NUMERICAL METHODS

$$U^{n+1} = U^n + \Delta t\, f(U^n)$$

$$\frac{\partial v}{\partial t} + V \cdot \nabla v = \nabla \cdot (k \nabla v) + g(v)$$

$$(\lambda + \mu)\nabla(\nabla \cdot u) + \mu \nabla^2 u = \alpha(3\lambda + 2\mu)\nabla T - \rho b$$

Lecture 11

$$\rho\left(\frac{\partial u}{\partial t} + V \cdot \nabla u\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$\nabla^2 u = f$$

Dr. P V Ramana

Comparison

Method	Local truncation error	Global truncation error
Euler Method $y_{i+1} = y_i + h\ f(x_i, y_i)$	$O(h^2)$	$O(h)$
Heun's Method Predictor : $y_{i+1}^{0} = y_i + h\ f(x_i, y_i)$ Corrector : $y_{i+1}^{k+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{k})\right)$	$O(h^3)$	$O(h^2)$
Midpoint $y_{i+\frac{1}{2}} = y_i + \frac{h}{2}\ f(x_i, y_i)$ $y_{i+1} = y_i + h\ f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$	$O(h^3)$	$O(h^2)$

Heun's Method some what better than Euler Method

Heun's Method

Integrate $y' = 4e^{0.8x} - 0.5y$ from x=1 to x=4

(Using h=1 and Initial conditions: $x_0 = -1$, $y_0 = -0.393$ and $x_1 = 0$, $y_1 = 2$

Solution:

$$\textbf{\textit{Predictor}}: y_1^0 = -0.393 + [4e^{0.8(0)} - 0.5(2)]2 = 5.607$$

$$\textbf{\textit{Corrector}}: y_1^1 = 2 + \frac{4e^{0.8(0)} - 0.5(2) + 4e^{0.8(1)} - 0.5(5.607)}{2}1 = 6.5493$$

which represents a relative error of -5.73% (True value is 6.1946). one can apply the Multistep formula iteratively to improve this result:

$$y_1^2 = 2 + \frac{3 + 4e^{0.8(1)} - 0.5(6.5493)}{2}1 = 6.3137 \qquad \varepsilon_t = -1.92\% \qquad \varepsilon_a = 3.7\%$$

If one cancontinue the iterations, it converges to y=6.36086 when e_a=-2.68%. Then, for the second step:

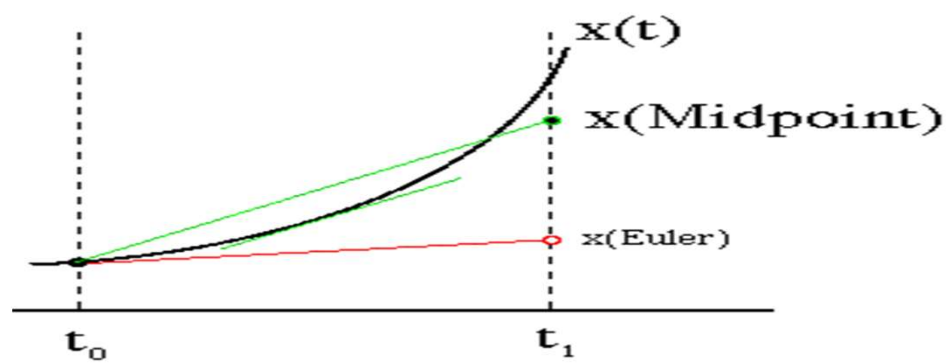
$$\textbf{\textit{Predictor}}: y_2^0 = 2 + [4e^{0.8(1)} - 0.5(6.36086)]2 = 13.44346 \qquad \varepsilon_t = 9.43\%$$

$$\textbf{\textit{Corrector}}: y_2^1 = 6.36086 + \frac{4e^{0.8(1)} - 0.5(6.36086) + 4e^{0.8(2)} - 0.5(13.44346)}{2}1 = 15.767$$

With the new method, one can get better error rates and faster convergence (compared to the Heun method)

Improvements of Euler's method

- A fundamental source of error in Euler's method is that the derivative at the beginning of the interval is assumed to apply across the entire interval.
- Two simple modifications are available to circumvent this shortcoming:
  - Heun's Method
  - The Midpoint (or Improved Polygon) Method

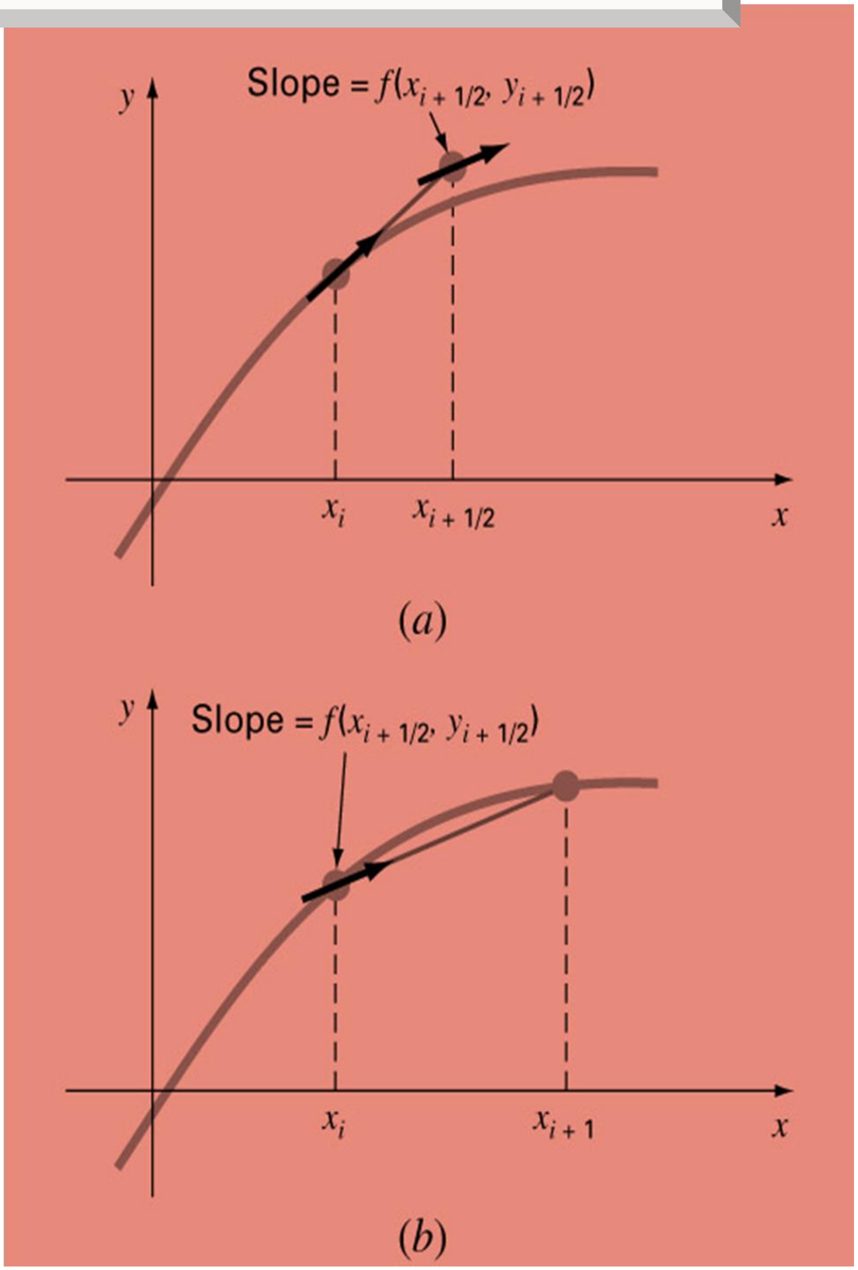


The Midpoint (or Improved Polygon) Method

- Uses Euler's method to predict a value of *y* using the slope value at the midpoint of the interval:

$$y_{i+1/2} = y_i + f(x_i, y_i)\frac{h}{2}$$

$$y_{i+1} = y_i + f(x_{i+1/2}, y_{i+1/2})h$$



Improved Polygon Method

- Another modification of Euler's Method
- Uses Euler's to predict a value of y at the midpoint of the interval

