NUMERICAL METHODS

Lecture 9LP

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- *Simplex*: a linear-programming algorithm that can solve problems having more than two decision variables.
- The simplex technique involves generating a series of solutions in tabular form, called tableaus. By inspecting the bottom row of each tableau, one can immediately tell if it represents the optimal solution. Each tableau corresponds to a corner point of the feasible solution space. The first tableau corresponds to the origin. Subsequent tableaus are developed by shifting to an adjacent corner point in the direction that yields the highest (smallest) rate of profit (cost). This process continues as long as a positive (negative) rate of profit (cost) exists.

Simplex Algorithm

The key solution concepts

- Solution Concept 1: the simplex method focuses on CPF solutions.
- Solution concept 2: the simplex method is an iterative algorithm (a systematic solution procedure that keeps repeating a fixed series of steps, called, an iteration, until a desired result has been obtained) with the following structure:

Simplex algorithm

Initialization: setup to start iterations, including finding an initial CPF solution

Optimality test: is the current CPF solution optimal?

if no if yes stop

Iteration: Perform an iteration to find a better CFP solution

Simplex algorithm

- Solution concept 3: whenever possible, the initialization of the simplex method chooses the origin point (all decision variables equal zero) to be the initial CPF solution.

- Solution concept 4: given a CPF solution, it is much quicker computationally to gather information about its adjacent CPF solutions than about other CPF solutions. Therefore, each time the simplex method performs an iteration to move from the current CPF solution to a better one, it always chooses a CPF solution that is adjacent to the current one.

Simplex algorithm

- Solution concept 5: After the current CPF solution is identified, the simplex method examines each of the edges of the feasible region that emanate from this CPF solution. Each of these edges leads to an adjacent CPF solution at the other end, but the simplex method doesn't even take the time to solve for the adjacent CPF solution. Instead it simply identifies the rate of improvement in Z that would be obtained by moving along the edge. And then chooses to move along the one with largest positive rate of improvement.

Simplex algorithm

- Solution concept 6: A positive rate of improvement in Z implies that the adjacent CPF solution is better than the current one, whereas a negative rate of improvement in Z implies that the adjacent CPF solution is worse. Therefore, the optimality test consists simply of checking whether any of the edges give a positive rate of improvement in Z. if none do, then the current CPF solution is optimal.

The simplex method in tabular form

- Steps:

1. Initialization:

   a. transform all the constraints to equality by introducing slack, surplus, and artificial variables as follows:

Constraint type	Variable to be added
Less than or equal	+ slack (s)
Greater than or equal	- Surplus (s) + artificial (A)
=	+ Artificial (A)

Simplex method in tabular form

b. Construct the initial simplex tableau

Basic variable	X_1	…	X_n	S_1	……	S_n	A_1	….	A_n	RHS
S	Coefficient of the constraints									b_1
⋮										⋮
A										b_m
Z	Objective function coefficient In different signs									Z value

Simplex method in tabular form

2. Test for optimality:

Case 1: Maximization problem

the current BF solution is optimal if every coefficient in the objective function row is nonnegative

Case 2: Minimization problem

the current BF solution is optimal if every coefficient in the objective function row is non positive.

Simplex method in tabular form

3. Iteration

Step 1: determine the entering basic variable by selecting the variable (automatically a non basic variable) with the most negative value (in case of maximization) or with the most positive (in case of minimization) in the last row (Z-row). Put a box around the column below this variable, and call it the "pivot column"

Simplex method in tabular form

- Step2: Determine the leaving basic variable by applying the minimum ratio test as following:

1. Pick out each coefficient in the pivot column that is strictly positive (>0)
2. Divide each of these coefficients into the right hand side entry for the same row
3. Identify the row that has the smallest of these ratios
4. The basic variable for that row is the leaving variable, so replace that variable by the entering variable in the basic variable column of the next simplex tableau. Put a box around this row and call it the "pivot row"

Simplex method in tabular form

- Step 3: Solve for the new BF solution by using elementary row operations (multiply or divide a row by a nonzero constant; add or subtract a multiple of one row to another row) to construct a new simplex tableau, and then return to the optimality test. The specific elementary row operations are:

1. Divide the pivot row by the "pivot number" (the number in the intersection of the pivot row and pivot column)
2. For each other row that has a negative coefficient in the pivot column, add to this row the product of the absolute value of this coefficient and the new pivot row.
3. For each other row that has a positive coefficient in the pivot column, subtract from this row the product of the absolute value of this coefficient and the new pivot row.

