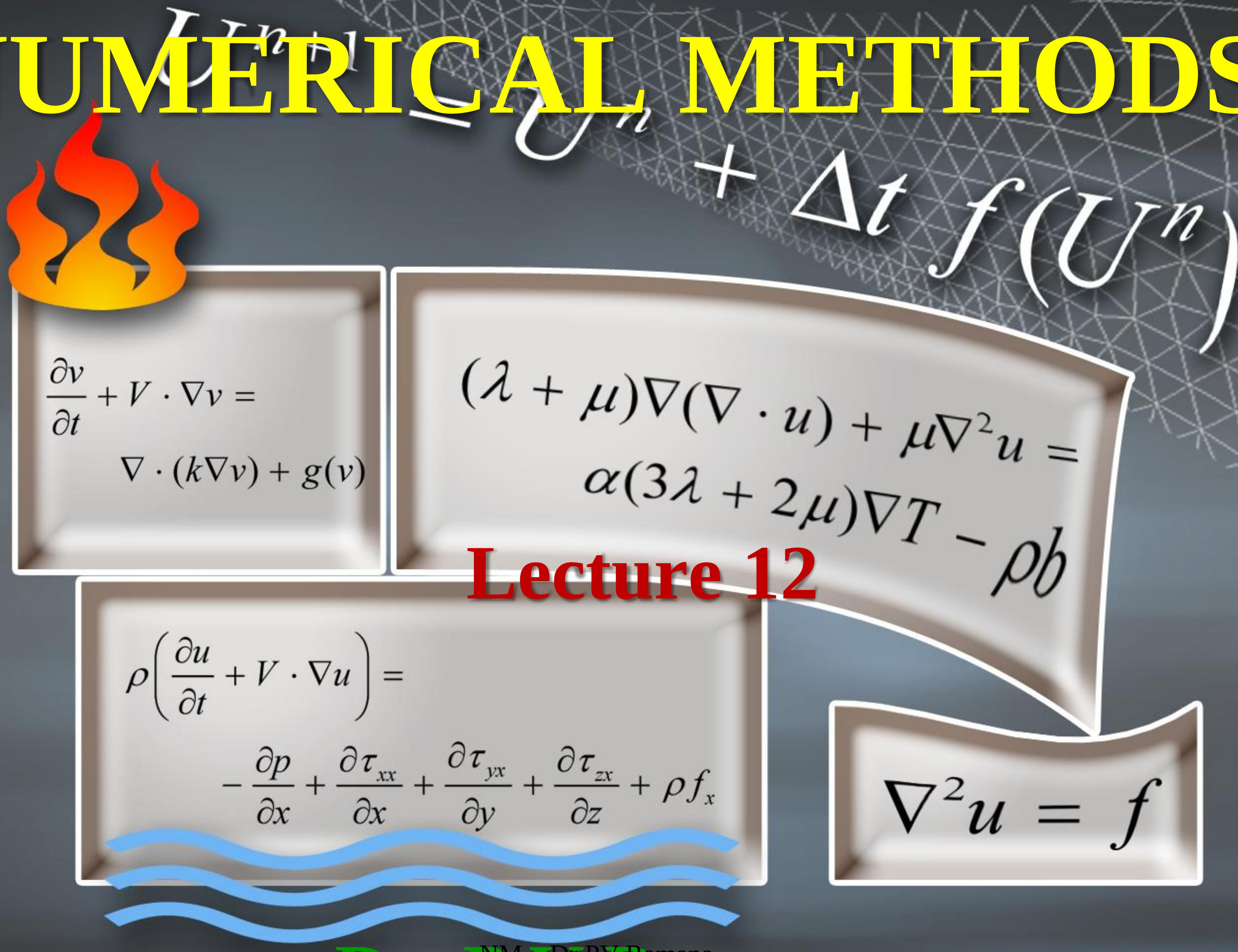
NUMERICAL METHODS



Lecture 12

Dr. P V Ramana

Numerical Methods for BVPs: 2-point BVP

- Three methods
  - **Finite difference**
  - **Finite element**
  - **Shooting**

Finite Difference Methods

- Central difference approximations for higher order derivatives:

$$y_i' = \frac{1}{2h}\left(y_{i+1} - y_{i-1}\right)$$

$$y_i'' = \frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right)$$

$$y_i''' = \frac{1}{2h^3}\left(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right)$$

$$y_i^{iv} = \frac{1}{h^4}\left(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right)$$

Notation

$$y = f(x)$$

$$y_i = f(x = i)$$

$$y_i' = f'(x = i)$$

$$y_i'' = f''(x = i) \quad \textit{and so on} \cdots$$

Implicit vs Explicit

An *explicit* method is obtained where the second order derivative is replaced by a central difference approximation and the first order derivative by a first forward difference approximation.

An *implicit* method is obtained where the second order derivative is replaced by a central difference approximation but using the unknown dependent values at some future time-row (rather than the known values at some present time-row as in the explicit case); the first order derivative is represented by the first backward difference approximation.

The explicit method has related stability criteria, which will restrict the space and time steps that may be used in a simulation. The implicit method is unconditionally stable but of lower accuracy.

Implicit Methods

- Euler's method is "explicit": only uses information a time t_k (or before)
  - Easy to use, but small stability region:

$$h < -2/J$$

- "Implicit" method: uses information at time t_{k+1}

$$y_{k+1} = y_k + f(t_{k+1}, y_{k+1})h_k$$

  - Harder to use, but larger stability region
- For example: "backward Euler" method:

More on Implicit Methods

- **Must solve (nonlinear) equation to determine y_{k+1}**
- **Use Newton's method, fsolve, fzero, etc.**
- **Can use an explicit method (such as Euler's method) to provide an initial guess**

$$\frac{dy}{dt} = -y^2,\ t \in [0, a] \qquad (2) \quad y(0) = 1.$$

$$y_{k+1} = y_k - \Delta t y_k^2 \qquad (3)$$

$$\left(\frac{dy}{dt}\right)_k \approx \frac{y_{k+1} - y_k}{\Delta t} = -y_k^2$$

$$y_{k+1} + \Delta t y_{k+1}^2 = y_k$$

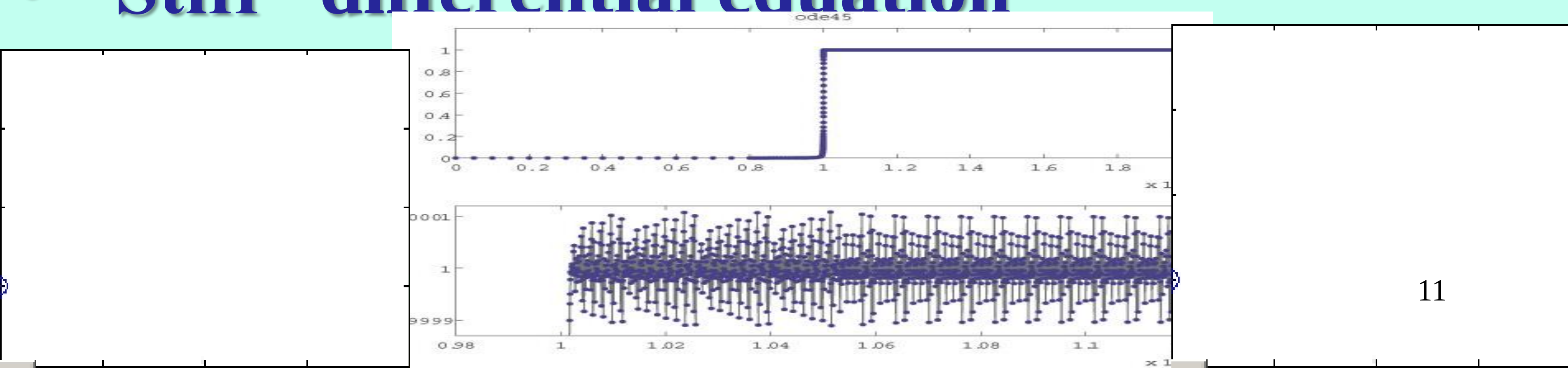
$$\frac{y_{k+1} - y_k}{\Delta t} = -y_{k+1}^2$$

Stiff Differential Equations

- **A stable differential equation is desirable:**
  - **Solution curves converge with time**

- **If it happens too rapidly, then numerical methods may have difficulty**

- **“Stiff” differential equation**



More on Stiffness

- **An ODE system is stiff if the eigen values of the Jacobian *J* differ greatly in magnitude**
  - **Large negative part (strongly damped component)**
  - **Large imaginary part (rapidly oscillating component)**
  - **Modelling physical process with disparate time scales**

Even more on Stiffness

- **A stiff ODE can cause a numerical method to take very small steps to maintain stability**
  - **Stiffness (typically) affects the growth factor more than the local error (accuracy)**
  - **Implicit methods are (typically) better suited for stiff problems**

Boundary-Value Problems

For example, Euler's method must take small steps, but backward Euler need not

- **Values of the solution are specified at more than one point (typically, at the initial and final points); # of conditions = # of equations**
- **For example: Solve**

$$y'' = f(t, y, y'), \quad a \le t \le b$$

with boundary conditions

$$y(a) = y_a, \quad y(b) = y_b$$

- **Two-point boundary-value problem**

Solving Boundary-Value Problems

- **In an initial-value problem (IVP), one can have full information at the initial point, one can need only march forward**

- **In a boundary-value problem (BVP), this is not possible; numerical methods are more complicated than for IVPs**

- **Proving existence and uniqueness of solutions is also more complicated**

Numerical Methods for BVPs

- **Focus on 2-point BVP (simplest case)**
- **Three methods**
  - **Shooting**
  - **Finite difference**
  - **Finite element**

Shooting Method

- **Given $y(a) = y_a$; if one can knew $y'(a)$ could use a method for an IVP**

- **Strategy: Guess values for $y'(a)$, and solve the resulting IVP, until one find one for which $y(b) = y_b$**

Shooting Method

- **For a given value of γ, solve**

$$y'' = f(t, y, y'), \quad y(a) = y_a, \; y'(a) = \gamma$$

to obtain the solution $y(b)$ at $t = b$

- **The results depends on γ so $y(b) = g(\gamma)$ for some function g**
- **Apply a nonlinear equation solver to $g(\gamma) = y_b$ to determine γ and solve BVP**

Finite-Difference Methods

- **Convert BVP into a system of algebraic equations by replacing derivatives with finite-difference approximations**
- **Introduce a (uniform) mesh: $t_i = a + ih$ for $i = 0,1,\ldots,n$, where $h = (b - a)/n$**
- **Solve for $y_i \approx y(t_i)$**

Finite Difference Methods

- **Central difference approximations for higher order derivatives:**

$$y_i' = \frac{1}{2h}\left(y_{i+1} - y_{i-1}\right)$$

$$y_i'' = \frac{1}{h^2}\left(y_{i+1} - 2y_i + y_{i-1}\right)$$

$$y_i''' = \frac{1}{2h^3}\left(y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}\right)$$

$$y_i^{iv} = \frac{1}{h^4}\left(y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}\right)$$

Notation

$y = f(x)$

$y_i = f(x = i)$

$y_i' = f'(x = i)$

$y_i'' = f''(x = i)$ *and so on* $\cdots$

Finite Difference Methods

- **BC's are given $y_0 = y_a$ and $y_n = y_b$**
- **Use the approximations**

$$y'(t_i) \approx (y_{i+1} - y_{i-1})/(2h)$$

$$y''(t_i) \approx (y_{i+1} - 2y_i + y_{i-1})/(h^2)$$

- **Substituting into BVP gives:**

- **System may be linear or nonlinear, but sparse**

$$y_{i+1} - 2y_i + y_{i-1} - h^2 f(t_i, y_i, (y_{i+1} - y_{i-1})/(2h)) = 0$$

Finite-Element Method

- **Approximate solution by a (finite) linear combination of basis functions ϕ_i, typically piecewise polynomials ("elements")**

- **This gives an approximation of the form**

$$y(t) \approx u(t) = \sum_{i=1}^{n} x_i \phi_i(t)$$

- **Solve for coefficients x_i (several alternatives)**

Finite-Element Method

- **Four popular approaches:**

– **Direct: stiffness based**

  – **Collocation: differential equation is satisfied exactly at *n* points**

  – **Galerkin: residual is orthogonal to space spanned by basis functions**

  – **Rayleigh-Ritz: residual is minimized in a (weighted) least-squares sense [in some cases, equivalent to Galerkin]**

- **Based on inner products of functions (not vectors):**
  – **Integrals must be computed**

1. **Hyperbolic equations-----e.g.:** $$\frac{\partial^2 F}{\partial t^2} - C^2 \frac{\partial^2 F}{\partial x^2} = 0$$
2. **Parabolic equations----e.g.:** $$\frac{\partial F}{\partial t} - A \frac{\partial^2 F}{\partial x^2} = 0$$
3. **Elliptic equations---e.g.:** . $$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = G(x, y)$$
4. **Advective equation—e.g:** $$\frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} = 0$$

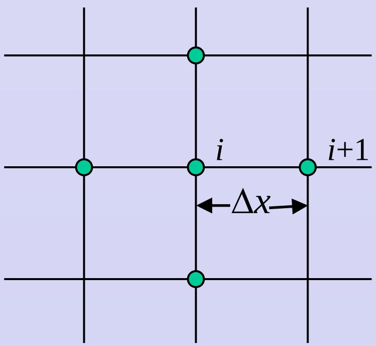
Classification of Discretization Methods

- Finite-difference methods---Oldest methods
- Finite-element methods----Popular in the last 10 years
- Finite-volume methods---New Methods

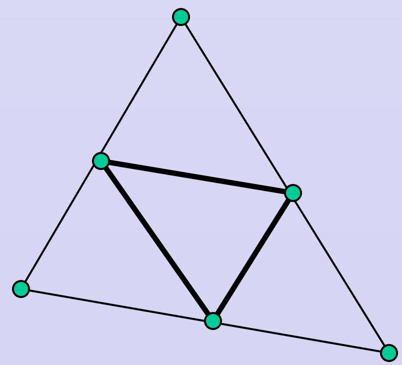
Difference between finite-difference, finite-element and finite-volume methods (FDM, FEM, and FVM)

$$\frac{\partial f}{\partial x} = C$$

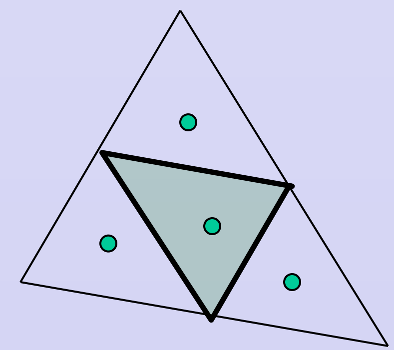
FDM



FEM



FVM



$$\frac{\partial f}{\partial x} = \frac{f_{i+1} - f_i}{\Delta x} = C$$

$$\oiint w_l \left(\frac{\partial \hat{f}}{\partial x} - C\right) = 0$$

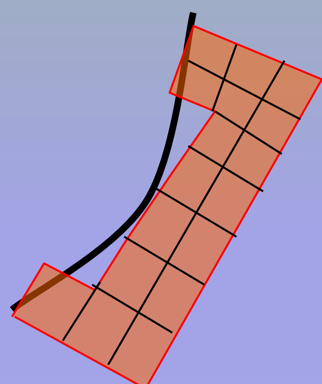
$$\oiint \frac{\partial f}{\partial x} dxdy = \oint f\, dy = C * Area$$

Difference

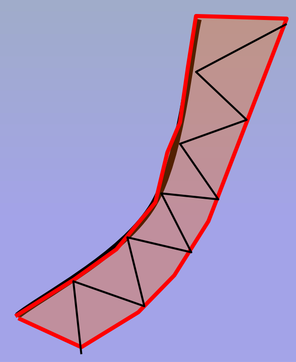
Variation

Integration

	Advantage:	Disadvantage:
FDM	1. Computational efficiency 2. Simple code structures 3. Mass conservation	Irregular geometric matching
FEM	Irregular geometric matching	1. Mass conservation 2. Complex code structures 3. Computational inefficiency
FVM	Combined the advantages of FDM and FEM	??



