NUMERICAL METHODS

$$U^{n+1} = U^n + \Delta t\, f(U^n)$$

$$\frac{\partial v}{\partial t} + V \cdot \nabla v = \nabla \cdot (k \nabla v) + g(v)$$

$$(\lambda + \mu)\nabla(\nabla \cdot u) + \mu \nabla^2 u = \alpha(3\lambda + 2\mu)\nabla T - \rho b$$

Lecture 7

$$\rho\left(\frac{\partial u}{\partial t} + V \cdot \nabla u\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$\nabla^2 u = f$$

Dr. P V Ramana

Upper and Lower Sums

The interval is divided into subintervals.

$$Partition\ P = \{a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b\}$$

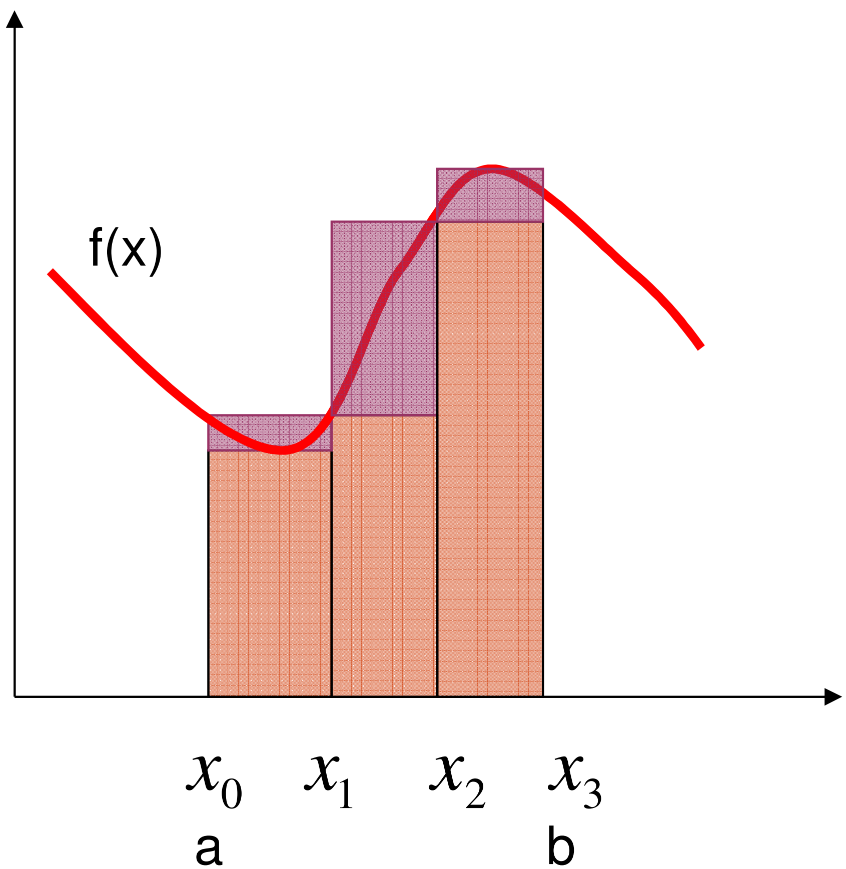
Define

$$m_i = \min\{f(x) : x_i \le x \le x_{i+1}\}$$

$$M_i = \max\{f(x) : x_i \le x \le x_{i+1}\}$$

Lower sum $\quad L(f,P) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$

Upper sum $\quad U(f,P) = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$



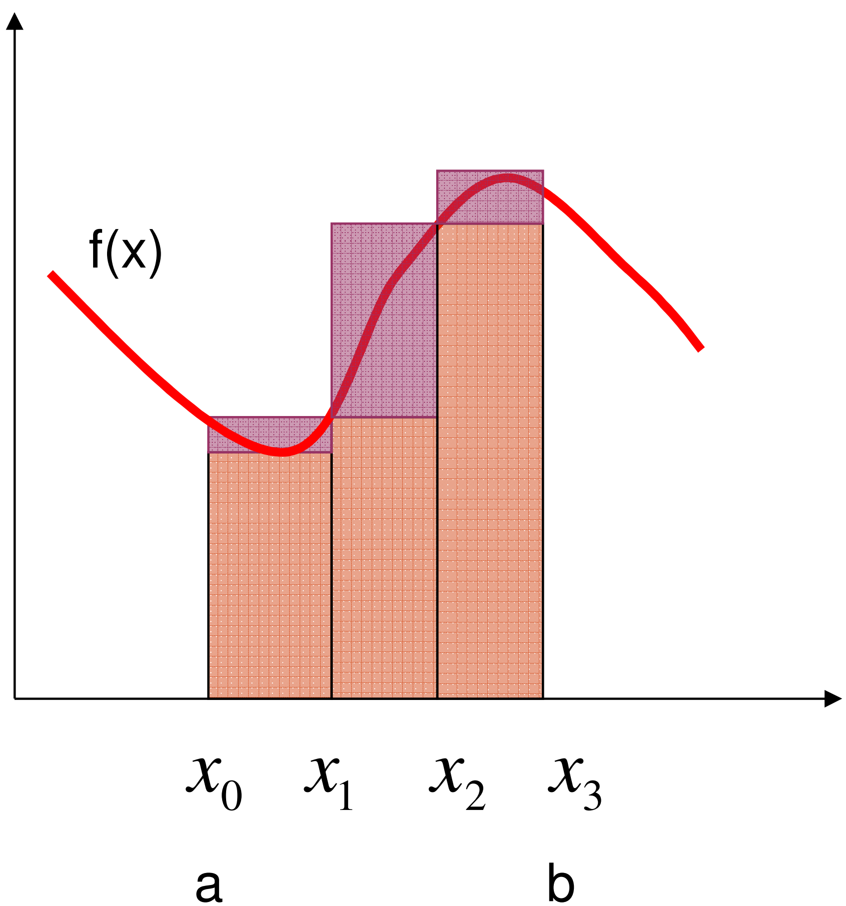
Upper and Lower Sums

Lower sum $L(f,P)=\sum_{i=0}^{n-1} m_i\left(x_{i+1}-x_i\right)$

Upper sum $U(f,P)=\sum_{i=0}^{n-1} M_i\left(x_{i+1}-x_i\right)$

$$\text{Estimate of the integral} = \frac{L+U}{2}$$

$$Error \le \frac{U-L}{2}$$



Example

$$\int_0^1 x^2 \, dx$$

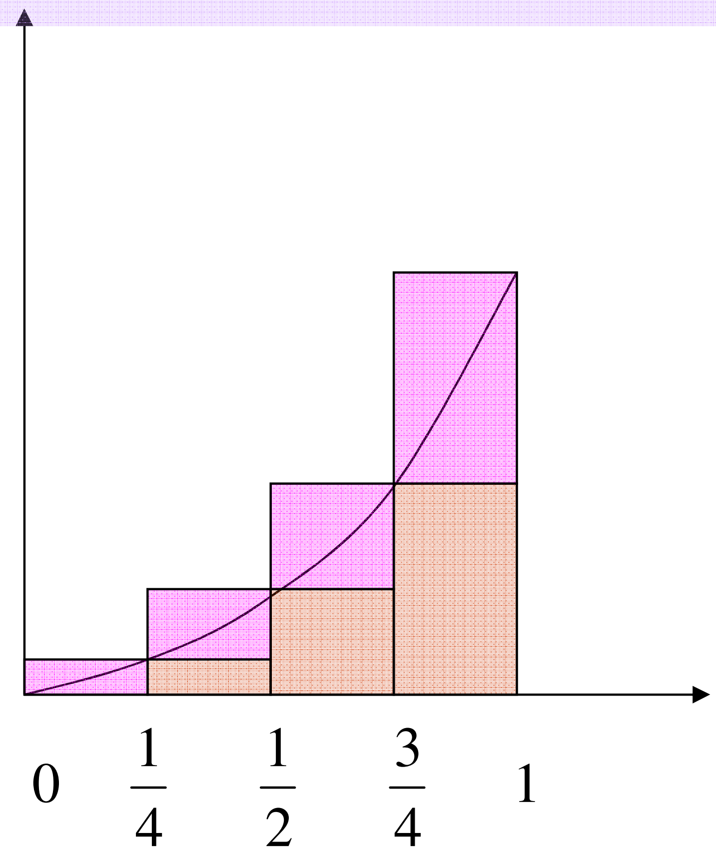
$$Partition: \; P = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$$

$$n = 4 \text{ (four equal intervals)}$$

$$m_0 = 0, \quad m_1 = \frac{1}{16}, \quad m_2 = \frac{1}{4}, \quad m_3 = \frac{9}{16}$$

$$M_0 = \frac{1}{16}, \quad M_1 = \frac{1}{4}, \quad M_2 = \frac{9}{16}, \quad M_3 = 1$$

$$x_{i+1} - x_i = \frac{1}{4} \qquad for \quad i = 0, 1, 2, 3$$



Example

$$\int_0^1 x^2\,dx \qquad Partition\ P=\left\{0,\frac{1}{4},\frac{2}{4},\frac{3}{4},1\right\}$$

$$m_0=0,\quad m_1=\frac{1}{16},\quad m_2=\frac{1}{4},\quad m_3=\frac{9}{16}$$

$$M_0=\frac{1}{16},\quad M_1=\frac{1}{4},\quad M_2=\frac{9}{16},\quad M_3=1$$

$$Lower\ \ sum \quad L(f,P)=\sum_{i=0}^{n-1} m_i\left(x_{i+1}-x_i\right)$$

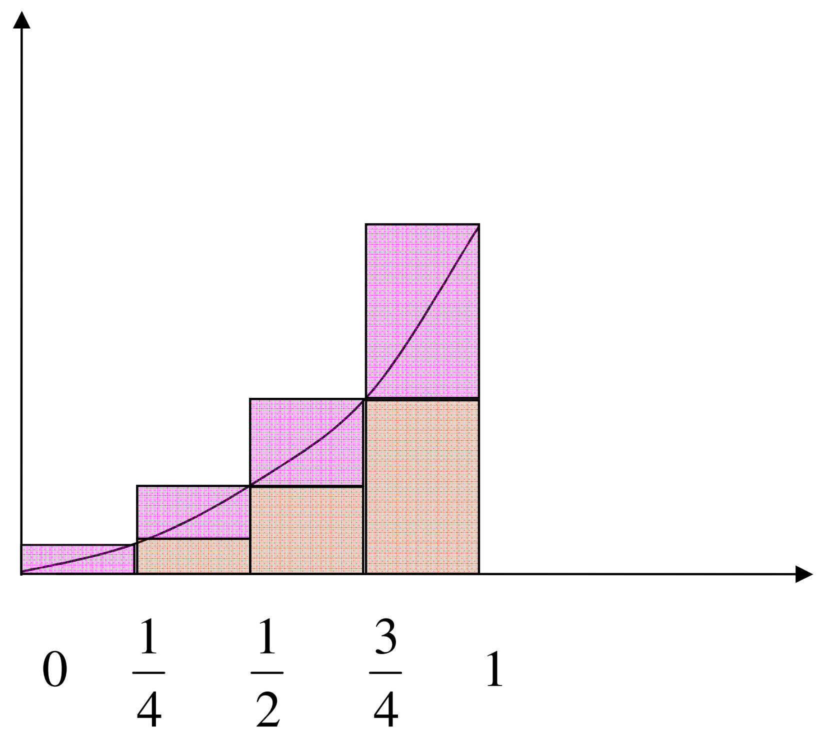
$$L(f,P)=\frac{1}{4}\left[0+\frac{1}{16}+\frac{1}{4}+\frac{9}{16}\right]=\frac{14}{64}$$

$$Upper\ sum \quad U(f,P)=\sum_{i=0}^{n-1} M_i\left(x_{i+1}-x_i\right)$$

$$U(f,P)=\frac{1}{4}\left[\frac{1}{16}+\frac{1}{4}+\frac{9}{16}+1\right]=\frac{30}{64}$$

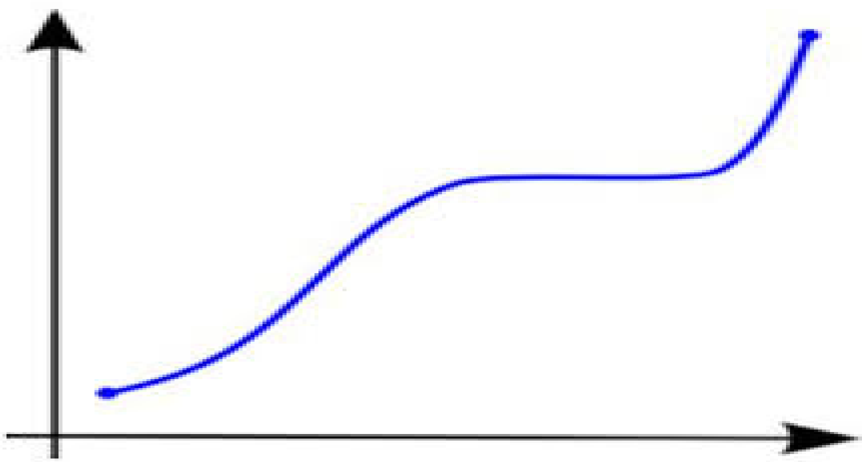
$$\text{Estimate of the integral}=\frac{1}{2}\left(\frac{30}{64}+\frac{14}{64}\right)=\frac{11}{32}$$

