NUMERICAL METHODS

$$U^{n+1} = U^n + \Delta t\, f(U^n)$$

$$\frac{\partial v}{\partial t} + V \cdot \nabla v = \nabla \cdot (k \nabla v) + g(v)$$

$$(\lambda + \mu)\nabla(\nabla \cdot u) + \mu \nabla^2 u = \alpha(3\lambda + 2\mu)\nabla T - \rho b$$

Lecture 7

$$\rho\left(\frac{\partial u}{\partial t} + V \cdot \nabla u\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$

$$\nabla^2 u = f$$

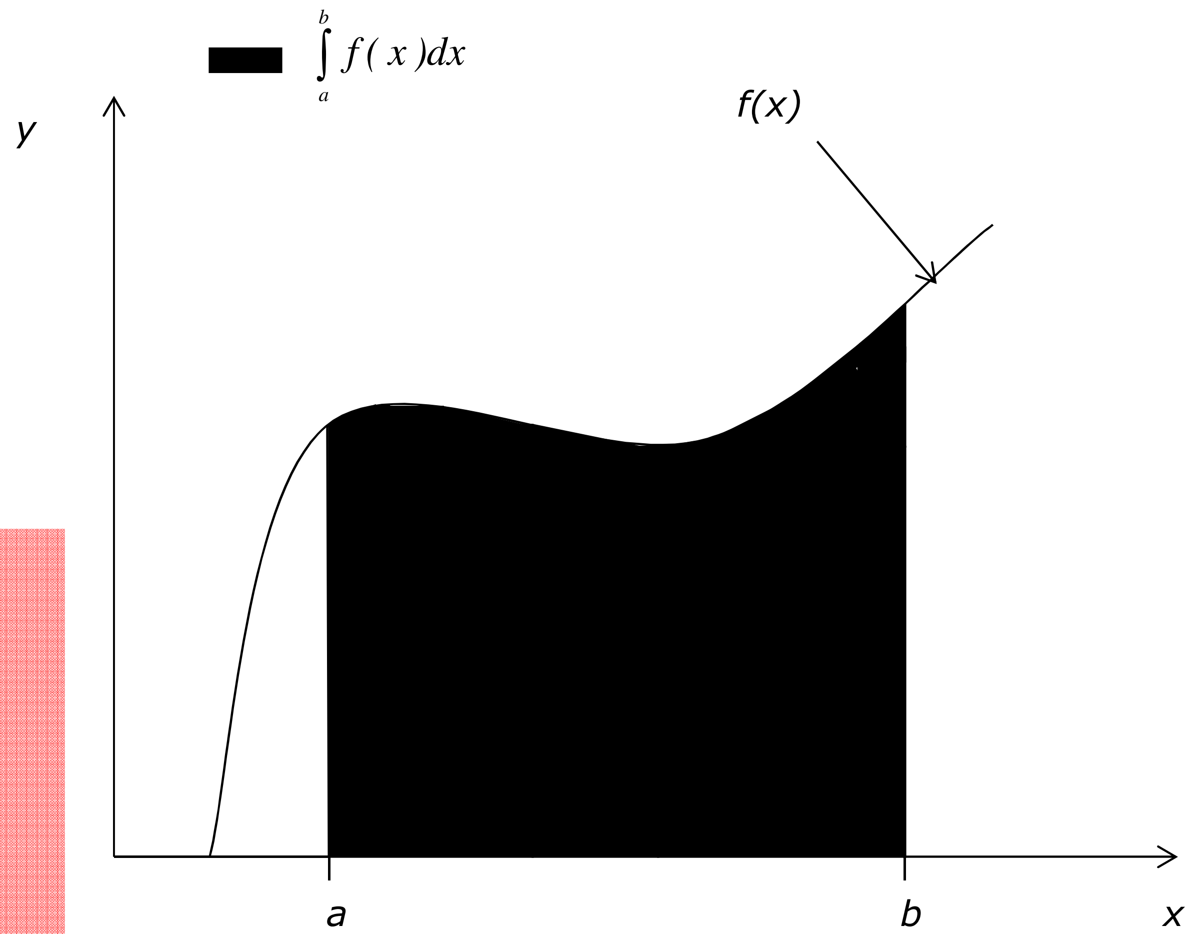
Dr. P V Ramana

Numerical Integration

What is Integration?

Integration

The process of measuring the area under a curve.



$$I = \int_a^b f(x)dx$$

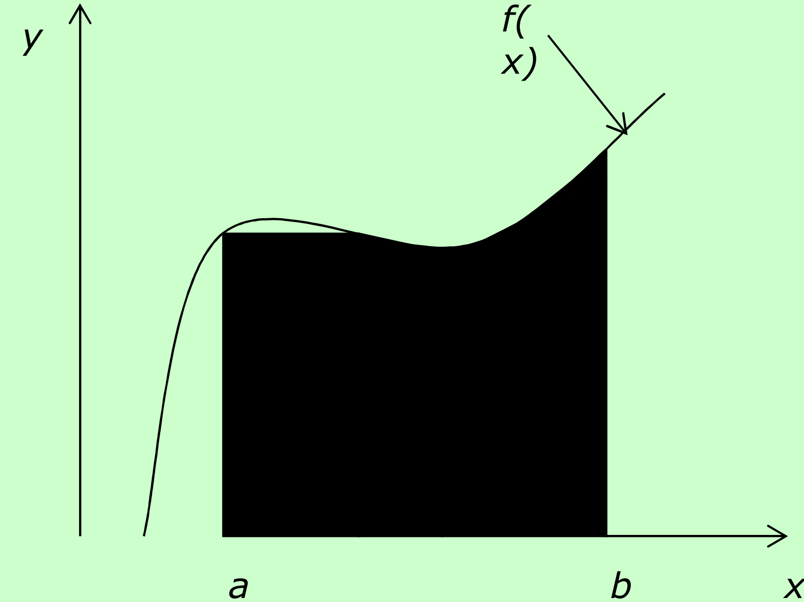
Where:

f(x) is the integrand

a= lower limit of integration

b= upper limit of integration

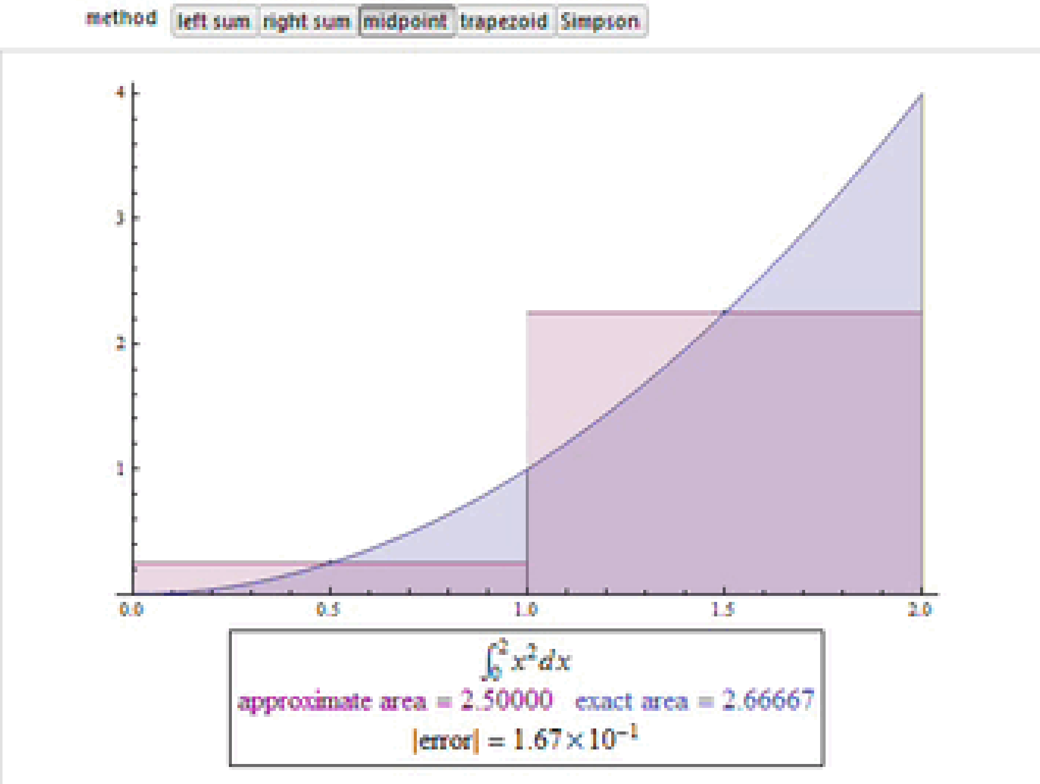
$$y = f(x)$$

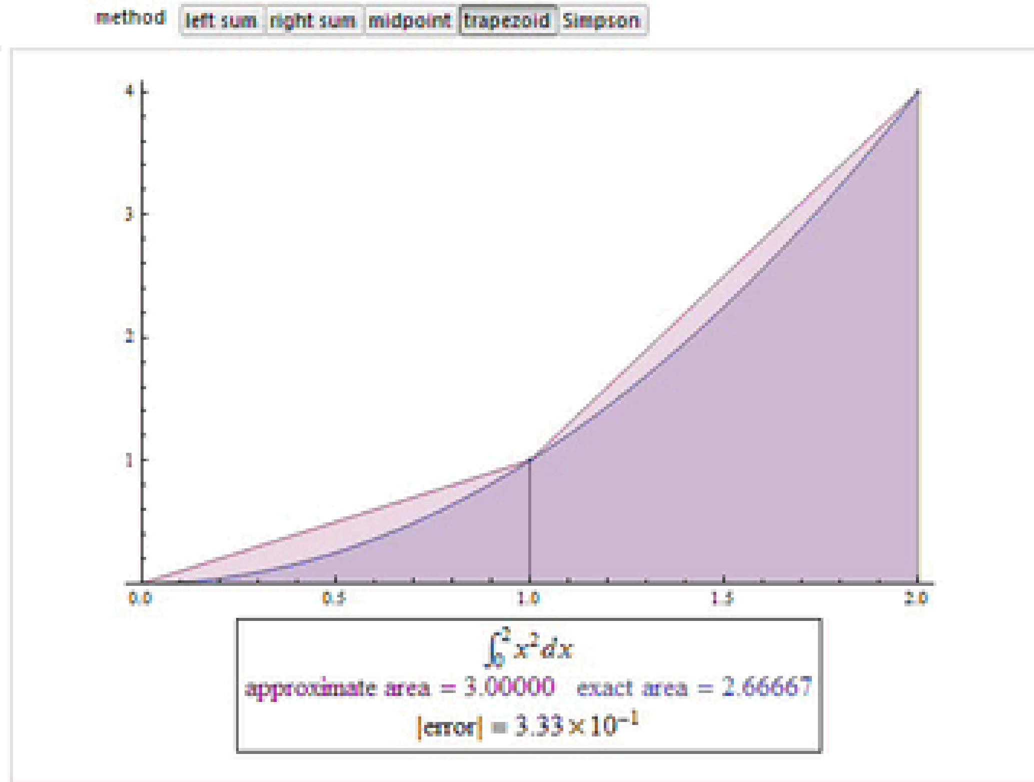


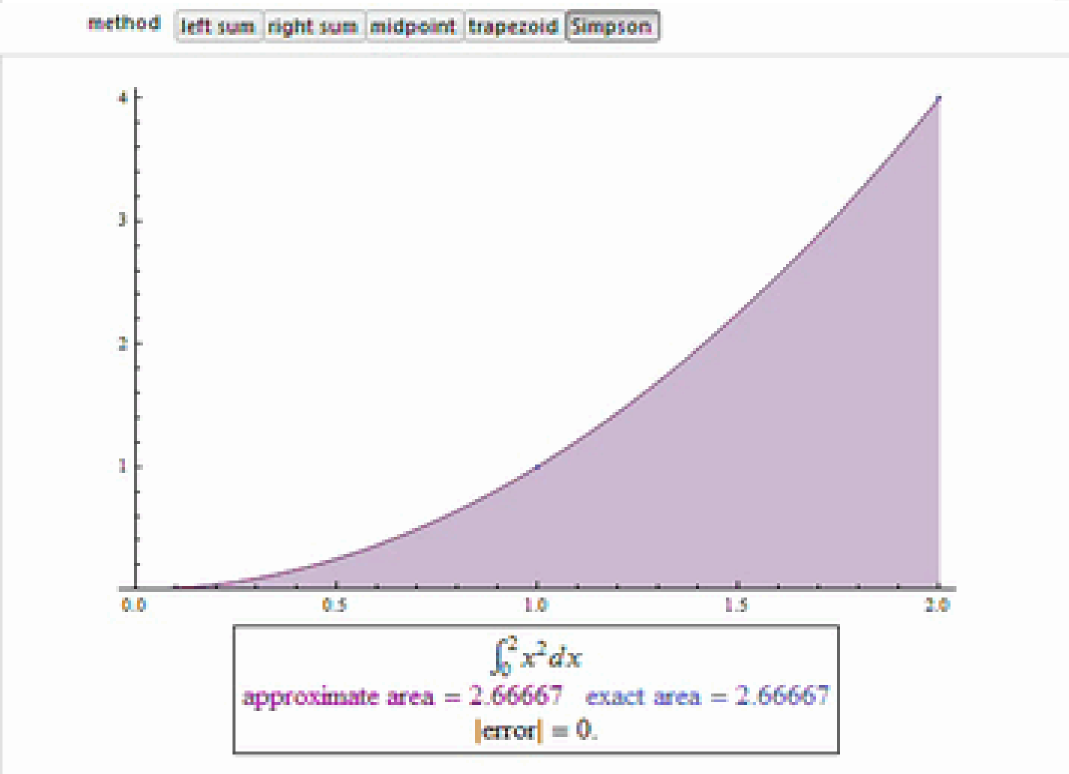
Integration

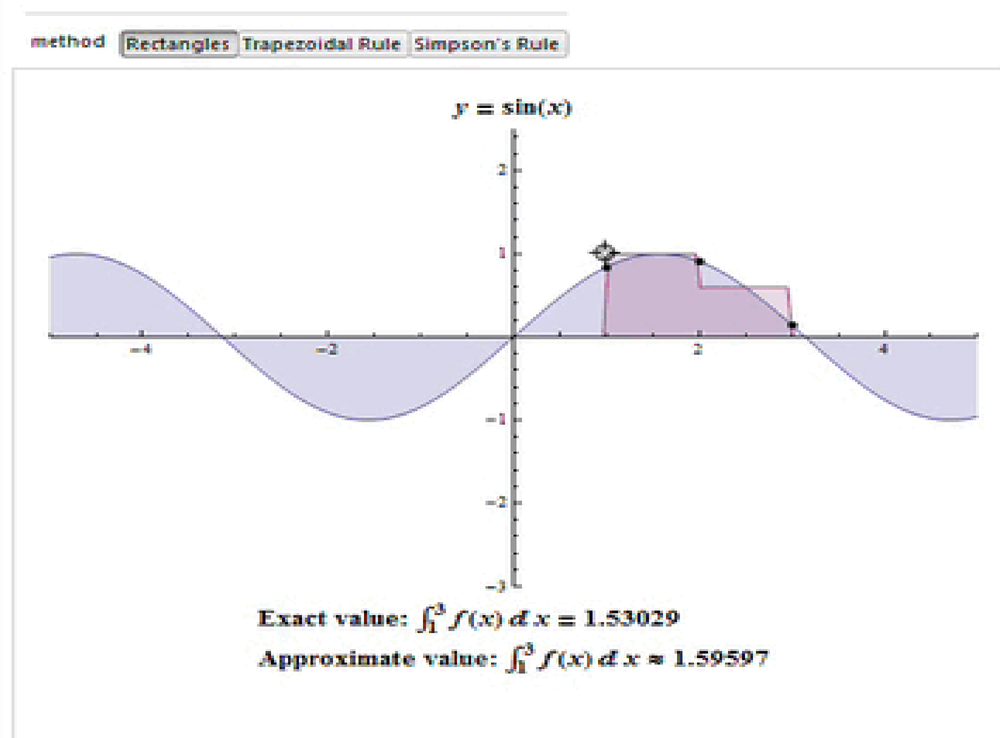
$$I = \lim_{|\max \Delta x| \to 0} \sum_{i=1}^{M} f(x_i)\Delta x_i = \int_a^b f(x)dx$$