FDM



FEM and FVM

Numerical methods: properties

Finite differences →

- time-dependent PDEs
- seismic wave propagation
- geophysical fluid dynamics
- Maxwell's equations
- Ground penetrating radar

-> robust, simple concept, easy to parallelize, regular grids, explicit method

Finite elements →

- static and time-dependent PDEs
- seismic wave propagation
- geophysical fluid dynamics
- all problems

-> implicit approach, matrix inversion, well founded, irregular grids, more complex algorithms, engineering problems

Finite volumes →

- time-dependent PDEs
- seismic wave propagation
- mainly fluid dynamics

-> robust, simple concept, irregular grids, explicit method

Comparing

FD and FEM (hat basis)

1D problem

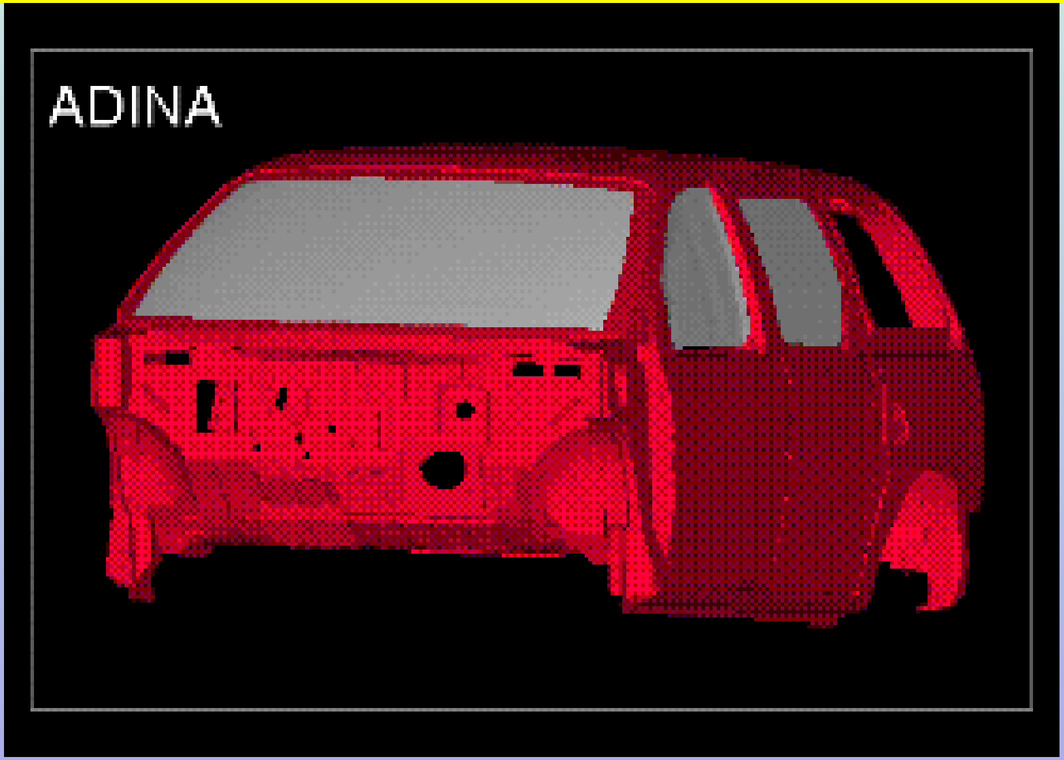
FEM

$$\frac{1}{\Delta x}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix} = \begin{bmatrix} \int_0^1 \varphi_1(x) f(x)dx \\ \int_0^1 \varphi_2(x) f(x)dx \\ \vdots \\ \int_0^1 \varphi_n(x) f(x)dx \end{bmatrix}$$

FD

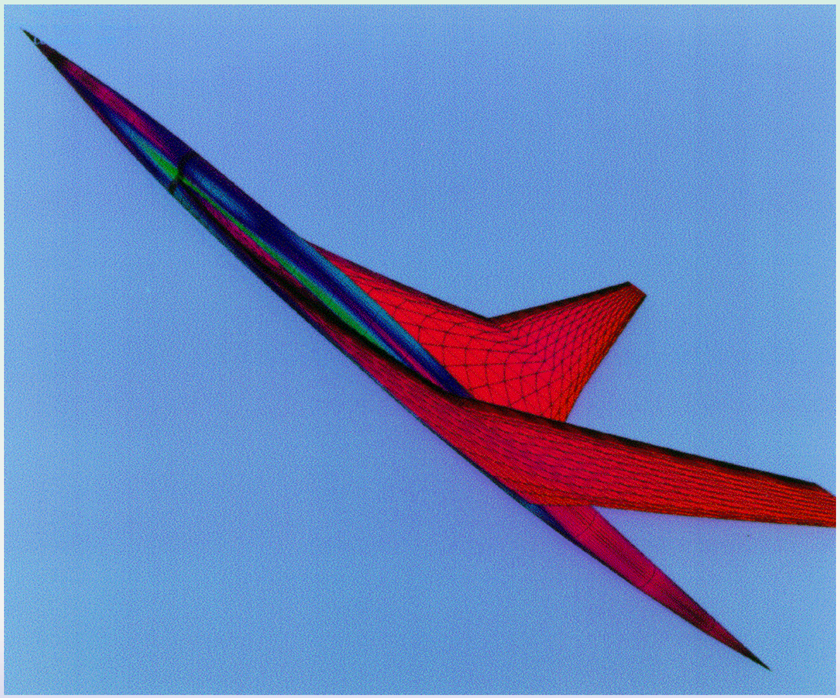
$$\frac{1}{\Delta x^2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{bmatrix}$$

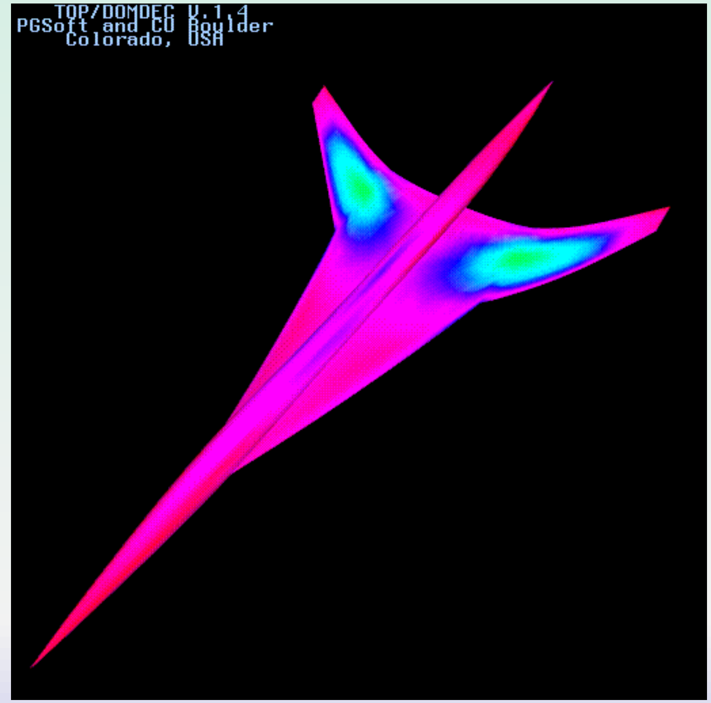
3-D Problems **Structural Analysis of Automobiles**



- Equations
  - **Force-displacement relationships for mechanical elements (plates, beams, shells) and sum of forces = 0.**
  - **Partial Differential Equations of Continuum Mechanics**

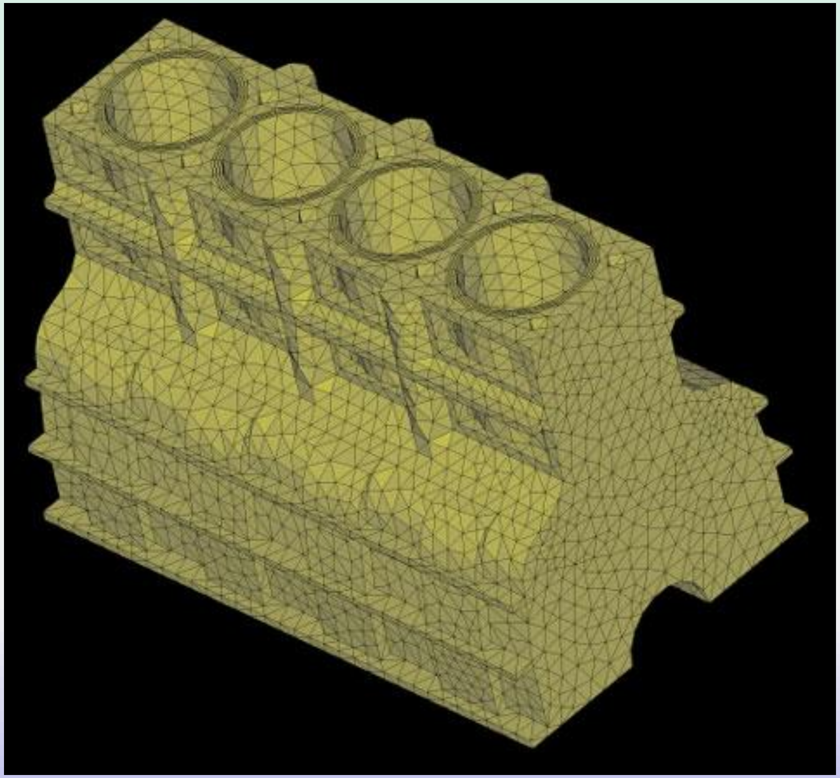
Drag Force Analysis of Aircraft





- Equations
  - Navier-Stokes Partial Differential Equations.

Engine Thermal Analysis



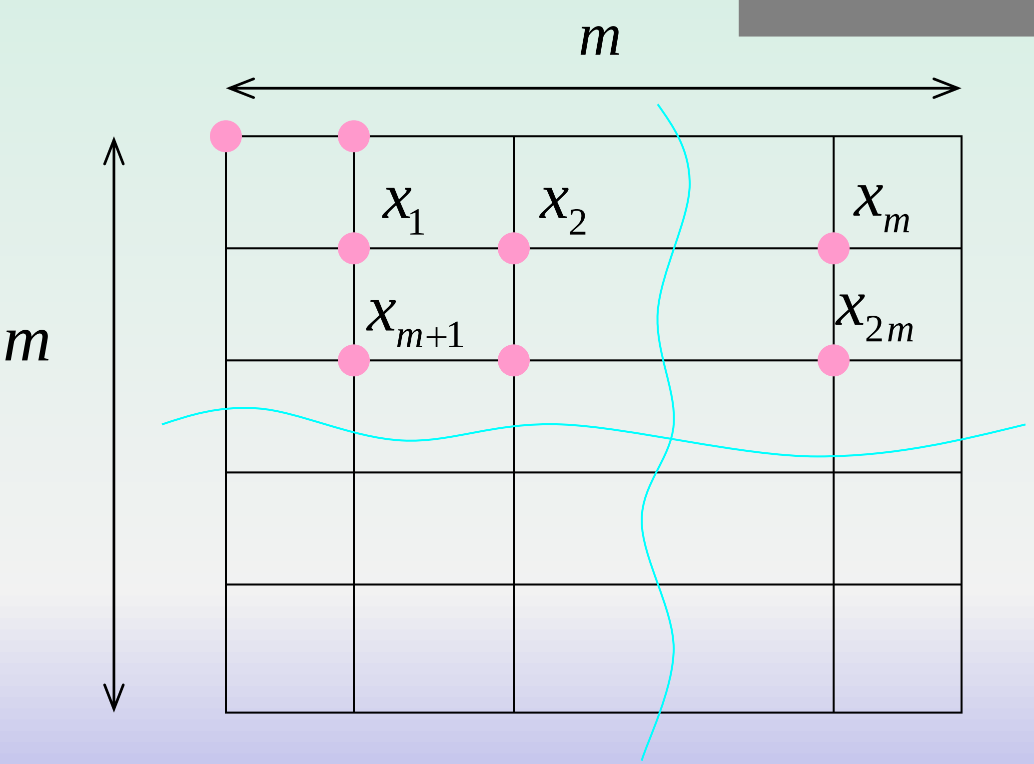
Picture from www.adina.com

- Equations
  - The Poisson Partial Differential Equation.