Simplex method

- Example (All constraints are $\leq$)

Solve the following problem using the simplex method

- Maximize

$$Z = 3X_1 + 5X_2$$

Subject to

$$X_1 \leq 4$$

$$2X_2 \leq 12$$

$$3X_1 + 2X_2 \leq 18$$

$$X_1, X_2 \geq 0$$

- Solution
- Initialization

1. Standard form

Maximize Z,

Subject to

$$Z - 3X_1 - 5X_2 = 0$$

$$X_1 + S_1 = 4$$

$$2X_2 + S_2 = 12$$

$$3X_1 + 2X_2 + S_3 = 18$$

$$X_1, X_2, S_1, S_2, S_3 \geq 0$$

Sometimes it is called the augmented form of the problem because the original form has been augmented by some supplementary variables needed to apply the simplex method

Definitions

- A basic solution is an augmented corner point solution.
- A basic solution has the following properties:

1. Each variable is designated as either a nonbasic variable or a basic variable.
2. The number of basic variables equals the number of functional constraints. Therefore, the number of nonbasic variables equals the total number of variables minus the number of functional constraints.
3. The nonbasic variables are set equal to zero.
4. The values of the basic variables are obtained as simultaneous solution of the system of equations (functional constraints in augmented form). The set of basic variables are called "basis"
5. If the basic variables satisfy the nonnegativity constraints, the basic solution is a Basic Feasible (BF) solution.

Initial tableau

2. Initial tableau

Entering variable

Basic variable	X_1	X_2	S_1	S_2	S_3	RHS (b)	Ratio b/pc
S_1	1	0	1	0	0	4	Inf
S_2	0	2	0	1	0	12	6
S_3	3	2	0	0	1	18	9
Z	-3	-5	0	0	0	0	0

Leaving variable

Pivot column (pc)

Pivot Element

Pivot row (pr)

$$
\begin{aligned}
Z - 3X_1 - 5X_2 &= 0 \\
X_1 + S_1 &= 4 \\
2X_2 + S_2 &= 12 \\
3X_1 + 2X_2 + S_3 &= 18
\end{aligned}
$$

Notes:

- The basic feasible solution at the initial tableau is (0, 0, 4, 12, 18) where:

$X_1 = 0$, $X_2 = 0$, $S_1 = 4$, $S_2 = 12$, $S_3 = 18$, and $Z = 0$

Where S_1, S_2, and S_3 are basic variables

X_1 and X_2 are non basic variables

- The solution at the initial tableau is associated to the origin point at which all the decision variables are zero.

By investigating the last row of the initial tableau, we find that there are some negative numbers. Therefore, the current solution is not optimal

Iteration

- Step 1: Determine the entering variable by selecting the variable with the most negative in the last row.
- From the initial tableau, in the last row (Z row), the coefficient of X_1 is -3 and the coefficient of X_2 is -5; therefore, the most negative is -5. consequently, X_2 is the entering variable.
- X_2 is surrounded by a box and it is called the pivot column

Iteration

- Step 2: Determining the leaving variable by using the minimum ratio test as following:

Basic variable	Entering variable X_2 (1)	RHS (2)	Ratio (2)÷(1)
S_1	0	4	Inf (None)
S_2 Leaving	**2**	**12**	**6 Smallest ratio**
S3	2	18	9

Iteration

BV	X_1	X_2	S_1	S_2	S_3	b
S_1	1	0	1	0	0	4
S_2	0	2	0	1	0	12
S_3	3	2	0	0	1	18
Z	-3	-5	0	0	0	0

- Step 3: solving for the new BF solution by using the eliminatory row operations as following:

1. New pivot row = old pivot row ÷ pivot number

Basic variable	X_1	X_2	S_1	S_2	S_3	RHS
S_1						
X_2	0	1	0	1/2	0	6
S_3						
Z						

Note that X_2 becomes in the basic variables list instead of S_2

Iteration

2. For the other row apply this rule:

New row = old row – the coefficient of this row in the pivot column (new pivot row).

