Back-Substitution

– determine x_n from $$x_n = b_n^{(n-1)} / a_{nn}^{(n-1)}$$

– put x_n into $(n\text{-}1)^{th}$ equation, solve for x_{n-1}

– repeat from (b), moving back to n-2, n-3, etc. until all equations are solved

Operation Count

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22} & a_{23} & \cdots & a_{2n} \\ & & a_{33} & \cdots & a_{3n} \\ & & & \ddots & \vdots \\ & & & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \cdot \\ b_n \end{bmatrix}$$

Important as matrix gets large for Gauss elimination

Elimination routine uses on the order of $O(n^3/3)$ operations

Back-substitution uses $O(n^2/2)$

$$x_n = \frac{b_n}{a_{n,n}}$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n}x_n - a_{n-2,n-1}}{a_{n-2,n-2}}$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} a_{i,j}x_j}{a_{i,i}}$$

Operation Count

Outer Loop k	*Inner Loop* i	*Addition/Subtraction flops*	*Multiplication/Division flops*
1	$2, n$	$(n-1)(n)$	$(n-1)(n+1)$
2	$3, n$	$(n-2)(n-1)$	$(n-2)(n)$
M	M	M	M
k	$k+1, n$	$(n-k)(n-k+1)$	$(n-k)(n-k+2)$
M	M	M	M
$n-1$	n, n	$(1)(2)$	$(1)(3)$

Total operation counts for elimination stage = $2n^3/3 + O(n^2)$

Total operation counts for back substitution stage = $n^2 + O(n)$

Operation Count

$$T = \begin{bmatrix} a & * & * & * & * & * & * & * & * \\ 0 & 0 & b & * & * & * & * & * & * \\ 0 & 0 & 0 & c & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & d & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

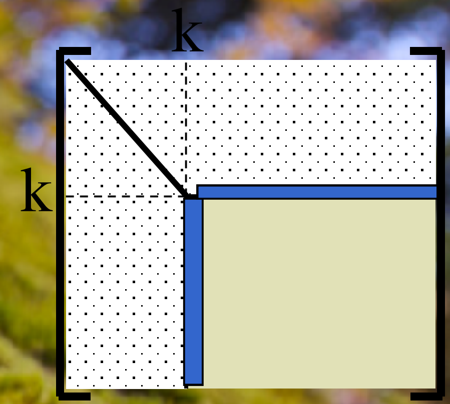
- **Number of flops (floating-point operations) for Gauss elimination**

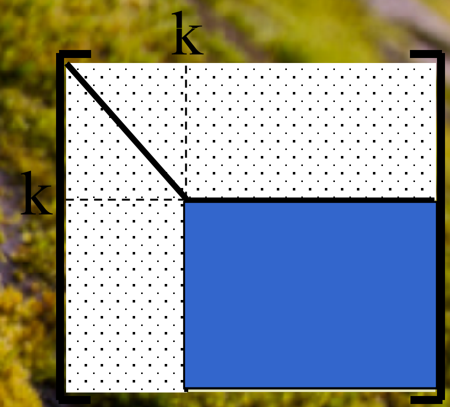
n	*Elimination*	*Back Sbustitution*	*Total Flops*	$\frac{2n^3}{3}$	*Percentage Due to Elimination*
10	*705*	*100*	*805*	*667*	*87.58%*
100	*671550*	*10000*	*681550*	*666667*	*98.53%*
1000	6.67×10^8	*1000000*	6.68×10^6	6.67×10^8	*99.85%*

- **Computation time increase rapidly with n**
- **Most of the effort is incurred in the elimination step**
- **Improve efficiency by reducing the elimination effort**

Pivoting strategies

- Partial Pivoting:
  - Only row interchange
- Complete (Full) Pivoting
  - Row and Column interchange
- Threshold Pivoting
  - Only if prospective pivot is found to be smaller than a certain threshold