FD Matrix properties

2-D Discretized Problem

Discretized Poisson



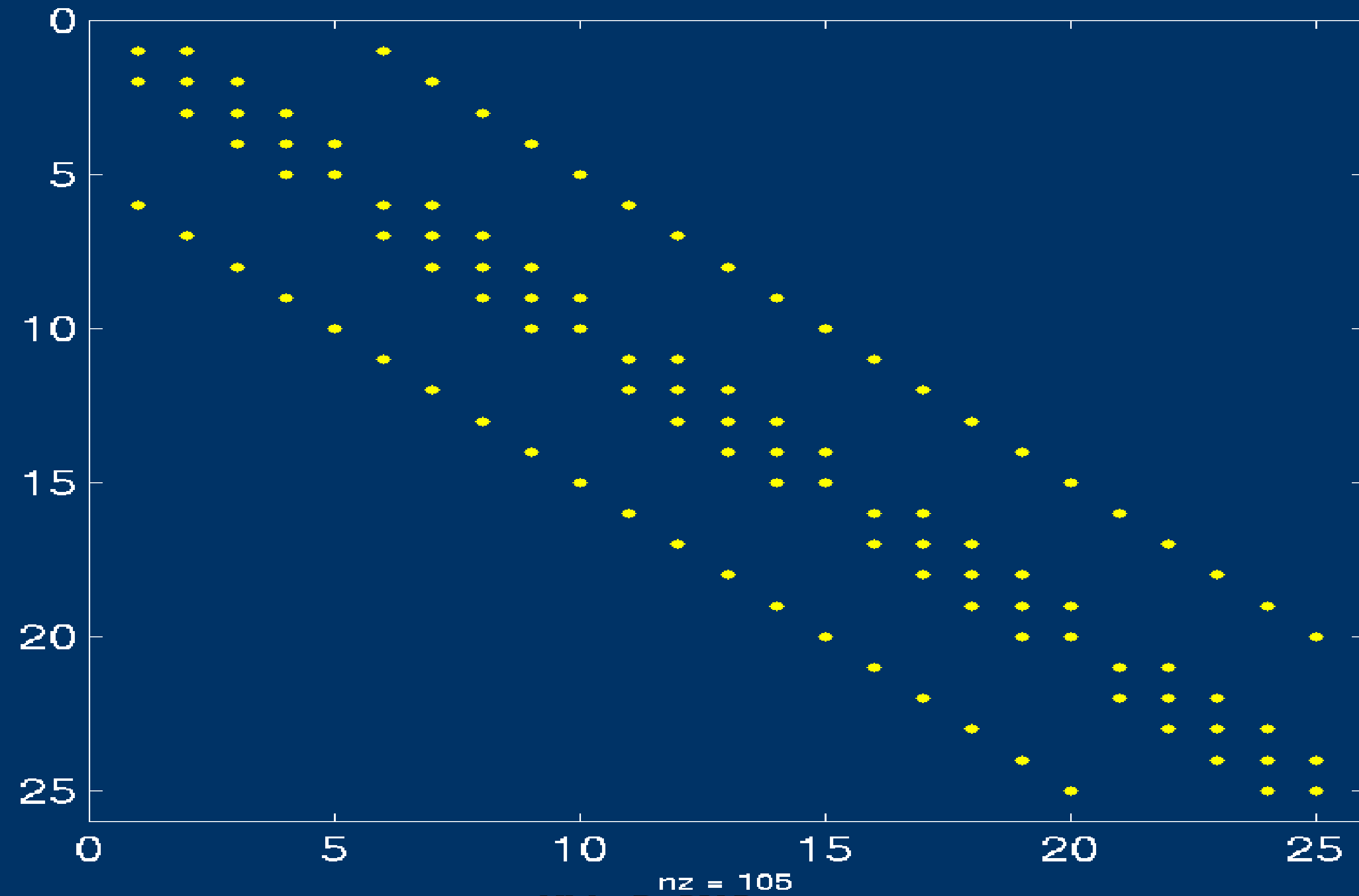
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x)$$

$$-\underbrace{\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}}_{u_{xx}} - \underbrace{\frac{u_{j+m} - 2u_j + u_{j-m}}{\Delta y^2}}_{u_{yy}} = f(x_j)$$

FD Matrix properties

2-D Discretized Problem

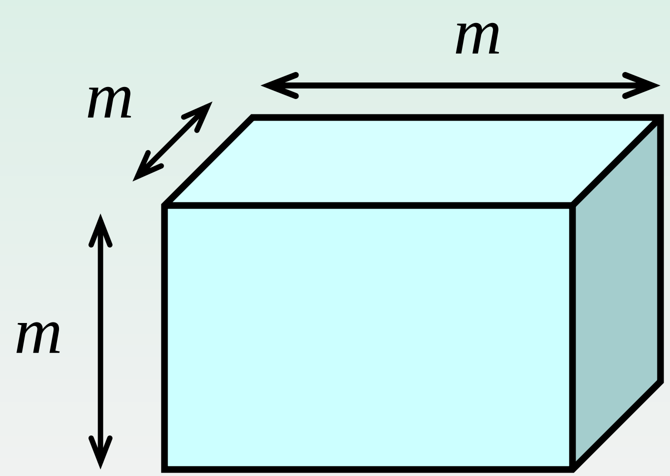
Matrix Nonzeros, 5x5 example

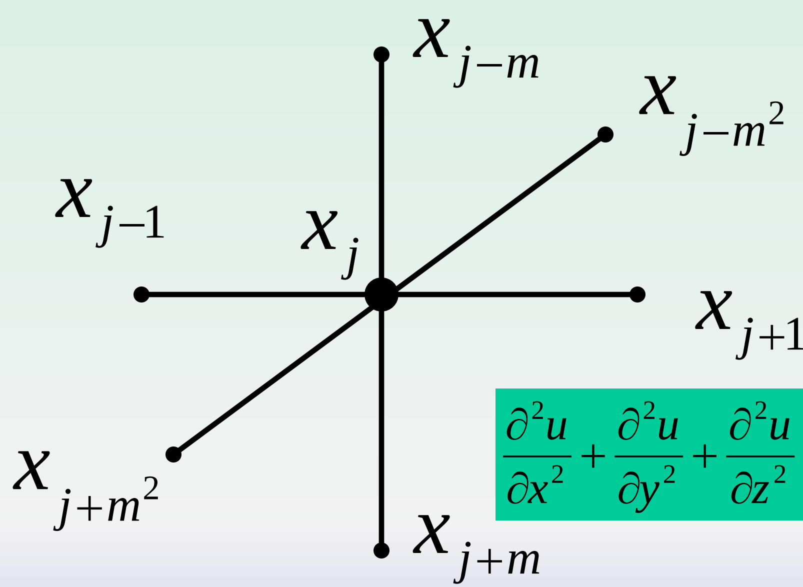


FD Matrix properties

3-D Discretization

Discretized Poisson





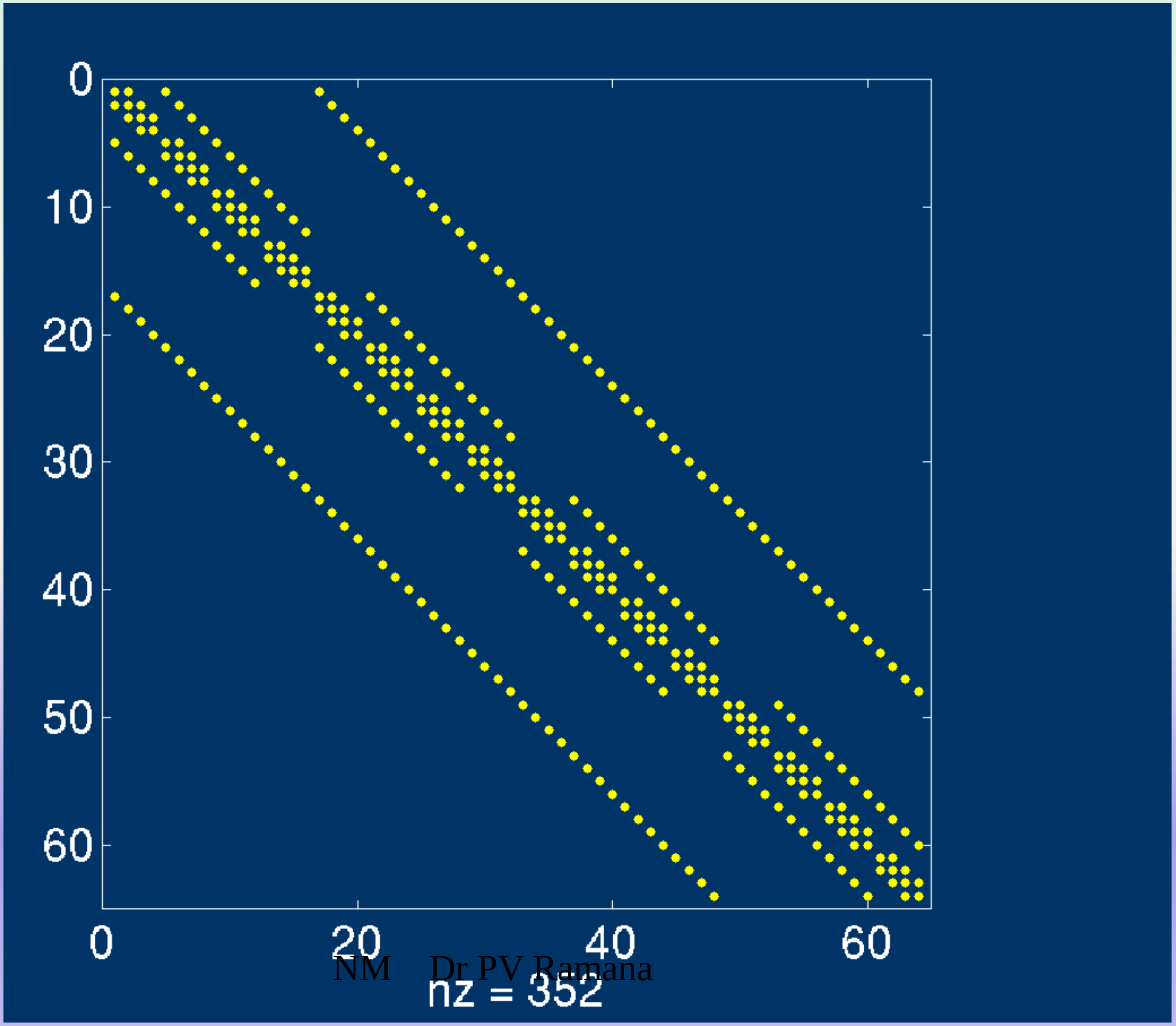
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x)$$

$$-\underbrace{\frac{\hat{u}_{j+1} - 2\hat{u}_j + \hat{u}_{j-1}}{(\Delta x)^2}}_{u_{xx}} - \underbrace{\frac{\hat{u}_{j+m} - 2\hat{u}_j + \hat{u}_{j-m}}{(\Delta y)^2}}_{u_{yy}} - \underbrace{\frac{\hat{u}_{j+m^2} - 2\hat{u}_j + \hat{u}_{j-m^2}}{(\Delta z)^2}}_{u_{zz}} = f(x_j)$$

FD Matrix properties

3-D Discretization

Matrix nonzeros, m = 4 example

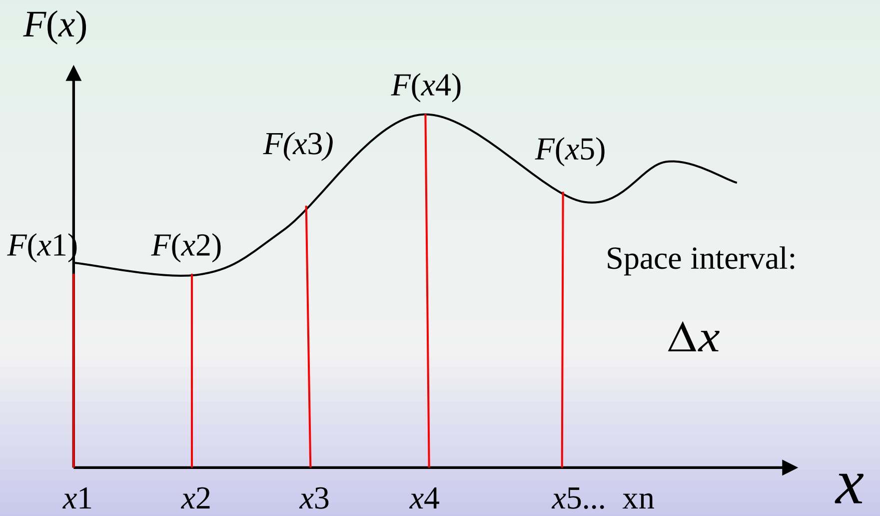


Key Properties of Numerical Methods

1. Consistency

Definition: The Discretization should approach the exact function as the discrete interval approach zero.

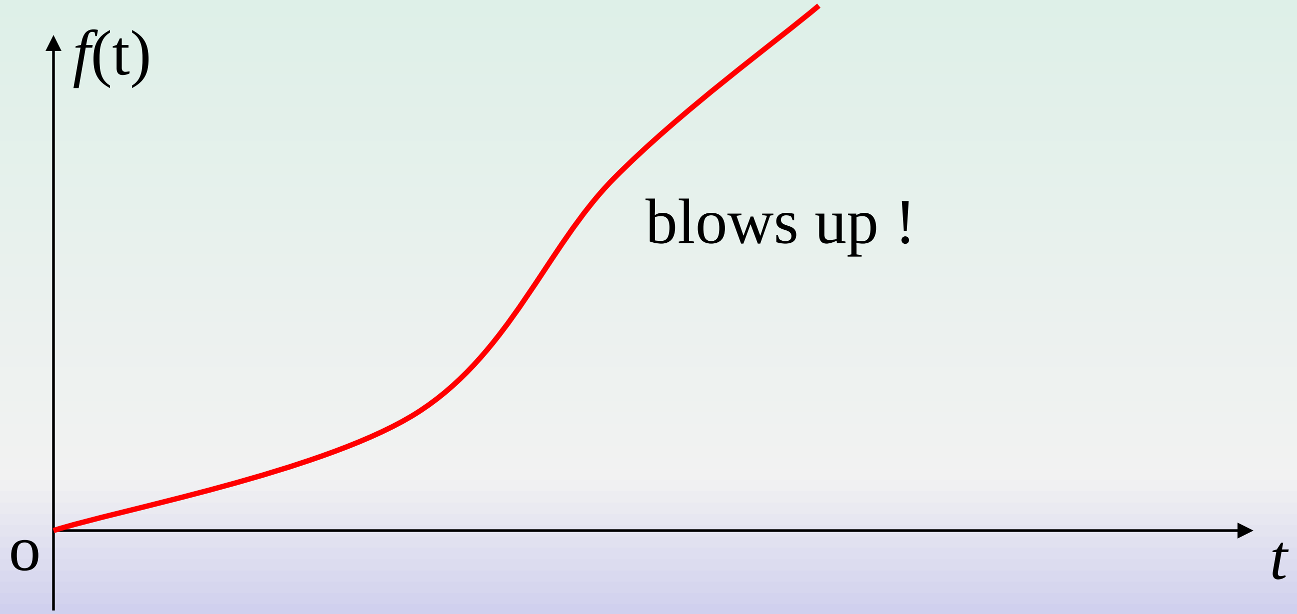
Example:



$$\{F_i(xi)\} \to F(x) \quad \text{as} \quad \Delta x \to 0$$

2. Stability

Definition: A numerical method is defined to be stable if the numerical solution does not grow up an unreasonable big value or becomes infinite during the time integration.

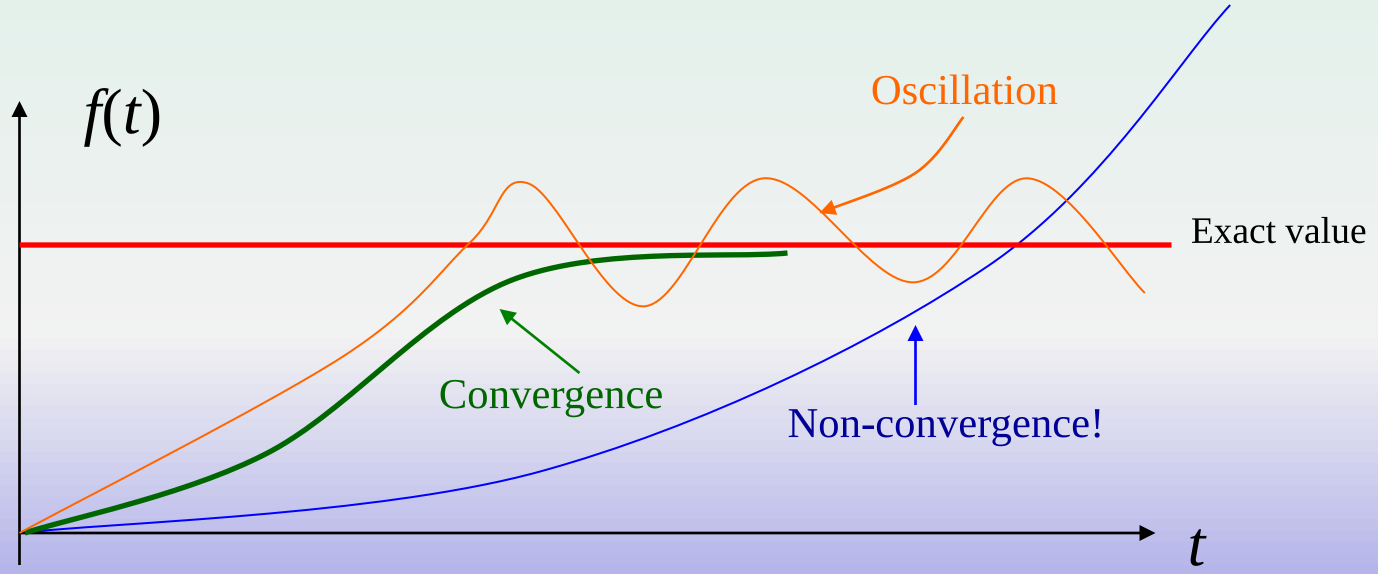


Depending on: 1) time step/space resolution (linear), mass conservation and boundary conditions, etc

Comments: A stable model does not means that is mass conservative.

