NUMERICAL METHODS

$$U^{n+1} = U^n + \Delta t\, f(U^n)$$

$$\frac{\partial v}{\partial t} + V \cdot \nabla v = \nabla \cdot (k \nabla v) + g(v)$$

$$(\lambda + \mu)\nabla(\nabla \cdot u) + \mu \nabla^2 u = \alpha(3\lambda + 2\mu)\nabla T - \rho b$$

Lecture 5

$$\rho\left(\frac{\partial u}{\partial t} + V \cdot \nabla u\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$\nabla^2 u = f$$

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Iterative Methods

- ❑ **Jacobi Iteration Method**
- ❑**Gauss – Siedel Method**

Conditions for Convergence

(1)

A sufficient condition for convergence is given by

$$|a_{ii}| > \sum_{\substack{j=1 \\ i \neq i}}^{n} |a_{ij}|$$

for two equations, n=3 and the following three conditions are sufficient to get convergence:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

Iterative Methods

- □ **Jacobi Iteration Method**
- □**Gauss – Siedel Method**

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

Iterative Methods

These methods generate a sequence of approximate solutions

$$x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}, \Lambda\ \Lambda$$

$$\text{Good method : How quickly} \quad x^{(k)} \rightarrow x^{*} = A^{-1}b$$

$$Ax = b$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$approx$ $\quad x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}, \Lambda\ \Lambda$

$residual$ $\quad r^{(0)}, r^{(1)}, r^{(2)}, r^{(3)}, \Lambda\ \Lambda$

$error$ $\quad e^{(0)}, e^{(1)}, e^{(2)}, e^{(3)}, \Lambda\ \Lambda$

$$r^{(k)} = b - Ax^{(k)}$$

$$e^{(k)} = x^{*} - x^{(k)}$$

Remark: $r^{(k)} = 0 \quad \Leftrightarrow \quad x^{(k)}$ is the exact solution

JACOBI ITERATION Method

Step 1:

-Algebraically solve each linear equation for x_i

Step 2:

-Assume an initial guess solution array

Step 3:

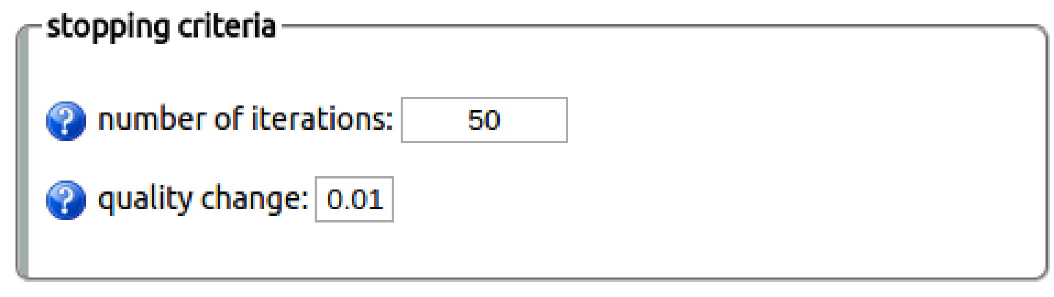
-Solve for each x_i and repeat

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Step 4:

-Use absolute relative approximate error after each iteration to check if error is within a pre-specified tolerance.

Stopping Criteria



stopping criteria

number of iterations: 50

quality change: 0.01

- $Ax=b$
- At any iteration k, the residual term is

  $r^k=b-Ax^k$

- Check the norm of the residual term

  $||b-Ax^k||$

- If it is less than a threshold value stop

Convergence of JACOBI ITERATION Method iteration

- **JACOBI** iteration converges for any initial vector if A is a *diagonally dominant matrix*

- **JACOBI** iteration converges for any initial vector if A is a *symmetric* and *positive definite matrix*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Matrix A is positive definite if

  $x^TAx>0$ for every nonzero x vector

JACOBI ITERATION Method

The JACOBI ITERATION Method allows the user to control round-off error.

Elimination methods such as Gaussian Elimination and LU Decomposition are prone to prone to round-off error.