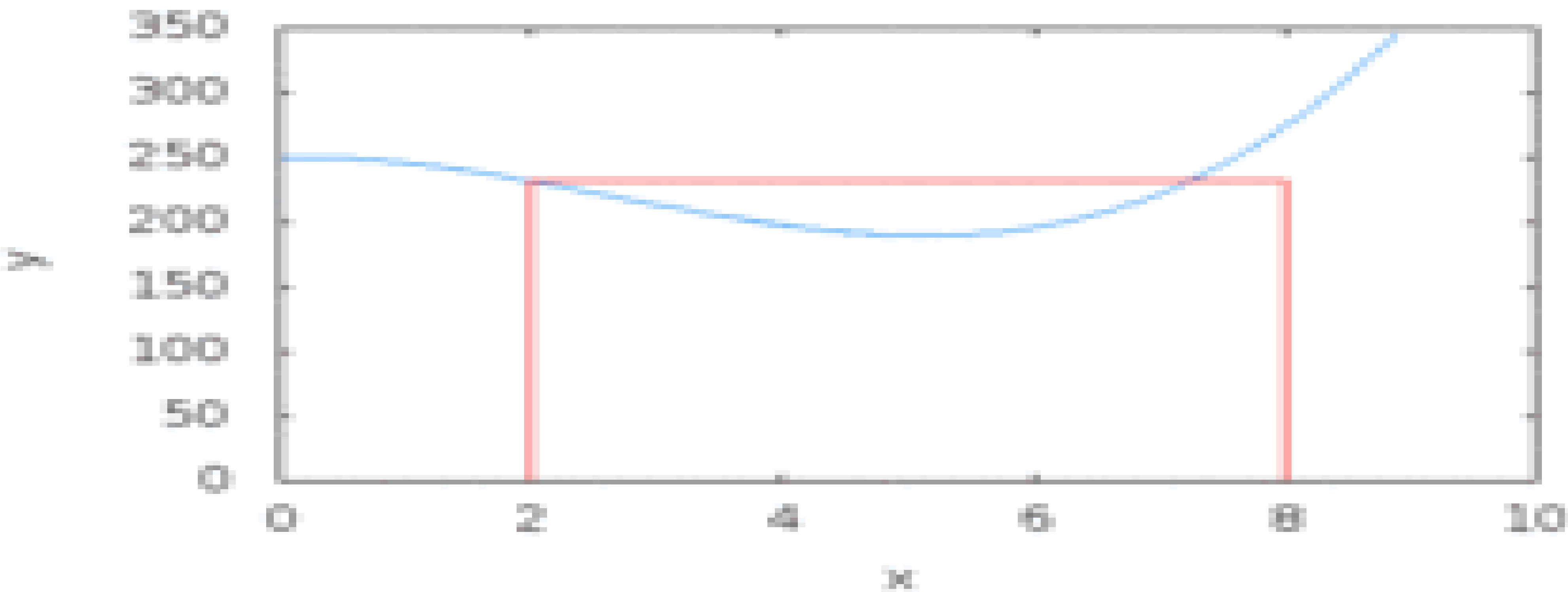
$$A = \sum_{i=1}^{M} f(x_i)\Delta x_i \approx I$$

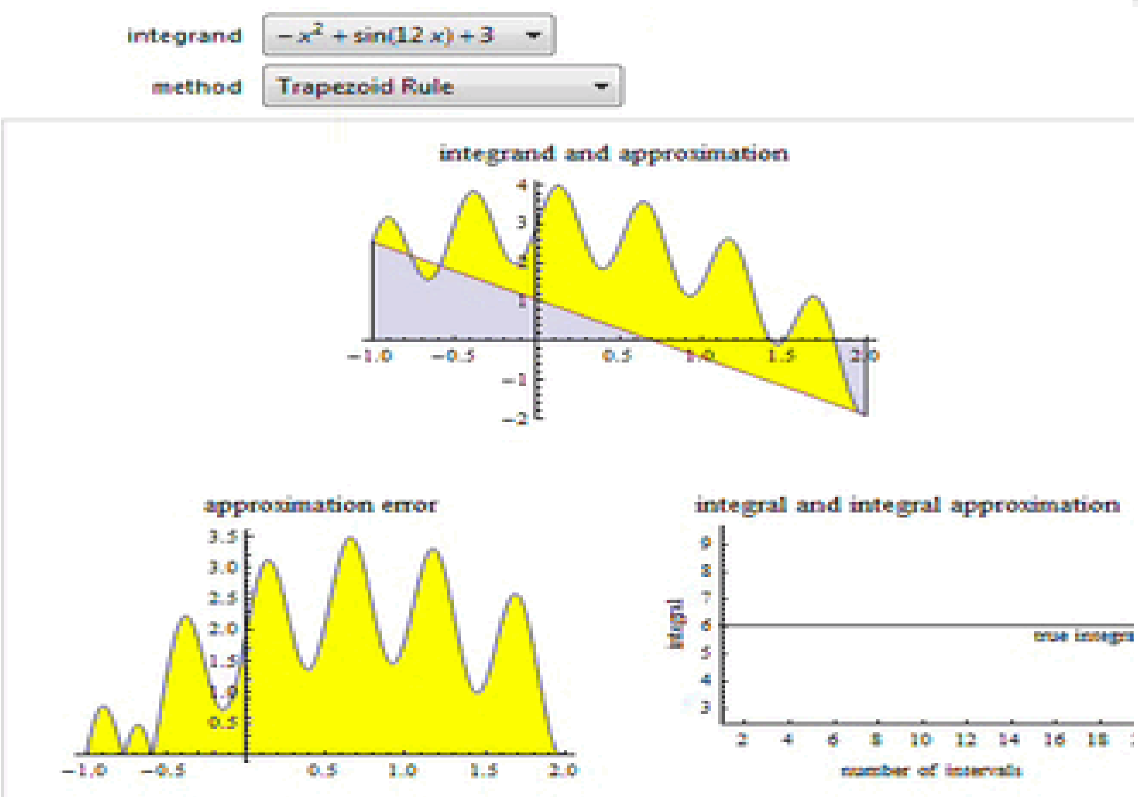

method
left sum
right sum
midpoint
trapezoid
Simpson
$\int_0^2 x^2 dx$
approximate area = 2.50000 exact area = 2.66667
|error| = 1.67×10^{-1}

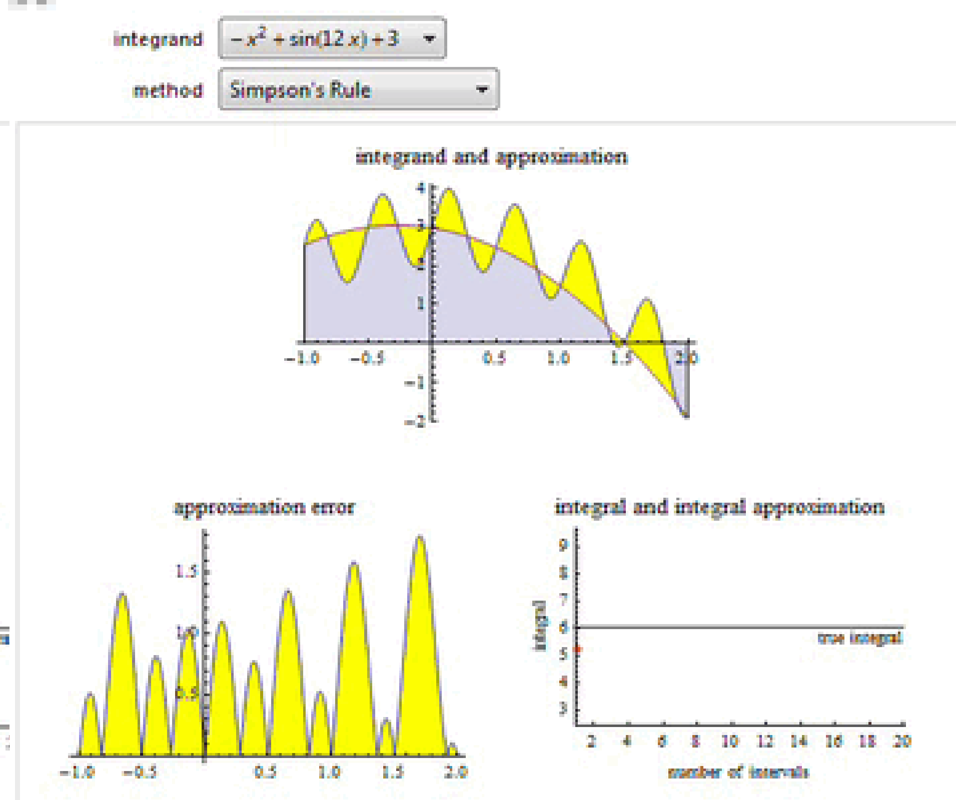

method
left sum
right sum
midpoint
trapezoid
Simpson
$\int_0^2 x^2 dx$
approximate area = 3.00000 exact area = 2.66667
|error| = 3.33×10^{-1}


method
left sum
right sum
midpoint
trapezoid
Simpson
$\int_0^2 x^2 dx$
approximate area = 2.66667 exact area = 2.66667
|error| = 0.