$$y_{i+1/2} = y_i + f(x_i, y_i)\frac{h}{2}$$

- This predicted value is used to estimate the slope at the midpoint

$$y'_{i+1/2} = f(x_{i+1/2}, y_{i+1/2})$$

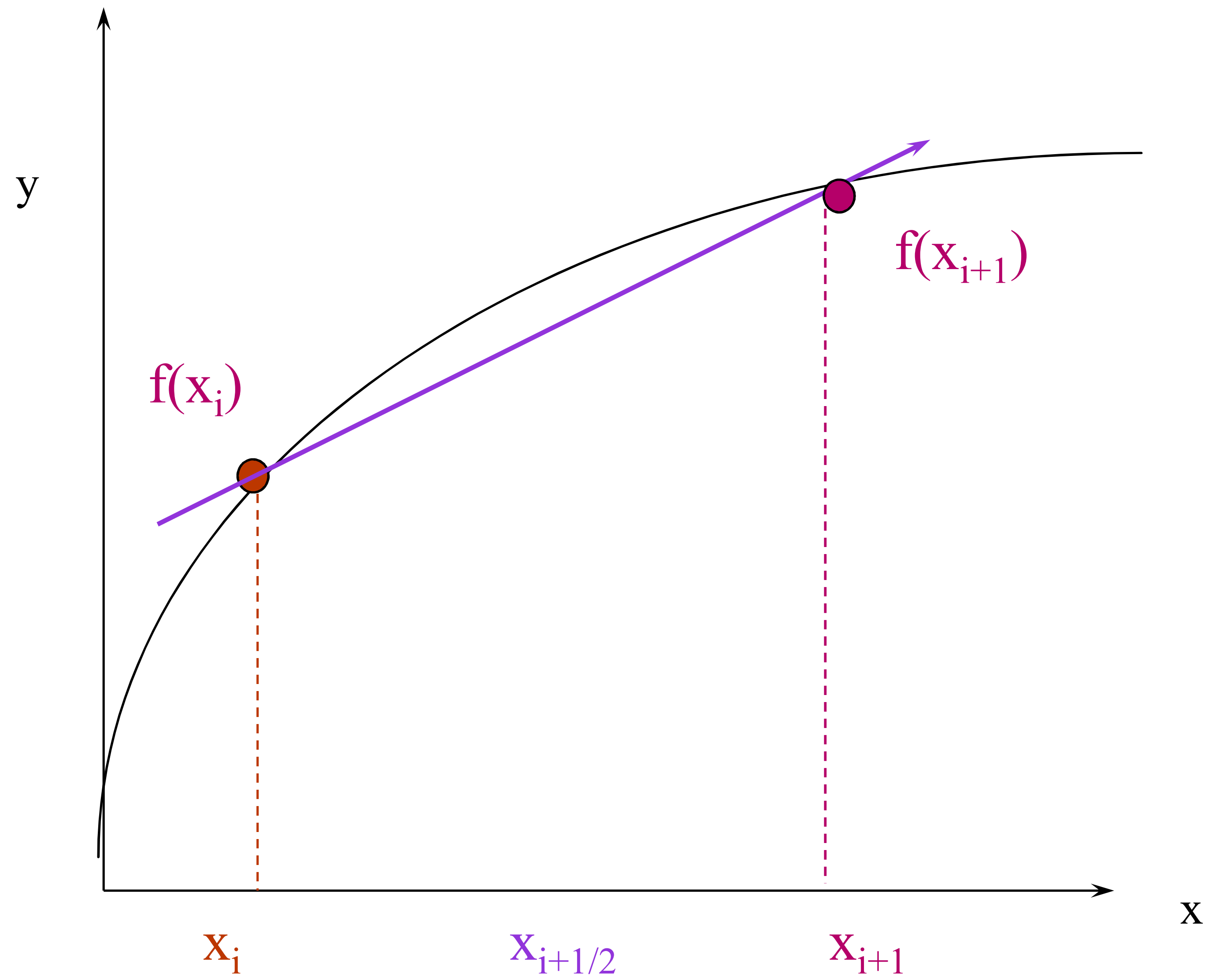
Improved Polygon Method

- One can then assume that this slope represents a valid approximation of the average slope for the entire interval
- Use this slope to extrapolate linearly from x_i to x_{i+1} using Euler's algorithm

$$y_{i+1} = y_i + f\left(x_{i+1/2}, y_{i+1/2}\right)h$$

- One can could also get this algorithm from substituting a forward difference in f to i+1/2 into the Taylor expansion for f', i.e.

$$y_{i+1} = y_i + f_i\,h + \left(\frac{f_{i+1/2} - f_i}{h/2}\right)\frac{h^2}{2} = y_i + f_{i+1/2}\,h$$


y
x
f(x_i)
f(x_{i+1})
x_i
x_{i+1/2}
x_{i+1}

Improved Polygon Method

Midpoint Method

Problem

$$\dot{y}(x) = f(x, y)$$

$$y(x_0) = y_0$$

Midpoint Method

$$y_0 = y(x_0)$$

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i)$$

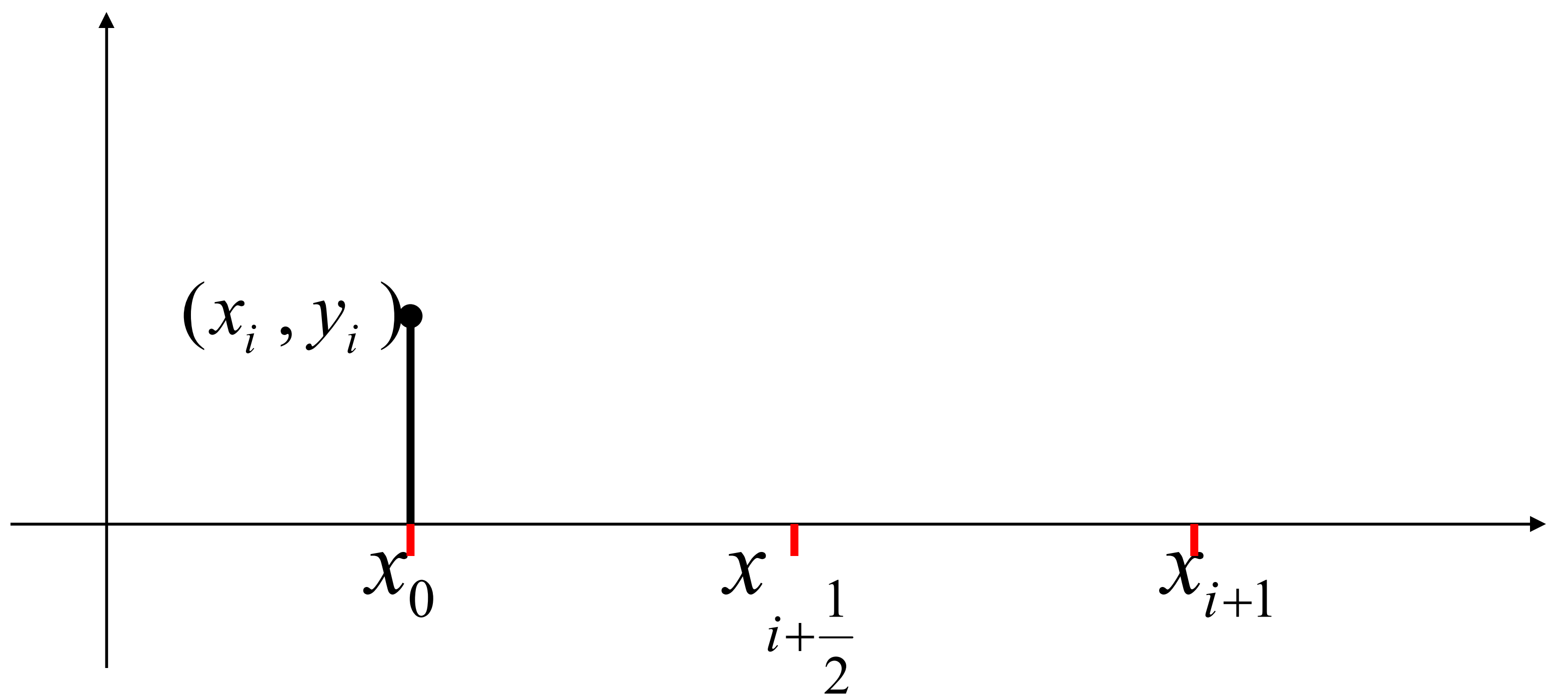
$$y_{i+1} = y_i + h\ f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Local Truncation Error $O(h^3)$

