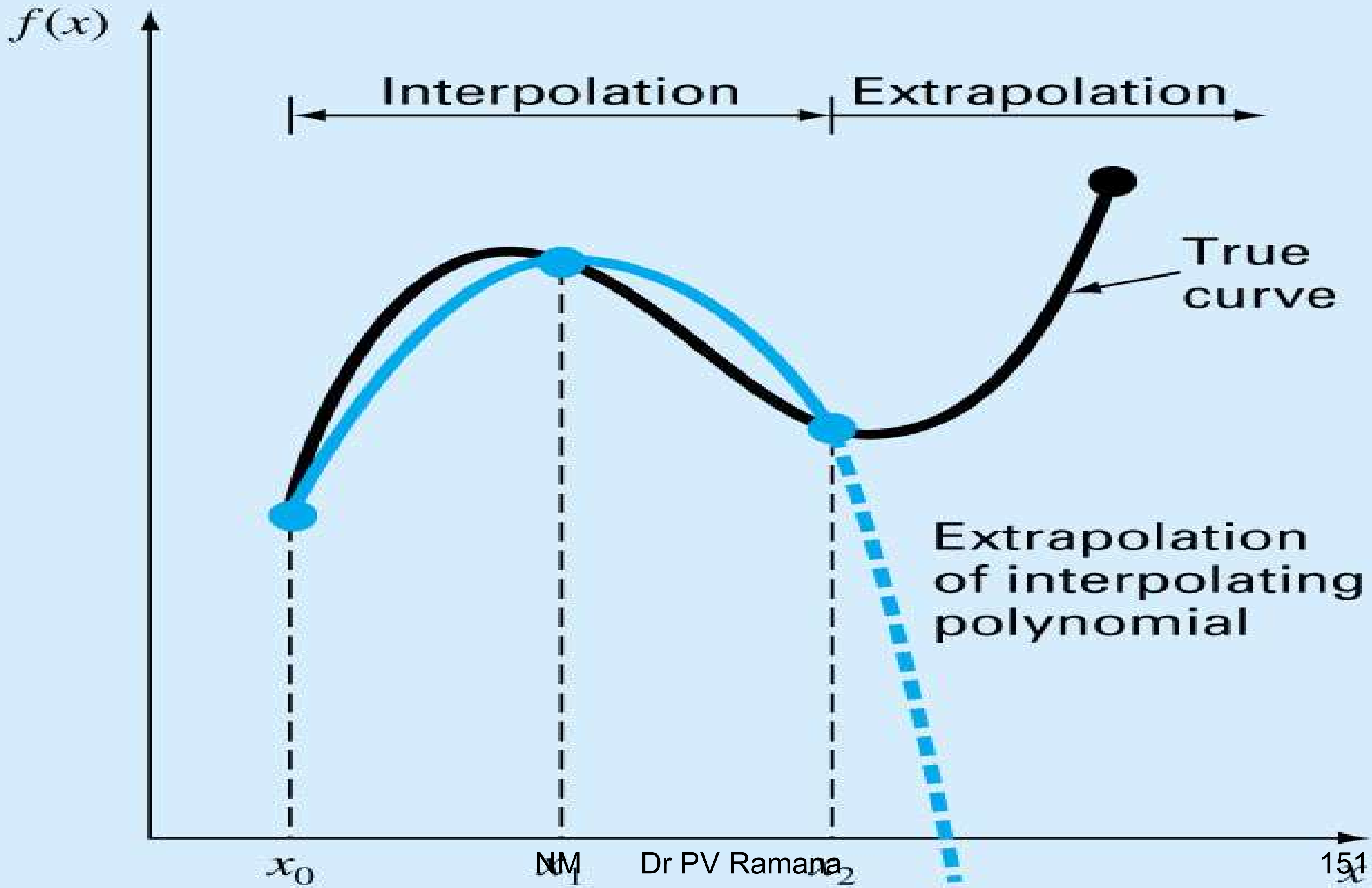
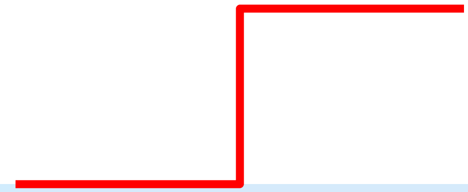
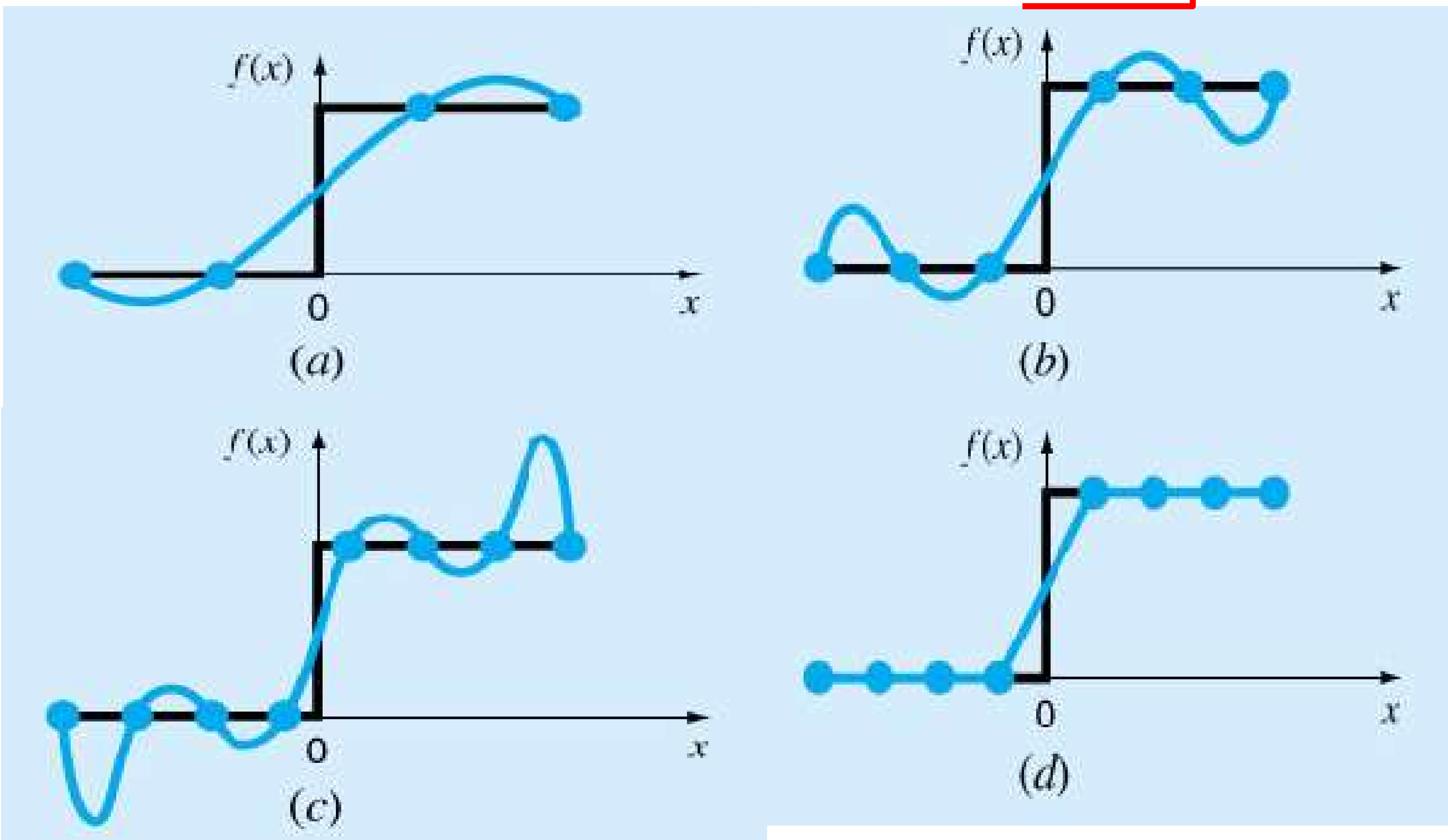
Possible divergence of an extrapolated production



Why Spline Interpolation?





Apply lower-order polynomials to subsets of data points. Spline provides a superior approximation of the behavior of functions that have local, abrupt changes.

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

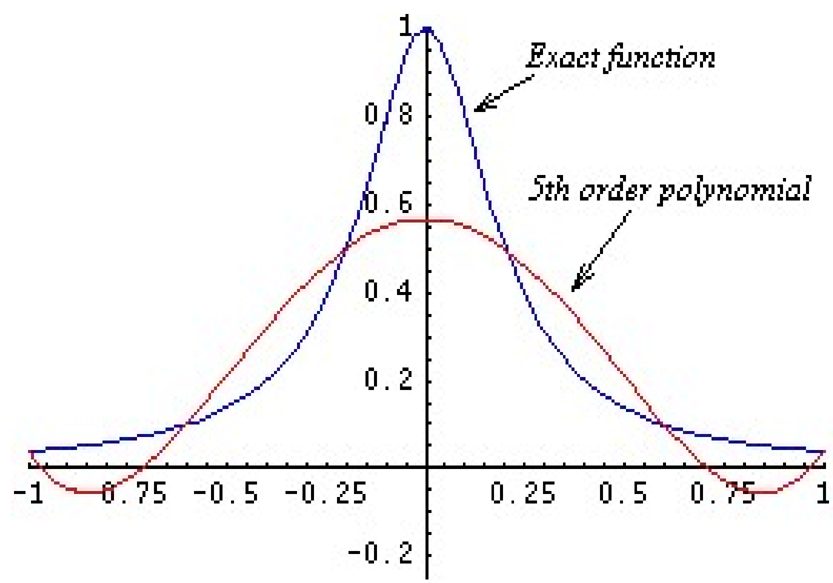
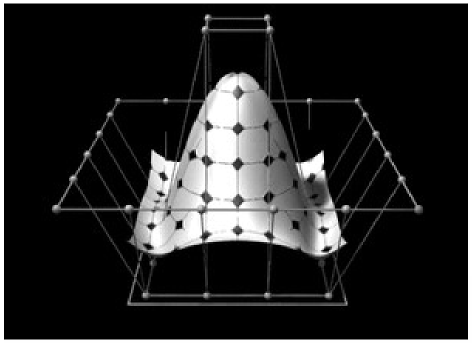


Figure : 5th order polynomial vs. exact function



Why Splines ?

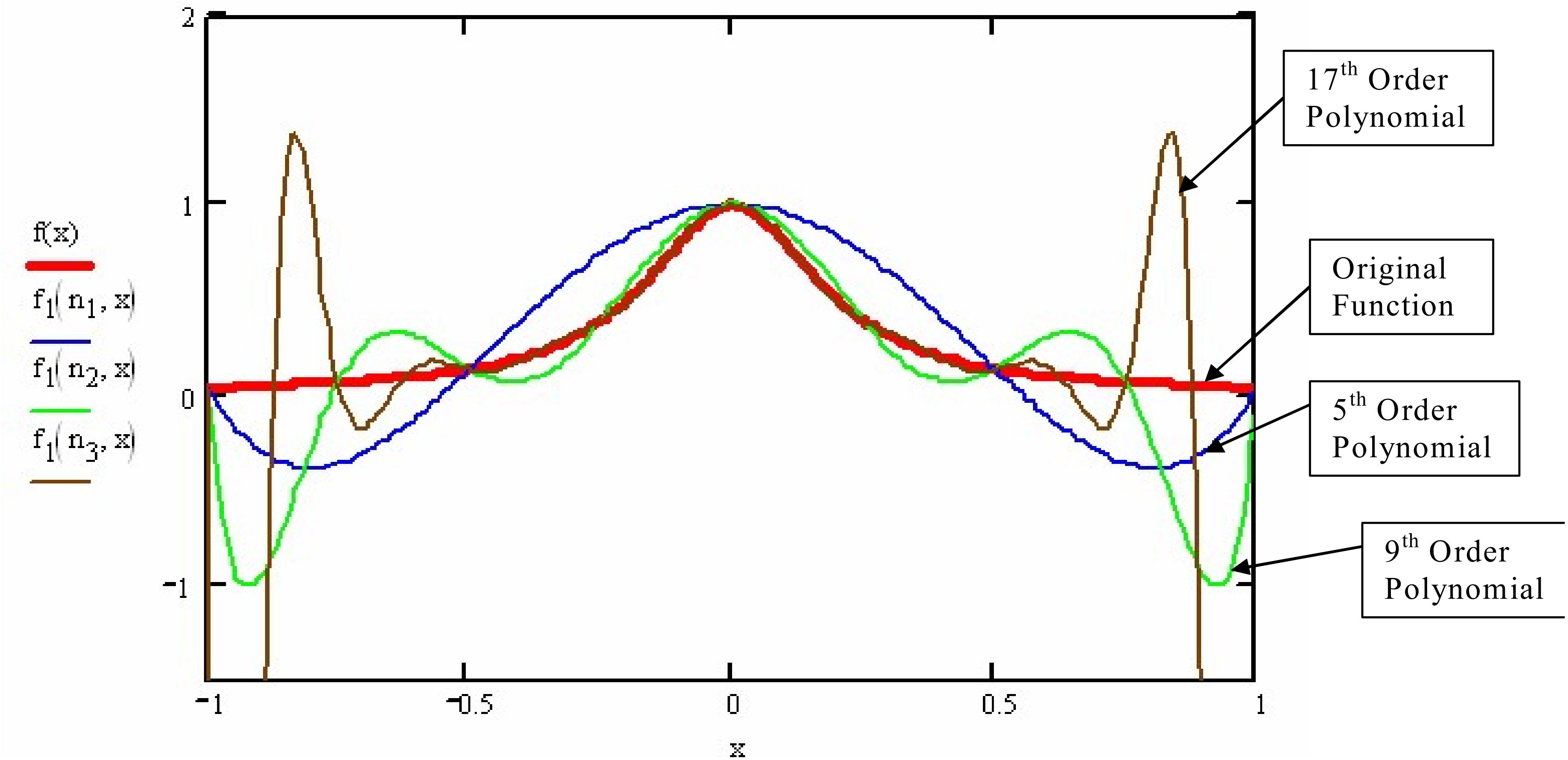
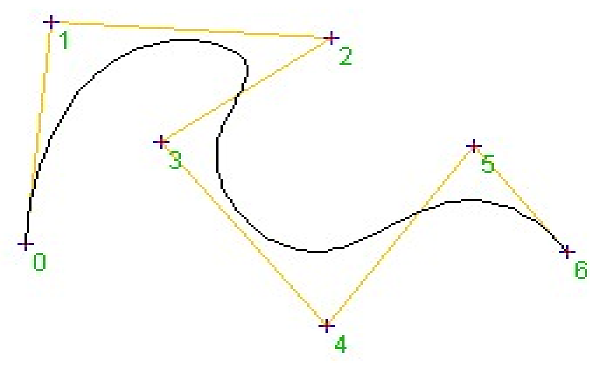


Figure : Higher order polynomial interpolation is a bad idea

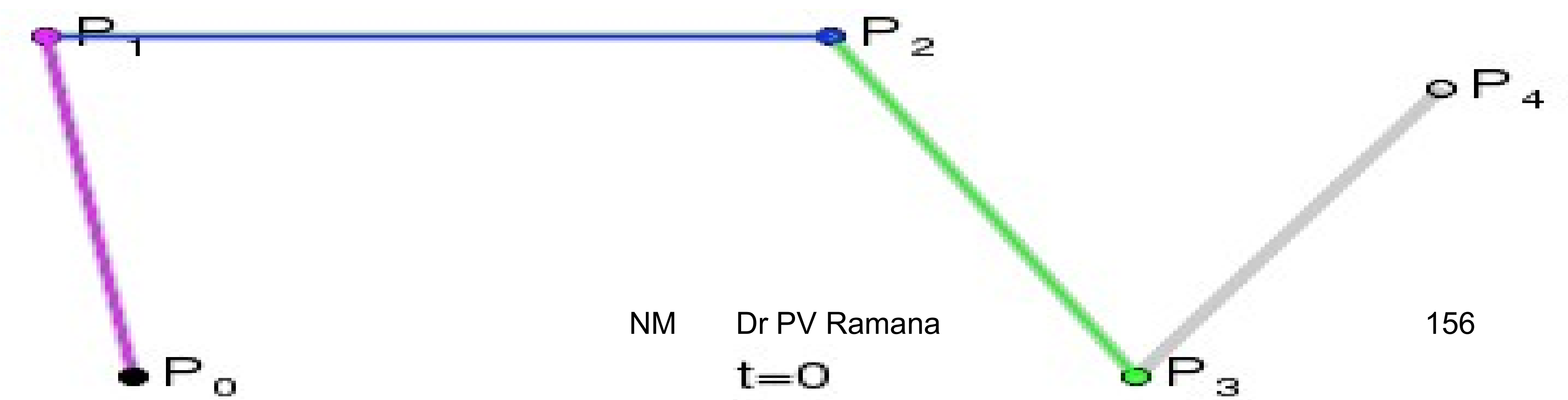
Spline Interpolation

- Our previous approach was to derive an *nth* order polynomial for *n+1* data points.
- An alternative approach is to apply lower-order polynomials to subset of data points.
- Such connecting polynomials are called spline functions.
- Adaptation of drafting techniques



Piecewise Polynomials and Splines

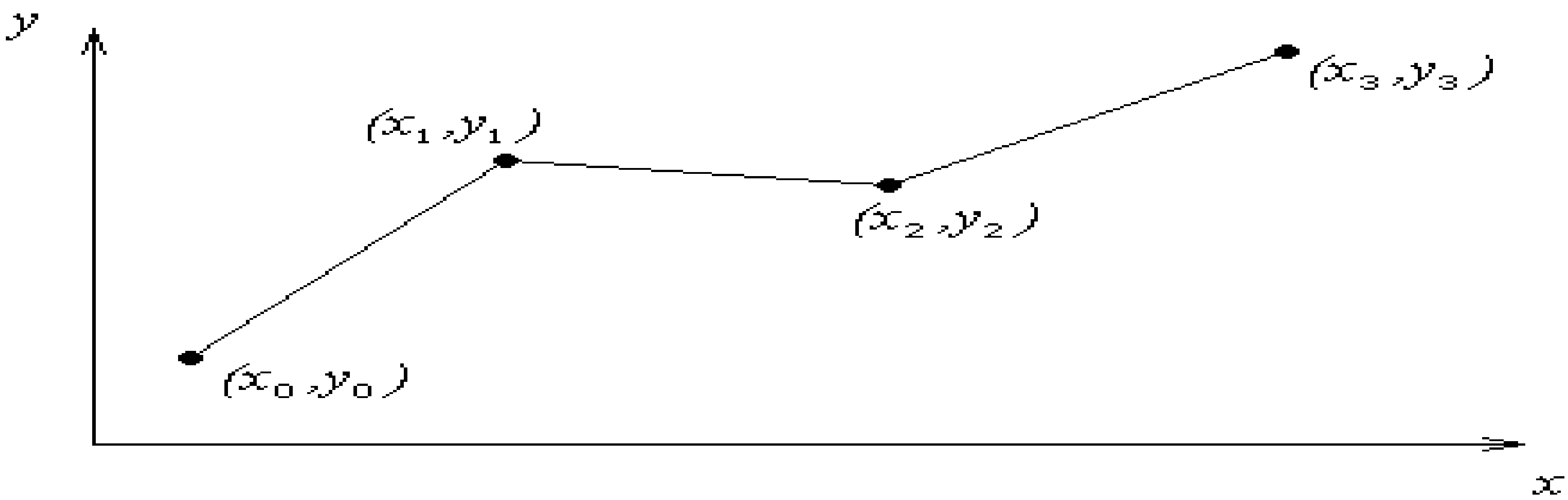
- Spline:
  - In Mathematics, a spline is a special function defined piecewise by polynomials;
  - In Computer Science, the term spline more frequently refers to a piecewise polynomial (parametric) curve.
- Simple construction, ease and accuracy of evaluation, capacity to approximate complex shapes through curve fitting and interactive curve design.



Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $(y_i = f(x_i))$

Figure : Linear splines



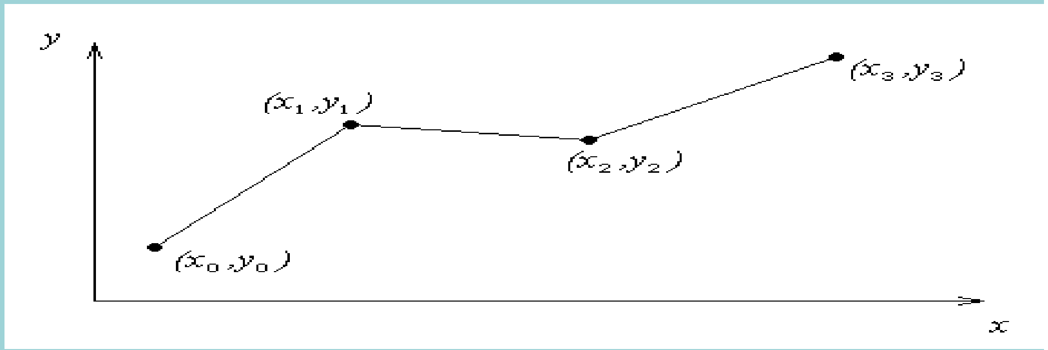
Linear Interpolation (contd)

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

$$\vdots$$

$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$



Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

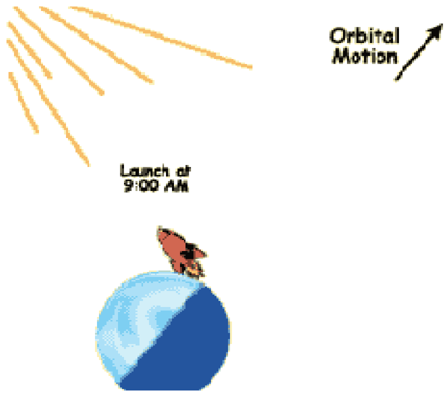
in the above function are simply slopes between x_{i-1} and x_i.

Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at t=16 seconds using linear splines.