If the physics of the problem are understood, a close initial guess can be made, decreasing the number of iterations needed.

JACOBI ITERATION Method

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Algorithm

A set of *n* equations and *n* unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

- **Maximum coefficient of each variable and the diagonal elements are non-zero**
- **Rewrite each equation solving for the corresponding unknown**
- **ex:**

First equation, solve for x_1

Second equation, solve for x_2

JACOBI ITERATION Method

Algorithm

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

Rewriting each equation

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots \ldots - a_{1n}x_n}{a_{11}}$$

From Equation 1

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots \ldots - a_{2n}x_n}{a_{22}}$$

From equation 2

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots \ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

From equation n-1

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots \ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

From equation n

JACOBI ITERATION Method

Algorithm

General Form of each equation

$$x_1 = \frac{c_1 - \sum\limits_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j} x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum\limits_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j} x_j}{a_{22}}$$

$$x_{n-1} = \frac{c_{n-1} - \sum\limits_{\substack{j=1 \\ j\neq n-1}}^{n} a_{n-1,j} x_j}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - \sum\limits_{\substack{j=1 \\ j\neq n}}^{n} a_{nj} x_j}{a_{nn}}$$

JACOBI ITERATION Method

Algorithm

General Form for any row ‘i’

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1,2,\mathrm{K}\ ,n.$$

JACOBI ITERATION Method

Step 1: Solve for the unknowns

Step 2: Assume an initial guess for [X]

Step 3: Use rewritten equations to solve for each value of x_i.

$$\begin{bmatrix} x_1 \\ x_2 \\ \mathrm{M} \\ x_{n\text{-}1} \\ x_n \end{bmatrix}$$

Important: Remember to use the most recent value of x_i. Which means to apply values calculated to the calculations remaining in the current iteration.

JACOBI ITERATION Method

Step 4:

Calculate the Absolute Relative Approximate Error

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{\text{new}} - x_i^{\text{old}}}{x_i^{\text{new}}}\right| \times 100$$

So when has the answer been found?

The iterations are stopped when the absolute relative approximate error is less than a prespecified tolerance for all unknowns.

Convergence Criterion for JACOBI ITERATION Method

- Iterations are repeated until the convergence criterion is satisfied:

$$\left|\varepsilon_{a,i}\right| = \left|\frac{x_i^j - x_i^{j-1}}{x_i^j}\right| 100\% \ \pi \ \varepsilon_s$$

For all *i*, where *j* and *j-1* are the *current* and *previous* iterations.

- As any other iterative method, the JACOBI ITERATION Method has problems:
  - It may not converge or it converges very slowly.

- If the coefficient matrix A is **Diagonally Dominant** JACOBI is guaranteed to converge.

Diagonally Dominant ➔

For each equation i :

$$\left|a_{ii}\right| \ \phi \ \sum_{\substack{j=1\\ j\neq i}}^{n} \left|a_{i,j}\right|$$

- Note that this is not a necessary condition, i.e. the system *may* still have a chance to converge even if A is not diagonally dominant.

Time Complexity: Each iteration takes $O(n^2)$

Solve $6x_1 - 2x_2 + x_3 = 11$ (1)

$x_1 + 2x_2 - 5x_3 = -1$ (2)

$-2x_1 + 7x_2 + 2x_3 = 5$ (3)

$6x_1 - 2x_2 + x_3 = 11$ (1)

$-2x_1 + 7x_2 + 2x_3 = 5$ (2)

$x_1 + 2x_2 - 5x_3 = -1$ (3)

Step 1:

Re-write the equations such that each equation has the unknown with largest coefficient on the left hand side:

from eq. (1)

$$x_1 = \frac{2x_2 - x_3 + 11}{6}$$

from eq. (3)

$$x_2 = \frac{2x_1 - 2x_3 + 5}{7}$$

from eq. (2)

$$x_3 = \frac{x_1 + 2x_2 + 1}{5}$$

Step 2:

Assume the initial guesses $(x_1)^0 = (x_2)^0 = (x_3)^0 = 0$, then calculate $(x_1)^1$, $(x_2)^1$ and $(x_3)^1$:

$$(x_1)^1 = \frac{2(x_2)^0 - (x_3)^0 + 11}{6} = \frac{2(0) - (0) + 11}{6} = 1.833$$

$$(x_2)^1 = \frac{2(x_1)^0 - 2(x_3)^0 + 5}{7} = \frac{2(0) - 2(0) + 5}{7} = 0.714$$

$$(x_3)^1 = \frac{(x_1)^0 + 2(x_2)^0 + 1}{5} = \frac{(0) + 2(0) + 1}{5} = 0.200$$

Solve $6x_1 - 2x_2 + x_3 = 11 \quad (1)$

$$\begin{aligned} 6x_1 - 2x_2 + x_3 &= 11 \quad (1) \\ x_1 + 2x_2 - 5x_3 &= -1 \quad (2) \\ -2x_1 + 7x_2 + 2x_3 &= 5 \quad (3) \end{aligned}$$

Step 3:

Use the values obtained in the first iteration, to calculate the values for the 2nd iteration

$$(x_1)^2 = \frac{2(x_2)^1 - (x_3)^1 + 11}{6} = \frac{2(0.714) - (0.200) + 11}{6} = 2.038$$

$$(x_2)^2 = \frac{2(x_1)^1 - 2(x_3)^1 + 5}{7} = \frac{2(1.833) - 2(0.200) + 5}{7} = 1.181$$

$$(x_3)^2 = \frac{(x_1)^1 + 2(x_2)^1 + 1}{5} = \frac{(1.833) + 2(0.714) + 1}{5} = 0.852$$

Step 4:

Continue the above iterative procedure until $[(x_k)^{i+1} - (x_k)^{i}]/ (x_k)^{i+1} < Є_s$ for k=1,2 and 3.

$$(x_1)^{i+1} = \frac{2(x_2)^i - (x_3)^i + 11}{6}$$

$$(x_2)^{i+1} = \frac{2(x_1)^i - 2(x_3)^i + 5}{7}$$

$$(x_3)^{i+1} = \frac{(x_1)^i + 2(x_2)^i + 1}{5}$$

Unknown → ↓ Iteration	x_1	x_2	x_3
1	1.833	0.714	0.200
2	2.038	1.181	0.852
3	2.085	1.053	1.080
4	2.004	1.001	1.038
.	.	.	.
.	.	.	.
9	2.000	1.000	1.000

JACOBI ITERATION Method: Example 2

Given the system of equations

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The coefficient matrix is:

$$[A] = \begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix}$$

Will the solution converge using the **JACOBI ITERATION** method?

JACOBI ITERATION Method : Example 2

Checking if the coefficient matrix is diagonally dominant

$$[A]=\begin{bmatrix}12 & 3 & -5\\ 1 & 5 & 3\\ 3 & 7 & 13\end{bmatrix}$$

$$|a_{11}|=|12|=12\geq|a_{12}|+|a_{13}|=|3|+|-5|=8$$

$$|a_{22}|=|5|=5\geq|a_{21}|+|a_{23}|=|1|+|3|=4$$

$$|a_{33}|=|13|=13\geq|a_{31}|+|a_{32}|=|3|+|7|=10$$

The inequalities are all true and at least one row is *strictly* greater than:

Therefore: The solution should converge using the **JACOBI ITERATION** Method

JACOBI ITERATION Method : Example 2

Rewriting each equation

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}$$

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

JACOBI ITERATION Method : Example 2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$\mathrm{x}_1 = 0.50000$

$\mathrm{x}_2 = 4.9000$

$\mathrm{x}_3 = 3.0923$

The absolute relative approximate error

$$\left|\in_{\mathrm{a}}\right|_1 = \left|\frac{0.50000 - 1.0000}{0.50000}\right| \times 100 = 67.662\%$$

$$\left|\in_{\mathrm{a}}\right|_2 = \left|\frac{4.9000 - 0}{4.9000}\right| \times 100 = 100.00\%$$

$$\left|\in_{\mathrm{a}}\right|_3 = \left|\frac{3.0923 - 1.0000}{3.0923}\right| \times 100 = 67.662\%$$