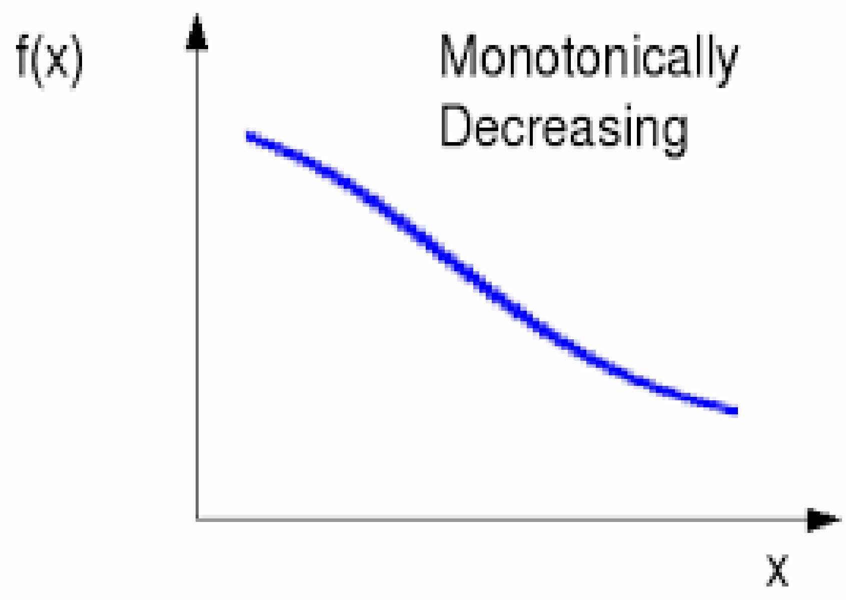
$$Error<\frac{1}{2}\left(\frac{30}{64}-\frac{14}{64}\right)=\frac{1}{8}$$



Upper and Lower Sums

- Estimates based on Upper and Lower Sums are easy to obtain for **monotonic** functions (always increasing or always decreasing).
- For non-monotonic functions, finding maximum and minimum of the function can be difficult and other methods can be more attractive.





Newton-Cotes Methods

- In Newton-Cote Methods, the function is approximated by a polynomial of order n.
- Computing the integral of a polynomial is easy.

$$\int_a^b f(x)dx \approx \int_a^b \left(a_0 + a_1 x + \ldots + a_n x^n\right)dx$$

$$\int_a^b f(x)dx \approx a_0(b-a) + a_1\frac{(b^2-a^2)}{2} + \ldots + a_n\frac{(b^{n+1}-a^{n+1})}{n+1}$$

$$\int_a^b f(x)dx \approx \int_a^b \left(a_0 + a_1 x + \ldots + a_n x^n\right)dx$$

$$\int_a^b f(x)dx \approx a_0(b-a) + a_1 \frac{(b^2 - a^2)}{2} + \ldots + a_n \frac{(b^{n+1} - a^{n+1})}{n+1}$$

Newton-Cotes Methods

- Lower & Upper Method (Zeroth Order Polynomials are used)

$$\int_a^b f(x)dx \approx \int_a^b (a_0)dx = a_0(b-a)$$

- Trapezoid Method (First Order Polynomials are used)

$$\int_a^b f(x)dx \approx \int_a^b (a_0 + a_1 x)dx$$

- Simpson 1/3 Rule (Second Order Polynomials are used)

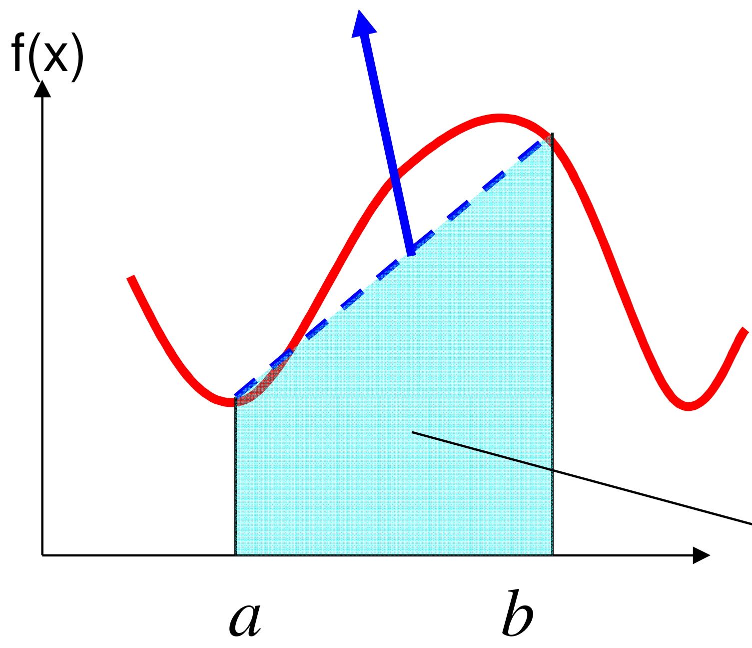
$$\int_a^b f(x)dx \approx \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

Trapezoid Method

- Derivation-One Interval
- Multiple Application Rule
- Estimating the Error
- Recursive Trapezoid Method

Trapezoid Method

$$f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$$



$$I=\int_a^b f(x)dx$$

$$I\approx\int_a^b\left(f(a)+\frac{f(b)-f(a)}{b-a}(x-a)\right)dx$$

$$=\left(f(a)-a\frac{f(b)-f(a)}{b-a}\right)x\Big|_a^b$$

$$+\frac{f(b)-f(a)}{b-a}\frac{x^2}{2}\Big|_a^b$$

$$=(b-a)\frac{f(b)+f(a)}{2}$$

Trapezoid Method

Derivation-One Interval

$$I = \int_a^b f(x)dx \approx \int_a^b \left(f(a) + \frac{f(b) - f(a)}{b-a}(x-a) \right) dx$$

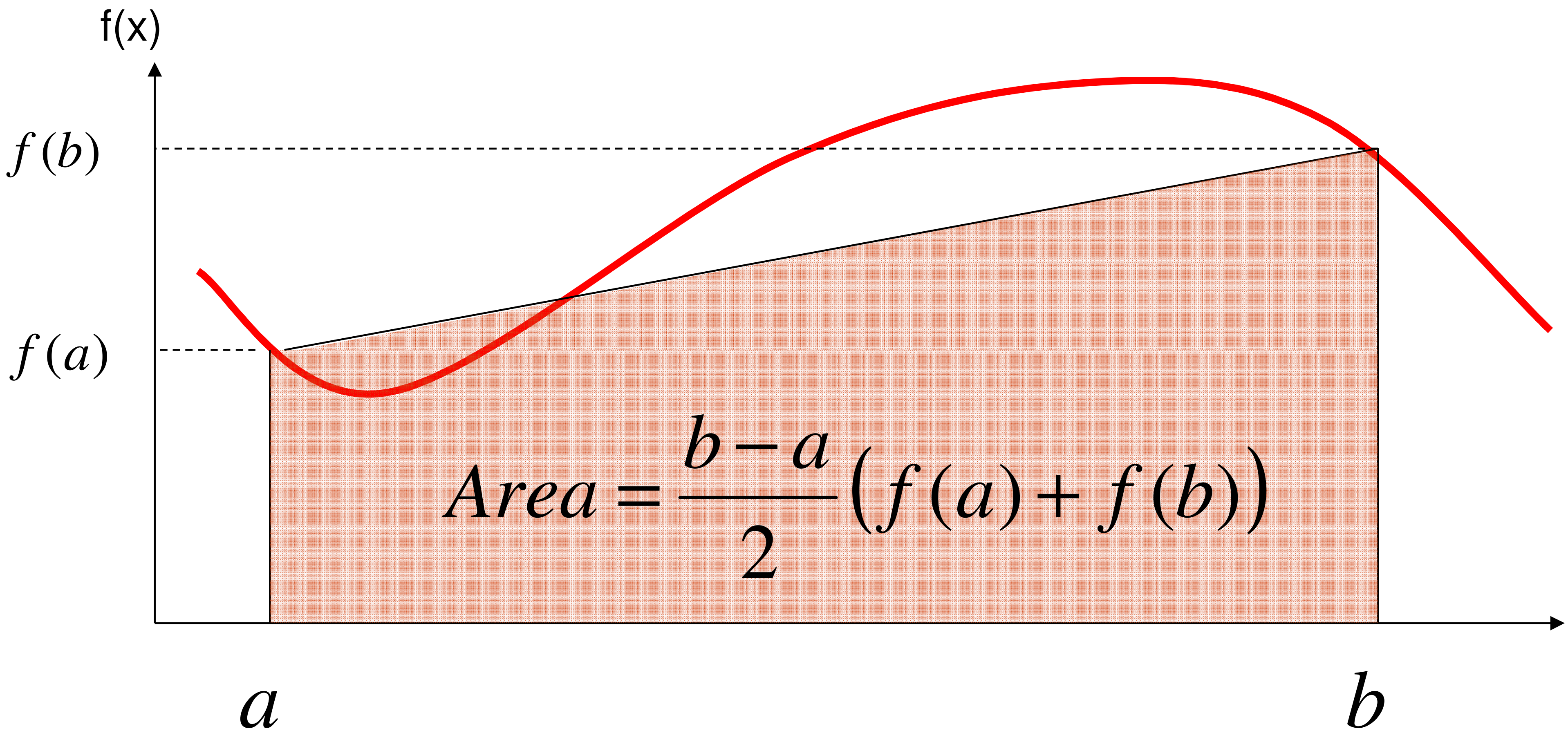
$$I \approx \int_a^b \left(f(a) - a\frac{f(b) - f(a)}{b-a} + \frac{f(b) - f(a)}{b-a}x \right) dx$$

$$= \left(f(a) - a\frac{f(b) - f(a)}{b-a} \right) x \Big|_a^b + \frac{f(b) - f(a)}{b-a} \frac{x^2}{2} \Big|_a^b$$

$$= \left(f(a) - a\frac{f(b) - f(a)}{b-a} \right) (b-a) + \frac{f(b) - f(a)}{2(b-a)} (b^2 - a^2)$$

$$= (b-a)\frac{f(b) + f(a)}{2}$$

Trapezoid Method



$$Area = \frac{b-a}{2}\left(f(a)+f(b)\right)$$

Trapezoid Method

Multiple Application Rule

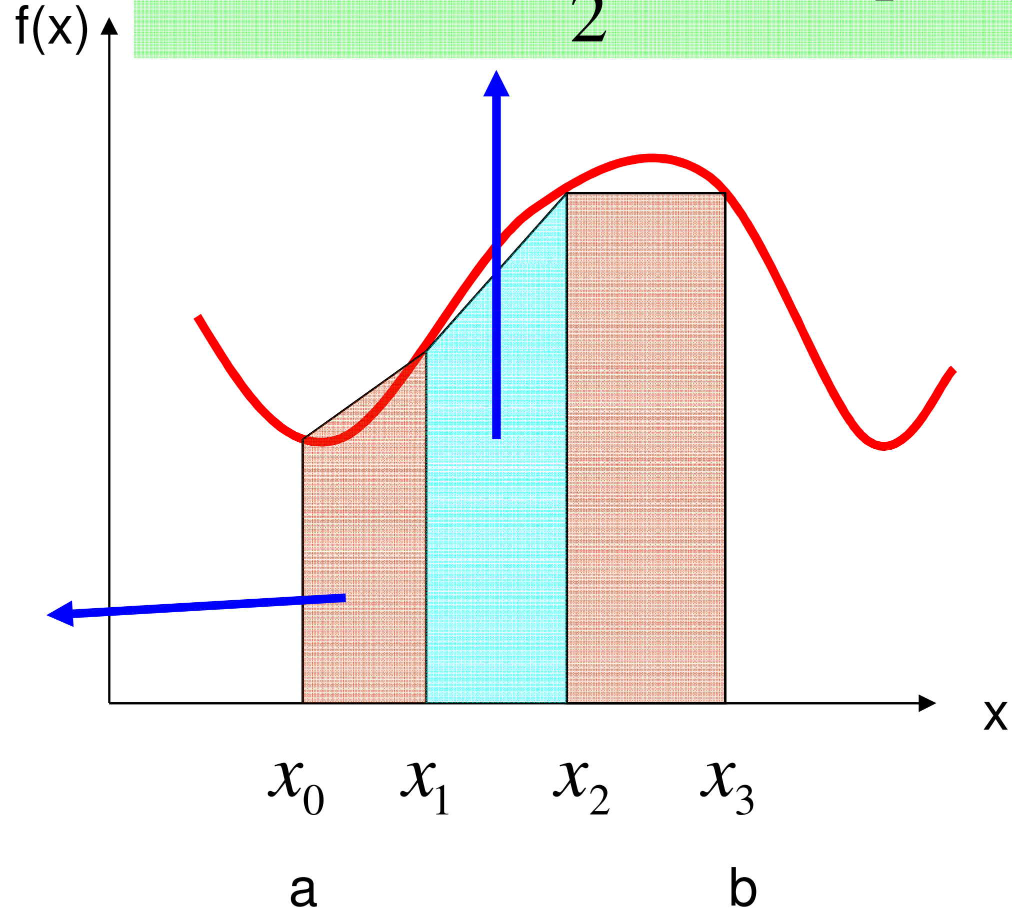
The interval $[a, b]$ is partitioned into n segments

