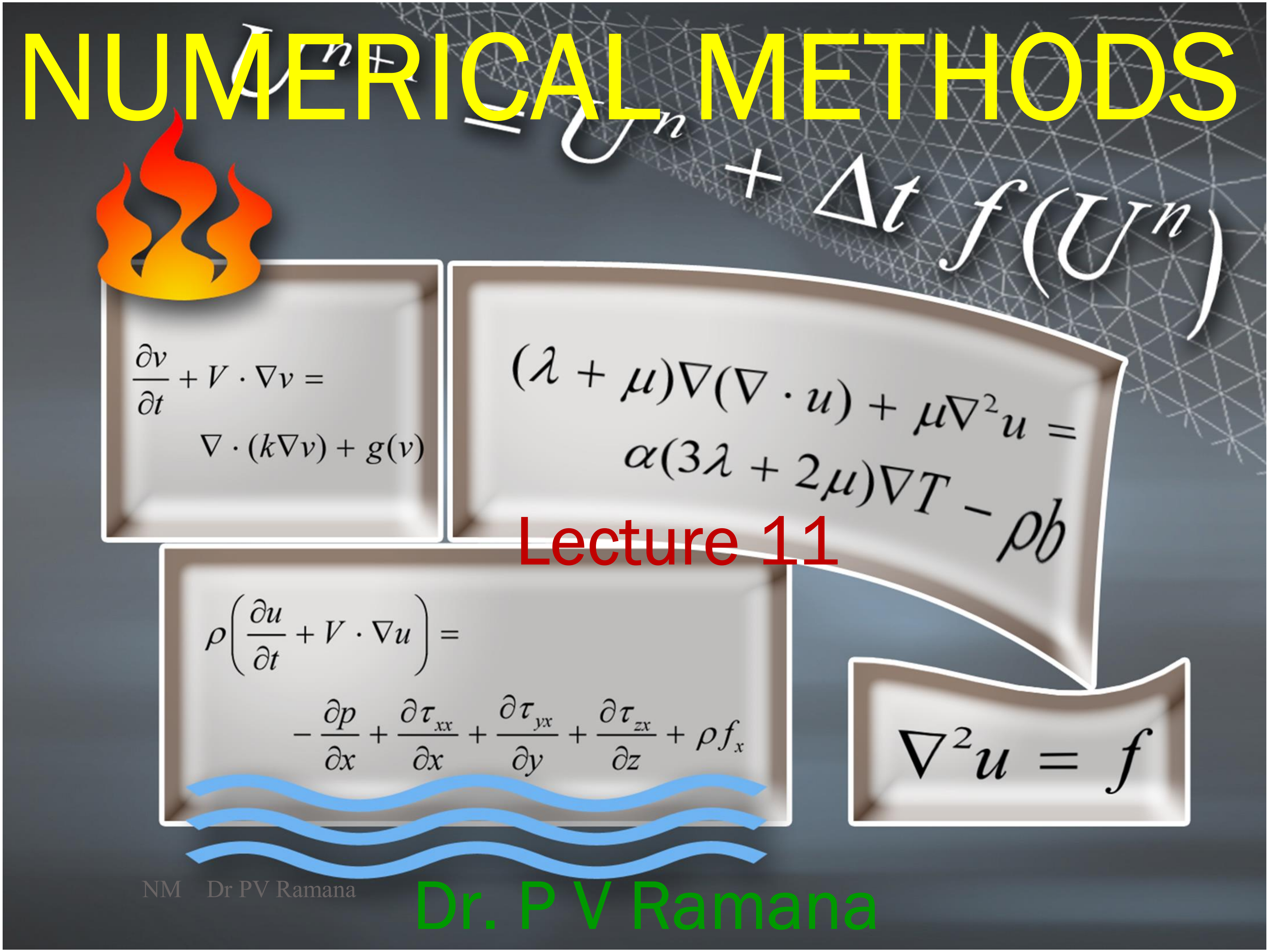
NUMERICAL METHODS

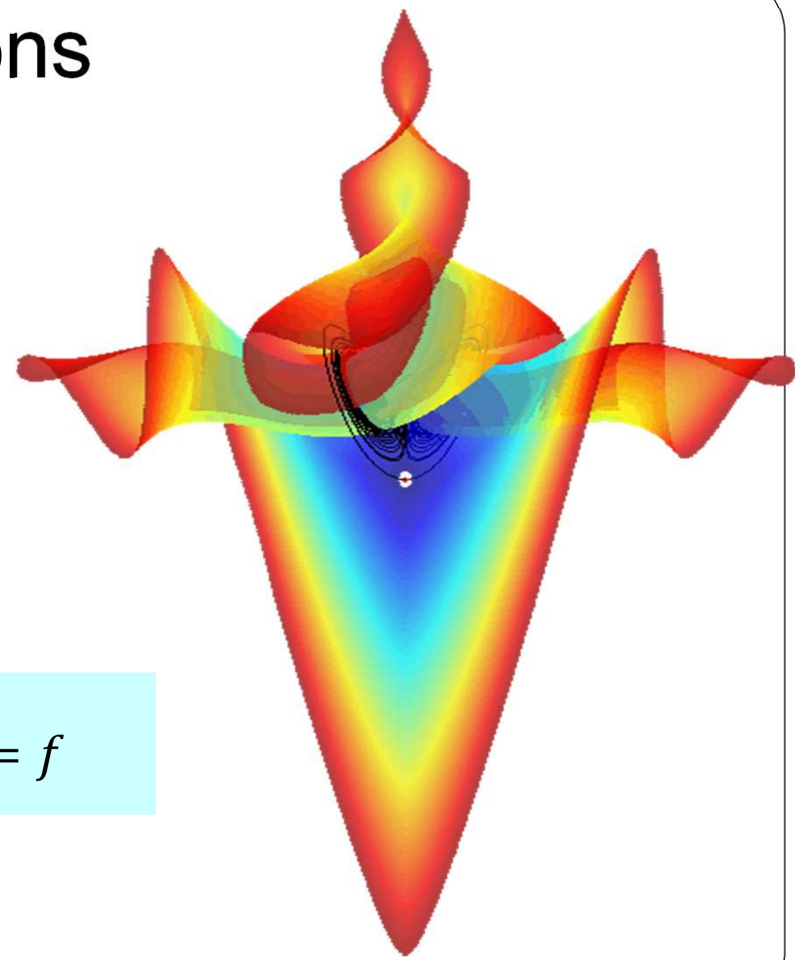


Lecture 11

Dr. P V Ramana

Ordinary Differential Equations

A relationship between an unknown function (Dependent variable) and one or more of its derivatives

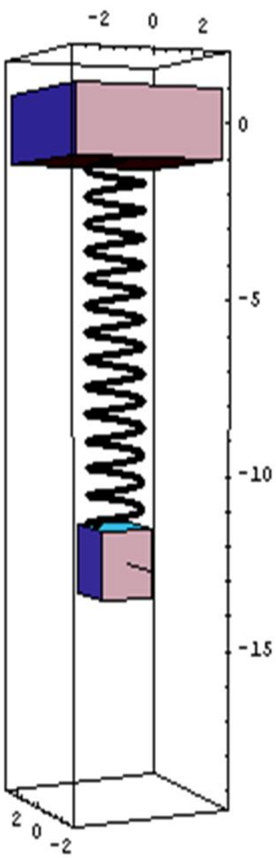


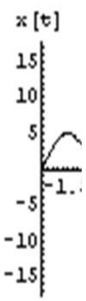
$$a\frac{d^n u}{dx^n} + b\frac{d^{n-1} u}{dx^{n-1}} + c\frac{d^{n-2} u}{dx^{n-2}} + \ldots + d\frac{du}{dx} + eu = f$$

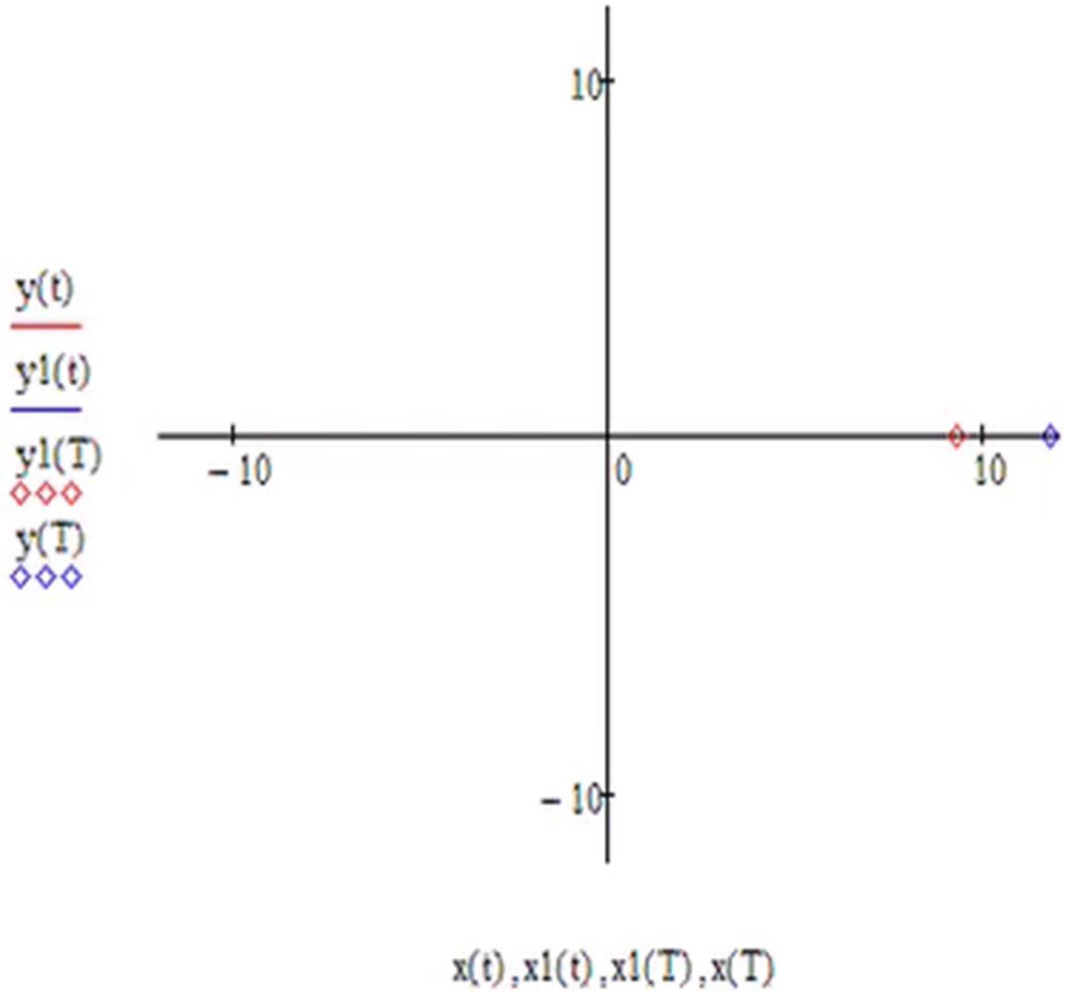
Physical problems using differential equations