Global Truncation Error $O(h^2)$

Improved Polygon Method

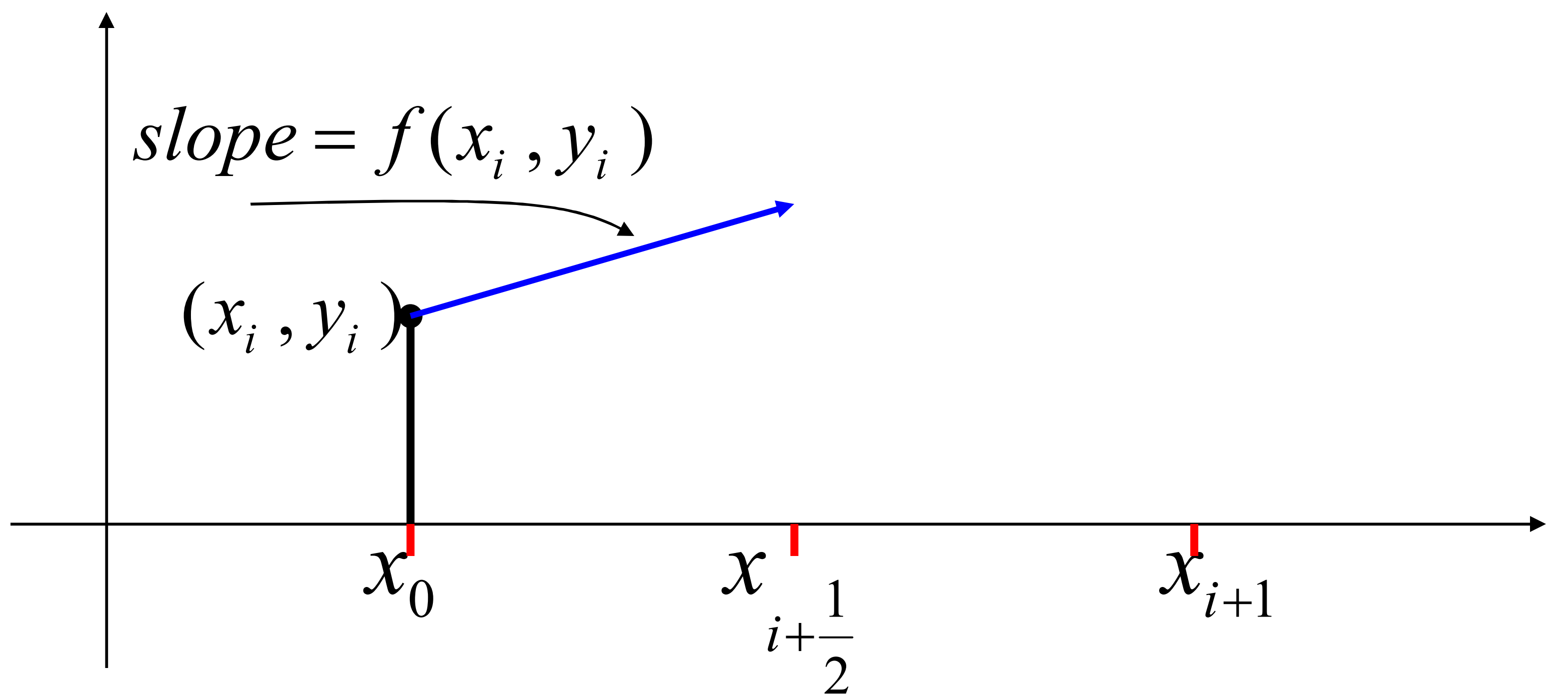
Midpoint Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Improved Polygon Method

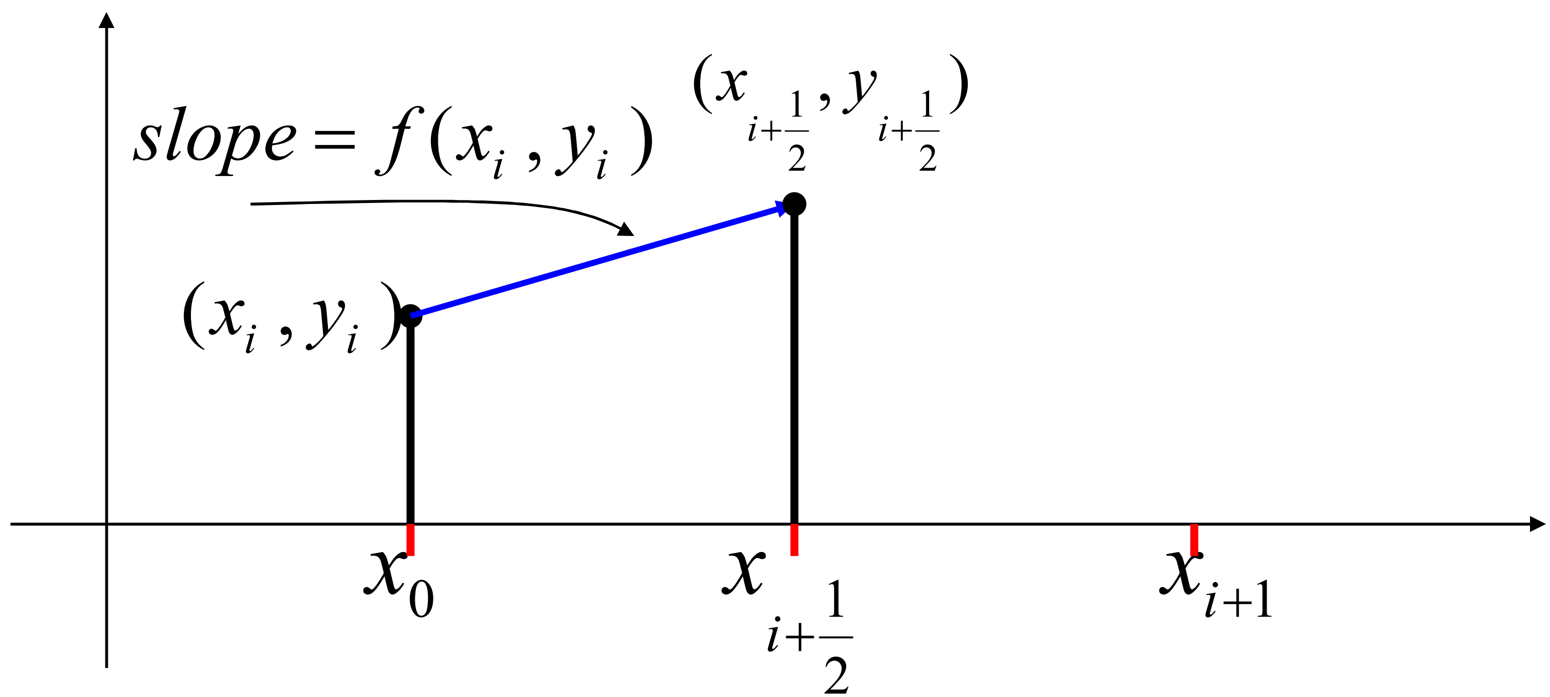
Midpoint Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Improved Polygon Method

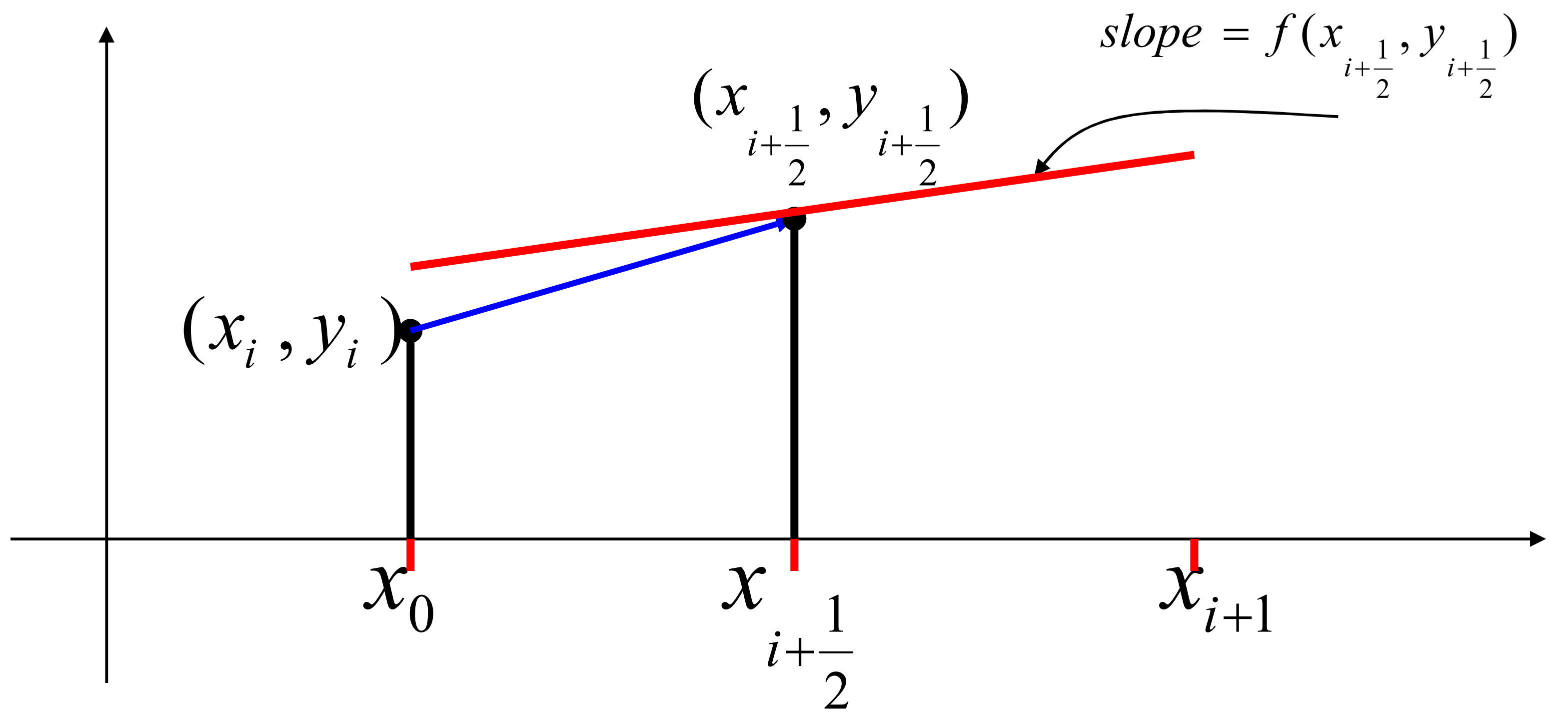
Midpoint Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Improved Polygon Method