For S_1

1	0	1	0	0	4	-
0	1	0	1/2	0	6	-
3	2	0	0	1	18	-

For S_3

3	2	0	0	1	18	R3 = R3-2R2
3	0	0	-1	1	6	

for Z

-3	-5	0	0	0	0 -	R4 = R4+5R2
-3	0	0	5/2	0	30	

BV	X_1	X_2	S_1	S_2	S_3	b
S_1	1	0	1	0	0	4
X_2	0	1	0	1/2	0	6
S_3	3	2	0	0	1	18
Z	-3	-5	0	0	0	0

Substitute this values in the table

1	**0**	**1**	**0**	**0**	**4**	**-**
0	**1**	**0**	**½**	**0**	**6**	**-**
3	**0**	**0**	**-1**	**1**	**6**	**-**
-3	**0**	**0**	**5/2**	**0**	**30**	

Iteration

1	0	1	0	0	4	-
0	1	0	½	0	6	-
3	0	0	-1	1	6	-
-3	0	0	5/2	0	30	-

This solution is not optimal, since there is a negative numbers in the last row

Basic variable	X_1	X_2	S_1	S_2	S_3	RHS	Ratio b/pc
S_1	1	0	1	0	0	4	4
X_2	0	1	0	1/2	0	6	Inf
S_3	3	0	0	-1	1	6	2
Z	-3	0	0	5/2	0	30	-10

Pivot Element

Pivot row (pr)

Pivot column (pc)

The most negative value; therefore, X_1 is the entering variable

The smallest ratio is 6/3 =2; therefore, S_3 is the leaving variable

Iteration

- Apply the same rules we will obtain this solution:

Basic variable	X_1	X_2	S_1	S_2	S_3	RHS
S_1	0	0	1	1/3	-1/3	2
X_2	0	1	0	1/2	0	6
X1	1	0	0	-1/3	1/3	2
Z	0	0	0	3/2	1	36

This solution is optimal; since there is no negative solution in the last row: basic variables are $X_1 = 2$, $X_2 = 6$ and $S_1 = 2$; the nonbasic variables are $S_2 = S_3 = 0$

S_3	3	0	0	-1	1	6

$Z = 3X_1 + 5X_2 = 3(2) + 5(6) = 36$

BV	X_1	X_2	S_1	S_2	S_3	b
S_1	1	0	1	0	0	4
X_2	0	1	0	1/2	0	6
X_1	1	0	0	-1/3	1/3	2
Z	-3	0	0	5/2	0	30

The Simplex Method

The geometric method of solving linear programming problems presented before. The graphical method is useful only for problems involving two decision variables and relatively few problem constraints.

What happens when we need more decision variables and more problem constraints?

We use an algebraic method called the simplex method, which was developed by George B. DANTZIG (1914-2005) in 1947 while on assignment with the U.S. Department of the air force.

Standard Maximization Problems in Standard Form

A linear programming problem is said to be a standard maximization problem in standard form if its mathematical model is of the following form:

Maximize the objective function

$$Z_{\max} = P = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to problem constraints of the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \leq b \qquad , b \geq 0$$

With non-negative constraints

$$x_1, x_2, \ldots, x_n \geq 0$$

Slack Variables

"A mathematical representation of surplus resources." In real life problems, it's unlikely that all resources will be used completely, so there usually are unused resources.

Slack variables represent the unused resources between the left-hand side and right-hand side of each inequality.