3. Convergence

A numerical method is defined to be convergent if the numerical solution of the discretization equation tends to reach the exact solution of the differential equation as grid spacing approaches zero.



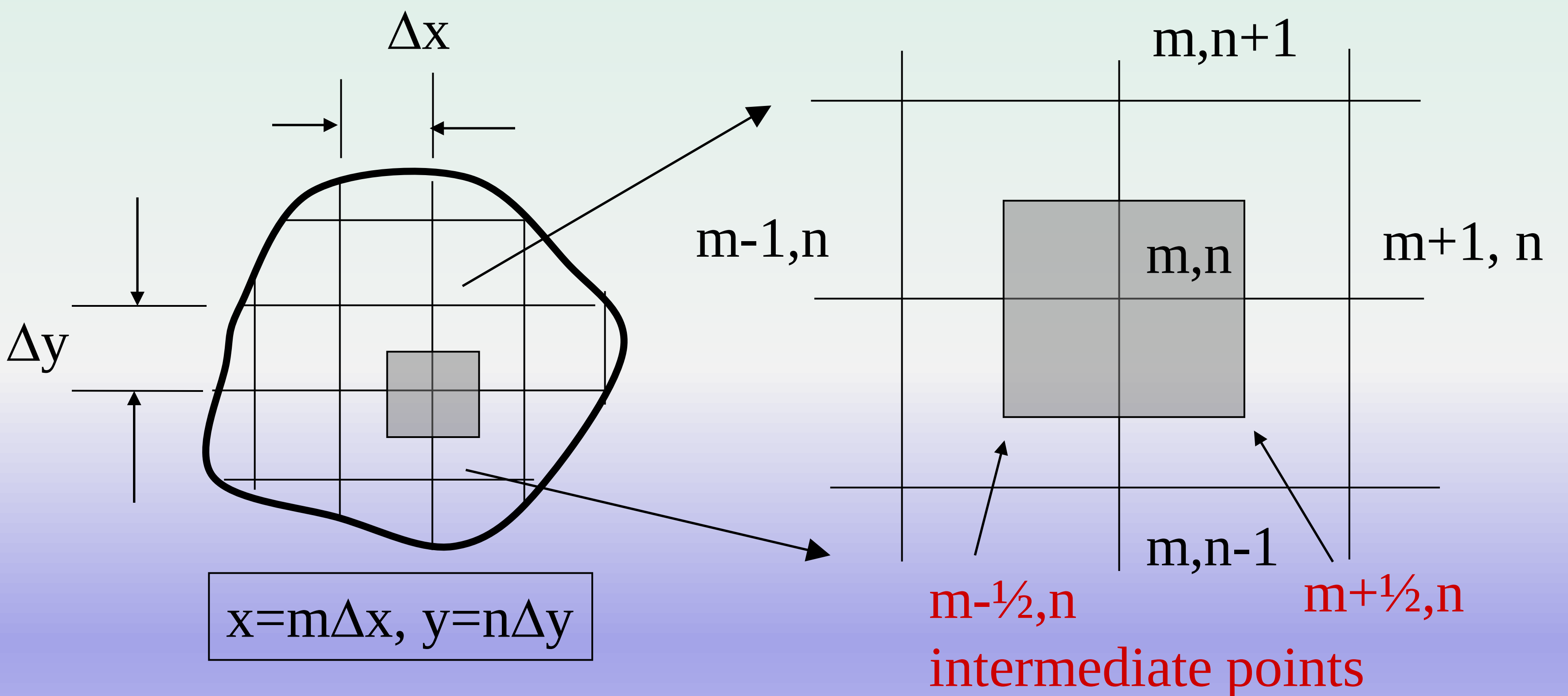
4. Accuracy

Once the equations are discretized and solved numerically, they only provided an approximate solution. The accuracy of this solution depends on grid resolution and the orders of the approximation.

Examples:

Coarse grids: low accuracy

High order approximation: high accuracy but probably cause boundedness problems.



$x = m\Delta x,\ y = n\Delta y$

Finite Difference Approximation

Heat Diffusion Equation: $\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$,

where $\alpha = \frac{\mathrm{k}}{\rho \mathrm{C_P} V}$ is the thermal diffusivity

No generation and steady state: $\dot{\mathrm{q}}=0$ and $\frac{\partial}{\partial \mathrm{t}} = 0, \Rightarrow \nabla^2 T = 0$

First, approximated the first order differentiation

at intermediate points (m+1/2,n) & (m-1/2,n)

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{(m+1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m+1/2,n)} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial \mathrm{T}}{\partial \mathrm{x}}\right|_{(m-1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m-1/2,n)} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Finite Difference Approximation (contd...)

Next, approximate the second order differentiation at m,n

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\left.\partial T/\partial x\right|_{m+1/2,n} - \left.\partial T/\partial x\right|_{m-1/2,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

Similarly, the approximation can be applied to

the other dimension y

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

Finite Difference Approximation (contd...)

$$\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)_{m,n} \approx \frac{T_{m+1,n}+T_{m-1,n}-2T_{m,n}}{(\Delta x)^2}+\frac{T_{m,n+1}+T_{m,n-1}-2T_{m,n}}{(\Delta y)^2}$$

To model the steady state, no generation heat equation: $\nabla^2 T = 0$

This approximation can be simplified by specify Δx=Δy

and the nodal equation can be obtained as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

This equation approximates the nodal temperature distribution based on the heat equation. This approximation is improved when the distance between the adjacent nodal points is decreased:

Since $\lim(\Delta x \to 0)\dfrac{\Delta T}{\Delta x}=\dfrac{\partial T}{\partial x}, \lim(\Delta y \to 0)\dfrac{\Delta T}{\Delta y}=\dfrac{\partial T}{\partial y}$

A System of Algebraic Equations

- The nodal equations derived previously are valid for all interior points satisfying the steady state, no generation heat equation. For each node, there is one such equation.

For example: for nodal point m=3, n=4, the equation is

$$T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5} - 4T_{3,4} = 0$$

$$T_{3,4} = (1/4)(T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5})$$

- Nodal relation table for exterior nodes (boundary conditions) can be found in standard heat transfer textbooks (Table 4.2 in this presentation).
- Derive one equation for each nodal point (including both interior and exterior points) in the system of interest. The result is a system of N algebraic equations for a total of N nodal points.

Matrix Form

The system of equations:

$$
\begin{aligned}
a_{11}T_1 + a_{12}T_2 + \cdots + a_{1N}T_N &= C_1 \\
a_{21}T_1 + a_{22}T_2 + \cdots + a_{2N}T_N &= C_2 \\
\vdots \qquad\quad \vdots \qquad\quad \vdots \quad \vdots \qquad &\quad \vdots \\
a_{N1}T_1 + a_{N2}T_2 + \cdots + a_{NN}T_N &= C_N
\end{aligned}
$$

A total of N algebraic equations for the N nodal points and the system can be expressed as a matrix formulation: [**A**][**T**]=[**C**]

$$
where\ A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}
$$

- Matrix form: [**A**][**T**]=[**C**].

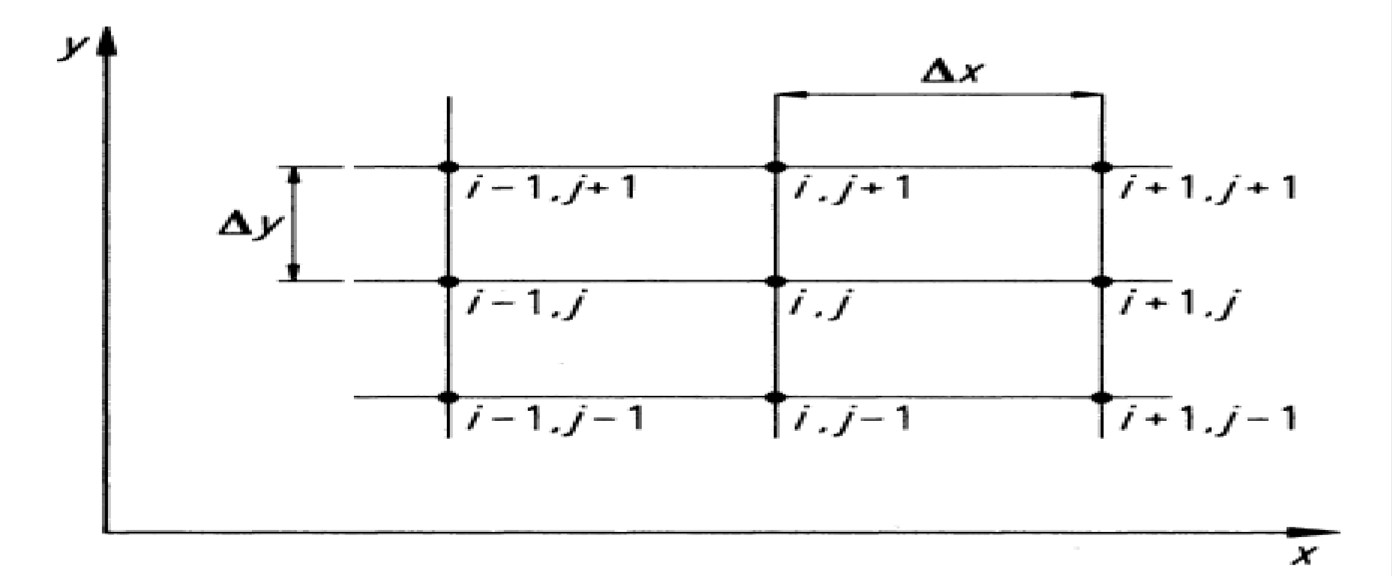
From linear algebra: [**A**]$^{-1}$[**A**][**T**]=[**A**]$^{-1}$[**C**], [**T**]=[**A**]$^{-1}$[**C**]

where [**A**]$^{-1}$ is the inverse of matrix [**A**]. [**T**] is the solution vector.

Components of numerical methods

Discretization methods (Finite Difference)

- Numerical solutions can give answers at only discrete points in the domain, called grid points.



- If the PDEs are totally replaced by a system of algebraic equations which can be solved for the values of the flow-field variables at the discrete points only, in this sense, the original PDEs have been discretized. Moreover, this method of discretization is called the method of finite differences.

6-5

Components of numerical methods

Discretization methods (Finite Difference)

- Taylor's series expansion:
- A partial derivative replaced with a suitable algebraic difference quotient is called finite difference. Most finite-difference representations of derivatives are based on Taylor's series expansion.

Consider a continuous function of x, namely, f(x), with all derivatives defined at *x*. Then, the value of f at a location $x+\Delta x$ can be estimated from a Taylor series expanded about point x, that is,

$$f(x+\Delta x) = f(x) + \frac{\partial f}{\partial x}\Delta x + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{1}{3!}\frac{\partial^3 f}{\partial x^3}(\Delta x)^3 + \ldots + \frac{1}{n!}\frac{\partial^n f}{\partial x^n}(\Delta x)^n + \ldots$$

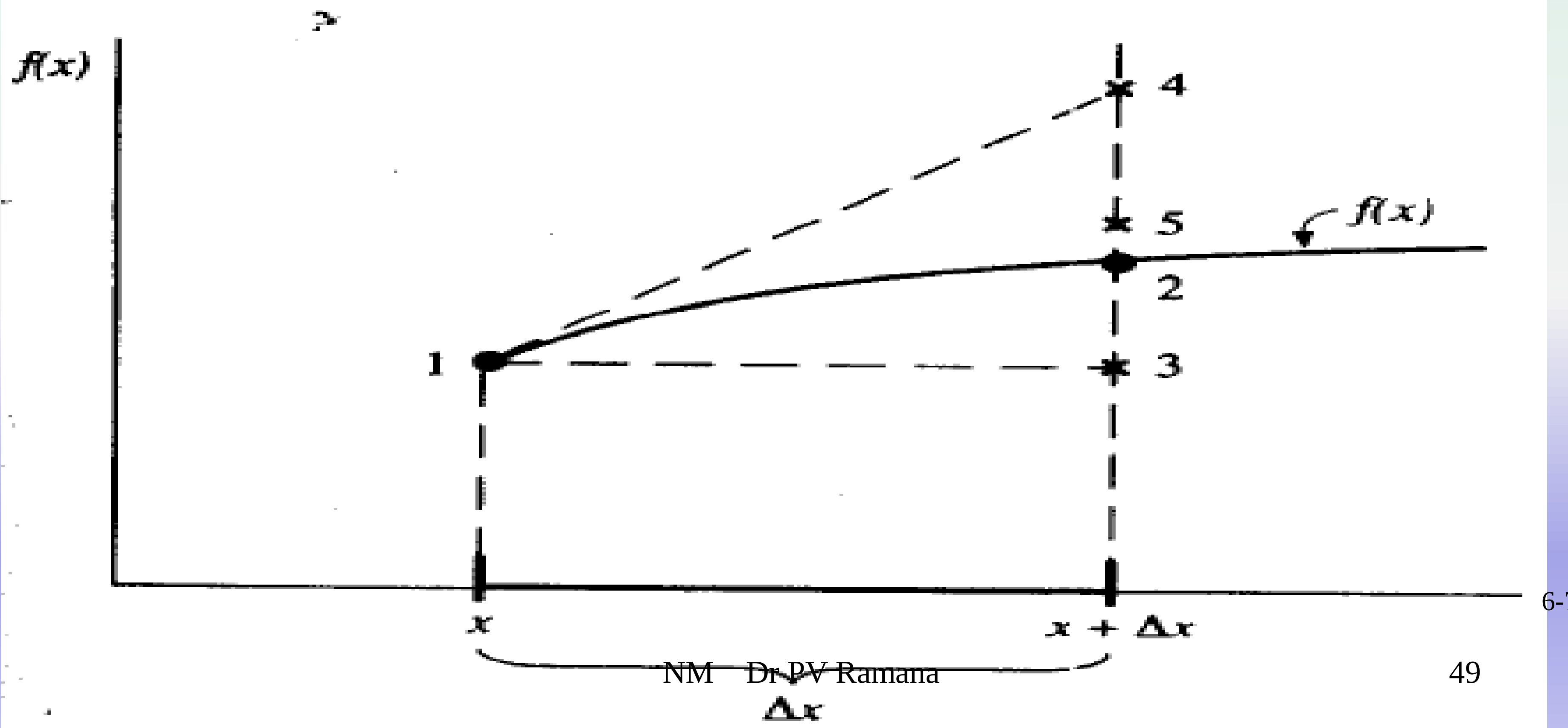
- In general, to obtain more accuracy, additional higher-order terms must be included.