$$a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b$$

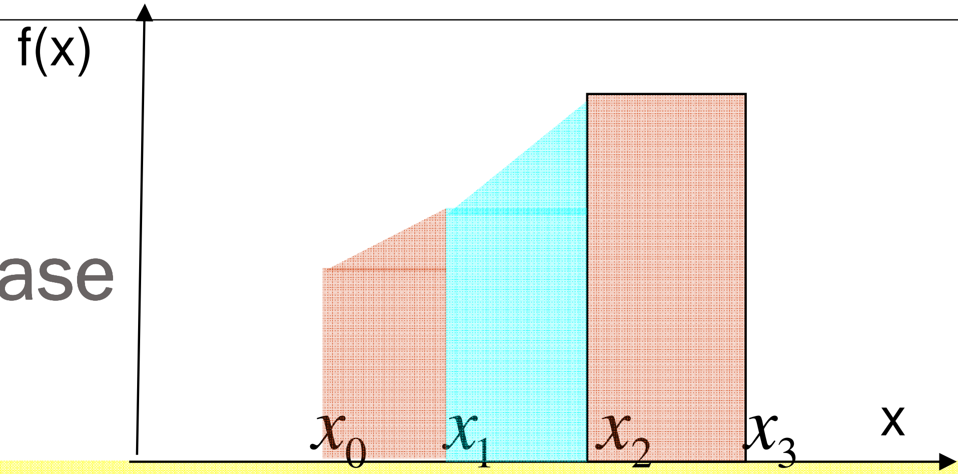
$$\int_a^b f(x)dx = \text{sum of the areas of the trapezoids}$$

$$Area = \frac{f(x_1) + f(x_0)}{2}(x_1 - x_0)$$

$$Area = \frac{f(x_2) + f(x_1)}{2}(x_2 - x_1)$$



Trapezoid Method



General Formula and Special Case

If the interval is divided into n segments (not necessarily equal)

$$a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b$$

$$\boxed{\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \frac{1}{2}(x_{i+1} - x_i)\big(f(x_{i+1}) + f(x_i)\big)}$$

Special Case (Equaliy spaced base points)

$$x_{i+1} - x_i = h \quad \text{for all} \quad i$$

$$\int_a^b f(x)dx \approx h\left[\frac{1}{2}\left[f(x_0) + f(x_n)\right] + \sum_{i=1}^{n-1} f(x_i)\right]$$

Example 1

Given a tabulated values of the velocity of an object.

Time (s)	0.0	1.0	2.0	3.0
Velocity (m/s)	0.0	10	12	14

Obtain an estimate of the distance traveled in the interval [0,3].

Distance = integral of the velocity

$$\text{Distance} = \int_0^3 V(t)\, dt$$

Example 1

Special Case (Equaliy spaced base points)

$x_{i+1} - x_i = h \quad \text{for all} \quad i$

$$\int_a^b f(x)dx \approx h\left[\frac{1}{2}[f(x_0) + f(x_n)] + \sum_{i=1}^{n-1} f(x_i)\right]$$

The interval is divided into 3 subintervals

Base points are $\{0,1,2,3\}$

Time (s)	0.0	1.0	2.0	3.0
Velocity (m/s)	0.0	10	12	14

Trapezoid Method

$$h = x_{i+1} - x_i = 1$$

$$T = h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\left(f(x_0) + f(x_n)\right)\right]$$

$$\text{Distance} = 1\left[(10+12) + \frac{1}{2}(0+14)\right] = 29$$

Error in estimating the integral Theorem

Assumption : $f''(x)$ is continuous on $[a,b]$

Equal intervals (width $= h$)

Theorem : If Trapezoid Method is used to

approximate $\int_a^b f(x)\,dx$ then

$$Error = -\frac{b-a}{12} h^2 f''(\xi) \quad where \quad \xi \in [a,b]$$

$$|Error| \le \frac{b-a}{12} h^2 \max_{x \in [a,b]} |f''(x)|$$

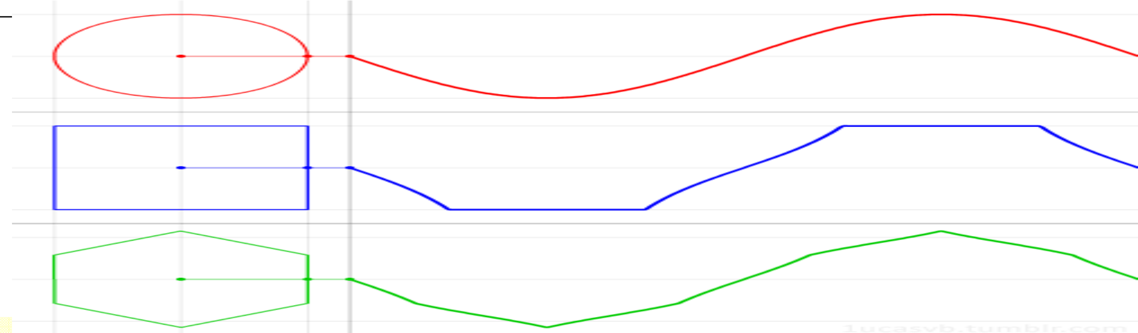
Theorem Let $f(x)$ have two continuous derivatives on the interval $a \le x \le b$. Then

$$E_n^T(f) \equiv \int_a^b f(x)\,dx - T_n(f) = -\frac{h^2(b-a)}{12} f''(c_n)$$

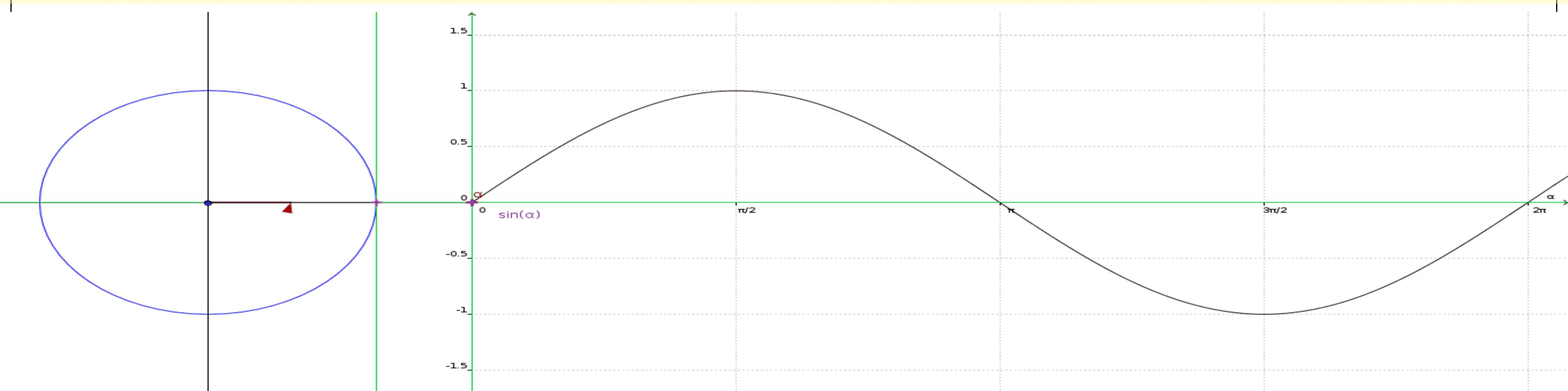
for some c_n in the interval $[a,b]$.

Estimating the Error

For Trapezoid Method



How many equally spaced intervals are needed to compute $\int_0^{\pi} \sin(x)dx$ to 5 decimal digit accuracy ?



Example 1a

$$\int_0^{\pi} \sin(x)\,dx, \quad find \quad h \quad so \quad that \quad |\text{error}| \le \frac{1}{2} \times 10^{-5}$$

$$|Error| \le \frac{b-a}{12} h^2 \max_{x \in [a,b]} |f''(x)|$$

$$b = \pi; \quad a = 0; \quad f'(x) = \cos(x); \quad f''(x) = -\sin(x)$$

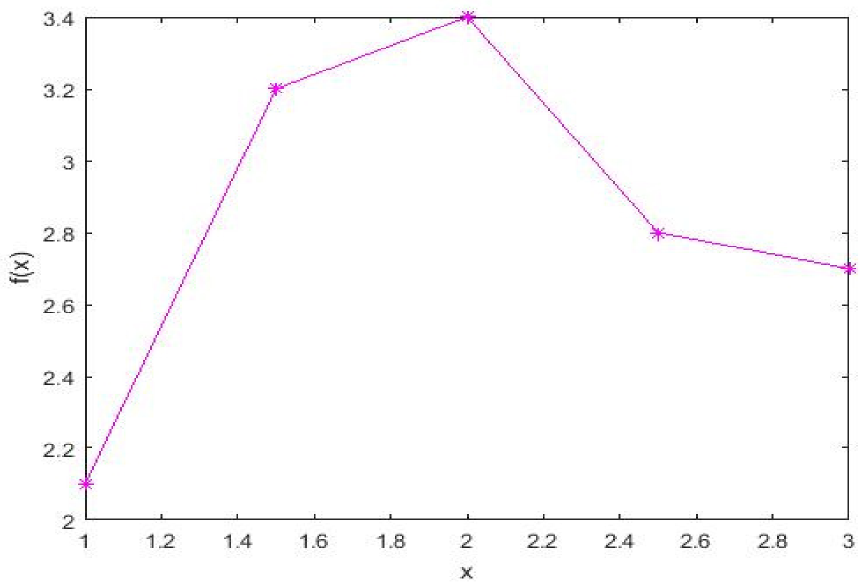
$$|f''(x)| \le 1 \quad \Rightarrow \quad |Error| \le \frac{\pi}{12} h^2 \le \frac{1}{2} \times 10^{-5}$$

$$\Rightarrow \quad h^2 \le \frac{6}{\pi} \times 10^{-5} \quad \Rightarrow \quad h \le 0.00437$$

$$\Rightarrow \quad n \ge \frac{(b-a)}{h} = \frac{\pi}{0.00437} = 719 \quad \text{intervals}$$

Example 2

x	1.0	1.5	2.0	2.5	3.0
f(x)	**2.1**	**3.2**	**3.4**	**2.8**	**2.7**



Use Trapezoid method to compute : $\int_1^3 f(x)\,dx$

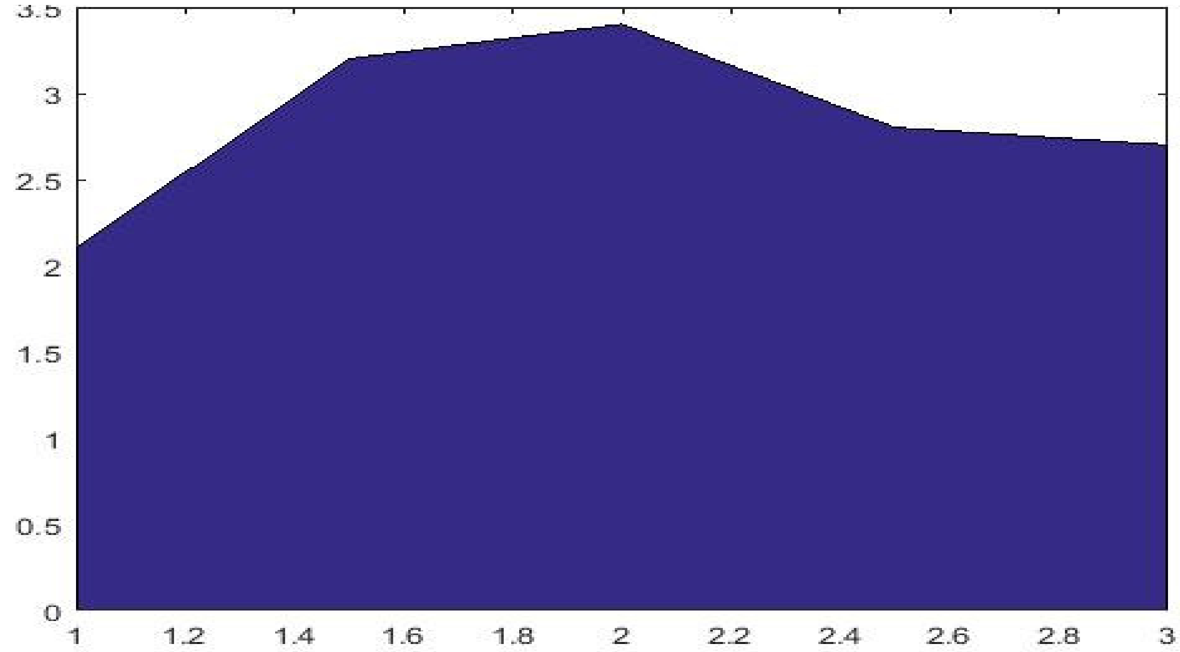
$$Trapezoid \qquad T(f,P) = \sum_{i=0}^{n-1} \frac{1}{2}(x_{i+1} - x_i)\big(f(x_{i+1}) + f(x_i)\big)$$

$$Special \quad Case: \quad h = x_{i+1} - x_i \quad \text{for all} \quad i,$$

$$T(f,P) = h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\big(f(x_0) + f(x_n)\big)\right]$$

