Taylor Series

n^{th} ***order Taylor series:*** $$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2 + \frac{f'''(x_i)}{3!}(x_{i+1} - x_i)^3$$

$$+ \ldots \frac{f^n(x_i)}{n!}(x_{i+1} - x_i)^n$$

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots . \quad \rightleftharpoons \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \cdots + \frac{h^{(n-1)}}{(n-1)!}f^{(n-1)}(x) + \frac{h^n}{n!}f^n(x+\lambda h).$$

Zero order Taylor series: $$\boxed{f(x_{i+1}) \approx f(x_i)}$$

1^{st} ***order Taylor series:*** $$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

2^{nd} ***order Taylor series:*** $$f(x_{i+1}) \approx f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all } x$$

$$\log(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{for } -1 \le x < 1$$

$$\log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \text{for } -1 < x \le 1$$

$$\frac{1 - x^{m+1}}{1 - x} = \sum_{n=0}^{m} x^n \quad \text{for } x \ne 1 \text{ and } m \in \mathbb{N}_0$$

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{for all } x$$

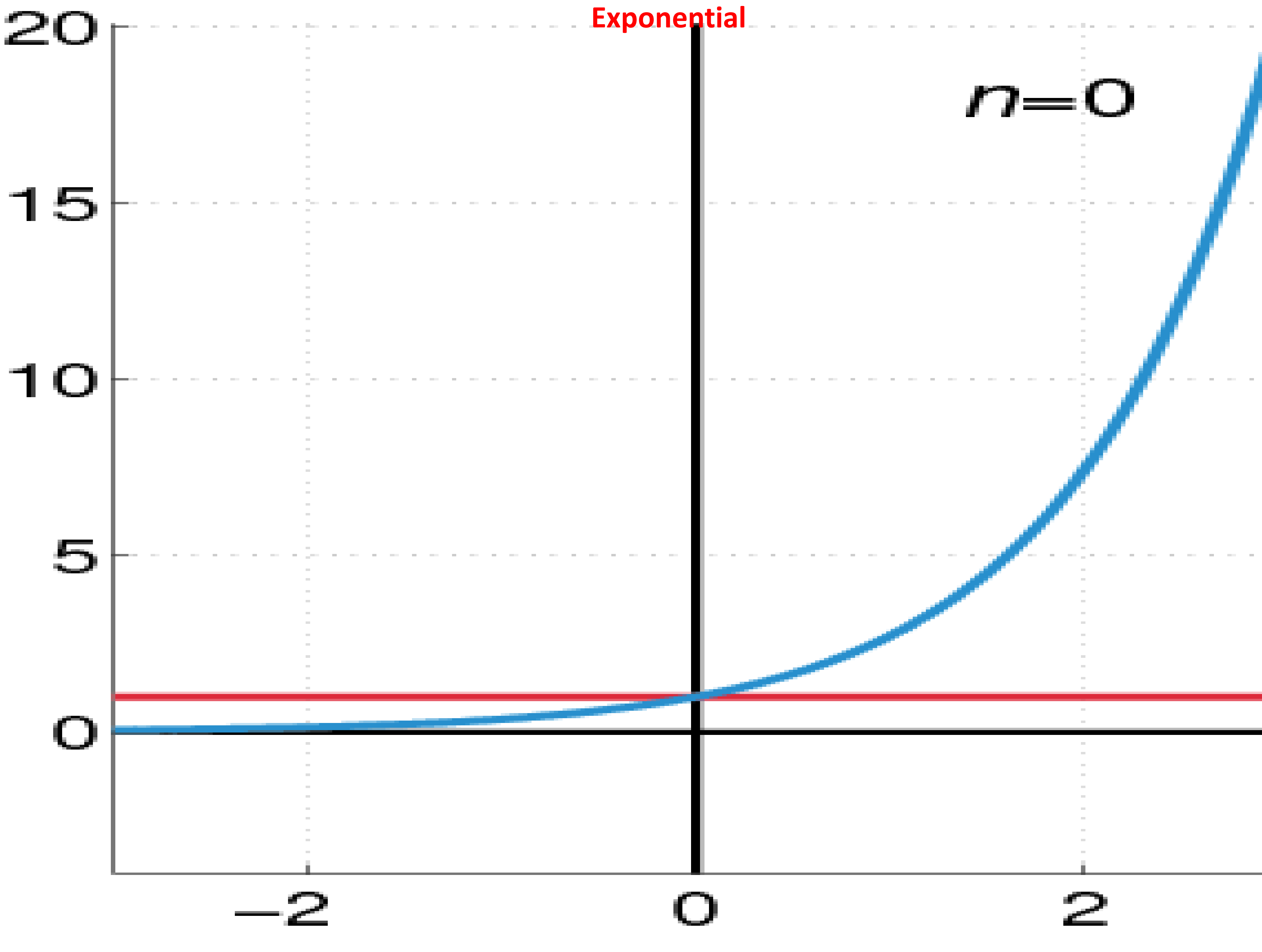
$$\tan x = \sum_{n=1}^{\infty} \frac{B_{2n}(-4)^n(1-4^n)}{(2n)!} x^{2n-1} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$$

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} x^{2n} \quad \text{for } |x| < \frac{\pi}{2}$$

$$\arcsin x = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \quad \text{for } |x| \le 1$$

$$\arccos x = \frac{\pi}{2} - \arcsin x = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} \quad \text{for } |x| \le 1$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{for } |x| < 1$$


Exponential
20
15
10
5
0
n=0
−2
0
2

Taylor Series

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots,$$

$$f(x) = 3 + 5x + 7x^2 + 9x^3 + 11x^4 + \cdots,$$

$$f(0) = 3 + 0 + 0 + 0 + 0 + \cdots = 3.$$

$$f'(x) = 5 + 14x + 27x^2 + 44x^3 + \cdots,$$

$$\begin{array}{cccccccccc} f(x) & = & 3 & + & 5x & + & 7x^2 & + & 9x^3 & + \cdots \\ & & & & \downarrow & & \downarrow & & \downarrow & \cdots \\ f'(x) & = & & & 5 & + & 14x & + & 27x^2 & + \cdots \end{array}$$

FORMULA FOR THE COEFFICIENTS

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be a power series. Then:

$$c_n = \frac{f^{(n)}(0)}{n!}$$

$$\begin{aligned}
f(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots, \\
f'(x) &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \cdots, \\
f''(x) &= 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots, \\
f^{(3)}(x) &= 6c_3 + 24c_4 x + 60c_5 x^2 + \cdots, \\
f^{(4)}(x) &= 24c_4 + 120c_5 x + \cdots, \\
f^{(5)}(x) &= 120c_5 + \cdots, \\
&\vdots \qquad \ddots
\end{aligned}$$

$$f^{(n)}(0) = n!\, c_n$$

Taylor Series

The Taylor series expansion of $f(x)$ about x_0:

$$f(x_0) + f^{'}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \dots$$

or

$$Taylor\ Series = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)\ (x - x_0)^k$$

If the series converge, one can write:

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)\ (x - x_0)^k$$

Taylor Series – Example 1

Obtain Taylor series expansion of $f(x) = e^x$ about $x = 0$:

$f(x) = e^x \qquad f(0) = 1$

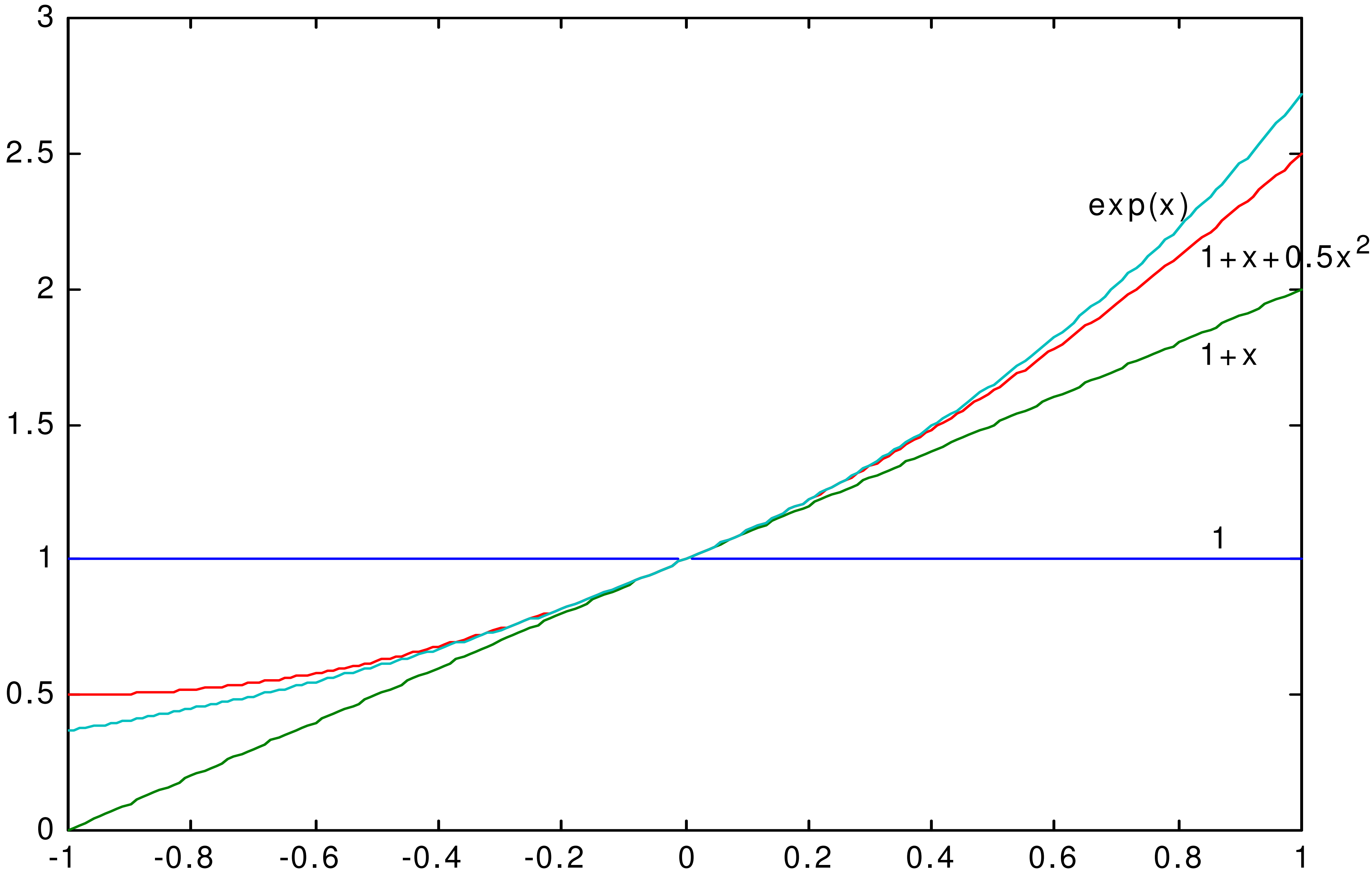
$f'(x) = e^x \qquad f'(0) = 1$

$f^{(2)}(x) = e^x \qquad f^{(2)}(0) = 1$

$f^{(k)}(x) = e^x \qquad f^{(k)}(0) = 1 \quad for\, k \geq 1$

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)\,(x - x_0)^k = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

The series converges for $|x| < \infty$.



The Taylor series converges fast (few terms are needed) when ***x*** is near the point of expansion. If ***|x-c|*** is large then more terms are needed to get a good approximation.

TAYLOR SERIES FORMULA

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots,$$

Find the Taylor series for e^x.