6-6

Components of numerical methods

Discretization methods (Finite Difference)

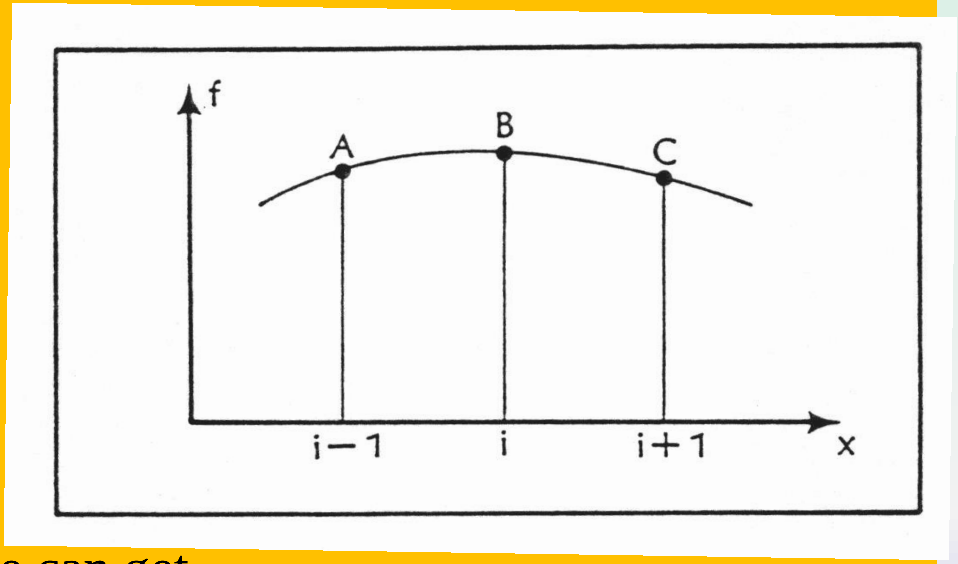
$$f(x+\Delta x) = \underbrace{f(x)}_{\substack{\text{First guess}\\ \text{(not very good)}}} + \underbrace{\frac{\partial f}{\partial x}\Delta x}_{\substack{\text{Add to capture}\\ \text{slope}}} + \underbrace{\frac{\partial^2 f}{\partial x^2}\,\frac{(\Delta x)^2}{2}}_{\substack{\text{Add to account}\\ \text{for curvature}}} + \cdots$$



Components of numerical methods

Discretization methods (Finite Difference)

- Forward, Backward and Central Differences:



(1) Forward difference:

Neglecting higher-order terms, we can get

$$f(x_{i+1}) = f(x_i) + (\frac{\partial f}{\partial x})_i (x_{i+1} - x_i) + \frac{(x_{i+1} - x_i)^2}{2!}(\frac{\partial^2 f}{\partial x^2})_i + \frac{(x_{i+1} - x_i)^3}{3!}(\frac{\partial^3 f}{\partial x^3})_i + \ldots + \frac{(x_{i+1} - x_i)^n}{n!}(\frac{\partial^n f}{\partial x^n})_i + \ldots \quad \text{(a)}$$

$$(\frac{\partial f}{\partial x})_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{f(x_{i+1}) - f(x_i)}{\Delta x_{i+1}}; \quad \Delta x_{i+1} = x_{i+1} - x_i$$

6-8

Finite Difference Method

An example of a boundary value ordinary differential equation is

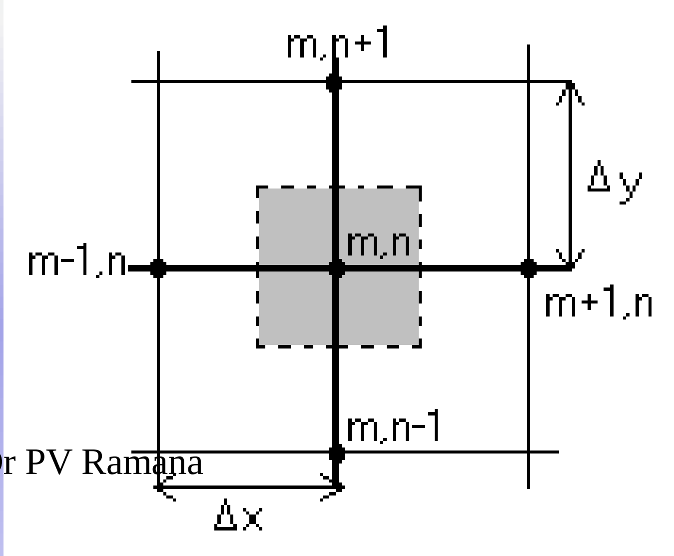
$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0,\quad u(5)=0.008731'',\ u(8)=0.0030769''$$

The derivatives in such ordinary differential equation are substituted by finite divided differences approximations, such as

$$T_{m,n+1}+T_{m,n-1}+T_{m+1,n}+T_{m-1,n}-4T_{m,n}=0$$

$$\frac{dy}{dx}\approx\frac{y_{i+1}-y_i}{\Delta x}$$

$$\frac{d^2y}{dx^2}\approx\frac{y_{i+1}-2y_i+y_{i-1}}{(\Delta x)^2}$$



Solution of Boundary-Value Problems

Finite Difference Method

Boundary-Value Problems

convert

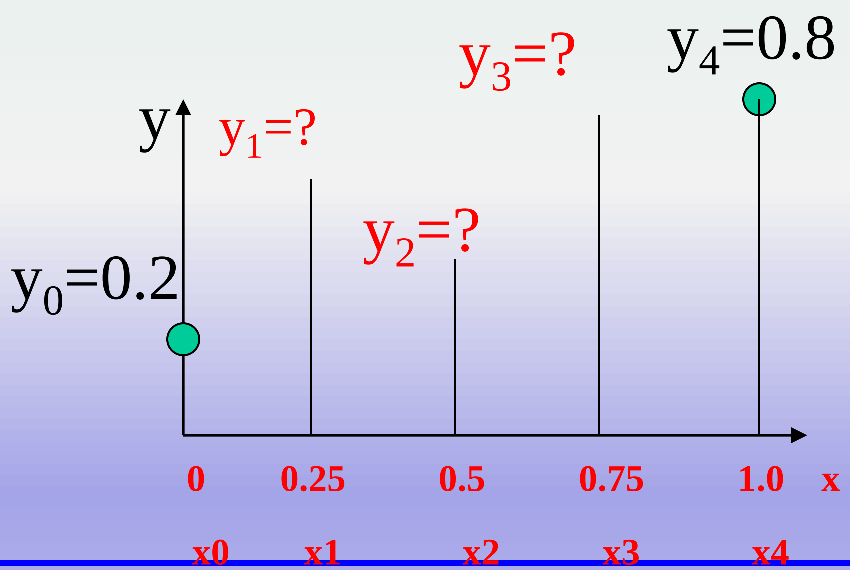
Algebraic Equations

Find the unknowns y_1, y_2, y_3

$$Find\ y(x)\ to\ solve\ BVP$$

$$\ddot{y} + 2\dot{y} + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$



Solution of Boundary-Value Problems
Finite Difference Method

- Divide the interval into *n* sub-intervals.
- The solution of the BVP is converted to the problem of determining the value of function at the base points.
- Use finite approximations to replace the derivatives.
- This approximation results in a set of algebraic equations.
- Solve the equations to obtain the solution of the BVP.

Finite Difference Method
Example 1

$$\ddot{y} + 2\dot{y} + y = x^2$$
$$y(0) = 0.2, \quad y(1) = 0.8$$

Divide the interval [0,1] into n = 4 intervals

Base points are

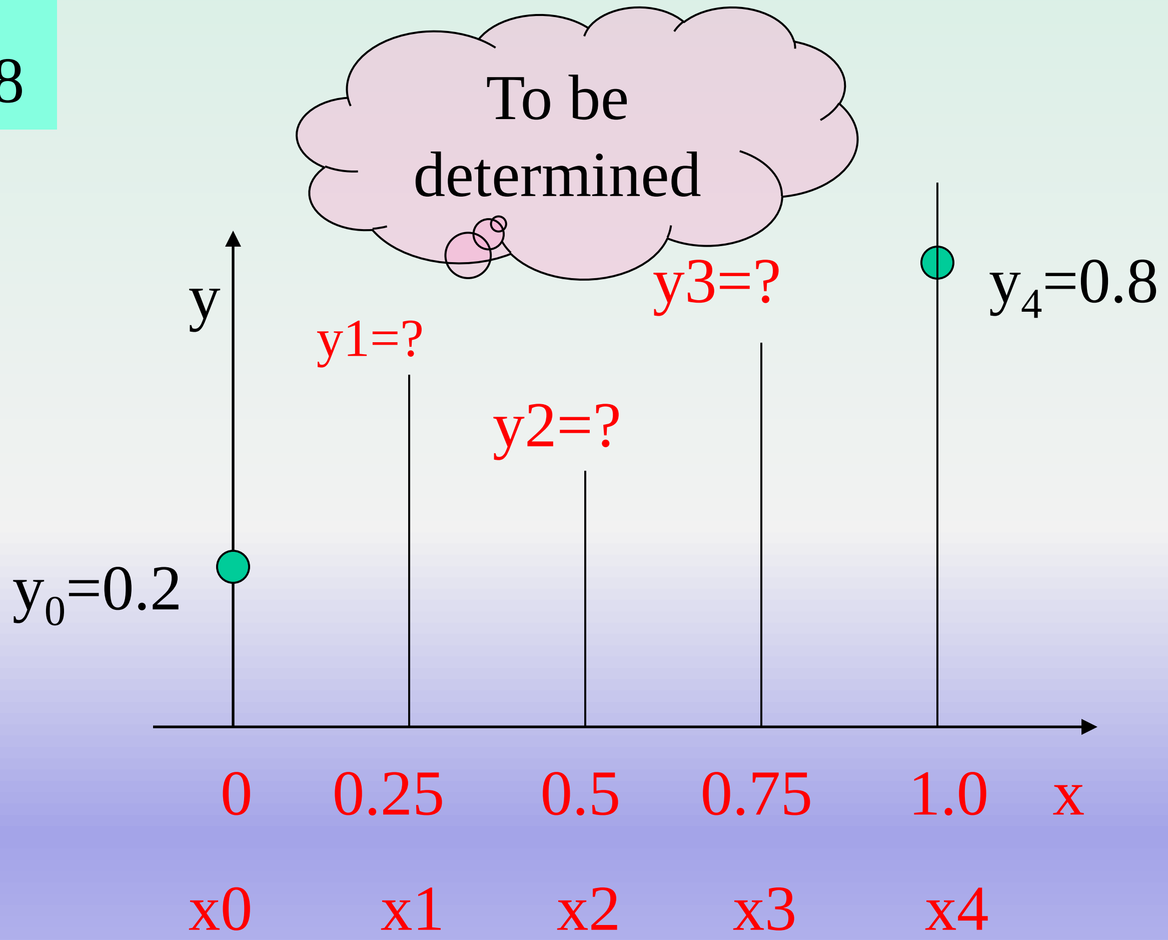
x0=0

x1=0.25

x2=.5

x3=0.75

x4=1.0



Finite Difference Method
Example 1

$$\ddot{y} + 2\dot{y} + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

Divide the interval [0,1] into n = 4 intervals

Base points are

x0=0

x1=0.25

x2=.5

x3=0.75

x4=1.0

Replace

$$\ddot{y} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \qquad \textit{central difference formula}$$

$$\dot{y} = \frac{y_{i+1} - y_{i-1}}{2h} \qquad \textit{central difference formula}$$

$$\ddot{y} + 2\dot{y} + y = x^2$$

Becomes

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + 2\frac{y_{i+1} - y_{i-1}}{2h} + y_i = x_i^2$$

Second Order BVP

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 \quad with \ \ y(0) = 0.2, \qquad y(1) = 0.8$$

$$Let \quad h = 0.25$$

Base Points

$$x_0 = 0, \ x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$$

$$\frac{dy}{dx} \approx \frac{y(x+h) - y(x)}{h} = \frac{y_{i+1} - y_i}{h}$$

$$\frac{d^2y}{dx^2} \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Second Order BVP

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x^2$$

$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+2\frac{y_{i+1}-y_i}{h}+y_i=x_i^{\,2}\quad i=1,2,3$$

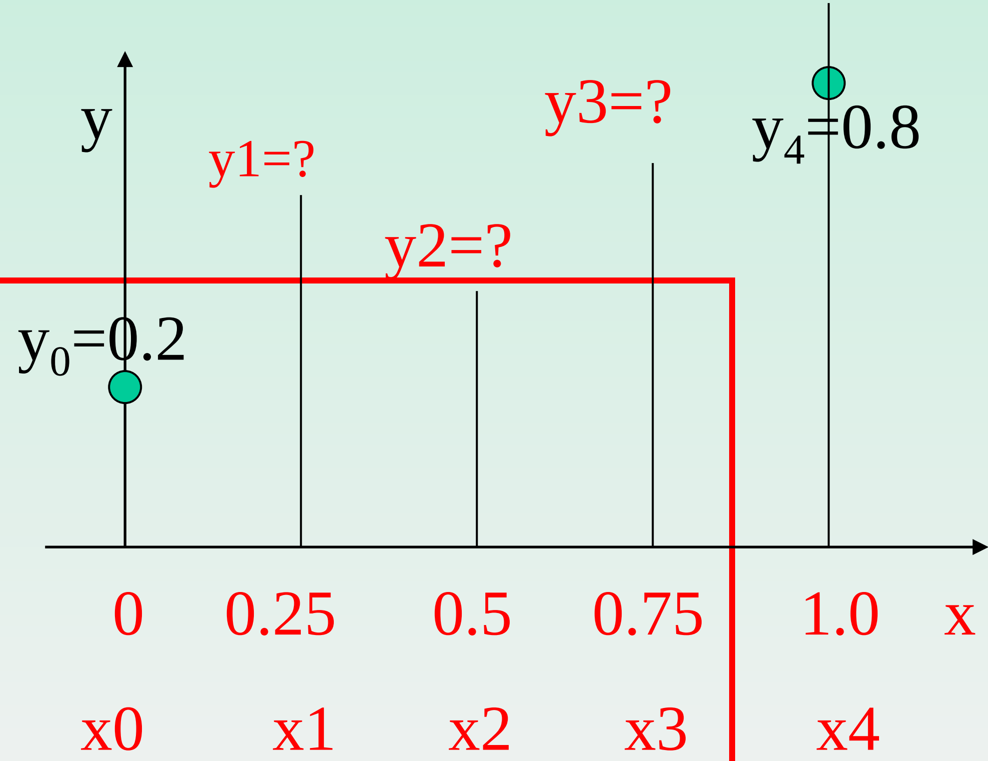
$$x_0=0,\ x_1=0.25, x_2=0.5, x_3=0.75, x_4=1$$