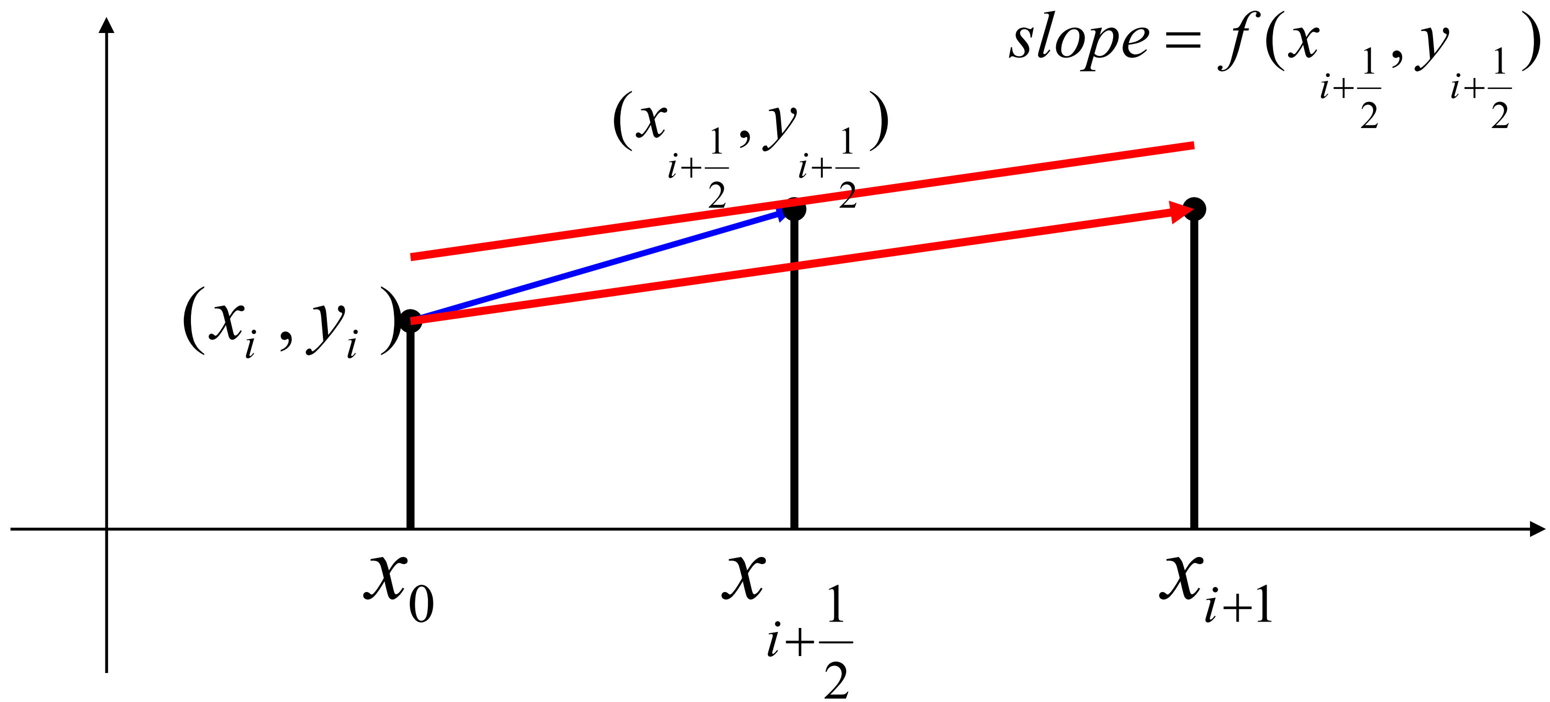
Midpoint Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Improved Polygon Method

Midpoint Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Improved Polygon Method

Use the Midpoint Method to solve the ODE

$$\dot{y}(x) = 1 + x^2 + y$$

$$y(0) = 1$$

Use $h = 0.1$. Determine y(0.1) and y(0.2)

Improved Polygon Method

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i),$$

$$y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Problem : $f(x, y) = 1 + x^2 + y,\ y_0 = y(0) = 1, h = 0.1$

Step 1 :

$$y_{0+\frac{1}{2}} = y_0 + \frac{h}{2} f(x_0, y_0) = 1 + 0.05(1 + 0 + 1) = 1.1$$

$$y_1 = y_0 + h\ f(x_{0+\frac{1}{2}}, y_{0+\frac{1}{2}}) = 1 + 0.1(1 + 0.0025 + 1.1) = 1.2103$$

Step 2 :

$$y_{1+\frac{1}{2}} = y_1 + \frac{h}{2} f(x_1, y_1) = 1.2103 + .05(1 + 0.01 + 1.2103) = 1.3213$$

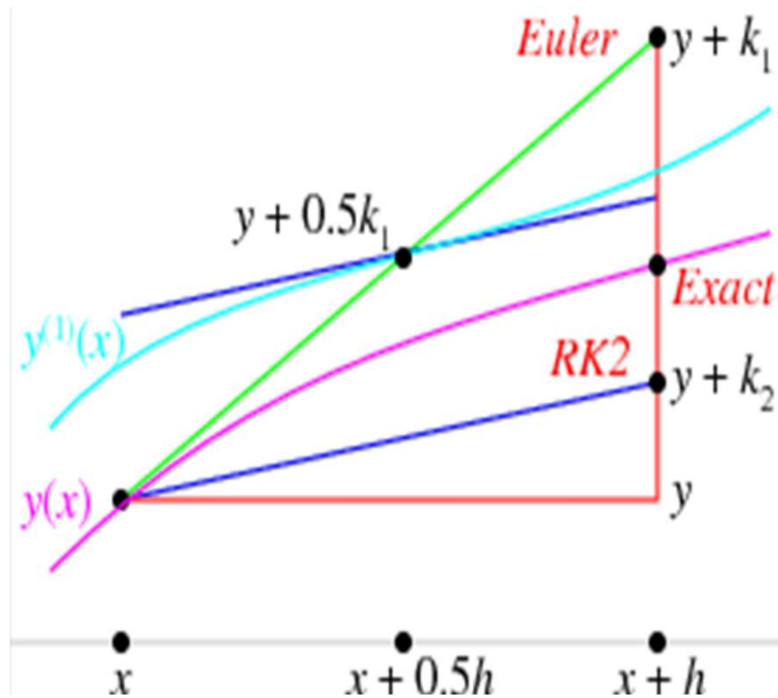
$$y_2 = y_1 + h\ f(x_{1+\frac{1}{2}}, y_{1+\frac{1}{2}}) = 1.2103 + 0.1(2.3438) = 1.4446$$

Runge-Kutta Methods

- RK methods achieve the accuracy of a Taylor series approach without requiring the calculation of a higher derivative
- Many variations exist but all can be cast in the generalized form:

$$y_{i+1} = y_i + \phi(\underbrace{x_i, y_i}, h)h$$

ϕ is called the incremental function



Runge-Kutta Methods (RK)

- Runge-Kutta methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives.