Example 2

x	1.0	1.5	2.0	2.5	3.0
f(x)	**2.1**	**3.2**	**3.4**	**2.8**	**2.7**



$$\int_1^3 f(x)dx \approx h\left[\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\left(f(x_0) + f(x_n)\right)\right]$$

$$= 0.5\left[3.2 + 3.4 + 2.8 + \frac{1}{2}(2.1 + 2.7)\right]$$

$$= 5.9$$

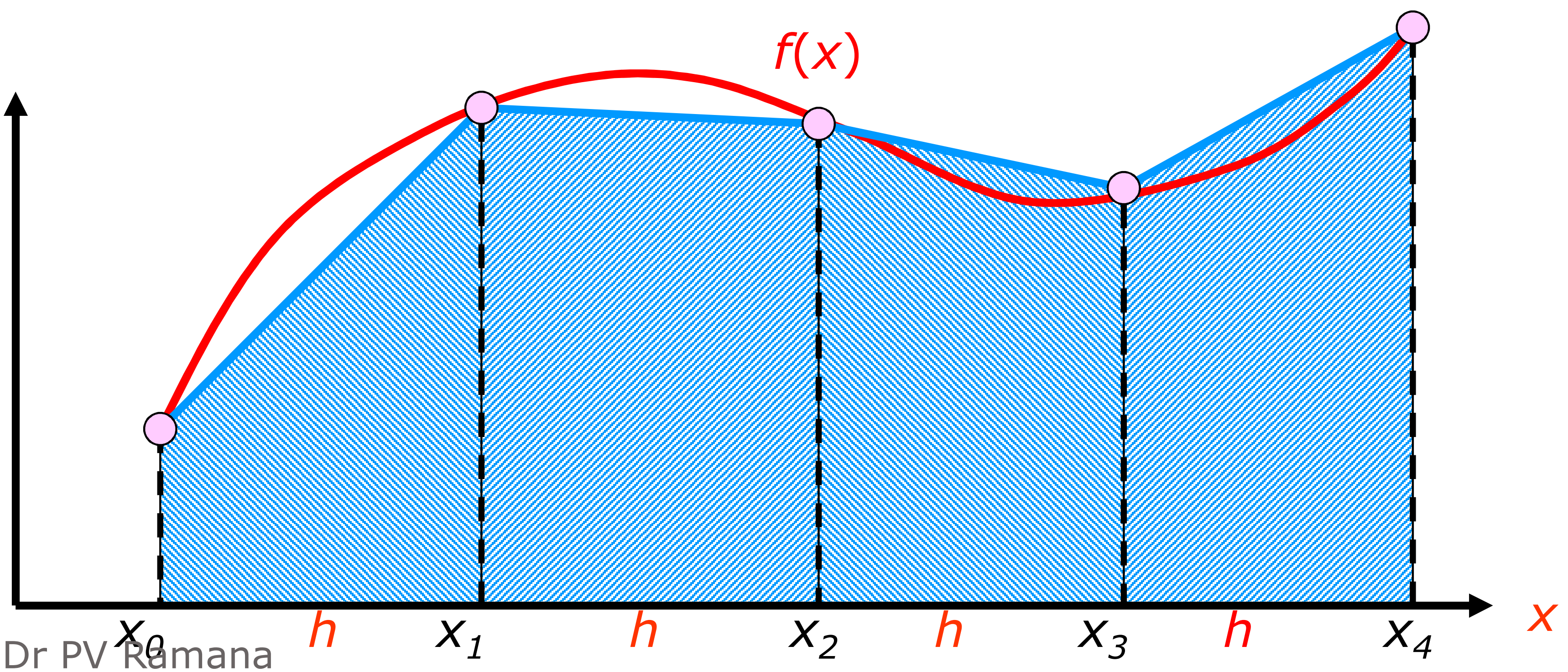
Composite Trapezoidal Rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \Lambda\ \Lambda\ + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right] + \frac{h}{2}\left[f(x_1) + f(x_2)\right] + \Lambda\ + \frac{h}{2}\left[f(x_{n-1}) + f(x_n)\right]$$

$$= \frac{h}{2}\left[f(x_0) + 2f(x_1) + \Lambda\ + 2f(x_i) + \Lambda\ + 2f(x_{n-1}) + f(x_n)\right]$$

$$\boxed{h = \frac{b-a}{n}}$$



Composite Trapezoidal Rule

- **Evaluate the integral** $I = \int_0^4 xe^{2x}dx = 5216.92$

$$n = 1, h = 4 \Rightarrow I = \frac{h}{2}\left[f(0) + f(4)\right] = 23847.66 \qquad \varepsilon = -357.12\,\%$$

$$n = 2, h = 2 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(2) + f(4)\right] = 12142.23 \qquad \varepsilon = -132.75\,\%$$

$$\begin{aligned} n = 4, h = 1 \Rightarrow I &= \frac{h}{2}\left[f(0) + 2f(1) + 2f(2)\right. \\ &\quad \left. + 2f(3) + f(4)\right] = 7288.79 \qquad \varepsilon = -39.71\,\% \end{aligned}$$

$$\begin{aligned} n = 8, h = 0.5 \Rightarrow I &= \frac{h}{2}\left[f(0) + 2f(0.5) + 2f(1)\right. \\ &\quad + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) \\ &\quad \left. + 2f(3.5) + f(4)\right] = 5764.76 \qquad \varepsilon = -10.50\,\% \end{aligned}$$

$$\begin{aligned} n = 16, h = 0.25 \Rightarrow I &= \frac{h}{2}\left[f(0) + 2f(0.25) + 2f(0.5) + \Lambda\right. \\ &\quad \left. + 2f(3.5) + 2f(3.75) + f(4)\right] = 5355.95 \qquad \varepsilon = -2.66\,\% \end{aligned}$$

Composite Trapezoidal Rule

```
function I = Trap(func, a, b, n)
% trap(func, a, b, n) :
%   composite trapezoidal rule.
% input:
%   func = name of function to be integrated
%   a, b = integration limits
%   n = number of segments
% output:
%   I = integral estimate

x = a;
h=(b-a)/n;
S = feval(func,a);
for j = 1 : n-1
    x = x + h;
    S = S + 2 * feval(func,x);
end
S = S + feval(func,b);
I = (b-a)*S / (2*n);
```

$$\int_0^{\pi} x^2 \sin(2x)\,dx$$

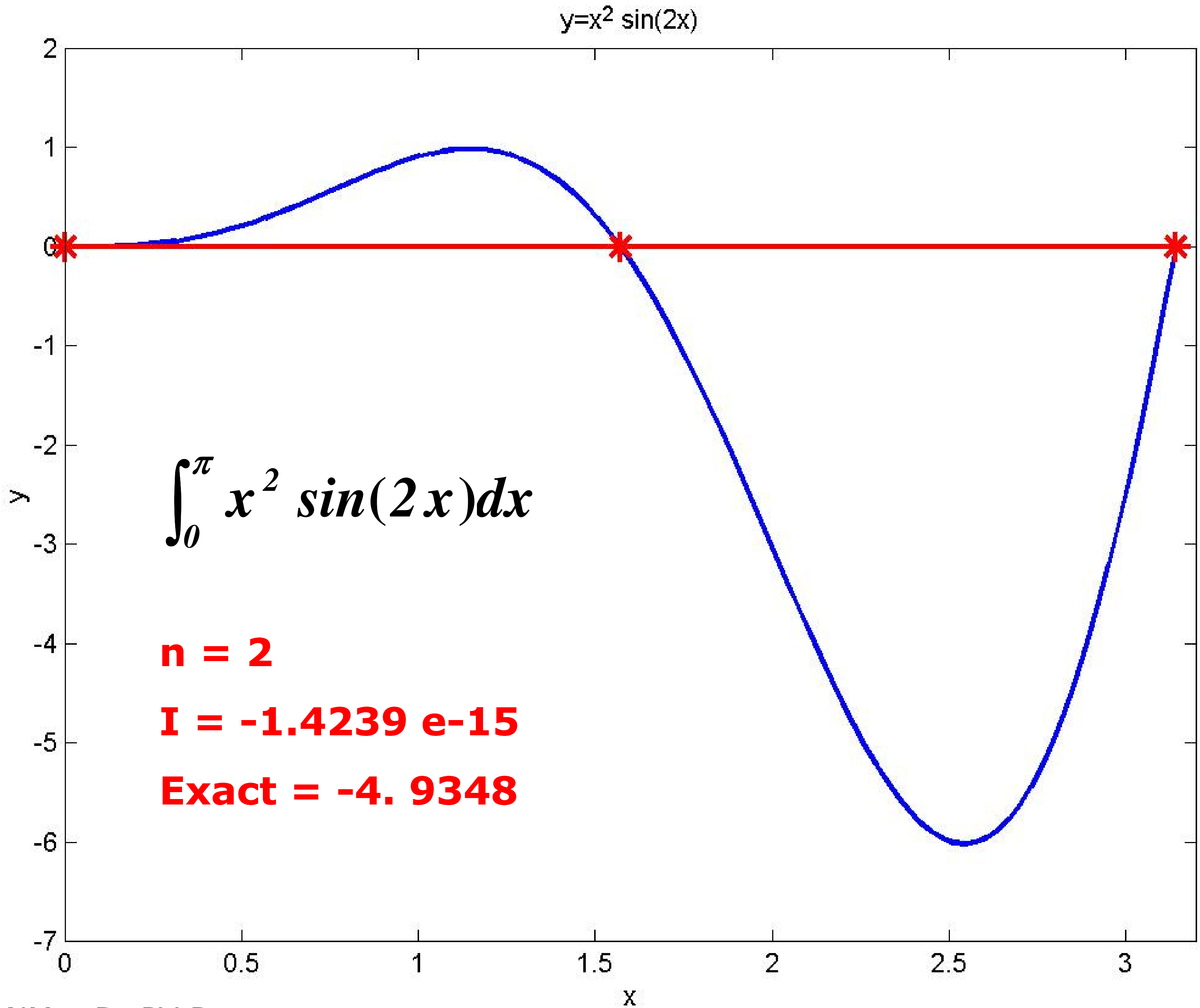
```
function f = example1(x)
% a = 0, b = pi
f=x.^2.*sin(2*x);
```