Partial Pivoting

Problems with Gauss elimination

- Division by zero
- Round off errors
- ill conditioned systems

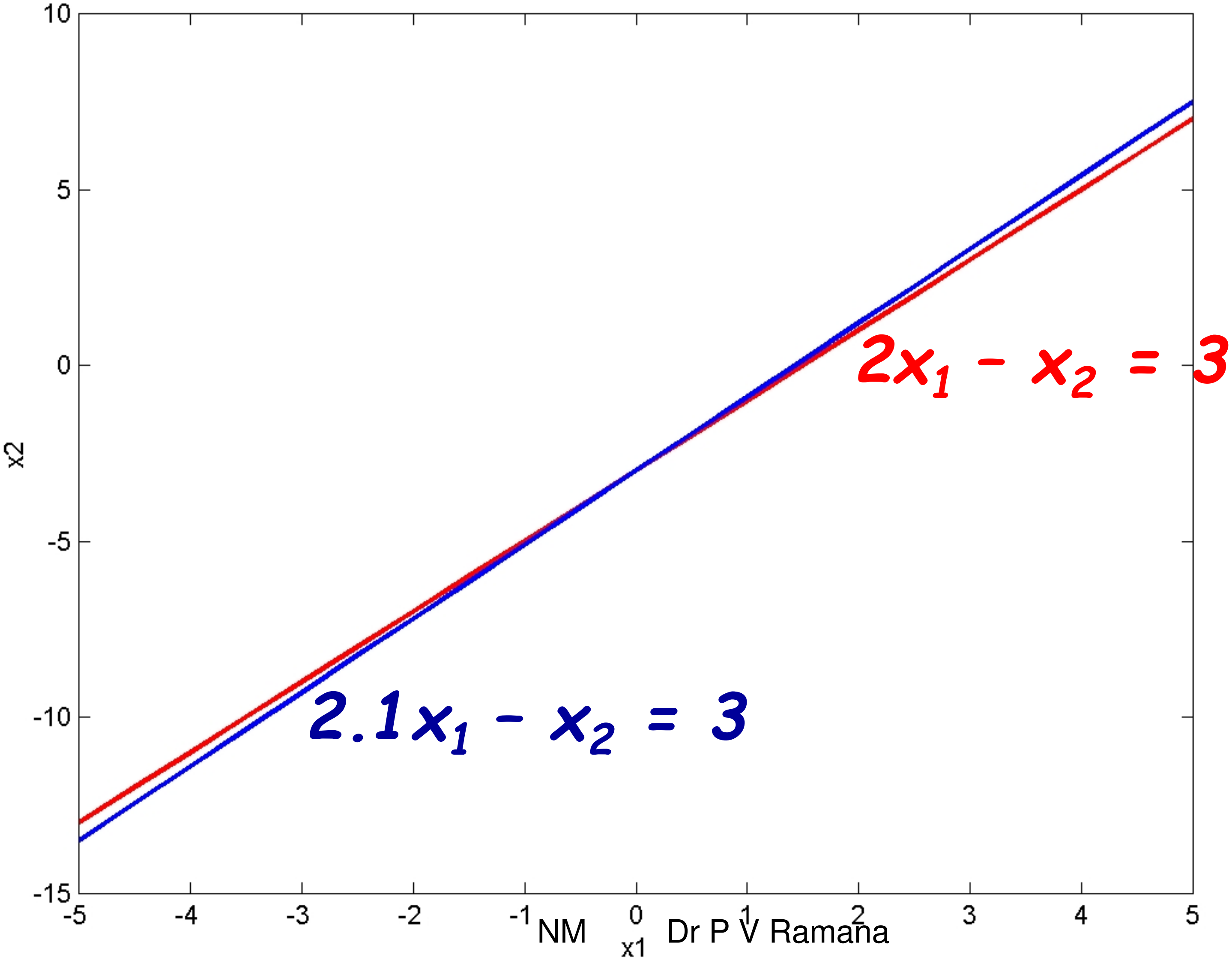
Use “Pivoting” to avoid this

- Find the row with largest absolute coefficient below the pivot element
- Switch rows (“partial pivoting”)
- Complete pivoting switch columns also (rarely used)

Round-off Errors

- A lot of chopping with more than $n^3/3$ operations
- More important - error is propagated
- For large systems (more than 100 equations), round-off error can be very important (machine dependent)
- Ill conditioned systems - small changes in coefficients lead to large changes in solution
- Round-off errors are especially important for ill-conditioned systems