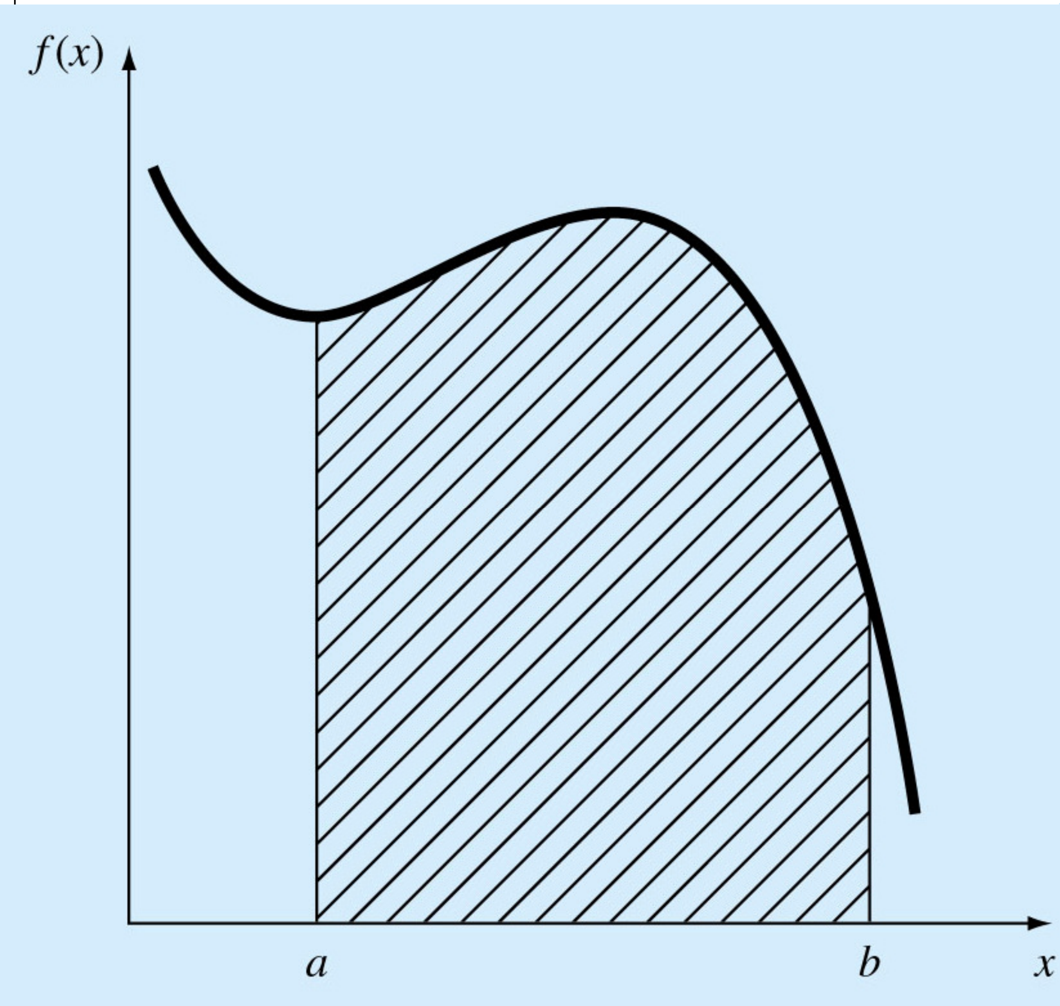

method
Rectangles
Trapezoidal Rule
Simpson's Rule
$y = \sin(x)$
Exact value: $\int_1^3 f(x)\,dx = 1.53029$
Approximate value: $\int_1^3 f(x)\,dx \approx 1.59597$

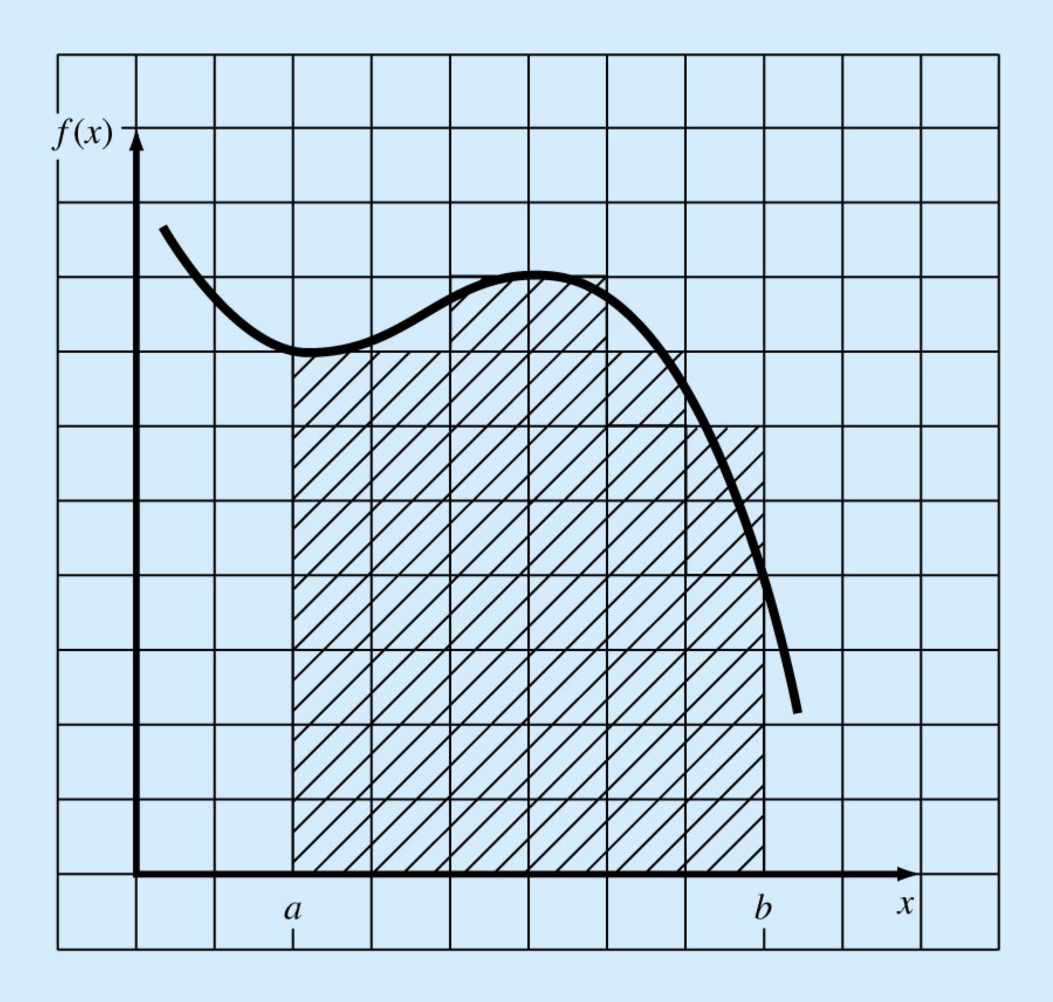

y
x
0
50
100
150
200
250
300
350
2
4
6
8
10


integrand
$-x^2 + \sin(12\,x) + 3$
method
Trapezoid Rule
integrand and approximation
approximation error
integral and integral approximation
integral
number of intervals


integrand
$-x^2 + \sin(12\,x) + 3$
method
Simpson's Rule
integrand and approximation
approximation error
integral and integral approximation
integral
true integral
number of intervals