$$y_0=0.2,\ y_1=?, y_2=?, y_3=?,\quad y_4=0.8$$

$$16\left(y_{i+1}-2y_i+y_{i-1}\right)+8\left(y_{i+1}-y_i\right)+y_i=x_i^{\,2}$$

$$24y_{i+1}-39y_i+16y_{i-1}=x_i^{\,2}$$

Second Order BVP



$$24y_{i+1} - 39y_i + 16y_{i-1} = x_i^{\,2}$$

$$i = 1 \qquad 24y_2 - 39y_1 + 16y_0 = x_1^{\,2}$$

$$i = 2 \qquad 24y_3 - 39y_2 + 16y_1 = x_2^{\,2}$$

$$i = 3 \qquad 24y_4 - 39y_3 + 16y_2 = x_3^{\,2}$$

$$\begin{bmatrix} -39 & 24 & 0 \\ 16 & -39 & 24 \\ 0 & 16 & -39 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0.25^2 - 16(0.2) \\ 0.5^2 \\ 0.75^2 - 24(0.8) \end{bmatrix}$$

$$Solution \quad y_1 = 0.4791, y_2 = 0.6477, y_3 = 0.7436$$

Second Order BVP

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x^2$$

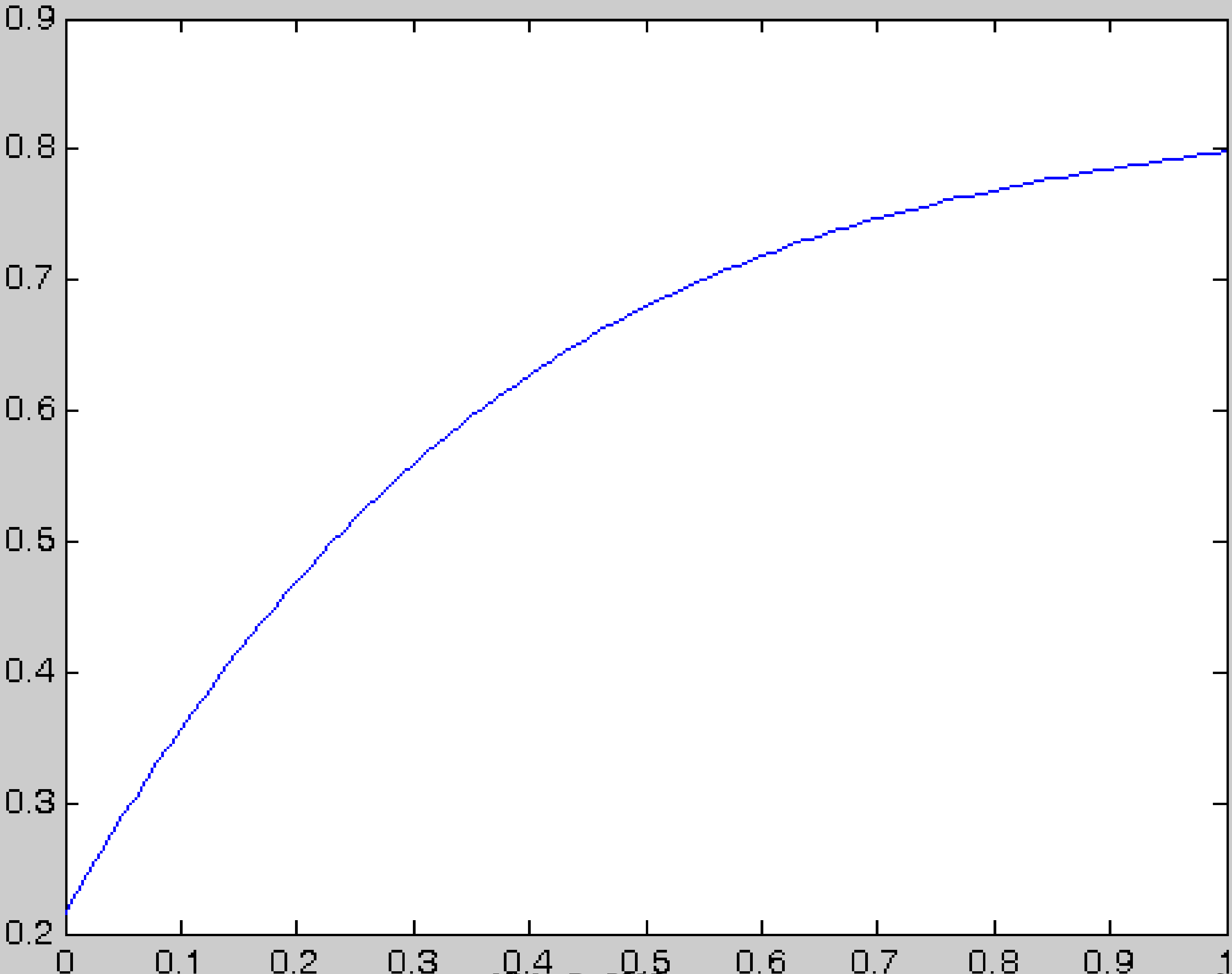
$$\frac{y_{i+1}-2y_i+y_{i-1}}{h^2}+2\frac{y_{i+1}-y_i}{h}+y_i=x_i^{\,2}\quad i=1,2,\ldots,100$$

$$x_0=0,\ x_1=0.01,x_2=0.02\quad\ldots\quad x_{99}=0.99,x_{100}=1$$

$$y_0=0.2,\ y_1=?,y_2=?,\ \ldots\ \ y_{99}=?,\ \ y_{100}=0.8$$

$$10000\left(y_{i+1}-2y_i+y_{i-1}\right)+200\left(y_{i+1}-y_i\right)+y_i=x_i^{\,2}$$

$$10200\,y_{i+1}-20199\,y_i+10000\,y_{i-1}=x_i^{\,2}$$


0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Solution of Boundary-Value Problems

Finite Difference Method

Boundary-Value Problems

convert

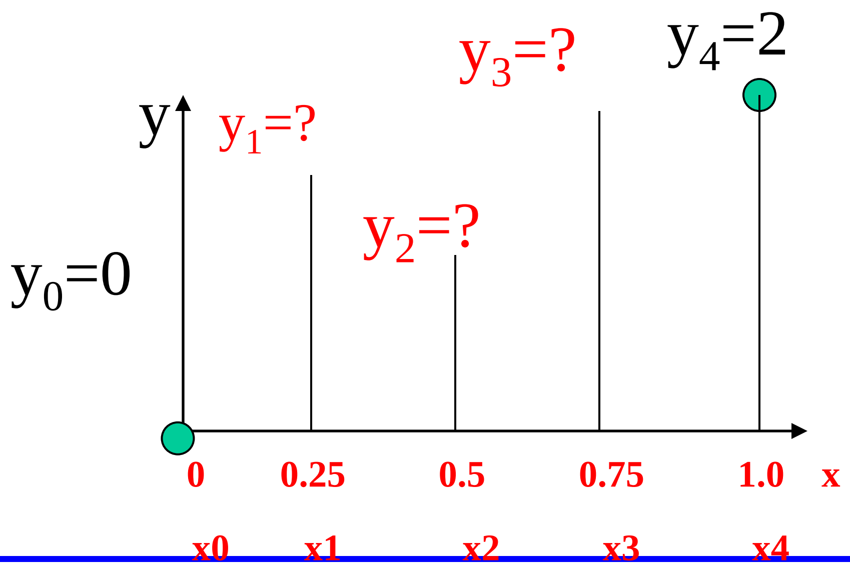
Algebraic Equations

Find the unknowns y_1, y_2, y_3

$$Find\ y(x)\ to\ solve\ BVP$$

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$



Finite Difference Method
Example

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

Divide the interval [0,1] into n = 4 intervals

Base points are

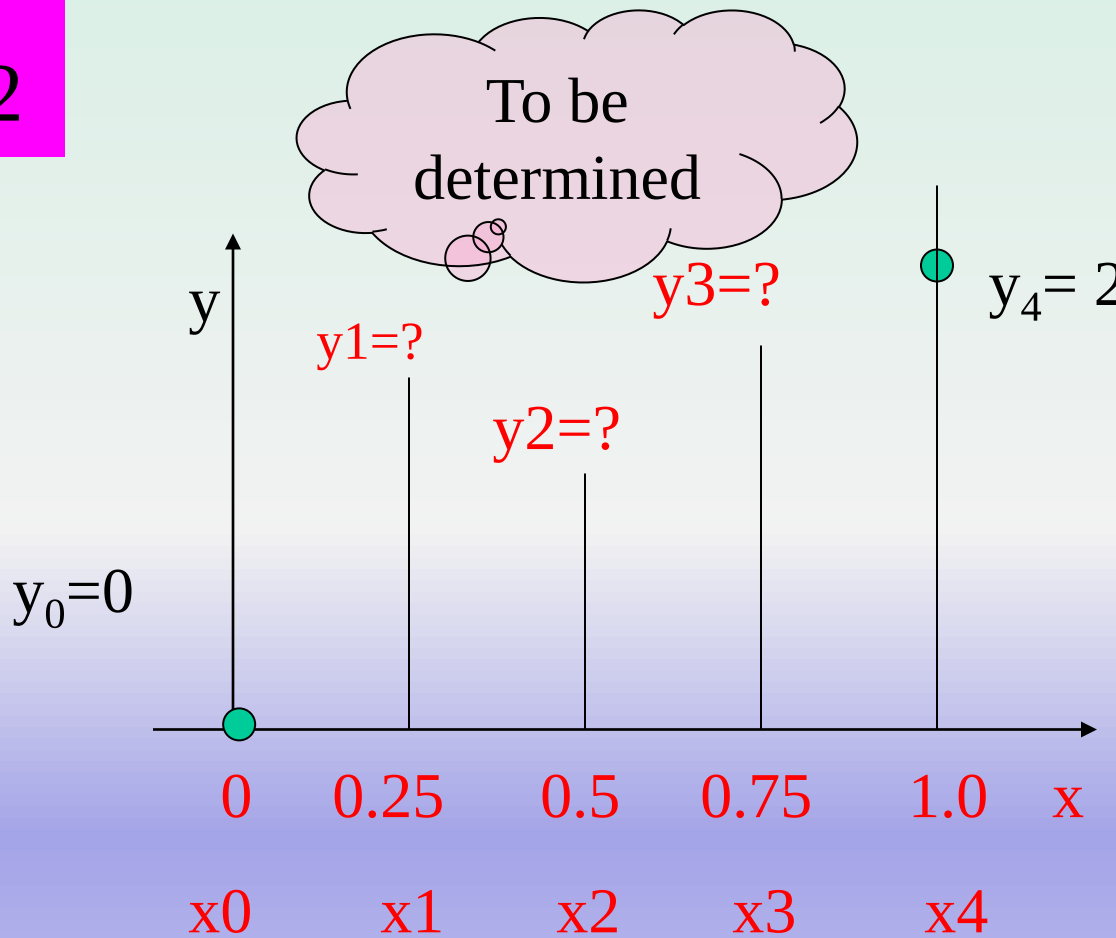
x0=0

x1=0.25

x2=.5

x3=0.75

x4=1.0



Finite Difference Method
Example

$$\ddot{y} - 4y + 4x = 0$$
$$y(0) = 0, \quad y(1) = 2$$

Divide the interval [0,1] into n = 4 intervals

Base points are

x0=0

x1=0.25

x2=.5

x3=0.75

x4=1.0

Replace

$$\ddot{y} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \qquad \textit{central difference formula}$$

$$\dot{y} = \frac{y_{i+1} - y_{i-1}}{2h} \qquad \textit{central difference formula}$$

$$y'' - 4y + 4x = 0$$

Becomes

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - 4y_i = -4x_i$$

Second Order BVP

$$\frac{d^2y}{dx^2} - 4y = -4x \quad with \; y(0) = 0, \qquad y(1) = 2$$

$$Let \quad h = 0.25$$

Base Points

$$x_0 = 0,\; x_1 = 0.25,\; x_2 = 0.5,\; x_3 = 0.75,\; x_4 = 1$$

$$\frac{dy}{dx} \approx \frac{y(x+h) - y(x)}{h} = \frac{y_{i+1} - y_i}{h}$$

$$\frac{d^2y}{dx^2} \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

Second Order BVP

$$\frac{d^2y}{dx^2} - 4y = -4x$$

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - 4y_i = -4x_i \quad i = 1,2,3$$

$$x_0 = 0,\ x_1 = 0.25, x_2 = 0.5, x_3 = 0.75, x_4 = 1$$

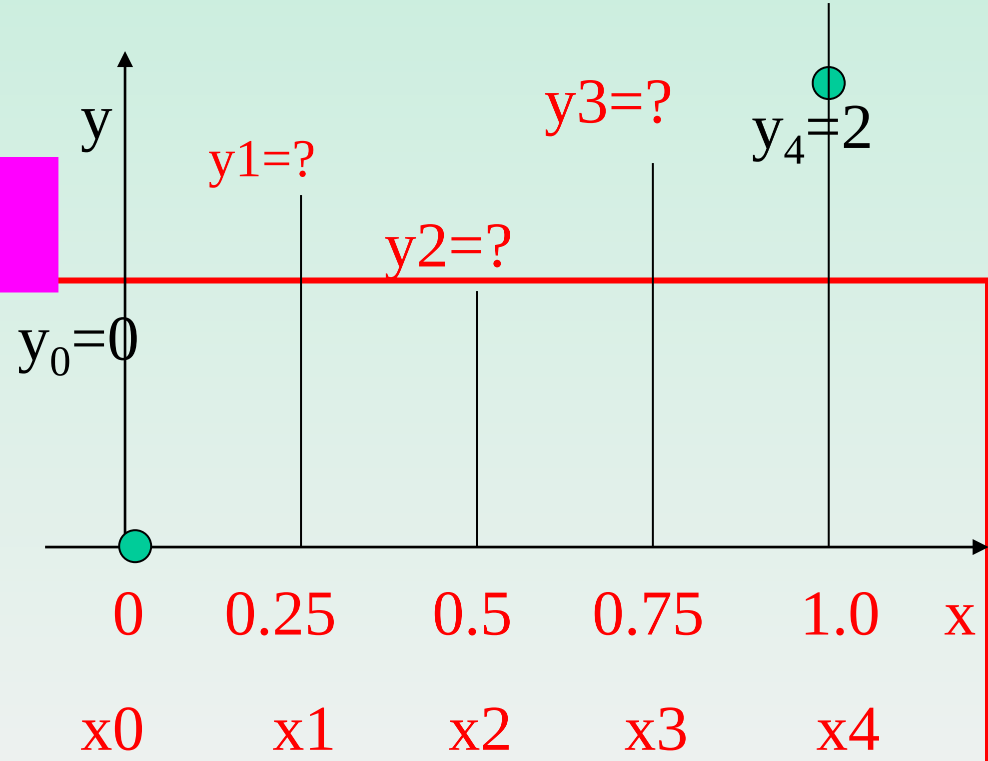
$$y_0 = 0,\ y_1 = ?, y_2 = ?, y_3 = ?,\ \ y_4 = 2$$