Ill-conditioned System



Ill-Conditioned System

- **Consider**

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases} \Rightarrow \begin{cases} x_2 = -\dfrac{a_{11}}{a_{12}}x_1 + \dfrac{b_1}{a_{12}} \\ x_2 = -\dfrac{a_{21}}{a_{22}}x_1 + \dfrac{b_2}{a_{22}} \end{cases}$$

- **Since slopes are almost equal**

$$\boxed{\frac{a_{11}}{a_{12}} \cong \frac{a_{21}}{a_{22}}}$$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \approx 0$$

Divided by small number

Determinant

- **Calculate determinant using Gauss elimination**

$$U = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ & & a''_{33} & \cdots & a''_{3n} \\ & & & & \vdots \\ & & & & a_{nn}^{(n-1)} \end{bmatrix}$$

$$\boxed{\det(A) = \det(U) = a_{11}a'_{22}a''_{33} \cdots a_{nn}^{(n-1)}}$$

Gauss Elimination with Partial Pivoting

- **<u>Forward elimination</u>**
- **for each equation *j*, $j = 1$ to $n-1$**
  - **First scale each equation *k* greater than *j***
  - **Then pivot (switch rows)**
  - **Now perform the elimination**

  (a) multiply equation *j* by a_{kj} / a_{jj}

  (b) subtract the result from equation

Example 5

Partial (Row) Pivoting

$$\begin{aligned} x_1 \quad\quad\quad & + 2x_3 + 3x_4 = 1 \\ -x_1 + 2x_2 & + 2x_3 - 3x_4 = -1 \\ x_2 & + x_3 + 4x_4 = 2 \\ 6x_1 + 2x_2 & + 2x_3 + 4x_4 = 1 \end{aligned}$$

$$[A \quad b] = \begin{bmatrix} 1 & 0 & 2 & 3 & 1 \\ -1 & 2 & 2 & -3 & -1 \\ 0 & 1 & 1 & 4 & 2 \\ 6 & 2 & 2 & 4 & 1 \end{bmatrix}$$

Forward Elimination

$$\begin{bmatrix} 1 & 0 & 2 & 3 & 1 \\ -1 & 2 & 2 & -3 & -1 \\ 0 & 1 & 1 & 4 & 2 \\ 6 & 2 & 2 & 4 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 6 & 2 & 2 & 4 & 1 \\ -1 & 2 & 2 & -3 & -1 \\ 0 & 1 & 1 & 4 & 2 \\ 1 & 0 & 2 & 3 & 1 \end{array}\right]$$

Interchange rows 1 & 4

$$f_{21} = -1/6$$

$$f_{31} = 0$$

$$f_{41} = 1/6$$

$$\left[\begin{array}{cccc|c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & -1/3 & 5/3 & 7/3 & 5/6 \end{array}\right]$$

$$(2) - (1) \times f_{21}$$

$$(3) - (1) \times f_{31}$$

$$(4) - (1) \times f_{41}$$

Forward Elimination

$$\left[\begin{array}{cccc|c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 1 & 1 & 4 & 2 \\ 0 & -1/3 & 5/3 & 7/3 & 5/6 \end{array}\right]$$

No interchange required

$$f_{32} = 3/7$$
$$f_{42} = 1/7$$

$$\left[\begin{array}{cccc|c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 0 & 0 & 5 & 33/14 \\ 0 & 0 & 2 & 2 & 5/7 \end{array}\right]$$

$$(3)-(2)\times f_{32}$$
$$(4)-(2)\times f_{42}$$

Back-Substitution

$$\left[\begin{array}{cccc|c} 6 & 2 & 2 & 4 & 1 \\ 0 & 7/3 & 7/3 & -7/3 & -5/6 \\ 0 & 0 & 2 & 2 & 5/7 \\ 0 & 0 & 0 & 5 & 33/14 \end{array}\right]$$