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



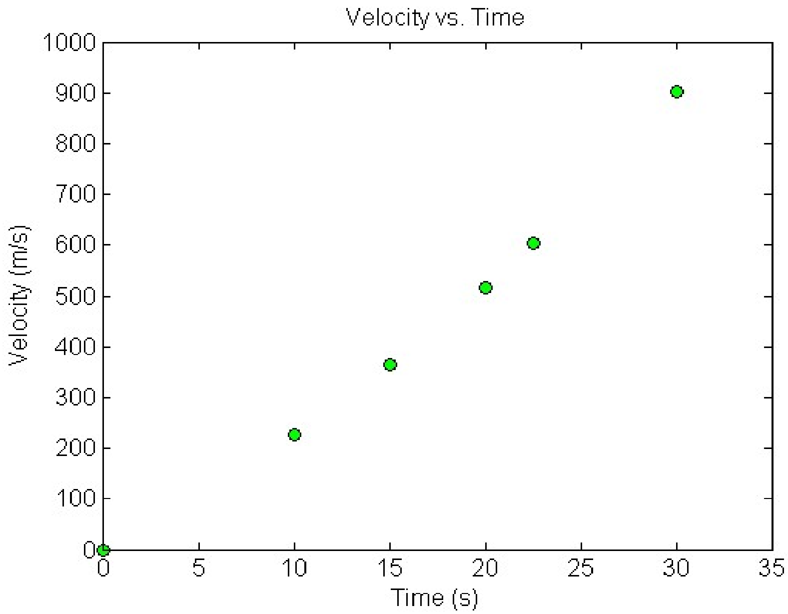
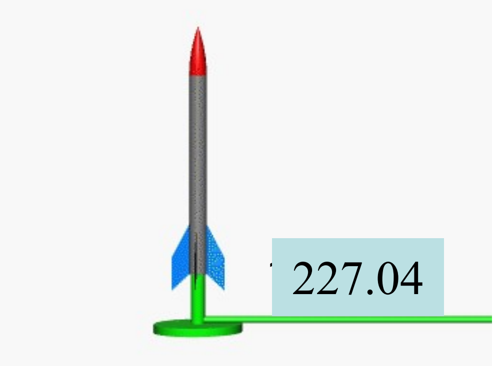


Figure. Velocity vs. time data for the rocket example



Linear Interpolation

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$t_0 = 15, \quad v(t_0) = 362.78$

$t_1 = 20, \quad v(t_1) = 517.35$

$$v(t) = v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0}(t - t_0)$$

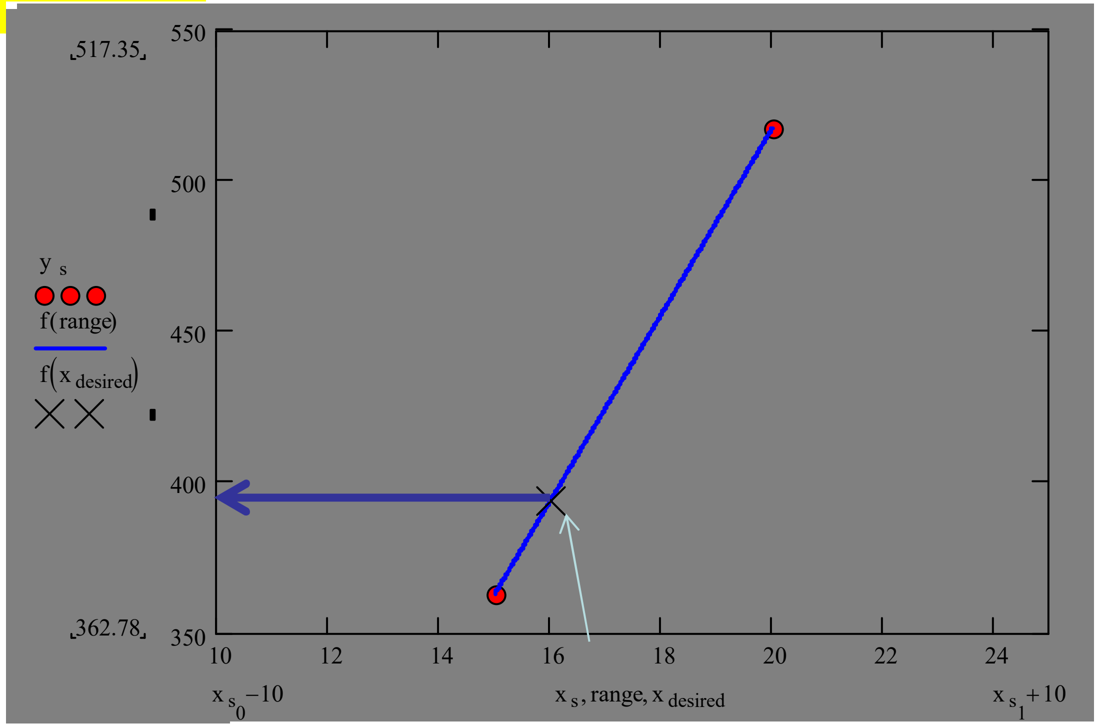
$$= 362.78 + \frac{517.35 - 362.78}{20 - 15}(t - 15)$$

$$v(t) = 362.78 + 30.913(t - 15)$$

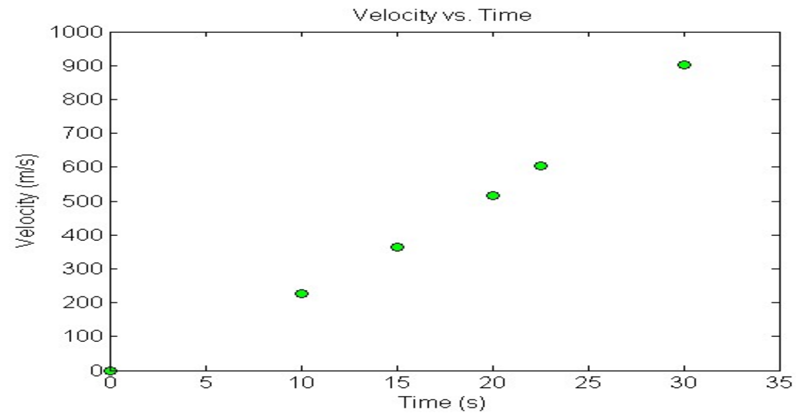
At $t = 16$,

$$v(16) = 362.78 + 30.913(16 - 15)$$

$$= 393.7 \text{ m/s}$$



t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



$$f_1(x) = 0 + 22.704(x - 0)$$

$$f_2(x) = 0 + 22.704(x - 0) + 0.296(x)(x - 10)$$

$$f_3(x) = 0 + 22.704(x - 0) + 0.296(x)(x - 10) + 4.05\left(10^{-3}\right)(x)(x - 10)(x - 15)$$

x (s)	$v(x)$ (m/s)					
0	0	22.704	0.296	**4.05×10^{-3}**	5.82×10^{-5}	1.79×10^{-6}
10	227.04	27.148	0.377	**5.36×10^{-3}**	11.2×10^{-5}	
15	362.78	30.914	0.444	**7.60×10^{-3}**		
20	517.35	34.248	0.558			
22.5	602.97	39.827				
30	901.67					

X = 16 s
1^{st} =363.264
2^{nd}=391.680
3^{rd}=392.069
4^{th}=392.046
5^{th}=392.051

$$f_4(x) = 0 + 22.704(x - 0) + 0.296(x)(x - 10) + 4.05\left(10^{-3}\right)(x)(x - 10)(x - 15) + 5.82\left(10^{-5}\right)(x)(x - 10)(x - 15)(x - 20)$$

$$f_5(x) = 0 + 22.704(x - 0) + 0.296(x)(x - 10) + 4.05\left(10^{-3}\right)(x)(x - 10)(x - 15) + 5.82\left(10^{-5}\right)(x)(x - 10)(x - 15)(x - 20) + 1.79\left(10^{-6}\right)(x)(x - 10)(x - 15)(x - 20)(x - 22.5)$$

Linear Splines

- Connect each two points with straight line functions connecting each pair of points

$$s_1(x) = a_1 + b_1(x - x_1)$$
$$s_2(x) = a_2 + b_2(x - x_2)$$
$$\vdots$$
$$s_i(x) = a_i + b_i(x - x_i)$$
$$\vdots$$
$$s_{n-1}(x) = a_{n-1} + b_{n-1}(x - x_{n-1})$$

b is the <u>slope</u> between points

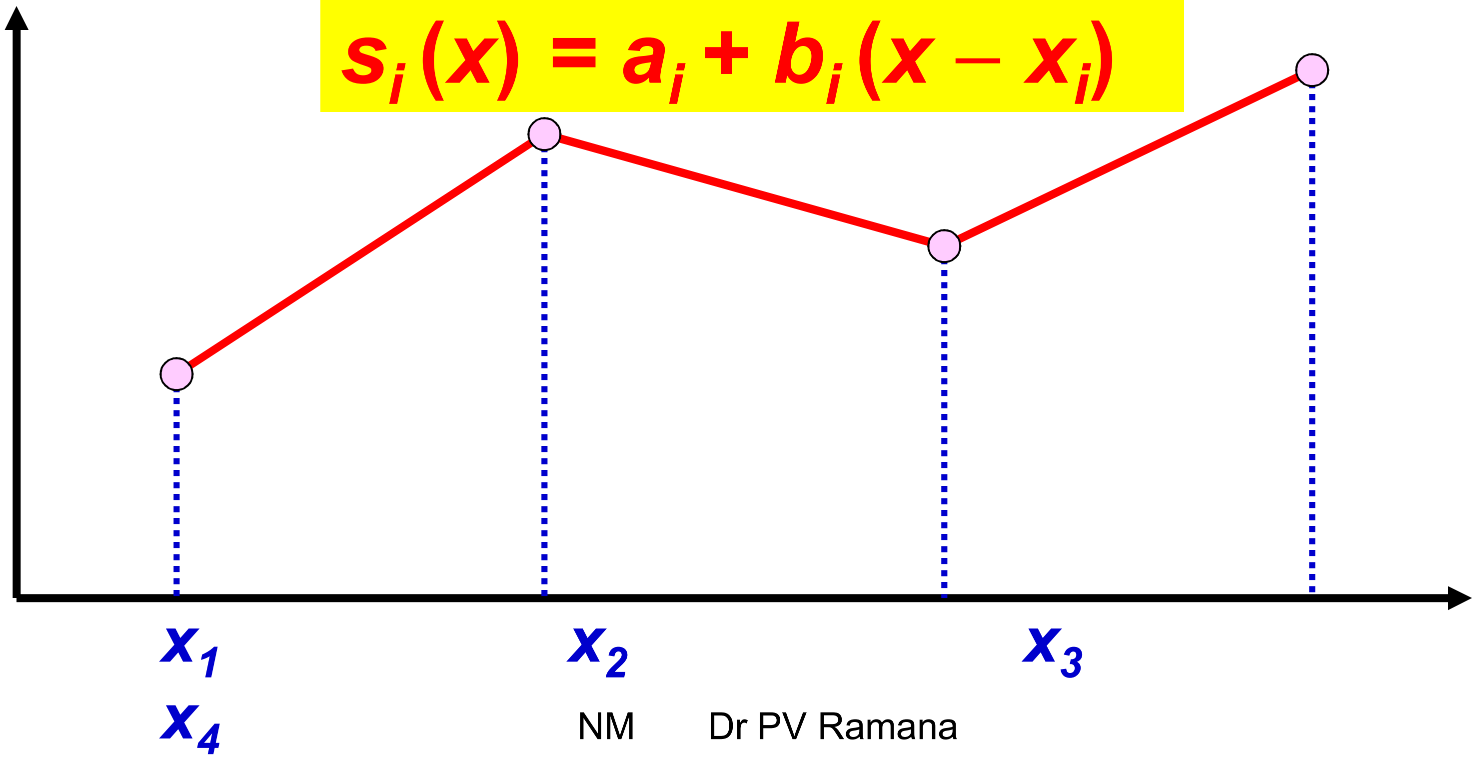
$$a_i = f_i$$
$$b_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

Linear Splines

$$data\ points: (x_{1,}y_1),(x_{2,}y_2),(x_{3,}y_3),\cdots,(x_{n,}y_n)$$

$$interval: \quad I_1=[x_1,x_2], I_2=[x_2,x_3],\cdots,I_{n-1}=[x_{n-1},x_n]$$

$$s_i(x) = a_i + b_i(x - x_i)$$



Linear Splines

$$data\ points : (x_{1,} y_1), (x_{2,} y_2), (x_{3,} y_3), \cdots, (x_{n,} y_n)$$

$$interval : \quad I_1 = [x_1, x_2], I_2 = [x_2, x_3], \cdots, I_{n-1} = [x_{n-1}, x_n]$$

$$s_i(x) = \begin{cases} \left(\dfrac{x - x_2}{x_1 - x_2}\right) f(x_1) + \left(\dfrac{x - x_1}{x_2 - x_1}\right) f(x_2), \; x_1 \le x \le x_2 \\ \left(\dfrac{x - x_3}{x_2 - x_3}\right) f(x_2) + \left(\dfrac{x - x_2}{x_3 - x_2}\right) f(x_3), \; x_2 \le x \le x_3 \\ \quad \vdots \\ \left(\dfrac{x - x_n}{x_{n-1} - x_n}\right) f(x_{n-1}) + \left(\dfrac{x - x_{n-1}}{x_n - x_{n-1}}\right) f(x_n), \; x_{n-1} \le x \le x_n \end{cases}$$

Identical to Lagrange interpolating polynomials

Linear splines

- Connect each two points with straight line
- Functions connecting each pair of points are

$$
\begin{aligned}
s_1(x) &= a_1 + b_1(x - x_1)\ ; && x_1 \le x \le x_2 \\
s_2(x) &= a_2 + b_2(x - x_2)\ ; && x_2 \le x \le x_3 \\
&\vdots \\
s_i(x) &= a_i + b_i(x - x_i)\ ; && x_i \le x \le x_{i+1} \\
&\vdots \\
s_{n-1}(x) &= a_{n-1} + b_{n-1}(x - x_{n-1})\ ; && x_{n-1} \le x \le x_n
\end{aligned}
$$

- **slope** $$b_i = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$

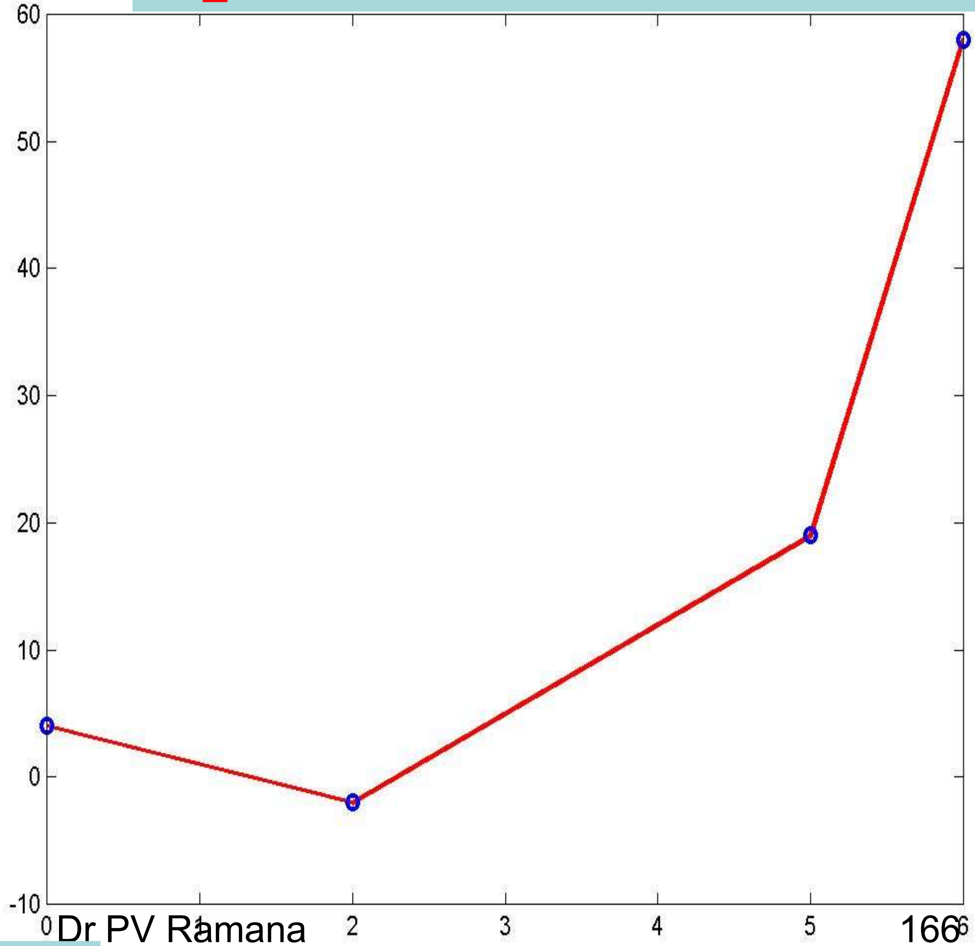
Linear splines are exactly the same as linear interpolation!

$s_1(x) = 4\text{-}3(x)$

$s_2(x) = 4\text{-}3(x)+2(x)(x\text{-}2)$

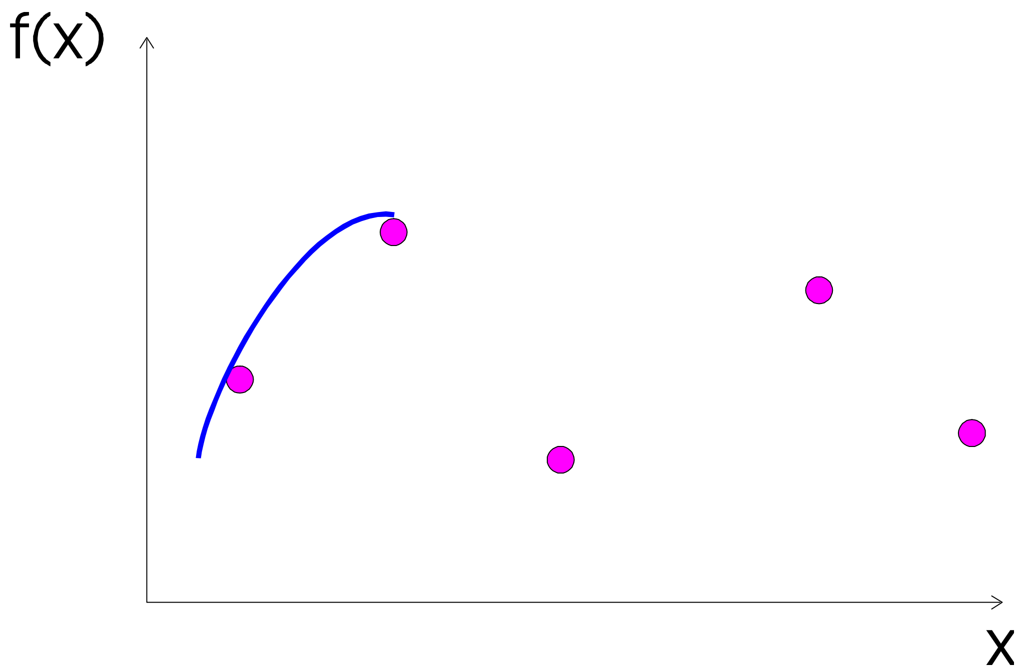
Example:

x	$f(x)$	b_i		
0	4			
		−3		
2	−2		2	
		7		1
5	19		8	
		39		
6	58			



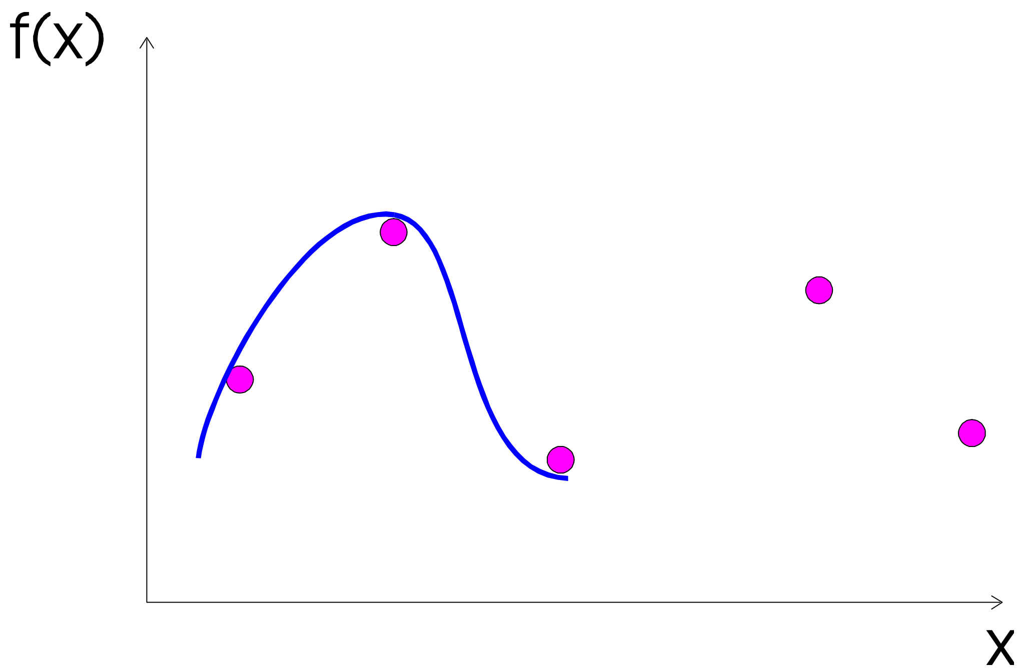
$s_3(x) = 4\text{-}3(x)+2(x)(x\text{-}2) +1(x)(x\text{-}2)(x\text{-}5)$

Quadratic Spline



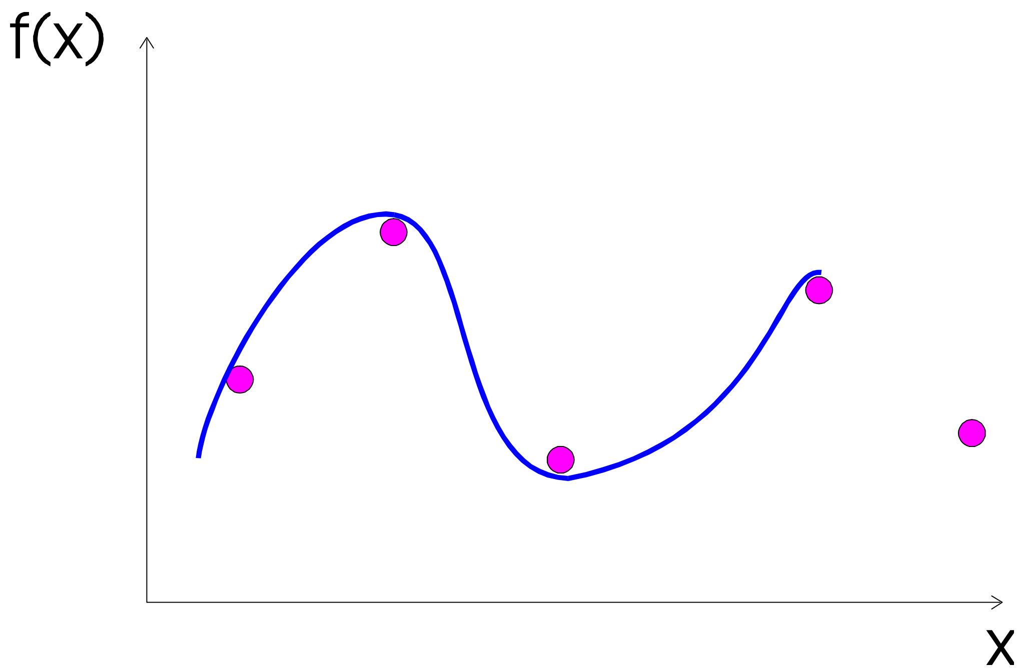
Spline interpolation is an adaptation of the drafting technique of using a spline to draw smooth curves through a series of points

Quadratic Spline



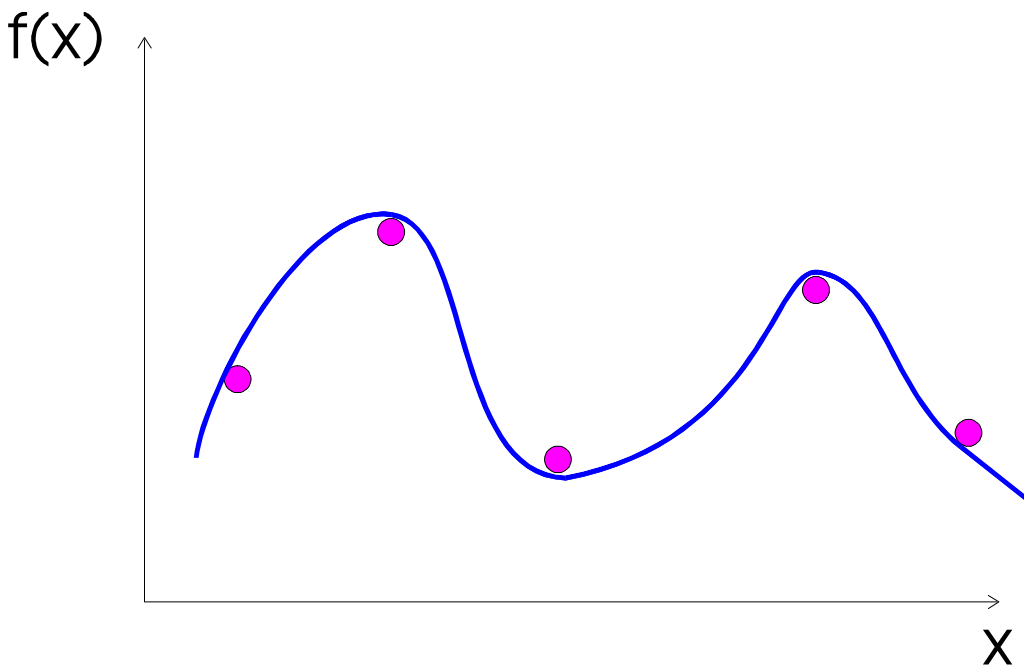
Spline interpolation is an adaptation of the drafting technique of using a spline to draw smooth curves through a series of points