Let $f(x) = e^x$. Then $f'(x) = e^x$, $f''(x) = e^x$, and so on, so

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad \ldots$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

Taylor Series – Example 2

Obtain Taylor series expansion of $f(x) = \sin(x)$ about $x = 0$:

$f(x) = \sin(x) \qquad f(0) = 0$

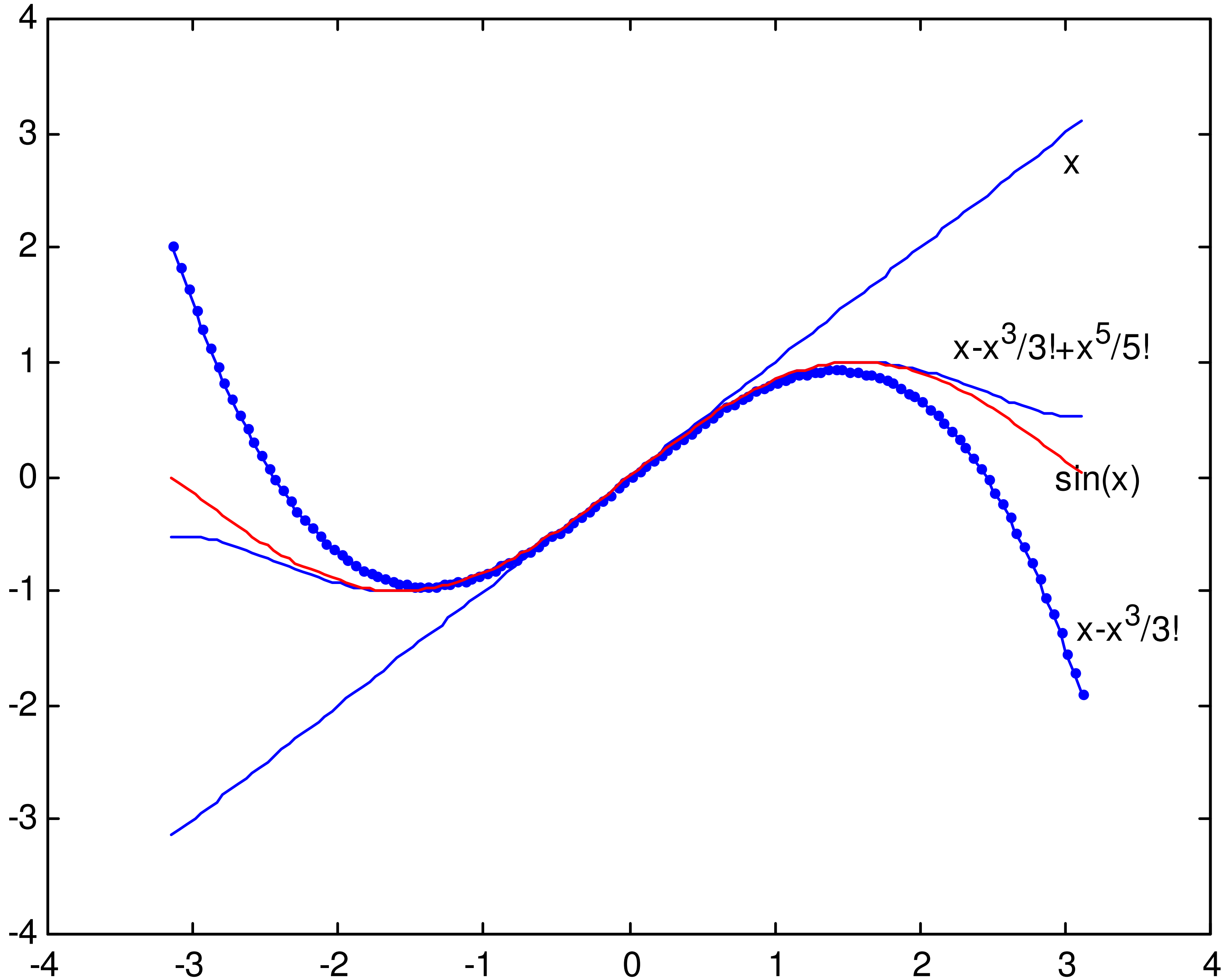
$f'(x) = \cos(x) \qquad f'(0) = 1$

$f^{(2)}(x) = -\sin(x) \qquad f^{(2)}(0) = 0$

$f^{(3)}(x) = -\cos(x) \qquad f^{(3)}(0) = -1$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.$$

The series converges for $|x| < \infty$.


4
3
2
1
0
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4
x
x-x^3/3!+x^5/5!
sin(x)
x-x^3/3!

Find Taylor series for **sin(x)**

$$f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots.$$

$$\begin{array}{rclrcrcr}
f(x) & = & \sin x & f(0) & = & \sin 0 & = & 0 \\
f'(x) & = & \cos x & f'(0) & = & \cos 0 & = & 1 \\
f''(x) & = & -\sin x & f''(0) & = & -\sin 0 & = & 0 \\
f^{(3)}(x) & = & -\cos x & f^{(3)}(0) & = & -\cos 0 & = & -1 \\
f^{(4)}(x) & = & \sin x & f^{(4)}(0) & = & \sin 0 & = & 0 \\
f^{(5)}(x) & = & \cos x & f^{(5)}(0) & = & \cos 0 & = & 1 \\
f^{(6)}(x) & = & -\sin x & f^{(6)}(0) & = & -\sin 0 & = & 0 \\
f^{(7)}(x) & = & -\cos x & f^{(7)}(0) & = & -\cos 0 & = & -1
\end{array}$$

$$\sin x = 0 + 1x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \cdots$$

$$= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

Taylor Series – Example 3

Obtain Taylor series expansion of $f(x) = \dfrac{1}{1-x}$ about $x = 0$:

$$f(x) = \frac{1}{1-x} \qquad\qquad f(0) = 1$$

$$f'(x) = \frac{1}{(1-x)^2} \qquad\qquad f'(0) = 1$$

$$f^{(2)}(x) = \frac{2}{(1-x)^3} \qquad\qquad f^{(2)}(0) = 2$$

$$f^{(3)}(x) = \frac{6}{(1-x)^4} \qquad\qquad f^{(3)}(0) = 6$$

Taylor Series Expansion of : $\dfrac{1}{1-\mathrm{x}} = 1 + \mathrm{x} + \mathrm{x}^2 + \mathrm{x}^3 + \ldots.$

Taylor's Theorem

If a function f(x) possesses continuous derivatives of orders $1, 2, \ldots, (n+1)$ in a closed interval $[a, b]$, then for any $c \in [a, b]$:

(n+1) terms Truncated Taylor Series

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + E_{n+1}$$

Remainder

where:

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1} \quad \textit{and } \xi \text{ is between } c \text{ and } x.$$

Taylor's Theorem

One can apply Taylor's theorem for:

$$f(x) = \frac{1}{1-x} \quad \text{with the point of expansion } c = 0 \text{ if } |x| < 1.$$

If $[a,b]$ includes $x = 1$, then the function and its derivatives are not defined.

$\Rightarrow$ Taylor Theorem is not applicable.

Error Term

To get an idea about the approximation error, we can derive an upper bound on:

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - c)^{n+1}$$

for all *values of* ξ between c and x.

Error Term – Example 4

How large is the error if we replaced $f(x)=e^{x}$ by the first 4 terms (n = 3) of its Taylor series expansion about $c=0$ $\quad when\ x=0.2$?

$f^{(k)}(x)=e^{x}$ $\qquad f^{(k)}(\xi)\le\ e^{0.2}\quad for\ k\ \ge\ 1$

$$E_{n+1}=\frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-c)^{n+1}$$

$$\left|E_{n+1}\right|\le\frac{e^{0.2}}{(n+1)!}(0.2)^{n+1}\Rightarrow\ \left|E_{4}\right|\le 8.14268E-05$$

Alternative form of Taylor's Theorem

Let $f(x)$ have continuous derivatives of orders $1, 2, \ldots, (\mathrm{n}+1)$ on an interval $[a,b]$, and $x \in [a,b]$ and $x+h \in [a,b]$, then :

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1}$$

$$E_{n+1} = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1} \quad \textit{where } \xi \text{ is between } x \text{ and } x+h$$

Taylor's Theorem – Alternative forms

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

where ξ is between c and x.

$$x \to x+h, \quad c \to x$$

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

where ξ is between x and $x+h$.

Find the Taylor series for $f(x) = \dfrac{1}{(1+x)^2}$

$$f(x) = \frac{1}{(1+x)^2} \qquad f(0) = 1$$

$$f'(x) = -2(1+x)^{-3} \qquad f'(0) = -2$$

$$f''(x) = 6(1+x)^{-4} \qquad f''(0) = 6$$

$$f^{(3)}(x) = -24(1+x)^{-5} \qquad f^{(3)}(0) = -24$$

$$f^{(4)}(x) = 120(1+x)^{-6} \qquad f^{(4)}(0) = 120$$

$$\frac{1}{(1+x)^2} = 1 + -2x + \frac{6}{2!}x^2 + \frac{-24}{3!}x^3 + \frac{120}{4!}x^4 + \cdots$$

$$\boxed{f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots}$$

$$= 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots$$

Find the first three terms of the Taylor series for $f(x) = \sqrt{1+x}$.

$$f(x) = (1+x)^{1/2} \qquad f(0) = 1$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2} \qquad f''(0) = -\frac{1}{4}$$

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{1}{2}x + \frac{-1/4}{2}x^2$$

$$\boxed{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,}$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

General Taylor Series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

General Taylor Series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots,$$

$$\boxed{f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,}$$

Find the Taylor series for $f(x) = 1/x^2$ centered at $x = 1$.

$$\begin{aligned} f(x) &= 1/x^2 & f(1) &= 1 \\ f'(x) &= -2x^{-3} & f'(1) &= -2 \\ f''(x) &= 6x^{-4} & f''(1) &= 6 \\ f^{(3)}(x) &= -24x^{-5} & f^{(3)}(1) &= -24 \\ f^{(4)}(x) &= 120x^{-6} & f^{(4)}(1) &= 120 \end{aligned}$$

$$\frac{1}{x^2} = 1 + -2(x-1) + \frac{6}{2!}(x-1)^2 + \frac{-24}{3!}(x-1)^3 + \frac{120}{4!}(x-1)^4 + \cdots$$

$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - \cdots$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \cdots$$

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \frac{1}{5}(x-1)^5 - \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{for } |x| < 1$$

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n \quad \text{for } |x| < 1$$

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} (n-1) n x^{n-2} \quad \text{for } |x| < 1$$

$$\frac{2x^2}{(1-x)^3} = \sum_{n=0}^{\infty} (n-1) n x^n \quad \text{for } |x| < 1$$

Alternating Series – Example 5

sin(1) can be computed using : $\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \Lambda$

This is a convergent alternatin g series since :

$a_1 \geq a_2 \geq a_3 \geq a_4 \geq \Lambda \quad and \quad \lim_{n \to \infty} a_n = 0$

Then :

$$\left| \sin(1) - \left(1 - \frac{1}{3!} \right) \right| \leq \frac{1}{5!}$$

$$\left| \sin(1) - \left(1 - \frac{1}{3!} + \frac{1}{5!} \right) \right| \leq \frac{1}{7!}$$

Obtain the Taylor series expansion of $f(x) = e^{2x+1}$ about $c = 0.5$ (the center of expansion)
How large can the error be when $(n+1)$ terms are used to approximate e^{2x+1} with $x = 1$?