- Structural elements
- Electrical circuits
- Heat transfer
- Motion

Animated Short Story

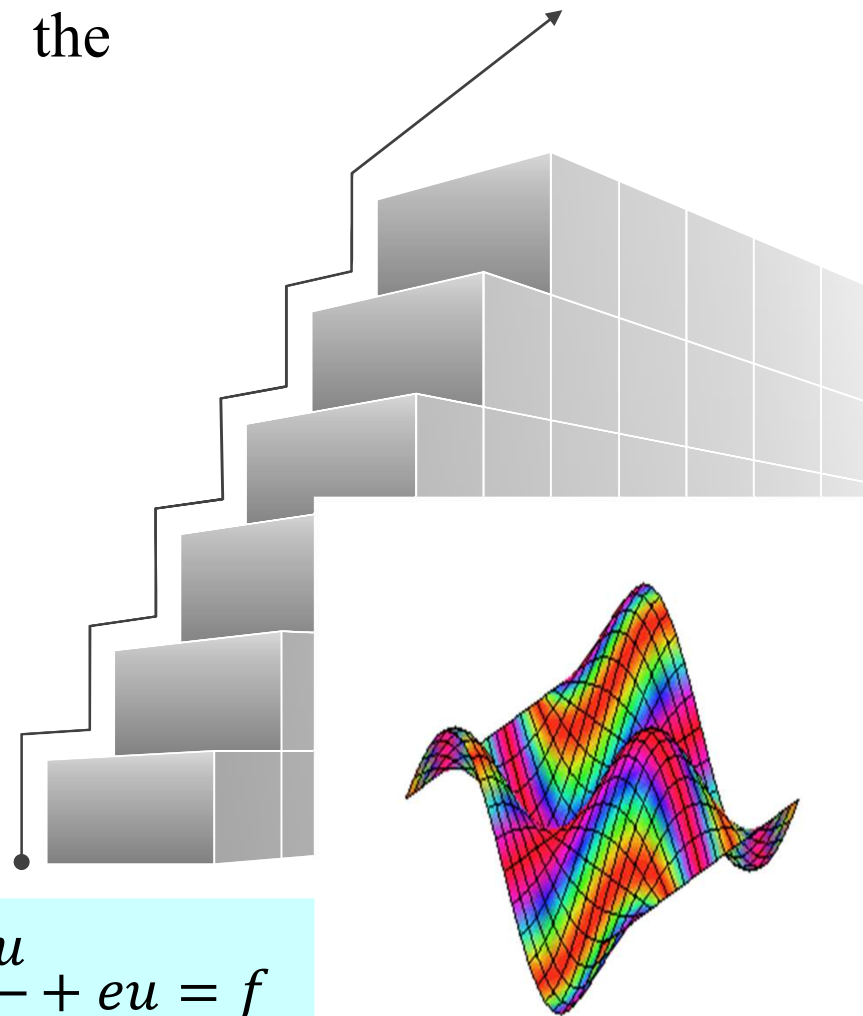

-2 0 2
0
-5
-10
-15
2 0 -2


x[t]
15
10
5
-5
-10
-15


y(t)
y1(t)
y1(T)
y(T)
10
-10
-10
0
10
x(t),x1(t),x1(T),x(T)

Ordinary Differential Equations

- The derivatives are of the dependent variable with respect to the independent variable



- First order differential equation with y as the dependent variable and x as the independent variable would be:
  - dy/dx = du/dx = f(x,y)

$$a\frac{d^n u}{dx^n} + b\frac{d^{n-1} u}{dx^{n-1}} + c\frac{d^{n-2} u}{dx^{n-2}} + \ldots + d\frac{du}{dx} + eu = f$$

Ordinary Differential Equations

- A second order differential equation would have the form:

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

does not necessarily have to include all of these variables

Ordinary Differential Equations

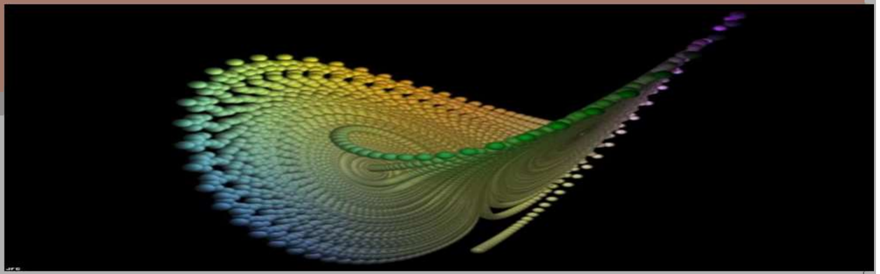
An ordinary differential equation is one with a single independent variable.

- The following is not:

$$\frac{dy}{dx_1} = f(x_1, x_2, y)$$

Ordinary Differential Equations

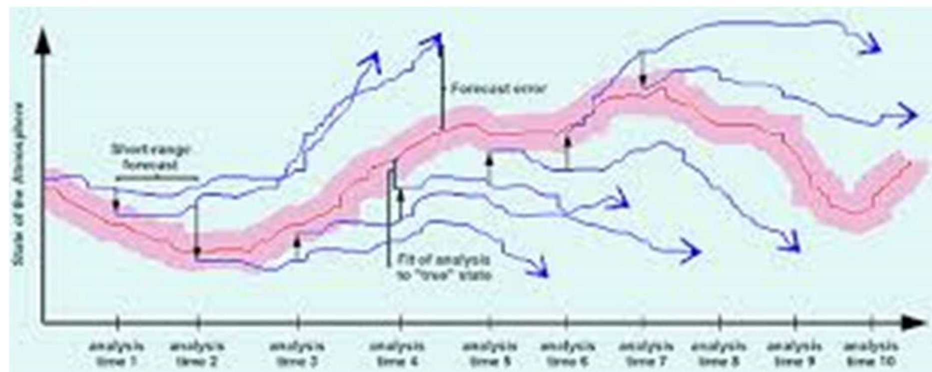
The analytical solution of ordinary differential equation as well as partial differential equations is called the “closed form solution”



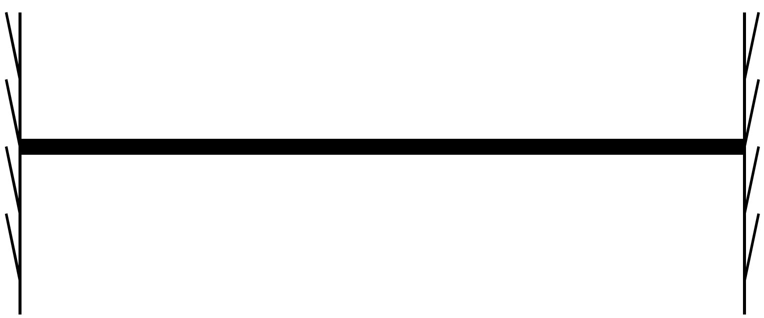
This solution requires that the constants of integration be evaluated using prescribed values of the independent variable(s).

Ordinary Differential Equations

- **An ordinary differential equation of order *n* requires that *n* conditions be specified.**

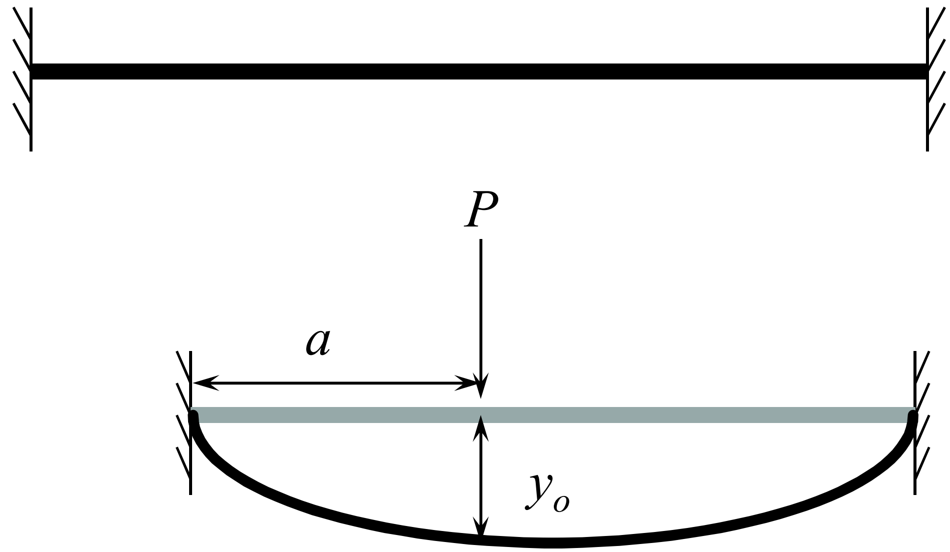


- Boundary conditions
- Initial conditions



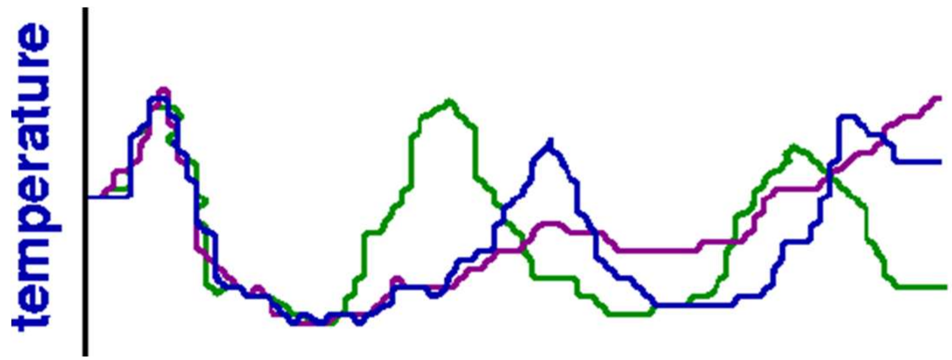
consider this beam where the deflection is zero at the boundaries $x = 0$ and $x = L$
These are boundary conditions

[redacted]



Consider this beam where the deflection is zero at the boundaries $x = 0$ and $x = L$

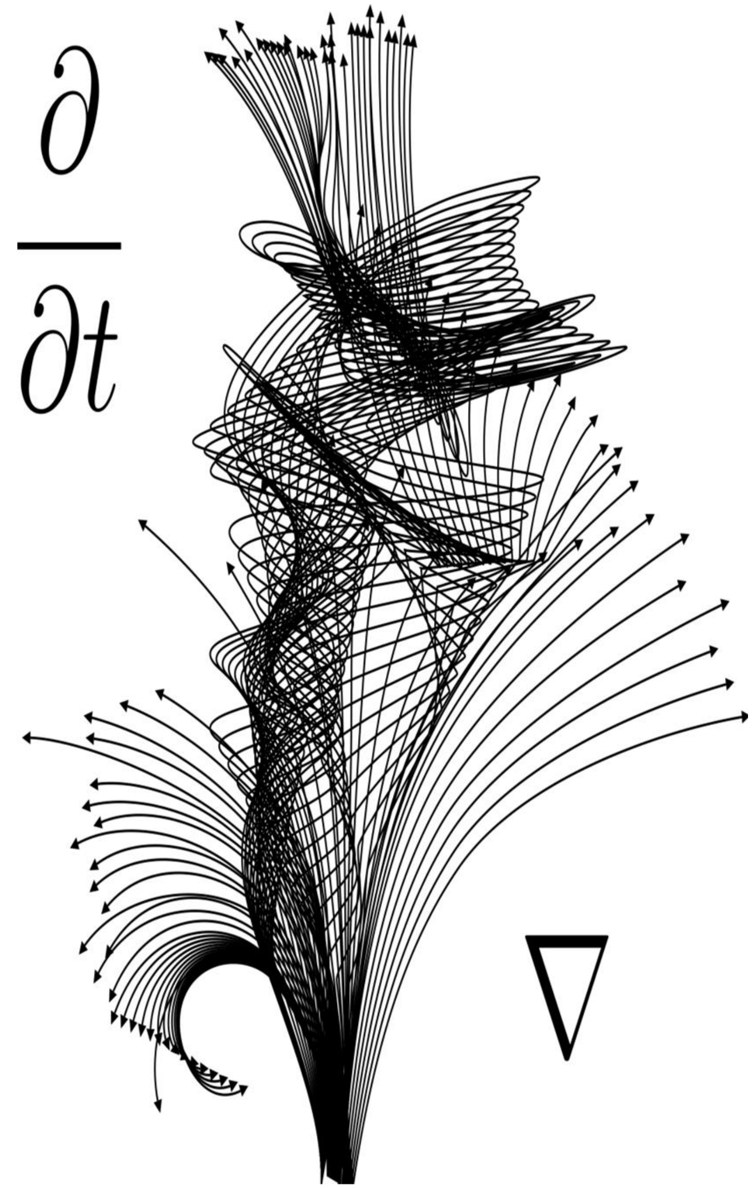
These are boundary conditions



In some cases, the specific behavior of a system(s) is known at a particular time. Consider how the deflection of a beam at $x = a$ is shown at time $t = 0$ to be equal to y_o. Being interested in the response for $t > 0$, this is called the initial condition.