Composite Trapezoidal Rule

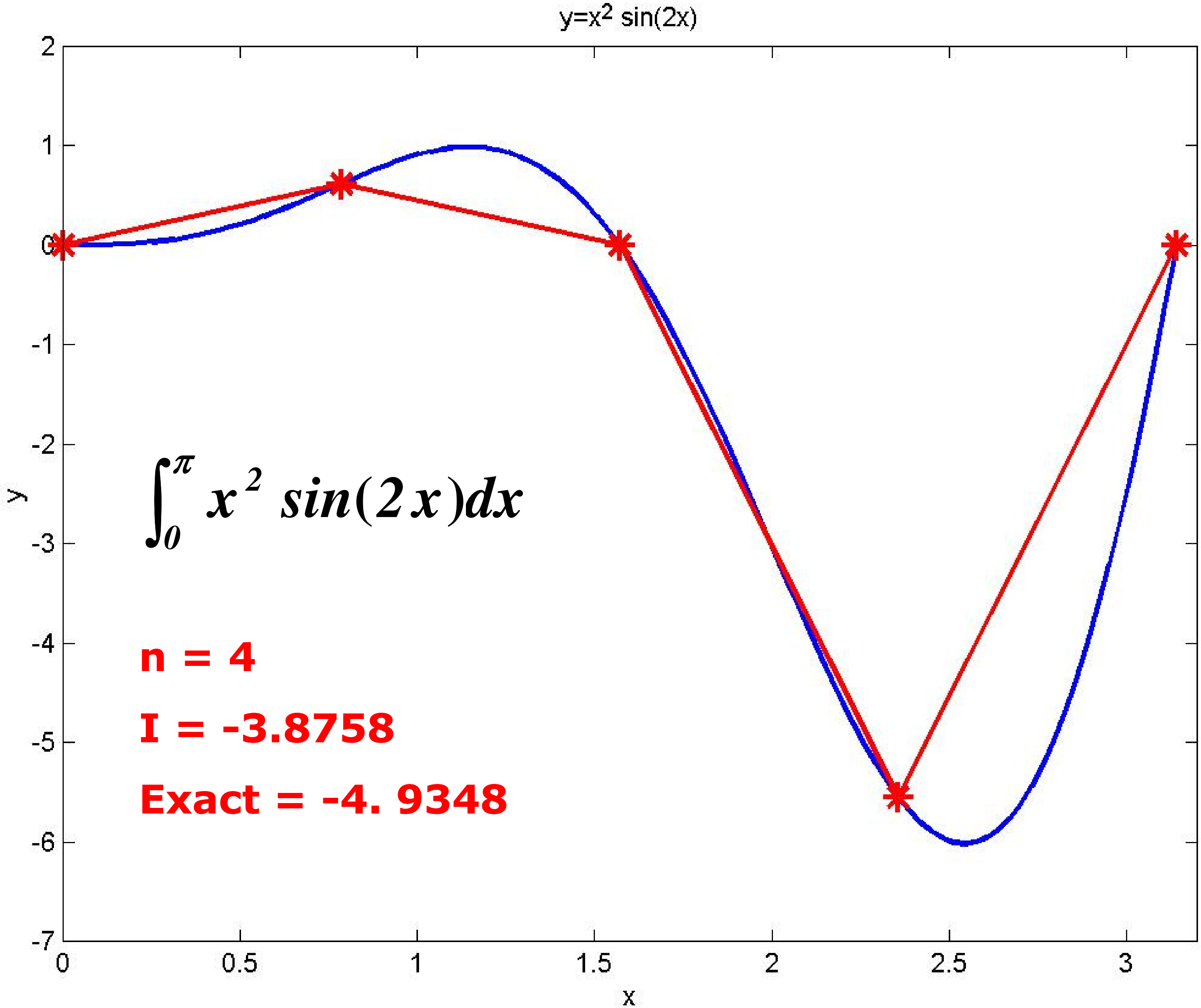
```
» a=0; b=pi; dx=(b-a)/100;
» x=a:dx:b; y=example1(x);
» I=trap('example1',a,b,1)
I =
 -3.7970e-015
» I=trap('example1',a,b,2)
I =
 -1.4239e-015
» I=trap('example1',a,b,4)
I =
   -3.8758
» I=trap('example1',a,b,8)
I =
   -4.6785
» I=trap('example1',a,b,16)
I =
   -4.8712
» I=trap('example1',a,b,32)
I =
   -4.9189
```

```
» I=trap('example1',a,b,64)
I =
   -4.9308
» I=trap('example1',a,b,128)
I =
   -4.9338
» I=trap('example1',a,b,256)
I =
   -4.9346
» I=trap('example1',a,b,512)
I =
   -4.9347
» I=trap('example1',a,b,1024)
I =
   -4.9348
» Q=quad8('example1',a,b)
Q =
   -4.9348
```

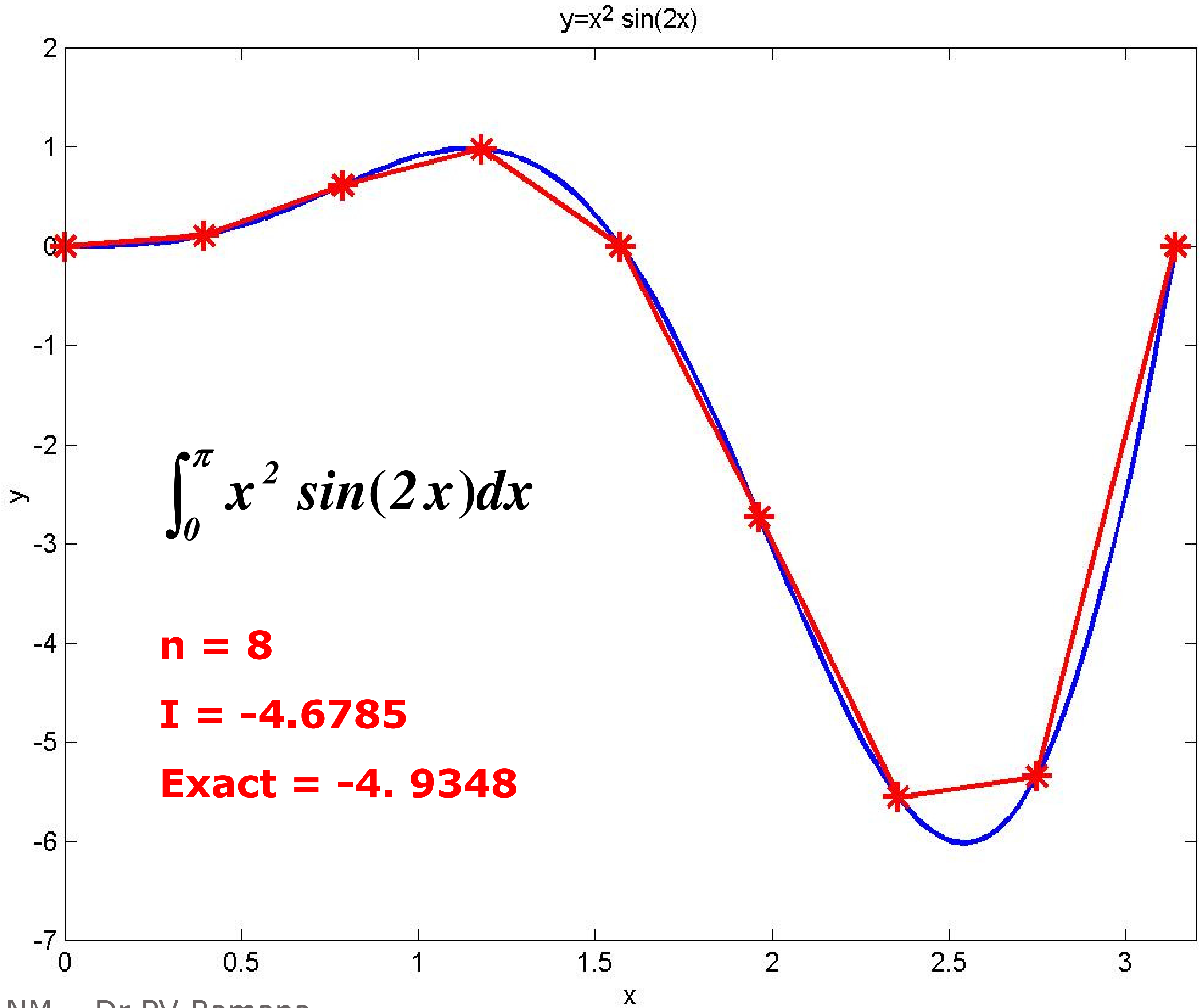
MATLAB function

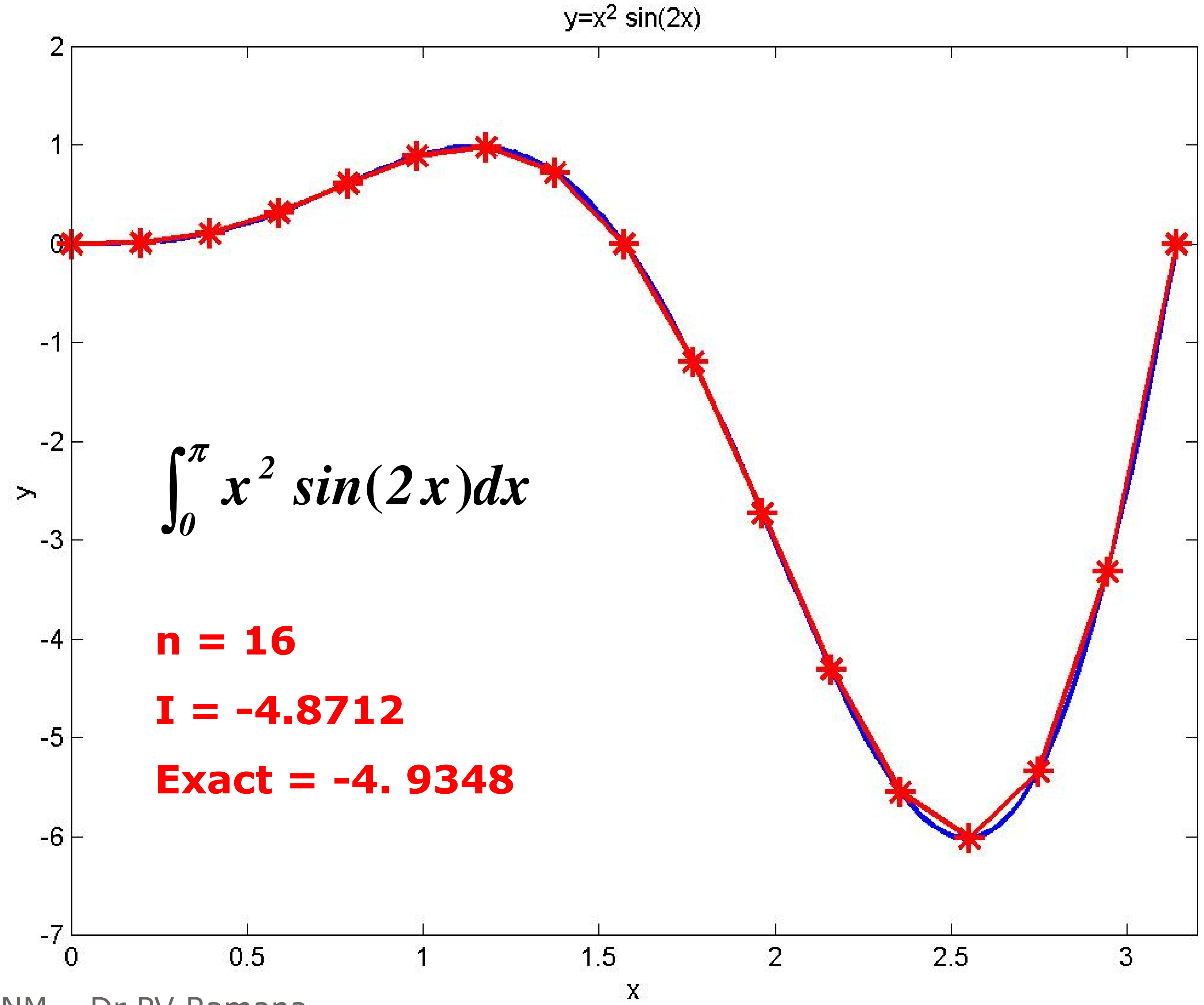


y=x² sin(2x)
$\int_0^{\pi} x^2 \sin(2x)dx$
n = 2
I = -1.4239 e-15
Exact = -4. 9348
y
x
2
1
0
-1
-2
-3
-4
-5
-6
-7
0
0.5
1
1.5
2
2.5
3



y=x2 sin(2x)
$\int_0^{\pi} x^2 \sin(2x)dx$
n = 4
I = -3.8758
Exact = -4. 9348
y
x
2
1
0
-1
-2
-3
-4
-5
-6
-7
0
0.5
1
1.5
2
2.5
3


y=x2 sin(2x)
$\int_0^\pi x^2 \sin(2x)dx$
n = 8
I = -4.6785
Exact = -4. 9348
y
x
2
1
0
-1
-2
-3
-4
-5
-6
-7
0
0.5
1
1.5
2
2.5
3



$$\int_0^{\pi} x^2 \sin(2x)dx$$

n = 16

I = -4.8712

Exact = -4. 9348

Composite Trapezoidal Rule

- **Evaluate the integral** $I = \int_0^4 xe^{2x}dx = 5216.92$

$$n = 1, h = 4 \Rightarrow I = \frac{h}{2}\left[f(0) + f(4)\right] = 23847.66 \qquad \varepsilon = -357.12\%$$

$$n = 2, h = 2 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(2) + f(4)\right] = 12142.23 \qquad \varepsilon = -132.75\%$$

$$n = 4, h = 1 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right] = 7288.79 \qquad \varepsilon = -39.71\%$$

$$n = 8, h = 0.5 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + f(4)\right] = 5764.76 \qquad \varepsilon = -10.50\%$$