Obtain Taylor series expansion of $f(x) = e^{2x+1}$, $c = 0.5$

$$f(x) = e^{2x+1} \qquad f(0.5) = e^2$$

$$f'(x) = 2e^{2x+1} \qquad f'(0.5) = 2e^2$$

$$f^{(2)}(x) = 4e^{2x+1} \qquad f^{(2)}(0.5) = 4e^2$$

$$f^{(k)}(x) = 2^k \ e^{2x+1} \qquad f^{(k)}(0.5) = 2^k e^2$$

$$e^{2x+1} = \sum_{k=0}^{\infty} \frac{f^{(k)}(0.5)}{k!} (x-0.5)^k$$

$$= e^2 + 2e^2(x-0.5) + 4e^2 \frac{(x-0.5)^2}{2!} + \ldots + 2^k e^2 \frac{(x-0.5)^k}{k!} + \ldots$$

Example 6 – Error Term

$$f^{(k)}(x) = 2^k \; e^{2x+1}$$

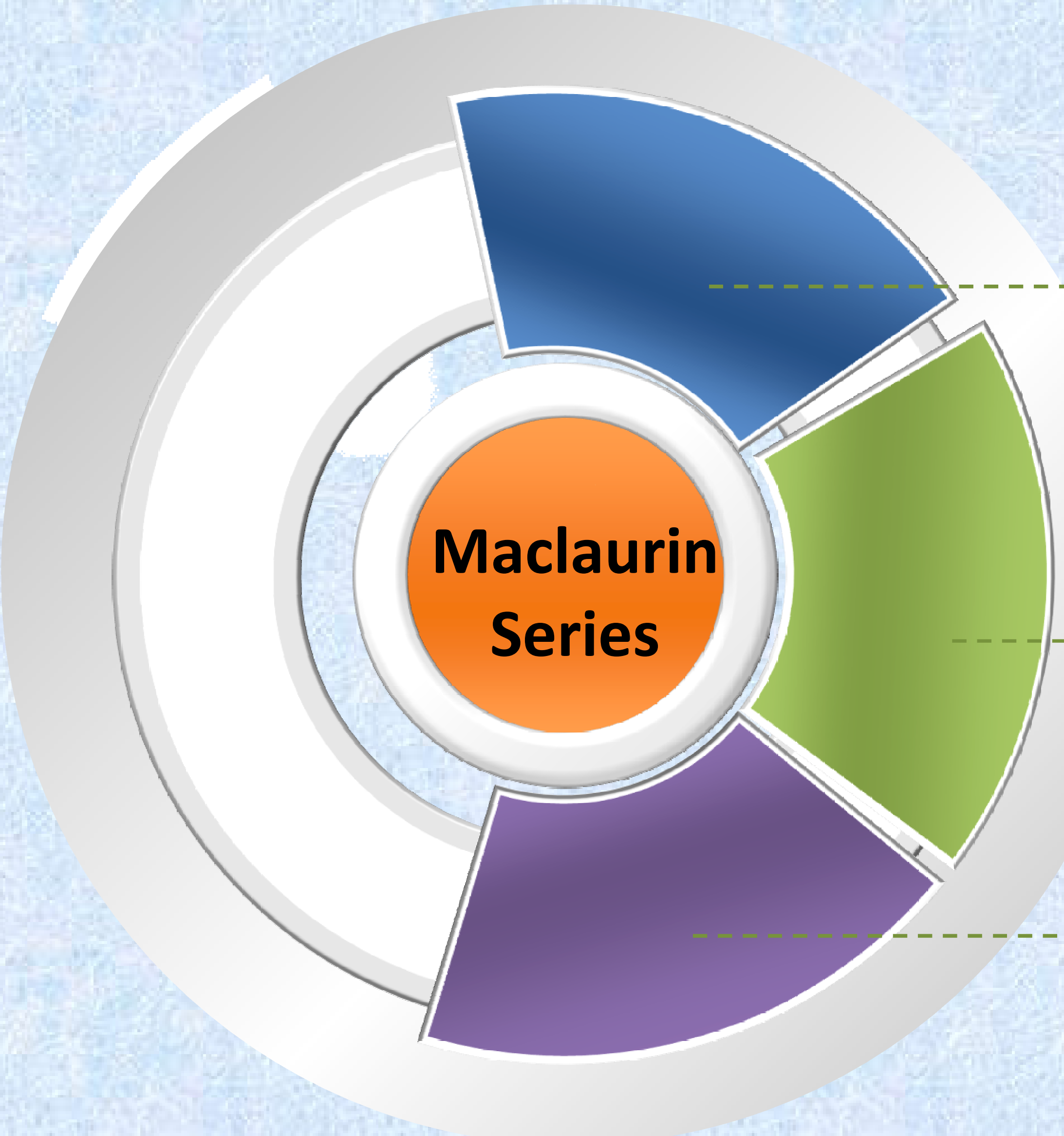
$$Error = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-0.5)^{n+1}$$

$$\left|Error\right| = \left| 2^{n+1} e^{2\xi+1} \quad \frac{(1-0.5)^{n+1}}{(n+1)!} \right|$$

$$\left|Error\right| \le 2^{n+1} \frac{(0.5)^{n+1}}{(n+1)!} \max_{\xi\in[0.5,1]} \left| e^{2\xi+1} \right|$$

$$\left|Error\right| \le \frac{e^3}{(n+1)!}$$

Maclaurin Series



Find Maclaurin series expansion of *cos (x)*.

- Maclaurin series is a special case of Taylor series with the center of expansion c = 0.

Taylor series is an **expansion** of any function which could be expanded about any point and **Maclaurin series** is the same **expansion** expanded about only one point, i.e, zero. infinite **series** that can start at any point.

Maclaurin Series – Example 7

Obtain Maclaurin series expansion of : $f(x) = \cos(x)$

$$f(x) = \cos(x) \qquad f(0) = 1$$

$$f'(x) = -\sin(x) \qquad f'(0) = 0$$

$$f^{(2)}(x) = -\cos(x) \qquad f^{(2)}(0) = -1$$

$$f^{(3)}(x) = \sin(x) \qquad f^{(3)}(0) = 0$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (x)^k = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.$$

The series converges for $|\mathrm{x}| < \infty$.

Maclaurin Series – Example 8

Use one of the Taylor Series derived in the notes to determine the Taylor Series for $f(x)=\frac{7}{x^4}$

$$n=0: \quad f(x)=\frac{7}{x^4}=7x^{-4}$$

$$n=1: \quad f'(x)=-7(4)x^{-5}$$

$$n=2: \quad f''(x)=7(4)(5)x^{-6}$$

$$n=3: \quad f^{(3)}(x)=-7(4)(5)(6)x^{-7}$$

$$n=4: \quad f^{(4)}(x)=7(4)(5)(6)(7)x^{-8}$$

about $x=-3$

$$f^{(n)}(x) = 7(-1)^n \frac{(2)(3)}{(2)(3)}(4)(5)(6)\cdots(n+3)x^{-8}$$

$$= 7(-1)^n \frac{(2)(3)(4)(5)(6)\cdots(n+3)}{6}x^{-8}$$

$$= \frac{7}{6}(-1)^n (n+3)!x^{-(n+4)} \qquad n = 0,1,2,3,\ldots$$

$$f^{(n)}(-3) = \frac{7}{6}(-1)^n (n+3)!(-3)^{-(n-4)}$$

$$= \frac{7(-1)^n (n+3)!}{6(-3)^{n+4}}$$

$$= \frac{7(-1)^n (n+3)!}{6(-1)^{n+4}(3)^{n+4}}$$

$$= \frac{7(n+3)!}{6(-1)^4 (3)^{n+4}}$$

$$= \frac{7(n+3)!}{6(3)^{n+4}} \qquad n = 1,2,3,\ldots$$

$$\frac{7}{x^4} = \sum_{n=0}^{\infty} \frac{f^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{7(n-3)!}{6(3)^{n+4}\, n!}(x+3)^n = \boxed{\sum_{n=0}^{\infty} \frac{7(n+3)(n+2)(n+1)}{6(3)^{n+4}}(x+3)^n}$$

Mean Value Theorem

If f(x) is a continuous function on a closed interval [a, b] and its derivative is defined on the open interval (a, b) then there exists $\xi \in [a, b]$

$$\frac{df(\xi)}{dx} = \frac{f(b) - f(a)}{(b\text{-}a)}$$

Proof : Use Taylor' s Theorem for $n = 0, x = a, \; x + h = b$

$$f(b) = f(a) + \frac{df(\xi)}{dx}\,(b\text{-}a)$$

Alternating Series Theorem

Consider the alternating series :

$$S = a_1 - a_2 + a_3 - a_4 + \Lambda$$

$$\text{If } \boxed{\begin{cases} a_1 \ge a_2 \ge a_3 \ge a_4 \ge \Lambda \\ \quad and \\ \lim_{n\to\infty} a_n = 0 \end{cases}} \quad \text{then } \boxed{\begin{cases} \text{The series converges} \\ \quad and \\ \left|S - S_n\right| \le a_{n+1} \end{cases}}$$

S_n : Partial sum (sum of the first n terms)

a_{n+1} : First omitted term

Binomial Series

Before presenting the basic list for elementary functions, you will develop one more series—for a function of the form $f(x) = (1 + x)^k$. This produces the **binomial series.**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

TAYLOR & MACLAURIN SERIES

- Arranging our work in columns, we have:

$f(x)=(1+x)^k$	$f(0)=1$
$f'(x)=k(1+x)^{k-1}$	$f'(0)=k$
$f''(x)=k(k-1)(1+x)^{k-2}$	$f''(0)=k(k-1)$
$f'''(x)=k(k-1)(k-2)(1+x)^{k-3}$	$f'''(0)=k(k-1)(k-2)$
.	.
.	.
.	.
$f^{(n)}=k(k-1)\cdots(k-n+1)(1+x)^{k-n}$	$f^{(n)}(0)=k(k-1)\cdots(k-n+1)$

Problem 1– *Binomial Series*

Find the Maclaurin series for $f(x) = (1 + x)^k$ and determine its radius of convergence.

Assume that k is not a positive integer and $k \neq 0$.

Solution:

By successive differentiation, you have

$$f(x) = (1 + x)^k \qquad f(0) = 1$$

$$f'(x) = k(1 + x)^{k-1} \qquad f'(0) = k$$

$$f''(x) = k(k-1)(1 + x)^{k-2} \qquad f''(0) = k(k-1)$$

$$f'''(x) = k(k-1)(k-2)(1 + x)^{k-3} \qquad f'''(0) = k(k-1)(k-2)$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = k\cdots(k-n+1)(1 + x)^{k-n} \qquad f^{(n)}(0) = k(k-1)\cdots(k-n+1)$$

Problem 1– Binomial Series

cont'd

which produces the series

$$1 + kx + \frac{k(k-1)x^2}{2} + \cdot \cdot \cdot + \frac{k(k-1) \cdot \cdot \cdot (k-n+1)x^n}{n!} + \cdot \cdot \cdot .$$

Because $a_{n+1}/a_n \to 1$, you can apply the Ratio Test to conclude that the radius of convergence is $R = 1$.

So, the series converges to some function in the interval (–1, 1).

Deriving Taylor Series from a Basic List

POWER SERIES FOR ELEMENTARY FUNCTIONS

Function	Interval of Convergence
$\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \cdots + (-1)^n(x-1)^n + \cdots$	$0 < x < 2$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots + (-1)^n x^n + \cdots$	$-1 < x < 1$
$\ln x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots$	$0 < x \le 2$
$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots + \frac{x^n}{n!} + \cdots$	$-\infty < x < \infty$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots$	$-\infty < x < \infty$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$	$-\infty < x < \infty$
$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \cdots$	$-1 \le x \le 1$
$\arcsin x = x + \frac{x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{(2n)!x^{2n+1}}{(2^n n!)^2(2n+1)} + \cdots$	$-1 \le x \le 1$
$(1+x)^k = 1 + kx + \frac{k(k-1)x^2}{2!} + \frac{k(k-1)(k-2)x^3}{3!} + \frac{k(k-1)(k-2)(k-3)x^4}{4!} + \cdots$	$-1 < x < 1$*

* The convergence at $x = \pm 1$ depends on the value of k.