Ordinary Differential Equations

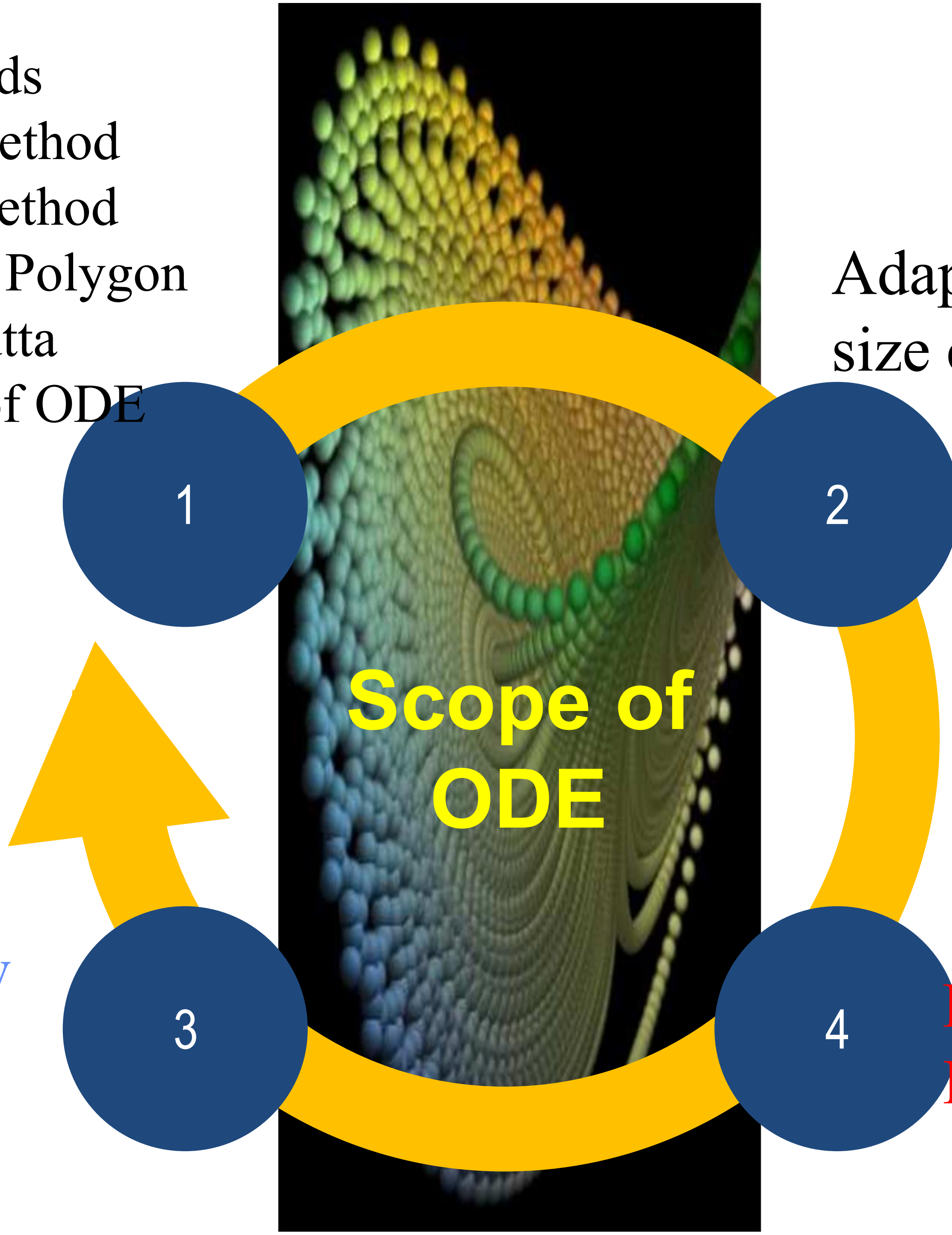
- At best, only a few differential equations can be solved analytically in a closed form.
- Solutions of most practical engineering problems involving differential equations require the use of numerical methods.



One Step Methods

- Euler’s Method
- Heun’s Method
- Improved Polygon
- Runge Kutta
- Systems of ODE

Adaptive step size control



Boundary Value Problems

Initial Valu Problems

Specific Study Objectives

- Understand the visual representation of Euler's, Heun's and the improved polygon methods.
- Understand the difference between local and global truncation errors.
- Know the general form of the Runge-Kutta methods.
- Understand the derivation of the second-order RK method and how it relates to the Taylor series expansion.

Specific Study Objectives

Realize that there are an infinite number of possible versions for second- and higher-order RK methods

Know how to apply any of the RK methods to systems of equations

Understand the difference between initial value and boundary value problems

Review of Analytical Solution

$$\frac{dy}{dx} = 4x^2$$

$$\int dy = \int 4x^2 dx$$

$$y = \frac{4x^3}{3} + C$$

At this point lets consider initial conditions. y(0)=1

and
y(0)=2
and
y(0)=3
and
y(0)=4

$$y = \frac{4x^3}{3} + C$$

for $y(0) = 1$

$$1 = \frac{4(0)^3}{3} + C$$

then $C = 1$

for $y(0) = 2$

$$2 = \frac{4(0)^3}{3} + C$$

and $C = 2$

What one can see are different values of C for the two different initial conditions.

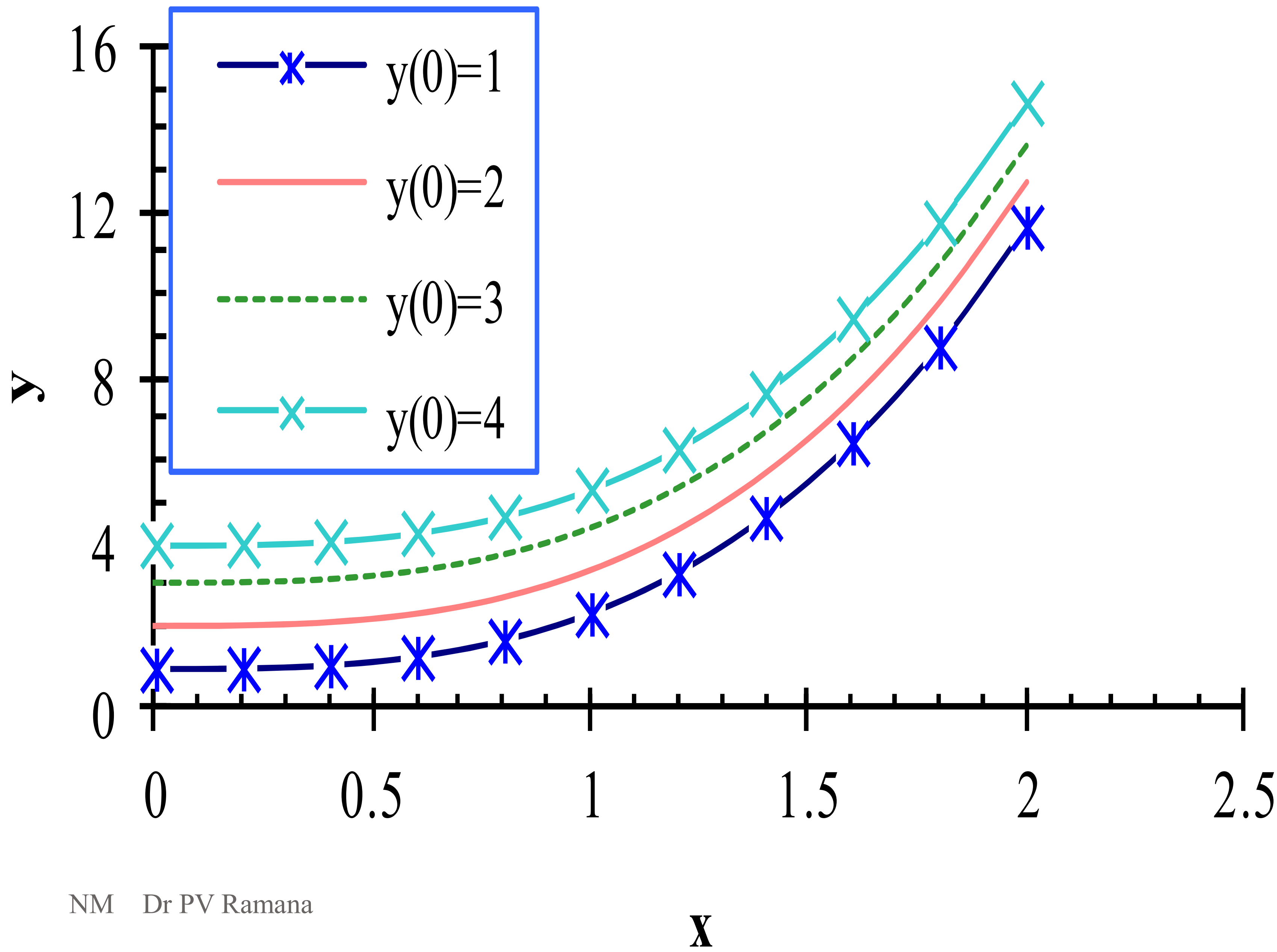
The resulting equations are:

$$y = \frac{4x^3}{3} + 1$$

$$y = \frac{4x^3}{3} + 2$$

$$y = \frac{4x^3}{3} + 3$$

$$y = \frac{4x^3}{3} + 4$$



y(0)=1
y(0)=2
y(0)=3
y(0)=4
16
12
8
4
0
y
0
0.5
1
1.5
2
2.5
x

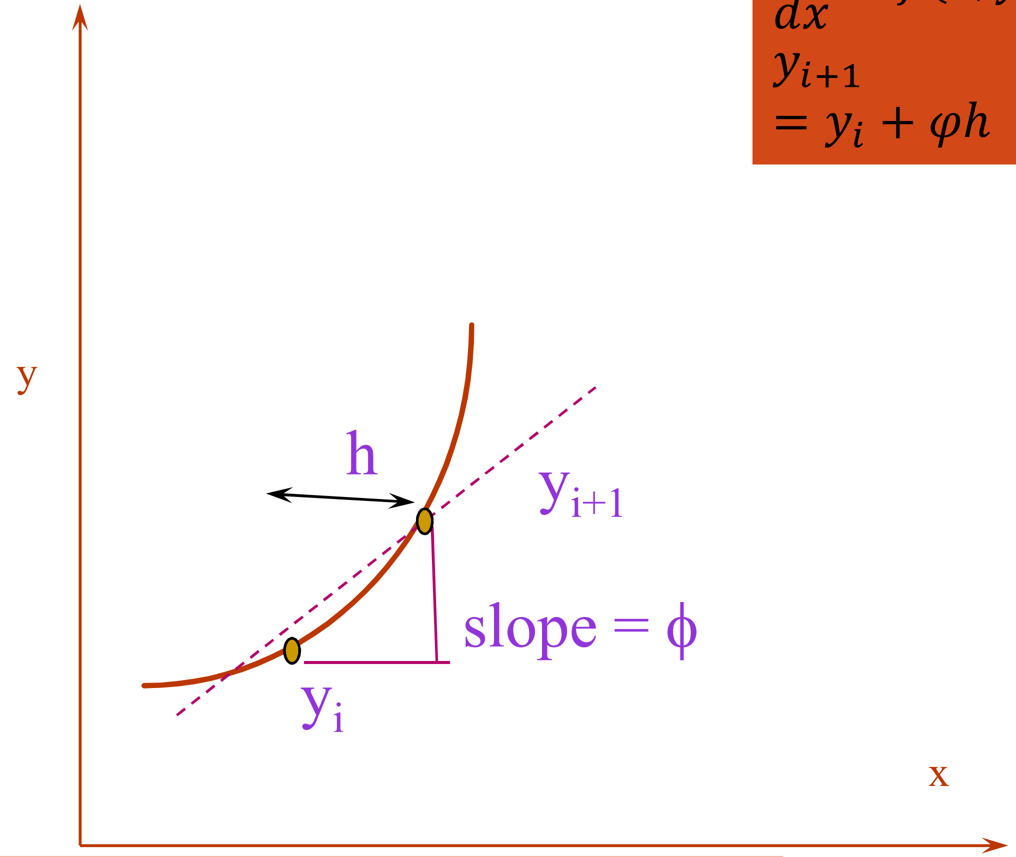
One Step Methods

- Focus is on solving ODE in the form

$$\frac{dy}{dx} = f(x,y)$$

$$y_{i+1} = y_i + \phi h$$

$$\frac{dy}{dx} = f(x,y)$$
$$y_{i+1} = y_i + \varphi h$$



This is the same as saying:
new value = old value + slope x step size

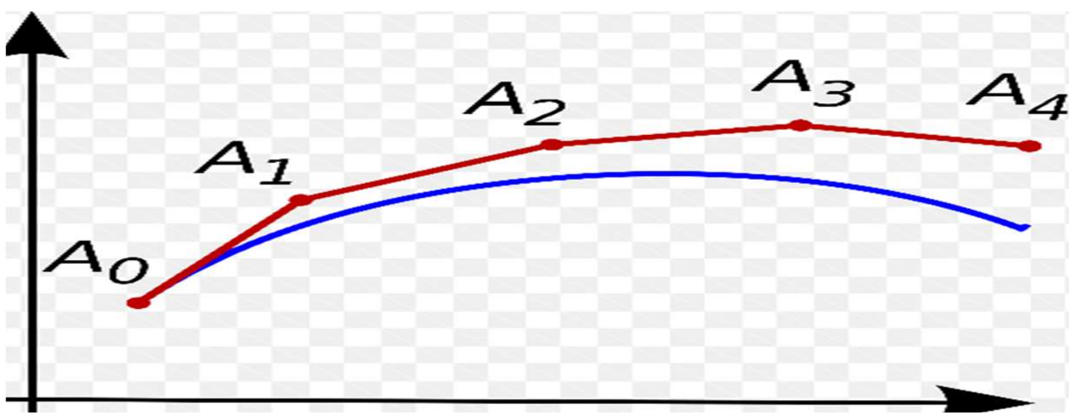
Euler's Method

$$\frac{dy}{dx} = f(x, y)$$
$$y_{i+1} = y_i + \varphi h$$

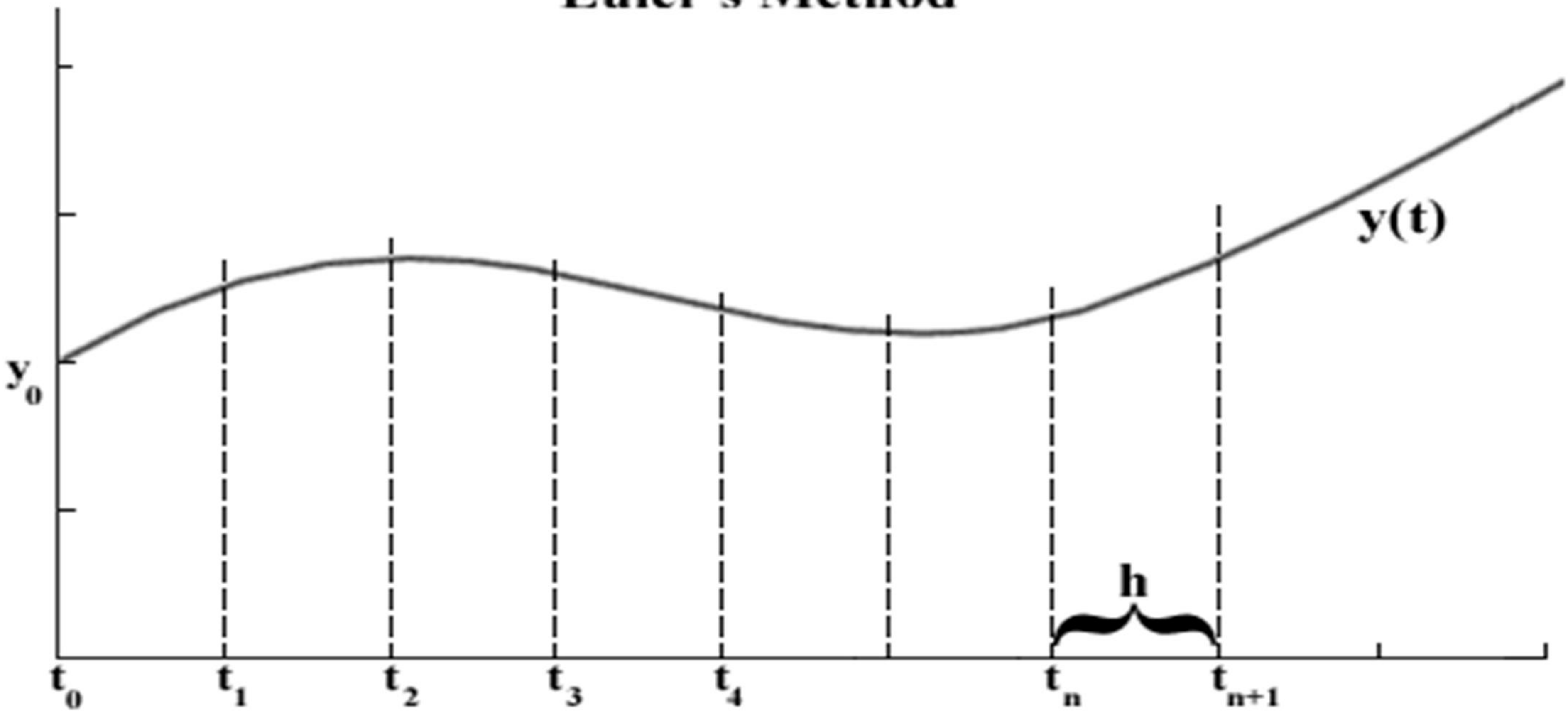
The first derivative provides a direct estimate of the slope at x_i

The equation is applied iteratively, or one step at a time, over small distance in order to reduce the error

Hence this is often referred to as Euler's One-Step Method



Euler's Method



$$\frac{dy}{dt} = f(t,y)$$

$$y(t_0) = y_0$$

y(t) is the solution of this differential equation

This is Euler's formula to approximate the solutions.

$$\boxed{y_{n+1} = y_n + hf(t_n, y_n)}$$

EXAMPLE

For the initial condition y(1)=1, determine y for h = 0.1 analytically and using Euler's method given:

$$\frac{dy}{dx} = 4x^2$$

First Order Taylor Series

$$y_{i+1} = y_i + f(x_i, y_i)h$$

$$dy = 4x^2 dx$$

$$\int dy = \int 4x^2 dx \Rightarrow y = \frac{4x^3}{3} + c_1$$

$$y(1) = \frac{4(1)^3}{3} + c_1 \Rightarrow c_1 = 1 - \frac{4}{3} = -\frac{1}{3}$$

$$y = \frac{4x^3}{3} - \frac{1}{3}$$

Error Analysis for Euler's Method

- Numerical solutions of ODEs involves two types of error:
  - *Truncation* error
    - ***Local*** *truncation error*
    - *Propagated truncation error*

      The sum of the two is the *total or **global** truncation error*
  - *Round-off* errors (due to limited digits in representing numbers in a computer)
- One can use Taylor series to quantify the ***local truncation error*** in Euler's method.

Given $y' = f(x, y)$

$$y_{i+1} = y_i + y_i' h + \frac{y_i''}{2!} h^2 + \ldots + \frac{y_i^{(n)}}{n!} h^n + R_n$$

$$\boxed{y_{i+1} = y_i + f(x_i, y_i) h} + \boxed{\frac{f'(x_i, y_i)}{2!} h^2 + \ldots + \frac{f^{(n-1)}(x_i, y_i)}{n!} h^n + O(h^{n+1})}$$

EULER

Local Truncation ERROR

$$E_a = \frac{f'(x_i, y_i)}{2!} h^2 + R_3$$

$$E_a \cong \frac{f'(x_i, y_i)}{2!} h^2 = O(h^2)$$

- **The error is reduced by 4 times if the step size is halved ➔ *O(h²)*.**
- In real problems, the derivatives used in the Taylor series are not easy to obtain.
- If the solution to the differential equation is *linear*, the method will provide error free predictions (2nd derivative is **zero** for a straight line).

Euler's Method

Recall that a first order initial value problem has the form

$$\frac{dy}{dt} = f(t, y), y(t_0) = y_0$$

If f and $\partial f / \partial y$ are continuous, then this IVP has a unique solution $y = \phi(t)$ in some interval about t_0.

When the differential equation is linear, separable or exact, one can find the solution by symbolic manipulations.

Solutions for most differential equations of this form cannot be found by analytical means.

Therefore it is important to be able to approach the problem in other ways.

Euler's Method

For our first order initial value problem

$$y' = f(t, y), y(t_0) = y_0,$$

an alternative is to compute approximate values of the solution $y = \phi(t)$ at a selected set of t-values.

Ideally, the approximate solution values will be accompanied by error bounds that ensure the level of accuracy.

There are many numerical methods that produce numerical approximations to solutions of differential equations.

This method also call tangent line method.

Euler's Method

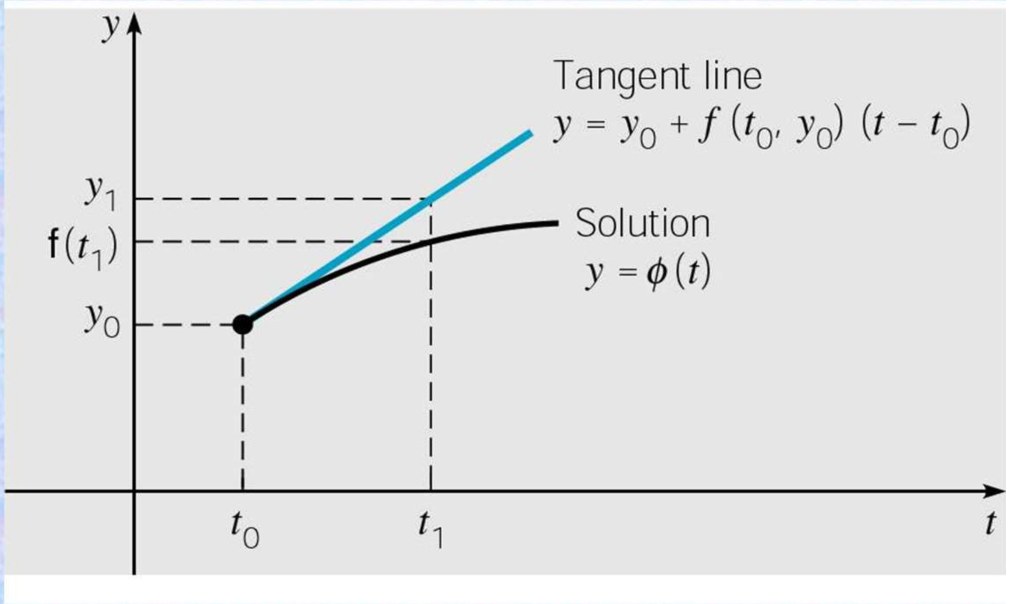
- **For the initial value problem** $y' = f(t, y), y(t_0) = y_0,$

  one can begin by approximating solution $y = \phi(t)$ at initial point t_0.
- **The solution passes through initial point (t_0, y_0) with slope $f(t_0,y_0)$. The line tangent to solution at initial point is thus**

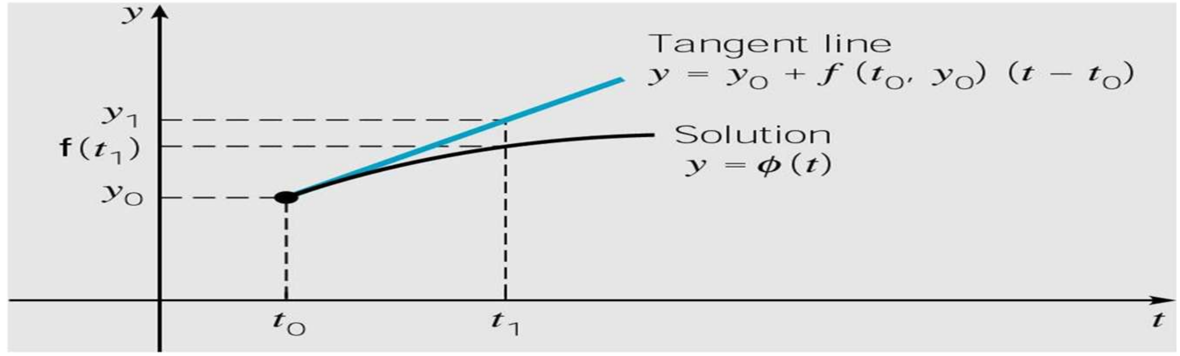
$$y = y_0 + f(t_0, y_0)(t - t_0)$$

- **The tangent line is a good approximation to solution curve on an interval short enough.**
- **Thus if t_1 is close enough to t_0, one can approximate $\phi(t_1)$ by**

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$



Euler's Formula



- For a point t_2 close to t_1, one can approximate $\phi(t_2)$ using the line passing through (t_1, y_1) with slope $f(t_1, y_1)$:

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$

- Thus one can create a sequence y_n of approximations to $\phi(t_n)$:

$$\begin{aligned} y_1 &= y_0 + f_0 \cdot (t_1 - t_0) \\ y_2 &= y_1 + f_1 \cdot (t_2 - t_1) \\ &\vdots \\ y_{n+1} &= y_n + f_n \cdot (t_{n+1} - t_n) \end{aligned}$$

where $f_n = f(t_n, y_n)$.