Interchange rows 3 & 4

$$f_{43} = 0$$

$$x_4 = (33/14)/5 = 33/70$$

$$x_3 = (5/7 - 2\,x_4)/2 = -4/35$$

$$x_2 = (-5/6 + 7/3\,x_4 - 7/3\,x_3)/(7/3) = 8/35$$

$$x_1 = (1 - 4\,x_4 - 2\,x_3 - 2\,x_2)/6 = -13/70$$

$$\vec{x} = \begin{bmatrix} -13/70 \\ 8/35 \\ -4/35 \\ 33/70 \end{bmatrix}$$

MATLAB M-File: GaussPivot

```
function x = GaussPivot(A,b)
% GaussPivot(A,b) :
% Solve Ax =b using Gaussian elimination with pivoting
% Input:
%    A = coefficient matrix
%    b = right-hand-side matrix
%
% Output:
%    x = solution matrix

% compute the matrix sizes
[m, n] = size(A);
if m ~= n, error('Matrix A must be square'); end
nb = n + 1;
Aug = [A b];

% forward elimination
for k = 1 : n-1
   % partial pivoting
   [big, i] = max(abs(Aug(k:n,k)));
   ipr = i+k-1;
   if ipr ~= k
      %pivot the rows
      Aug([k,ipr],:) = Aug([ipr,k],:);
   end
  for i = k+1 : n
     factor = Aug(i,k) / Aug(k,k);
     Aug(i,k:nb) = Aug(i,k:nb) - factor*Aug(k,k:nb);
  end;
end

% back-substitution
x = zeros(n,1);
x(n) = Aug(n,nb) / Aug(n,n);
for i = n-1 : -1 : 1
     x(i) = (Aug(i,nb) - Aug(i,i+1:n)*x(i+1:n)) / Aug(i,i);
end
```

Partial Pivoting

Partial Pivoting (switch rows)

largest element in {x}

`[big,i] = max(x)`

index of the largest element

```
>> format short
>> x=GaussPivot0(A,b)
Aug =
     1     0     2     3     1
    -1     2     2    -3    -1
     0     1     1     4     2
     6     2     2     4     1
big =
     6
i =
     4
ipr =
     4
Aug =
     6     2     2     4     1
    -1     2     2    -3    -1
     0     1     1     4     2
     1     0     2     3     1
factor =
   -0.1667
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0    1.0000    1.0000    4.0000    2.0000
    1.0000         0    2.0000    3.0000    1.0000
factor =
     0
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0    1.0000    1.0000    4.0000    2.0000
    1.0000         0    2.0000    3.0000    1.0000
factor =
    0.1667
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0    1.0000    1.0000    4.0000    2.0000
         0   -0.3333    1.6667    2.3333    0.8333
```

Aug = [A b]

Find the first pivot element and its index

Interchange rows 1 and 4

Eliminate first column

No need to interchange

```
big =
    2.3333
i =
     1
ipr =
     2
factor =
    0.4286
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0         0    5.0000    2.3571
         0   -0.3333    1.6667    2.3333    0.8333
```

Second pivot element and index

No need to interchange

```
factor =
   -0.1429
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0         0    5.0000    2.3571
         0         0    2.0000    2.0000    0.7143
```

Eliminate second column

```
big =
     2
i =
     2
ipr =
     4
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0    2.0000    2.0000    0.7143
         0         0         0    5.0000    2.3571
```

Third pivot element and index

Interchange rows 3 and 4

```
factor =
     0
Aug =
    6.0000    2.0000    2.0000    4.0000    1.0000
         0    2.3333    2.3333   -2.3333   -0.8333
         0         0    2.0000    2.0000    0.7143
         0         0         0    5.0000    2.3571
```

Eliminate third column

Back substitution

```
x =
         0
         0
         0
    0.4714
x =
         0
         0
   -0.1143
    0.4714
x =
         0
    0.2286
   -0.1143
    0.4714
x =

   -0.1857
    0.2286
   -0.1143
    0.4714
```

Save factors f_{ij} for *LU* Decomposition

Example: Rocket Velocity

The upward velocity of a rocket is given at three different times

Time, t	Velocity, v
s	m/s
5	106.8
8	177.2
12	279.2

The velocity data is approximated by a polynomial as:

$$v(t) = a_1 t^2 + a_2 t + a_3 \ , \qquad 5 \le t \le 12.$$

Find: The Velocity at t = 6, 7.5, 9, and 11 seconds.