$$16\left(y_{i+1} - 2y_i + y_{i-1}\right) - 4y_i = -4x_i$$

$$16y_{i+1} - 36y_i + 16y_{i-1} = -4x_i$$

Second Order BVP

$$16y_{i+1} - 36y_i + 16y_{i-1} = -4x_i$$



$$i=1 \quad 16y_2 - 36y_1 + 16y_0 = -4x_1$$

$$i=2 \quad 16y_3 - 36y_2 + 16y_1 = -4x_2$$

$$i=3 \quad 16y_4 - 36y_3 + 16y_2 = -4x_3$$

$$\begin{bmatrix} -36 & 16 & 0 \\ 16 & -36 & 16 \\ 0 & 16 & -36 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1-16(0) \\ -2 \\ -3-16(2) \end{bmatrix}$$

$$Solution \;\; y_1 = 0.3951, y_2 = 0.8265, y_3 = 1.3396$$

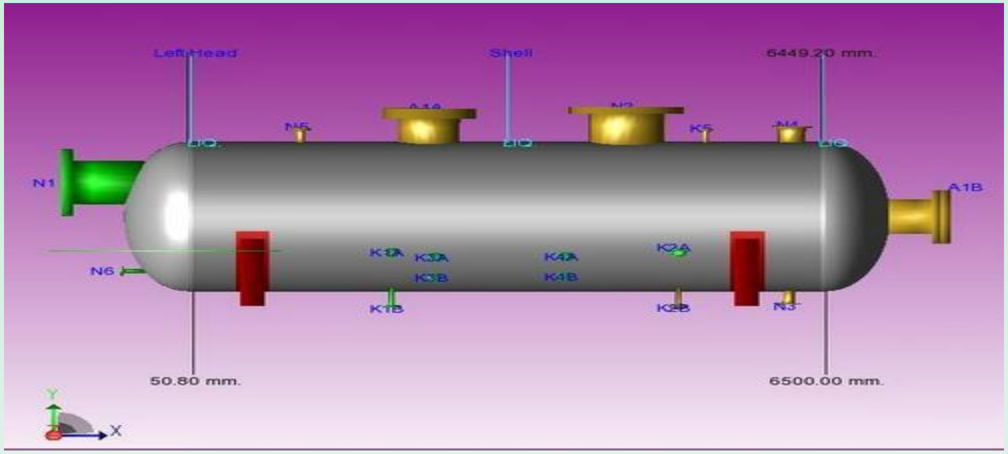
Summary of the Discretiztion Methods

- Select the base points.
- Divide the interval into *n* sub-intervals.
- Use finite approximations to replace the derivatives.
- This approximation results in a set of algebraic equations.
- Solve the equations to obtain the solution of the BVP.

Remarks

Finite Difference Method :

- Different formulas can be used for approximating the derivatives.
- Different formulas lead to different solutions. All of them are approximate solutions.
- For linear second order cases, this reduces to tri-diagonal system.



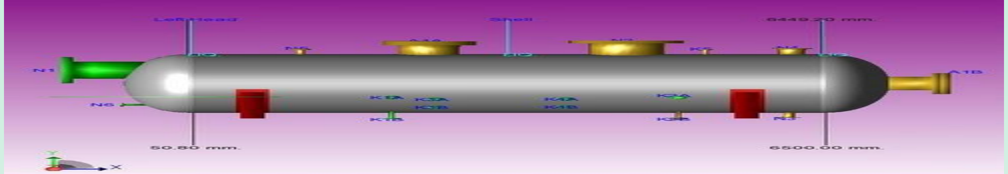
Using the approximation of

$$\frac{d^2y}{dx^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} \qquad \text{and} \qquad \frac{dy}{dx} \approx \frac{y_{i+1} - y_{i-1}}{2(\Delta x)}$$

Gives you

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2} + \frac{1}{r_i}\frac{u_{i+1} - u_{i-1}}{2(\Delta r)} - \frac{u_i}{r_i^{\,2}} = 0$$

$$\left(-\frac{1}{2r_i(\Delta r)} + \frac{1}{(\Delta r)^2}\right)u_{i-1} + \left(-\frac{2}{(\Delta r)^2} - \frac{1}{r_i^{\,2}}\right)u_i + \left(\frac{1}{(\Delta r)^2} + \frac{1}{2r_i\Delta r}\right)u_{i+1} = 0$$



Example 3

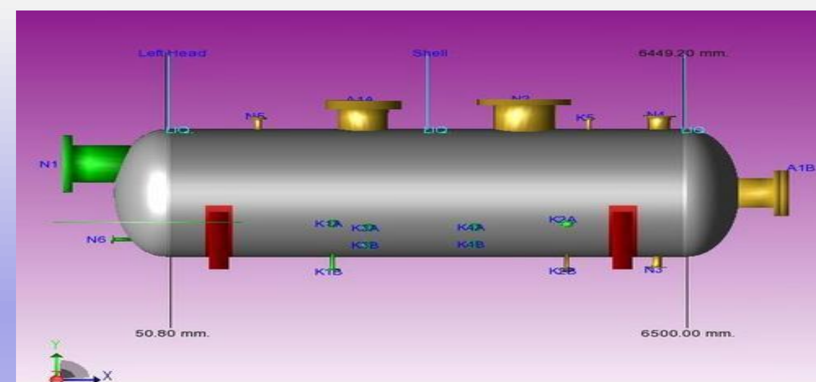
Take the case of a pressure vessel that is being tested in the laboratory to check its ability to withstand pressure. For a thick pressure vessel of inner radius *a* and outer radius *b*, the differential equation for the radial displacement *u* of a point along the thickness is given by

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0$$

The pressure vessel can be modeled as,and limits are 5 to 8 at dr= 0.6

$$\frac{d^2u}{dr^2} \approx \frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta r)^2}$$

$$\frac{du}{dr} \approx \frac{u_{i+1}-u_i}{\Delta r}$$



Substituting these approximations gives,

$$\frac{u_{i+1}-2u_i+u_{i-1}}{(\Delta r)^2}+\frac{1}{r_i}\frac{u_{i+1}-u_i}{\Delta r}-\frac{u_i}{r_i^2}=0$$

$$\left(\frac{1}{(\Delta r)^2}+\frac{1}{r_i\Delta r}\right)u_{i+1}+\left(-\frac{2}{(\Delta r)^2}-\frac{1}{r_i\Delta r}-\frac{1}{r_i^2}\right)u_i+\frac{1}{(\Delta r)^2}u_{i-1}=0$$

Solution

$$\left(\frac{1}{(\Delta r)^2}+\frac{1}{r_i\Delta r}\right)u_{i+1}+\left(-\frac{2}{(\Delta r)^2}-\frac{1}{r_i\Delta r}-\frac{1}{r_i^{\,2}}\right)u_i+\frac{1}{(\Delta r)^2}u_{i-1}=0$$

Step 1 At node $i=0,\quad r_0=a=5''\quad u_0=0.0038731''$

Step 2 At node $i=1,\quad r_1=r_0+\Delta r=5+0.6=5.6''$

$$\frac{1}{(0.6)^2}u_0+\left(-\frac{2}{(0.6)^2}-\frac{1}{(5.6)(0.6)}-\frac{1}{(5.6)^2}\right)u_1+\left(\frac{1}{0.6^2}+\frac{1}{(5.6)(0.6)}\right)u_2=0$$

$$2.7778u_0-5.8851u_1+3.0754u_2=0$$

Step 3 At node $i=2,\quad r_2=r_1+\Delta r=5.6+0.6=6.2''$

$$\frac{1}{0.6^2}u_1+\left(-\frac{2}{0.6^2}-\frac{1}{(6.2)(0.6)}-\frac{1}{6.2^2}\right)u_2+\left(\frac{1}{0.6^2}+\frac{1}{(6.2)(0.6)}\right)u_3=0$$

$$2.7778u_1-5.8504u_2+3.0466u_3=0$$

Solution Cont

$$\left(\frac{1}{(\Delta r)^2}+\frac{1}{r_i\Delta r}\right)u_{i+1}+\left(-\frac{2}{(\Delta r)^2}-\frac{1}{r_i\Delta r}-\frac{1}{r_i^2}\right)u_i+\frac{1}{(\Delta r)^2}u_{i-1}=0$$

Step 4 At node $i=3,\ r_3=r_2+\Delta r=6.2+0.6=6.8''$

$$\frac{1}{0.6^2}u_2+\left(-\frac{2}{0.6^2}-\frac{1}{(6.8)(0.6)}-\frac{1}{6.8^2}\right)u_3+\left(\frac{1}{0.6^2}+\frac{1}{(6.8)(0.6)}\right)u_4=0$$

$$2.7778u_2-5.8223u_3+3.0229u_4=0$$

Step 5 At node $i=4,\ r_4=r_3+\Delta r=6.8+0.6=7.4''$

$$\frac{1}{0.6^2}u_3+\left(-\frac{2}{0.6^2}-\frac{1}{(7.4)(0.6)}-\frac{1}{(7.4)^2}\right)u_4+\left(\frac{1}{0.6^2}+\frac{1}{(7.4)(0.6)}\right)u_5=0$$

$$2.7778u_3-5.7990u_4+3.0030u_5=0$$

Step 6 At node $i=5,\ r_5=r_4+\Delta r=7.4+0.6=8$

$$u_5=u|_{r=b}=0.0030769''$$

Solving system of equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2.7778 & -5.8851 & 3.0754 & 0 & 0 & 0 \\ 0 & 2.7778 & -5.8504 & 3.0466 & 0 & 0 \\ 0 & 0 & 2.7778 & -5.8223 & 3.0229 & 0 \\ 0 & 0 & 0 & 2.7778 & -5.7990 & 3.0030 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0.0038731 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0030769 \end{bmatrix}$$

$u_0 = 0.0038731$ $\qquad$ $u_3 = 0.0032743$

$u_1 = 0.0036165$ $\qquad$ $u_4 = 0.0031618$

$u_2 = 0.0034222$ $\qquad$ $u_5 = 0.0030769$

Solution Cont

$$\frac{du}{dr}\bigg|_{r=a} \approx \frac{u_1 - u_0}{\Delta r} = \frac{0.0036165 - 0.0038731}{0.6} \quad = -0.00042767$$

$$\sigma_{\max} = \frac{30\times10^6}{1-0.3^2}\left(\frac{0.0038731}{5} + 0.3(-0.00042767)\right) = 21307\, psi$$

$$FS = \frac{36\times10^3}{21307} = 1.6896$$

$$E_t = 20538 - 21307 = -768.59$$

$$\left|\in_t\right| = \left|\frac{20538-21307}{20538}\right| \times 100 = 3.744\,\%$$

Solution Cont

Step 1 At node $i = 0, r_0 = a = 5$

$$u_0 = 0.0038731$$

Step 2 At node $i = 1,\ r_1 = r_0 + \Delta r = 5 + 0.6 = 5.6''$

$$\left(-\frac{1}{2(5.6)(0.6)} + \frac{1}{(0.6)^2}\right)u_0 + \left(-\frac{2}{(0.6)^2} - \frac{1}{(5.6)^2}\right)u_1 + \left(\frac{1}{0.6^2} + \frac{1}{2(5.6)(0.6)}\right)u_2 = 0$$

$$2.6297u_0 - 5.5874u_1 + 2.9266u_2 = 0$$

Step 3 At node $i = 2,\quad r_2 = r_1 + \Delta r = 5.6 + 0.6 = 6.2$

$$\left(-\frac{1}{2(6.2)(0.6)} + \frac{1}{0.6^2}\right)u_1 + \left(-\frac{2}{0.6^2} - \frac{1}{6.2^2}\right)u_2 + \left(\frac{1}{0.6^2} + \frac{1}{2(6.2)(0.6)}\right)u_3 = 0$$

$$2.6434u_1 - 5.5816u_2 + 2.9122u_3 = 0$$

Solution Cont

Step 4 At node $i = 3,\ r_3 = r_2 + \Delta r = 6.2 + 0.6 = 6.8$

$$\left(-\frac{1}{2(6.8)(0.6)} + \frac{1}{0.6^2}\right)u_2 + \left(-\frac{2}{0.6^2} - \frac{1}{6.8^2}\right)u_3 + \left(\frac{1}{0.6^2} + \frac{1}{2(6.8)(0.6)}\right)u_4 = 0$$

$$2.6552u_2 - 5.5772u_3 + 2.9003u_4 = 0$$

Step 5 At node $i = 4,\ r_4 = r_3 + \Delta r = 6.8 + 0.6 = 7.4$

$$\left(-\frac{1}{2(7.4)(0.6)} + \frac{1}{0.6^2}\right)u_3 + \left(-\frac{2}{0.6^2} - \frac{1}{(7.4)^2}\right)u_4 + \left(\frac{1}{0.6^2} + \frac{1}{2(7.4)(0.6)}\right)u_5 = 0$$

$$2.6651u_3 - 5.5738u_4 + 2.8903u_5 = 0$$

Step 6 At node $i = 5,\ r_5 = r_4 + \Delta r = 7.4 + 0.6 = 8''$

$$u_5 = u|_{r=b} = 0.0030769''$$

Solving system of equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2.6297 & -5.5874 & 2.9266 & 0 & 0 & 0 \\ 0 & 2.6434 & -5.5816 & 2.9122 & 0 & 0 \\ 0 & 0 & 2.6552 & -5.5772 & 2.9003 & 0 \\ 0 & 0 & 0 & 2.6651 & -5.5738 & 2.8903 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 0.0038731 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.0030769 \end{bmatrix}$$

$u_0 = 0.0038731$ $\quad u_3 = 0.0032689$

$u_1 = 0.0036115$ $\quad u_4 = 0.0031586$

$u_2 = 0.0034159$ $\quad u_5 = 0.0030769$

Solution Cont

$$\frac{du}{dr}\bigg|_{r=a} \approx \frac{-3u_0 + 4u_0 - u_2}{2(\Delta r)} = \frac{-3\times 0.0038731 + 4\times 0.0036115 - 0.0034159}{2(0.6)} = -0.0004925$$

$$\sigma_{\max} = \frac{30\times 10^6}{1-0.3^2}\left(\frac{0.0038731}{5} + 0.3(-0.0004925)\right) = 20666\,psi$$

$$FS = \frac{36\times 10^3}{20666} = 1.7420$$

$$E_t = 20538 - 20666 = -128$$

$$|\in_t| = \left|\frac{20538 - 20666}{20538}\right| \times 100 = 0.62323\ \%$$

Comparison of radial displacements

Table 1 Comparisons of radial displacements from two methods

r	u_{exact}	$u_{1st\ order}$	$\lvert\epsilon_t\rvert$	$u_{2nd\ order}$	$\lvert\epsilon_t\rvert$
5	0.0038731	0.0038731	0.0000	0.0038731	0.0000
5.6	0.0036110	0.0036165	1.5160×10^{-1}	0.0036115	1.4540×10^{-2}
6.2	0.0034152	0.0034222	2.0260×10^{-1}	0.0034159	1.8765×10^{-2}
6.8	0.0032683	0.0032743	1.8157×10^{-1}	0.0032689	1.6334×10^{-2}
7.4	0.0031583	0.0031618	1.0903×10^{-1}	0.0031586	9.5665×10^{-3}
8	0.0030769	0.0030769	0.0000	0.0030769	0.0000