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

$$\phi = a_1k_1 + a_2k_2 + \cdots + a_nk_n$$

Increment Function

***a*'s are constants**

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1h, y_i + q_{11}k_1h)$$

***p*'s and *q*'s are constants**

$$k_3 = f(x_i + p_3h, y_i + q_{21}k_1h + q_{22}k_2h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1}h, y_i + q_{n-1}k_1h + q_{n-1,2}k_2h + \cdots + q_{n-1,n-1}k_{n-1}h)$$

Runge-Kutta Methods

- **Various types of RK methods can be devised by employing different number of terms in the increment function as specified by *n*.**
- **First order** RK method with $n=1$ and $a_1=1$ is in fact **Euler's method**.

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

$$k_1 = f(x_i, y_i)$$

choose $\quad n = 1 \quad$ and $\quad a_1 = 1,$ one can obtain

$$y_{i+1} = y_i + f(x_i, y_i)h \qquad \textbf{(Euler's Method)}$$

Runge-Kutta Methods

Second-order Runga-Kutta Methods:

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

- Values of $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{p_1}$, and $\mathbf{q_{11}}$ are evaluated by setting the above equation equal to a *Taylor series expansion* to the second order term. This way, three equations can be derived

$$\begin{aligned} y_{i+1} &= y_i + a_1 k_1 h + a_2 k_2 h \\ &= y_i + a_1 f(x_i, y_i) h + a_2 k_2 h \end{aligned}$$

where k_2 can be expanded in Taylor series as

$$\begin{aligned} k_2 &= f(x_i + p_1 h, y_i + q_{11} k_1 h) = f(x_i, y_i) + \frac{\partial f(x_i, y_i)}{\partial x} p_1 h + \frac{\partial f(x_i, y_i)}{\partial y} q_{11} k_1 h \\ & \qquad \text{(ignore higher order terms)} \\ &= f(x_i, y_i) + \frac{\partial f(x_i, y_i)}{\partial x} p_1 h + \frac{\partial f(x_i, y_i)}{\partial y} q_{11} f(x_i, y_i) h \end{aligned}$$

Substituting k_2 in (2) by (3), we have

$$\begin{aligned} y_{i+1} &= y_i + a_1 f(x_i, y_i) h + a_2 f(x_i, y_i) h + a_2 \frac{\partial f(x_i, y_i)}{\partial x} p_1 h^2 + a_2 \frac{\partial f(x_i, y_i)}{\partial y} q_{11} f(x_i, y_i) h^2 \\ &= y_i + (a_1 + a_2) f(x_i, y_i) h + a_2 \frac{\partial f(x_i, y_i)}{\partial x} p_1 h^2 + a_2 \frac{\partial f(x_i, y_i)}{\partial y} q_{11} f(x_i, y_i) h^2 \end{aligned}$$

A value is assumed for one of the unknowns to solve for the other three.

$$\left\{ \begin{aligned} a_1 + a_2 &= 1 \\ a_2 p_1 &= \frac{1}{2} \\ a_2 q_{11} &= \frac{1}{2} \end{aligned} \right.$$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

- Because onecan choose an infinite number of values for a_2, there are an infinite number of second-order RK methods.

Three of the most commonly used methods are:

- **Huen's Method** with a Single Corrector (a_2=1/2)
- **The Midpoint Method** (a_2=1)
- **Ralston's Method** (a_2=2/3)

Huen's Method (a_2= 1/2) ➔ a_1= 1/2 p_1= 1 q_{11}= 1

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h = y_i + (\frac{1}{2}k_1 + \frac{1}{2}k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + h, y_i + k_1 h)$$

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

The Midpoint Method ($a_2 = 1$)

➔➔➔ $a_1 = 0$ $p_1 = 1/2$ $q_{11} = 1/2$

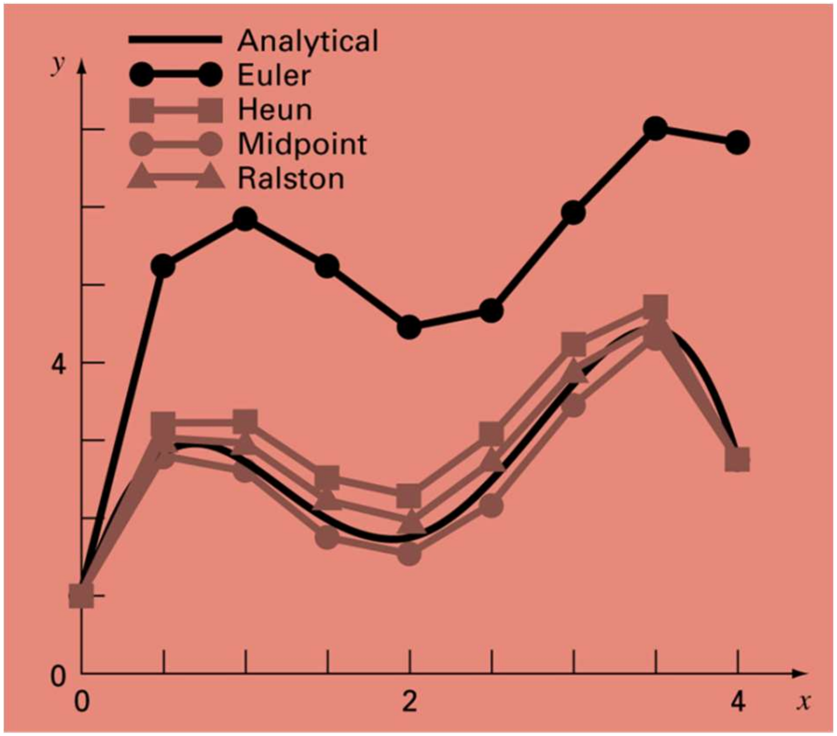
$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h = y_i + (k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

$$y_{i+1} = y_i + (k_2)h = y_i + f(x_i + \frac{h}{2}, y_i + \frac{h}{2} f(x_i, y_i))h$$

- Three most commonly used methods:
  - **Huen Method** with a Single Corrector (a_2=1/2)
  - **The Midpoint Method** (a_2=1)
  - **Ralston's Method** (a_2=2/3)

Comparison of Various Second-Order RK Methods



Ralston's Method (a_2=2/3)

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h = y_i + (\frac{1}{3}k_1 + \frac{2}{3}k_2)h$$

$$k_1 = f(x_i, y_i) \qquad k_2 = f(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h)$$

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

$$where\ the\ a's\ are\ constant\ and\ the\ k's\ are:$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$k_3 = f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$\vdots$$

$$k_n = f(x_i + p_n h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)$$

NOTE:

k's are recurrence relationships, that is k_1 appears in the equation for k_2 which appears in the equation for k_3

This recurrence makes RK methods efficient for computer calculations

Ralston's Method

Ralston (1962) and Ralston and Rabinowitiz (1978) determined that choosing $a_2 = 2/3$ provides a minimum bound on the truncation error for the second order RK algorithms.

This results in $a_1 = 1/3$ and $p_1 = q_{11} = 3/4$

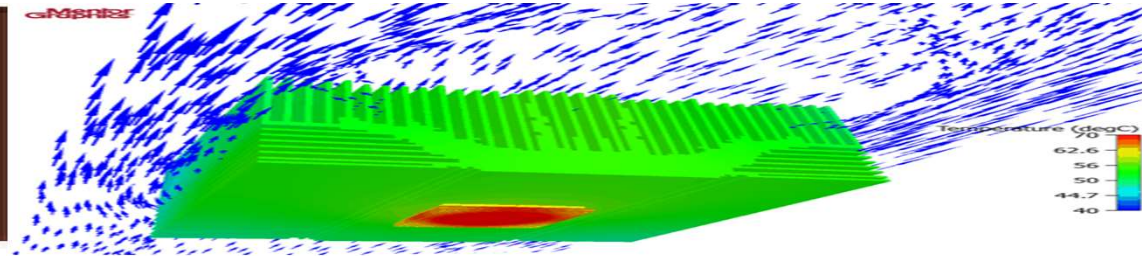
$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$$

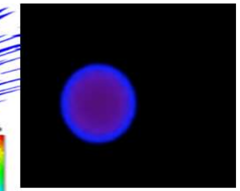
where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

EXAMPLE





A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right),\ \theta(0) = 1200K$$

Find the temperature at $t = 480$ **seconds using Heun's method. Assume a step size of** $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

$$(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

Solution

$$\frac{d\theta}{dt} = -2.2067 \times 10^{-12}\left(\theta^4 - 81 \times 10^8\right)$$

Step 1:

$$i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200 \ K$$

$$k_1 = f(t_0, \theta_o)$$
$$= f(0, 1200)$$
$$= -2.2067 \times 10^{-12}\left(1200^4 - 81 \times 10^8\right)$$
$$= -4.5579$$

$$k_2 = f(t_0 + h, \theta_0 + k_1 h)$$
$$= f(0 + 240, 1200 + (-4.5579)240)$$
$$= f(240, 106.09)$$
$$= -2.2067 \times 10^{-12}\left(106.09^4 - 81 \times 10^8\right)$$
$$= 0.017595$$