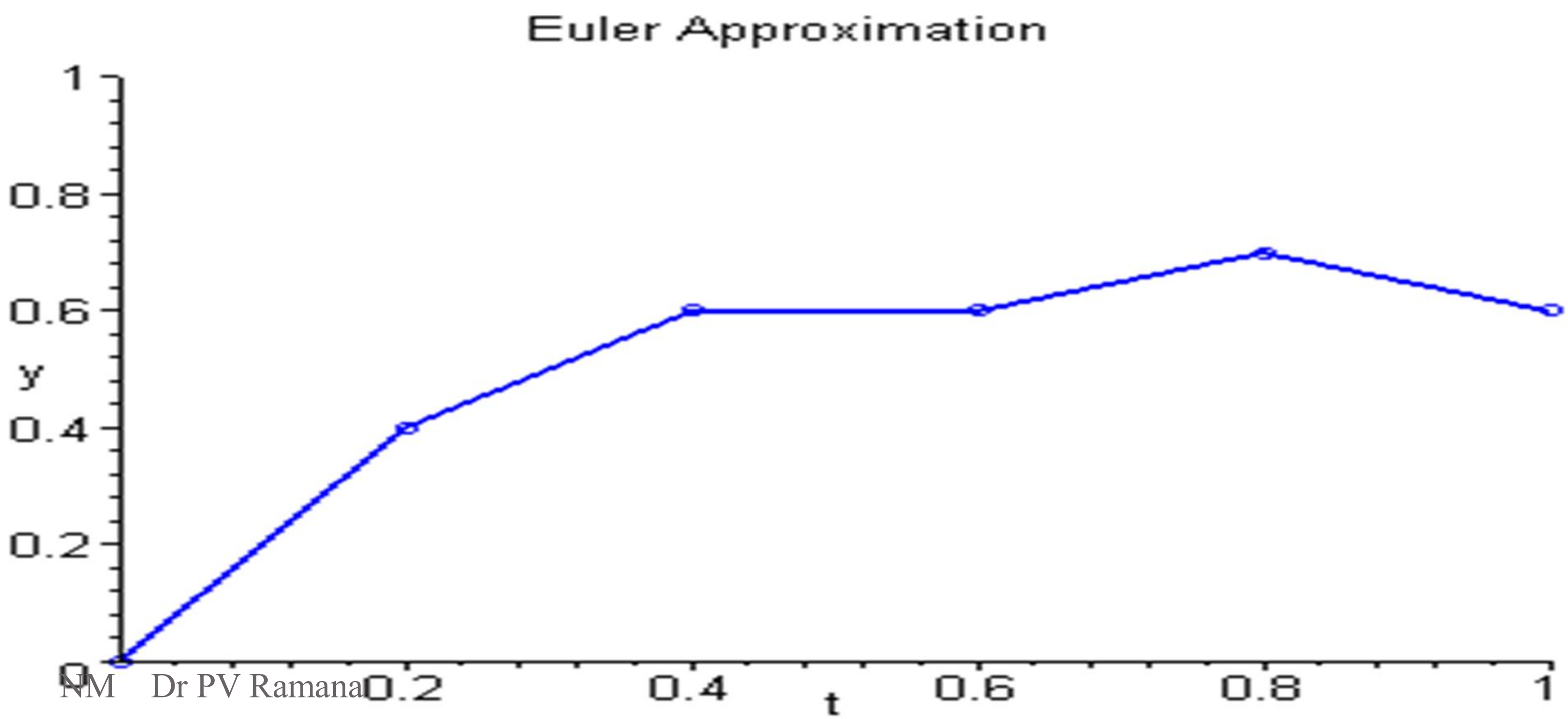
- For a uniform step size $h = t_n - t_{n-1}$, Euler's formula becomes

$$y_{n+1} = y_n + f_n h, n = 0,1,2, \ldots$$

To graph an Euler approximation, one canplot the points (t_0, y_0), (t_1, y_1),…, (t_n, y_n), and then connect these points with line segments.

Euler Approximation

$$y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n), \text{where } f_n = f(t_n, y_n)$$



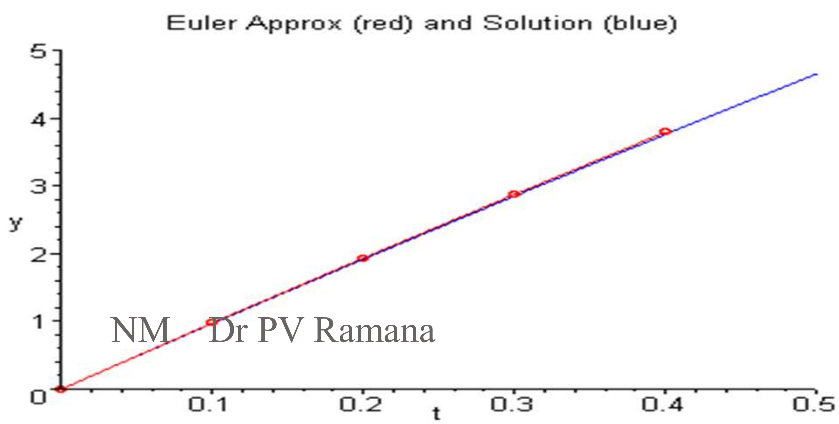
Example 1: Euler's Method

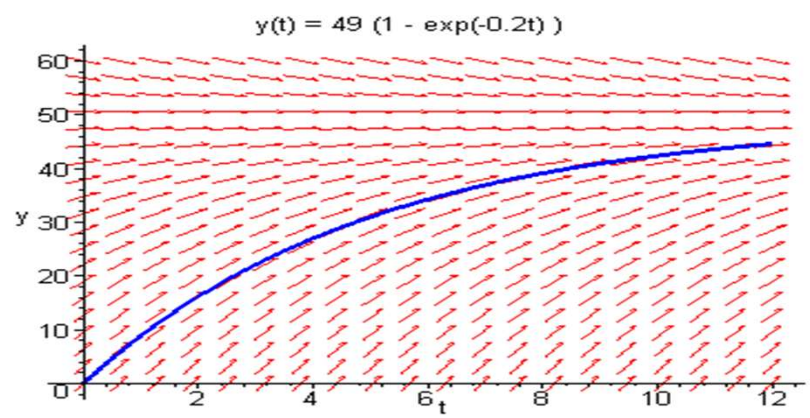
$$\frac{dy}{dx} = \varphi$$
$$y_{i+1} = y_i + \varphi h$$

- **For the initial value problem** $\dot{y} = 9.8 - 0.2y, y(0) = 0$

One can use Euler's method with $h = 0.1$ to approximate the solution at $t = 0.1, 0.2, 0.3, 0.4$, as shown below.

$$y_1 = y_0 + f_0 \cdot h = 0 + \big(9.8 - (0.2)(0)\big)(0.1) = .98$$
$$y_2 = y_1 + f_1 \cdot h = .98 + \big(9.8 - (0.2)(.98)\big)(0.1) \approx 1.94$$
$$y_3 = y_2 + f_2 \cdot h = 1.94 + \big(9.8 - (0.2)(1.94)\big)(0.1) \approx 2.88$$
$$y_4 = y_3 + f_3 \cdot h = 2.88 + \big(9.8 - (0.2)(2.88)\big)(0.1) \approx 3.80$$





Example 1: Euler's Method

- The exact solution:

$$y' = 9.8 - 0.2y, \quad y(0) = 0$$

$$y' = -0.2(y - 49)$$

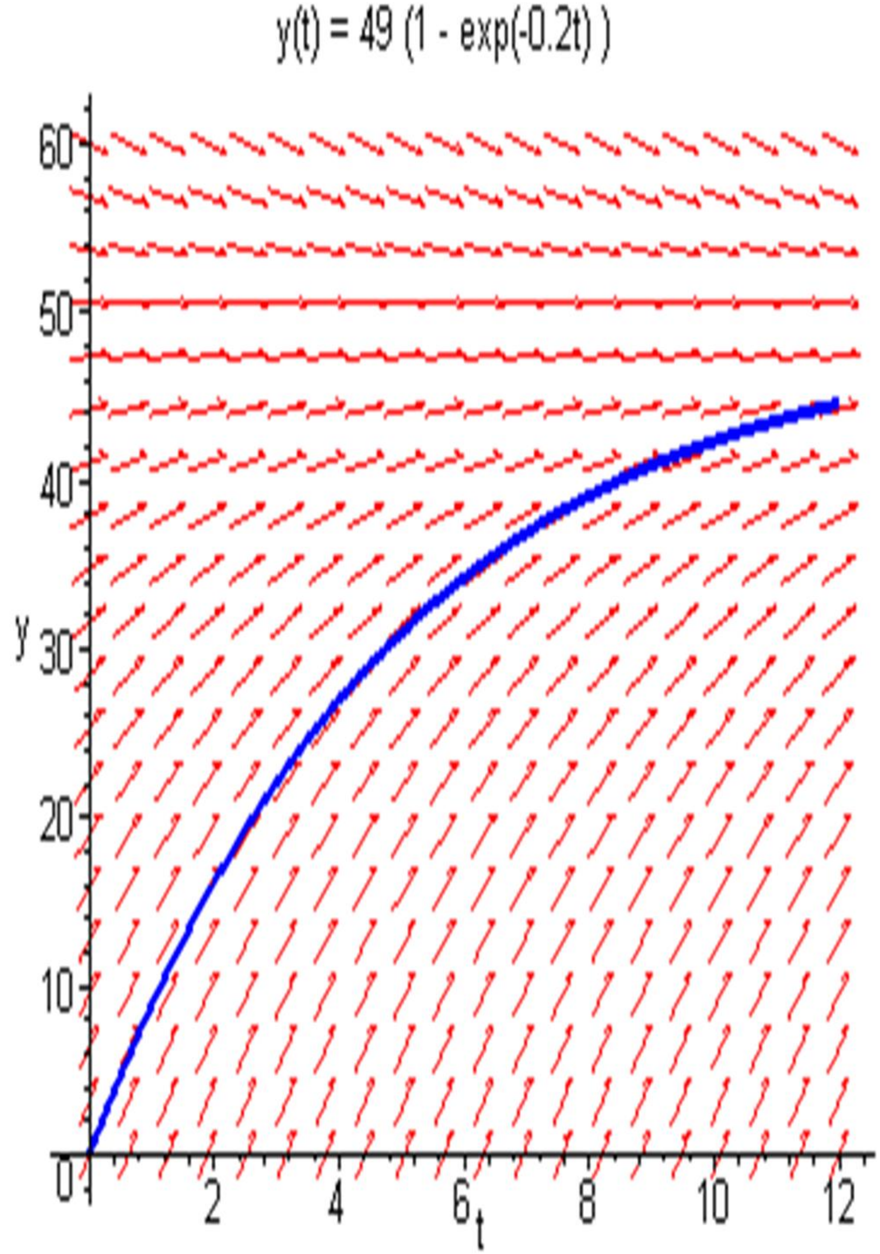
$$\frac{dy}{y - 49} = -0.2dt$$

$$\ln|y - 49| = -0.2t + C$$

$$y = 49 + ke^{-0.2t}, \quad k = \pm e^{C}$$

$$y(0) = 1 \Rightarrow k = -49$$

$$\Rightarrow y = 49\left(1 - e^{-0.2t}\right)$$

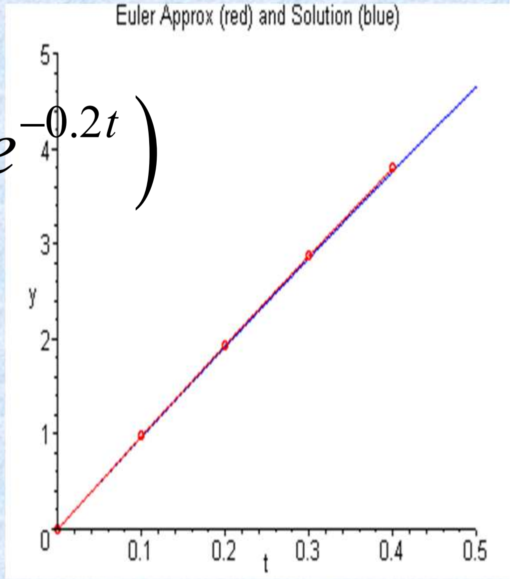


Example 1: Euler's Method Error Analysis

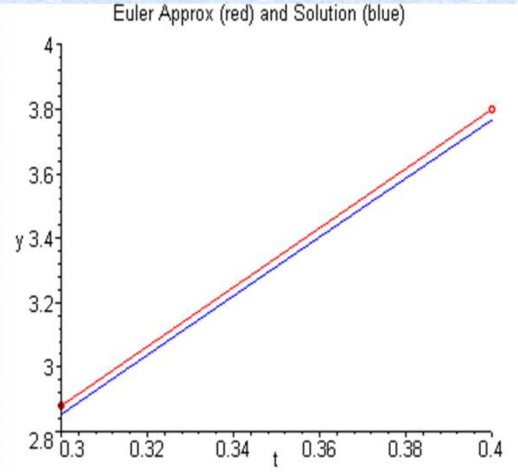
- The errors are small. This is most likely due to round-off error and the fact that the exact solution is approximately linear on [0, 0.4].

$$\text{Percent Relative Error} = \frac{y_{exact} - y_{approx}}{y_{exact}} \times 100$$

$$y = 49\left(1 - e^{-0.2t}\right)$$



t	Exact y	Approx y	Error	% Rel Error
0.00	0	0.00	0.00	0.00
0.10	0.97	0.98	-0.01	-1.03
0.20	1.92	1.94	-0.02	-1.04
0.30	2.85	2.88	-0.03	-1.05
0.40	3.77	3.8	-0.03	-0.80



Example 2: Euler's Method

$y_{i+1} = y_i + \varphi h$

- For the initial value problem

$$\dot{y} = 4 - t + 2y, y(0) = 1$$

Use Euler's method with $h = 0.1$ to approximate the solution at $t = 1, 2, 3,$ and 4, as shown below.

$$y_1 = y_0 + f_0 \cdot h = 1 + (4 - 0 + (2)(1))(0.1) = 1.6$$
$$y_2 = y_1 + f_1 \cdot h = 1.6 + (4 - 0.1 + (2)(1.6))(0.1) = 2.31$$
$$y_3 = y_2 + f_2 \cdot h = 2.31 + (4 - 0.2 + (2)(2.31))(0.1) \approx 3.15$$
$$y_4 = y_3 + f_3 \cdot h = 3.15 + (4 - 0.3 + (2)(3.15))(0.1) \approx 4.15$$
$$\vdots$$

- Exact solution

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

Euler's Method

$$y_{i+1} = y_i + \varphi h$$

- The exact solution:

$$\dot{y} = 4 - t + 2y, y(0) = 1$$
$$\dot{y} = 4 + 2y - t$$

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

Euler's Method Error Analysis

$y_{i+1} = y_i + \varphi h$

- The first ten Euler approxs are given in table below on left. A table of approximations for $t = 0, 1, 2, 3$ is given on right.
- The errors are small initially, but quickly reach an unacceptable level. This suggests a nonlinear solution.

Exact Solution:

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
0.10	1.66	1.60	0.06	3.55
0.20	2.45	2.31	0.14	5.81
0.30	3.41	3.15	0.26	7.59
0.40	4.57	4.15	0.42	9.14
0.50	5.98	5.34	0.63	10.58
0.60	7.68	6.76	0.92	11.96
0.70	9.75	8.45	1.30	13.31
0.80	12.27	10.47	1.80	14.64
0.90	15.34	12.89	2.45	15.96
1.00	19.07	15.78	3.29	17.27

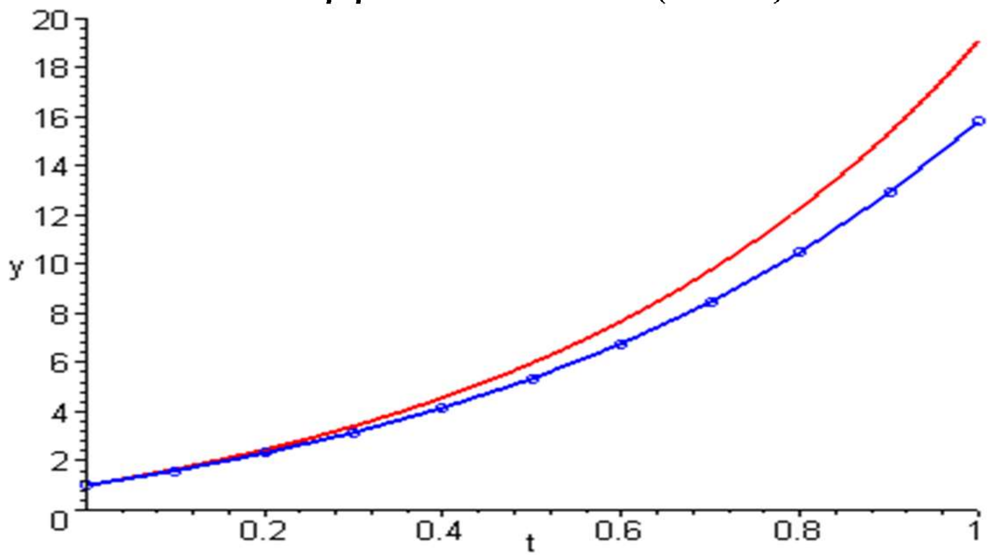
t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

Euler's Method Error Analysis

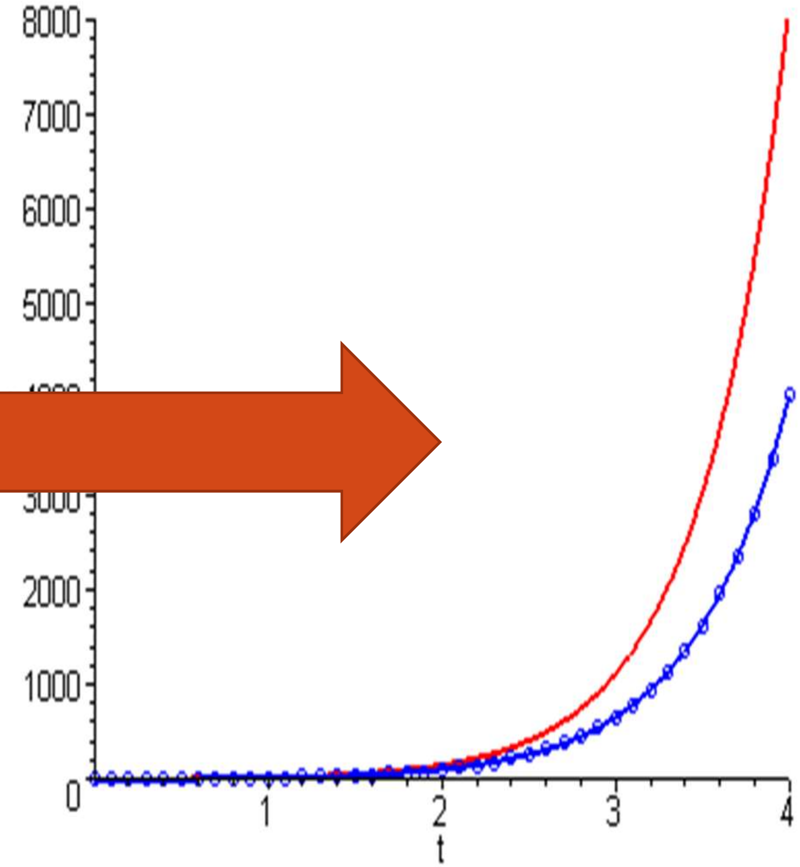
- Given below are graphs showing the exact solution (red) plotted together with the Euler approximation (blue).

Exact Solution:

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$



t	Exact y	Approx y	Error	% Rel Err
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69



Euler's Method EX3

Solve numerically : $\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$

From **x=0** to **x=4** with step size **h=0.5**

initial condition: (x=0 ; y=1)

<u>Exact Solution</u>: $y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$

Numerical Solution: $y_{i+1} = y_i + f(x_i, y_i)h$

y(0.5) = y(0)+*f*(0, 1)0.5 = 1+8.5*0.5 = 5.25

(true solution at x=0.5 is y(0.5) = 3.22 and ε_t = 63%)

y(1) = y(0.5)+*f*(0.5, 5.25)0.5

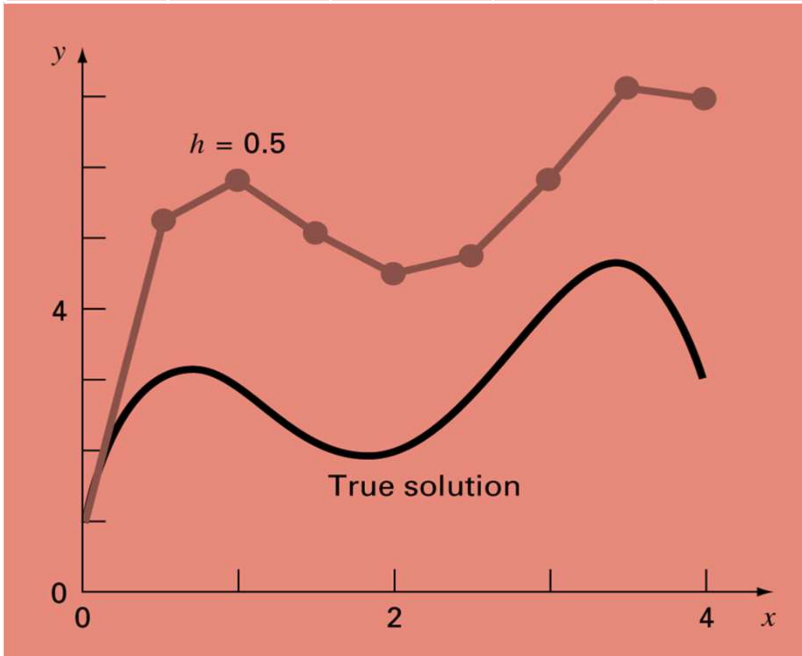
= $5.25+[-2(0.5)^3+12(0.5)^2-20(0.5)+8.5]*0.5$