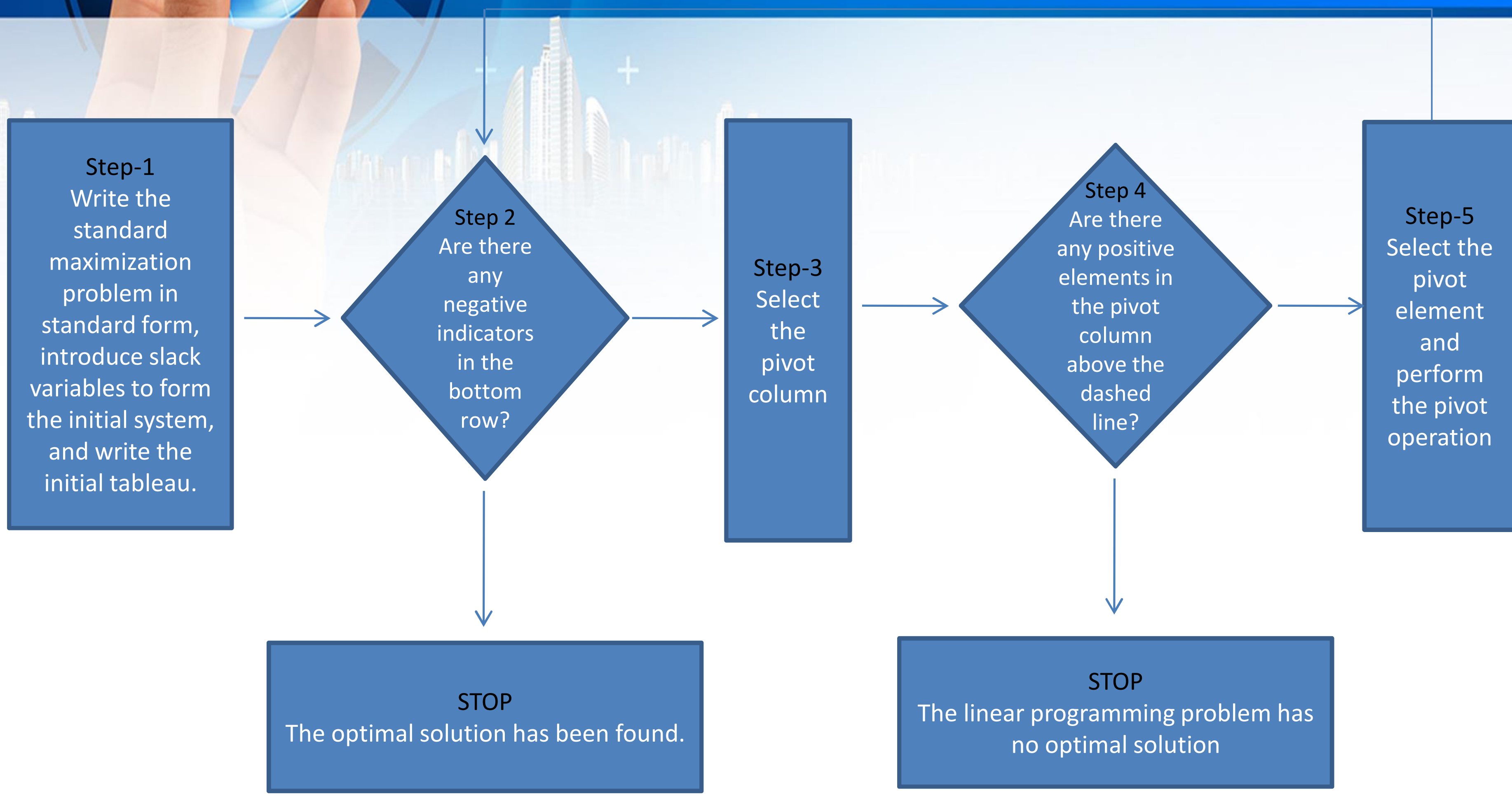
Basic and Nonbasic Variables

Basic variables are selected arbitrarily with the restriction that there be as many basic variables as there are equations. The remaining variables are non-basic variables.