The maximum absolute relative error after the first iteration is 100%

JACOBI ITERATION Method : Example 2

After Iteration #1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

Substituting the x values into the equations

$$x_1 = \frac{1-3(4.9000)+5(3.0923)}{12} = 0.14679$$

$$x_2 = \frac{28-(0.14679)-3(3.0923)}{5} = 3.7153$$

$$x_3 = \frac{76-3(0.14679)-7(4.900)}{13} = 3.8118$$

After Iteration #2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

JACOBI ITERATION Method : Example 2

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.14679 \\ 3.7153 \\ 3.8118 \end{bmatrix}$$

Iteration #2 absolute relative approximate error

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

$$\left|\in_a\right|_1 = \left|\frac{0.14679 - 0.50000}{0.14679}\right| \times 100 = 240.62\%$$

$$\left|\in_a\right|_2 = \left|\frac{3.7153 - 4.9000}{3.7153}\right| \times 100 = 31.887\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.8118 - 3.0923}{3.8118}\right| \times 100 = 18.876\%$$

The maximum absolute relative error after the first iteration is 240.62%.

This is much larger than the maximum absolute relative error obtained in iteration #1. Is this a problem?

JACOBI ITERATION Method: Example 2

Repeating more iterations, the following values are obtained

Iteration	a_1	$\lvert\varepsilon_a\rvert_1$	a_2	$\lvert\varepsilon_a\rvert_2$	a_3	$\lvert\varepsilon_a\rvert_3$
1	0.50000	67.662	4.900	100.00	3.0923	67.662
2	0.14679	240.62	3.7153	31.887	3.8118	18.876
3	0.74275	80.23	3.1644	17.409	3.9708	4.0042
4	0.94675	21.547	3.0281	4.5012	3.9971	0.65798
5	0.99177	4.5394	3.0034	0.82240	4.0001	0.07499
6	0.99919	0.74260	3.0001	0.11000	4.0001	0.00000

The solution obtained

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$$

is close to the exact solution of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

JACOBI ITERATION Method

Example

Consider 4x4 case

$$\begin{bmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 25 \\ -11 \\ 15 \end{bmatrix}$$

$$\begin{aligned} 10x_1 - x_2 + 2x_3 &= 6 \\ -x_1 + 11x_2 - x_3 + 3x_4 &= 25 \\ 2x_1 - x_2 + 10x_3 - x_4 &= -11 \\ 3x_2 - x_3 - 8x_4 &= 15 \end{aligned}$$

$$\begin{aligned} x_1 &= (x_2 - 2x_3 + 6)/10 \\ x_2 &= (x_1 + x_3 - 3x_4 + 25)/11 \\ x_3 &= (-2x_1 + x_2 + x_4 - 11)/10 \\ x_4 &= (-3x_2 + x_3 + 15)/(-8) \end{aligned}$$

$$\text{given} \quad x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1^{(1)} &= (x_2^{(0)} - 2x_3^{(0)} + 6)/10 \\ x_2^{(1)} &= (x_1^{(0)} + x_3^{(0)} - 3x_4^{(0)} + 25)/11 \\ x_3^{(1)} &= (-2x_1^{(0)} + x_2^{(0)} + x_4^{(0)} - 11)/10 \\ x_4^{(1)} &= (-3x_2^{(0)} + x_3^{(0)} + 15)/(-8) \end{aligned}$$

$$x^{(1)} = \begin{bmatrix} 0.6000 \\ 2.2727 \\ -1.1000 \\ 1.8750 \end{bmatrix}$$