Quadratic Spline



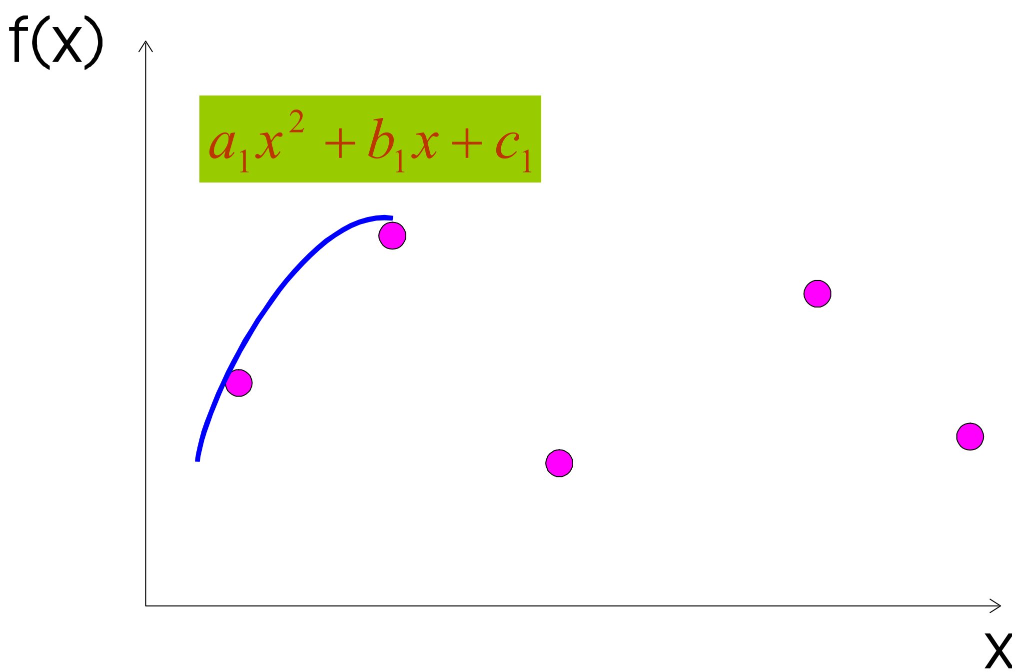
Spline interpolation is an adaptation of the drafting technique of using a spline to draw smooth curves through a series of points

Quadratic Spline

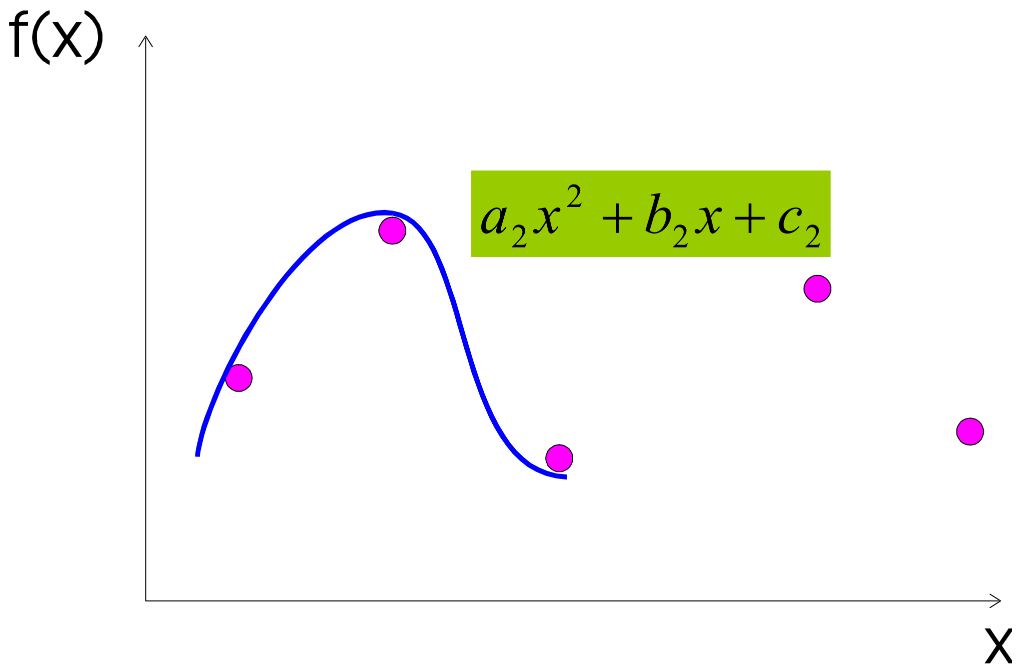


Spline interpolation is an adaptation of the drafting technique of using a spline to draw smooth curves through a series of points

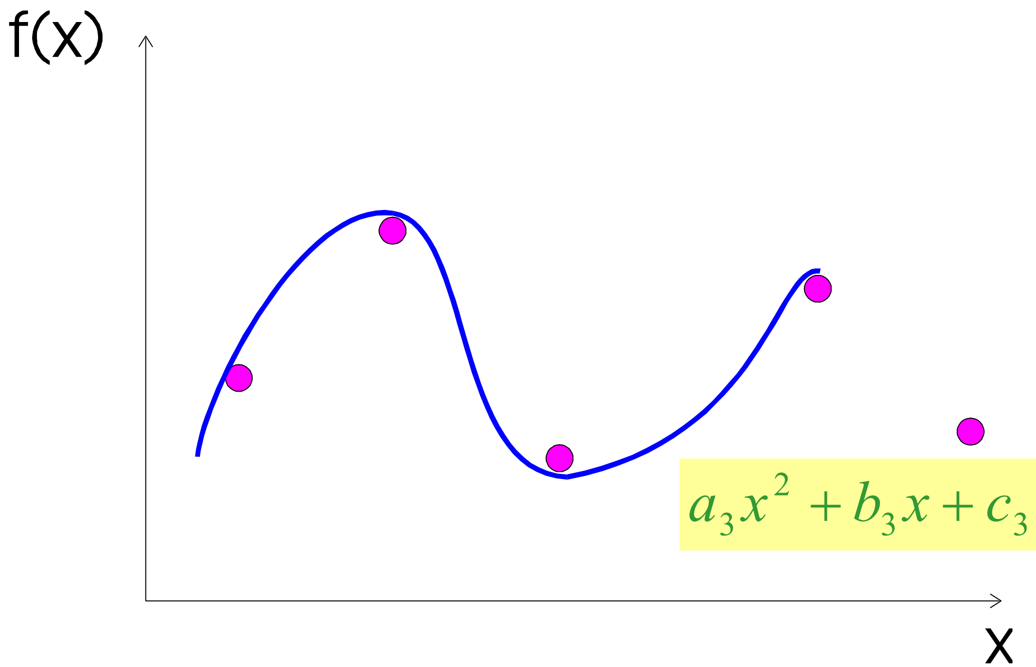
Quadratic Spline



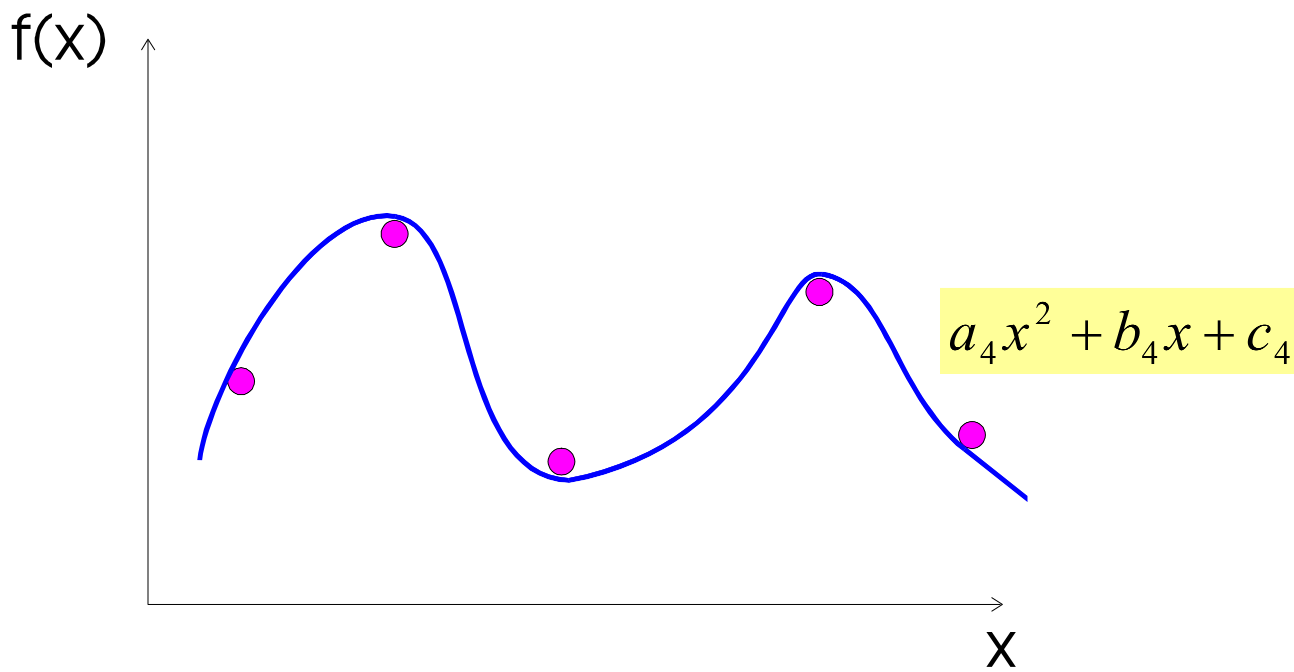
Quadratic Spline



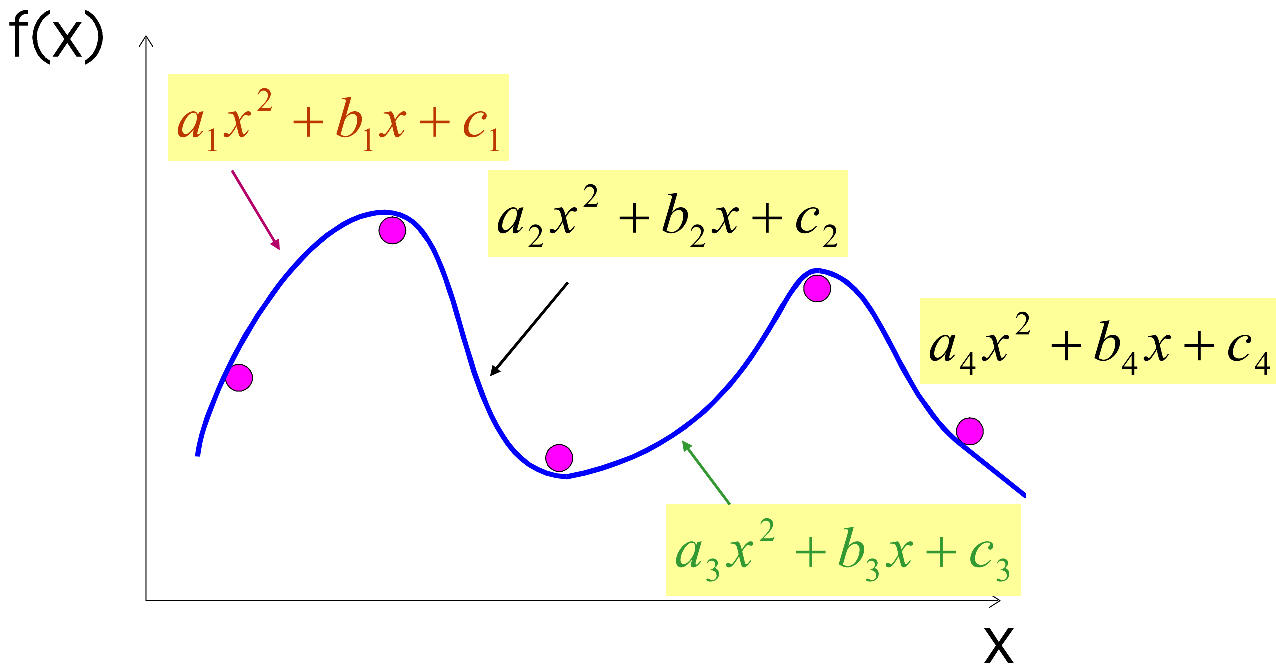
Quadratic Spline



Quadratic Spline



Quadratic Spline



Quadratic Splines

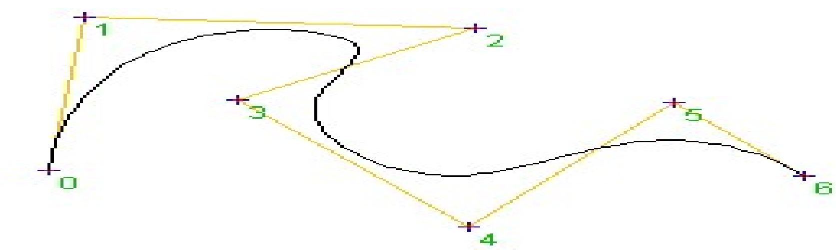
Objective: To derive a second order polynomial for each interval between data points.

Terms: Interior knots and end points

$$f_i(x) = a_i x^2 + b_i x + c_i$$

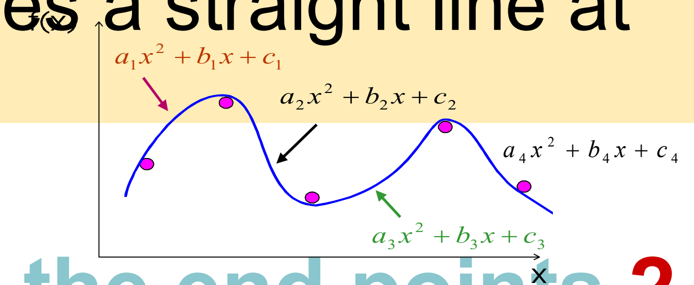
For n+1 data points:

- ***i* = (0, 1, 2, …*n*),**
- ***n* intervals,**
- ***3n* unknown constants (a's, b's & c's)**



Quadratic Splines

- The function values must be equal at the interior knots **(2n-2).**
- The first & last fun must pass through the end point **(2).**
- The first derivatives at the interior knots must be equal **(n-1).**
- The second derivatives at the start/end knot is zero **(1),** (the 2nd derivative function becomes a straight line at the start/end point)



•Equal interior points (2n-2)

•First and last functions pass the end points 2

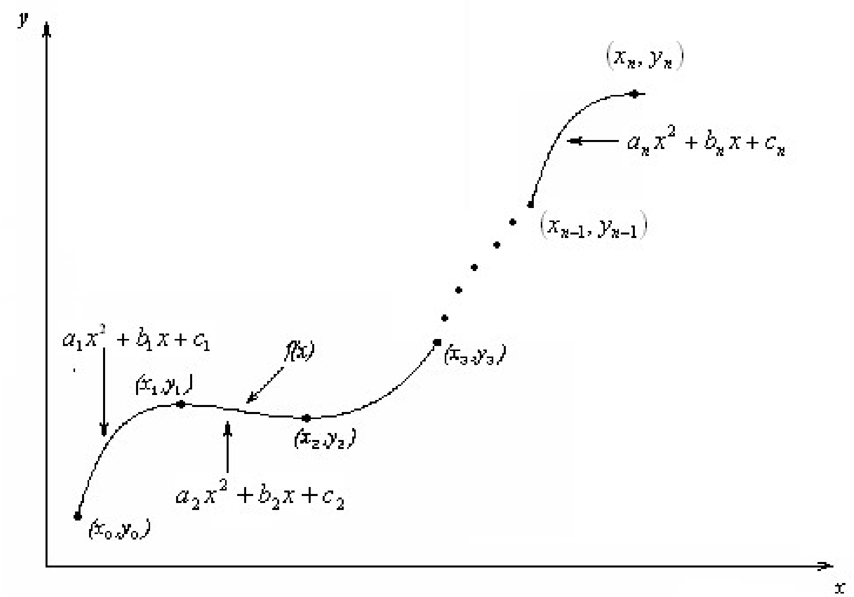
•Equal derivatives at the interior knots (n-1)

•Second derivative at the first point is 0 1

Quadratic Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, && x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, && x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_n x^2 + b_n x + c_n, && x_{n-1} \le x \le x_n
\end{aligned}
$$



Find a_i, b_i, c_i, $i = 1, 2, \ldots,$ n

Quadratic Interpolation (contd)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^2 + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^2 + b_1 x_1 + c_1 = f(x_1)$$

.

.

.

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

$$a_i x_i^2 + b_i x_i + c_i = f(x_i)$$

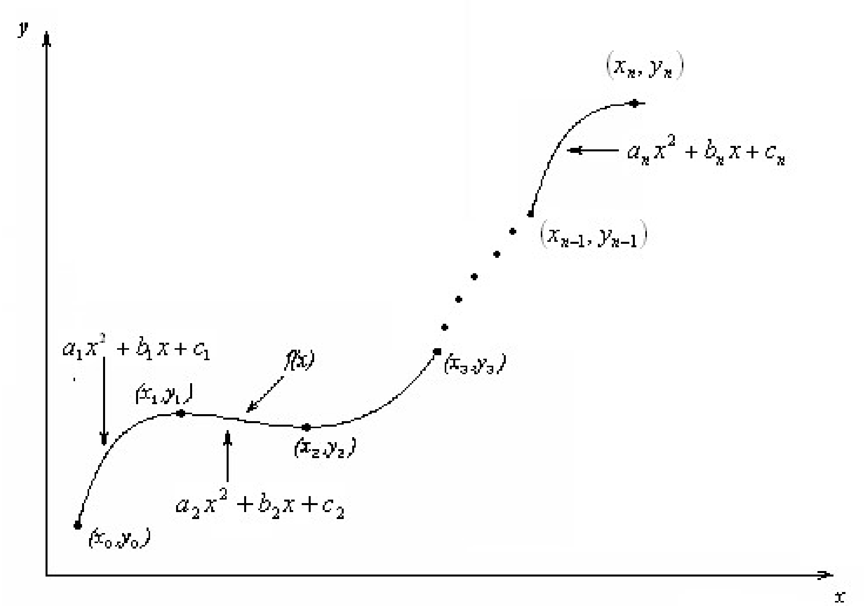
.

.

.

$$a_n x_{n-1}^2 + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^2 + b_n x_n + c_n = f(x_n)$$



This condition gives 2n equations

Quadratic Splines (contd)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1 \quad \text{is} \quad 2a_1x + b_1$$

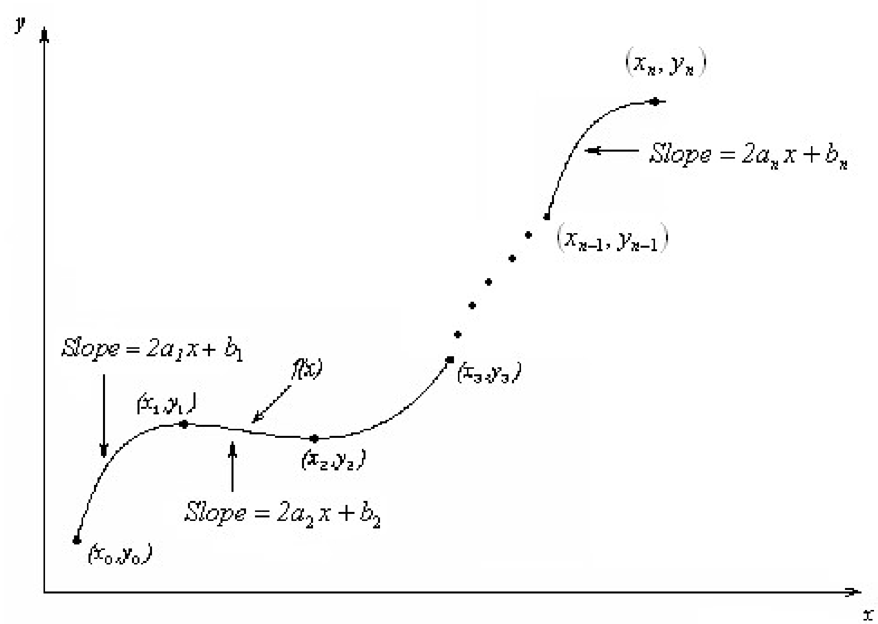
The derivative of the second spline

$$a_2x^2 + b_2x + c_2 \quad \text{is} \quad 2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$



Quadratic Splines (contd)

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

.

.

.

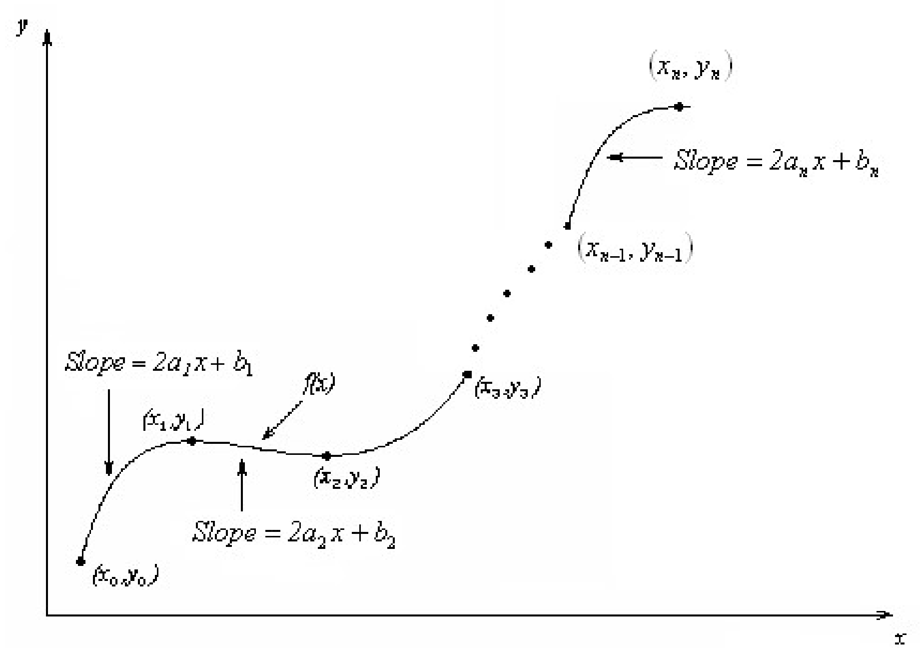
$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

.

.

.

