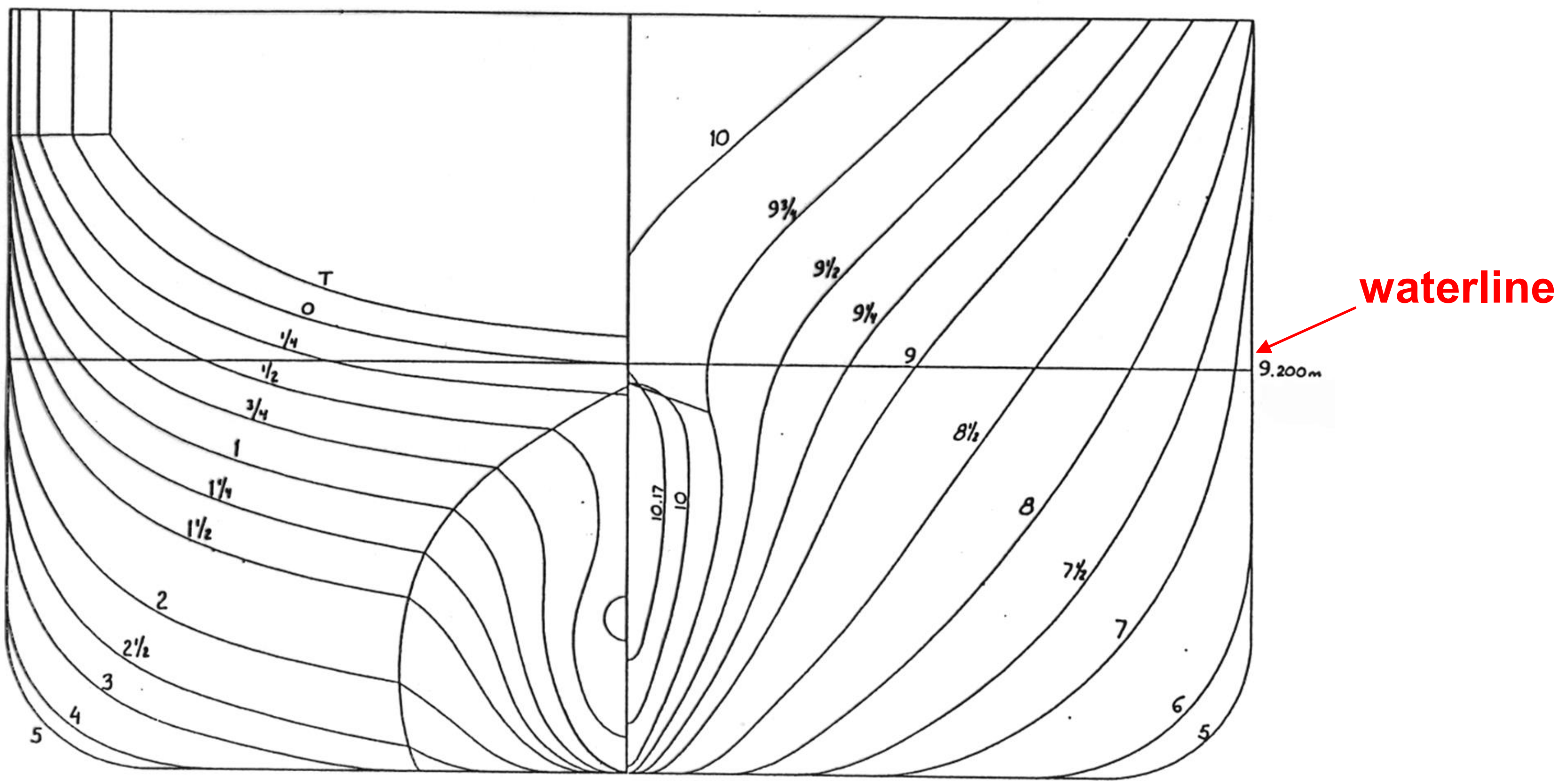
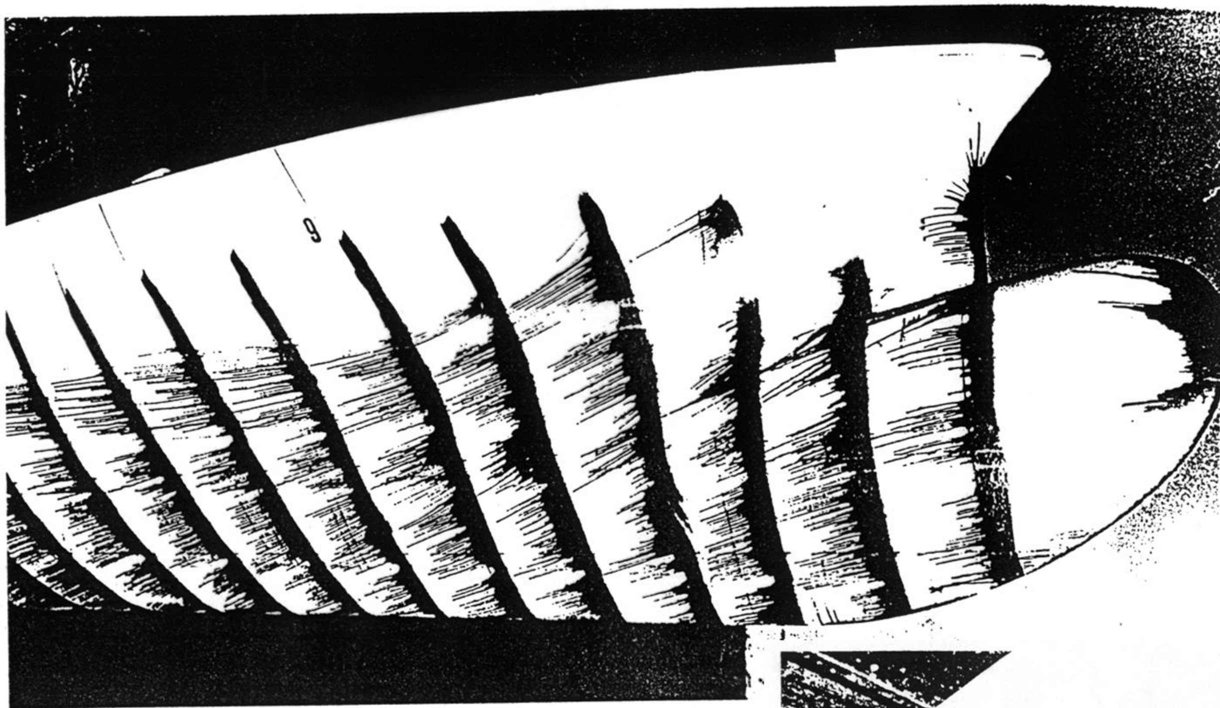