Example: Rocket Velocity

Assume $v(t) = a_1 t^2 + a_2 t + a_3 , \quad 5 \le t \le 12.$

Time, t	Velocity, v
s	m/s
5	106.8
8	177.2
12	279.2

Results in a matrix template of the form:

$$\begin{bmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

Using date from the time / velocity table, the matrix becomes:

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Example: Rocket Velocity

Forward Elimination: Step 1

$$Row2 - \left[\frac{Row1}{25}\right] \times (64) =$$

Yields

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.81 \\ -96.21 \\ 279.2 \end{bmatrix}$$

Example: Rocket Velocity

Forward Elimination: Step 1

$$Row3 - \left[\frac{Row1}{25}\right] \times (144) =$$

Yields

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & -16.8 & -4.76 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ -336.0 \end{bmatrix}$$

Example: Rocket Velocity

Forward Elimination: Step 2

$$Row3 - \left[\frac{Row2}{-4.8}\right] \times (-16.8) =$$

Yields

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

This is now ready for Back Substitution

Example: Rocket Velocity

Back Substitution: Solve for a_3 using the third equation

$$0.7a_3 = 0.735$$

$$a_3 = \frac{0.735}{0.7}$$

$$a_3 = 1.050$$

Example: Rocket Velocity

Back Substitution: Solve for a_2 using the second equation

$$-4.8a_2 - 1.56a_3 = -96.21$$

$$a_2 = \frac{-96.21 + 1.56a_3}{-4.8}$$

$$a_2 = \frac{-96.21 + 1.56(1.050)}{-4.8}$$

$$a_2 = 19.70$$

Example: Rocket Velocity

Back Substitution: Solve for a_1 using the first equation

$$25a_1 + 5a_2 + a_3 = 106.8$$

$$a_1 = \frac{106.8 - 5a_2 - a_3}{25}$$

$$a_1 = \frac{106.8 - 5(19.70) - 1.050}{25}$$

$$a_1 = 0.2900$$

Example: Rocket Velocity

Solution:

The solution vector is

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

The polynomial that passes through the three data points is then:

$$v(t) = a_1 t^2 + a_2 t + a_3$$

$$= 0.2900t^2 + 19.70t + 1.050, \; 5 \le t \le 12$$

Example: Rocket Velocity

Solution: $$\mathrm{v(t)} = 0.2900t^2 + 19.70t + 1.050,\ 5 \le t \le 12$$

Substitute each value of t to find the corresponding velocity

$$v(6) = 0.2900(6)^2 + 19.70(6) + 1.050$$
$$= 129.69\ m/s.$$

$$\mathrm{v}(7.5) = 0.2900(7.5)^2 + 19.70(7.5) + 1.050$$
$$= 165.1\ \mathrm{m/s.}$$

$$\mathrm{v}(9) = 0.2900(9)^2 + 19.70(9) + 1.050$$
$$= 201.8\ \mathrm{m/s.}$$

$$\mathrm{v}(11) = 0.2900(11)^2 + 19.70(11) + 1.050$$
$$= 252.8\ \mathrm{m/s.}$$

Limitations of Gaussian Elimation

- The ***naïve*** implementation of Gaussian Elimination is not robust and can suffer from severe round-off errors due to:
  - Dividing by zero
  - Dividing by small numbers and adding.
- Both can be solved with ***pivoting***