Graphical Representation of Integral

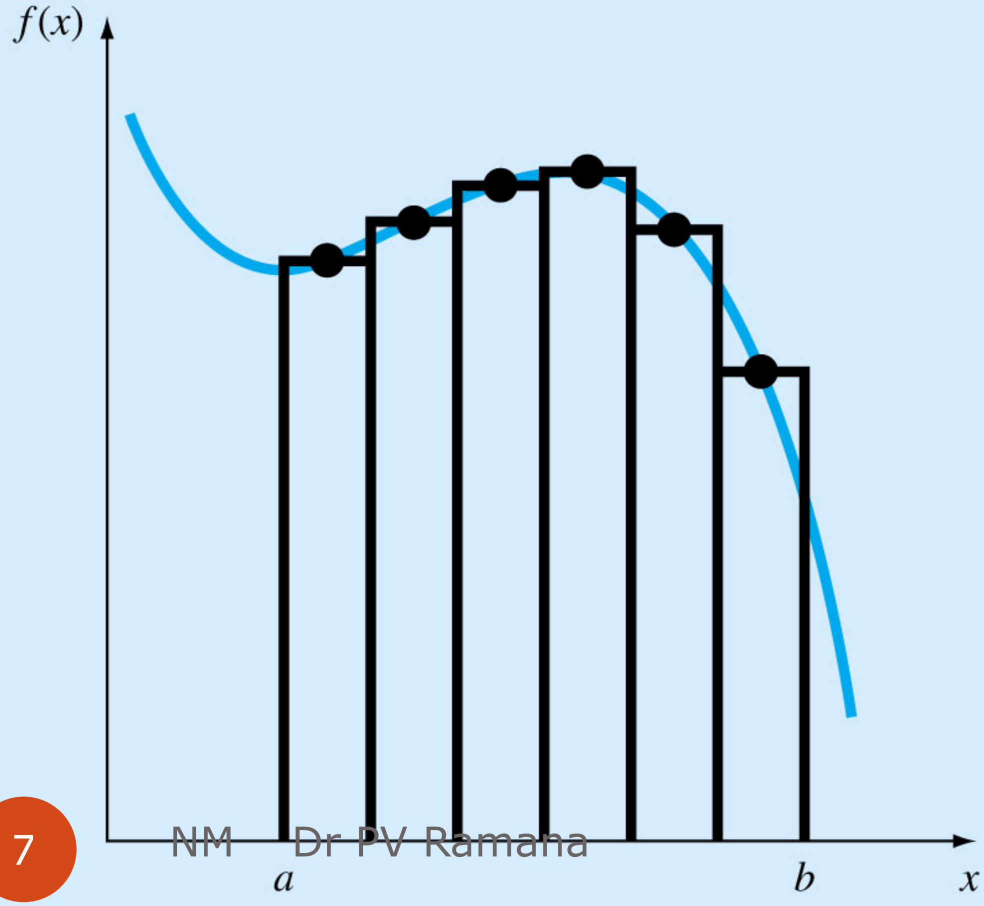




Integral = area under the curve

Use of a grid to approximate an integral

Use of strips to approximate an integral

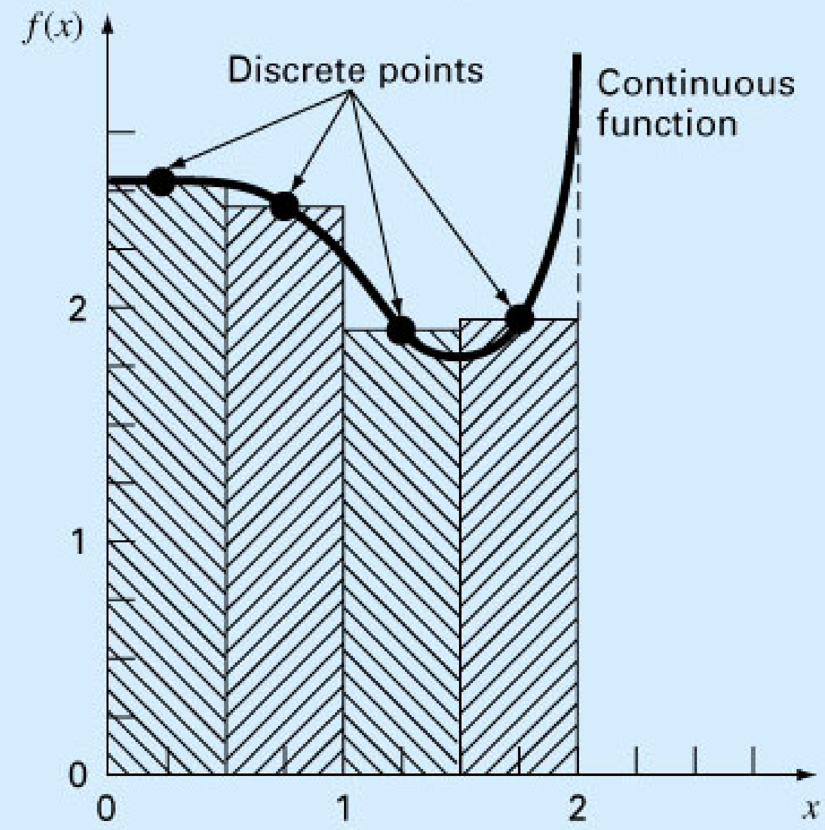


(a) $$\int_0^2 \frac{2 + \cos(1 + x^{3/2})}{\sqrt{1 + 0.5 \sin x}} e^{0.5x}\, dx$$

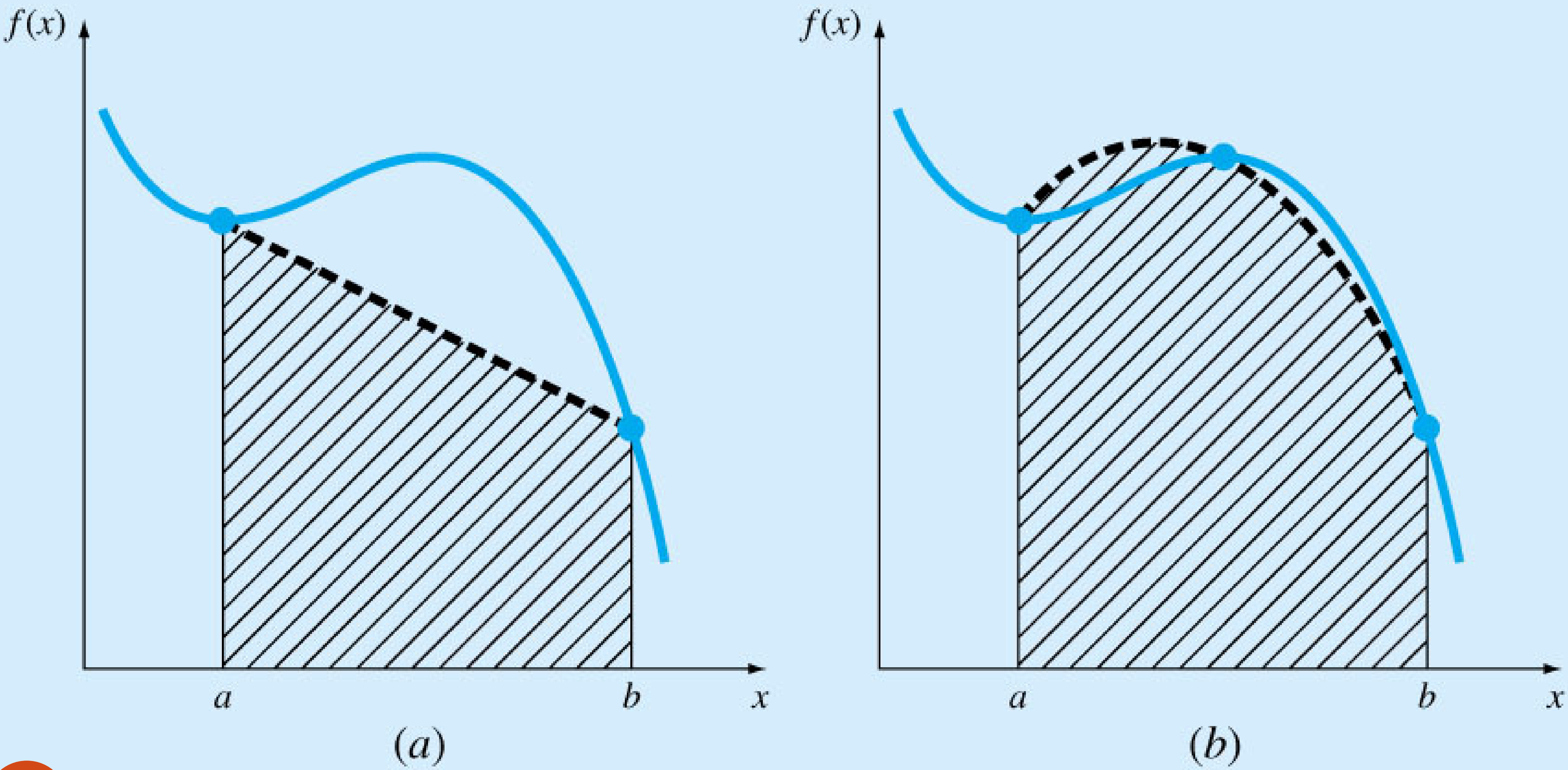
(b)