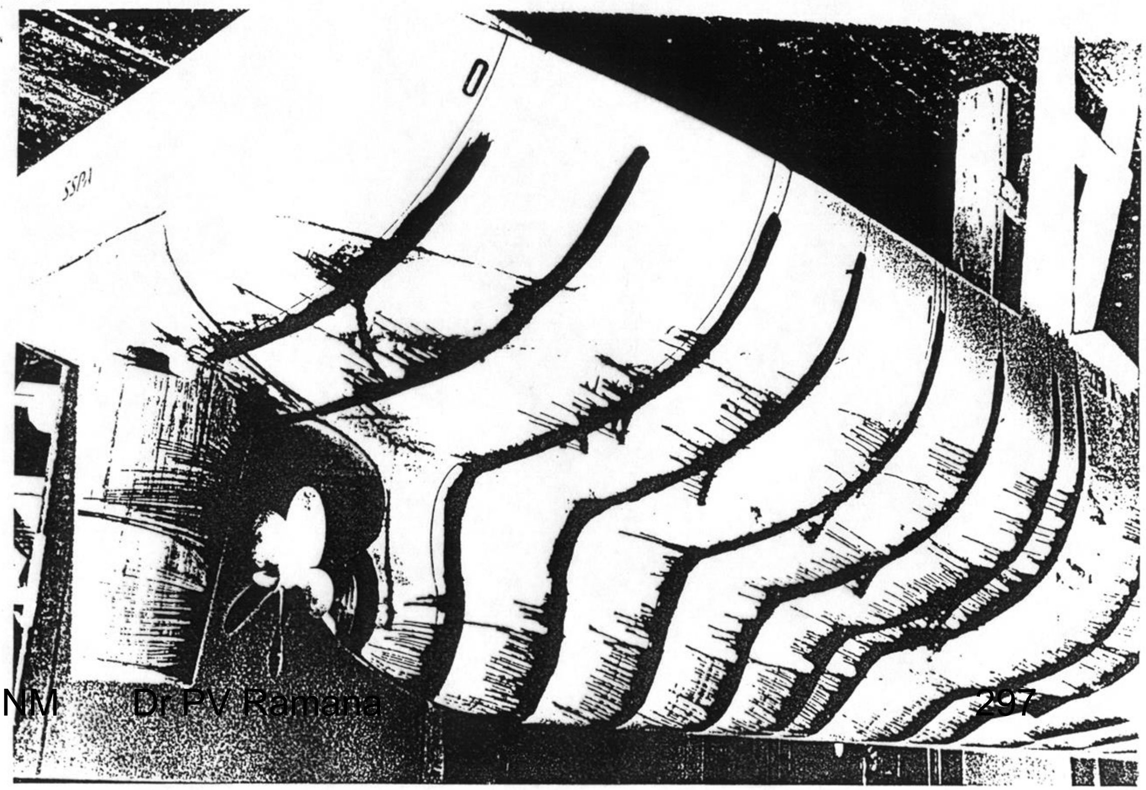
Example: Ship Lines