JACOBI ITERATION Method

$$\begin{aligned}
x_1^{(k+1)} &= (\qquad x_2^{(k)} - 2x_3^{(k)} \qquad +6)/10 \\
x_2^{(k+1)} &= (\quad x_1^{(k)} \qquad + x_3^{(k)} - 3x_4^{(k)} + 25)/11 \\
x_3^{(k+1)} &= (-2x_1^{(k)} + \ x_2^{(k)} \qquad + \ x_4^{(k)} - 11)/10 \\
x_4^{(k+1)} &= (\qquad -3x_2^{(k)} + \ x_3^{(k)} \qquad +15)/(-8)
\end{aligned}$$

Note that in the Jacobi iteration one does not use the most recently available information.

$$\begin{aligned}
x_1^{(k+1)} &= (\qquad x_2^{(k)} - \quad 2x_3^{(k)} \qquad +6)/10 \\
x_2^{(k+1)} &= (\quad x_1^{(k+1)} \qquad + \quad x_3^{(k)} - 3x_4^{(k)} + 25)/11 \\
x_3^{(k+1)} &= (-2x_1^{(k+1)} + \ x_2^{(k+1)} \qquad + \quad x_4^{(k)} - 11)/10 \\
x_4^{(k+1)} &= (\qquad -3x_2^{(k+1)} + \ x_3^{(k+1)} \qquad +15)/(-8)
\end{aligned}$$

	K=1	K=2	K=3	K=4	K=5
x1	0.6000	1.0302	1.0066	1.0009	1.0001
x2	2.3273	2.0369	2.0036	2.0003	2.0000
x3	-0.9873	-1.0145	-1.0025	-1.0003	-1.0000
X4	0.8789	0.9843	0.9984	0.9998	1.0000
$\left\| r^{(k)} \right\|$	5.6930	0.4300	0.0662	0.0082	0.0009

$$x^* = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

JACOBI ITERATION Method

$$\begin{bmatrix} a_{11} & \Lambda & a_{1n} \\ \mathrm{M} & \mathrm{O} & \mathrm{M} \\ a_{n1} & \Lambda & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \mathrm{M} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \mathrm{M} \\ b_n \end{bmatrix}$$

Gauss-Seidel iteration for general n:

```
for i = 1 : n
    $x_i^{(k+1)} = \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right) \Big/ a_{ii}$
end
```

MATLAB CODE

Ex:
Write a Matlab function for JI

```
function [sol,X]=gs(A,b,x0)
n=length(b);
maxiter=10;
x=x0;
for k=1:maxiter
for i=1:n
    sum1=0;
    for j=1:i-1
        sum1=sum1+A(i,j)*x(j);
    end
     sum2=0;
    for j=i+1:n
        sum2=sum2+A(i,j)*x(j);
    end
    x(i)=(b(i)-sum1-sum2)/A(i,i)
end
X(1:n,k)=x;
end
sol=x;
```

JACOBI ITERATION Method

iteration for general n:

```
for i = 1 : n
      $x_i^{(k+1)} = \left( b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right) \Big/ a_{ii}$
end
```

Iterative Methods

- ❑ **Jacobi Iteration Method**
- ❑ **Gauss – Siedel Method**

Gauss – Siedel Method

GS Iterative methods provide an **alternative** to the *elimination methods*.

$$Ax = b \qquad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$[D+(A-D)]x = b \quad \Rightarrow \quad Dx = b-(A-D)x \quad \Rightarrow \quad x = D^{-1}[b-(A-D)x]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/a_{11} & 0 & 0 \\ 0 & 1/a_{22} & 0 \\ 0 & 0 & 1/a_{33} \end{bmatrix} * \left(\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right)$$

$$x_1^k = \frac{b_1 - a_{12}x_2^{k-1} - a_{13}x_3^{k-1}}{a_{11}} \qquad x_2^k = \frac{b_2 - a_{21}x_1^{k-1} - a_{23}x_3^{k-1}}{a_{22}} \qquad x_3^k = \frac{b_3 - a_{31}x_1^{k-1} - a_{32}x_2^{k-1}}{a_{33}}$$

Choose an initial guess (i.e. all zeros) and Iterate until the equality is satisfied.
No guarantee for convergence! Each iteration takes O(n^2) time!