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



We have (n-1) such equations. The total number of equations is $(2n)+(n-1)=(3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

Quadratic Splines (contd)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$f(x) = a_1x^2 + b_1x + c_1, \qquad x_0 \le x \le x_1$$

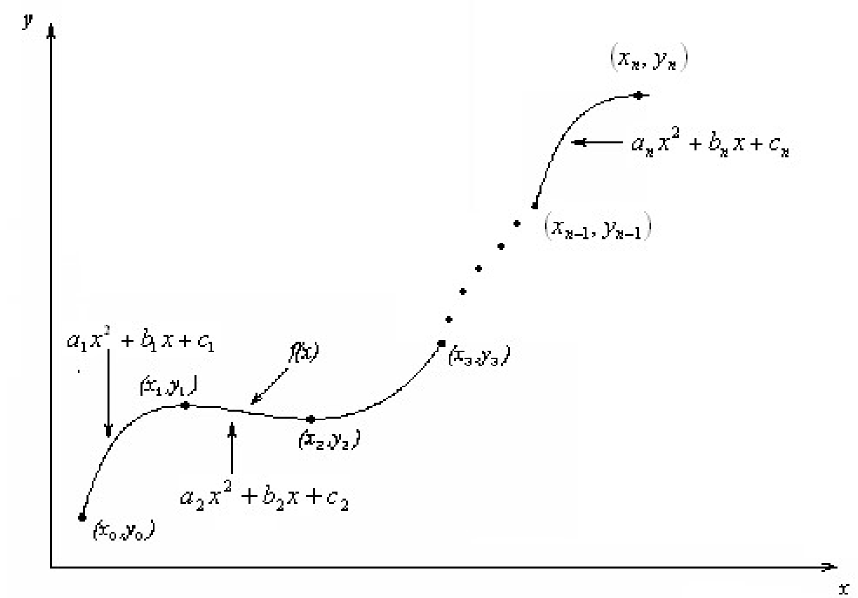
$$= a_2x^2 + b_2x + c_2, \qquad x_1 \le x \le x_2$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$= a_nx^2 + b_nx + c_n, \qquad x_{n-1} \le x \le x_n$$



Quadratic Spline Example

The upward velocity of a rocket is given as a function of time. Using quadratic splines

a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

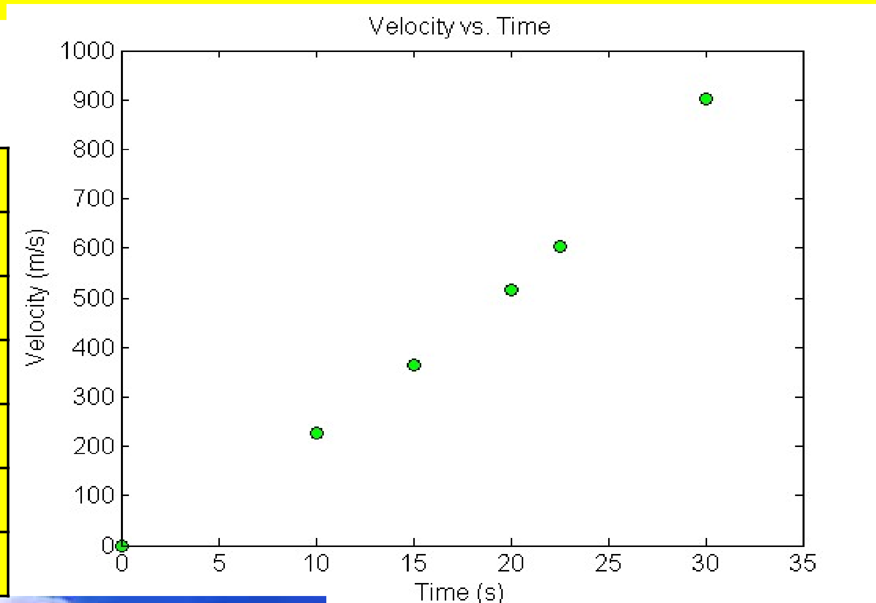
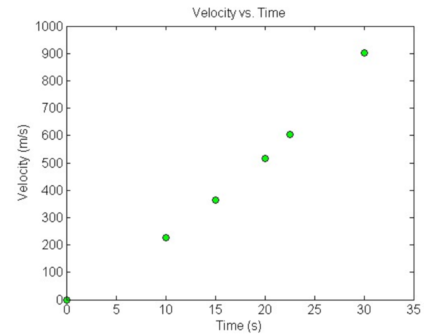


Figure. Velocity vs. time data for the rocket example

Solution

1. Equal interior points



t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$v(t) = a_1t^2 + b_1t + c_1, \qquad 0 \le t \le 10$$

$$= a_2t^2 + b_2t + c_2, \qquad 10 \le t \le 15$$

$$= a_3t^2 + b_3t + c_3, \qquad 15 \le t \le 20$$

$$= a_4t^2 + b_4t + c_4, \qquad 20 \le t \le 22.5$$

$$= a_5t^2 + b_5t + c_5, \qquad 22.5 \le t \le 30$$

Let us set up the equations

Each Spline Goes Through Two Consecutive Data Points

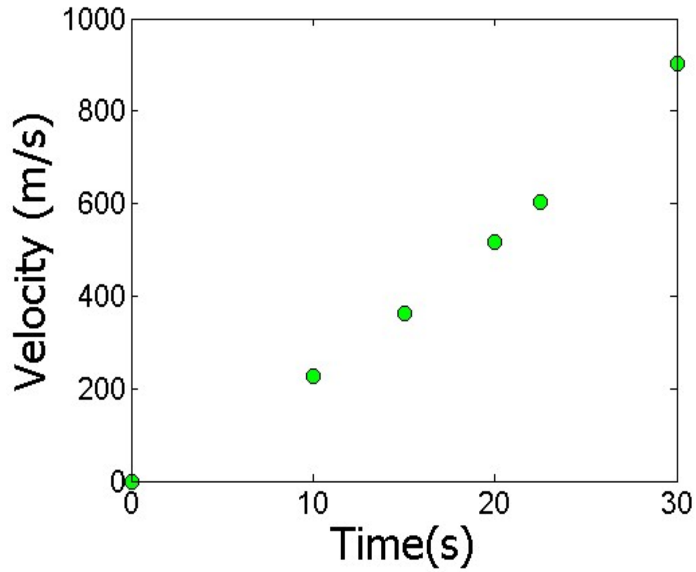
$$v(t) = a_1 t^2 + b_1 t + c_1, \; 0 \le t \le 10$$

$$a_1(0)^2 + b_1(0) + c_1 = 0 \quad \Rightarrow \quad c_1 = 0$$

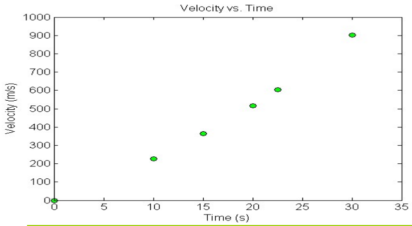
$$a_1(10)^2 + b_1(10) + c_1 = 227.04$$

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Each Spline Goes Through Two Consecutive Data Points



$$v(t) = a_i t^2 + b_i t + c_i$$

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$
$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$

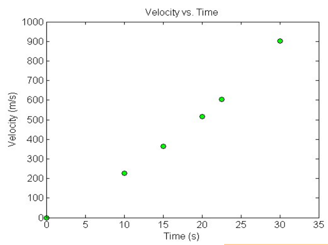
$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$
$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$
$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

2. Equal derivatives at the interior knots

Derivatives are Continuous at Interior Data Points



$$v(t) = a_1 t^2 + b_1 t + c_1, \qquad 0 \le t \le 10$$

$$= a_2 t^2 + b_2 t + c_2, \qquad 10 \le t \le 15$$

$$\left.\frac{d}{dt}\left(a_1 t^2 + b_1 t + c_1\right)\right|_{t=10} = \left.\frac{d}{dt}\left(a_2 t^2 + b_2 t + c_2\right)\right|_{t=10}$$

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

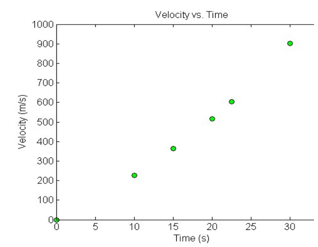
$$\left.\left(2a_1 t + b_1\right)\right|_{t=10} = \left.\left(2a_2 t + b_2\right)\right|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$

Derivatives are continuous at Interior Data Points

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



At t=10

$$2a_1(10)+b_1-2a_2(10)-b_2=0$$

At t=15

$$2a_2(15)+b_2-2a_3(15)-b_3=0$$

At t=20

$$2a_3(20)+b_3-2a_4(20)-b_4=0$$

At t=22.5

$$2a_4(22.5)+b_4-2a_5(22.5)-b_5=0$$

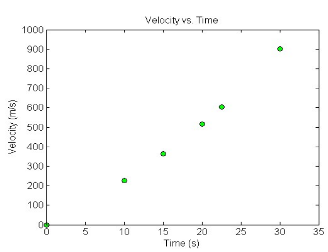
3. **First and last functions pass the end points**

$$a_1(0)^2 + b_1(0) + c_1 = 0 \quad \Rightarrow \quad c_1 = 0$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

4. Second derivative at the first point is 0



Last Equation $a_1 = 0$

0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	a1	0
100	10	1	0	0	0	0	0	0	0	0	0	0	0	0	b1	227.04
0	0	0	100	10	1	0	0	0	0	0	0	0	0	0	c1	227.04
0	0	0	225	15	1	0	0	0	0	0	0	0	0	0	a2	362.78
0	0	0	0	0	0	225	15	1	0	0	0	0	0	0	b2	362.78
0	0	0	0	0	0	400	20	1	0	0	0	0	0	0	c2	517.38
0	0	0	0	0	0	0	0	0	400	20	1	0	0	0	a3	517.38
0	0	0	0	0	0	0	0	0	506.3	22.5	1	0	0	0	b3	602.97
0	0	0	0	0	0	0	0	0	0	0	0	506.3	22.5	1	c3	602.97
0	0	0	0	0	0	0	0	0	0	0	0	900	30	1	a4	901.67
20	1	0	-20	-1	0	0	0	0	0	0	0	0	0	0	b4	0
0	0	0	30	1	0	-30	-1	0	0	0	0	0	0	0	c4	0
0	0	0	0	0	0	40	1	0	-40	-1	0	0	0	0	a5	0
0	0	0	0	0	0	0	0	0	45	1	0	-45	-1	0	b5	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c5	0

a1	b1	c1	a2	b2	c2	a3	b3	c3	a4	b4	c4	a5	b5	c5

a1	b1	c1	a2	b2	c2	a3	b3	c3	a4	b4	c4	a5	b5	c5			
0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	a1	0	$a_1(0)^2 + b_1(0) + c_1 = 0$
100	10	1	0	0	0	0	0	0	0	0	0	0	0	0	b1	227.04	$a_1(10)^2 + b_1(10) + c_1 = 227.04$
0	0	0	100	10	1	0	0	0	0	0	0	0	0	0	c1	227.04	$a_2(10)^2 + b_2(10) + c_2 = 227.04$
0	0	0	225	15	1	0	0	0	0	0	0	0	0	0	a2	362.78	$a_2(15)^2 + b_2(15) + c_2 = 362.78$
0	0	0	0	0	0	225	15	1	0	0	0	0	0	0	b2	362.78	$a_3(15)^2 + b_3(15) + c_3 = 362.78$
0	0	0	0	0	0	400	20	1	0	0	0	0	0	0	c2	517.38	$a_3(20)^2 + b_3(20) + c_3 = 517.35$
0	0	0	0	0	0	0	0	0	400	20	1	0	0	0	a3	517.38	$a_4(20)^2 + b_4(20) + c_4 = 517.35$
0	0	0	0	0	0	0	0	0	506.3	22.5	1	0	0	0	b3	602.97	$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.9$
0	0	0	0	0	0	0	0	0	0	0	0	506.3	22.5	1	c3	602.97	$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.9$
0	0	0	0	0	0	0	0	0	0	0	0	900	30	1	a4	901.67	$a_5(30)^2 + b_5(30) + c_5 = 901.67$
20	1	0	-20	-1	0	0	0	0	0	0	0	0	0	0	b4	0	$20a_1 + b_1 - 20a_2 - b_2 = 0$
0	0	0	30	1	0	-30	-1	0	0	0	0	0	0	0	c4	0	$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0$
0	0	0	0	0	0	40	1	0	-40	-1	0	0	0	0	a5	0	$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0$
0	0	0	0	0	0	0	0	0	45	1	0	-45	-1	0	b5	0	$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0$
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c5	0	$a_1 = 0$

Final Set of Equations

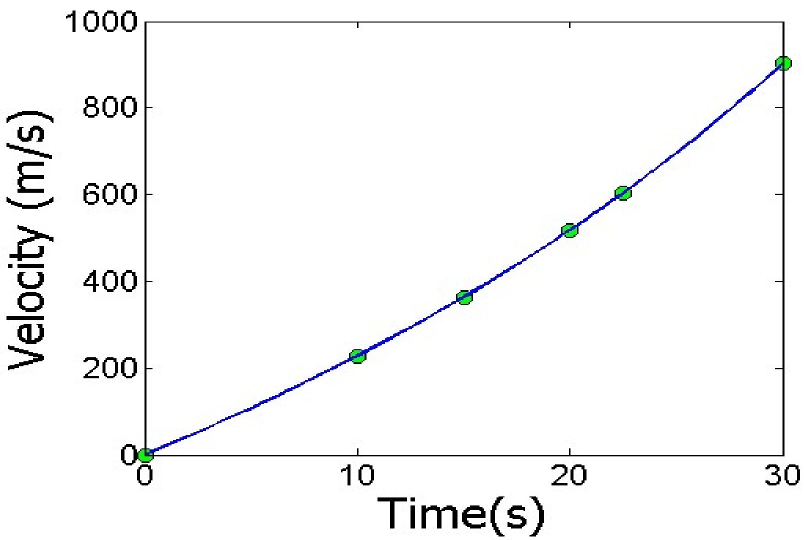
$$
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Coefficients of Spline

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	−0.1356	35.66	−141.61
4	1.6048	−33.956	554.55
5	0.20889	28.86	−152.13

Final Solution

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$



i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	−0.1356	35.66	−141.61
4	1.6048	−33.956	554.55
5	0.20889	28.86	−152.13

Velocity at a Particular Point

a) Velocity at t=16

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$

$$
\begin{aligned}
v(16) &= -0.1356(16)^2 + 35.66(16) - 141.61 \\
&= 394.24 \text{ m/s}
\end{aligned}
$$

The upward velocity of a rocket is given as a function of time. Using quadratic splines

a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

$$
\begin{aligned}
v(t) &= 22.704t, && 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, && 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, && 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, && 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, && 22.5 \le t \le 30
\end{aligned}
$$

$$a(t) = \frac{dv}{dt} = -0.2712t + 35.66$$

$$a(16) = -0.2712(16) + 35.66 = 31.32 \text{ m/s}^2$$

The upward velocity of a rocket is given as a function of time. Using quadratic splines

a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

$$v(t) = 22.704t, \qquad 0 \le t \le 10$$

$$= 0.8888t^2 + 4.928t + 88.88, \qquad 10 \le t \le 15$$

$$= -0.1356t^2 + 35.66t - 141.61, \qquad 15 \le t \le 20$$

$$= 1.6048t^2 - 33.956t + 554.55, \qquad 20 \le t \le 22.5$$

$$= 0.20889t^2 + 28.86t - 152.13, \qquad 22.5 \le t \le 30$$

$$d(t) = \int_{t_1}^{t_2} v\,dt = \int_{11}^{16} \left(0.8\bar{8}t^2 + 4.928t + 88.\bar{8}\right)dt$$

$$Dis\tan ce = 0.8\bar{8}t^3/3 + 4.928t^2/2 + 88.\bar{8}t \,\Big|_{11}^{16}$$

$$d(t) = 0.29627t^3 + 2.464t^2 + 88.\bar{8}t \,\Big|_{11}^{16}$$

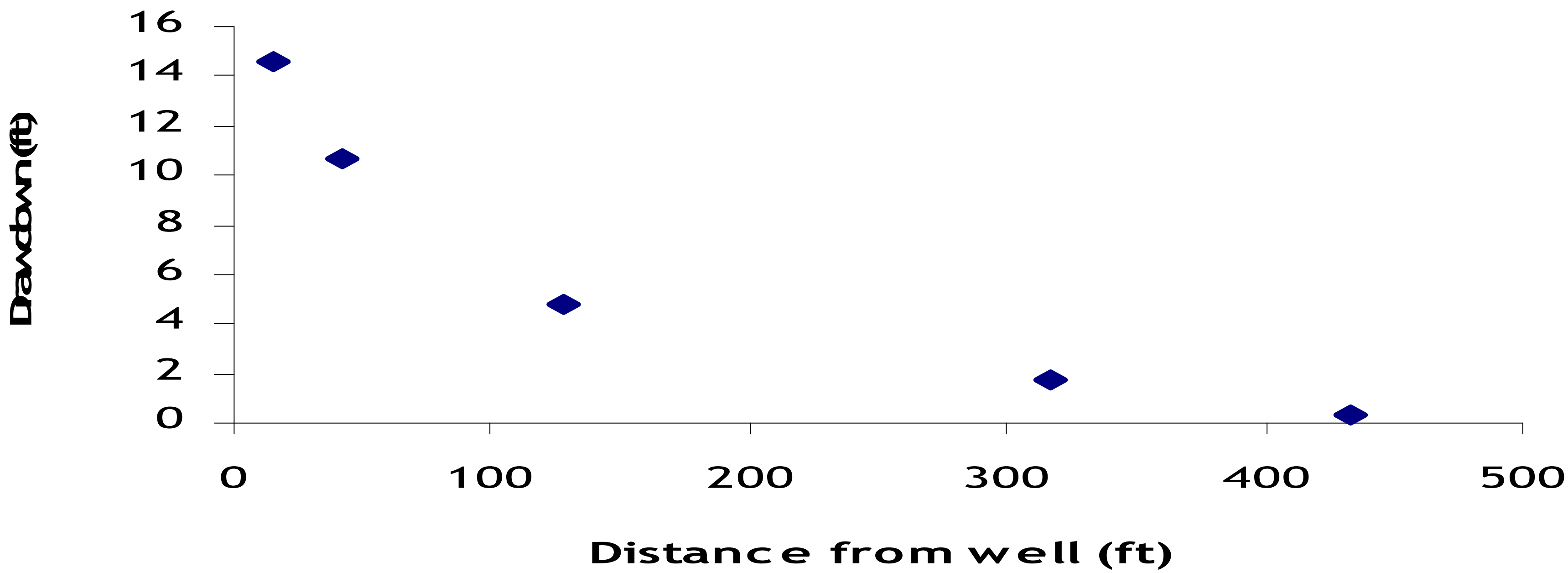
$$d\,\Big|_{11}^{16} = 1591.8265m$$

$$d(t) = \int_{t_1}^{t_2} v\,dt = \int_{11}^{15} \left(0.8888t^2 + 4.928t + 88.88\right)dt + \int_{15}^{16} \left(-0.1356t^2 + 35.66t - 141.61\right)dt$$

$$Dis\tan ce = 1602.67 \quad m$$

Example 3

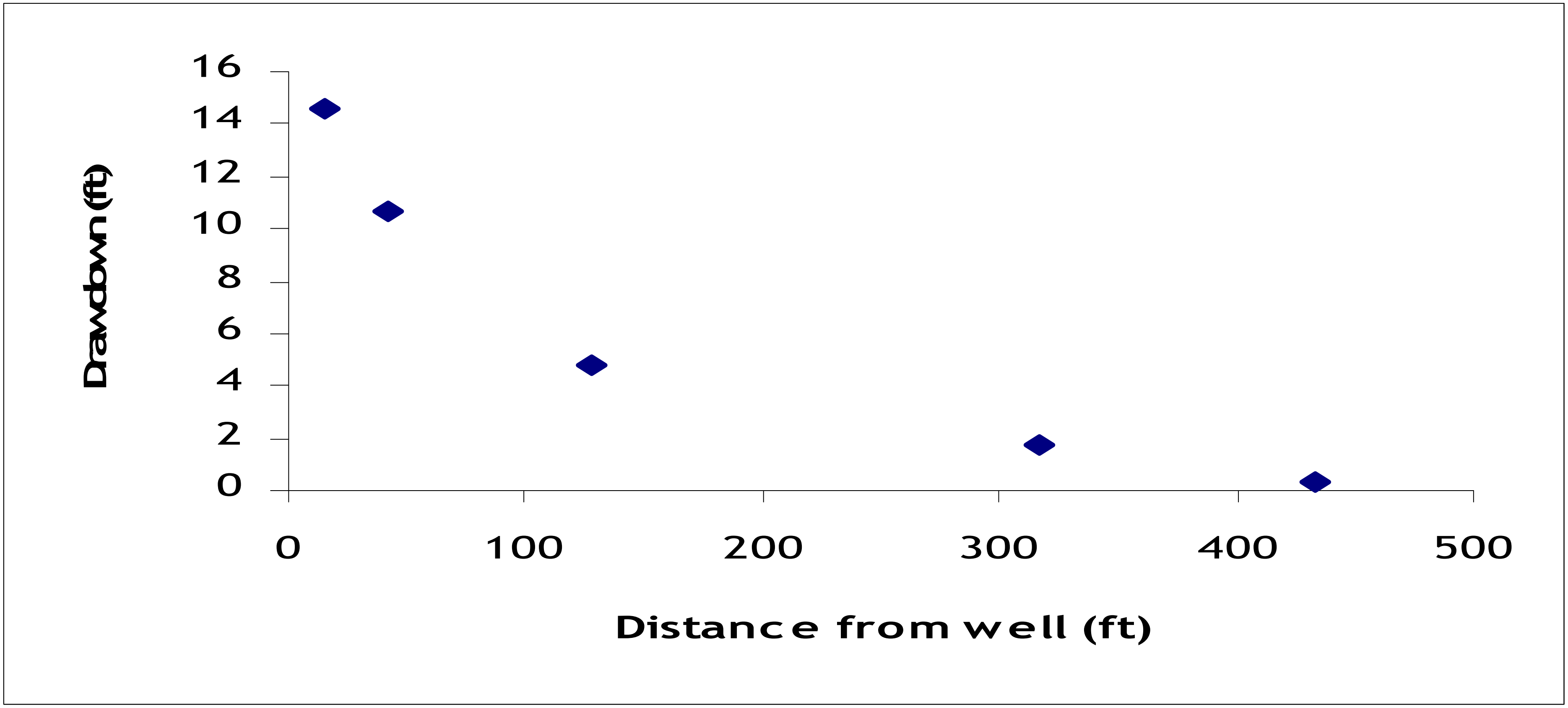
A well pumping at 250 gallons per minute has observation wells located at 15, 42, 128, 317 and 433 ft away along a straight line from the well.
After three hours of pumping, the following drawdown's in the five wells were observed: 14.6, 10.7, 4.8 1.7 and 0.3 ft respectively. Derive equations of each quadratic spline.



15	14.6
42	10.7
128	4.8
317	1.7
433	0.3

$$a_{i-1}x_{i-1}^2 + b_{i-1}x_{i-1} + c_{i-1} = f(x_{i-1})$$

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

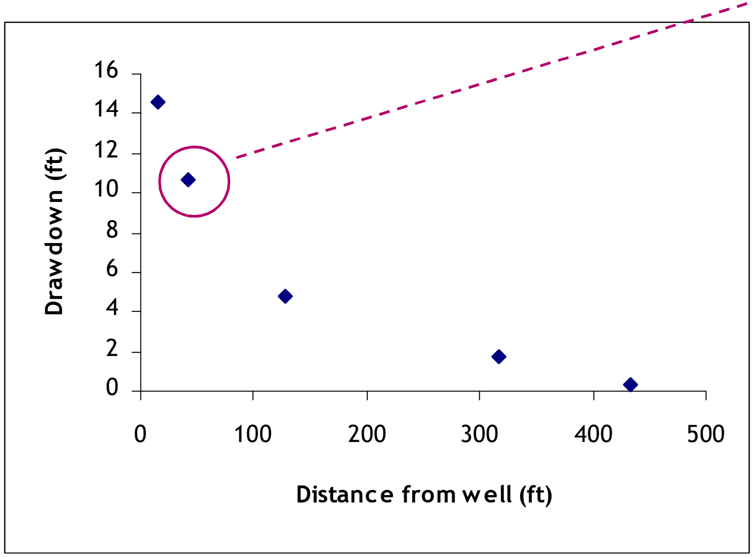


Solution

$$a_{i-1}x_{i-1}^2 + b_{i-1}x_{i-1} + c_{i-1} = f\left(x_{i-1}\right)$$

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f\left(x_{i-1}\right)$$

$$(42)^2 a_2 + 42\, b_2 + c_2 = 10.7$$



15	14.6
42	10.7
128	4.8
317	1.7
433	0.3

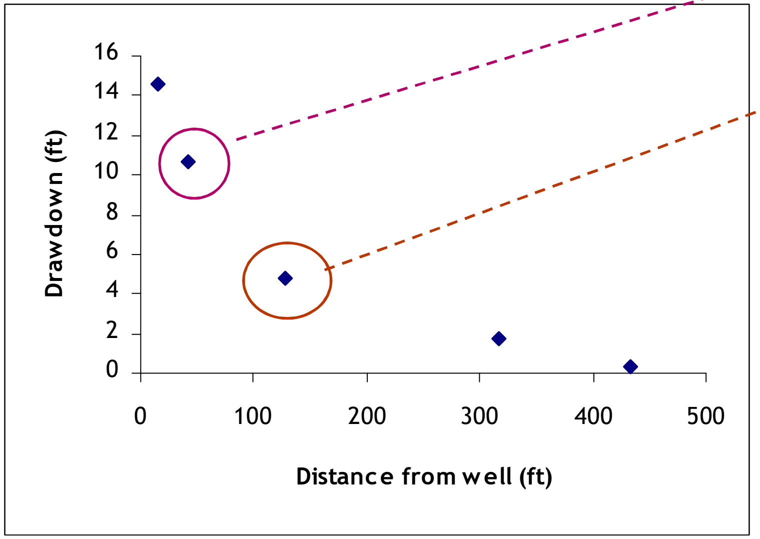
Solution

$$a_{i-1}x_{i-1}^2 + b_{i-1}x_{i-1} + c_{i-1} = f(x_{i-1})$$

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

$(42)^2 a_2 + 42\, b_2 + c_2 = 10.7$

$16{,}384\, a_3 + 128\, b_3 + c_3 = 4.8$



15	14.6
42	10.7
128	4.8
317	1.7
433	0.3

Solution

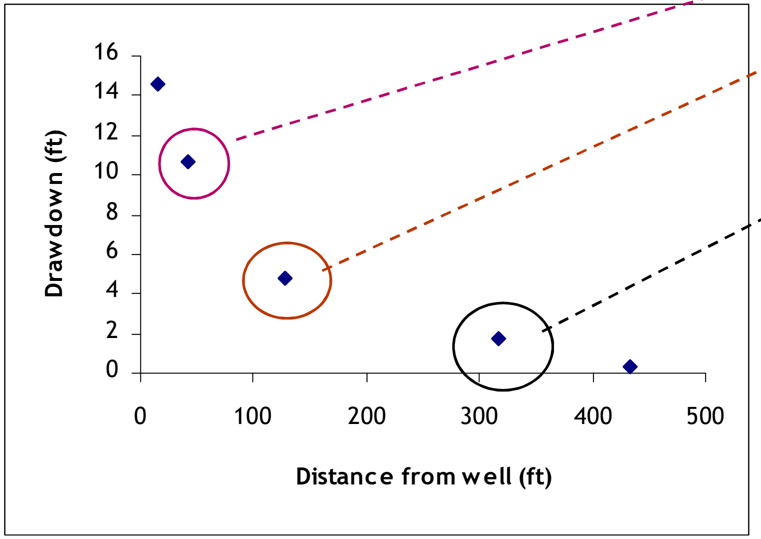
$$a_{i-1}x_{i-1}^2 + b_{i-1}x_{i-1} + c_{i-1} = f(x_{i-1})$$