Problem 2– *Deriving a Power Series from a Basic List*

Find the power series for $f(x) = \cos\sqrt{x}.$

Solution:

Using the power series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

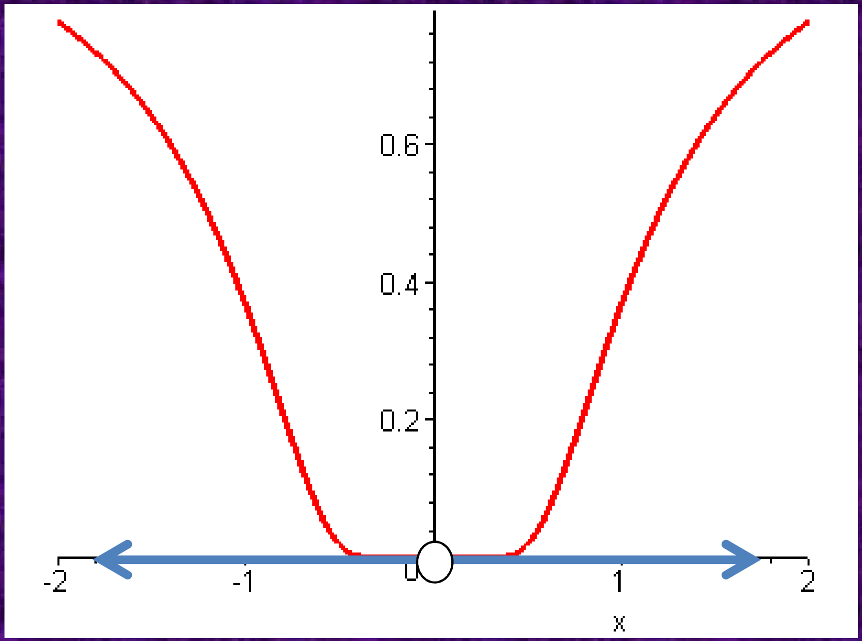
you can replace *x* by $\sqrt{x}$ to obtain the series

$$\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} - \cdots.$$

This series converges for all *x* in the domain of $\cos\sqrt{x}$ that is, for $x \geq 0$.

So where were we?

$$f(x)=\begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Facts:

- f is continuous and has derivatives of all orders at $x = 0$.
- $f^{(n)}(0)=0$ for all n.

This tells us that the Maclaurin Series for f is zero everywhere!

The Maclaurin Series for f converges everywhere, but is equal to f only at $x = 0$!

Why the Taylor series, then?

Power series as functions

First a series . . .

$$\boxed{\sum_{n=0}^{\infty} a_n (x-x_0)^n}$$

. . . Then a function

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n}$$

Guarantees that the power series we started with is, in fact, the *TAYLOR SERIES FOR f*.

Taylor Series

First a function . . .

$$\boxed{f(x)}$$

. . . Then a series

$$\boxed{\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n}$$

If *f* is equal to any power series at all, that power series must be the Taylor series for *f*.

That's why that's were we look!

Problem 3- TAYLOR & MACLAURIN SERIES

$$\cos x = \frac{d}{dx}(\sin x)$$

$$= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)$$

$$= 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \frac{7x^6}{7!} + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Problem 4- USES OF TAYLOR SERIES

- Now, we integrate term by term:

$$\int e^{-x^2}\,dx = \int\left(1-\frac{x^2}{1!}+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots+(-1)^n\frac{x^{2n}}{n!}+\cdots\right)dx$$

$$= C + x - \frac{x^3}{3\cdot 1!}+\frac{x^5}{5\cdot 2!}-\frac{x^7}{7\cdot 3!}$$

$$+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)n!}+\cdots$$

– This series converges for all *x* because the original series for e^{-x^2} converges for all *x*.

Problem 4- USES OF TAYLOR SERIES

$$\int_0^1 x^{-x^2}\,dx = \left[x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \cdots \right]_0^1$$

$$= 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216} - \cdots$$

$$\approx 1 - \tfrac{1}{3} + \tfrac{1}{10} - \tfrac{1}{42} + \tfrac{1}{216}$$

$$\approx 0.7475$$

Problem 5—Taylor Series

Find the value of $f(6)$ given that $f(4)=125,\ f'(4)=74,$ $f''(4)=30,\ f'''(4)=6$ and all other higher order derivatives of $f(x)$ at $x=4$ are zero.

Solution:

$$f(x+h)=f(x)+f'(x)h+f''(x)\frac{h^2}{2!}+f'''(x)\frac{h^3}{3!}+\Lambda$$

$$x=4$$

$$h=6-4=2$$

Problem 6—Taylor Series

Solution: (cont.)

Since the higher order derivatives are zero,

$$f(4+2) = f(4) + f'(4)2 + f''(4)\frac{2^2}{2!} + f'''(4)\frac{2^3}{3!}$$

$$f(6) = 125 + 74(2) + 30\left(\frac{2^2}{2!}\right) + 6\left(\frac{2^3}{3!}\right)$$

$$= 125 + 148 + 60 + 8$$

$$= 341$$

Note that to find $f(6)$ exactly, we only need the value of the function and all its derivatives at some other point, in this case

$$x = 4$$

Derivation for Maclaurin Series for e^x

Derive the Maclaurin series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \Lambda$$

The Maclaurin series is simply the Taylor series about the point x=0

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + f''''(x)\frac{h^4}{4} + f'''''(x)\frac{h^5}{5} + \Lambda$$

$$f(0+h) = f(0) + f'(0)h + f''(0)\frac{h^2}{2!} + f'''(0)\frac{h^3}{3!} + f''''(0)\frac{h^4}{4} + f'''''(0)\frac{h^5}{5} + \Lambda$$

Problem 7

- Find the truncation error in approximating the function

$$y_2(x) = \ln(1+x)$$

- (a) $y_1(x) = x$
- (b) $y_2(x) = x - \frac{1}{2}x^2$
- (c) $y_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3$
- (d) $y_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$
- over the range $0 \le x \le 1$

Problem 7- Solution

- Consider a representative value of x, x = 0.5, and find the truncation error. The exact value of the function is given by

$$y_2(x) = \ln(1+x) = \ln(1.5) = 0.405465108$$

- (a) $y_1(0.5) = 0.5$

$$T_e = 0.405465108 - 0.5 = -0.0094544892$$

- (b) $y_2(0.5) = (0.5) - \frac{1}{2}(0.5)^2 = 0.375$

$$T_e = 0.405465108 - 0.375 = -0.030465108$$

Problem 7- Solution

- Consider a representative value of x, x = 0.5, and find the truncation error. The exact value of the function is given by

$$y_2(x) = \ln(1+x) = \ln(1.5) = 0.405465108$$

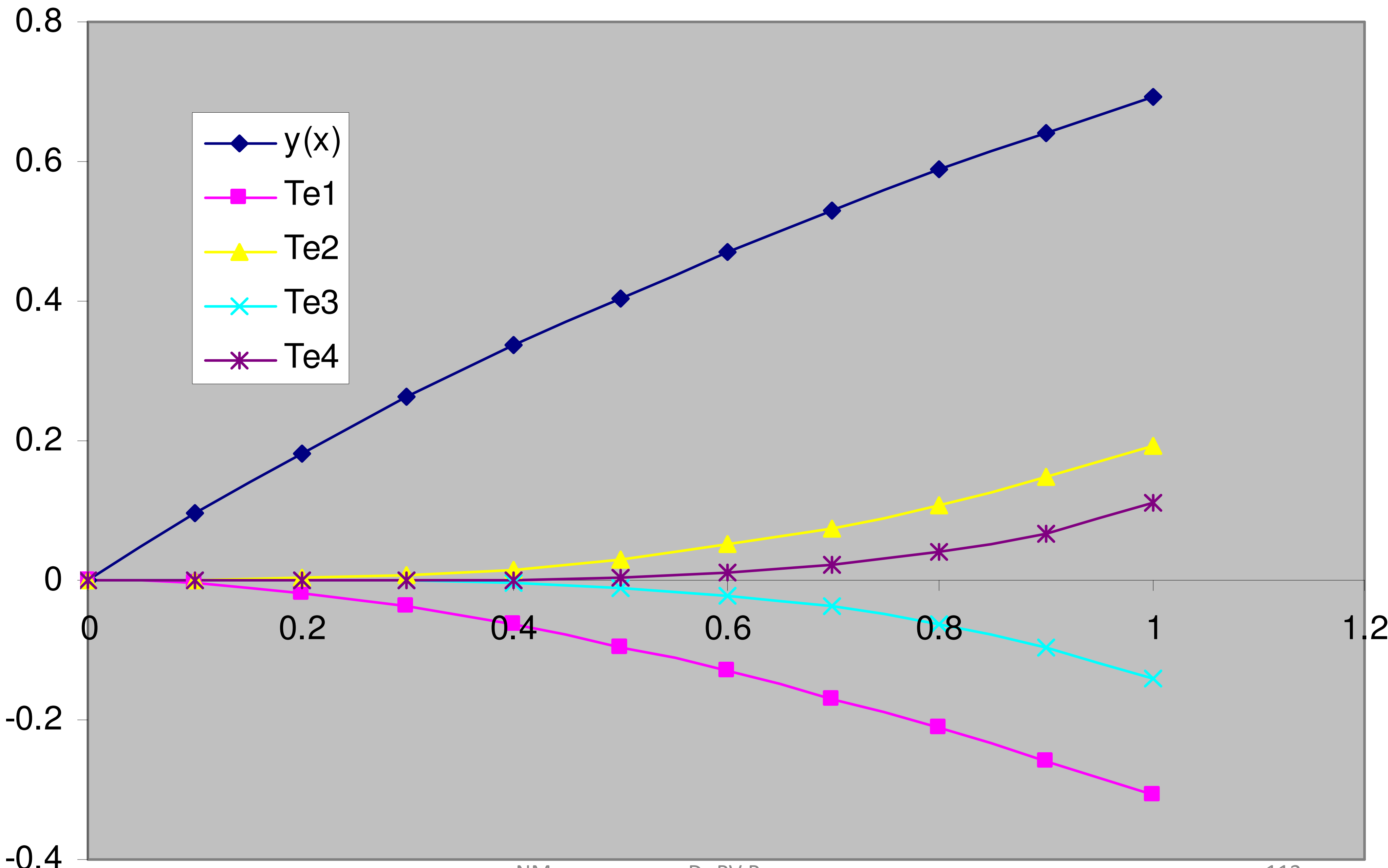
- (c) $$y_3(0.5) = (0.5) - \frac{1}{2}(0.5)^2 + \frac{1}{3}(0.5)^3 = 0.416666667$$

$$T_e = 0.405465108 - 0.416666667 = -0.011201559$$

- (d) $$y_4(0.5) = (0.5) - \frac{1}{2}(0.5)^2 + \frac{1}{3}(0.5)^3 - \frac{1}{4}(0.5)^4 = 0.401041667$$

$$T_e = 0.405465108 - 0.401041667 = 0.004423441$$

Truncation error



Problem 8 - Use the table of Maclaurin series to find the first 3 nonzero terms and the general term for the Maclaurin series generated by the following functions:

$\cos x^2$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots$$

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \ldots + (-1)^n \frac{x^{4n}}{(2n)!} + \ldots$$

xe^{-x}

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \ldots + (-1)^n \frac{x^n}{n!} + \ldots$$

$$xe^{-x} = x - x^2 + \frac{x^3}{2!} - \ldots + (-1)^n \frac{x^{n+1}}{n!} + \ldots$$

Problem 9- Given the Taylor series below for the following functions centered at x = 0, find $f^{(8)}(0)$.