Gauss – Siedel Method

- The *Gauss-Seidel* method is a commonly used *iterative method*.
- It is same as **Jacobi technique** except with one important difference:

  A newly computed *x* value (say x_k) is substituted in the subsequent equations (equations *k+1*, *k+2*, …, *n*) **in the same iteration**.

Example: Consider the *3x3* system below:

$$x_1^{new} = \frac{b_1 - a_{12}x_2^{old} - a_{13}x_3^{old}}{a_{11}}$$

$$x_2^{new} = \frac{b_2 - a_{21}x_1^{new} - a_{23}x_3^{old}}{a_{22}}$$

$$x_3^{new} = \frac{b_3 - a_{31}x_1^{new} - a_{32}x_2^{new}}{a_{33}}$$

$$\{X\}_{old} \leftarrow \{X\}_{new}$$

- First, choose initial guesses for the x's.
- A simple way to obtain initial guesses is to assume that they are all **zero**.
- Compute **new** $\mathbf{x_1}$ using the previous iteration values.
- **New** $\boldsymbol{x_1}$ is substituted in the equations to calculate x_2 and x_3
- The process is repeated for $\mathbf{x_2}$, $\mathbf{x_3}$, …

Gauss – Siedel Method

$$
\begin{aligned}
a_{11}\boxed{x_1} + a_{12}x_2 + \Lambda + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}\boxed{x_2} + \Lambda + a_{2n}x_n &= b_2\\
&\mathrm{M}\\
a_{n1}x_1 + a_{n2}x_2 + \Lambda + a_{nn}\boxed{x_n} &= b_n
\end{aligned}
$$

$$x^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \mathrm{M} \\ x_n^0 \end{bmatrix}$$

$$x_1^1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2^0 - \Lambda - a_{1n}x_n^0)$$

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=i+1}^{n} a_{ij}x_j^k\right]$$

$$x_2^1 = \frac{1}{a_{22}}(b_2 - a_{21}x_1^0 - a_{23}x_3^0 - \Lambda - a_{2n}x_n^0)$$

$$x_n^1 = \frac{1}{a_{nn}}(b_n - a_{n1}x_1^0 - a_{n2}x_2^0 - \Lambda - a_{nn-1}x_{n-1}^0)$$

Gauss – Siedel Method

$x^{k+1}=Ex^{k}+f$ iteration for Jacobi method

A can be written as A=L+D+U (*not decomposition*)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

$Ax=b \Rightarrow (L+D+U)x=b$

$Dx^{k+1} = -(L+U)x^{k}+b$

$x^{k+1}=-D^{-1}(L+U)x^{k}+D^{-1}b$

$E=-D^{-1}(L+U)$

$f=D^{-1}b$

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \underbrace{\sum_{j=1}^{i-1} a_{ij}x_j^k}_{Lx^k} - \underbrace{\sum_{j=i+1}^{n} a_{ij}x_j^k}_{Ux^k} \right]$$