$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

$(42)^2 a_2 + 42 b_2 + c_2 = 10.7$

$16{,}384 a_3 + 128 b_3 + c_3 = 4.8$

$100{,}489 a_4 + 317 b_4 + c_4 = 1.7$



15	14.6
42	10.7
128	4.8
317	1.7
433	0.3

Solution

$$a_{i-1}x_{i-1}^2 + b_{i-1}x_{i-1} + c_{i-1} = f(x_{i-1})$$

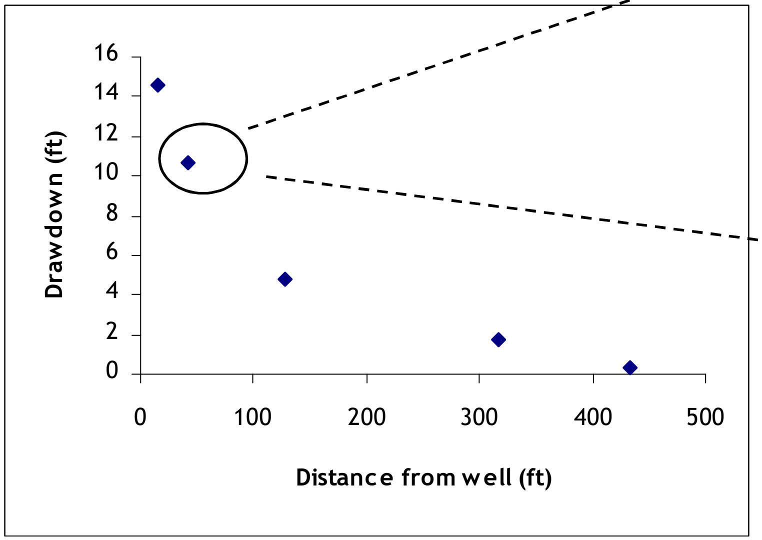
$$a_i x_{i-1}^2 + b_i x_{i-1} + c_i = f(x_{i-1})$$

$(42)^2\, a_2 + 42\, b_2 + c_2 = 10.7$

$16{,}384\, a_3 + 128\, b_3 + c_3 = 4.8$

$100{,}489 a_4 + 317 b_4 + c_4 = 1.7$

$(42)^2\, a_1 + 42 b_1 + c_1 = 10.7$

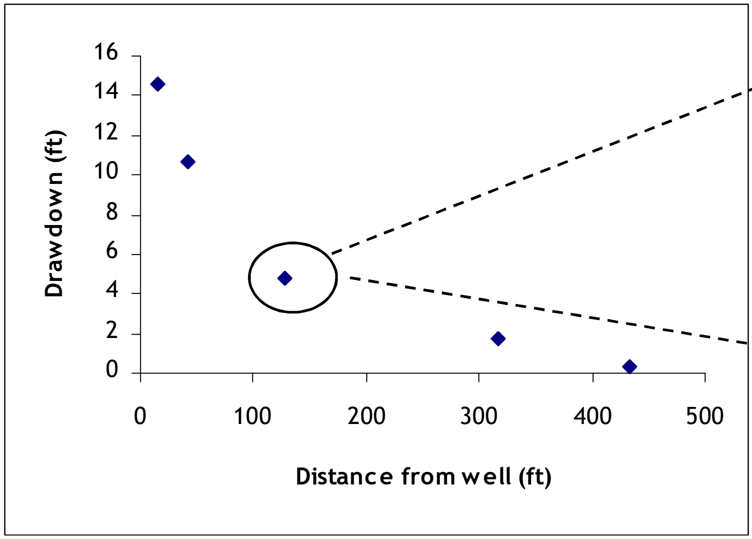


15	14.6
42	10.7
128	4.8
317	1.7
433	0.3

Note: This point is in the first and second polynomial

Solution

15	14.6
42	10.7
128	4.8
317	1.7
433	0.3



$(42)^2 a_2 + 42\, b_2 + c_2 = 10.7$

$16{,}384\, a_3 + 128\, b_3 + c_3 = 4.8$

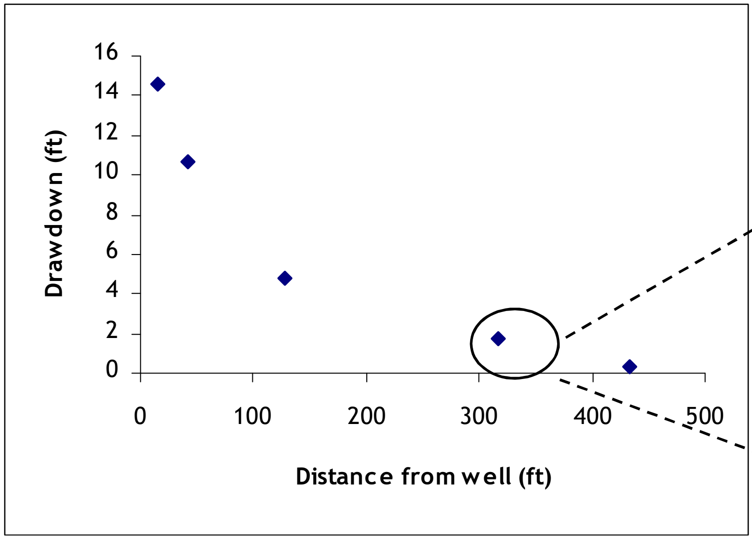
$100{,}489 a_4 + 317 b_4 + c_4 = 1.7$

$(42)^2 a_1 + 42 b_1 + c_1 = 10.7$

$16{,}384 a_2 + 128 b_2 + c_2 = 4.8$

Solution

15	14.6
42	10.7
128	4.8
317	1.7
433	0.3



$(42)^2\ a_2 + 42\ b_2 + c_2 = 10.7$

$16{,}384\ a_3 + 128\ b_3 + c_3 = 4.8$

$\mathbf{100{,}489 a_4 + 317 b_4 + c_4 = 1.7}$

$(42)^2\ a_1 + 42 b_1 + c_1 = 10.7$

$16{,}384 a_2 + 128 b_2 + c_2 = 4.8$

$\mathbf{100{,}489 a_3 + 317 b_3 + c_3 = 1.7}$

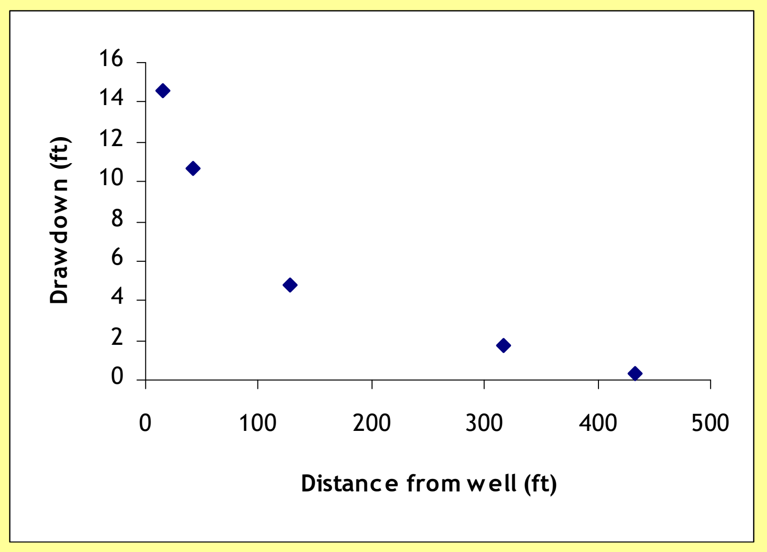
Solution

15	14.6
42	10.7
128	4.8
317	1.7
433	0.3

Similarly, the equations include the end points

$$(15)^2 a_1 + 15\, b_1 + c_1 = 14.6$$

$$187{,}489 a_4 + 433 b_4 + c_4 = 0.3$$



Solution

15	14.6
42	10.7
128	4.8
317	1.7
433	0.3

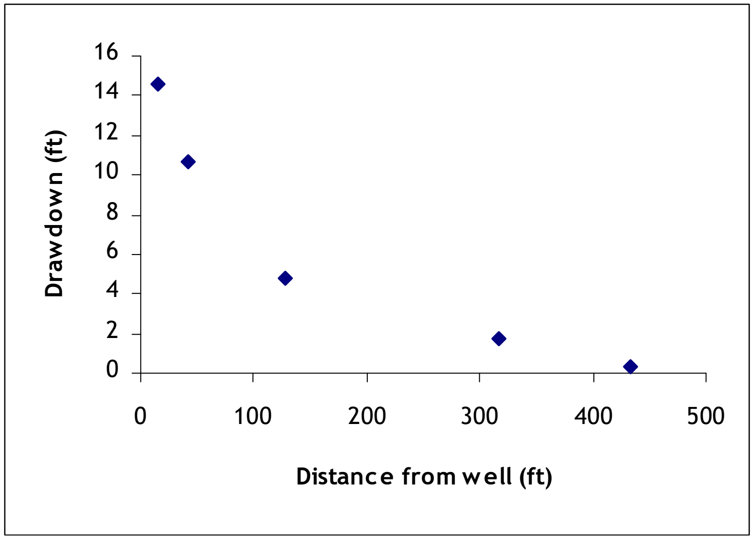
$$2a_{i-1}x_{i-1} + b_{i-1} = 2a_i x_{i-1} + b$$

The first derivative at the interior knots must be equal.

$2a_1 (42) + b_1 = 2a_2 (42) + b_2$

$2a_2 (128) + b_2 = 2a_3 (128) + b_3$

$2a_3 (317) + b_3 = 2a_4 (317) + b_4$



Solution

15	14.6
42	10.7
128	4.8
317	1.7
433	0.3

Addition the last condition $a_1 = 0$

One should be able to set these equations into a matrix to solve for a_i , b_i, and c_i for i = 1,3

Calculate a_i , b_i, and c_i

$(42)^2\ a_2 + 42\ b_2 + c_2 = 10.7$

$16{,}384\ a_3 + 128\ b_3 + c_3 = 4.8$

$100{,}489a_4 + 317b_4 + c_4 = 1.7$

$(42)^2\ a_1 + 42b_1 + c_1 = 10.7$

$16{,}384a_2 + 128b_2 + c_2 = 4.8$

$100{,}489a_3 + 317b_3 + c_3 = 1.7$

$(15)^2 a_1 + 15\ b_1 + c_1 = 14.6$

$187{,}489a_4 + 433b_4 + c_4 = 0.3$

15	14.6
42	10.7
128	4.8
317	1.7
433	0.3

$$2a_{i-1}x_{i-1} + b_{i-1} = 2a_i x_{i-1} + b$$

$2a_1\ (42) + b_1 = 2a_2\ (42) + b_2$

$2a_2\ (128) + b_2 = 2a_3\ (128) + b_3$

$2a_3\ (317) + b_3 = 2a_4\ (317) + b_4$

225	15	1	0	0	0	0	0	0	0	0	0	a1	14.6
$(42)^2$	42	1	0	0	0	0	0	0	0	0	0	b1	10.7
0	0	0	$(42)^2$	42	1	0	0	0	0	0	0	c1	10.7
0	0	0	128^2	128	1	0	0	0	0	0	0	A2	4.8
0	0	0	0	0	0	128^2	128	1	0	0	0	B2	4.8
0	0	0	0	0	0	317^2	317	1	0	0	0	c2	1.7
0	0	0	0	0	0	0	0	0	317^2	317	1	a3	1.7
0	0	0	0	0	0	0	0	0	433^2	433	1	a4	0.3
84	1	0	-84	-1	0	0	0	0	0	0	0	b4	0
0	0	0	256	1	0	-256	-1	0	0	0	0	c4	0
0	0	0	0	0	0	634	1	0	-634	-1	0	a5	0

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c5	0

Addition the last condition $a_1 = 0$

15	14.(
42	10.7
128	4.8
317	1.7
433	0.3

a1	b1	c1	a2	b2	c2	a3	b3	c3	a4	b4	c4		
225	15	1	0	0	0	0	0	0	0	0	0	a1	14.6
$(42)^2$	42	1	0	0	0	0	0	0	0	0	0	b1	10.7
0	0	0	$(42)^2$	42	1	0	0	0	0	0	0	c1	10.7
0	0	0	128^2	128	1	0	0	0	0	0	0	A2	4.8
0	0	0	0	0	0	128^2	128	1	0	0	0	B2	4.8
0	0	0	0	0	0	317^2	317	1	0	0	0	c2	1.7
0	0	0	0	0	0	0	0	0	317^2	317	1	a3	1.7
0	0	0	0	0	0	0	0	0	433^2	433	1	a4	0.3
84	1	0	-84	-1	0	0	0	0	0	0	0	b4	0
0	0	0	256	1	0	-256	-1	0	0	0	0	c4	0
0	0	0	0	0	0	634	1	0	-634	-1	0	a5	0

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	c5	0

$$a_1(15)^2 + b_1(15) + c_1 = 0$$

$$a_1(42)^2 + b_1(42) + c_1 = 10.7$$

$$a_2(42)^2 + b_2(42) + c_2 = 10.7$$

$$a_2(128)^2 + b_2(128) + c_2 = 4.8$$

$$a_3(128)^2 + b_3(128) + c_3 = 4.8$$

$$a_3(317)^2 + b_3(317) + c_3 = 1.7$$

$$a_4(317)^2 + b_4(317) + c_4 = 1.7$$

$$a_5(433)^2 + b_5(433) + c_5 = 0.3$$

$$84a_1 + b_1 - 84a_2 - b_2 = 0$$

$$2a_2(128) + b_2 - 2a_3(128) - b_3 = 0$$

$$2a_3(317) + b_3 - 2a_4(317) - b_4 = 0$$

$$a_1 = 0$$

Quadric Splines – Example 4

Fit the following data with **quadratic splines**. Estimate the value at $x = 5$.

x	3.0	4.5	7.0	9.0
f(x)	2.5	1.0	2.5	0.5

Solutions:

There are **3** intervals (n=3), **9** unknowns.

Quadric Splines - Example

1. Equal interior points:

- **For first interior point (4.5, 1.0)**

x	3.0	4.5	7.0	9.0
f(x)	2.5	1.0	2.5	0.5

The 1st equation:

$$x_1^2 a_1 + x_1 b_1 + c_1 = f(x_1)$$

$$(4.5)^2 a_1 + 4.5 b_1 + c_1 = f(4.5) \longrightarrow \boxed{20.25 a_1 + 4.5 b_1 + c_1 = 1.0}$$

The 2nd equation:

$$x_1^2 a_2 + x_1 b_2 + c_2 = f(x_1)$$

$$(4.5)^2 a_2 + 4.5 b_2 + c_2 = f(4.5) \longrightarrow \boxed{20.25 a_2 + 4.5 b_2 + c_2 = 1.0}$$

Quadric Splines - Example

- For second interior point **(7.0, 2.5)**

The 3rd equation:

x	3.0	4.5	7.0	9.0
f(x)	2.5	1.0	2.5	0.5

$$x_2^2 a_2 + x_2 b_2 + c_2 = f(x_2)$$

$$(7)^2 a_2 + 7b_2 + c_2 = f(7) \rightarrow \boxed{49a_2 + 7b_2 + c_2 = 2.5}$$

The 4th equation:

$$x_2^2 a_3 + x_2 b_3 + c_3 = f(x_2)$$

$$(7)^2 a_3 + 7b_3 + c_3 = f(7) \rightarrow \boxed{49a_3 + 7b_3 + c_3 = 2.5}$$

Quadric Splines - Example

- **Equal derivatives at the interior knots:**

x	3.0	4.5	7.0	9.0
f(x)	2.5	1.0	2.5	0.5

For first interior point **(4.5, 1.0)**

$$2x_1 a_1 + b_1 = 2x_1 a_2 + b_2 \rightarrow \boxed{9a_1 + b_1 = 9a_2 + b_2}$$

For second interior point **(7.0, 2.5)**

$$2x_2 a_2 + b_2 = 2x_3 a_3 + b_3 \rightarrow \boxed{14a_2 + b_2 = 14a_3 + b_3}$$

Quadric Splines - Example

x	3.0	4.5	7.0	9.0
f(x)	2.5	1.0	2.5	0.5

- **First and last functions pass the end points**

For the start point **(3.0, 2.5)**

$$x_0^2\, a_1 + x_0\, b_1 + c_1 = f(x_0) \rightarrow \boxed{9a_1 + 3b_1 + c_1 = 2.5}$$

For the end point **(9, 0.5)**

$$x_3^2 a_1 + x_3 b_3 + c_3 = f(x_3) \rightarrow \boxed{81a_3 + 9b_3 + c_3 = 0.5}$$

- **Second derivative at the first point is 0** $\boxed{f''(x_0) = a_1 = 0}$

Quadric Splines - Example

$$\begin{bmatrix} 4.5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 20.25 & 4.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 49 & 7 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 49 & 7 & 1 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 81 & 9 & 1 \\ 1 & 0 & -9 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 14 & 1 & 0 & -14 & -1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2.5 \\ 2.5 \\ 2.5 \\ 0.5 \\ 0 \\ 0 \end{bmatrix}$$

Quadric Splines - Example

x	3.0	4.5	7.0	9.0
f(x)	2.5	1.0	2.5	0.5

Solving these 8 equations with 8 unknowns

$$a_1 = 0, \quad b_1 = -1, \quad c_1 = 5.5$$
$$a_2 = 0.64, \; b_2 = -6.76, \; c_2 = 18.46$$
$$a_3 = -1.6, \; b_3 = 24.6, \quad c_3 = -91.3$$

$$f_2(x) = f_2(5) = 0.5$$

$$f_2(x) = f_2(5) = -3.84$$

$$f_2(x) = f_2(5) = -8.3$$

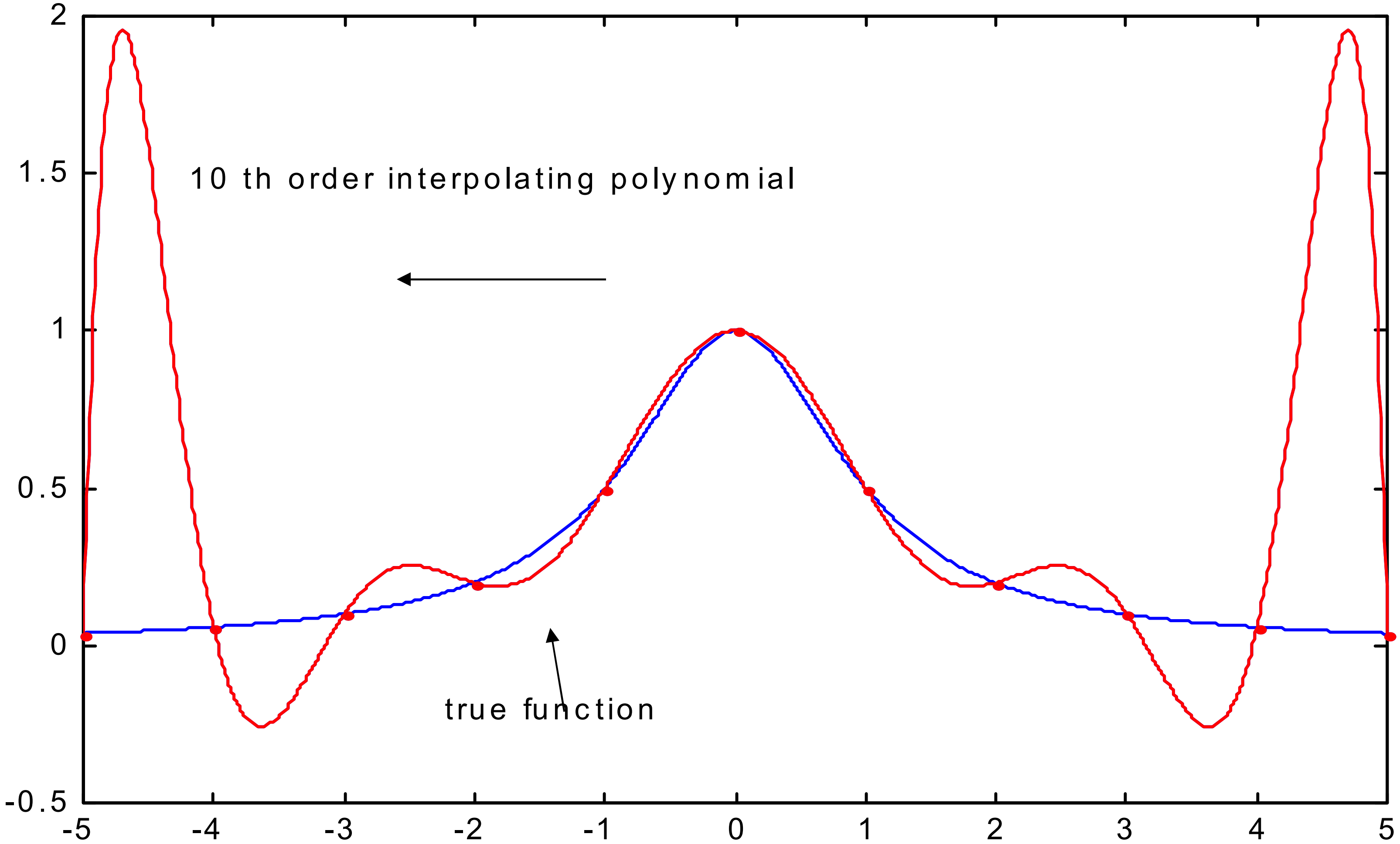
$$f_1(x) = -x + 5.5, \qquad 3.0 \le x \le 4.5$$

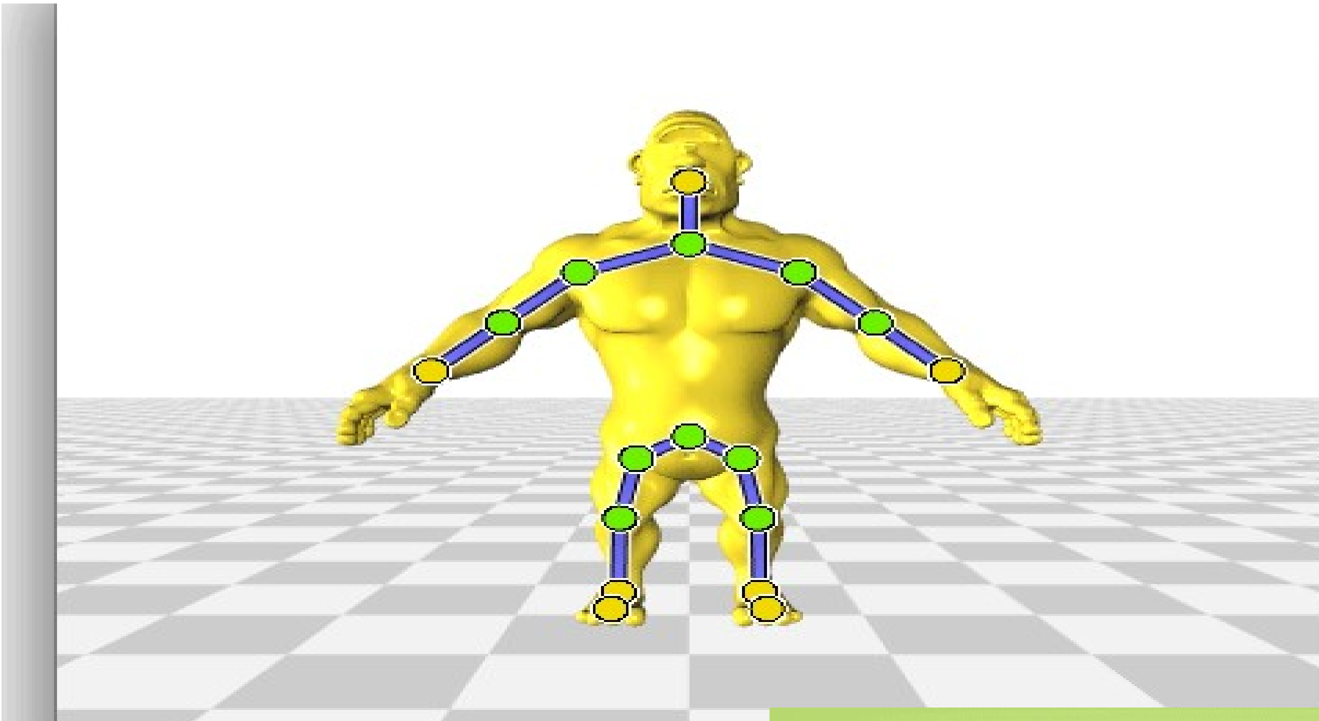
$$f_2(x) = 0.46x^2 - 6.76x + 18.46, \qquad 4.5 \le x \le 7.0$$