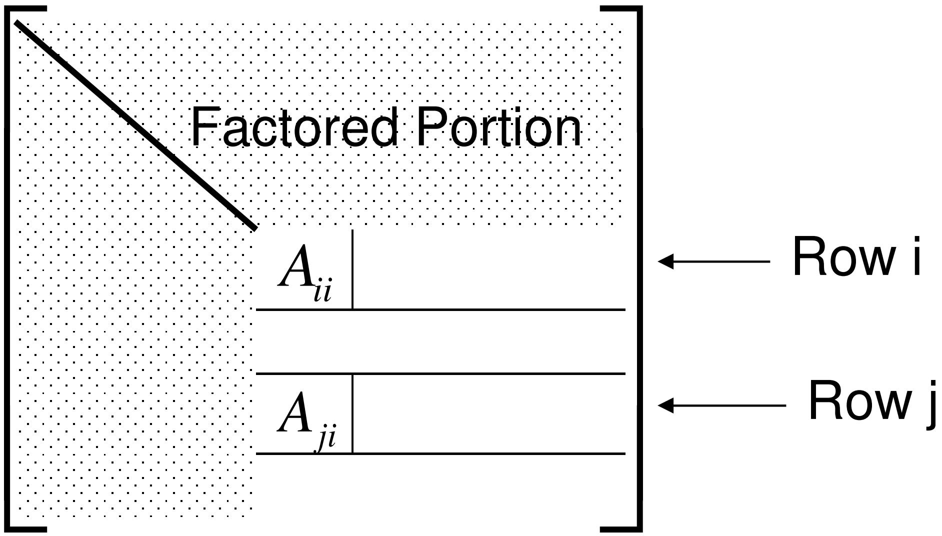
Partial Pivoting

- What if at step i, $A_{ii} = 0$?
- Simple Fix:

If $A_{ii} = 0$

Find $A_{ji} \neq 0$ $j > i$

Swap Row j with i



Example – Partial Pivoting

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

Forward Elimination

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 0 & 12.5 - 1.25\cdot 10^{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 - 6.25\cdot 10^{5} \end{bmatrix}$$

Example – Partial Pivoting

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{5\,digits}=\begin{bmatrix} 1.0001 \\ 4.9999 \end{bmatrix}$$

Forward Elimination

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 0 & 12.5-1.25\cdot 10^{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 6.25 \\ 75-6.25\cdot 10^{5} \end{bmatrix}$$

Rounded to 3 digits

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 0 & -1.25\cdot 10^{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 6.25 \\ -6.25\cdot 10^{5} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{3\,digits}=\begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

Better Pivoting

- Partial Pivoting to mitigate round-off error

$$\text{If } |A_{ii}| < \max_{j>i} |A_{ji}|$$

$$\text{Swap row } i \text{ with } \arg(\max_{j>i} |A_{ij}|)$$

Avoids Small Multipliers

- Adds an **O**(*n*) search.

Partial Pivoting

$$\begin{bmatrix} 1.25\cdot 10^{-4} & 1.25 \\ 12.5 & 12.5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6.25 \\ 75 \end{bmatrix}$$

swap

$$\begin{bmatrix} 12.5 & 12.5 \\ 1.25\cdot 10^{-4} & 1.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

Forward Elimination

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 - 12.5\cdot 10^{-5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 - 75\cdot 10^{-5} \end{bmatrix}$$

Rounded to 3 digits

$$\begin{bmatrix} 12.5 & 12.5 \\ 0 & 1.25 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 75 \\ 6.25 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{3digits} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Two Potential Pitfalls

-Division by zero: May occur in the forward elimination steps. Consider the set of equations:

$$10x_1 - 7x_2 + 0x_3 = 7$$
$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$
$$5x_1 - x_2 + 5x_3 = 6$$

- Round-off error: Prone to round-off errors.

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix}$$

Pitfalls: Example

$$10x_1 - 7x_2 + 0x_3 = 7$$
$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$
$$5x_1 - x_2 + 5x_3 = 6$$

Consider the system of equations: Use five significant figures with chopping

$$\begin{bmatrix} 10 & -7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

At the end of Forward Elimination

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

Pitfalls: Example

Back Substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 0 & 15005 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 15004 \end{bmatrix}$$

$$x_3 = \frac{15004}{15005} = 0.99993$$

$$x_2 = \frac{6.001 - 6x_3}{-0.001} = -1.5$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = -0.3500$$

Pitfalls: Example

$$10x_1 - 7x_2 + 0x_3 = 7$$
$$-3x_1 + 2.099x_2 + 6x_3 = 3.901$$
$$5x_1 - x_2 + 5x_3 = 6$$

Compare the calculated values with the exact solution

$$[X]_{exact} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[X]_{calculated} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -1.5 \\ 0.99993 \end{bmatrix}$$

Improvements

Increase the number of significant digits

Decreases round off error

Does not avoid division by zero

Gaussian Elimination with Partial Pivoting

Avoids division by zero

Reduces round off error


Thank You!