= 5.25+0.625 = 5.875

(true solution at x=1 is y(1) = 3 and ε_t = 96%)

y(1.5) = y(1)+*f*(1, 5.875)0.5 = 5.125

X	y_{euler}	y_{true}	Error Global	Error Local
0	1	1	%	%
0.5	5.250	3.218	63.1	63.1
1.0	5.875	3.000	95.8	28
1.5	5.125	2.218	131.0	1.41
2.0	4.500	2.000	125.0	20.5

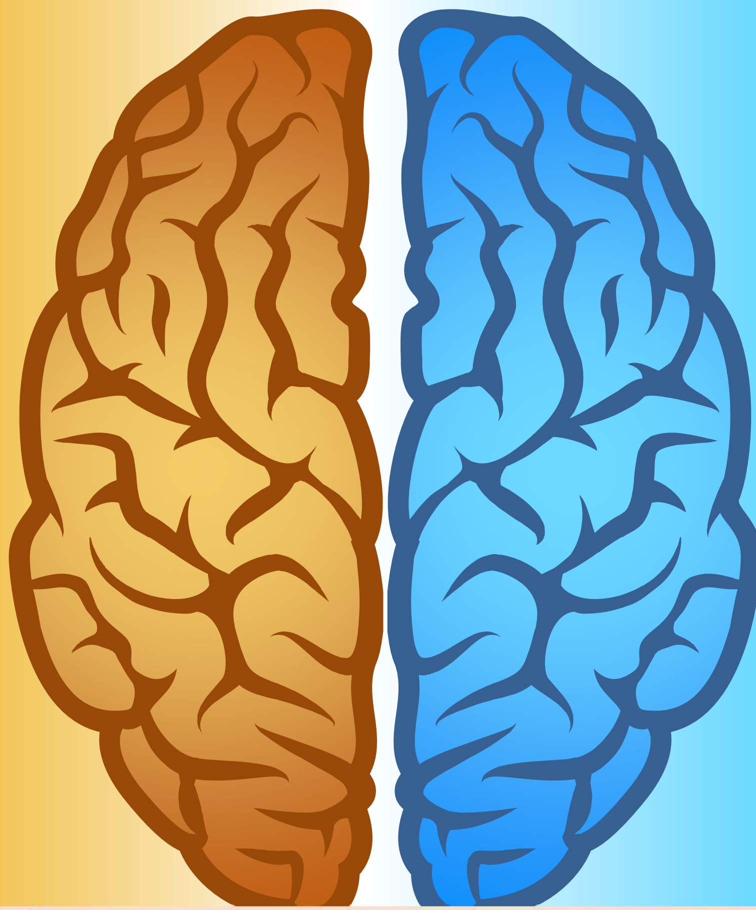


Error Analysis of Euler’s Method

Truncation error

caused by the nature of the techniques employed to approx values of *y*

- local truncation error (from Taylor Series)
- propagated truncation error
- sum of the two = global truncation error



Round off error

caused by the limited number of significant digits that can be retained by a computer or calculator

Higher Order Taylor Series Methods

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{f'(x_i, y_i)}{2}h^2$$

$$y_{i+1} = y_i + \varphi h + \varphi' \, h^2/_{2!}$$

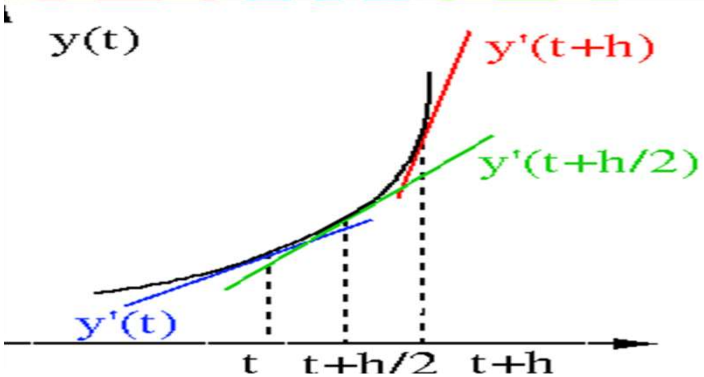
- **This is simple enough to implement with polynomials**
- **Not so trivial with more complicated ODE**
- **In particular, ODE that are functions of both dependent and independent variables require chain-rule differentiation**
- **Alternative one-step methods are needed**

$$+\frac{f(t)''\Delta t^2}{2}+\frac{f(t)'''\Delta t^3}{3!}+$$

A fundamental error in Euler's method is that the derivative at the beginning of the interval is assumed to apply across the entire interval

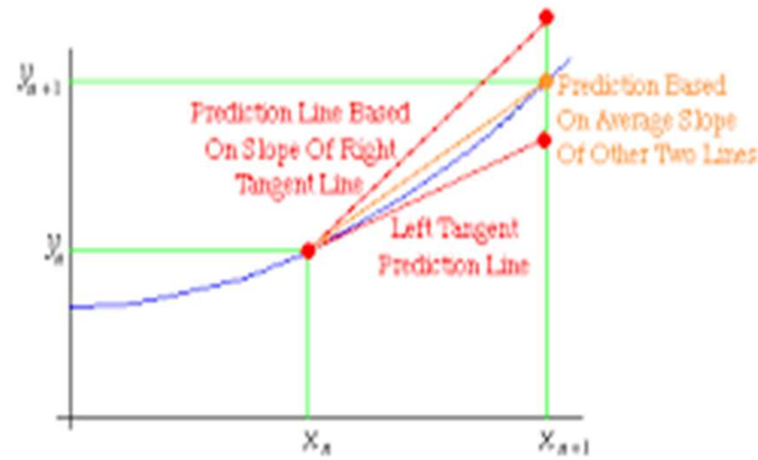
Modification of Euler's Methods

Two simple modifications will be demonstrated



These modification actually belong to a larger class of solution techniques called Runge-Kutta which one can will explore later.

Heun's Method



- Consider our Taylor expansion

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{f'(x_i, y_i)}{2} h^2$$

- Approximate f' as a simple forward difference

$$f'(x_i, y_i) \cong \frac{f(x_{i+1}, y_{i+1}) - f(x_i, y_i)}{h}$$

- Substituting into the expansion

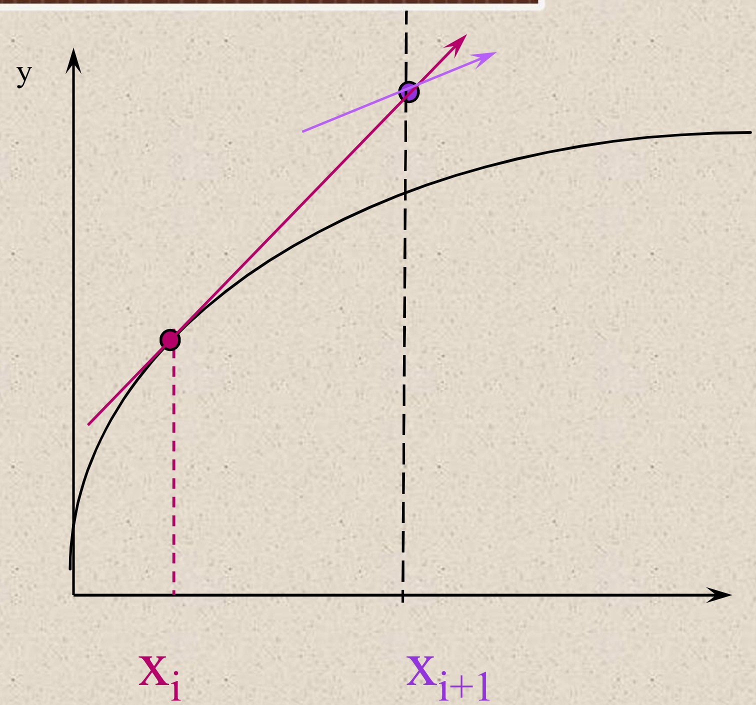
$$y_{i+1} = y_i + f_i h + \left(\frac{f_{i+1} - f_i}{h}\right) \rightleftarrows \frac{h^2}{2} = y_i + \left(\frac{f_{i+1} + f_i}{2}\right) \rightleftarrows h$$

Heun’s Method

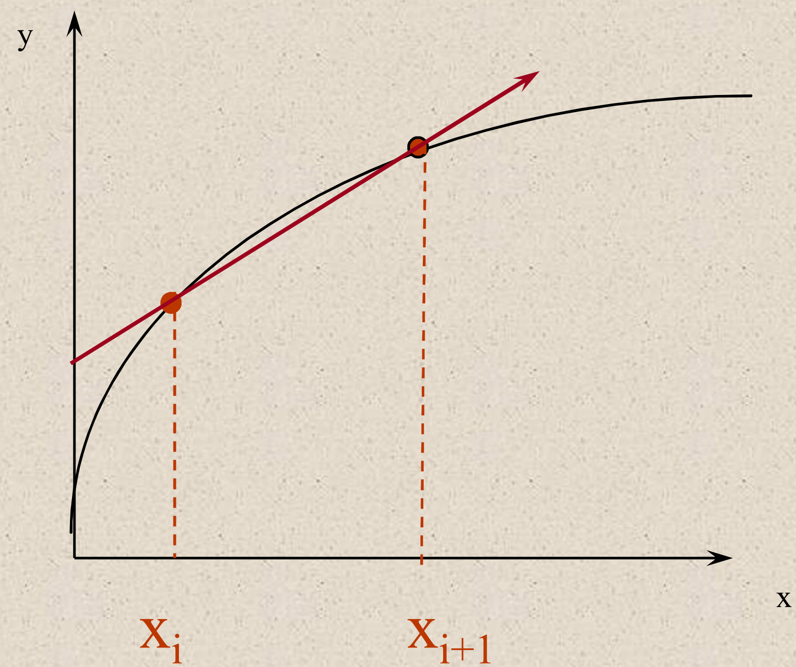
$$y_{i+1} = y_i + f_i h + \left(\frac{f_{i+1} - f_i}{h}\right) \rightleftarrows \frac{h^2}{2} = y_i + \left(\frac{f_{i+1} + f_i}{2}\right) \rightleftarrows h$$

- Determine the derivatives for the interval @
  - the initial point
  - end point (based on Euler step from initial point)
- Use the average to obtain an improved estimate of the slope for the entire interval