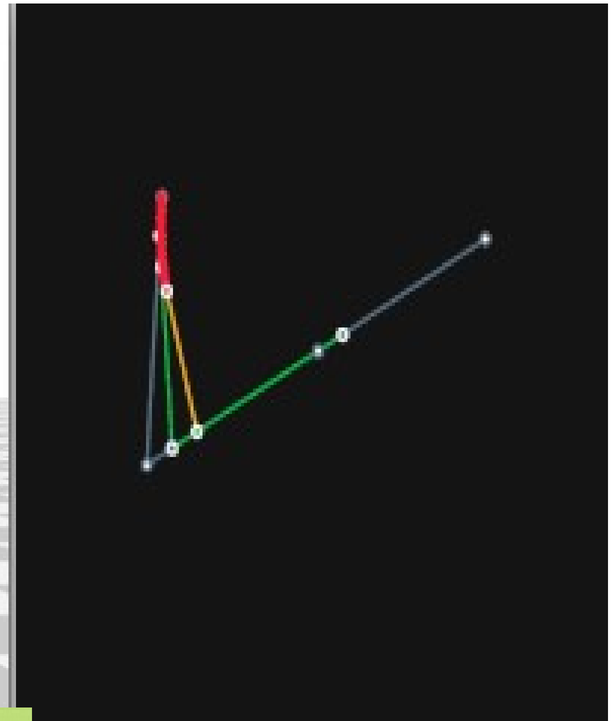
$$f_3(x) = -1.6x^2 + 24.6x - 91.3, \qquad 7.0 \le x \le 9.0$$

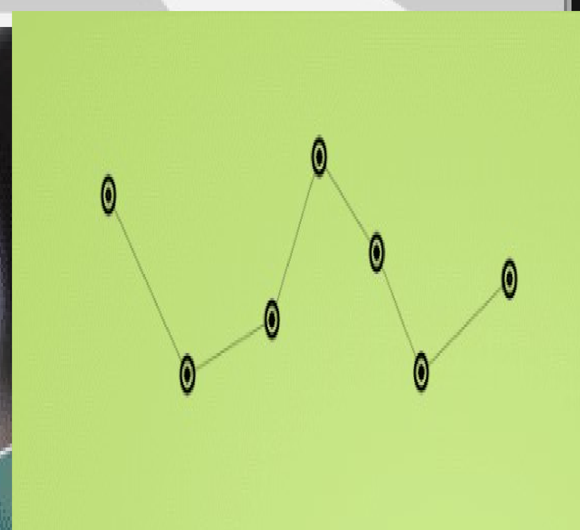
$$f_2(x) = f_2(5) = -3.84$$

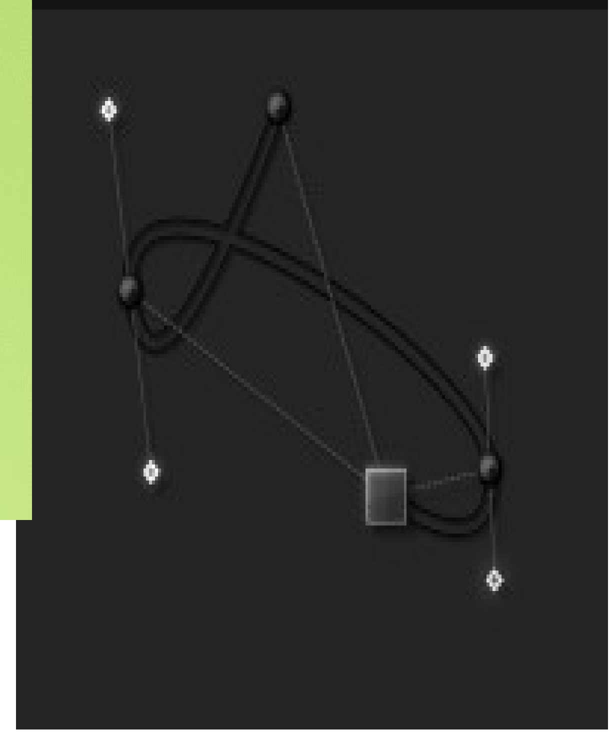
10^{th} Order Polynomial Interpolation











Cubic Splines

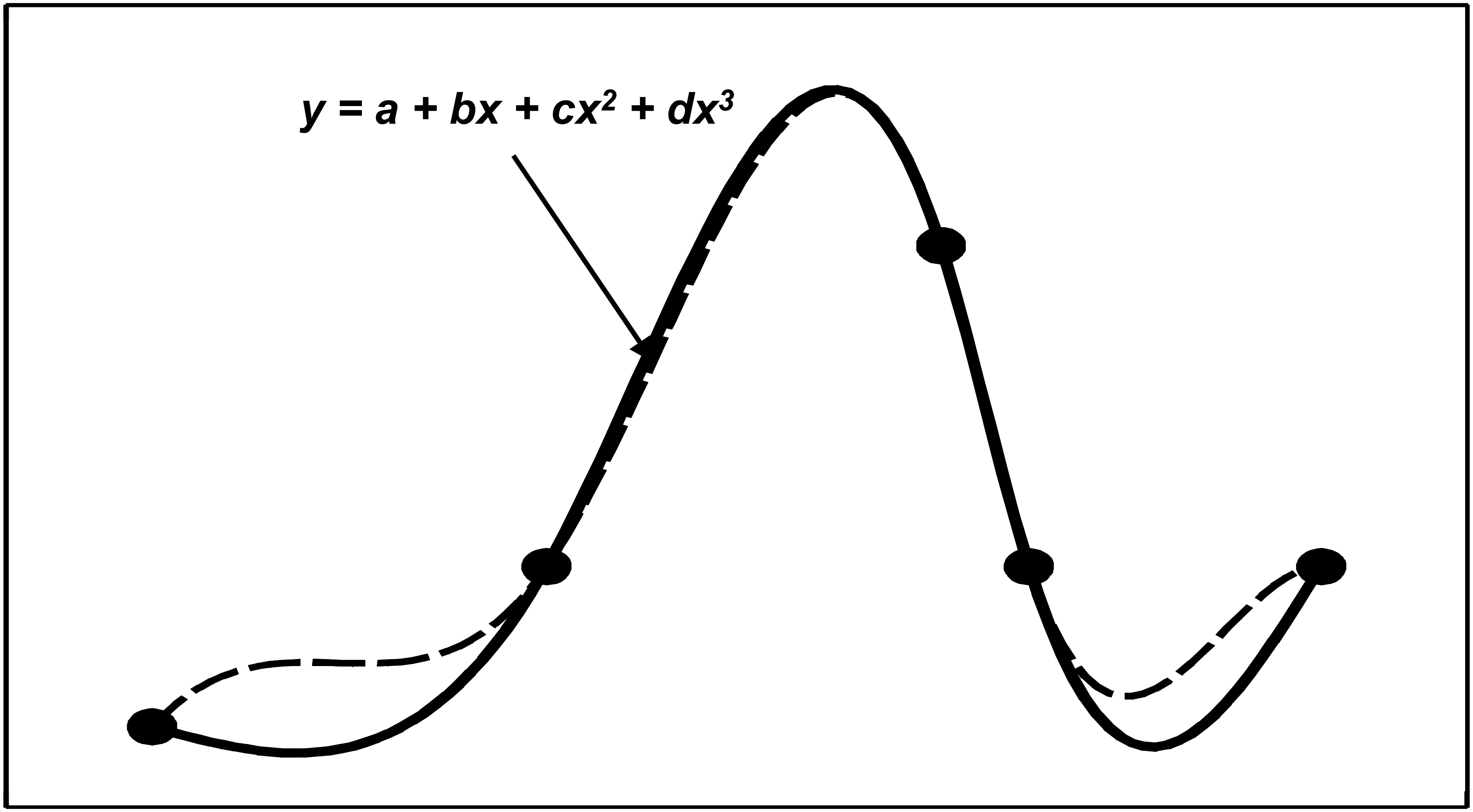
Objective: To derive a third order polynomial for each interval between data points.

Terms: Interior knots and end points

$$f_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$$

For *n*+1 data points:

- ***i* = (0, 1, 2, …*n*),**
- ***n* intervals,**
- ***4n unknown constants (a's, b's ,c's and d's)***



In this example, five arbitrary points (defining four *intervals*) are joined by a cubic spline. In each interval the four coefficients ***a...d*** are different. Their 16 values are fixed by requiring that the curve *pass through the points* and that its *slope be continuous* at each point (no ‘corners’). At each endpoint we must therefore either *specify the first derivative* (dashed line) or require that the *second derivative be zero* (‘natural’ cubic spline, full line).

Chapter Objectives

- Understanding that splines minimize oscillations by fitting lower-order polynomials to data in a piecewise fashion.
- Knowing how to develop code to perform table lookup.
- Recognizing why cubic polynomials are preferable to quadratic and higher-order splines.
- Understanding the conditions that underlie a cubic fit.
- Understanding the differences between natural, clamped, and not-a-knot end conditions.
- Knowing how to fit a spline to data with MATLAB's built-in functions.
- Understanding how multidimensional interpolation is implemented with MATLAB.

Introduction to Splines

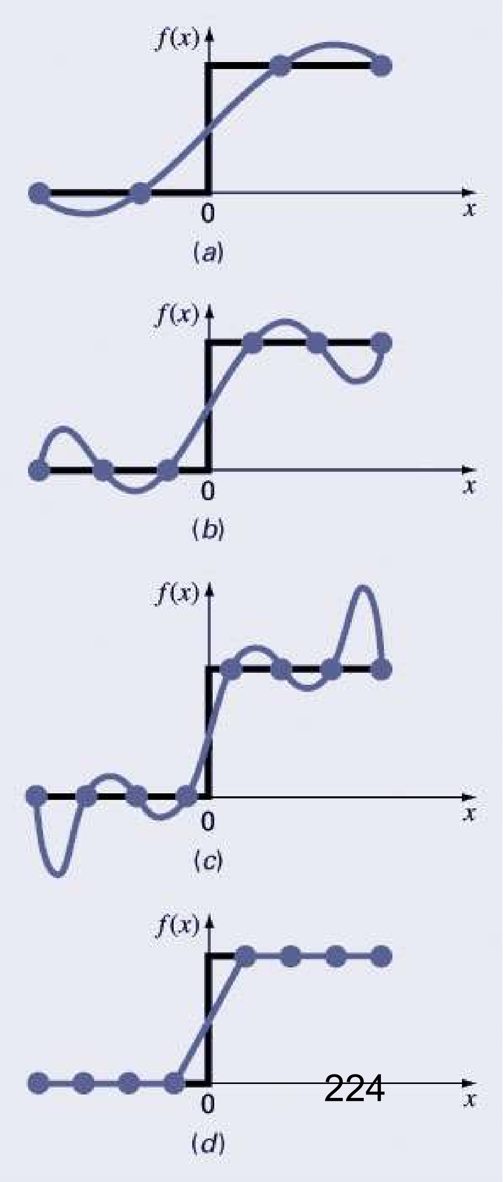
- An alternative approach to using a single $(n\text{-}1)^{th}$ order polynomial to interpolate between n points is to apply lower-order polynomials in a piecewise fashion to subsets of data points.

- These connecting polynomials are called *spline functions*.

- Splines minimize oscillations and reduce round-off error due to their lower-order nature.

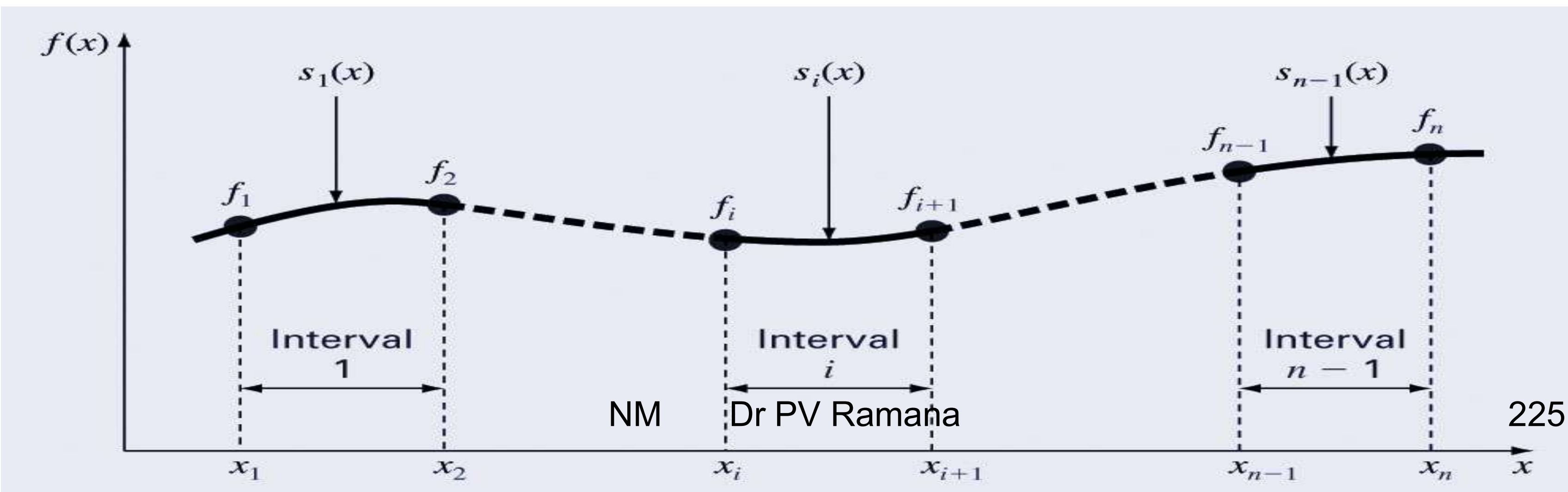
Higher Order vs. Splines

- Splines eliminate oscillations by using small subsets of points for each interval rather than every point. This is especially useful when there are jumps in the data:
  a) 3rd order polynomial
  b) 5th order polynomial
  c) 7th order polynomial
  d) Linear spline
    - seven 1st order polynomials generated by using pairs of points at a time



Spline Development

- Spline function ($s_i(x)$)coefficients are calculated for each interval of a data set.
- The number of data points (f_i) used for each spline function depends on the order of the spline function.



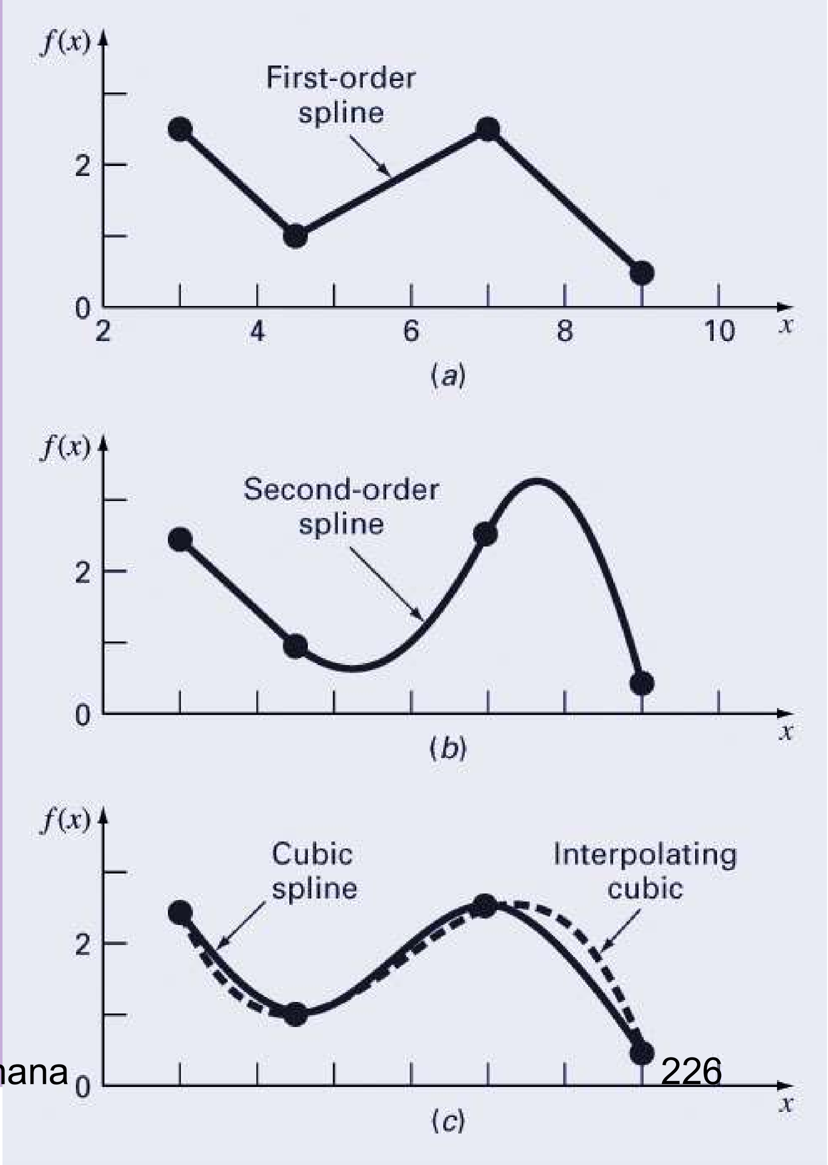
Spline Development

a) First-order splines find straight-line equations between each pair of points that
- Go through the points

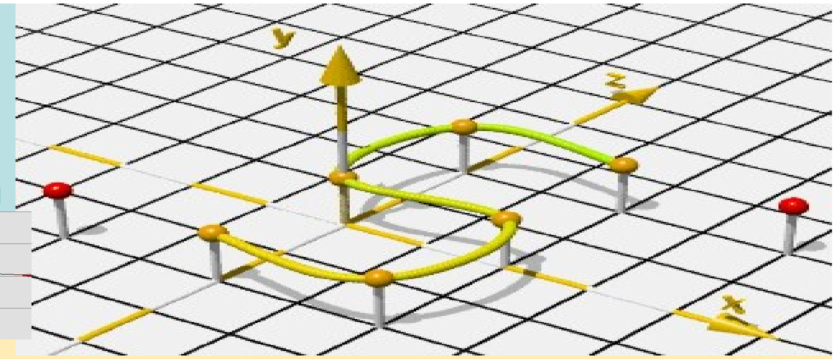
b) Second-order splines find quadratic equations between each pair of points that
- Go through the points
- Match first derivatives at the interior points

c) Third-order splines find cubic equations between each pair of points that
- Go through the points
- Match first and second derivatives at the interior points

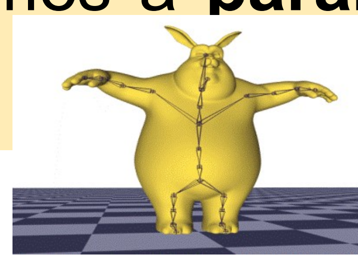
Note that the results of cubic spline interpolation are different from the results of an interpolating cubic.



Cubic Splines



- The function values must be equal at the interior knots **(2n-2).**
- The first and last functions must pass through the end points **(2).**
- The first derivatives at the interior knots must be equal **(n-1).**
- The second derivatives at the interior knots must be equal **(n-1).**
- The second derivatives at the end knots (both) are zero **(2),** (the 2nd derivative function becomes a **parabolic** line at the end points)



Cubic Spline Example

The upward velocity of a rocket is given as a function of time. Using cubic splines

a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

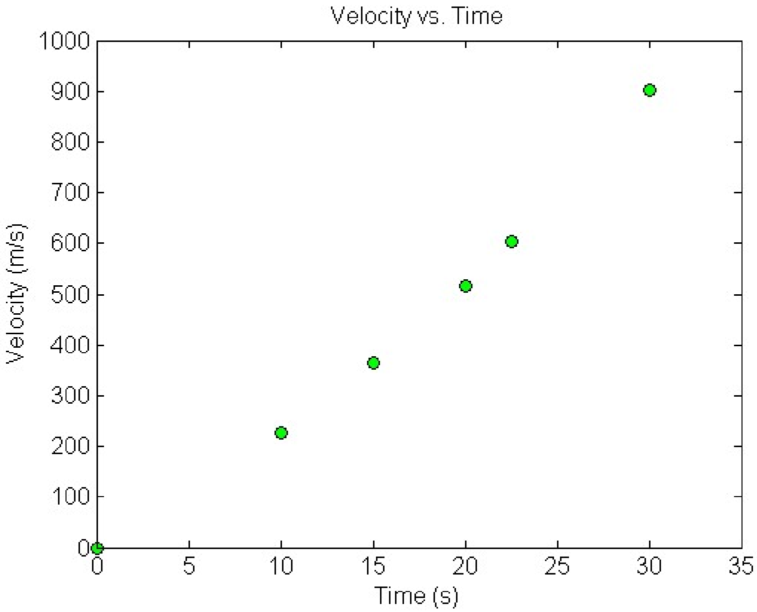


Figure. Velocity vs. time data for the rocket example

Solution

1. Equal interior points

t(s)	V(t) (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$v(t) = a_1 t^3 + b_1 t^2 + c_1 t + d_1, \quad 0 \le t \le 10$$

$$= a_2 t^3 + b_2 t^2 + c_2 t + d_2, \quad 10 \le t \le 15$$

$$= a_3 t^3 + b_3 t^2 + c_3 t + d_3, \quad 15 \le t \le 20$$

$$= a_4 t^3 + b_4 t^2 + c_4 t + d_4, \quad 20 \le t \le 22.5$$

$$= a_5 t^3 + b_5 t^2 + c_5 t + d_5, \quad 22.5 \le t \le 30$$

Let us set up the equations

Each Spline Goes Through Two Consecutive Data Points

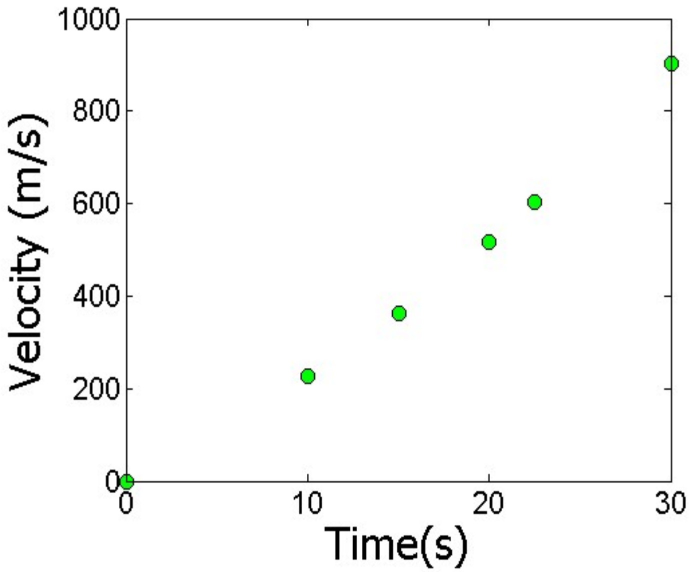
$$v(t) = a_1 t^3 + b_1 t^2 + c_1 t + d_1, \qquad 0 \le t \le 10$$