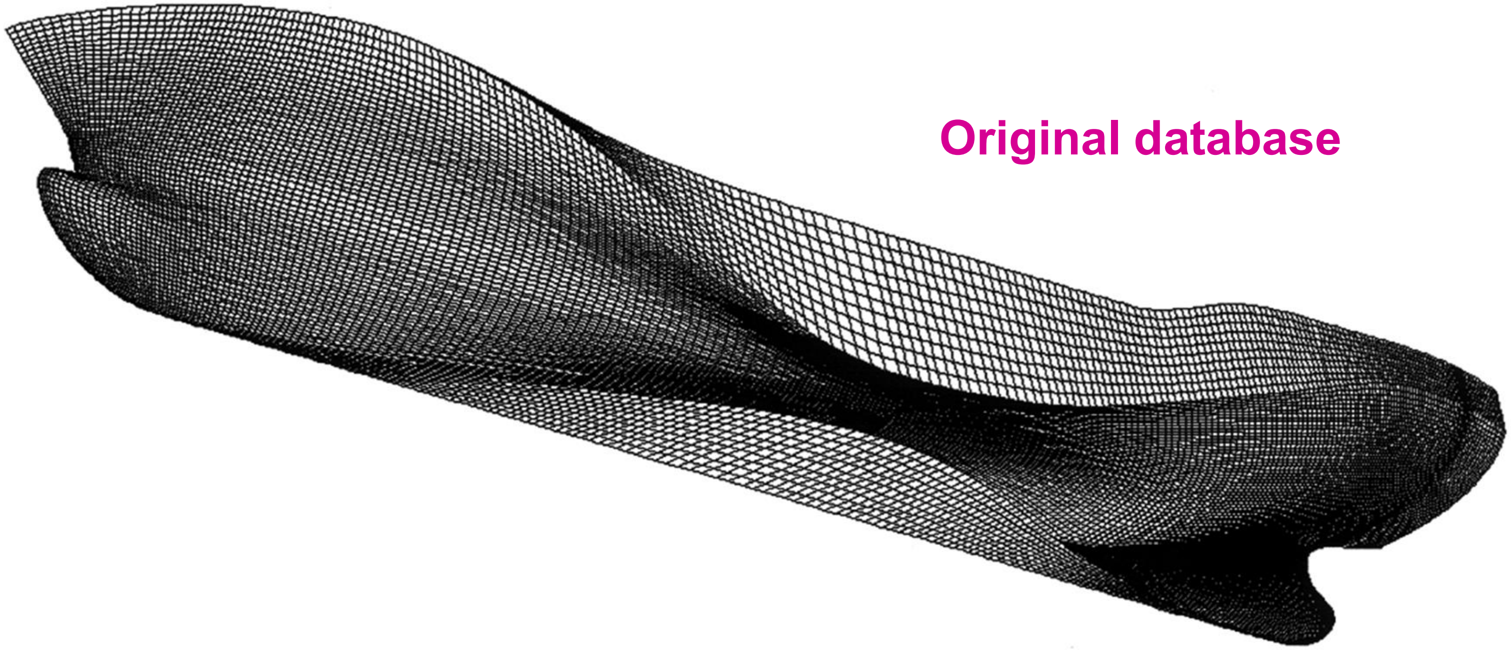




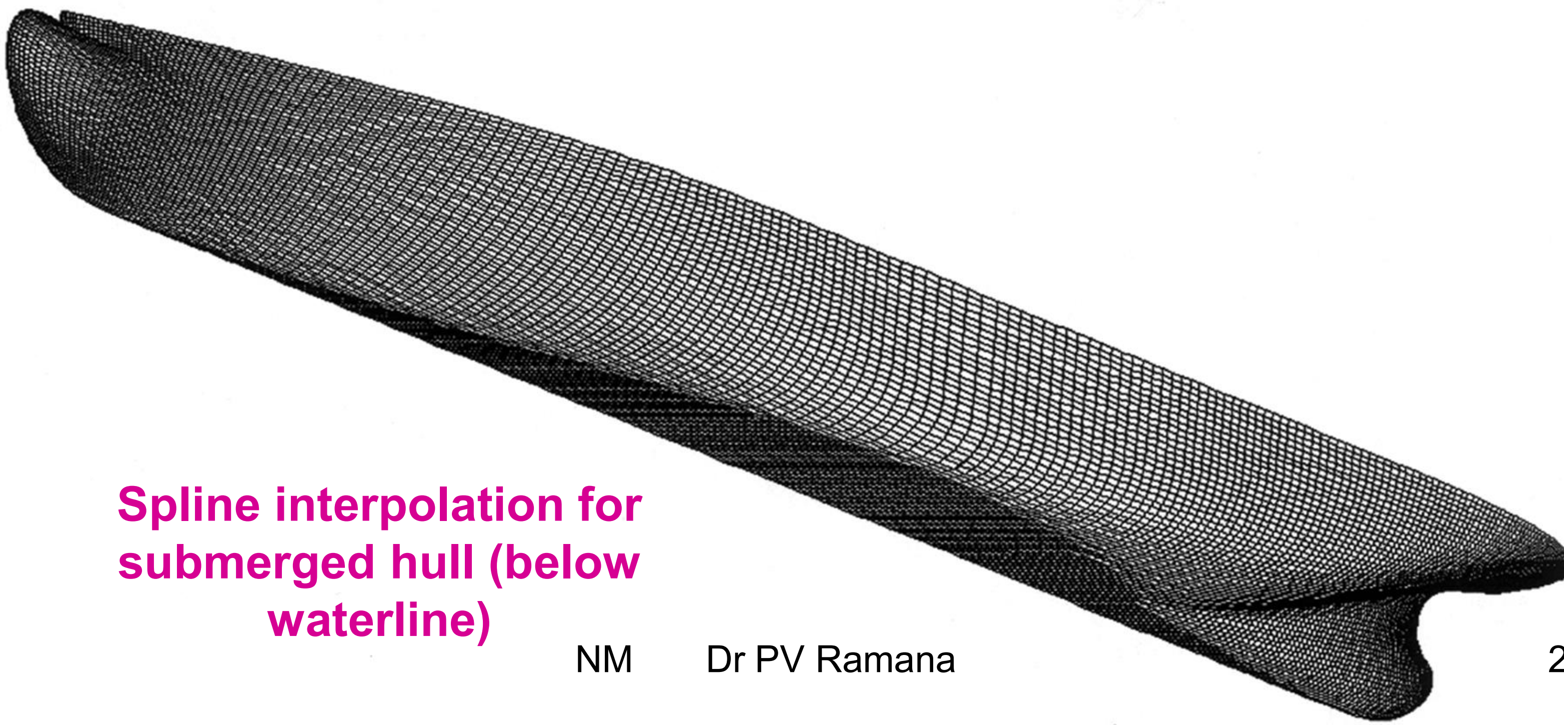
Bow



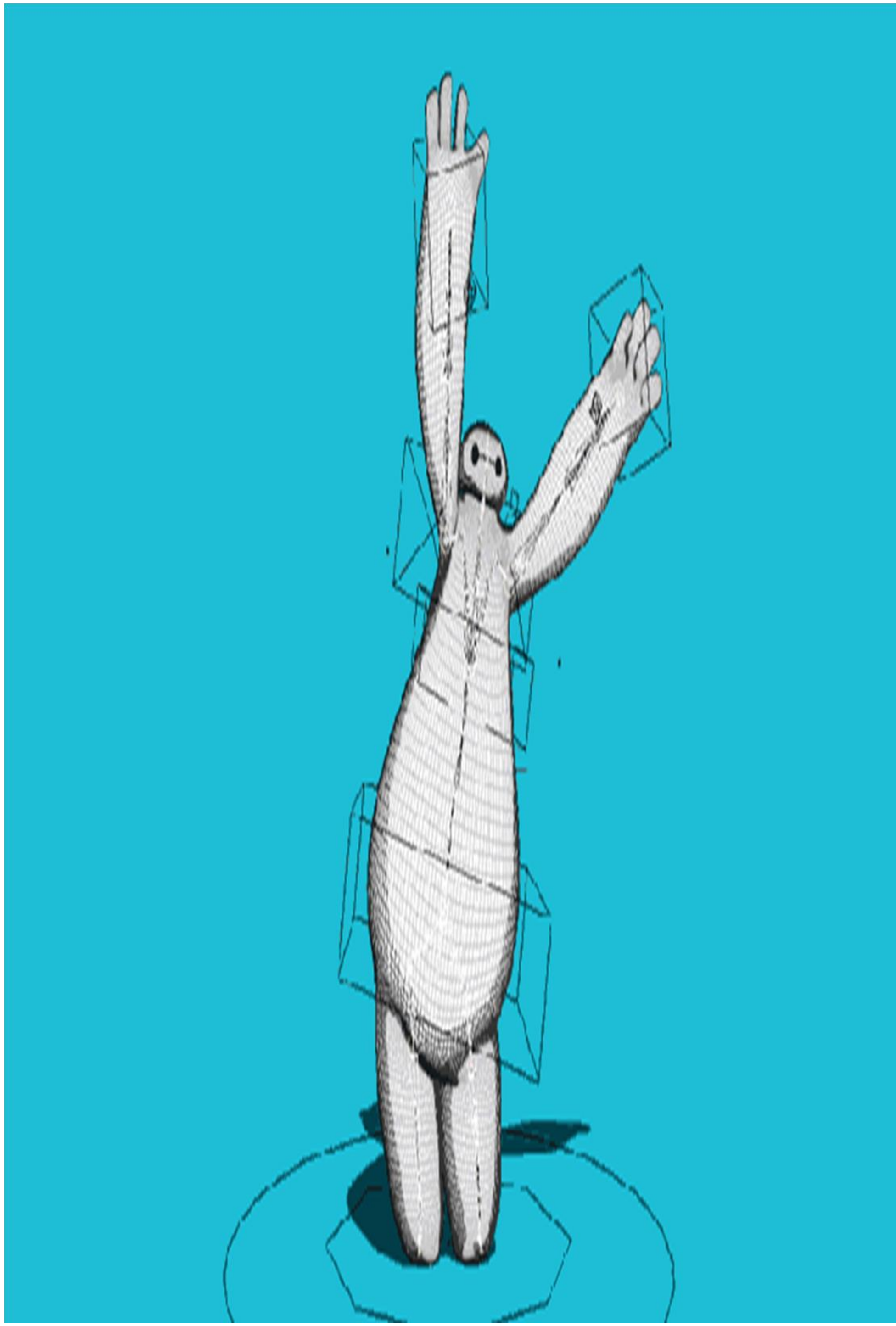
Stern



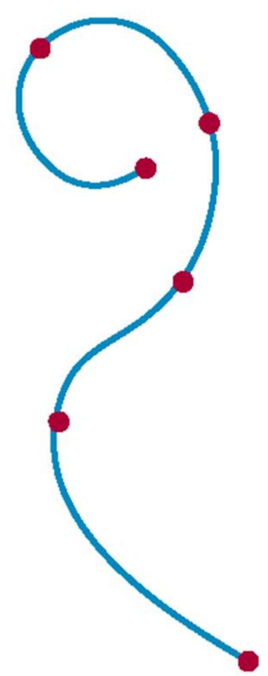
Original database



Spline interpolation for submerged hull (below waterline)



Interpolation and Approximation



A set of six control points approximated
with piecewise continuous polynomial
sections.

A set of six control points interpolated
with piecewise continuous polynomial sections.

Continuity Conditions

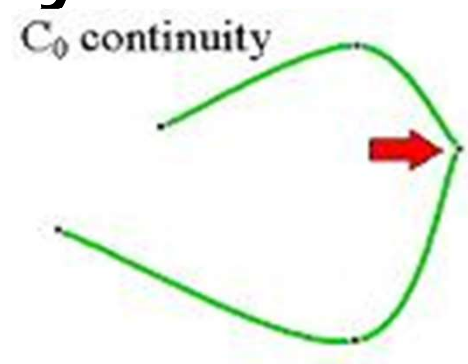
- **To ensure a smooth transition from one section of a piecewise parametric spline to the next, can impose various continuity conditions at the connection points**

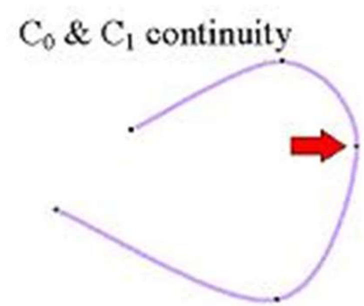
- **Parametric continuity**
  - **Matching the parametric derivatives of adjoining curve sections at their common boundary**

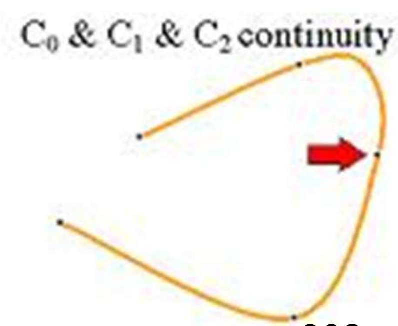
- **Geometric continuity**
  - **Geometric smoothness independent of parametrization**
  - **parametric continuity is sufficient, but not necessary, for geometric smoothness**

Parametric Continuity

- **Zero-order parametric continuity**
  - C^0 **-continuity**
  - **Means simply that the curves meet**
- **First-order parametric continuity**
  - C^1 **-continuity**
  - **The first derivatives of two adjoining curve functions are equal**
- **Second-order parametric continuity**
  - C^2 **-continuity**
  - **Both the first and the second derivatives of two adjoining curve functions are equal**