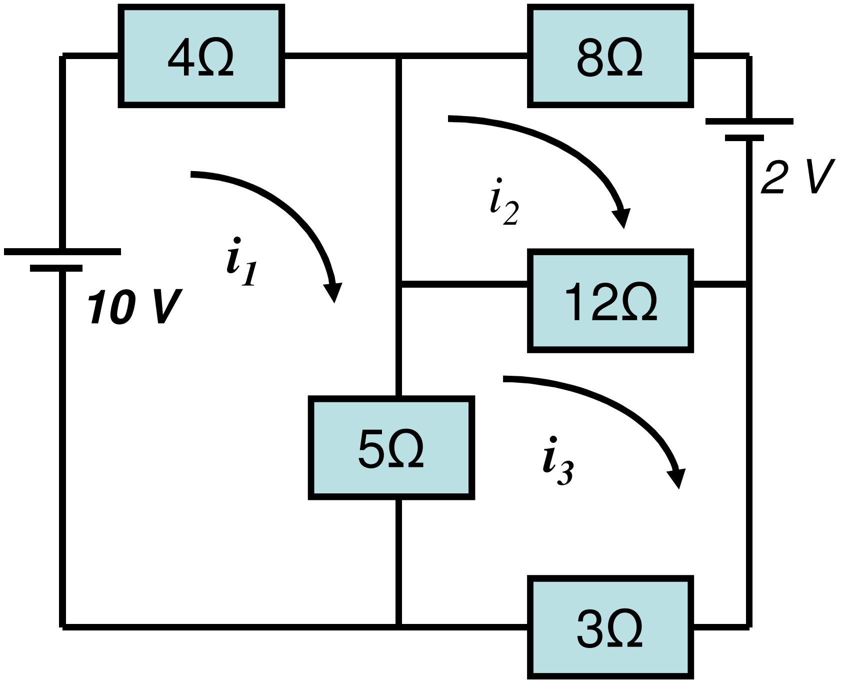
Dx^{k+1}

Gauss – Siedel Method: Example 1

Consider a circuit shown in figure here; currents i_1, i_2, and i_3 are given by

$$4i_1 + 0i_2 + 5(i_1 - i_3) = 10$$
$$0i_1 + 8i_2 + 12(i_2 - i_3) = -2$$
$$5(i_3 - i_1) + 12(i_3 - i_2) + 3i_3 = 0$$

$$9i_1 + 0i_2 - 5i_3 = 10$$
$$0i_1 + 20i_2 - 12i_3 = -2$$
$$-5i_1 - 12i_2 + 20i_3 = 0$$



The matrix form is:

$$\begin{bmatrix} 9 & 0 & -5 \\ 0 & 20 & -12 \\ -5 & -12 & 20 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 0 \end{bmatrix}$$

Notice that magnitude of any diagonal element is greater than the sum of the magnitudes of other elements in that row

A matrix with this property is said to be Diagonally dominant.

Gauss – Siedel Method: Example 1

The set of equations:

$$9i_1 + 0i_2 - 5i_3 = 10$$

$$0i_1 + 20i_2 - 12i_3 = -2$$

$$-5i_1 - 12i_2 + 20i_3 = 0$$

Let us write for i_1, i_2 and i_3 as

$$i_1 = (10 + 5i_3)/9 = 1.1111 + 0.5556i_2 \qquad (1)$$

$$i_2 = (-2 + 12i_3)/20 = -0.1000 + 0.6000i_2 \qquad (2)$$

$$i_3 = (5i_1 + 12i_3)/20 = 0.2500\, i_1 + 0.6000\, i_2 \qquad (3)$$

Let us make an initial guess as $i_1 = 0.0;$ $i_2 = 0.0$ *and* $i_3 = 0.0$

First iteration results: $i_1 = 1.1111;$ $i_2 = -0.1000$ *and* $i_3 = 0.0$

Gauss – Siedel Method: Example 1

$$i_1 = (10 + 5i_3)/9 = 1.1111 + 0.5556i_2 \qquad (1)$$

$$i_2 = (-2 + 12i_3)/20 = -0.1000 + 0.6000i_2 \qquad (2)$$

$$i_3 = (5i_1 + 12i_3)/20 = 0.2500\, i_1 + 0.6000\, i_2 \qquad (3)$$

First iteration results: $i_1 = 1.1111;$ $i_2 = -0.1000$ *and* $i_3 = 0.0$

2nd iteration results: $i_1 = 1.1111;$ $i_2 = -0.1000$ *and* $i_3 = 0.22$

3rd iteration results: $i_1 = 1.23;$ $i_2 = 0.03$ *and* $i_3 = 0.22$

4th iteration results: $i_1 = 1.23\ ;$ $i_2 = 0.03$ *and* $i_3 = 0.33$

5th iteration results: $i_1 = 1.29;$ $i_2 = 0.1$ *and* $i_3 = 0.33$

6th iteration results: $i_1 = 1.29;$ $i_2 = 0.1$ *and* $i_3 = 0.38$