$$a_1(0)^3 + b_1(0)^2 + c_1(0) + d_1 = 0 \quad \Rightarrow \quad d_1 = 0$$

$$a_1(10)^3 + b_1(10)^2 + c_1(10) + d_1 = 227.04$$

$$a_2(10)^3 + b_2(10)^2 + c_2(10) + d_2 = 227.04$$

t (s)	V(t) (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Each Spline Goes Through Two Consecutive Data Points

$$v(t) = a_i t^3 + b_i t^2 + c_i t + d_i$$

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$a_2(15)^3 + b_2(15)^2 + c_2(15) + d_2 = 362.78$$

$$a_3(15)^3 + b_3(15)^2 + c_3(15) + d_3 = 362.78$$

$$a_3(20)^3 + b_3(20)^2 + c_3(20) + d_3 = 517.35$$

$$a_4(20)^3 + b_4(20)^2 + c_4(20) + d_4 = 517.35$$

$$a_4(22.5)^3 + b_4(22.5)^2 + c_4(22.5) + d_4 = 602.97$$

$$a_5(22.5)^3 + b_5(22.5)^2 + c_5(22.5) + d_5 = 602.97$$

$$a_5(30)^3 + b_5(30)^2 + c_5(30) + d_5 = 901.67$$

2. Equal derivatives at the interior knots

Derivatives are Continuous at Interior Data Points

$$v(t) = a_1 t^3 + b_1 t^2 + c_1 t + d_1, \qquad 0 \le t \le 10$$

$$= a_2 t^3 + b_2 t^2 + c_2 t + d_2, \qquad 10 \le t \le 15$$

$$\frac{d}{dt}\left(a_1 t^3 + b_1 t^2 + c_1 t + d_1\right)\Big|_{t=10} = \frac{d}{dt}\left(a_2 t^3 + b_2 t^2 + c_2 t + d_2\right)\Big|_{t=10}$$

$$3a_1 t^2 + 2b_1 t + c_1 = 3a_2 t^2 + 2b_2 t + c_2$$

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$\left(3a_1 t^2 + 2b_1 t + c_1\right)\Big|_{t=10} = \left(3a_2 t^2 + 2b_2 t + c_2\right)\Big|_{t=10}$$

$$\left(300a_1 + 20b_1 + c_1\right)\Big| = \left(300a_2 + 20b_2 + c_2\right)\Big|$$

$$300(a_1 - a_2) + 20(b_1 - b_2) + (c_1 - c_2) = 0$$

Derivatives are continuous at Interior Data Points

At t=10

$$300(a_1 - a_2) + 20(b_1 - b_2) + (c_1 - c_2) = 0$$

At t=15

$$675(a_2 - a_3) + 30(b_2 - b_3) + (c_2 - c_3) = 0$$

At t=20

$$1200(a_3 - a_4) + 40(b_3 - b_4) + (c_3 - c_4) = 0$$

At t=22.5

$$1518.7(a_4 - a_5) + 45(b_4 - b_5) + (c_4 - c_5) = 0$$

t	**v(t)**
s	**m/s**
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

3. Equal Second derivatives at the interior knots

Derivatives are Continuous at Interior Data Points

$$v(t) = a_1 t^3 + b_1 t^2 + c_1 t + d_1, \quad 0 \le t \le 10$$

$$= a_2 t^3 + b_2 t^2 + c_2 t + d_2, \quad 10 \le t \le 15$$

$$\left.\frac{d^2}{dt^2}\left(a_1 t^3 + b_1 t^2 + c_1 t + d_1\right)\right|_{t=10} = \left.\frac{d^2}{dt^2}\left(a_2 t^3 + b_2 t^2 + c_2 t + d_2\right)\right|_{t=10}$$

$$6a_1 t + 2b_1 = 6a_2 t + 2b_2$$

$$\left.\left(6a_1 t + 2b_1\right)\right|_{t=10} = \left.\left(6a_2 t + 2b_2\right)\right|_{t=10}$$

$$60\left(a_1 - a_2\right) + 2\left(b_1 - b_2\right) = 0$$

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Derivatives are continuous at Interior Data Points

At t=10

$$60(a_1 - a_2) + 2(b_1 - b_2) = 0$$

At t=15

$$90(a_2 - a_3) + 2(b_2 - b_3) = 0$$

At t=20

$$120(a_3 - a_4) + 2(b_3 - b_4) = 0$$

At t=22.5

$$135(a_4 - a_5) + 2(b_4 - b_5) = 0$$

t	**v(t)**
s	**m/s**
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

3. First and last functions pass the end points

$$a_1(0)^3 + b_1(0)^2 + c_1{}^1 + d_1 = 0 \quad \Longrightarrow \quad d_1 = 0$$

$$a_5(30)^3 + b_5(30)^2 + c_5(30) + d_5 = 901.67$$

t	v(t)
s	m/s
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

4. Second derivative at the first point is 0

Last Equation $a_1 = a_5 = 0$

$$
\begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1000 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1000 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3375 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3375 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8000 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8000 & 400 & 20 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1139.625 & 506.2 & 22.5 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1139.6 & 506.2 & 22.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27000 & 900 & 30 & 1 \\
300 & 20 & 1 & 0 & -300 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 675 & 30 & 1 & 0 & -675 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1200 & 40 & 1 & 0 & -1200 & -40 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1519 & 45 & 1 & 0 & -1519 & -45 & -1 & 0 \\
60 & 2 & 0 & 0 & -60 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 90 & 2 & 0 & 0 & -90 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 120 & 2 & 0 & 0 & -120 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 135 & 2 & 0 & 0 & -135 & -2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{pmatrix}
a_1 \\ b_1 \\ c_1 \\ d_1 \\ a_2 \\ b_2 \\ c_2 \\ d_2 \\ a_3 \\ b_3 \\ c_3 \\ d_3 \\ a_4 \\ b_4 \\ c_3 \\ d_4 \\ a_5 \\ b_5 \\ c_5 \\ d_5
\end{pmatrix}
=
\begin{bmatrix}
0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

Coefficients of Spline

i	a_i	b_i	c_i	d_i
1	0	0.2854	19.85	0
2	0.0065	0.0902	21.8021	-6.5076
3	0.0039	0.206	20.0646	2.18
4	0.0169	-0.5718	35.6211	-101.53
5	0	0.5688	9.9625	90.9032

Velocity at a Particular Point

a) Velocity at t=16

$$v(t) = 0.2854\,t^2 + 19.85\,t, \qquad 0 \le t \le 10$$

$$= 0.0065\,t^3 + 0.0902\,t^2 + 21.8021\,t - 6.5076, \qquad 10 \le t \le 15$$

$$= 0.0039t^3 + 0.206t^2 + 20.0646t + 2.18, \qquad 15 \le t \le 20$$

$$= 0.0169\,t^3 - 0.5718\,t^2 + 35.6211\,t - 101.53, \qquad 20 \le t \le 22.5$$

$$= 0.5688t^2 + 9.9625t + 90.9032, \qquad 22.5 \le t \le 30$$

$$v(16) = 0.0039(16)^3 + 0.206(16)^2 + 20.0646(16) + 2.18 = 391.924$$

The upward velocity of a rocket is given as a function of time. Using quadratic splines

a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

a) Acceleration at t=16

$v(t) = 0.2854t^2 + 19.85t,$	$0 \le t \le 10$
$= 0.0065t^3 + 0.0902t^2 + 21.8021t - 6.5076,$	$10 \le t \le 15$
$= 0.0039t^3 + 0.206t^2 + 20.0646t + 2.18,$	$15 \le t \le 20$
$= 0.0169t^3 - 0.5718t^2 + 35.6211t - 101.53,$	$20 \le t \le 22.5$
$= 0.5688t^2 + 9.9625t + 90.9032,$	$22.5 \le t \le 30$

$$a(t) = \frac{dv}{dt} = 0.0117t^2 + 0.412t + 20.0646$$

$$a(16) = 0.0117(16)^2 + 0.412(16) + 20.0646 = 29.65 \text{ m/s}^2$$

The upward velocity of a rocket is given as a function of time. Using quadratic splines

a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

$v(t) = 0.2854t^2 + 19.85t,$	$0 \le t \le 10$
$= 0.0065t^3 + 0.0902t^2 + 21.8021t - 6.5076,$	$10 \le t \le 15$
$= 0.0039t^3 + 0.206t^2 + 20.0646t + 2.18,$	$15 \le t \le 20$
$= 0.0169t^3 - 0.5718t^2 + 35.6211t - 101.53,$	$20 \le t \le 22.5$
$= 0.5688t^2 + 9.9625t + 90.9032,$	$22.5 \le t \le 30$

$$d(t) = \int_{t_1}^{t_2} v\,dt = \int_{11}^{16} \left(0.0039t^3 + 0.206t^2 + 20.064t + 2.18\right) dt$$

$$Distance = 0.0039t^4/4 + 0.206t^3/3 + 20.064t^2/2 + 2.18t \Big|_{11}^{16}$$

$$d(t) = 0.000975t^4 + 0.0686t^3 + 10.032t^2 + 2.18t \Big|_{11}^{16}$$

$$d\Big|_{11}^{16} = 1604.708\, m$$

Cubic Splines

- While data of a particular size presents many options for the order of spline functions, cubic splines are preferred because they provide the simplest representation that exhibits the desired appearance of smoothness.
  - Linear splines have discontinuous first derivatives
  - Quadratic splines have discontinuous second derivatives and require setting the second derivative at some point to a pre-determined value
  - Quartic or higher-order splines tend to exhibit the instabilities inherent in higher order polynomials (ill-conditioning or oscillations)

Cubic Splines (Method 2)

- In general, the i^{th} spline function for a cubic spline can be written as:

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

- For n data points, there are n-1 intervals and thus 4(n-1) unknowns to evaluate to solve all the spline function coefficients.

Solving Spline Coefficients

- One condition requires that the spline function goes through the first and last point of the interval, yielding 2(n-1) equations of the form:

$$s_i(x_i) = f_i \Rightarrow a_i = f_i$$

$$s_i(x_{i+1}) = f_i \Rightarrow s_i(x_{i+1}) = a_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 + d_i(x_{i+1} - x_i)^3 = f_i$$

- Another requires that the first derivative is continuous at each interior point, yielding n-2 equations of the form:

$$s_i'(x_{i+1}) = s_{i+1}'(x_{i+1}) \Rightarrow b_i + 2c_i(x_{i+1} - x_i) + 3d_i(x_{i+1} - x_i)^2 = b_{i+1}$$

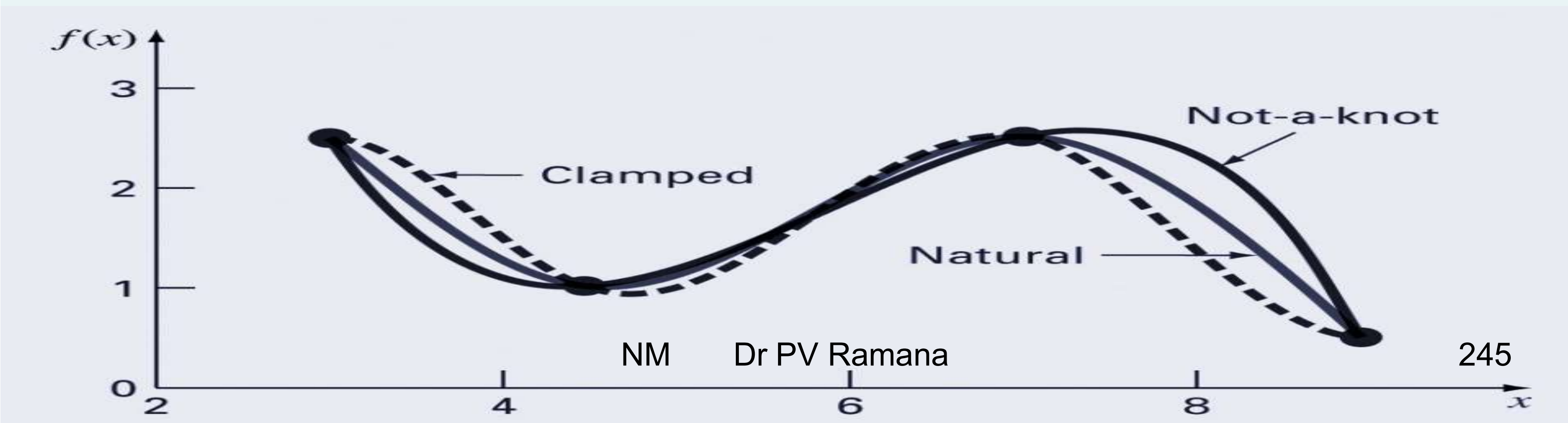
- A third requires that the *second* derivative is continuous at each interior point, yielding n-2 equations of the form:

$$s_i''(x_{i+1}) = s_{i+1}''(x_{i+1}) \Rightarrow 2c_i + 6d_i(x_{i+1} - x_i) = 2c_{i+1}$$

- These give 4n-6 total equations and 4n-4 are needed!

Two Additional Equations

- There are several options for the final two equations:
  - Natural end conditions - assume the second derivative at the end knots are zero.
  - Clamped end conditions - assume the first derivatives at the first and last knots are known.
  - "Not-a-knot" end conditions - force continuity of the *third* derivative at the second and penultimate points (results in the first two intervals having the same spline function and the last two intervals having the same spline function)

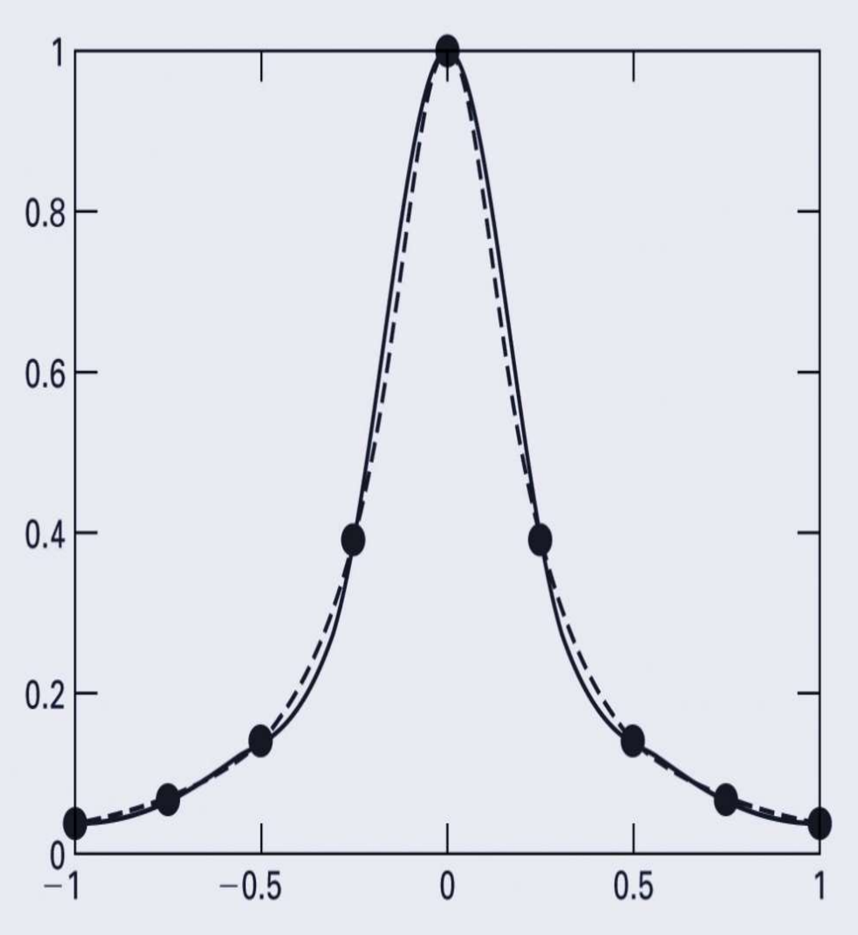


Piecewise Interpolation in MATLAB

- MATLAB has several built-in functions to implement piecewise interpolation.
- The first is spline:
  `yy=spline(x, y, xx)`
  This performs cubic spline interpolation, generally using not-a-knot conditions. If `y` contains two more values than `x` has entries, then the first and last value in `y` are used as the derivatives at the end points (i.e. clamped)

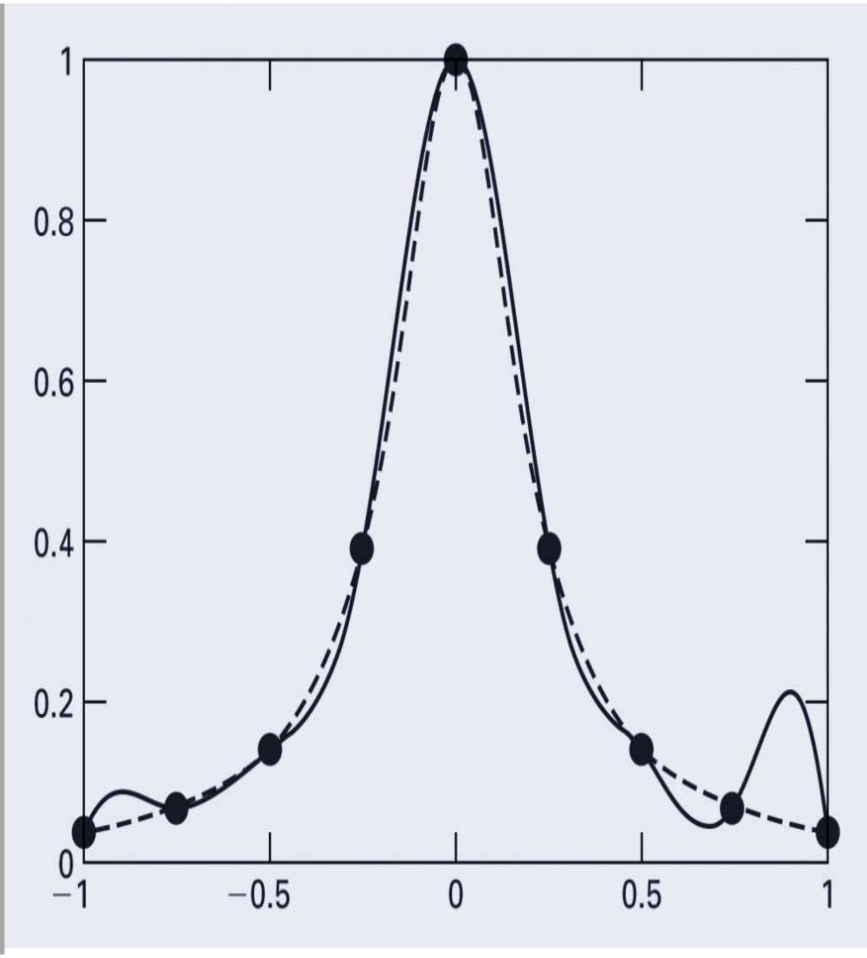
Not-a-knot Example

- Generate data:
```
x = linspace(-1, 1, 9);
y = 1./(1+25*x.^2);
```
- Calculate 100 model points and determine not-a-knot interpolation
```
xx = linspace(-1, 1);
yy = spline(x, y, xx);
```
- Calculate actual function values at model points and data points, the 9-point not-a-knot interpolation (solid), and the actual function (dashed),
```
yr = 1./(1+25*xx.^2)
plot(x, y, ‘o’, xx, yy, ‘-
’, xx, yr, ‘--’)
```



Clamped Example

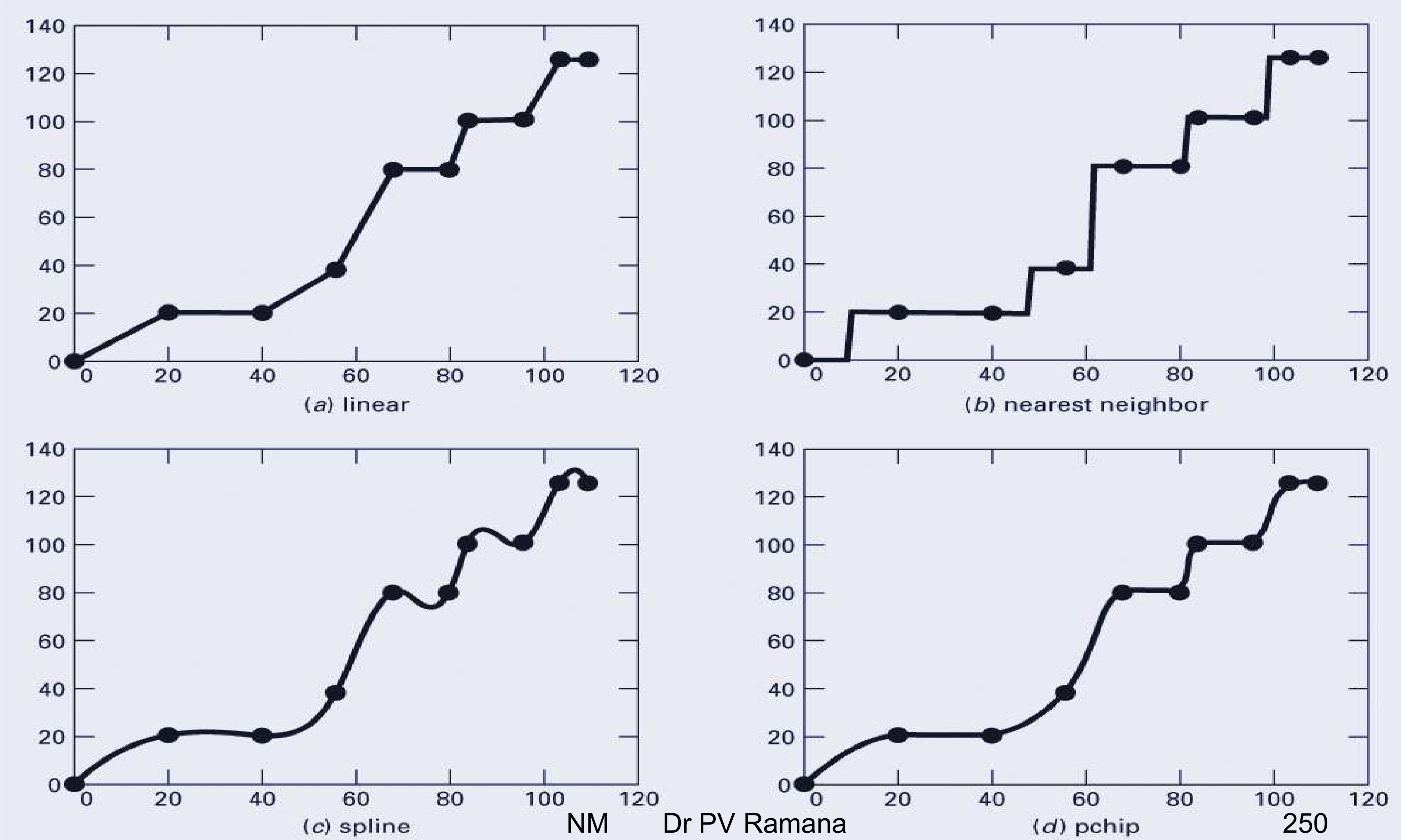
- Generate data w/ first derivative information:
```
x = linspace(-1, 1, 9);
y = 1./(1+25*x.^2);
yc = [1 y -4]
```
- Calculate 100 model points and determine not-a-knot interpolation
```
xx = linspace(-1, 1);
yyc = spline(x, yc, xx);
```
- Calculate actual function values at model points and data points, the 9-point clamped interpolation (solid), and the actual function (dashed),
```
yr = 1./(1+25*xx.^2)
plot(x, y, 'o', xx, yyc,
'-', xx, yr, '--')
```



MATLAB’s `interp1` Function

- While `spline` can only perform cubic splines, MATLAB’s `interp1` function can perform several different kinds of interpolation:
  `yi = interp1(x, y, xi, 'method')`
  - `x` & `y` contain the original data
  - `xi` contains the points at which to interpolate
  - `'method'` is a string containing the desired method:
    - `'nearest'` - nearest neighbor interpolation
    - `'linear'` - connects the points with straight lines
    - `'spline'` - not-a-knot cubic spline interpolation
    - `'pchip'` or `'cubic'` - piecewise cubic Hermite interpolation

Piecewise Polynomial Comparisons



(Method 2)