x	$f(x)$
0.25	2.599
0.75	2.414
1.25	1.945
1.75	1.993

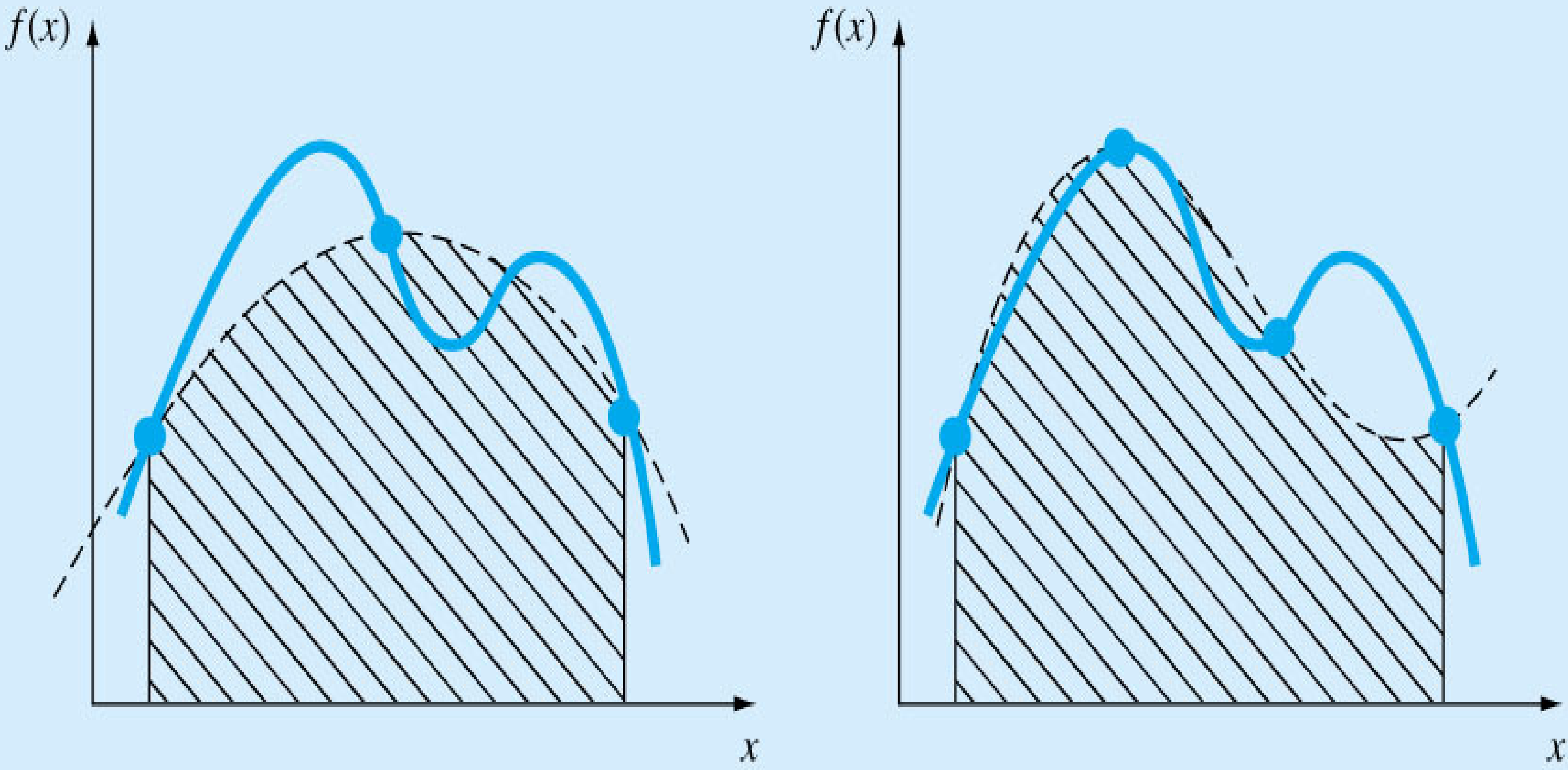
(c)



- $f_n(x)$ can be linear
- $f_n(x)$ can be quadratic

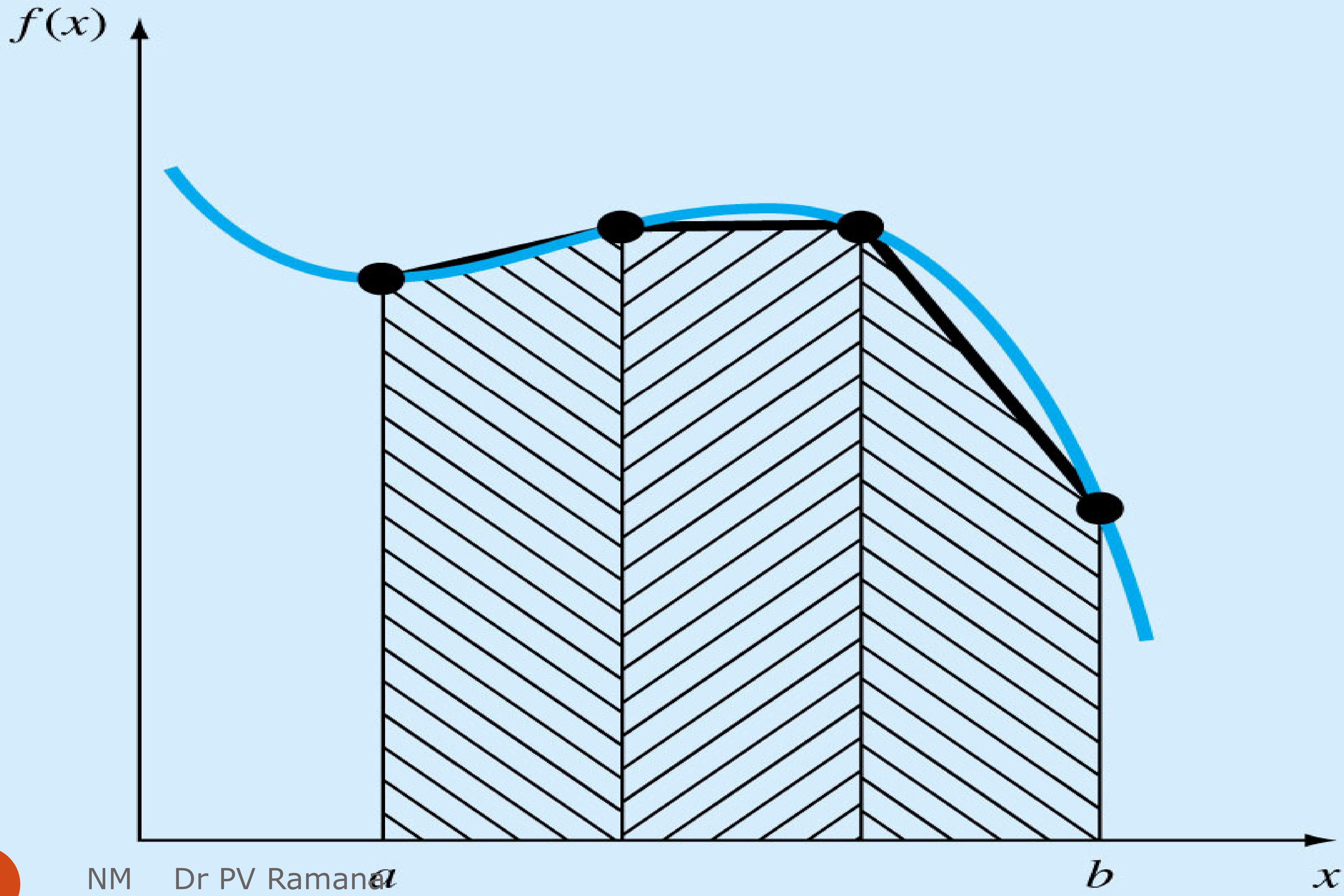


- **$f_n(x)$ can also be cubic or other higher-order polynomials**



(a) (b)