Recall that the term for n = 8 is $\dfrac{f^{(8)}(0)x^8}{8!}$, so find the coefficient of x^8 in the following functions:

$$f(x) = \cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \ldots + (-1)^n \frac{x^{4n}}{(2n)!} + \ldots$$

- Coefficient of x^8 = 1/(4!)

$$\frac{f^{(8)}(0)}{8!} = \frac{1}{4!} \Rightarrow f^{(8)}(0) = \frac{8!}{4!} = 8 \bullet 7 \bullet 6 \bullet 5 = 1680$$

$$f(x) = xe^{-x} = x - x^2 + \frac{x^3}{2!} - \ldots + (-1)^n \frac{x^{n+1}}{n!} + \ldots$$

- Coefficient of x^8 term = (-1)/(7!)

$$\frac{f^{(8)}(0)}{8!} = -\frac{1}{7!} \Rightarrow f^{(8)}(0) = -\frac{8!}{7!} = -8$$

Problem 11 - Use the definition of a Taylor series to construct a 4th-order Taylor polynomial and the Taylor series for the function **f(x)=lnx** at x=1.

$$P_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \ldots + \frac{f^{(4)}(1)}{5!}(x-1)^4$$

$$f'(1) = \left.\frac{1}{x}\right|_{x=1} = 1 \qquad f''(1) = -\left.\frac{1}{x^2}\right|_{x=1} = -1 \qquad f'''(1) = \left.\frac{2}{x^3}\right|_{x=1} = 2$$

$$f^{(4)}(1) = -\left.\frac{6}{x^4}\right|_{x=1} = -6$$

$$P_4(x) = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{2(x-1)^3}{6} - \frac{6(x-1)^4}{24}$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

- For the overall Taylor series, take a look at your polynomial and write the pattern:

$$\ln x = \sum_{k=0}^{\infty} (-1)^k \frac{(x-1)^{k+1}}{k+1}$$

Problem 12- Let f be a function with f(1)=2, f'(1)=3, f''(1) = -2, and f'''(1)=8. If f has derivatives for all orders for all real numbers, find the 3rd order Taylor polynomial for f at x = 1 and use it to approximate f(1.1).

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$$P_3(x) = 2 + 3(x-1) - \frac{2}{2}(x-1)^2 + \frac{8}{6}(x-1)^3$$

$$P_3(x) = 2 + 3(x-1) - (x-1)^2 + \frac{4(x-1)^3}{3}$$

- For the approximation, just plug in 1.1 for x:

$$f(1.1) = 2 + 3(.1) - (.1)^2 + \frac{4(.1)^3}{3}$$

$$= 2 + \frac{3}{10} - \frac{1}{100} + \frac{4}{3000} = \frac{3437}{1500}$$

SOLUTION OF NON-LINEAR EQUATIONS

- All equations used in horizontal adjustment are non-linear.
- Solution involves approximating solution using 1'st order Taylor series expansion, and
- Then solving system for corrections to approximate solution.
- Repeat solving system of linearized equations for corrections until corrections become small.
- This process of solving equations is known as: ITERATING

Taylor's Series

Given a function, *L = f(x,y)*

$$L = f(x,y) = f(x_0, y_0) + \frac{\left(\partial L/\partial x\right)_0 dx}{1!} + \frac{\left(\partial^2 L/\partial x^2\right)_0 dx^2}{2!} + \ldots$$

$$+ \frac{\left(\partial L/\partial y\right)_0 dy}{1!} + \frac{\left(\partial^2 L/\partial y^2\right)_0 dy^2}{2!} + \ldots$$

Taylor's Series

- The series is also non-linear (unknowns are the dx's, dy's, and higher order terms)
- Therefore, truncate the series after only the first order terms, which makes the equation an approximation
- Initial approximations generally need to be reasonably close in order for the solution to converge

$$L = f(x, y) \approx f(x_0, y_0) + \left(\frac{\partial L}{\partial x}\right)_0 dx + \left(\frac{\partial L}{\partial y}\right)_0 dy$$

Solution

- Determine initial approximations (closer is better)
- Form the (first order) equations
- Solve for corrections, dx and dy
- Add corrections to approximations to get improved values
- Iterate until the solution converges

Problem 13- Solve the following pair of non-linear equations. Use initial approximation of 1 (one) for both x and y.

$$F(x, y) = x + y - 2y^2 = -4$$

$$G(x, y) = x^2 + y^2 = 8$$

First, determine the partial derivatives

Partials

$$\frac{\partial F}{\partial x} = 1 \qquad \frac{\partial F}{\partial y} = 1 - 4y$$

$$\frac{\partial G}{\partial x} = 2x \qquad \frac{\partial G}{\partial y} = 2y$$

Write the Linearized Equations

$$F(x, y) = 1 + 1 - 2(1)^2 + dx + [1 - 4(1)]dy = -4$$

$$G(x, y) = (1)^2 + (1)^2 + 2(1)dx + 2(1)dy = 8$$

Simplify

$$dx - 3dy = -4$$

$$2dx + 2dy = 6$$

Solution

$$\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \dfrac{2(-4)+3(6)}{8} \\ \dfrac{-2(-4)+1(6)}{8} \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \end{bmatrix}$$

New approximations:

$$dx = 1 + 1.25 = 2.25$$

$$dy = 1 + 1.75 = 2.75$$

Linearized Equations – Iteration 2

$$F(x,y) = 2.25 + 2.75 - 2(2.75)^2 + dx + [1 - 4(2.75)]dy = -4$$

$$G(x,y) = (2.25)^2 + (2.75)^2 + 2(2.25)dx + 2(2.75)dy = 8$$

Simplify

$$dx - 10dy = 6.125$$

$$4.5dx + 5.5dy = -4.625$$

Solve – Iteration 2

$$\begin{bmatrix} 1 & -10 \\ 4.5 & 5.5 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 6.125 \\ -4.625 \end{bmatrix}$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \frac{1}{50.5} \begin{bmatrix} 5.5 & 10 \\ -4.5 & 1 \end{bmatrix} \begin{bmatrix} 6.125 \\ -4.625 \end{bmatrix}$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \dfrac{5.5(6.125)+10(-4.625)}{50.5} \\ \dfrac{-4.5(6.125)+1(-4.625)}{50.5} \end{bmatrix} = \begin{bmatrix} -0.25 \\ -0.64 \end{bmatrix}$$

New approximations:

$$dx = 2.25 - 0.25 = 2.00$$
$$dy = 2.75 - 0.64 = 2.11$$

Iteration 3

Same procedure yields: *dx* = 0.00 and *dy* = -0.11

This results in new approximations of *x* = 2.00 and *y* = 2.00

Further iterations are negligible

General Matrix Form

- The coefficient matrix formed by the partial derivatives of the functions with respect to the variables is the *Jacobian* matrix
- It can be used directly in a general matrix form

$$J = \begin{bmatrix} \partial F/\partial x & \partial F/\partial y \\ \partial G/\partial x & \partial G/\partial y \end{bmatrix}$$

General Form for problem 10

$JX = K$

$$J = \begin{bmatrix} 1 & 1-4y_0 \\ 2x_0 & 2y_0 \end{bmatrix}$$

$$X = \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$K = \begin{bmatrix} -4-F(x_0, y_0) \\ 8-G(x_0, y_0) \end{bmatrix}$$

Problem 11- Determine the equation of a circle that passes through the points (9.4, 5.6), (7.6, 7.2), and (3.8, 4.8).

Initial approximations for unknown and circle equation:
Center point: (7, 4.5), Radius: 3

Rearranged

$$(x-h)^2 + (y-k)^2 = r^2$$

$$C(h,k,r) = (x-h)^2 + (y-k)^2 - r^2 = 0$$

Linearizing

$$-2(x-h_o)dh - 2(y-k_o)dk - 2r_o dr = -[(x-h_o)^2 + (y-k_o)^2 - r_o^2]$$

$$(x-h_o)dh + (y-k_o)dk + r_o dr = -\frac{1}{2}[(x-h_o)^2 + (y-k_o)^2 - r_o^2]$$

Set Up General Matrix Form

Setup linearized matrix equations for 3 points.

$$\begin{bmatrix} (x_1 - h_o) & (y_1 - k_o) & r_o \\ (x_2 - h_o) & (y_2 - k_o) & r_o \\ (x_3 - h_o) & (y_3 - k_o) & r_o \end{bmatrix} \begin{bmatrix} dh \\ dk \\ dr \end{bmatrix} = \begin{bmatrix} 0.5\left((x_1 - h_o)^2 + (y_1 - k_o)^2 - r_o^2 \right) \\ 0.5\left((x_2 - h_o)^2 + (y_2 - k_o)^2 - r_o^2 \right) \\ 0.5\left((x_3 - h_o)^2 + (y_3 - k_o)^2 - r_o^2 \right) \end{bmatrix}$$

Substitute the Values and Solve

Substitute in values and solve system:

$$\begin{bmatrix} 2.4 & 1.1 & 3 \\ 0.6 & 2.7 & 3 \\ -3.2 & 0.3 & 3 \end{bmatrix} \begin{bmatrix} dh \\ dk \\ dr \end{bmatrix} = \begin{bmatrix} -1.015 \\ -0.675 \\ 0.665 \end{bmatrix}$$

First iteration solution:

$$\begin{bmatrix} dh \\ dk \\ dr \end{bmatrix} = \begin{bmatrix} -0.28462 \\ -0.10769 \\ -0.07115 \end{bmatrix}$$

Continue Until Converged

Initial approximations for second iteration:

$$h = 7.0 - 0.28462 = 6.7154$$

$$k = 4.5 - 0.10769 = 4.3923$$

$$r = 3.0 - 0.07115 = 2.9288$$

Repeat setup of matrices and solving again results in final solution of:

$$h = 6.72$$

$$k = 4.39$$

$$r = 2.94$$

Newton's 2nd law of Motion

- *"The time rate change of momentum of a body is equal to the resulting force acting on it."*

- Formulated as **F = m.a**

  F = net force acting on the body

  m = mass of the object (kg)

  a = its acceleration (m/s^2)

- Some complex models may require more sophisticated mathematical techniques than simple algebra
  - Example, modeling of a falling parachutist:

$$F = F_D + F_U$$

F_U = Force due to air resistance = $-cv$ ***(c = drag coefficient)***

F_D = Force due to gravity = mg

$$\frac{dv}{dt} = \frac{F}{m}$$

$$F = F_D + F_U$$

$$F_D = mg$$

$$F_U = -cv$$

$$\frac{dv}{dt} = \frac{mg - cv}{m}$$

- This is a first order ordinary **differential equation.** We would like to solve for v (velocity).

- It can **not** be solved using algebraic manipulation

- <u>Analytical Solution:</u>

  If the parachutist is initially at rest (v=0 at t=0), using calculus dv/dt can be solved to give the result:

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

Independent variable

Dependent variable

$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

Forcing function

Parameters

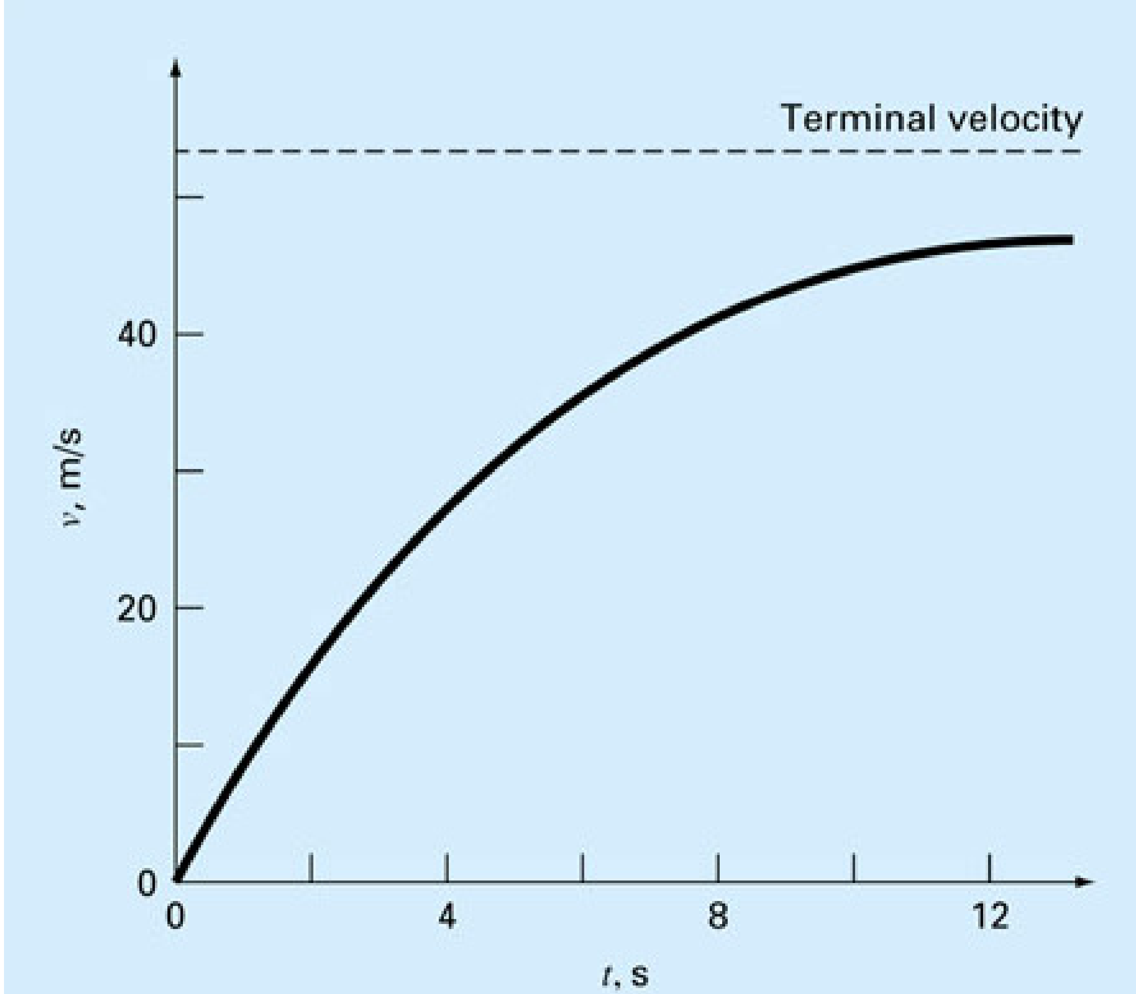
**here Analytical Solution

$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

If ***v(t)*** could not be solved **analytically**, then we need to use a numerical method to solve it

g = 9.8 m/s^2 c = 12.5 kg/s
m = 68.1 kg

t (sec.)	V (m/s)
0	0
2	16.40
4	27.77
8	41.10
10	44.87
12	47.49
∞	53.39



**Run *the programs through* MATLAB

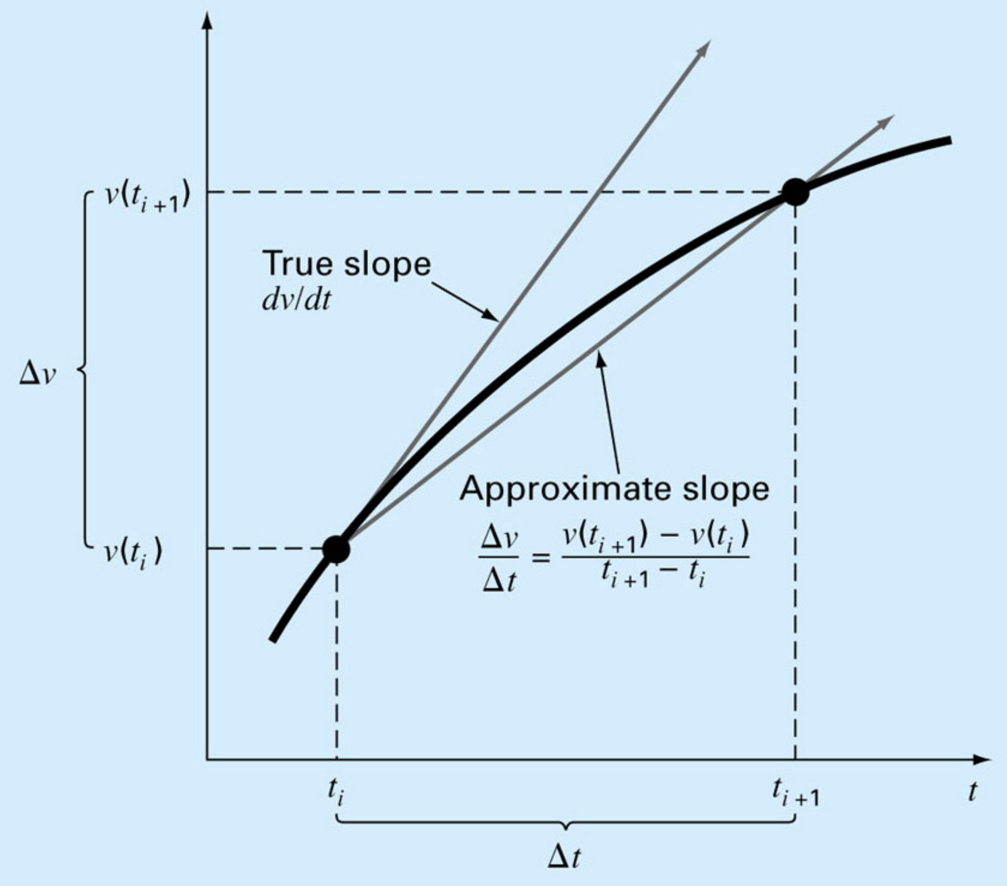
Numerical Solution

$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} \ldots\ldots\ldots \frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

$$\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c}{m} v(t_i)$$

This equation can be rearranged to yield

$$v(t_{i+1}) = v(t_i) + [g - \frac{c}{m} v(t_i)](t_{i+1} - t_i)$$



t (sec.)	V (m/s)
0	0
2	19.60
4	32.00
8	44.82
10	47.97
12	49.96
∞	53.39

Δt = 2 sec

To minimize the error, use a smaller step size, Δt
No problem, if you use a computer!

Analytical vs. Numerical solution

m=68.1 kg c=12.5 kg/s
g=9.8 m/s

t (sec.)	V (m/s)
0	0
2	16.40
4	27.77
8	41.10
10	44.87
12	47.49
∞	53.39

Δt = 2 sec

t (sec.)	V (m/s)
0	0
2	19.60
4	32.00
8	44.82
10	47.97
12	49.96
∞	53.39

Δt = 0.5 sec

t (sec.)	V (m/s)
0	0
2	17.06
4	28.67
8	41.95
10	45.60
12	48.09
∞	53.39

Δt = 0.01 sec

t (sec.)	V (m/s)
0	0
2	16.41
4	27.83
8	41.13
10	44.90
12	47.51
∞	53.39

$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

$$v(t_{i+1}) = v(t_i) + [g - \frac{c}{m} v(t_i)]\Delta t$$

**Run *the programs through* MATLAB

CONCLUSION: If you want to minimize the error, use a **smaller step size**, Δt

Mathematical Models

- **Modeling is the development of a mathematical representation of a physical/biological/chemical/ economic/etc. system**
- **Putting our understanding of a system into mathematical form**

- **<u>Problem Solving Tools</u>:**

  Analytic solutions, statistics, <u>numerical methods</u>, graphics, etc.
- **Numerical methods are one means by which mathematical models are solved**

Mathematical Models

Modeling is the development of a mathematical representation of a physical/ biological/ chemical/ economic/etc. system

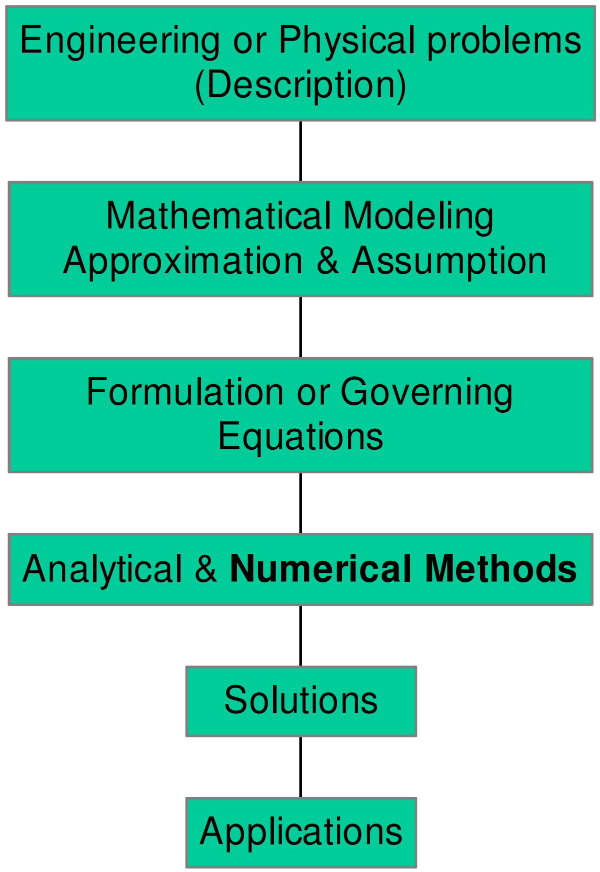
Putting our understanding of a system into math

<u>Problem Solving Tools</u>: Analytic solutions, statistics, <u>numerical methods</u>, graphics, etc.

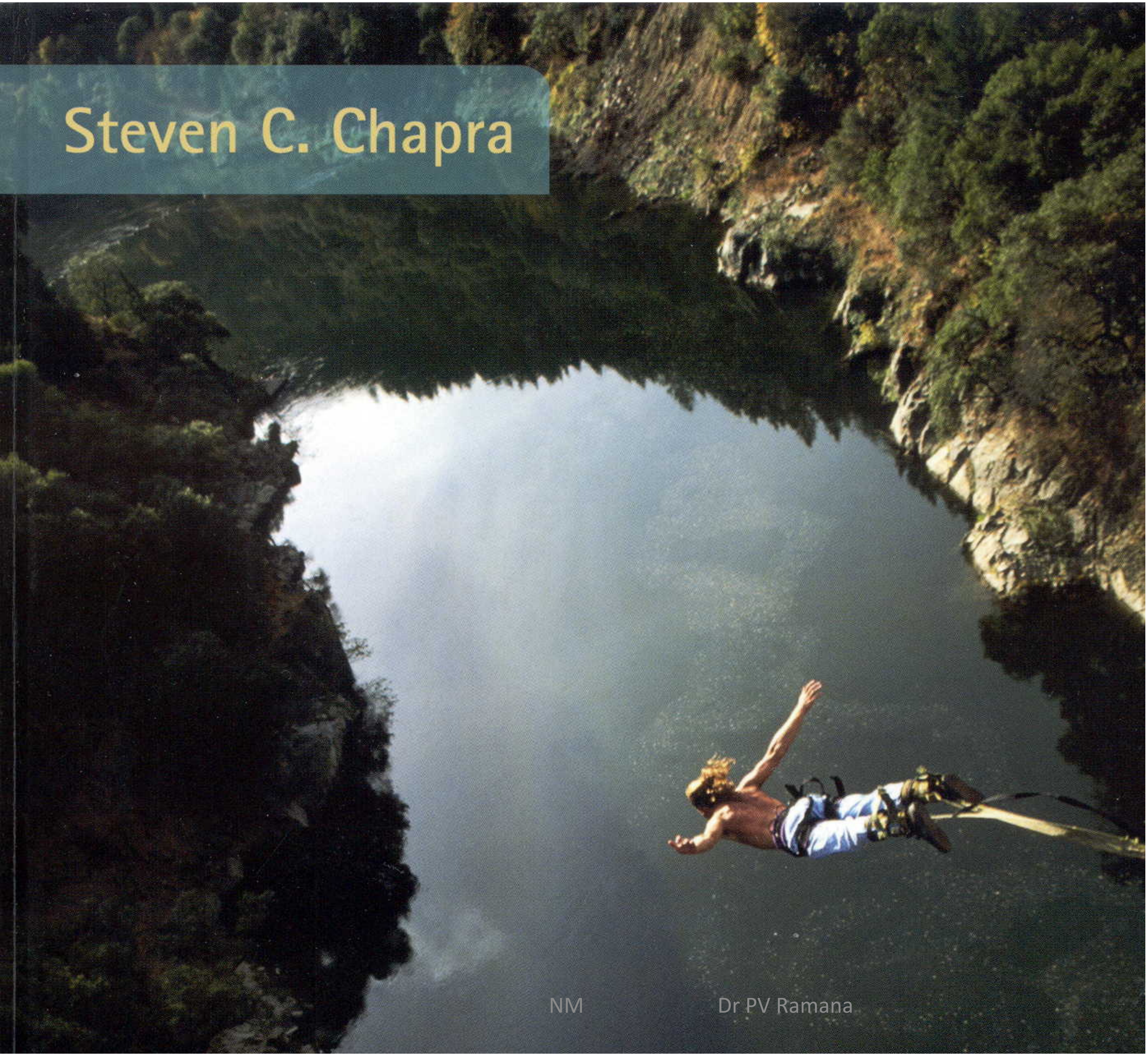
Numerical methods are one means by which mathematical models are solved