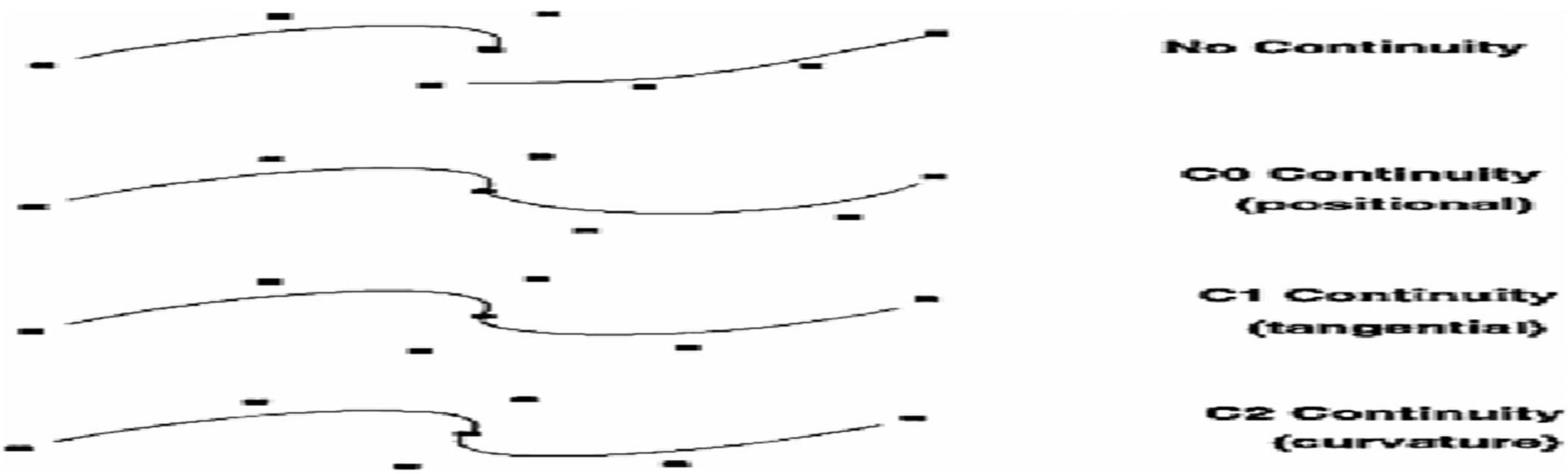




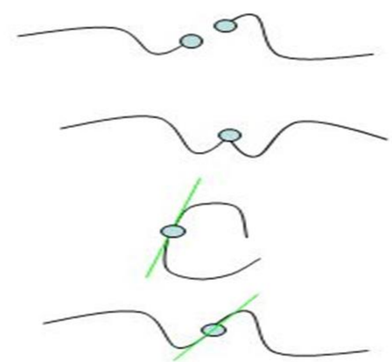


Geometric Continuity

- **Zero-order geometric continuity**
  - **Equivalent to G^0-continuity**
- **First-order geometric continuity**
  - G^1 **- continuity**
  - **The tangent directions at the ends of two adjoining curves are equal, but their magnitudes can be different**
- **Second-order geometric continuity**
  - G^2**-continuity**
  - **Both the tangent directions and curvatures at the ends of two adjoining curves are equal**



Continuity at Join Points



- Discontinuous: physical separation
- Parametric Continuity
  - Positional (C^0): no physical separation
  - C^1 : C^0 and matching first derivatives
  - C^2 : C^1 and matching second derivatives
- Geometric Continuity
  - Positional (G^0) $= C^0$
  - Tangential (G^1) : G^0 and tangents are proportional, point in same direction, but magnitudes may differ
  - Curvature (G^2) : G^1 and tangent lengths are the same and rate of length change is the same

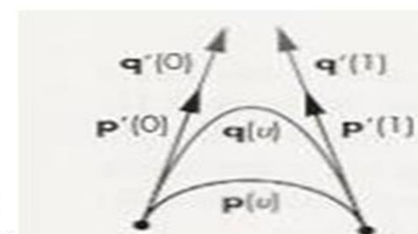


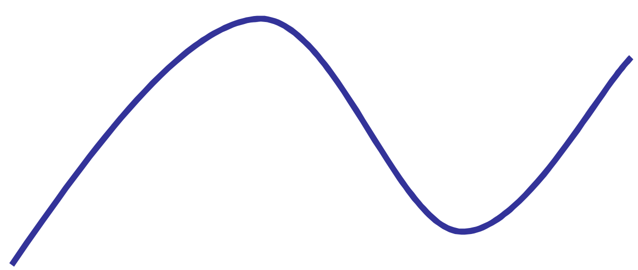
Figure 9.15 Change c
magnitude in G^1 con

Basis Functions

- A linear space of cubic polynomials
  - Monomial basis (t^3, t^2, t^1, t^0)