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$k_2 = f(x_i + h, y_i + k_1 h)$$

$$\theta_1 = \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$= 1200 + \left(\frac{1}{2}(-4.5579) + \frac{1}{2}(0.017595)\right)240$$
$$= 1200 + (-2.2702)240$$
$$= 655.16 K$$

Solution

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

Step 2:

$$i = 1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16K$$

$$\begin{aligned} k_1 &= f(t_1, \theta_1) \\ &= f(240, 655.16) \\ &= -2.2067\times10^{-12}\left(655.16^4 - 81\times10^8\right) \\ &= -0.38869 \end{aligned}$$

$$\begin{aligned} k_2 &= f(t_1 + h, \theta_1 + k_1 h) \\ &= f(240 + 240, 655.16 + (-0.38869)240) \\ &= f(480, 561.87) \\ &= -2.2067\times10^{-12}\left(561.87^4 - 81\times10^8\right) \\ &= -0.20206 \end{aligned}$$

$$\begin{aligned} \theta_2 &= \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\ &= 655.16 + \left(\frac{1}{2}(-0.38869) + \frac{1}{2}(-0.20206)\right)240 \\ &= 655.16 + (-0.29538)240 \\ &= 584.27K \end{aligned}$$

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$
$$k_2 = f(x_i + h, y_i + k_1 h)$$

Solution

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300} - 1.8519\tan^{-1}\left(0.00333333\theta\right) = -0.22067\times10^{-3}t - 2.9282$$

The solution to this nonlinear equation at t=480 seconds is $\theta(480) = 647.57K$

Comparison with exact results

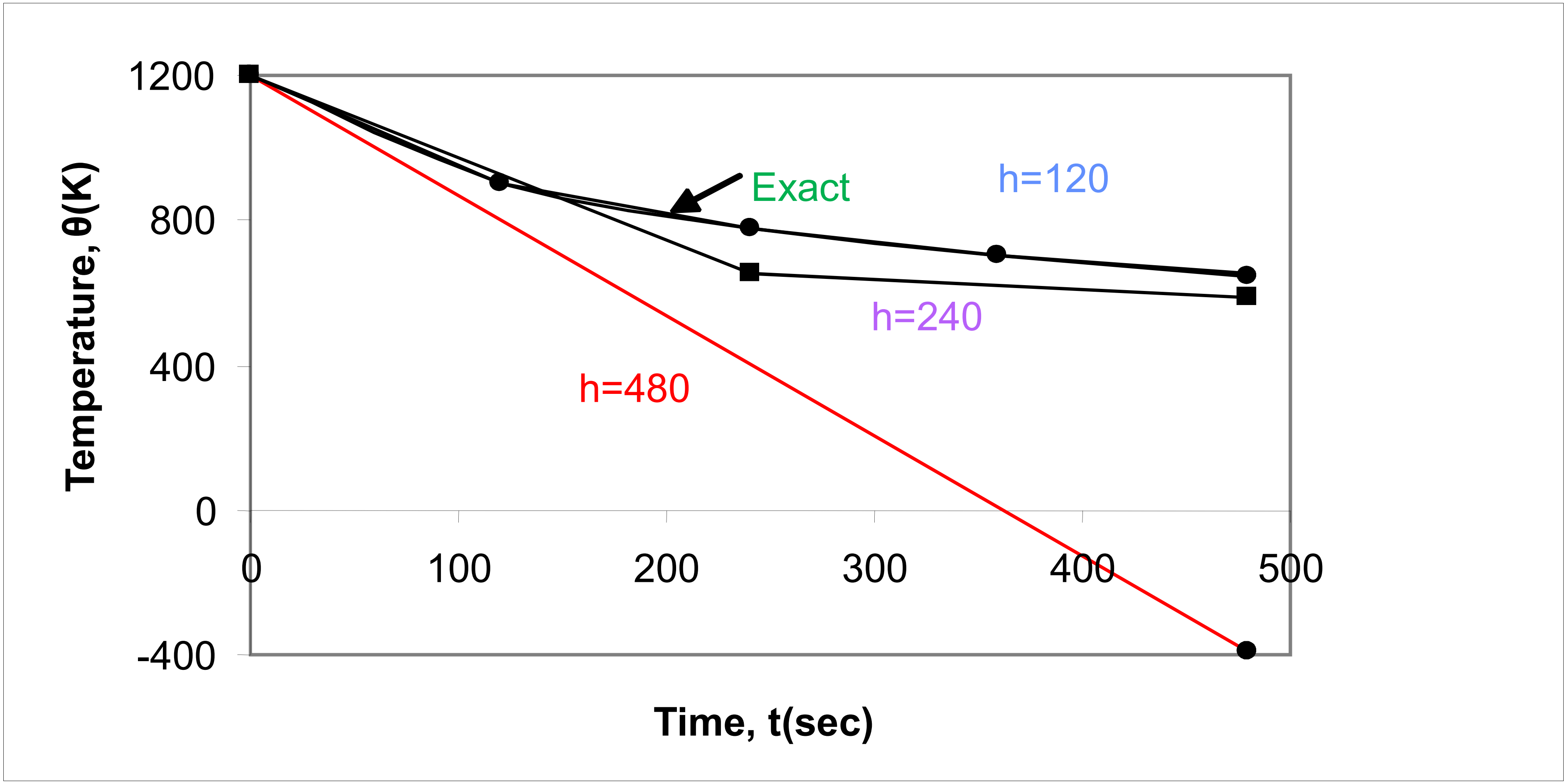


Figure . Heun's method results for different step sizes

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, h	$\theta(480)$	E_t	$\lvert \epsilon_t \rvert \%$
480	−393.87	1041.4	160.82
240	584.27	63.304	9.7756
120	651.35	−3.7762	0.58313
60	649.91	−2.3406	0.36145
30	648.21	−0.63219	0.097625

$$\theta(480) = 647.57K \quad \text{(exact)}$$

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	q(480)			
	Euler	Heun	Midpoint	Ralston
480	−987.84	−393.87	1208.4	449.78
240	110.32	584.27	976.87	690.01
120	546.77	651.35	690.20	667.71
60	614.97	649.91	654.85	652.25
30	632.77	648.21	649.02	648.61

$$\theta(480) = 647.57K \qquad \text{(exact)}$$

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\|\in_t\|\%$			
	Euler	Heun	Midpoint	Ralston
480	252.54	160.82	86.612	30.544
240	82.964	9.7756	50.851	6.5537
120	15.566	0.58313	6.5823	3.1092
60	5.0352	0.36145	1.1239	0.72299
30	2.2864	0.097625	0.22353	0.15940