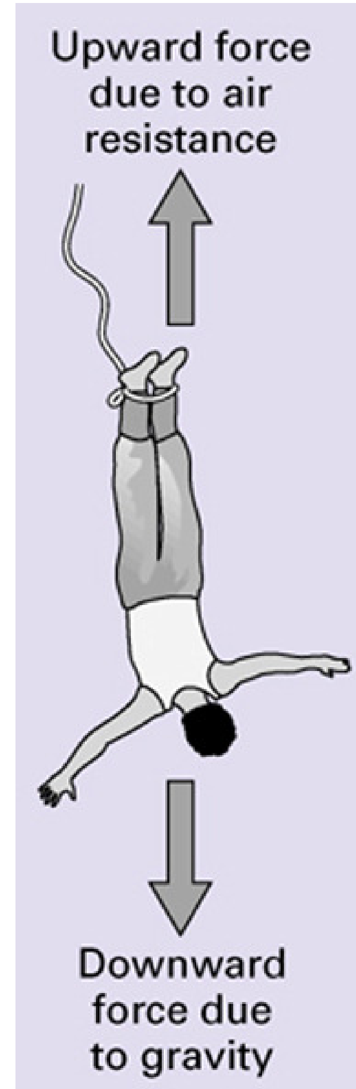
Mathematical Modeling

The process of solving an engineering or physical problem.



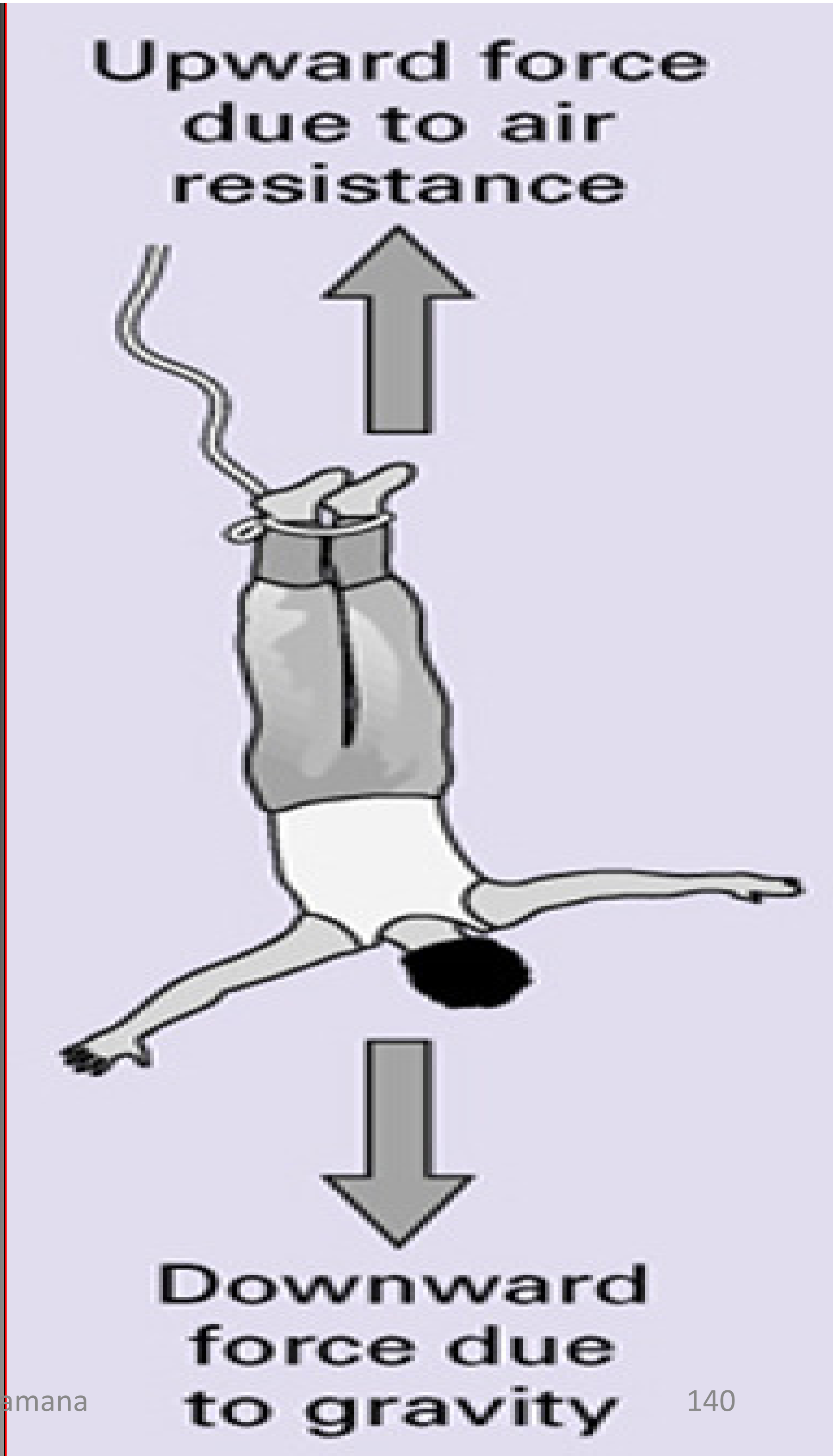
Common features
operation


Steven C. Chapra


Upward force
due to air
resistance
Downward
force due
to gravity

Bungee Jumper

- Use the information to determine the length and required strength of the bungee cord for jumpers of different mass
- The same analysis can be applied to a falling parachutist or a rain drop



Bungee Jumper / Falling Parachutist

Newton's Second Law

$$F = ma = F_{down} - F_{up}$$
$$= mg - c_d v^2$$

(gravity minus air resistance)

Observations / Experiments

Where does *mg* come from?

Where does $-c_d v^2$ come from?

Now we have fundamental physical laws, so we combine those with observations to model the system

A lot of what you will do is *"canned"* but need to know how to make use of observations

How have computers changed problem solving in engineering?

Allow us to focus more on the correct description of the problem at hand, rather than worrying about how to solve it.

Exact (Analytic) Solution

➢ Newton's Second Law

$$m\frac{dv}{dt} = mg - c_d v^2$$

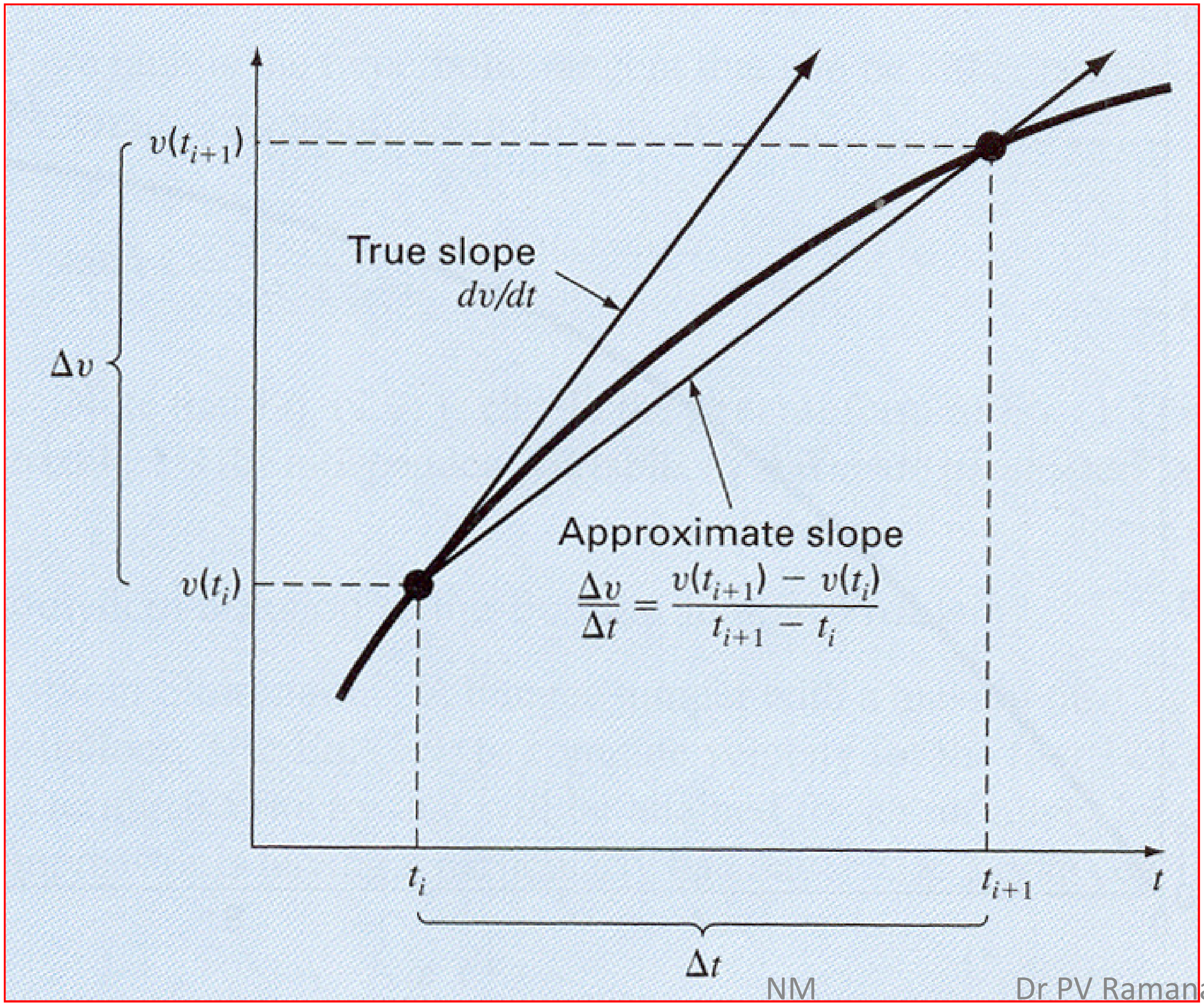
$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

➢ Exact Solution

$$v(t) = \sqrt{\frac{mg}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}} t\right)$$

Numerical Method

- *What if $c_d = c_d(v) \neq const$?*
- *Solve the ODE numerically!*



$$\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

$$\frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

Assume constant slope (i.e, constant drag force) over Δt

Numerical (Approximate) Solution

- Finite difference (Euler's) method

$$\frac{dv}{dt} \cong \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i}$$

$$\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c_d}{m} v(t_i)^2$$

- Numerical Solution

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c_d}{m} v(t_i)^2 \right] (t_{i+1} - t_i)$$

Home work: Hand Calculations

A stationary bungee jumper with m = 68.1 kg leaps from a stationary hot air balloon. Use the Euler's method with a time increment of 2 s to compute the velocity for the first 12 s of free fall. Assume a drag coefficient of 0.25 kg/m.