$$\begin{aligned} x_1 + 2x_2 + s_1 \quad\quad &= 32 \\ 3x_1 + 4x_2 \quad + s_2 &= 84 \end{aligned}$$

This system has two equations, we can select any two of the four variables as basic variables. The remaining two variables are then non-basic variables. A solution found by setting the two non-basic variables equal to 0 and solving for the two basic variables is a basic solution. If a basic solution has no negative values, it is a basic feasible solution.

SIMPLEX METHOD



Simplex algorithm for standard maximization problems

To solve a linear programming problem in standard form, use the following steps.

1- Convert each inequality in the set of constraints to an equation by adding slack variables.

2- Create the initial simplex tableau.

3- Select the pivot column. (The column with the “most negative value” element in the last row.)

4- Select the pivot row. (The row with the smallest non-negative result when the last element in the row is divided by the corresponding in the pivot column.)

5-Use elementary row operations calculate new values for the pivot row so that the pivot is 1 (Divide every number in the row by the pivot number.)

6- Use elementary row operations to make all numbers in the pivot column equal to 0 except for the pivot number. If all entries in the bottom row are zero or positive, this the final tableau. If not, go back to step 3.

7- If you obtain a final tableau, then the linear programming problem has a maximum solution, which is given by the entry in the lower-right corner of the tableau.

Pivot

Pivot Column: The column of the tableau representing the variable to be entered into the solution mix.

Pivot Row: The row of the tableau representing the variable to be replaced in the solution mix.

Pivot Number: The element in both the pivot column and the pivot row.

Simplex Tableau

Most real-world problems are too complex to solve graphically. They have too many corners to evaluate, and the algebraic solutions are lengthy. A simplex tableau is a way to systematically evaluate variable mixes in order to find the best one.

Initial Simplex Tableau

	All variables	Solution
Basic variables	coefficients	0

EXAMPLE 2

The Cannon Hill furniture Company produces **tables** and **chairs**. Each table takes four hours of labor from the carpentry department and two hours of labor from the finishing department. Each chair requires three hours of carpentry and one hour of finishing. During the current week, 240 hours of carpentry time are available and 100 hours of finishing time. Each table produced gives a profit of $70 and each chair a profit of $50. How many chairs and tables should be made?

STEP 1

All information about example			
Resource	Table s (x_1)	Chairs (x_2)	Constraints
Carpentry (hr)	4	3	240
Finishing (hr)	2	1	100
Unit Profit	$70	$50	

Objective Function $$P = 70x_1 + 50x_2$$

Carpentry Constraint $$4x_1 + 3x_2 \le 240$$

Finishing Constraint $$2x_1 + 1x_2 \le 100$$

Non-negativity conditions $$x_1, x_2 \ge 0$$

The first step of the simplex method requires that each inequality be converted into an equation. ”less than or equal to” inequalities are converted to equations by including slack variables.

Suppose s_1 carpentry hours and s_2 finishing hours remain unused in a week. The constraints become;

$$4x_1 + 3x_2 + s_1 = 240$$
$$2x_1 + x_2 + s_2 = 100$$

or

$$4x_1 + 3x_2 + s_1 + 0s_2 = 240$$
$$2x_1 + x_2 + 0s_1 + s_2 = 100$$

As unused hours result in no profit, the slack variables can be included in the objective function with zero coefficients:

$$P = 70x_1 + 50x_2 + 0s_1 + 0s_2$$
$$P - 70x_1 - 50x_2 - 0s_1 - 0s_2 = 0$$

$$P = 70x_1 + 50x_2 + 0s_1 + 0s_2$$
$$P - 70x_1 - 50x_2 - 0s_1 - 0s_2 = 0$$

$$4x_1 + 3x_2 + s_1 + 0s_2 = 240$$
$$2x_1 + x_2 + 0s_1 + s_2 = 100$$

The problem can now be considered as solving a system of 3 linear equations involving the 5 variables x_1, x_2, s_1, s_2, P in such a way that P has the maximum value;

$$4x_1 + 3x_2 + s_1 + 0s_2 = 240$$
$$2x_1 + x_2 + 0s_1 + s_2 = 100$$
$$P - 70x_1 - 50x_2 - 0s_1 - 0s_2 = 0$$