Heun's Method



$$y_{i+1} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1})}{2} h$$



Heun's Predictor Corrector Method

Problem Heun's Method

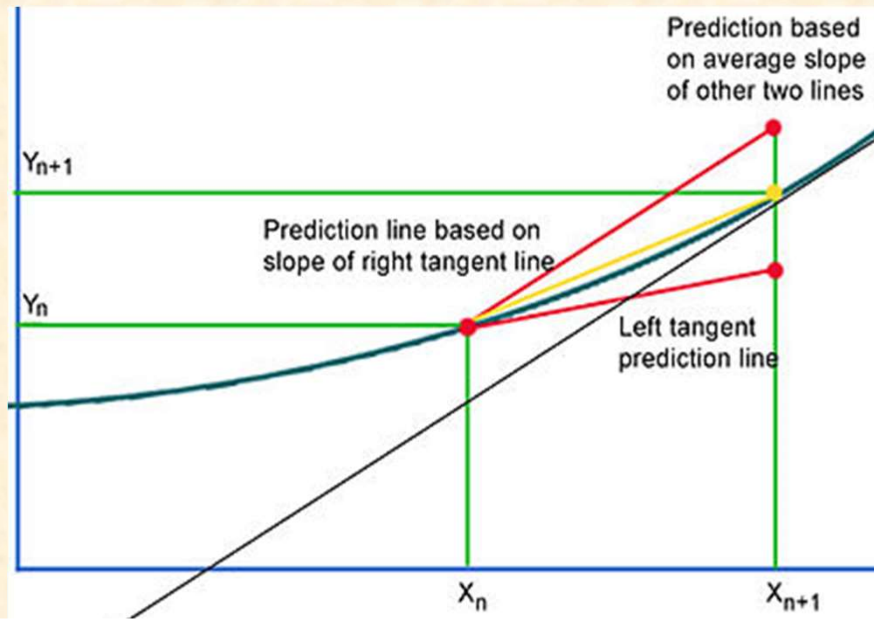
$\dot{y}(x) = f(x,y)$ $y_0 = y(x_0)$

$y(x_0) = y_0$ Predictor: $y_{i+1}^{0} = y_i + hf(x_i, y_i)$

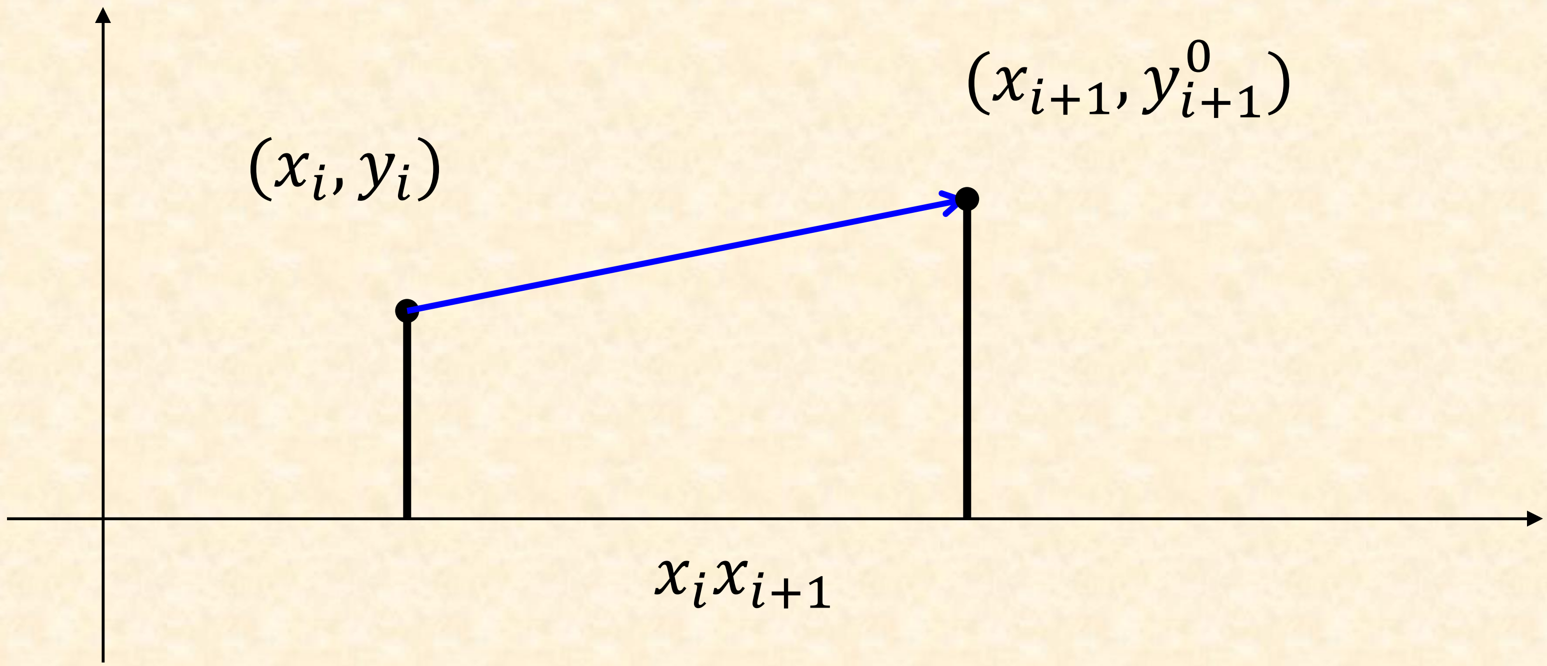
Corrector: $y_{i+1}^{1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{0})\right)$

Local Truncation Error $O(h^3)$

Global Truncation Error $O(h^2)$

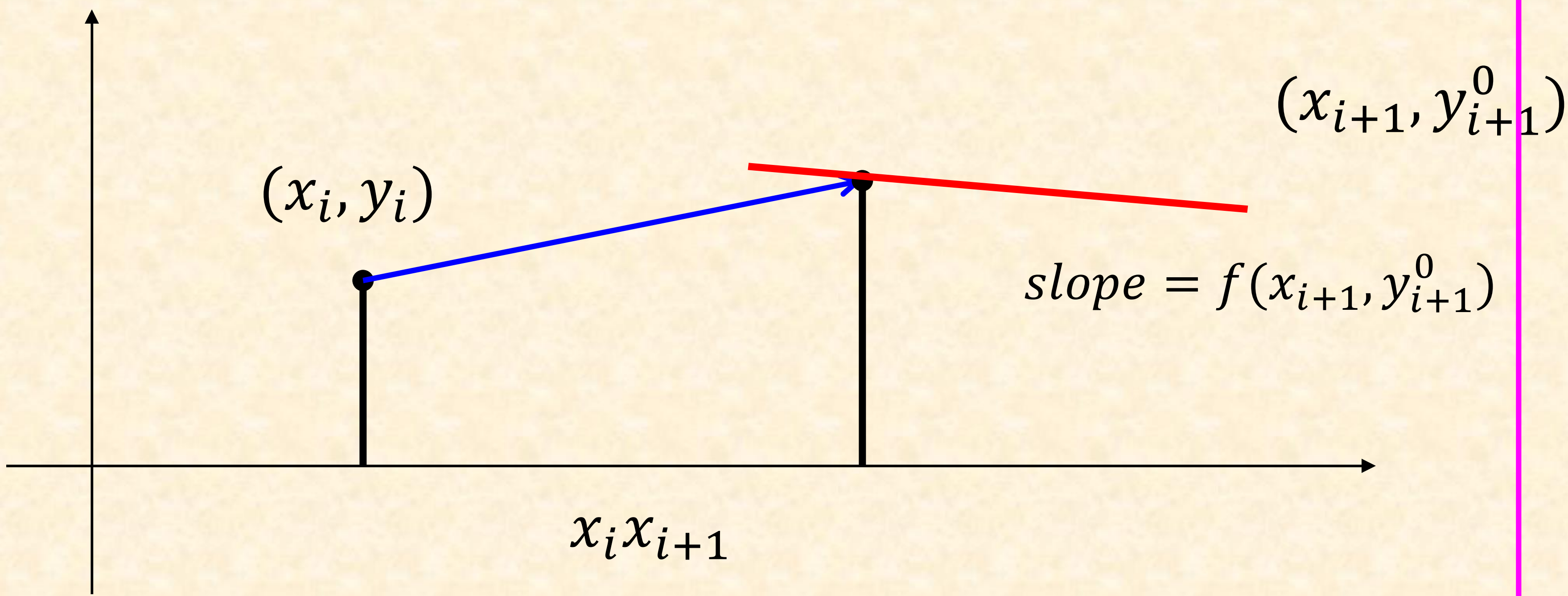


Heun's Predictor Corrector Method



Prediction$y_{i+1}^0 = y_i + hf(x_i, y_i)$

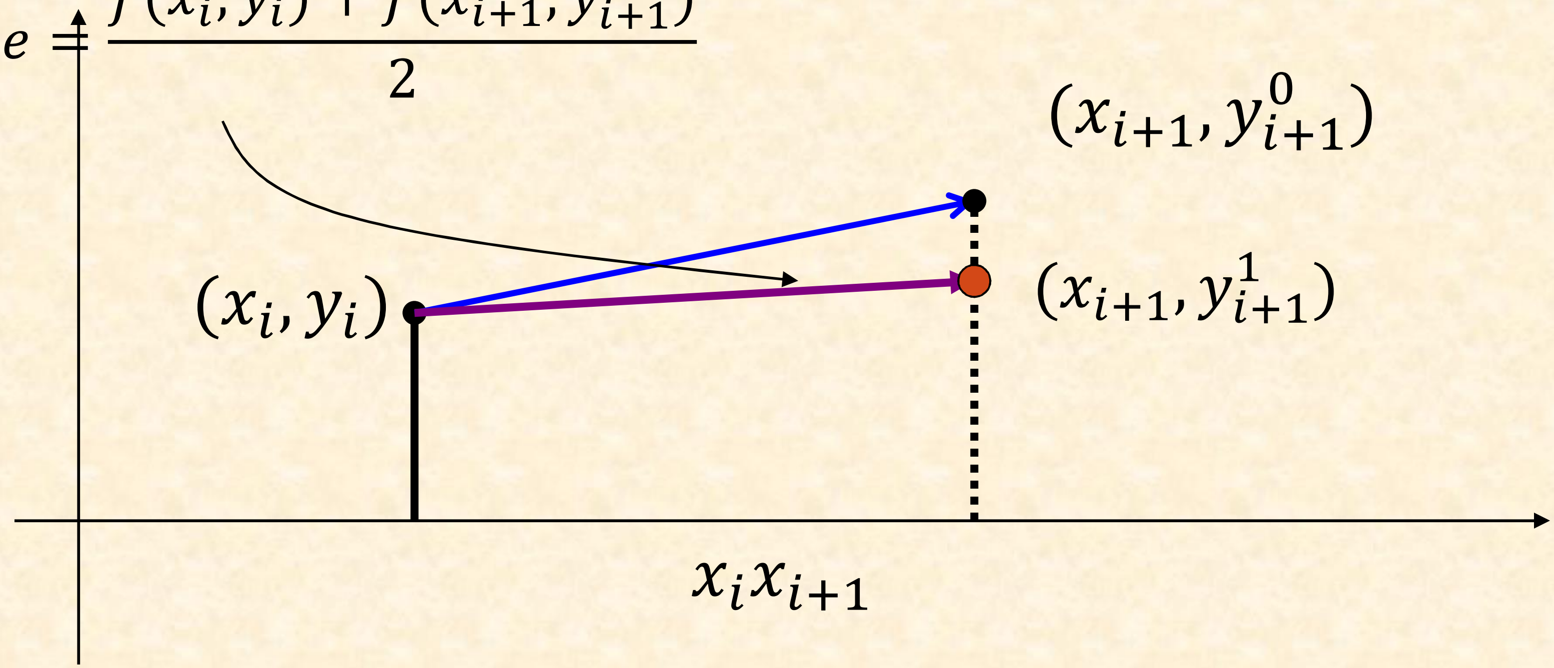
Heun's Predictor Corrector Method



Prediction $y_{i+1}^0 = y_i + hf(x_i, y_i)$

Heun's Predictor Corrector Method

$$slope = \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)}{2}$$



$$y_{i+1}^1 = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)\right)$$

Heun's Predictor Corrector Method

Use the Heun′s Method to solve the ODE

$\dot{y}(x) = 1 + x^2 + y$

$y(0) = 1$

Use $h = 0.1$. One correction only

Determine y(0.1) and y(0.2)

Heun's Predictor Corrector Method

Problem: $f(x,y) = 1 + y + x^2, y_0 = y(x_0) = 1, h = 0.1$

Step1:

Predictor: $y_1^0 = y_0 + hf(x_0, y_0) = 1 + 0.1(2) = 1.2$

Corrector: $y_1^1 = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1^0)) = 1.2105$

Step2:

Predictor: $y_2^0 = y_1 + hf(x_1, y_1) = 1.4326$

Corrector: $y_2^1 = y_1 + \frac{h}{2}(f(x_1, y_1) + f(x_2, y_2^0)) = 1.4452$

$$y_{i+1}^0 = y_i + hf(x_i, y_i) \qquad y_{i+1}^1 = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)\right)$$

Summary

- Euler and Heun's methods are similar in the following sense:

$$y_{i+1} = y_i + h \times slope$$

  - Different methods use different estimates of the slope.
- Heun's method is comparable in accuracy to the second order Taylor series method.

Comparison

Method	Local truncation error	Global truncation error
Euler Method $y_{i+1} = y_i + hf(x_i, y_i)$	$O(h^2)$	$O(h)$
Heun's Method Predictor: $y_{i+1}^{0} = y_i + hf(x_i, y_i)$ Corrector: $y_{i+1}^{k+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{k})\right)$	$O(h^3)$	$O(h^2)$
Midpoint $y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i)$ $y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$	$O(h^3)$	$O(h^2)$

Heun's Method some what better than Euler Method