➢ Polynomial can be piecewise over the data

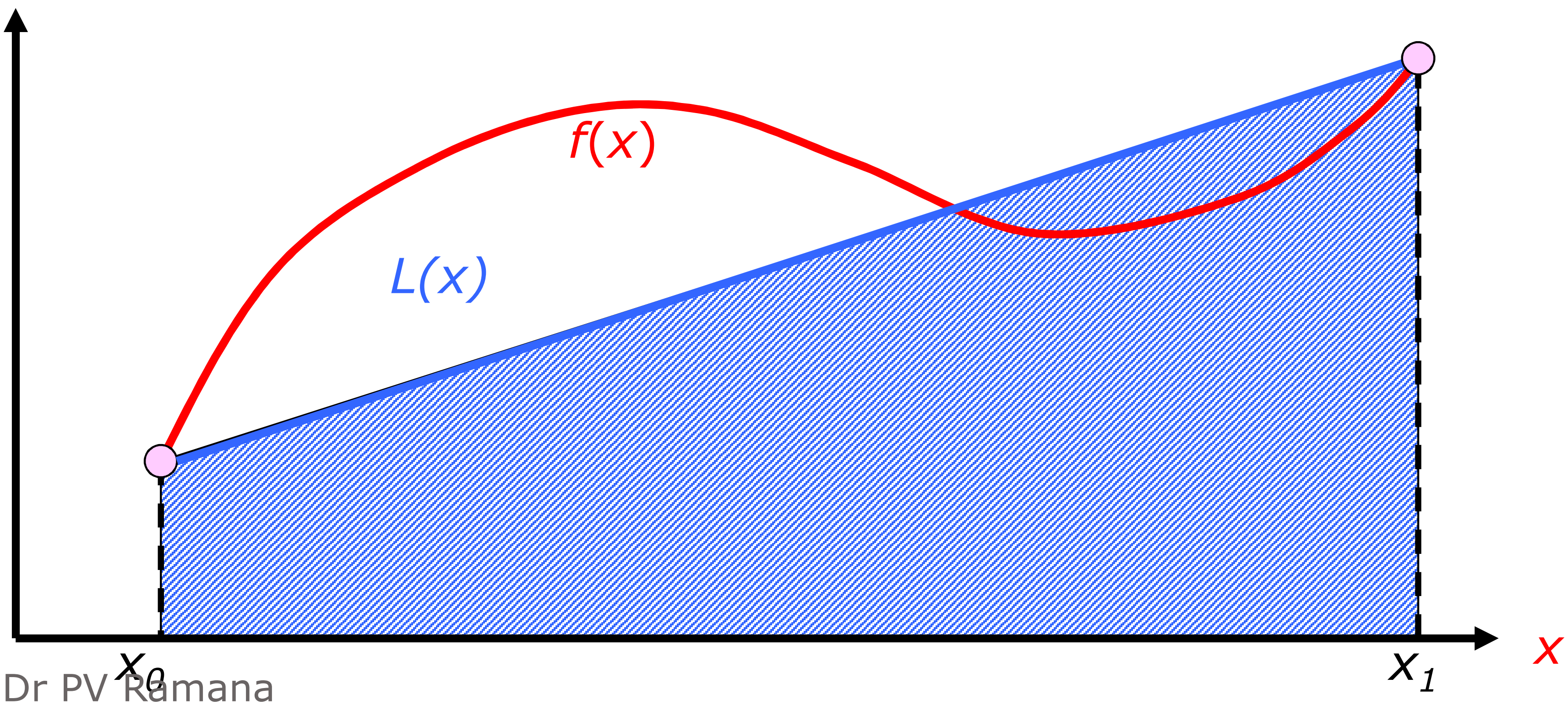


Trapezoidal Rule

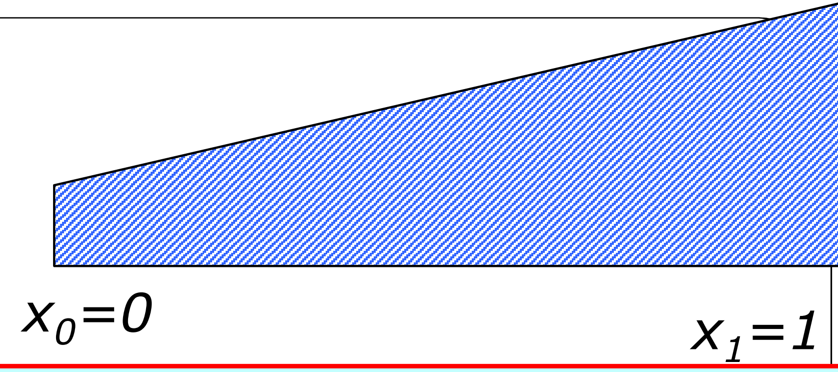
- **Straight-line approximation**

$$\int_a^b f(x)dx \approx \sum_{i=0}^{1} c_i f(x_i) = c_0 f(x_0) + c_1 f(x_1)$$

$$= \frac{h}{2}\left[f(x_0) + f(x_1)\right]$$



Trapezoidal Rule



- **Lagrange interpolation**

$$L(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$let \quad a = x_0,\ b = x_1,\ \xi = \frac{x - a}{b - a},\ d\xi = \frac{dx}{h};\ h = b - a$$

$$\left\{\begin{array}{lll} x = a & \Rightarrow & \xi = 0 \\ x = b & \Rightarrow & \xi = 1 \end{array}\right\} \Rightarrow L(\xi) = (1 - \xi) f(a) + (\xi) f(b)$$

$$\int_a^b f(x)dx \approx \int_a^b L(x)dx = h\int_0^1 L(\xi)d\xi$$

$$= f(a)h\int_0^1 (1 - \xi)d\xi + f(b)h\int_0^1 \xi d\xi$$

$$= f(a)h(\xi - \frac{\xi^2}{2})\Big|_0^1 + f(b)h\frac{\xi^2}{2}\Big|_0^1 = \frac{h}{2}\left[f(a) + f(b)\right]$$

Example:Trapezoidal Rule

- **Evaluate the integral** $\int_0^4 xe^{2x}dx$
- **Exact solution**

$$\int_0^4 xe^{2x}dx = \left[\frac{x}{2}e^{2x} - \frac{1}{4}e^{2x}\right]_0^4$$

$$= \frac{1}{4}e^{2x}(2x-1)\Big|_0^1 = 5216.926477$$

- **Trapezoidal Rule** $\frac{h}{2}[f(x_0) + f(x_1)]$

$$I = \int_0^4 xe^{2x}dx \approx \frac{4-0}{2}[f(0)+f(4)] = 2(0+4e^8) = 23847.66$$

$$\varepsilon = \frac{5216.926 - 23847.66}{5216.926} = -357.12\%$$

Introduction to Numerical Integration

- Definitions
- Upper and Lower Sums
- Trapezoid Method (Newton-Cotes Methods)
- Romberg Method
- Gauss Quadrature
- Examples

Integration

Indefinite Integrals

$$\int x \, dx = \frac{x^2}{2} + c$$

Indefinite Integrals of a function are <u>functions</u> that differ from each other by a constant.

Definite Integrals

$$\int_0^1 x dx = \frac{x^2}{2}\Bigg|_0^1 = \frac{1}{2}$$

Definite Integrals are <u>numbers.</u>

Fundamental Theorem of Calculus

If f is continuous on an interval $[a,b]$,

F is antiderivative of f (i.e., $F'(x) = f(x)$)

$$\boxed{\int_a^b f(x)dx = F(b) - F(a)}$$

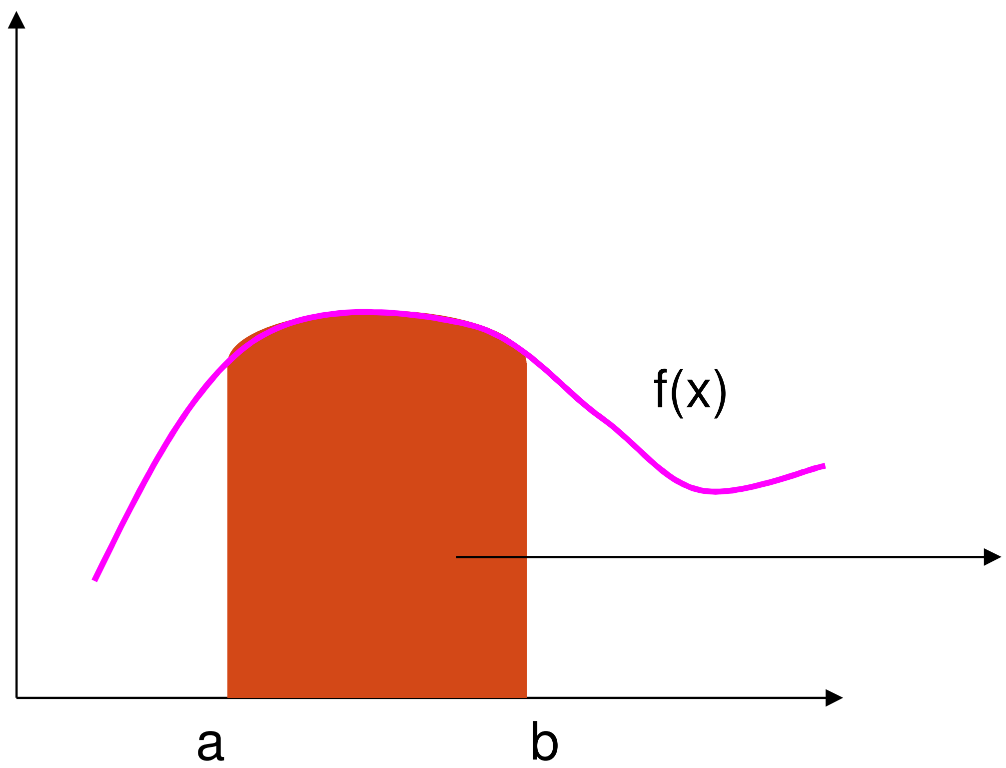
There is no antiderivative for: e^{x^2}

No closed form solution for: $\int_a^b e^{x^2} dx$

The Area Under the Curve

One interpretation of the definite integral is:

<u>Integral = area under the curve</u>



$$Area = \int_a^b f(x)dx$$

Upper and Lower Sums

The interval is divided into subintervals.