$$\theta(480) = 647.57K \quad \text{(exact)}$$

Comparison of Euler and Runge-Kutta 2nd Order Methods

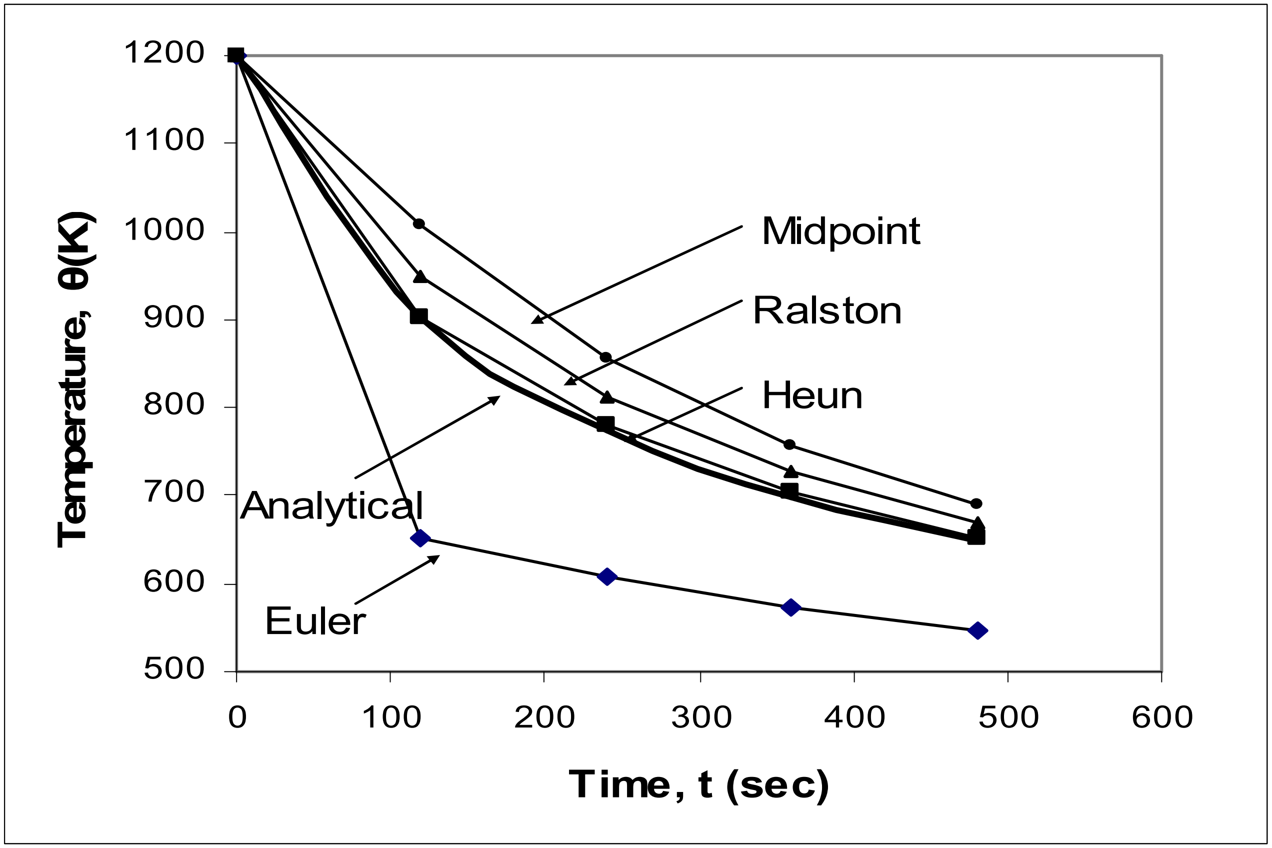


Figure . Comparison of Euler and Runge Kutta 2^{nd} order methods with exact results.

Third Order Runge-Kutta Methods

Less use in research

- Derivation is similar to the one for the second-order
- Results in six equations and eight unknowns.
- One common version results in the following

$$y_{i+1} = y_i + \left[\frac{1}{6}\left(k_1 + 4k_2 + k_3\right)\right]h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + h, y_i - hk_1 + 2hk_2\right)$$

Note the third term

NOTE: if the derivative is a function of x only, this reduces to Simpson's 1/3 Rule

Fourth Order Runge Kutta

- **The most popular**
- **The following is sometimes called the classical fourth-order RK method**

$$y_{i+1} = y_i + \left[\frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)\right]h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_2\right)$$

$$k_4 = f\left(x_i + h, y_i + hk_3\right)$$

- Note that for ODE that are a function of x alone that this is also the equivalent of Simpson's 1/3 Rule

$$y_{i+1} = y_i + \left[\frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)\right]h$$

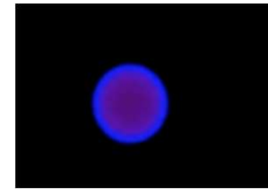
where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_2\right)$$

$$k_4 = f\left(x_i + h, y_i + hk_3\right)$$



Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right),\ \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Runge-Kutta 4th order method.

Assume a step size of $h = 240$ seconds.

$$y_{i+1} = y_i + \left[\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\right]h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_2\right)$$

$$k_4 = f(x_i + h, y_i + hk_3)$$

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

Solution

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200$

$$k_1 = f\left(t_0, \theta_o\right) = f\left(0,1200\right) = -2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right) = -4.5579$$

$$k_2 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_1 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-4.5579)240\right)$$

$$= f\left(120, 653.05\right) = -2.2067\times10^{-12}\left(653.05^4 - 81\times10^8\right) = -0.38347$$

$$k_3 = f\left(t_0 + \frac{1}{2}h, \theta_0 + \frac{1}{2}k_2 h\right) = f\left(0 + \frac{1}{2}(240), 1200 + \frac{1}{2}(-0.38347)240\right)$$

$$= f\left(120, 1154.0\right) = 2.2067\times10^{-12}\left(1154.0^4 - 81\times10^8\right) = -3.8954$$

$$k_4 = f\left(t_0 + h, \theta_0 + k_3 h\right) = f\left(0 + (240), 1200 + (-3.984)240\right)$$

$$= f\left(240, 265.10\right) = 2.2067\times10^{-12}\left(265.10^4 - 81\times10^8\right) = 0.0069750$$

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Solution

$$\theta_1 = \theta_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 1200 + \frac{1}{6}(-4.5579 + 2(-0.38347) + 2(-3.8954) + (0.069750))240$$

$$= 1200 + \frac{1}{6}(-2.1848)240$$

$$= 675.65K$$

θ_1 is the approximate temperature at

$$t = t_1 = t_0 + h = 0 + 240 = 240$$

$$\theta(240) \approx \theta_1 = 675.65K$$

$$y_{i+1} = y_i + \left[\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\right]h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_2\right)$$

$$k_4 = f(x_i + h, y_i + hk_3)$$

Solution

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Step 2: $i = 1, t_1 = 240, \theta_1 = 675.65K$

$$k_1 = f(t_1, \theta_1) = f(240, 675.65) = -2.2067\times10^{-12}\left(675.65^4 - 81\times10^8\right) = -0.44199$$

$$k_2 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_1 h\right) = f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.44199)240\right)$$

$$= f(360, 622.61) = -2.2067\times10^{-12}\left(622.61^4 - 81\times10^8\right) = -0.31372$$

$$k_3 = f\left(t_1 + \frac{1}{2}h, \theta_1 + \frac{1}{2}k_2 h\right) = f\left(240 + \frac{1}{2}(240), 675.65 + \frac{1}{2}(-0.31372)240\right)$$

$$= f(360, 638.00) = 2.2067\times10^{-12}\left(638.00^4 - 81\times10^8\right) = -0.34775$$

$$k_4 = f(t_1 + h, \theta_1 + k_3 h) = f(240 + (240), 675.65 + (-0.34775)240)$$

$$= f(480, 592.19) = 2.2067\times10^{-12}\left(592.19^4 - 81\times10^8\right) = -0.25351$$

Solution

$$\theta_2 = \theta_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)h$$

$$= 675.65 + \frac{1}{6}\left(-0.44199 + 2(-0.31372) + 2(-0.34775) + (-0.25351)\right)240$$

$$= 675.65 + \frac{1}{6}(-2.0184)240$$

$$= 594.91K$$

θ_2 is the approximate temperature at

$$t_2 = t_1 + h = 240 + 240 = 480$$

$$\theta(480) \approx \theta_2 = 594.91K$$

Solution

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.00333\theta) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480) = 647.57K$$