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c_d}{m} v(t_i)^2 \right](t_{i+1} - t_i)$$

$$t_0 = 0; \quad v(t_0) = 0$$

Explicit time-marching scheme

$m = 68.1$ kg, $g = 9.81$ m/s^2, $c_d = 0.25$ kg/m

Numerical Method

$$v(t_{i+1}) = v(t_i) + \left[g - \frac{c_d}{m} v(t_i)^2 \right] (t_{i+1} - t_i)$$

$$t_0 = 0; \quad v(t_0) = 0$$

- Use a constant time increment $\Delta t = 2$ s

Step 1 $\quad t = 2s; \quad v = 0 + \left[9.81 - \frac{0.25}{68.1} (0)^2 \right] (2 - 0) = 19.6200 m/s$

Step 2 $\quad t = 4s; \quad v = 19.6200 + \left[9.81 - \frac{0.25}{68.1} (19.6200)^2 \right] (4 - 2) = 36.4317 m/s$

Step 3 $\quad t = 6s; \quad v = 36.4137 + \left[9.81 - \frac{0.25}{68.1} (36.4137)^2 \right] (6 - 4) = 46.2983 m/s$

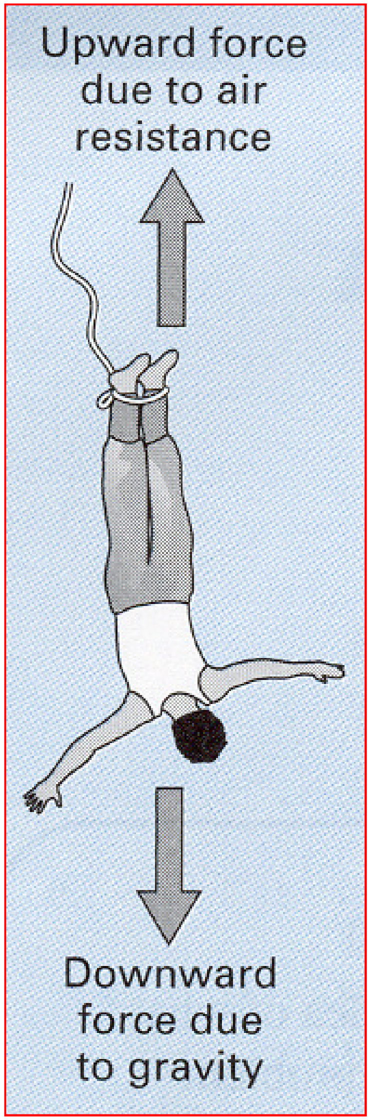
Step 4 $\quad t = 8s; \quad v = 46.2983 + \left[9.81 - \frac{0.25}{68.1} (46.2983)^2 \right] (8 - 6) = 50.1802 m/s$

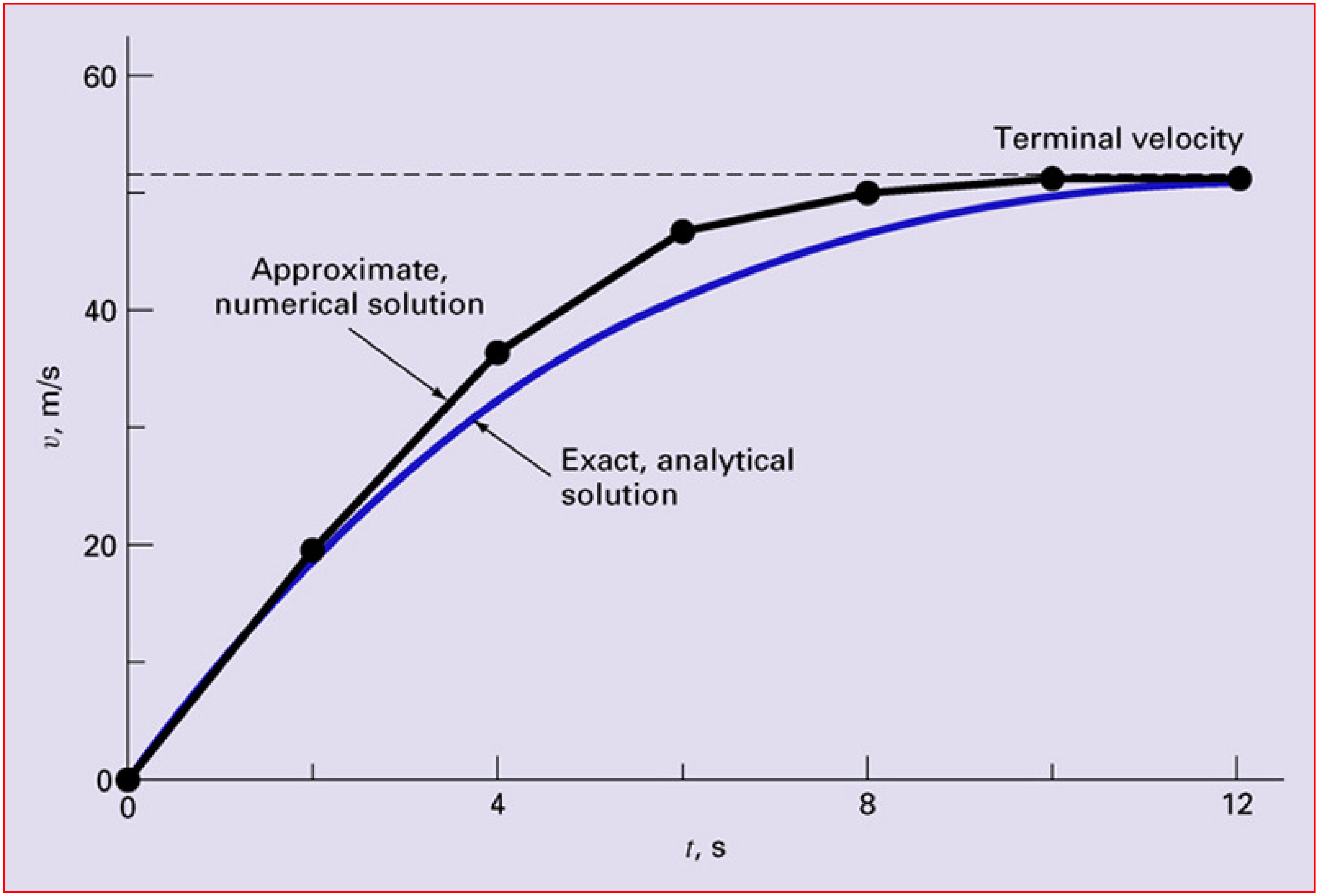
Step 5 $\quad t = 10s; \quad v = 50.1802 + \left[9.81 - \frac{0.25}{68.1} (50.1802)^2 \right] (10 - 8) = 51.3123 m/s$

Step 6 $\quad t = 12s; \quad v = 51.3123 + \left[9.81 - \frac{0.25}{68.1} (51.3123)^2 \right] (12 - 10) = 51.6008 m/s$

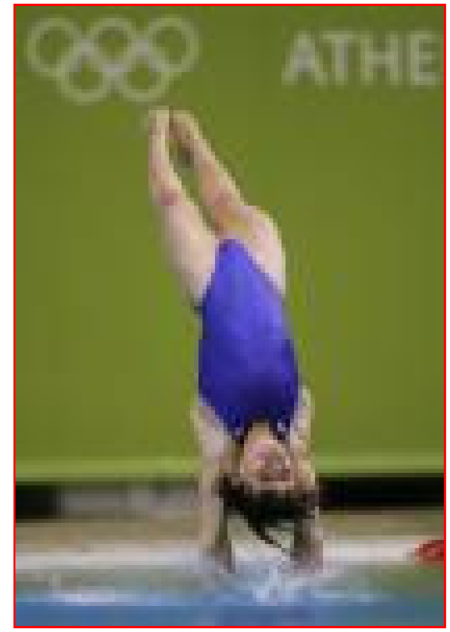
The solution accuracy depends on **time increment**

Example: Bungee Jumper





Olympic 10-m Platform Diving



$$\textbf{Air :} \quad m\frac{dv}{dt} = mg - c_{da}v^2 - \frac{\rho_a}{\rho}mg$$

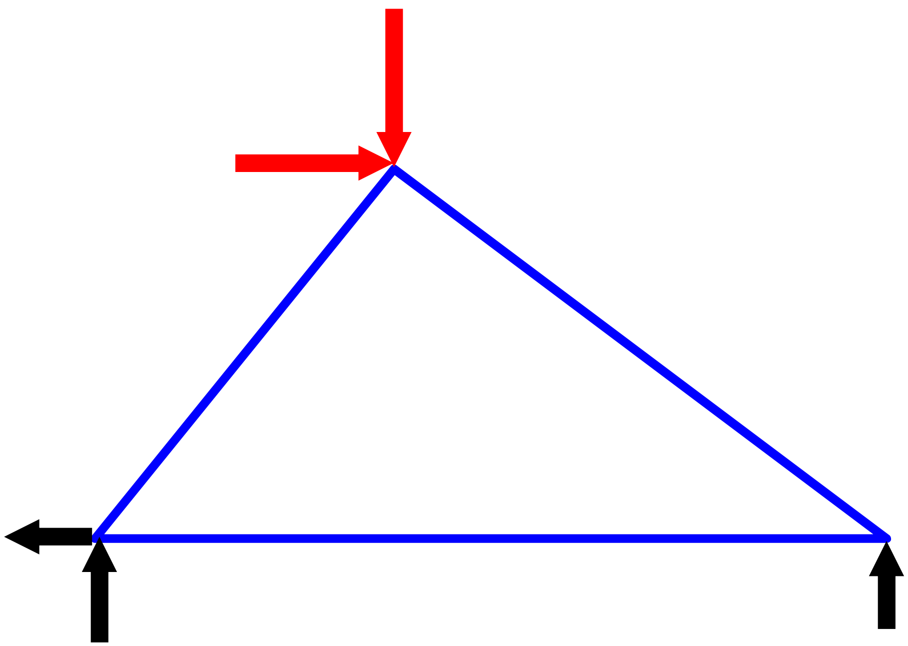
$$\textbf{Water :} \quad m\frac{dv}{dt} = mg - c_{dw}|v|v - \frac{\rho_w}{\rho}mg$$

Buoyant Force

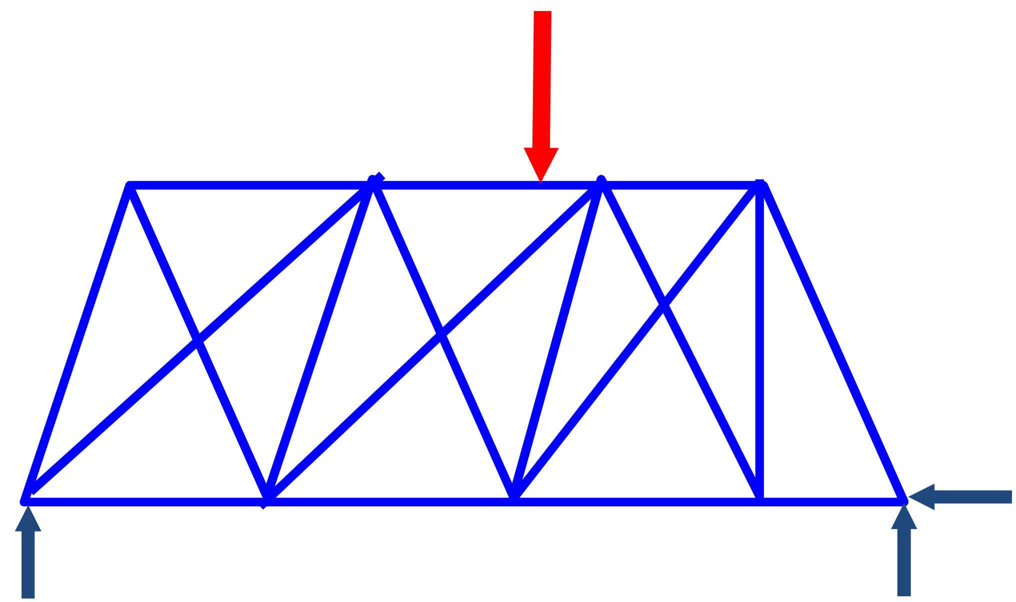
$$c_{da} \neq c_{dw}$$

Home Work: Structural Analysis

Simple truss - force balance



Complex truss



Instead of limiting our analysis to simple cases, numerical method allows us to work on realistic cases.

Thank you