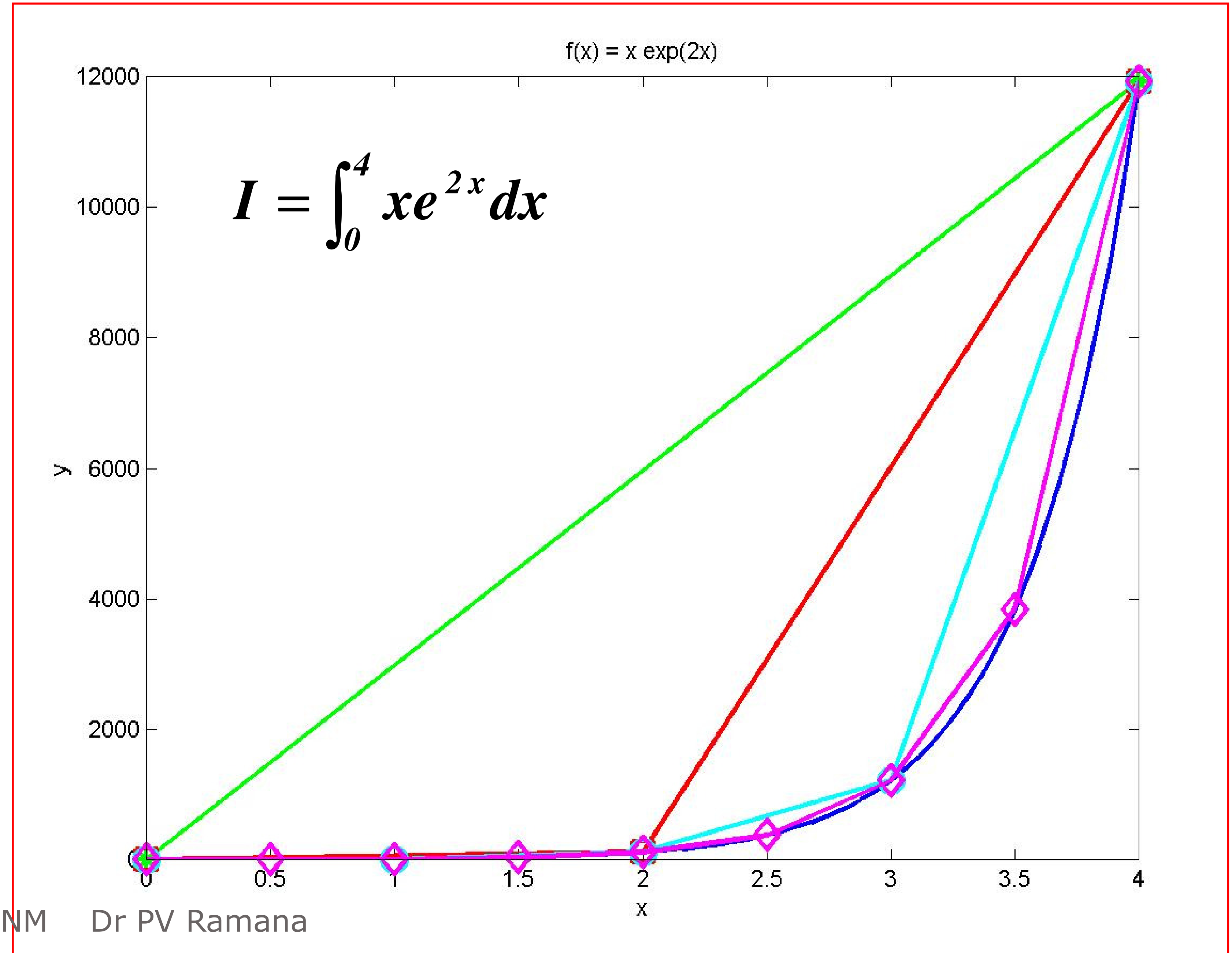
$$n = 16, h = 0.25 \Rightarrow I = \frac{h}{2}\left[f(0) + 2f(0.25) + 2f(0.5) + \Lambda + 2f(3.5) + 2f(3.75) + f(4)\right] = 5355.95 \qquad \varepsilon = -2.66\%$$

Composite Trapezoidal Rule

```
» x=0:0.04:4; y=example2(x);
» x1=0:4:4; y1=example2(x1);
» x2=0:2:4; y2=example2(x2);
» x3=0:1:4; y3=example2(x3);
» x4=0:0.5:4; y4=example2(x4);
» H=plot(x,y,x1,y1,'g-*',x2,y2,'r-s',x3,y3,'c-o',x4,y4,'m-d');
» set(H,'LineWidth',3,'MarkerSize',12);
» xlabel('x'); ylabel('y'); title('f(x) = x exp(2x)');

» I=trap('example2',0,4,1)
I =
  2.3848e+004
» I=trap('example2',0,4,2)
I =
  1.2142e+004
» I=trap('example2',0,4,4)
I =
  7.2888e+003
» I=trap('example2',0,4,8)
I =
  5.7648e+003
» I=trap('example2',0,4,16)
I =
  5.3559e+003
```

Composite Trapezoidal Rule

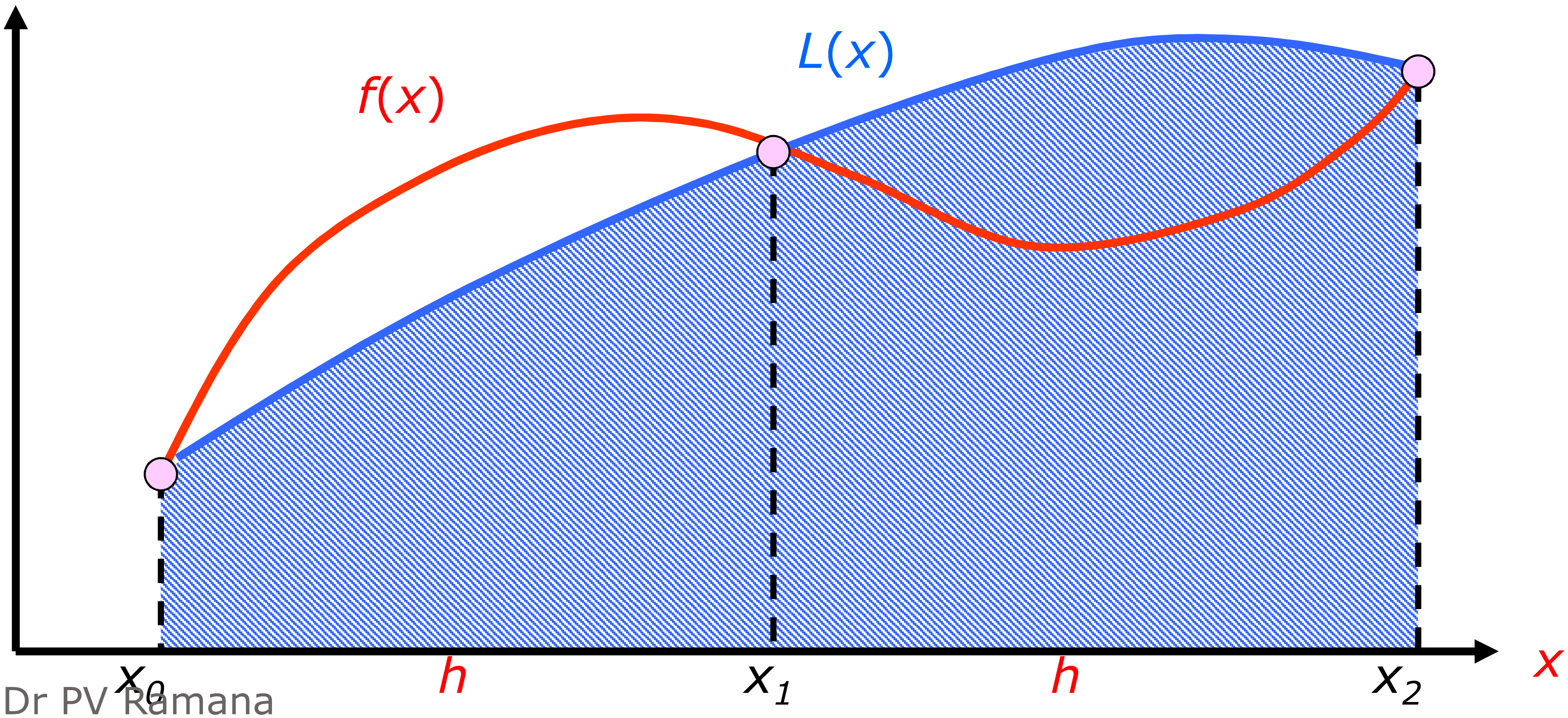


Simpson's 1/3 - Rule

- **Approximate the function by a parabola**

$$\int_a^b f(x)dx \approx \sum_{i=0}^{2} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2)$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$



Simpson's 1/3 - Rule

$$L(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

$$let \quad x_0 = a, x_2 = b, x_1 = \frac{a+b}{2}$$

$$h = \frac{b-a}{2}, \xi = \frac{x - x_1}{h}, \; d\xi = \frac{dx}{h}$$

$$\begin{cases} x = x_0 \Rightarrow \xi = -1 \\ x = x_1 \Rightarrow \xi = 0 \\ x = x_2 \Rightarrow \xi = 1 \end{cases}$$

$$\boxed{L(\xi) = \frac{\xi(\xi - 1)}{2} f(x_0) + (1 - \xi^2) f(x_1) + \frac{\xi(\xi + 1)}{2} f(x_2)}$$

Simpson's 1/3 - Rule

$$L(\xi) = \frac{\xi(\xi - 1)}{2} f(x_0) + (1 - \xi^2) f(x_1) + \frac{\xi(\xi + 1)}{2} f(x_2)$$

$$\int_a^b f(x)dx \approx h\int_{-1}^{1} L(\xi)d\xi = f(x_0)\frac{h}{2}\int_{-1}^{1} \xi(\xi - 1)d\xi$$

$$+ f(x_1)h\int_0^1 (1 - \xi^2)d\xi + f(x_2)\frac{h}{2}\int_{-1}^{1} \xi(\xi + 1)d\xi$$

$$= f(x_0)\frac{h}{2}\left(\frac{\xi^3}{3} - \frac{\xi^2}{2}\right)\Bigg|_{-1}^{1} + f(x_1)h\left(\xi - \frac{\xi^3}{3}\right)\Bigg|_{-1}^{1}$$

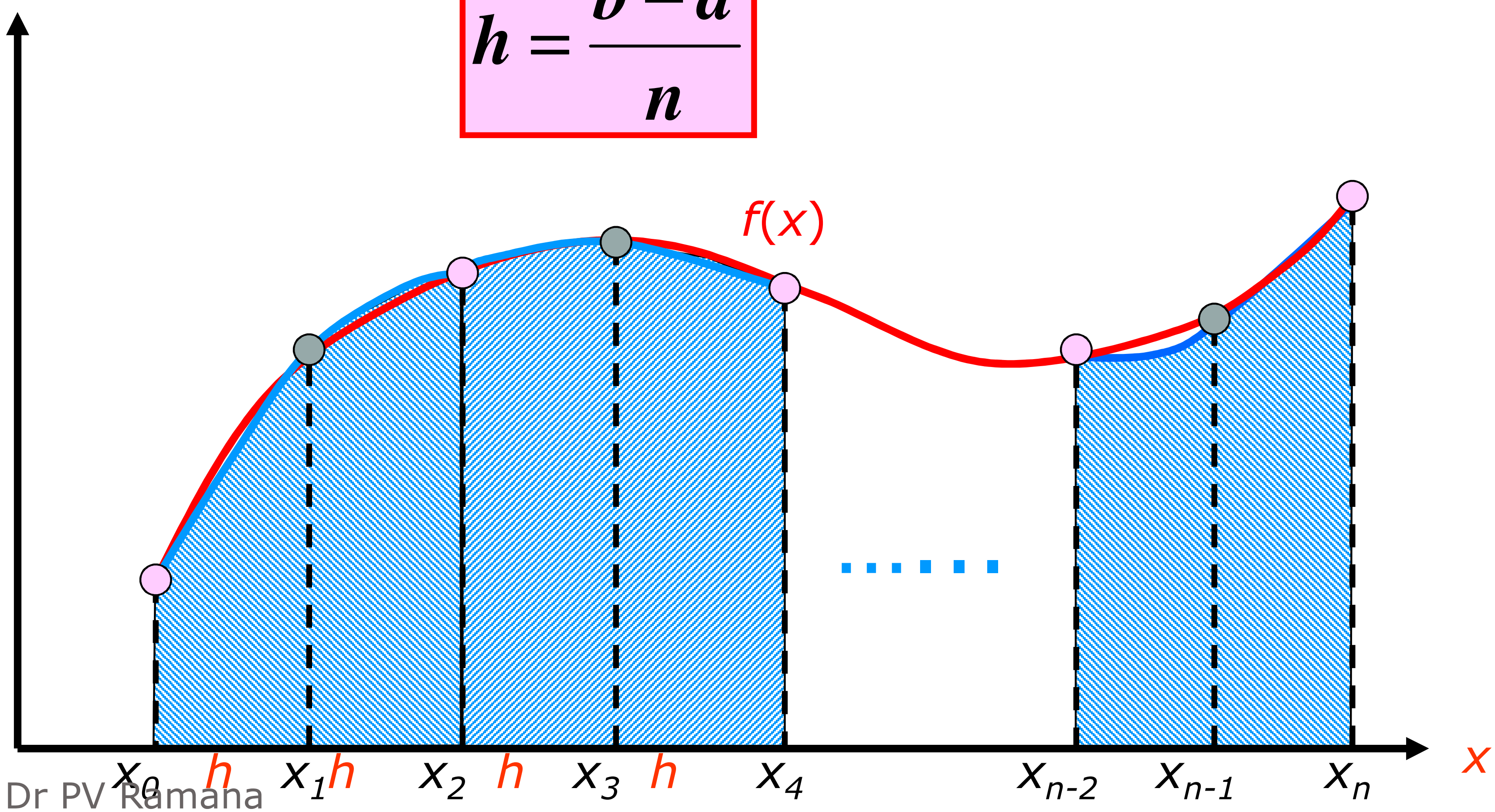
$$+ f(x_2)\frac{h}{2}\left(\frac{\xi^3}{3} + \frac{\xi^2}{2}\right)\Bigg|_{-1}^{1}$$