Now, the system of linear equations can be written in matrix form or as a 3x6 augmented matrix. The initial tableau is;

STEP 2

Basic Var.	x_1	x_2	S_1	S_2	RHS	Ratio
S_1	4	3	1	0	240	
S_2	2	1	0	1	100	
P	-70	-50	0	0	0	

The tableau represents the initial solution;

$$x_1 = 0, \quad x_2 = 0, \quad s_1 = 240, \quad s_2 = 100, \quad P = 0$$

The slack variables S_1 and S_2 form the initial solution mix. The initial solution assumes that all avaliable hours are unused. i.e. The slack variables take the largest possible values.

$$4x_1 + 3x_2 + s_1 + 0s_2 = 240$$
$$2x_1 + x_2 + 0s_1 + s_2 = 100$$

$$P = 70x_1 + 50x_2 + 0s_1 + 0s_2$$
$$P - 70x_1 - 50x_2 - 0s_1 - 0s_2 = 0$$

Variables in the solution mix are called basic variables. Each basic variables has a column consisting of all 0's except for a single 1. all variables not in the solution mix take the value 0.

The simplex process, a basic variable in the solution mix is replaced by another variable previously not in the solution mix. The value of the replaced variable is set to 0.

BV	x_1	x_2	S_1	S_2	RHS	Ratio
S_1	4	3	1	0	240	
S_2	2	1	0	1	100	
P	-70	-50	0	0	0	

STEP 3

Select the pivot column (determine which variable to enter into the solution mix). Choose the column with the "most negative" element in the objective function row.

Basic Variables	x_1	x_2	S_1	S_2	b	Ratio
S_1	4	3	1	0	240	60
S_2	2	1	0	1	100	50
P	-70	-50	0	0	0	0

Pivot column

x_1 should enter into the solution mix because each unit of x_1 (a table) contributes a profit of $70 compared with only $50 for each unit of x_1 (a chair)

Step 4: No, There aren't any positive elements in the pivot column above the dashed line.

STEP 5

Select the pivot row (determine which variable to replace in the solution mix). Divide the last element in each row by the corresponding element in the pivot column. The pivot row is the row with the smallest non-negative result.

Enter

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	4	3	1	0	0	240
S_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0

$240/4 = 60$

$100/2 = 50$

Exit

Pivot row

Pivot column

Pivot Element

Should be replaced by x_1 in the solution mix. 60 tables can be made with 240 unused carpentry hours but only 50 tables can be made with 100 finishing hours. Therefore we decide to make 50 tables.

Now calculate new values for the pivot row. Divide every number in the row by the pivot number.

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	4	3	1	0	0	240
x_1	1	1/2	0	1/2	0	50
P	-70	-50	0	0	1	0

$$\frac{R_2}{2}$$

BV	x_1	x_2	S_1	S_2	P	RHS
S_1	4	3	1	0	0	240
S_2	2	1	0	1	0	100
P	-70	-50	0	0	1	0

Use row operations to make all numbers in the pivot column equal to 0 except for the pivot number which remains as 1.

Basic Variables	x_1	x_2	S_1	S_2	RHS	Ratio	
S_1	0	1	1	-2	40		$-4.R_2 + R_1$
x_1	1	1/2	0	1/2	50		
P	0	-15	0	35	3500		$70.R_2 + R_3$

If 50 tables are made, then the unused carpentry hours are reduced by 200 hours (4 h/table multiplied by 50 tables); the value changes from 240 hours to 40 hours. Making 50 tables results in the profit being increased by $3500; the value changes from $0 to $3500.

$$x_1 = 50, \quad x_2 = 0, \quad s_1 = 40, \quad s_2 = 0, \quad P = 3500$$

In this case,
Now repeat the steps until there are no negative numbers in the last row.