$$x(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0$$

  - The coefficients a_i do not give tangible geometric meaning

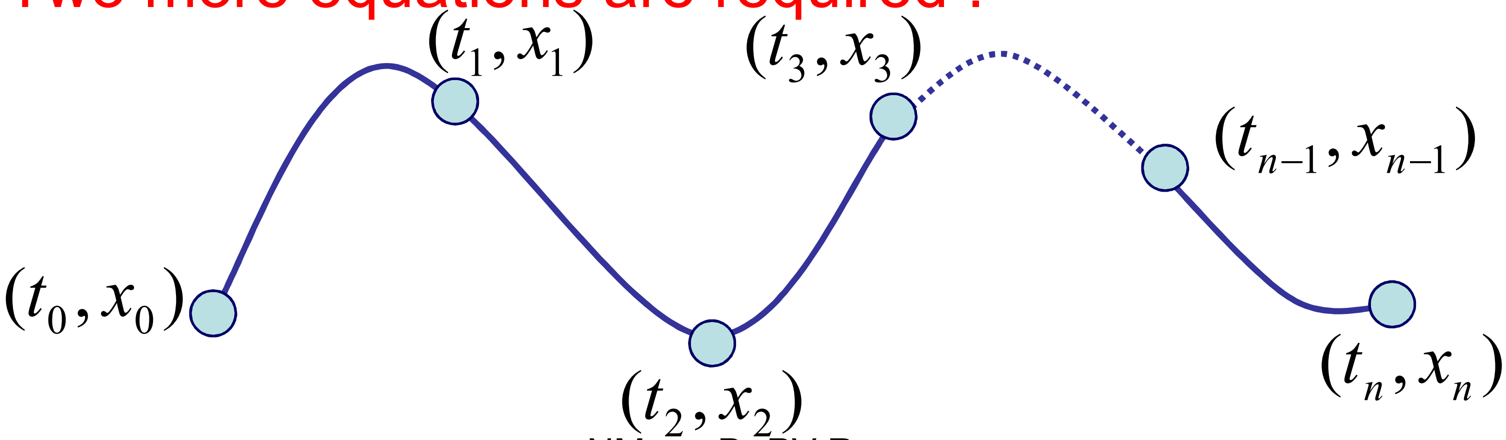


Natural Cubic Splines

- **Is it possible to achieve higher continuity ?**
  - C^{n-1}**-continuity can be achieved from splines of degree n**
- **Motivated by loftman's spline**
  - **Long narrow strip of wood or plastic**
  - **Shaped by lead weights (called ducks)**

Natural Cubic Splines

- **One have *4n* unknowns**
  - ***n* Bezier curve segments (4 control points per each segment)**
- **One have *(4n-2)* equations**
  - ***2n* equations for end point interpolation**
  - **(*n-1*) equations for tangential continuity**
  - **(*n-1*) equations for second derivative continuity**
- Two more equations are required !



Natural Cubic Splines

- **Natural spline boundary condition**

$$x''(t_0) = x''(t_n) = 0$$

- **Closed boundary condition**

$$x'(t_0) = x'(t_n) \quad \text{and} \quad x''(t_0) = x''(t_n)$$

- **High-continuity, but no local controllability**

B-spline Properties

- Convex hull
- Affine invariance
- Variation diminishing
- continuity
- Local controllability

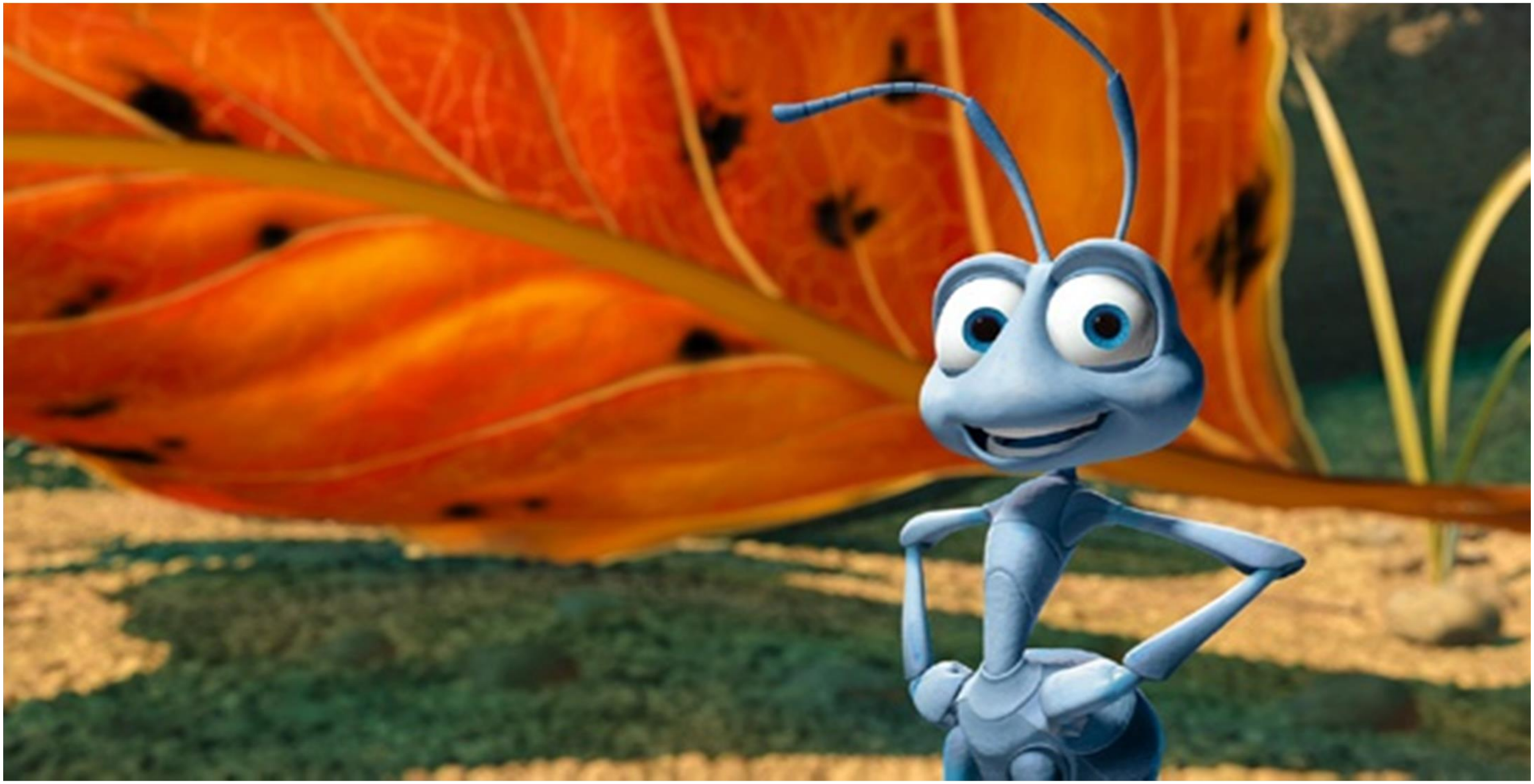
NURBS

$$\mathbf{p}(t) = \frac{\sum_{j=0}^{n} \omega_j \mathbf{b}_j B_j^n(t)}{\sum_{j=0}^{n} \omega_j B_j^n(t)}$$