$$\int_a^b f(x)dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

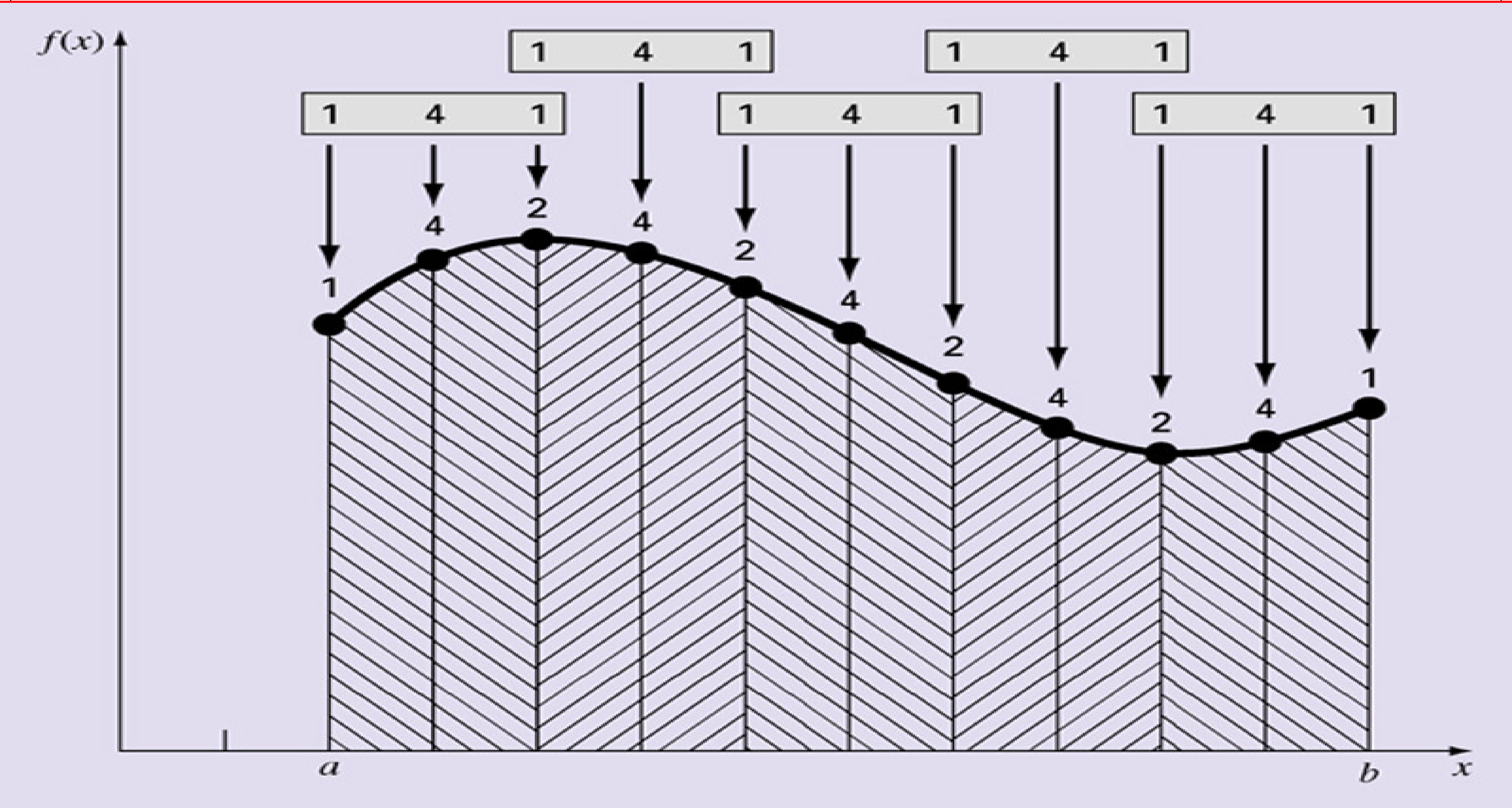
Simpson's 1/3 - Rule

Piecewise Quadratic approximations

$$h = \frac{b-a}{n}$$



Composite Simpson's 1/3 Rule



- Applicable only if the number of segments is even

Composite Simpson's 1/3 Rule

- **Applicable only if the number of segments is even**

$$I = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \Lambda + \int_{x_{n-2}}^{x_n} f(x)dx$$

- **Substitute Simpson's 1/3 rule for each integral**

$$I = 2h\frac{f(x_0)+4f(x_1)+f(x_2)}{6} + 2h\frac{f(x_2)+4f(x_3)+f(x_4)}{6} + \Lambda + 2h\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}$$

- **For uniform spacing (equal segments)**

$$I = \frac{(b-a)}{3n}\left\{ f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n) \right\}$$

Simpson's 1/3 Rule - Error

- **Truncation error (single application)**

$$\boxed{E_t = -\frac{1}{90} h^5 f^{(4)}(\xi) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)\,; \quad h = \frac{b-a}{2}}$$

- **Exact up to cubic polynomial ($f^{(4)} = 0$)**
- **Approximate error for ($n/2$) multiple applications**

$$\boxed{E_a = -\frac{(b-a)^5}{180 n^4} \bar{f}^{(4)}}$$

Composite Simpson's 1/3 Rule

- **Evaluate the integral** $I = \int_0^4 xe^{2x}dx$

- $n = 2, h = 2$

$$I = \frac{h}{3}\left[f(0) + 4f(2) + f(4)\right]$$
$$= \frac{2}{3}\left[0 + 4(2e^4) + 4e^8\right] = 8240.411 \quad \Rightarrow \varepsilon = -57.96\%$$

- $n = 4, h = 1$

$$I = \frac{h}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)\right]$$
$$= \frac{1}{3}\left[0 + 4(e^2) + 2(2e^4) + 4(3e^6) + 4e^8\right]$$
$$= 5670.975 \quad \Rightarrow \varepsilon = -8.70\%$$

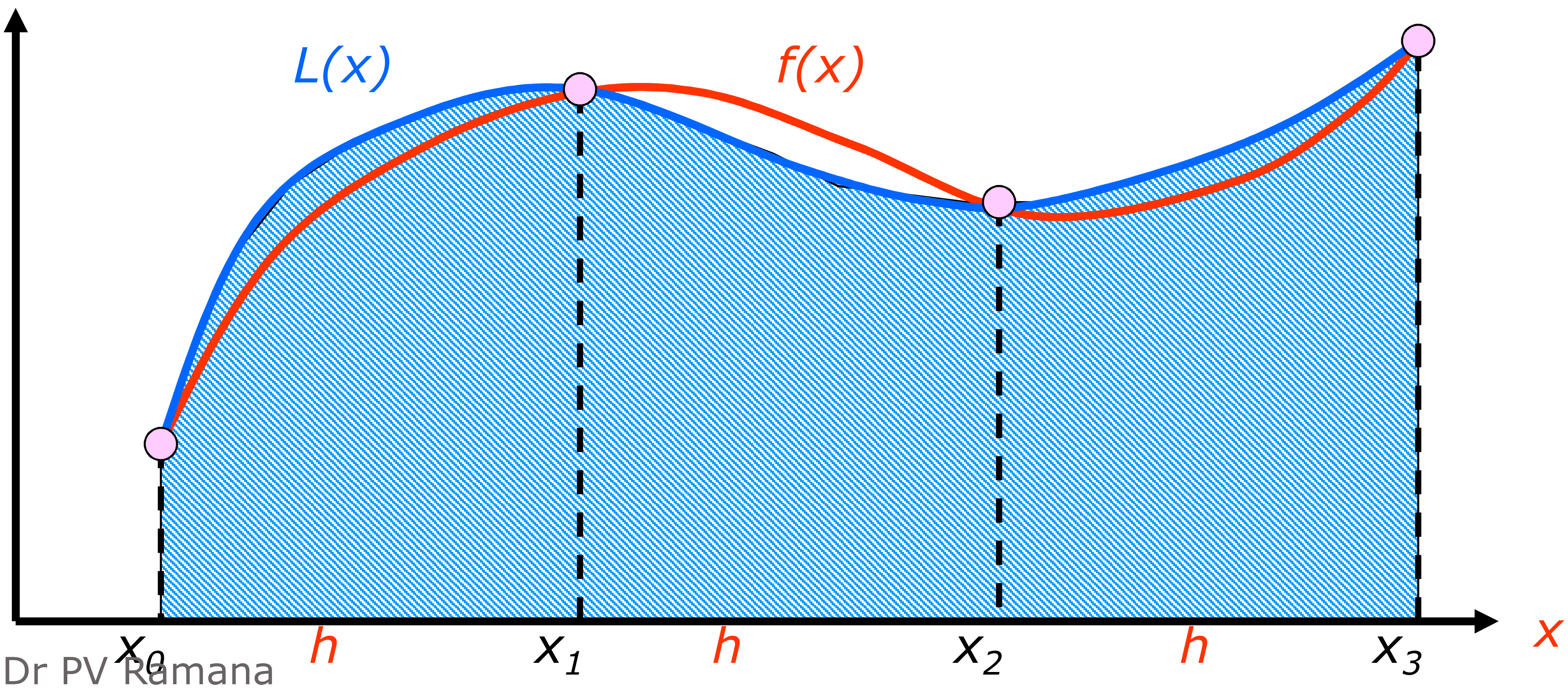
Simpson's 3/8-Rule

$$\int_a^b f(x)\,dx \approx \int_a^b \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3\right)dx$$

- **Approximate by a cubic polynomial**

$$\int_a^b f(x)dx \approx \sum_{i=0}^{3} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

$$= \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$



Simpson's 3/8- Rule

$$L(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$\int_a^b f(x)dx \approx \int_a^b L(x)dx \quad ; \quad h=\frac{b-a}{3}$$
$$=\frac{3h}{8}\left[f(x_0)+3f(x_1)+3f(x_2)+f(x_3)\right]$$

- Truncation error

$$E_t=-\frac{3}{80}h^5 f^{(4)}(\xi)=-\frac{(b-a)^5}{6480}f^{(4)}(\xi)\ ;\ h=\frac{b-a}{3}$$

Example: Simpson's Rules

- **Evaluate the integral** $\int_0^4 xe^{2x}dx$
- **Simpson's 1/3-Rule**

$$I=\int_0^4 xe^{2x}dx \approx \frac{h}{3}\left[f(0)+4f(2)+f(4)\right]$$

$$=\frac{2}{3}\left[0+4(2e^4)+4e^8\right]=8240.411$$

$$\varepsilon=\frac{5216.926-8240.411}{5216.926}=-57.96\%$$

- **Simpson's 3/8-Rule**

$$I=\int_0^4 xe^{2x}dx \approx \frac{3h}{8}\left[f(0)+3f(\frac{4}{3})+3f(\frac{8}{3})+f(4)\right]$$