Select the new pivot column. x_2 should enter into the solution mix.
Select the new pivot row. S_1 should be replaced by x_2 in the solution mix.

BV	x_1	x_2	S_1	S_2	RHS	Ratio
S_1	0	1	1	-2	40	40
x_1	1	1/2	0	1/2	50	100
P	0	-15	0	35	3500	-233x

Enter

Exit

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side
S_1	0	1	1	-2	0	40
x_1	1	1/2	0	1/2	0	50
P	0	-15	0	35	1	3500

$40/1 = 40$

$50/0.5 = 100$

New pivot Element

New pivot column

New pivot row

Calculate new values for the pivot row. As the pivot number is already 1, there is no need to calculate new values for the pivot row.

Use row operations to make all numbers in the pivot column equal to except for the pivot number.

Basic Variables	x_1	x_2	S_1	S_2	P	Right hand side	
x_2	0	1	1	-2	0	40	
x_1	1	0	-1/2	3/2	0	30	$-\frac{1}{2}.R_1 + R_2$
P	0	0	15	5	1	4100	$15.R_1 + R_3$

BV	x_1	x_2	S_1	S_2	P	b
S_1	0	1	1	-2	0	40
x_1	1	1/2	0	1/2	0	50
P	0	-15	0	35	1	3500

Result

If 40 chairs are made, then the number of tables are reduced by 20 tables (1/2 table/chair multiplied by 40 chairs); the value changes from 50 tables to 30 tables. The replacement of 20 tables by 40 chairs results in the profit being increased by $600; the value changes from $3500 to $4100.

As the last row contains no negative numbers, this solution gives the maximum value of P.

This simplex tableau represents the optimal solution to the LP problem and is interpreted as:

bv	x_1	x_2	S_1	S_2	P	B
x_2	0	1	1	-2	0	40
x_1	1	0	-1/2	3/2	0	30
P	0	0	15	5	1	4100

$$x_1 = 30, \quad x_2 = 40, \quad s_1 = 0, \quad s_2 = 0$$

and profit or P=$4100

The optimal solution (maximum profit to be made) is to company 30 tables and 40 chairs for a profit of $4100.

Special cases of linear programming

- Infeasible solution
- Multiple solution (infinitely many solution)
- Unbounded solution
- Degenerated solution

Notes on the Simplex tableau

1. In any Simplex tableau, the intersection of any basic variable with itself is always one and the rest of the column is zeroes.
2. In any simplex tableau, the objective function row (Z row) is always in terms of the non basic variables. This means that under any basic variable (in any tableau) there is a zero in the Z row. For the non basic there is no condition (it can take any value in this row).
3. If there is a zero under one or more non basic variables in the last tableau (optimal solution tableau), then there is a multiple optimal solution.
4. When determining the leaving variable of any tableau, if there is no positive ratio (all the entries in the pivot column are negative and zeroes), then the solution is unbounded.
5. If there is a tie (more than one variables have the same most negative or positive) in determining the entering variable, choose any variable to be the entering one.
6. If there is a tie in determining the leaving variable, choose any one to be the leaving variable. In this case a zero will appear in RHS column; therefore, a “cycle” will occur, this means that the value of the objective function will be the same for several iterations.
7. A Solution that has a basic variable with zero value is called a “degenerate solution”.
8. If there is no Artificial variables in the problem, there is no room for “infeasible solution”

Simplex method incase of Artificial variables "Big M method"

- Solve the following linear programming problem by using the simplex method:
- Min $Z = 2X_1 + 3X_2$

S.t.

$\frac{1}{2}X_1 + \frac{1}{4}X_2 \leq 4$

$X_1 + 3X_2 \geq 20$

$X_1 + X_2 = 10$

$X_1, X_2 \geq 0$