- Non-uniform Rational B-splines
  - Non-uniform knot spacing
  - Rational polynomial
    - A polynomial divided by a polynomial
    - Can represent conics (circles, ellipses, and hyperbolics)
    - Invariant under projective transformation
- Note
  - Uniform B-spline is a special case of non-uniform B-spline
  - Non-rational B-spline is a special case of rational B-spline

Subdivision in Action

- A Bug’s Life



Subdivision in Action

- Geri's Game

What are basis functions?

- **One need flexible method for constructing a function *f(t)* that can track local curvature.**
- **One pick a system of *K* *basis* functions $\varphi_k(t)$, and call this the *basis* for *f(t)*.**
- **One express *f(t)* as a weighted sum of these basis functions:**

$$f(t) = a_1\varphi_1(t) + a_2\varphi_2(t) + \ldots + a_K\varphi_K(t)$$

The coefficients $a_1, \ldots, a_K$ determine the shape of the function.

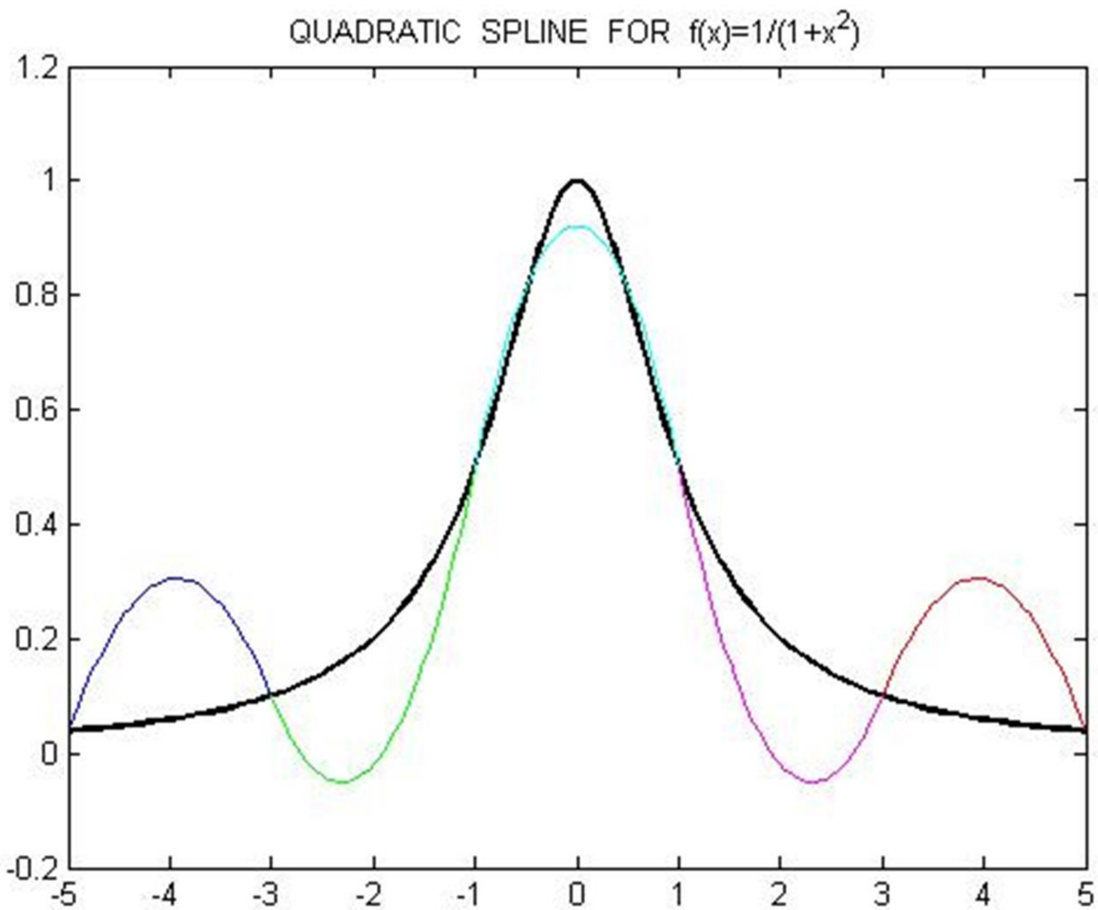
What do want from basis functions?

- Fast computation **of individual basis functions.**
- Flexible**: can exhibit the required curvature where needed, but also be nearly linear when appropriate.**
- Fast computation **of coefficients a_k: possible if matrices of values are diagonal, banded or sparse.**
- Differentiable **as required: make lots of use of derivatives in functional data analysis.**
- Constrained **as required, such as periodicity, positivity, monotonicity, asymptotes and etc.**

What are some commonly used basis functions?