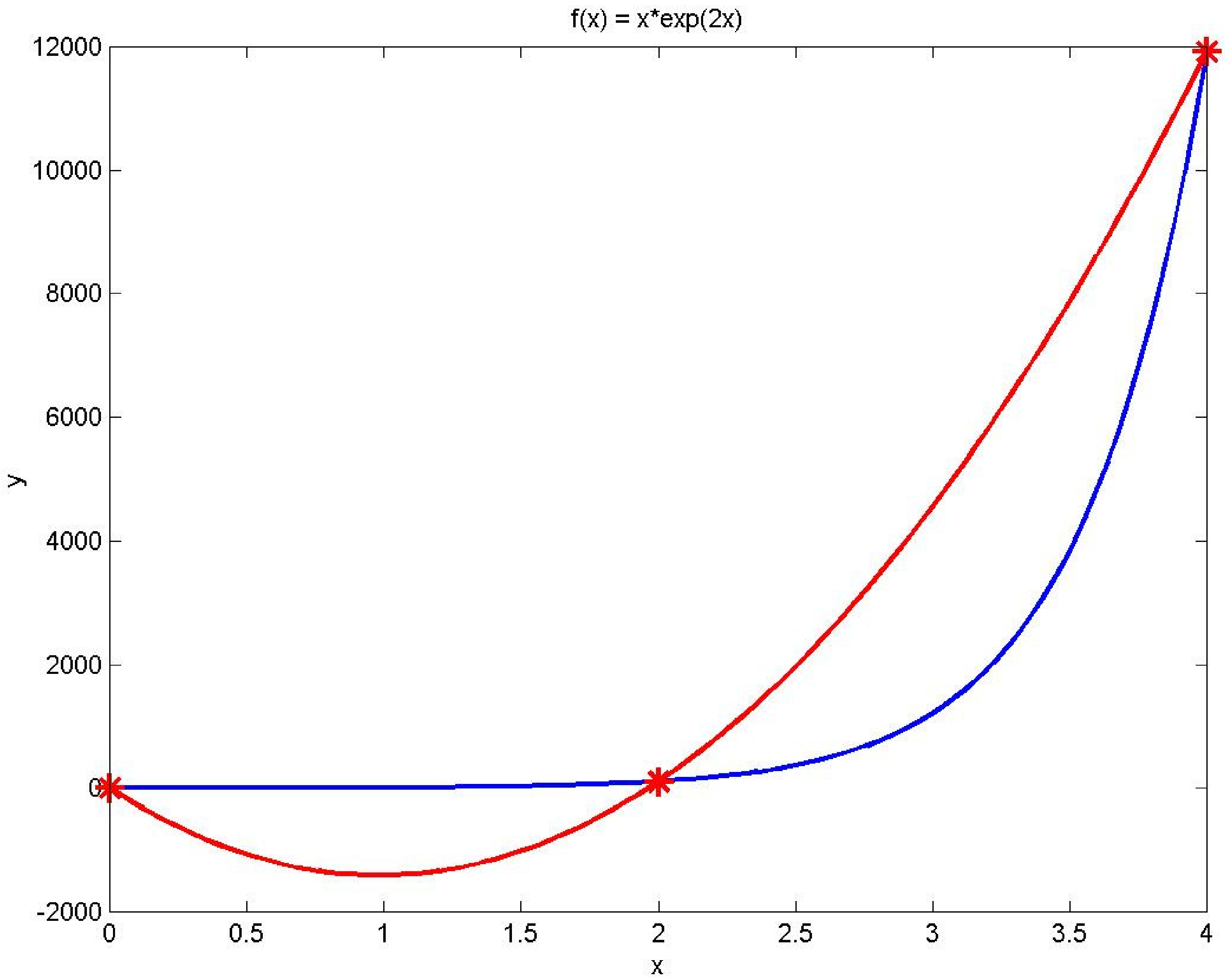
$$=\frac{3(4/3)}{8}\left[0+3(19.18922)+3(552.33933)+11923.832\right]=6819.209$$

$$\varepsilon=\frac{5216.926-6819.209}{5216.926}=-30.71\%$$

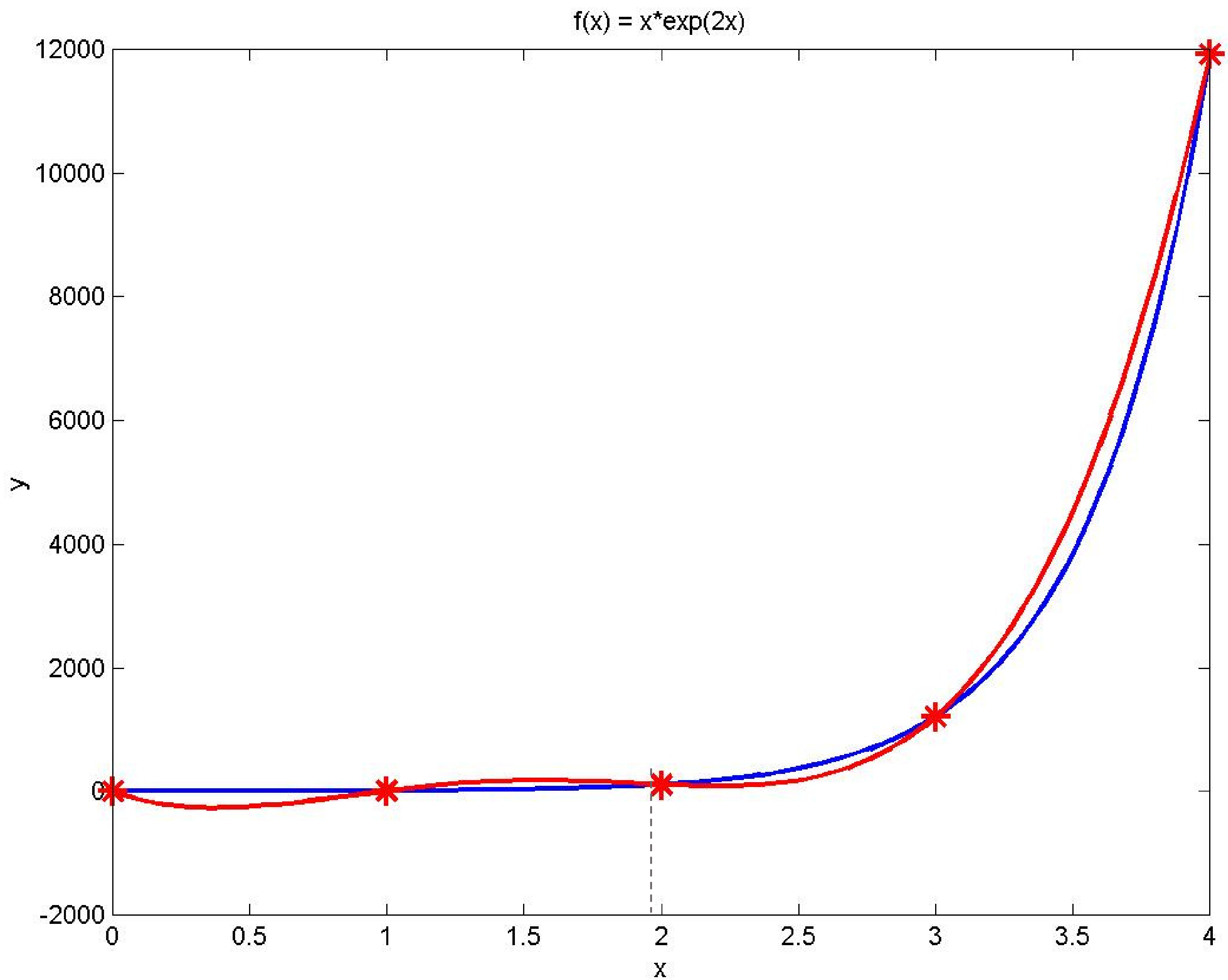
Matlab: Simpson’s Rules

```
function I = Simp(f, a, b, n)
% integral of f using composite Simpson rule
% n must be even
h = (b - a)/n;
S = feval(f,a);
for i = 1 : 2 : n-1
    x(i) = a + h*i;
    S = S + 4*feval(f, x(i));
end
for i = 2 : 2 : n-2
    x(i) = a + h*i;
    S = S + 2*feval(f, x(i));
end
S = S + feval(f, b); I = h*S/3;
```

Simpson's 1/3 Rule



Composite Simpson's 1/3 Rule



```
» x=0:0.04:4; y=example(x);
» x1=0:2:4; y1=example(x1);
» c=Lagrange_coef(x1,y1); p1=Lagrange_eval(x,x1,c);
» H=plot(x,y,x1,y1,'r*',x,p1,'r');
» xlabel('x'); ylabel('y'); title('f(x) = x*exp(2x)');
» set(H,'LineWidth',3,'MarkerSize',12);
» x2=0:1:4; y2=example(x2);
» c=Lagrange_coef(x2,y2); p2=Lagrange_eval(x,x2,c);
» H=plot(x,y,x2,y2,'r*',x,p2,'r');
» xlabel('x'); ylabel('y'); title('f(x) = x*exp(2x)');
» set(H,'LineWidth',3,'MarkerSize',12);
»
» I=Simp('example',0,4,2)
I =
  8.2404e+003
» I=Simp('example',0,4,4)
I =
  5.6710e+003
» I=Simp('example',0,4,8)
I =
  5.2568e+003
» I=Simp('example',0,4,16)
I =
  5.2197e+003
» Q=Quad8('example',0,4)
Q =
  5.2169e+003
```

n = 2

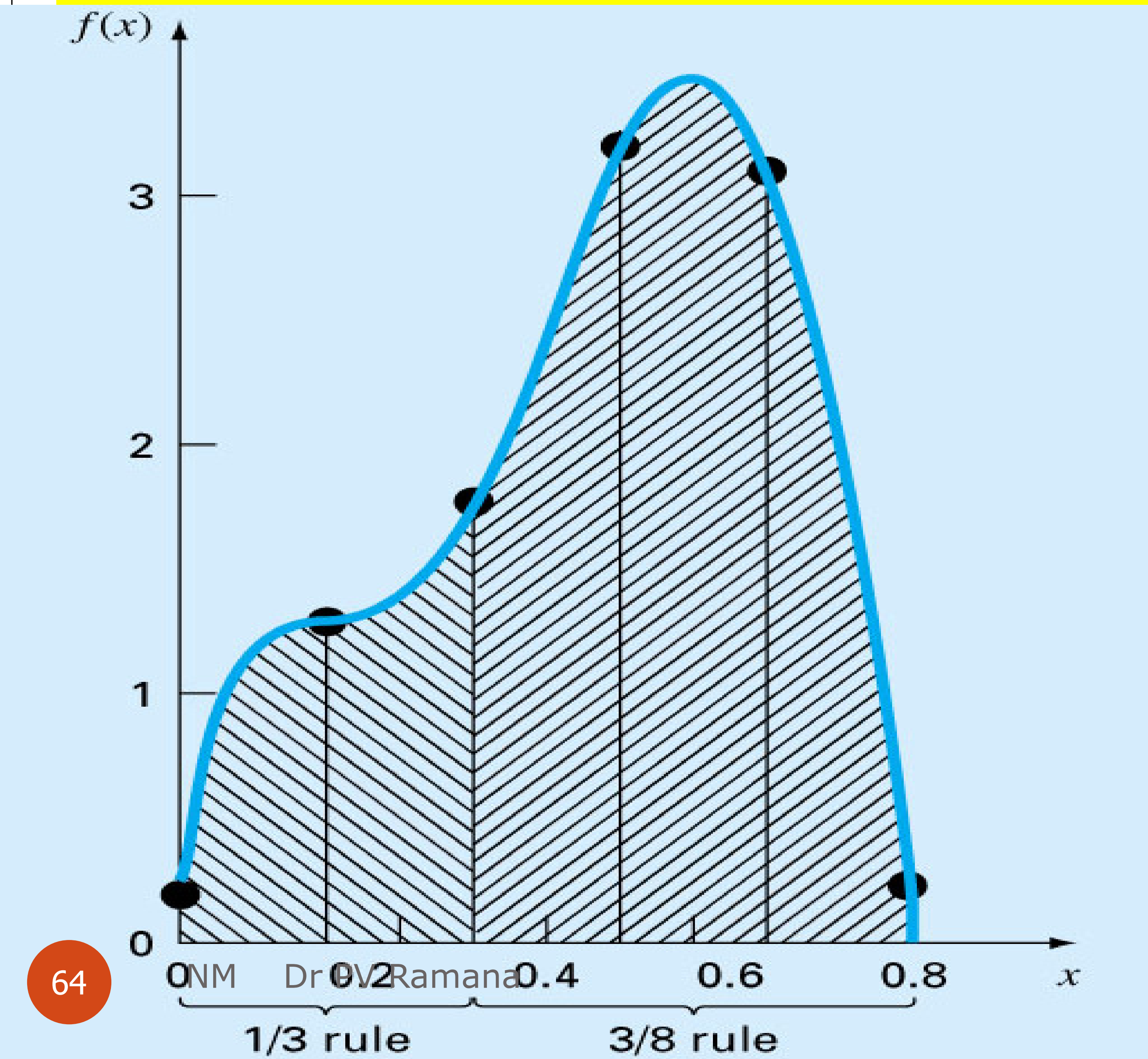
n = 4

n = 8

n = 16

MATLAB fun

Multiple applications of Simpson's rule with odd number of intervals



Hybrid Simpson's 1/3 & 3/8 rules

Newton-Cotes Closed Integration Formulae

n	Name	Formula	Truncation Error
1	Trapezoidal rule	$(b-a)\dfrac{f(x_0)+f(x_1)}{2}$	$-\dfrac{1}{12}h^3 f''(\xi)$
2	Simpson's 1/3 rule	$(b-a)\dfrac{f(x_0)+4f(x_1)+f(x_2)}{6}$	$-\dfrac{1}{90}h^5 f^{(4)}(\xi)$
3	Simpson's 3/8 rule	$(b-a)\dfrac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}$	$-\dfrac{3}{80}h^5 f^{(4)}(\xi)$
4	Boole's rule	$(b-a)\dfrac{7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)+7f(x_4)}{90}$	$-\dfrac{8}{945}h^7 f^{(6)}(\xi)$
5		$(b-a)\dfrac{19f(x_0)+75f(x_1)+50f(x_2)+50f(x_3)+75f(x_4)+19f(x_5)}{288}$	$-\dfrac{275}{12096}h^7 f^{(6)}(\xi)$

$$\boxed{h=\frac{b-a}{n}}$$

Recursive Trapezoid Method

Estimate based on one interval:

$$h = \frac{b-a}{2^0}$$

$$\mathrm{R}(0,0) = \frac{b-a}{2}\big(f(a) + f(b)\big)$$