Cubic Splines

Cubic splines avoid the straight line and the over-swing

$$f_i = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

Can develop method like did for quadratic

– 4(n–1) unknowns – 4(n–1) equations

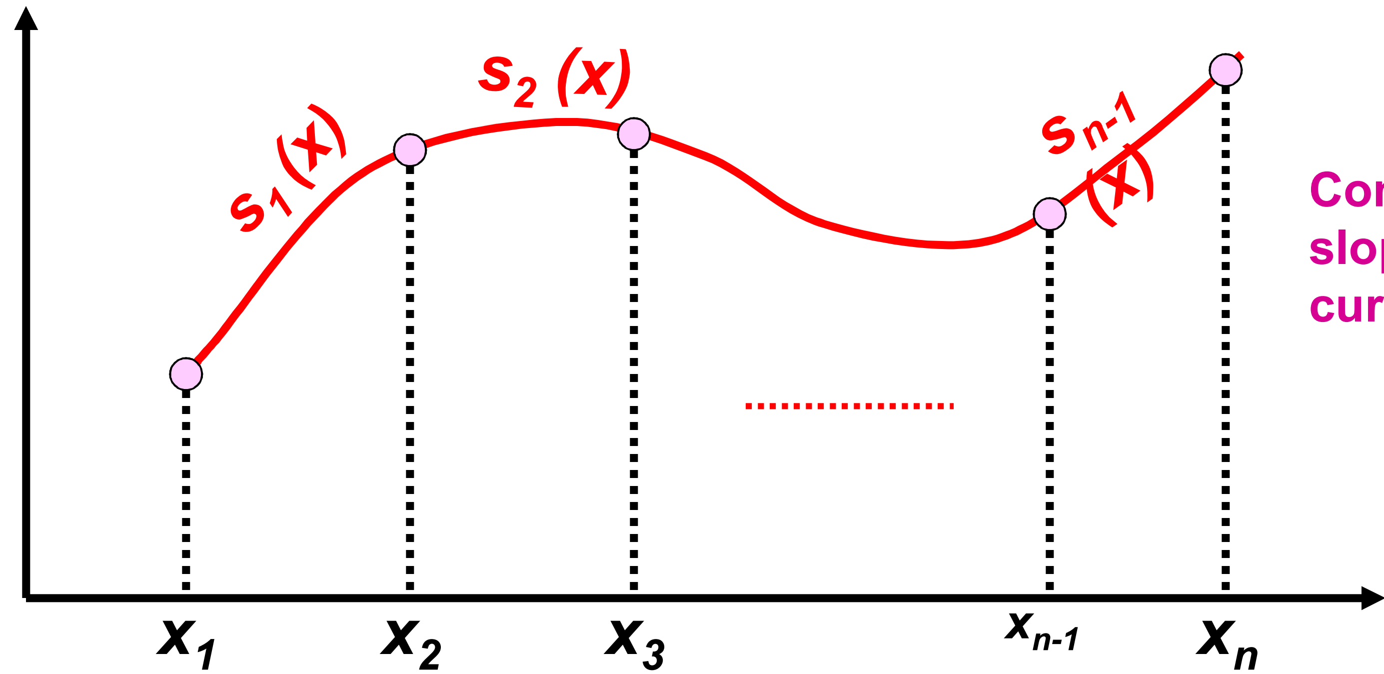
- interior knot equality
- end point fixed
- interior knot first derivative equality
- assume derivative value if needed

Piecewise Cubic Splines

(Method 2)

$$\text{data points}: (x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots, (x_n, y_n)$$

$$\text{interval}: \quad I_1 = [x_1, x_2], I_2 = [x_2, x_3], \cdots, I_{n-1} = [x_{n-1}, x_n]$$



Continuous slopes and curvatures

$$f_i = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

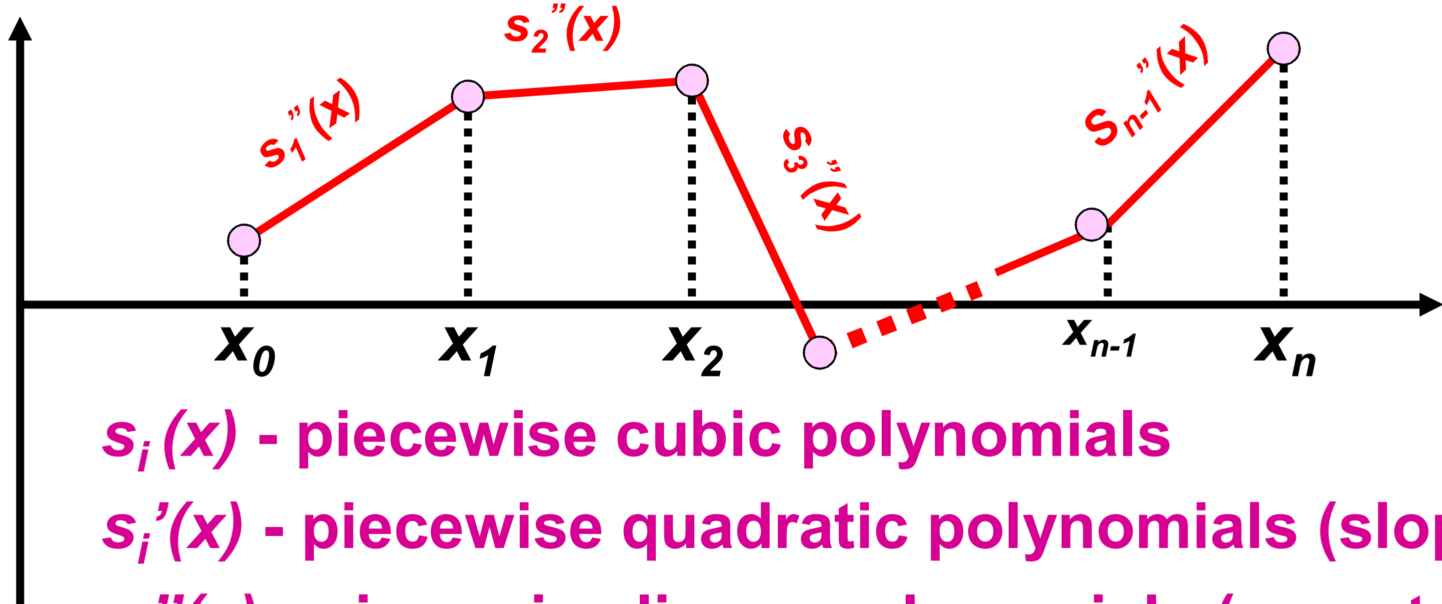
4(n–1) unknowns

Piecewise Cubic Splines

(Method 2)

$$data\ points: (x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots, (x_n, y_n)$$

$$interval: \quad I_1 = [x_1, x_2], I_2 = [x_2, x_3], \cdots, I_{n-1} = [x_{n-1}, x_n]$$



$s_i(x)$ - piecewise cubic polynomials

$s_i'(x)$ - piecewise quadratic polynomials (slope)

$s_i''(x)$ - piecewise linear polynomials (curvatures)

Reduce to (n-1) unknowns and (n-1) equations for s_i''

Cubic Splines (Method 2)

- Piecewise cubic polynomial with continuous derivatives up to order 2

1. The function must pass through all the data points

$$\begin{cases} x = x_i : s_i(x_i) = f_i = a_i + b_i(x_i - x_i) + c_i(x_i - x_i)^2 + d_i(x_i - x_i)^3 = a_i \\ x = x_{i+1} : s_i(x_{i+1}) = f_{i+1} = a_i + b_i(x_{i+1} - x_i) + c_i(x_{i+1} - x_i)^2 + d_i(x_{i+1} - x_i)^3 \end{cases}$$

gives 2(n-1) equations

$$h_i = x_{i+1} - x_i$$

$$\begin{cases} a_i = f_i \\ f_i + b_i h_i + c_i h_i^2 + d_i h_i^3 = f_{i+1} \end{cases}$$

i = 1,2,…, n-1

Cubic Splines

(Method 2)

2. First derivatives at the interior nodes must be equal:

$$s_i'(x) = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2$$

$$s_i'(x_{i+1}) = s_{i+1}'(x_{i+1})$$

$$b_i + 2c_ih_i + 3d_ih_i^2 = b_{i+1}$$

(n-2) equations

3. Second derivatives at the interior nodes must be equal:

$$s_i''(x) = 2c_i + 6d_i(x - x_i)$$

$$s_i''(x_{i+1}) = s_{i+1}''(x_{i+1})$$

$$c_i + 3d_ih_i = c_{i+1}$$

(n-2) equations

Cubic Splines

(Method 2)

4. <u>Two additional conditions are needed</u> (arbitrary)

$$\begin{cases} s_1''(x_1) = 0 = 2c_1 \\ s_{n-1}''(x_n) = 0 = 2c_{n-1} + 6d_{n-1}h_{n-1} \end{cases}$$

The last two equations

$$\begin{cases} s_1''(x_1) = 0 \\ s_{n-1}''(x_n) = 0 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_n = c_{n-1} + 3d_{n-1}h_{n-1} = 0 \end{cases}$$

Total equations: (n-1) + (n-2) + (n-2) +2 = 4(n-1)

Cubic Splines (Method 2)

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

- Solve for (a_i, b_i, c_i, d_i) –>

$$h_{i-1}c_{i-1} + 2(h_{i-1} - h_i)c_i + h_i c_{i+1} = 3\,\frac{f_{i+1} - f_i}{h_i} - 3\,\frac{f_i - f_{i-1}}{h_{i-1}}$$
$$= 3(f[x_{i+1}, x_i] - f[x_i, x_{i-1}])$$

- Tridiagonal system with boundary conditions $c_1 = c_n = 0$

$$a_i = f_i, \quad d_i = \frac{c_{i+1} - c_i}{3h_i}, \quad b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}\left(2c_i + c_{i+1}\right)$$

Cubic Splines

(Method 2)

$$
\begin{bmatrix}
1 & & & & & \\
h_1 & 2(h_1+h_2) & h_2 & & & \\
& h_2 & 2(h_2+h_3) & h_3 & & \\
& & \ddots & \ddots & \ddots & \\
& & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\
& & & & & 1
\end{bmatrix}
\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-1} \\ c_n \end{Bmatrix}
=
\begin{Bmatrix}
0 \\
3\left(f[x_3,x_2]-f[x_2,x_1]\right) \\
3\left(f[x_4,x_3]-f[x_3,x_2]\right) \\
\vdots \\
3\left(f[x_n,x_{n-1}]-f[x_{n-1},x_{n-2}]\right) \\
0
\end{Bmatrix}
$$

Tridiagonal matrix

Estimate f(4): Cubic Splines

(Method 1)

x	0	2	5	6
$f(x)$	4	-2	19	58

Solution

$$\text{Exact solution}: f(x) = x^3 - 5x^2 + 3x + 4, \quad f(4) = 0$$

1. Equal interior points

$$f(x) = a_1x^3 + b_1x^2 + c_1x + d_1, \quad 0 \le x \le 2$$

$$= a_2x^3 + b_2x^2 + c_2x + d_2, \quad 2 \le x \le 5$$

$$= a_3x^3 + b_3x^2 + c_3x + d_3, \quad 5 \le x \le 6$$

x	f (x)
0	4
2	-2
5	19
6	58

Let us set up the equations

(Method 1)

Each Spline Goes Through Two Consecutive Data Points

$$f(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1, \qquad 0 \le x \le 2$$

$$a_1(0)^3 + b_1(0)^2 + c_1(0) + d_1 = 4 \quad \Rightarrow \quad d_1 = 4$$

$$a_1(2)^3 + b_1(2)^2 + c_1(2) + d_1 = -2$$

$$a_2(2)^3 + b_2(2)^2 + c_2(2) + d_2 = -2$$

x	f (x)
0	4
2	-2
5	19
6	58

(Method 1)

Each Spline Goes Through Two Consecutive Data Points

$$f(x) = a_i x^3 + b_i x^2 + c_i x + d_i$$

x	f (x)
0	4
2	-2
5	19
6	58

$$a_2(5)^3 + b_2(5)^2 + c_2(5) + d_2 = 19$$

$$a_3(5)^3 + b_3(5)^2 + c_3(5) + d_3 = 19$$

$$a_3(6)^3 + b_3(6)^2 + c_3(6) + d_3 = 58$$

(Method 1)

2. Equal derivatives at the interior knots

Derivatives are Continuous at Interior Data Points

$$f(x) = a_1x^3 + b_1x^2 + c_1x + d_1,$$
$$= a_2x^3 + b_2x^2 + c_2x + d_2,$$

$$\frac{d}{dx}\left(a_1x^3 + b_1x^2 + c_1x + d_1\right)\Big|_{x=2} = \frac{d}{dx}\left(a_2x^3 + b_2x^2 + c_2x + d_2\right)\Big|_{x=2}$$

$$3a_1x^2 + 2b_1x + c_1 = 3a_2x^2 + 2b_2x + c_2$$

x	f (x)
0	4
2	-2
5	19
6	58

$$\left(3a_1x^2 + 2b_1x + c_1\right)\Big|_{x=2} = \left(3a_2x^2 + 2b_2x + c_2\right)\Big|_{x=2}$$

$$\left(12a_1 + 4b_1 + c_1\right)\Big| = \left(12a_2 + 4b_2 + c_2\right)\Big|$$

$$12\left(a_1 - a_2\right) + 4\left(b_1 - b_2\right) + \left(c_1 - c_2\right) = 0$$

Derivatives are continuous at Interior Data Points

At x= 2

$$12\left(a_1 - a_2\right) + 4\left(b_1 - b_2\right) + \left(c_1 - c_2\right) = 0$$

At x=5

$$75\left(a_2 - a_3\right) + 0\left(b_2 - b_3\right) + \left(c_2 - c_3\right) = 0$$

x	f (x)
0	4
2	-2
5	19
6	58

$$\left.\left(3a_1x^2 + 2b_1x + c_1\right)\right|_x = \left.\left(3a_2x^2 + 2b_2x + c_2\right)\right|_x$$

(Method 1)

3. Equal Second derivatives at the interior knots

Derivatives are Continuous at Interior Data Points

$$f(x) = a_1x^3 + b_1x^2 + c_1x + d_1,$$
$$= a_2x^3 + b_2x^2 + c_2x + d_2,$$

$$\frac{d^2}{dx^2}\left(a_1x^3 + b_1x^2 + c_1x + d_1\right)\Big|_{x=2} = \frac{d^2}{dx^2}\left(a_2x^3 + b_2x^2 + c_2x + d_2\right)\Big|_{x=2}$$

$$6a_1x + 2b_1 = 6a_2x + 2b_2$$

$$\left(6a_1x + 2b_1\right)\Big|_{x=2} = \left(6a_2x + 2b_2\right)\Big|_{x=2}$$

$$12\left(a_1 - a_2\right) + 2\left(b_1 - b_2\right) = 0$$

x	f (x)
0	4
2	-2
5	19
6	58

(Method 1)

$$\left(6a_1x+2b_1\right)\Big|_{x=2}=\left(6a_2x+2b_2\right)\Big|_{x=2}$$

Derivatives are continuous at Interior Data Points

At x=2

$$12\left(a_1-a_2\right)+2\left(b_1-b_2\right)=0$$

At x=5

$$30\left(a_2-a_3\right)+2\left(b_2-b_3\right)=0$$

x	f (x)
0	4
2	-2
5	19
6	58

4. Third derivative at first & last points are 0

First & Last Equation $a_1=a_5=0$

0	0	0	1	0	0	0	0	0	0	0	0	a1	4
8	4	2	1	0	0	0	0	0	0	0	0	b1	-2
0	0	0	0	8	4	2	1	0	0	0	0	c1	-2
0	0	0	0	125	25	5	1	0	0	0	0	d1	19
0	0	0	0	0	0	0	0	125	25	5	1	a2	19
0	0	0	0	0	0	0	0	216	36	6	1	b2	58
12	4	1	0	-12	-4	-1	0	0	0	0	0	c2	0
0	0	0	0	75	10	1	0	-75	-10	-1	0	d2	0
12	2	0	0	-12	-2	0	0	0	0	0	0	a3	0
0	0	0	0	30	2	0	0	-30	-2	0	0	b3	0
1	0	0	0	0	0	0	0	0	0	0	0	c3	0
0	0	0	0	0	0	0	0	1	0	0	0	d3	0
a1	b1	c1	d1	a2	b2	c2	d2	a3	b3	c3	d3		

$$f(x) = a_1 x^3 + b_1 x^2 + c_1 x + d_1,$$

$$\left(3a_1 x^2 + 2b_1 x + c_1\right)\Big| = \left(3a_2 x^2 + 2b_2 x + c_2\right)\Big|$$

$$\left(6a_1 x + 2b_1\right)\Big| = \left(6a_2 x + 2b_2\right)\Big|$$

x	f (x)
0	4
2	-2
5	19
6	58

Coefficients of Spline

i	a_i	b_i	c_i	d_i
1	0	-0.1818	-2.6364	4.000
2	1.2121	-7.4545	11.9091	-5.6970
3	0	10.7273	-79.0001	145.8182

$$f(x) = 0x^3 - 0.1818x^2 - 2.6364x + 4.000, \quad 0 \le x \le 2$$

$$= 1.2121x^3 - 7.4545x^2 + 11.9091x - 5.697, \quad 2 \le x \le 5$$

$$= 0x^3 + 10.7273x^2 - 79.001x + 145.8182; \quad 5 \le x \le 6$$

$$f(4) = 1.2121(4)^3 - 7.4545(4)^2 + 11.9091(4) - 5.697 = 0.24$$

Hand Calculations

(Method 2)

x	0	2	5	6
$f(x)$	4	-2	19	58

estimate f(4)

Exact solution : $f(x) = x^3 - 5x^2 + 3x + 4, \quad f(4) = 0$

$$h_1 = x_2 - x_1 = 2 - 0 = 2, h_2 = x_3 - x_2 = 5 - 2 = 3, h_3 = x_4 - x_3 = 6 - 5 = 1$$

$$f_1 = 4, \quad f_2 = -2, \quad f_3 = 19, \quad f_4 = 58$$

$$f[x_2, x_1] = \frac{f_2 - f_1}{h_1} = \frac{-2 - 4}{2} = -3$$

$$f[x_3, x_2] = \frac{f_3 - f_2}{h_2} = \frac{19 - (-2)}{3} = 7$$

$$f[x_4, x_3] = \frac{f_4 - f_3}{h_3} = \frac{58 - 19}{1} = 39$$

Hand Calculations

x	0	2	5	6
$f(x)$	4	-2	19	58

$h_1 = 2, h_2 = 3, h_3 = 1$

$f_1 = 4, \quad f_2 = -2, \quad f_3 = 19, \quad f_4 = 58$

$$f[x_2, x_1] = \frac{f_2 - f_1}{h_1} = \frac{-2-4}{2} = -3$$

$$f[x_3, x_2] = \frac{f_3 - f_2}{h_2} = \frac{19-(-2)}{3} = 7$$

$$f[x_4, x_3] = \frac{f_4 - f_3}{h_3} = \frac{58-19}{1} = 39$$

$$\begin{bmatrix} 1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & \\ & h_2 & 2(h_2+h_3) & h_3 \\ & & & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3\left(f[x_3,x_2] - f[x_2,x_1]\right) \\ 3\left(f[x_4,x_3] - f[x_3,x_2]\right) \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & & & \\ 2 & 10 & 3 & \\ & 3 & 8 & 1 \\ & & & 1 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3\left(7-(-3)\right) \\ 3\left(39-7\right) \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \\ 96 \\ 0 \end{Bmatrix}$$

- can be further simplified since $c_1 = c_4 = 0$ (natural spline)

$$\begin{bmatrix} 10 & 3 \\ 3 & 8 \end{bmatrix} \begin{Bmatrix} c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 30 \\ 96 \end{Bmatrix} \Rightarrow \begin{Bmatrix} c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} -0.676056 \\ 12.253521 \end{Bmatrix}$$

Cubic Spline Interpolation

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

$$a_i = f_i, \quad d_i = \frac{c_{i+1} - c_i}{3h_i}, \quad b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}\left(2c_i + c_{i+1}\right)$$

(Method 2)

x	0	2	5	6
$f(x)$	4	– 2	19	58

$$\begin{cases} a_1 = f_1 = 4 \\ a_2 = f_2 = -2 \\ a_3 = f_3 = 19 \\ a_4 = f_4 = 58 \end{cases} \qquad \begin{cases} c_1 = 0 \\ c_2 = -0.676056 \\ c_3 = 12.253521 \\ c_4 = 0 \end{cases}$$

$$\begin{cases} d_1 = \dfrac{c_2 - c_1}{3h_1} = -0.112676 \\[2ex] d_2 = \dfrac{c_3 - c_2}{3h_2} = 1.4366197 \\[2ex] d_3 = \dfrac{c_4 - c_3}{3h_3} = -4.0845070 \end{cases} \qquad \begin{cases} b_1 = \dfrac{f_2 - f_1}{h_1} - \dfrac{h_1}{3}(2c_1 + c_2) = -2.549296 \\[2ex] b_2 = \dfrac{f_3 - f_2}{h_2} - \dfrac{h_2}{3}(2c_2 + c_3) = -3.901408 \\[2ex] b_3 = \dfrac{f_4 - f_3}{h_3} - \dfrac{h_3}{3}(2c_3 + c_4) = 3.0830986 \end{cases}$$

Cubic Splines

$$\begin{cases} a_1 = f_1 = 4 \\ a_2 = f_2 = -2 \\ a_3 = f_3 = 19 \\ a_4 = f_4 = 58 \end{cases} \qquad \begin{cases} c_1 = 0 \\ c_2 = -0.676056 \\ c_3 = 12.253521 \\ c_4 = 0 \end{cases}$$

$$\begin{cases} d_1 = -0.112676 \\ d_2 = 1.4366197 \\ d_3 = -4.0845070 \end{cases} \qquad \begin{cases} b_1 = -2.549296 \\ b_2 = -3.901408 \\ b_3 = 3.0830986 \end{cases}$$

$$s_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

$$a_i = f_i, \quad d_i = \frac{c_{i+1} - c_i}{3h_i}, \quad b_i = \frac{f_{i+1} - f_i}{h_i} - \frac{h_i}{3}\left(2c_i + c_{i+1}\right)$$

x	0	2	5	6
$f(x)$	4	-2	19	58

- Piecewise cubic splines (cubic polynomials)

$$s_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3$$
$$= 4 - 2.549296x - 0.112676x^3$$
$$s_2(x) = a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3$$
$$= -2 - 3.901408(x - 2) - 0.676056(x - 2)^2 + 1.43661972(x - 2)^3$$
$$s_3(x) = a_3 + b_3(x - x_3) + c_3(x - x_3)^2 + d_3(x - x_3)^3$$
$$= 19 + 30.830986(x - 5) + 12.253521(x - 5)^2 - 4.0845070(x - 5)^3$$

(Method 2)

Cubic Spline Interpolation

x	0	2	5	6
$f(x)$	4	-2	19	58

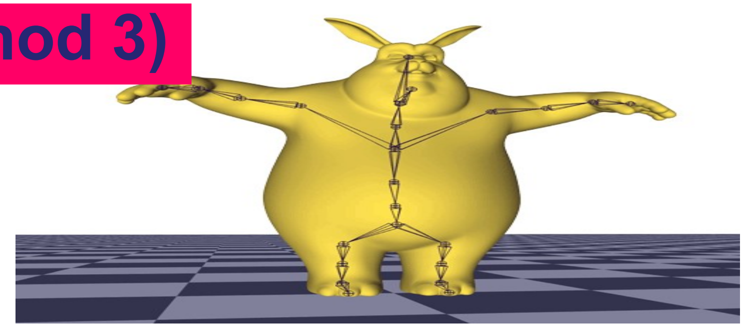
$$\textit{Exact solution}: f(x) = x^3 - 5x^2 + 3x + 4, \quad f(4) = 0$$

$$\textit{Cubic spline interpolation}: f(4) = s_2(4) = -1.0141$$

- The exact solution is a cubic function
- Why cubic spline interpolation does not give the exact solution for a cubic polynomial?
- Because the conditions on the end knots are different!
- ***In general, $f''(x_0) \neq 0$ and $f''(x_n) \neq 0$!!***

Alternative technique to get Cubic Splines

(Method 3)



- The second derivative within each interval $[x_{i-1}, x_i]$ is a **straight line**. (the 2nd derivatives can be represented by first order Lagrange interpolating polynomials.

$$f_i''(x) = f_i''(x_{i-1})\frac{x - x_i}{x_{i-1} - x_i} + f_i''(x_i)\frac{x - x_{i-1}}{x_i - x_{i-1}}$$

A straight line connecting the first knot $f''(x_{i-1})$ and the second knot $f''(x_i)$

The second derivative at any point x within the interval