Comparison with exact results

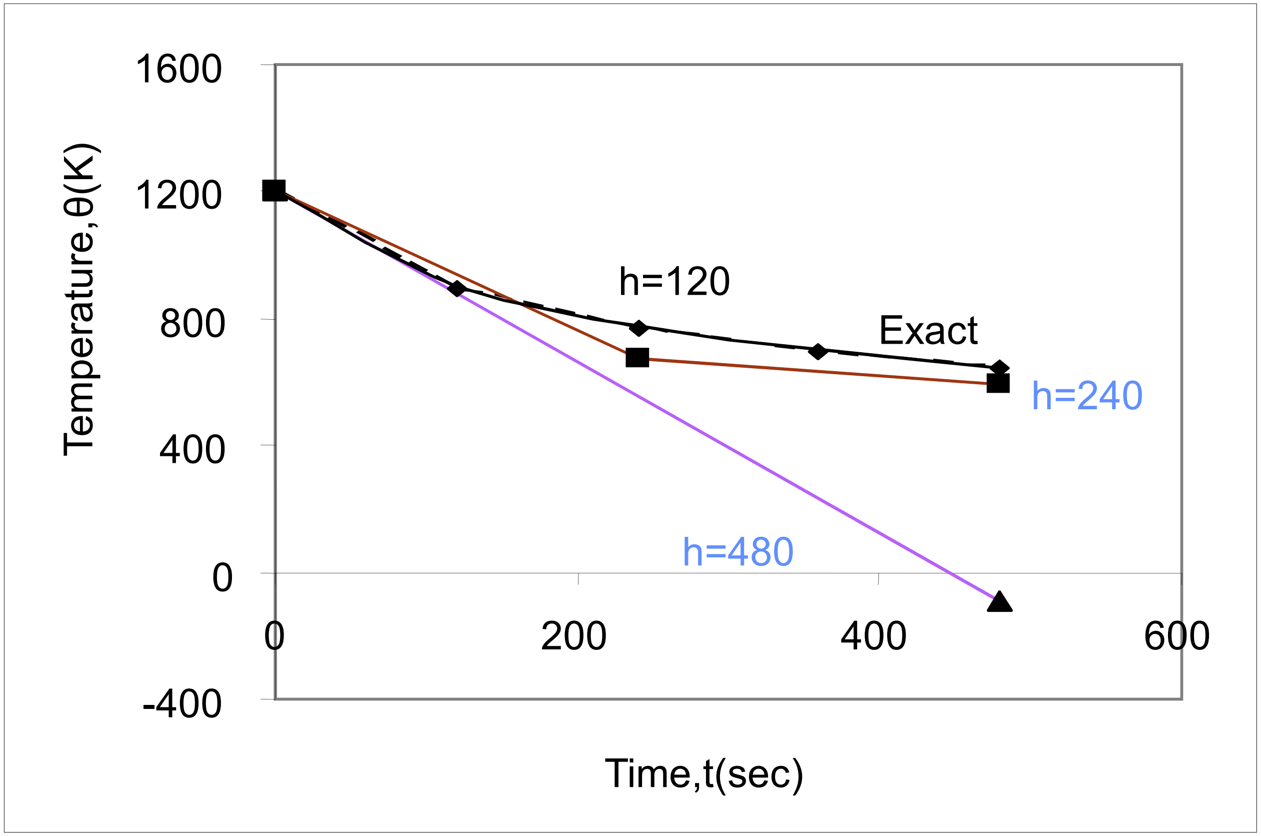


Figure 1. Comparison of Runge-Kutta 4th order method with exact solution

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, h	q (480)	E_t	$\|\epsilon_t\|\%$
480	−90.278	737.85	113.94
240	594.91	52.660	8.1319
120	646.16	1.4122	0.21807
60	647.54	0.033626	0.0051926
30	647.57	0.00086900	0.00013419

$$\theta(480) = 647.57K \quad \text{(exact)}$$

Effects of step size on Runge-Kutta 4th Order Method

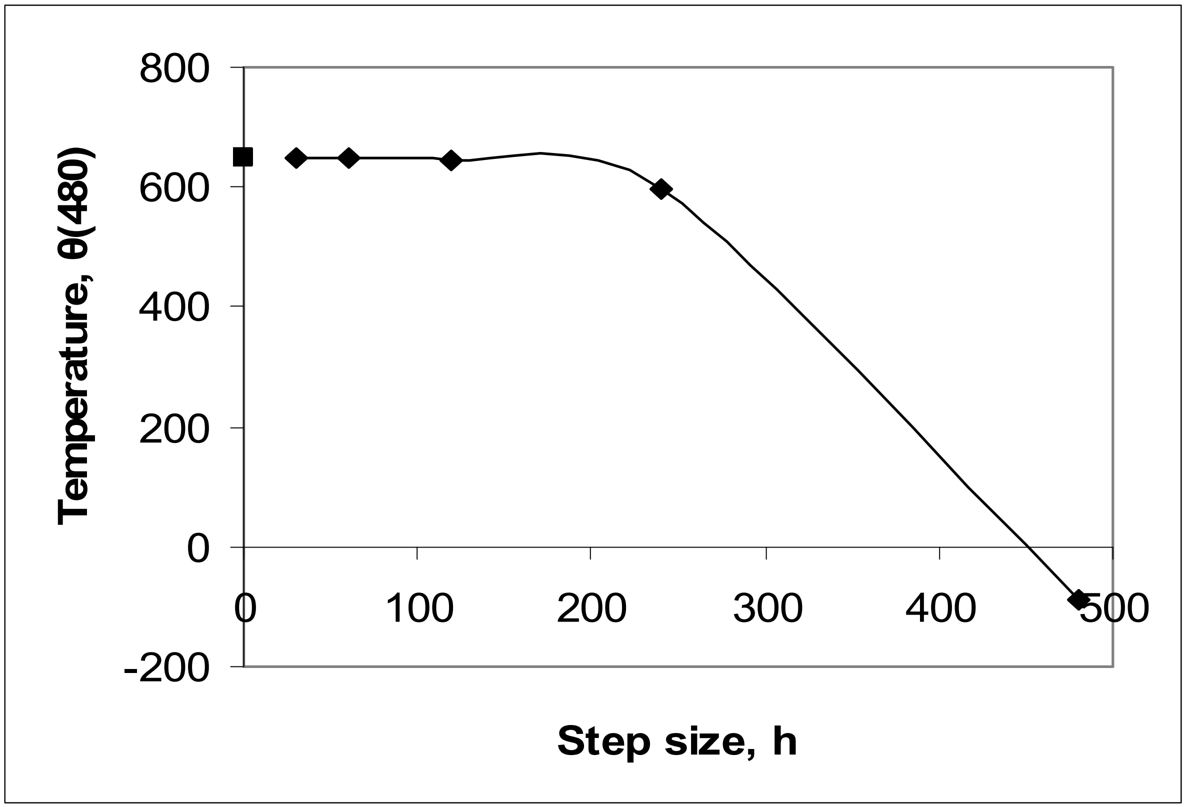


Figure . Effect of step size in Runge-Kutta 4th order method

Comparison of Euler and Runge-Kutta Methods

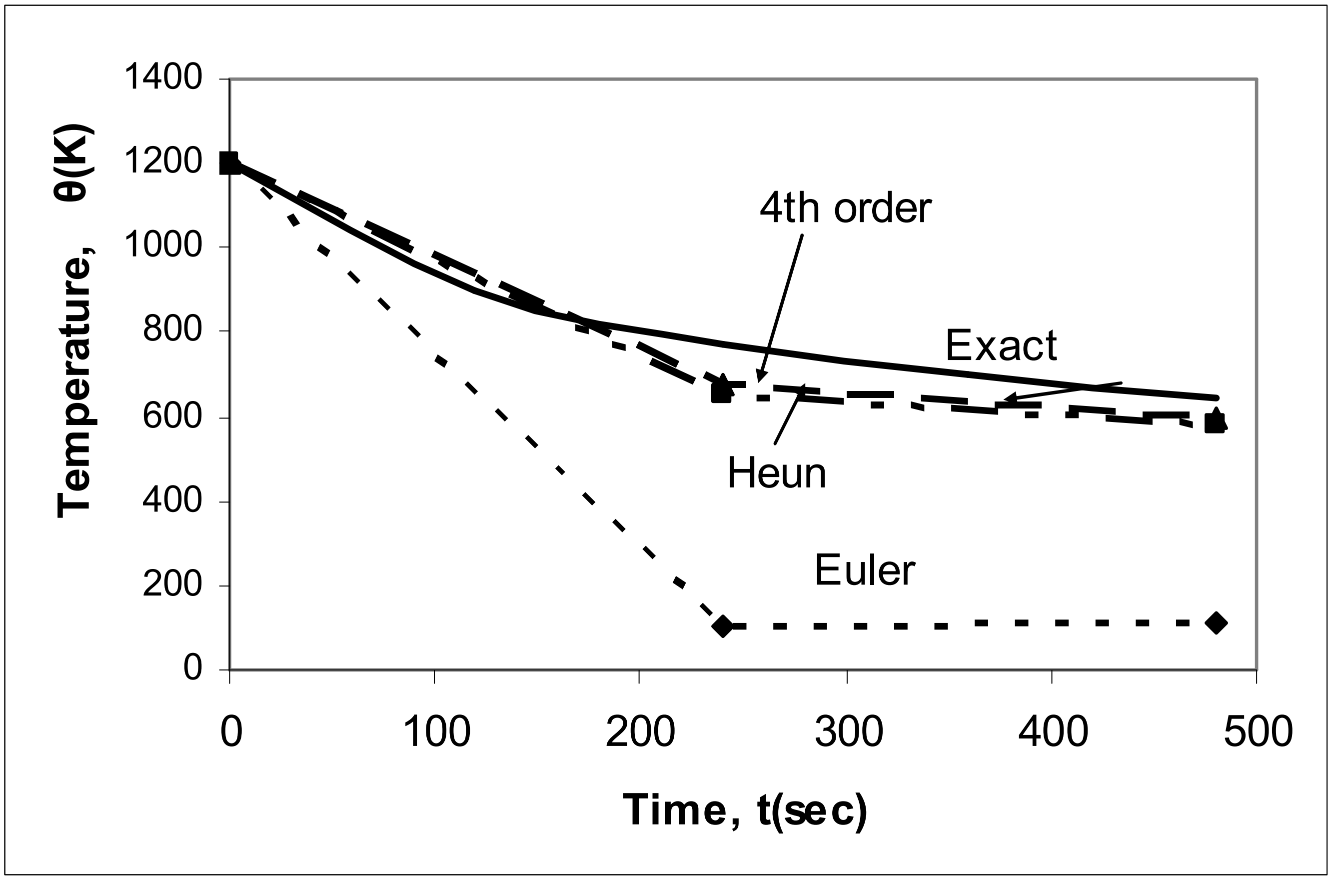


Figure . Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.


Thank You!