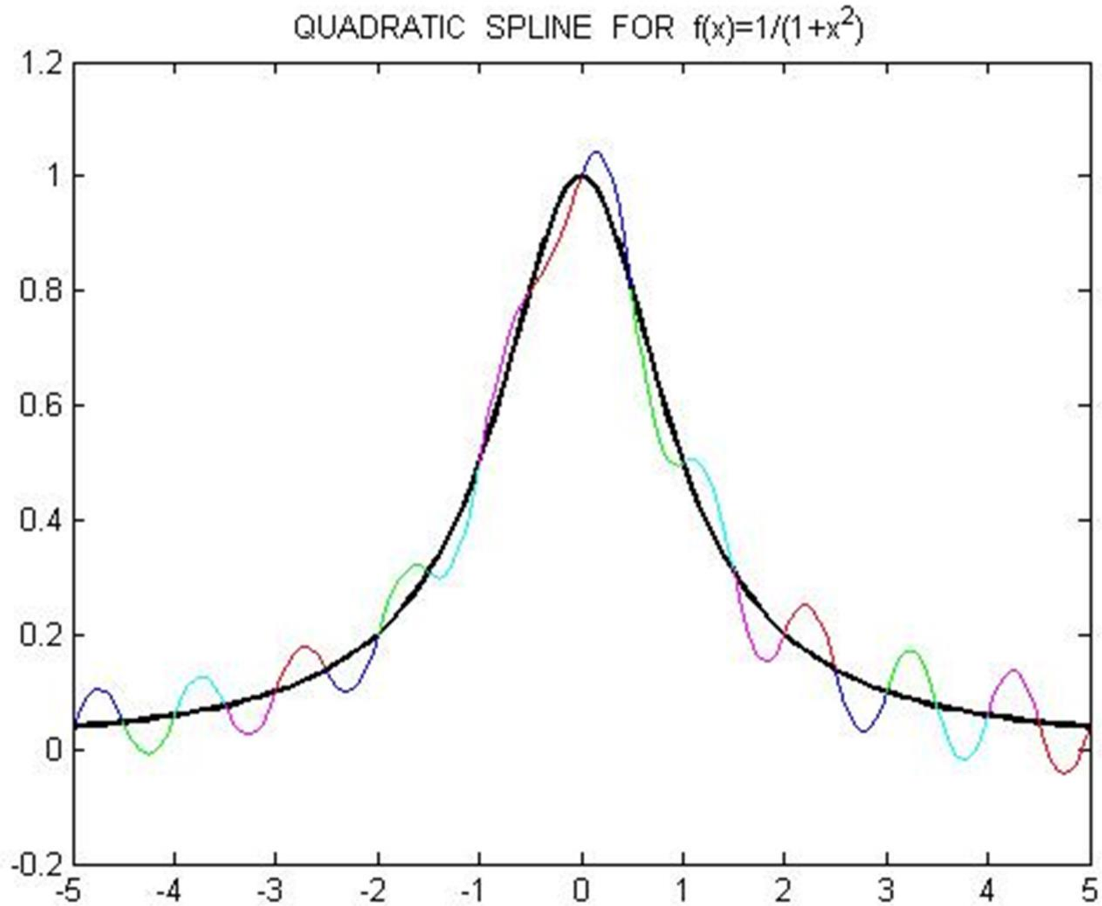
- **Powers:** $1, t, t^2$, **and so on. They are the basis functions for polynomials. These are not very flexible, and are used only for simple problems.**
- **Fourier series:** $1, \sin(\omega t), \cos(\omega t), \sin(2\omega t), \cos(2\omega t)$, **and so on for a fixed known frequency** ω**. These are used for periodic functions.**
- **Spline functions: These have now more or less replaced polynomials for non-periodic problems. Mostly expanded in Engineering field.**

Quadratic Spline Graph



t=a:2:b;

Quadratic Spline Graph



t=a:0.5:b;

Natural Cubic Spline Interpolation

SPLINE OF DEGREE k = 3

- The domain of S is an interval [a,b].
- S, S', S'' are all continuous functions on [a,b].
- There are points t_i (the knots of S) such that $a = t_0 < t_1 < .. t_n = b$ and such that S is a polynomial of degree at most k on each subinterval $[t_i, t_{i+1}]$.

x	t_0	t_1	…	t_n
y	y_0	y_1	…	y_n

t_i are knots

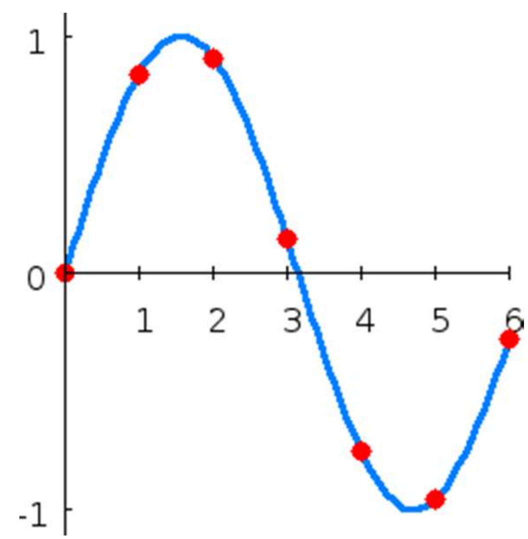
Natural Cubic Spline Interpolation

$$S(x) = \begin{cases} S_0(x), \ x \in [x_0, x_1] \\ S_1(x), \ x \in [x_1, x_2] \\ \cdots \\ S_{n-1}(x), \ x \in [x_{n-1}, x_n] \end{cases}$$

$S_i(x)$ *is a cubic polynomial that will be used on the subinterval* [$\boldsymbol{x_i}$, $\boldsymbol{x_{i+1}}$].

Natural Cubic Spline Interpolation

- $S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$
  - 4 Coefficients with n subintervals = 4n equations
  - There are 4_{n-2} conditions
    - Interpolation conditions
    - Continuity conditions
    - Natural Conditions
  - $S''(x_0) = 0$
  - $S''(x_n) = 0$



Summary

- Polynomial interpolation
  - Lagrange polynomial
- Spline interpolation
  - Piecewise polynomial
  - Knot sequence
  - Continuity across knots
    - Natural spline (C^2-continuity)
    - Catmull-Rom spline (C^1 -continuity)
  - Basis function
    - Bezier
    - B-spline


Thank You!