$$Partition\ P = \{a = x_0 \le x_1 \le x_2 \le \ldots \le x_n = b\}$$

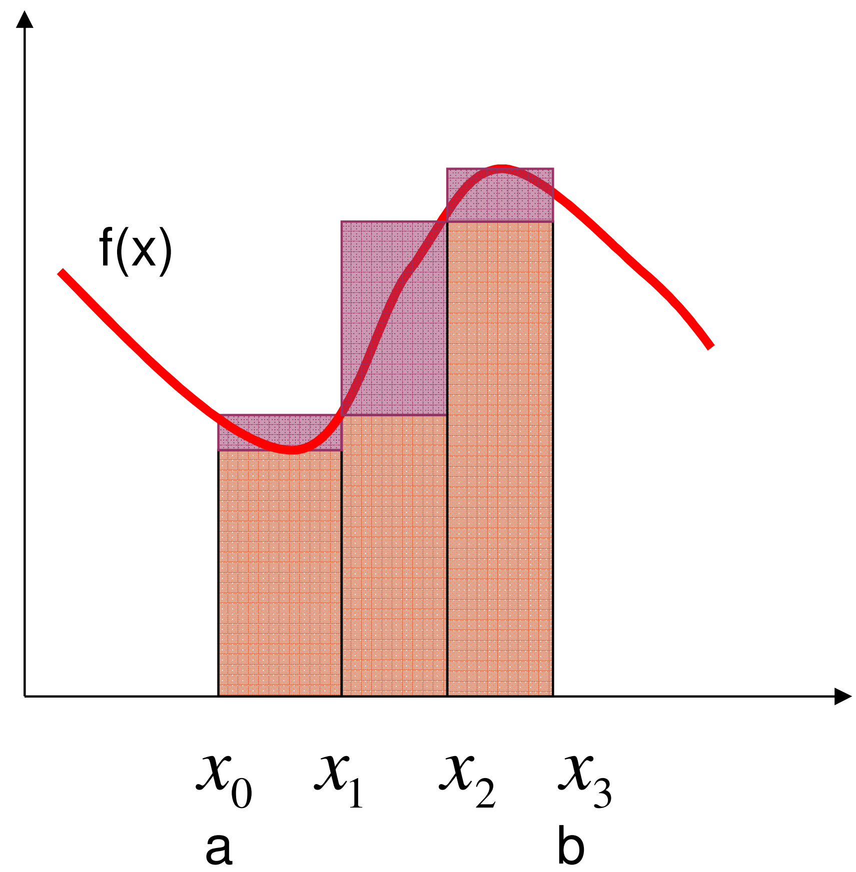
$$Define$$

$$m_i = \min\{f(x) : x_i \le x \le x_{i+1}\}$$

$$M_i = \max\{f(x) : x_i \le x \le x_{i+1}\}$$

$$Lower\ sum \quad L(f,P) = \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i)$$

$$Upper\ sum \quad U(f,P) = \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i)$$



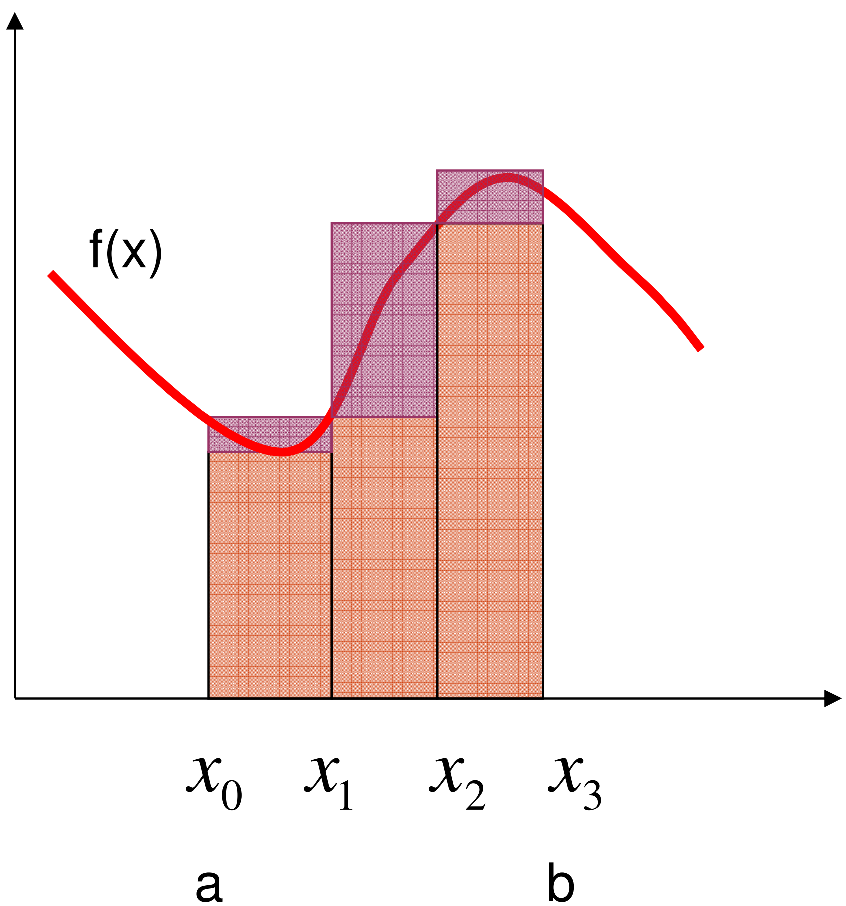
Upper and Lower Sums

Lower sum $\quad L(f,P)=\sum_{i=0}^{n-1} m_i\left(x_{i+1}-x_i\right)$

Upper sum $\quad U(f,P)=\sum_{i=0}^{n-1} M_i\left(x_{i+1}-x_i\right)$

$$\text{Estimate of the integral} = \frac{L+U}{2}$$

$$Error \le \frac{U-L}{2}$$



Example

$$\int_0^1 x^2 \, dx$$

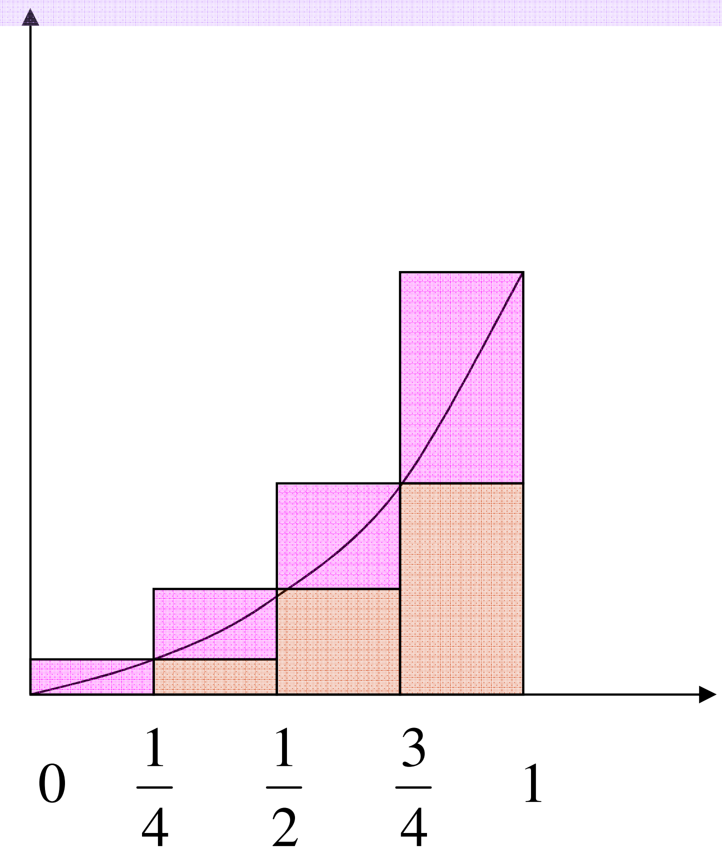
$$Partition: \; P = \left\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\right\}$$

$$n = 4 \text{ (four equal intervals)}$$

$$m_0 = 0, \quad m_1 = \frac{1}{16}, \quad m_2 = \frac{1}{4}, \quad m_3 = \frac{9}{16}$$

$$M_0 = \frac{1}{16}, \quad M_1 = \frac{1}{4}, \quad M_2 = \frac{9}{16}, \quad M_3 = 1$$

$$x_{i+1} - x_i = \frac{1}{4} \quad for \quad i = 0, 1, 2, 3$$



Example

$$\int_0^1 x^2\,dx \qquad Partition\ P=\left\{0,\frac{1}{4},\frac{2}{4},\frac{3}{4},1\right\}$$

$$m_0=0,\qquad m_1=\frac{1}{16},\qquad m_2=\frac{1}{4},\qquad m_3=\frac{9}{16}$$

$$M_0=\frac{1}{16},\quad M_1=\frac{1}{4},\quad M_2=\frac{9}{16},\quad M_3=1$$

$$Lower\ \ sum \qquad L(f,P)=\sum_{i=0}^{n-1} m_i\left(x_{i+1}-x_i\right)$$

$$L(f,P)=\frac{1}{4}\left[0+\frac{1}{16}+\frac{1}{4}+\frac{9}{16}\right]=\frac{14}{64}$$

$$Upper\ sum \qquad U(f,P)=\sum_{i=0}^{n-1} M_i\left(x_{i+1}-x_i\right)$$

$$U(f,P)=\frac{1}{4}\left[\frac{1}{16}+\frac{1}{4}+\frac{9}{16}+1\right]=\frac{30}{64}$$

$$\text{Estimate of the integral}=\frac{1}{2}\left(\frac{30}{64}+\frac{14}{64}\right)=\frac{11}{32}$$

$$Error<\frac{1}{2}\left(\frac{30}{64}-\frac{14}{64}\right)=\frac{1}{8}$$