Boundary-Value and Initial Value Problems

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad \dot{x}(0) = 2.5$$

same

Boundary-Value Problems

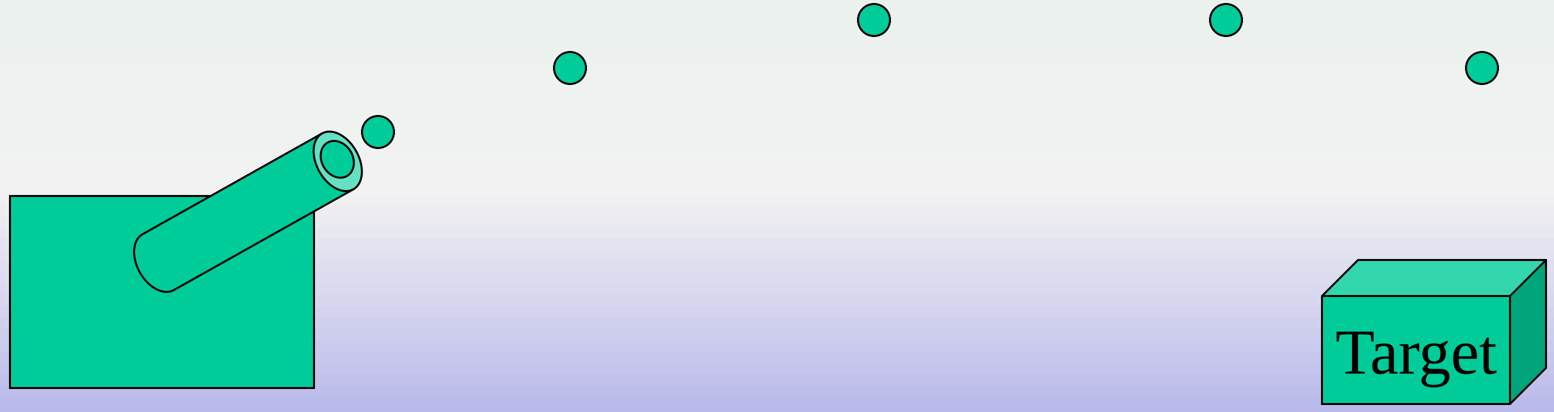
- The auxiliary conditions are not at one point of the independent variable
- More difficult to solve than initial value problem

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 1.5$$

different

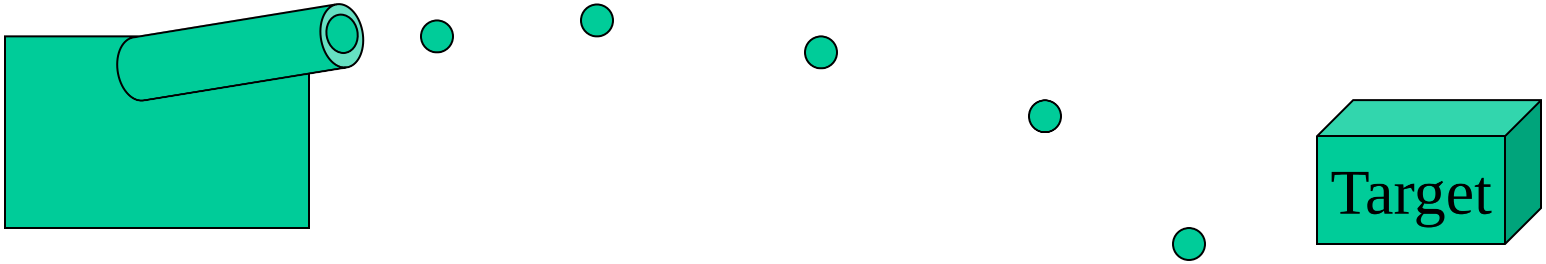
The Shooting Method



$y_0 = y_a$ assume $y'_a = \gamma$

$y_n = y_b$

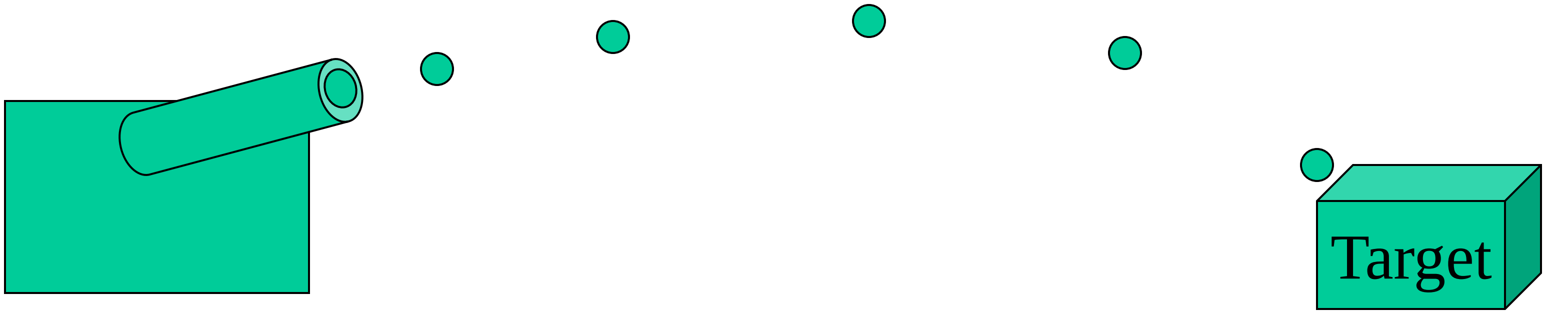
The Shooting Method



$y_0 = y_a$ **assume** $y'_a = \gamma$

$y_n = y_b$

The Shooting Method



$y_0 = y_a$ **assume** $y'_a = \gamma$

$y_n = y_b$

Solution of Boundary-Value Problems

Shooting Method for Boundary-Value Problems

1. Guess a value for the auxiliary conditions at one point of time.
2. Solve the initial value problem using Euler, Runge-Kutta, …
3. **Check if the boundary conditions are satisfied, otherwise modify the guess and resolve the problem.**

- **Use interpolation in updating the guess.**
- **It is an iterative procedure and can be efficient in solving the BVP.**

Solution of Boundary-Value Problems

Shooting Method

Boundary-Value Problem

convert

Initial-value Problem

$$Find\ y(x)\ to\ solve\ BVP$$

$$y'' + 2y' + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

1. **Convert the ODE to a system of first order ODEs.**
2. **Guess the initial conditions that are not available.**
3. **Solve the Initial-value problem.**
4. **Check if the known boundary conditions are satisfied.**
5. **If needed modify the guess and resolve the problem again.**

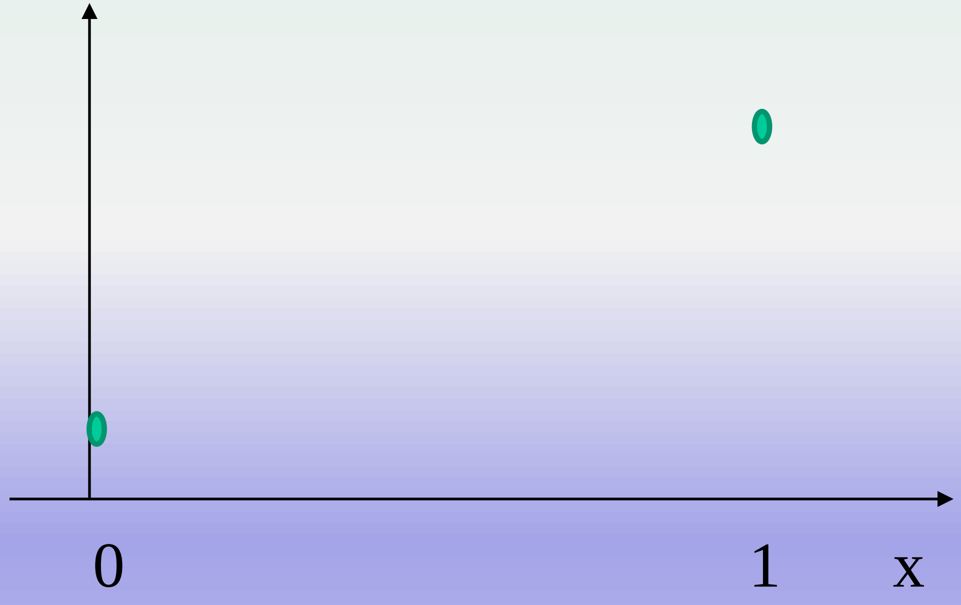
Example 0

Original BVP

$$Find\ y(x)\ to\ solve\ BVP$$

$$y'' + 2y' + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$



Example 0

Original BVP

$$y'' + 2y' + y = x^2$$

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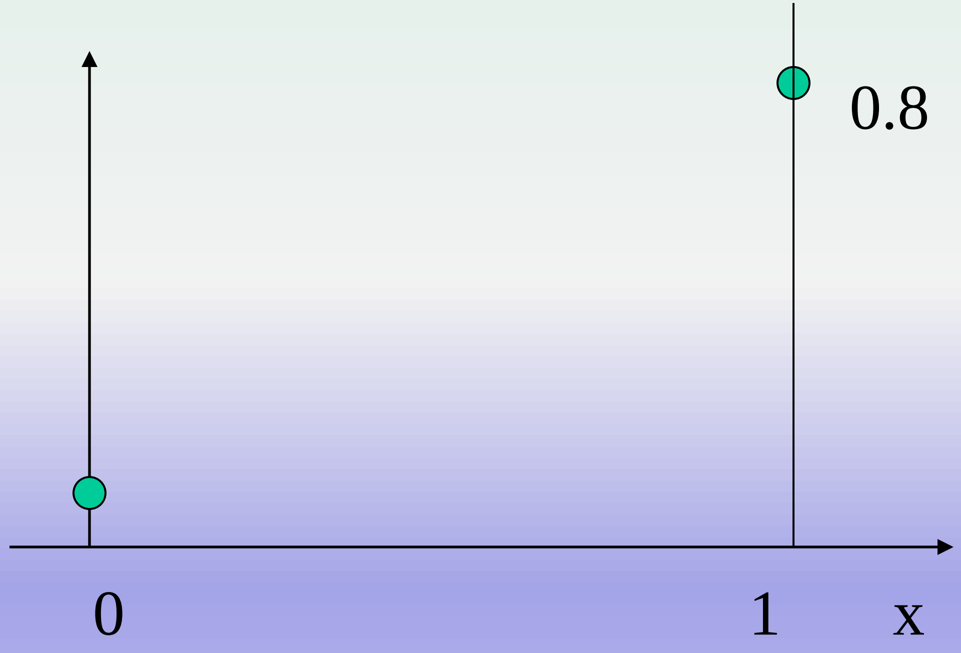


Example 0

Guess # 1

$$y'' + 2y' + y = x^2$$

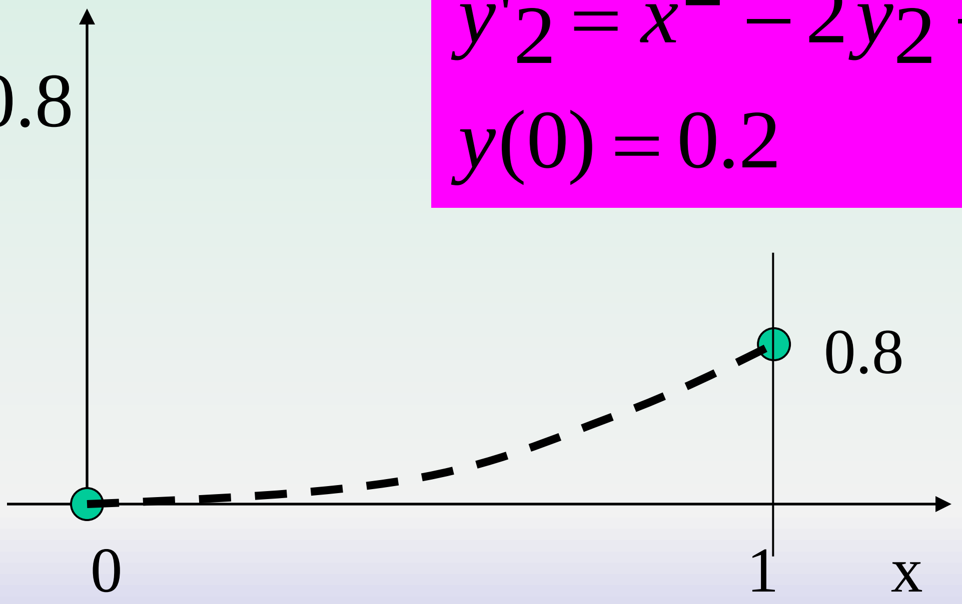
$$y(0) = 0.2, \quad y(1) = 0.8$$

$$y'_2 = x^2 - 2y_2 - y_1$$

$$y(0) = 0.2$$

$$Guess\#1$$

$$\dot{y}(0) = 0$$



Example 0

Guess # 1

$$y'' + 2y' + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

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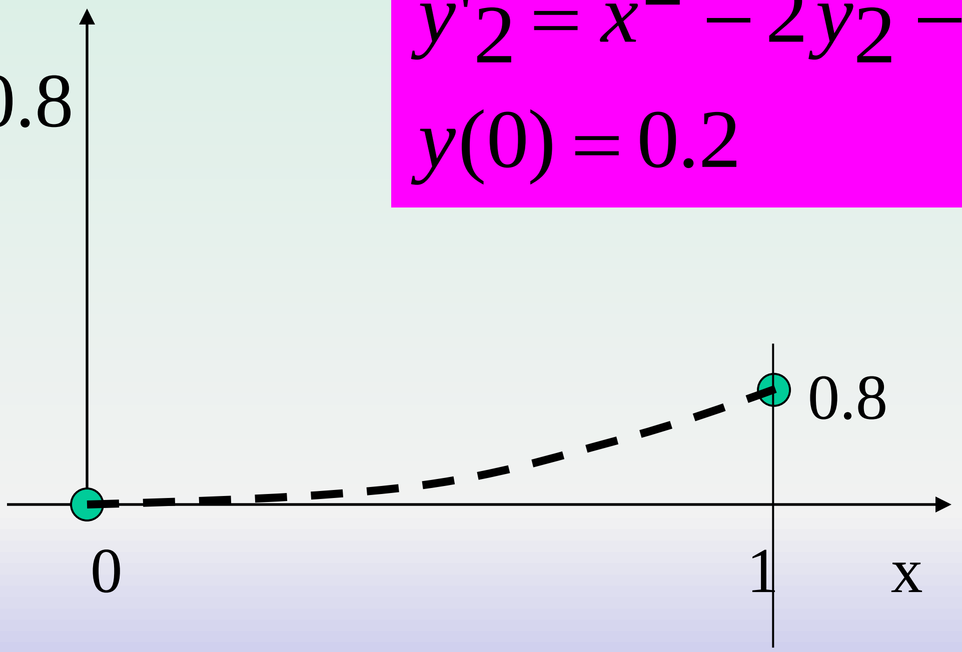
$$\dot{y}(0) = 0$$



Example 0

Guess # 2

$$\ddot{y} - 4y + 4x = 0$$

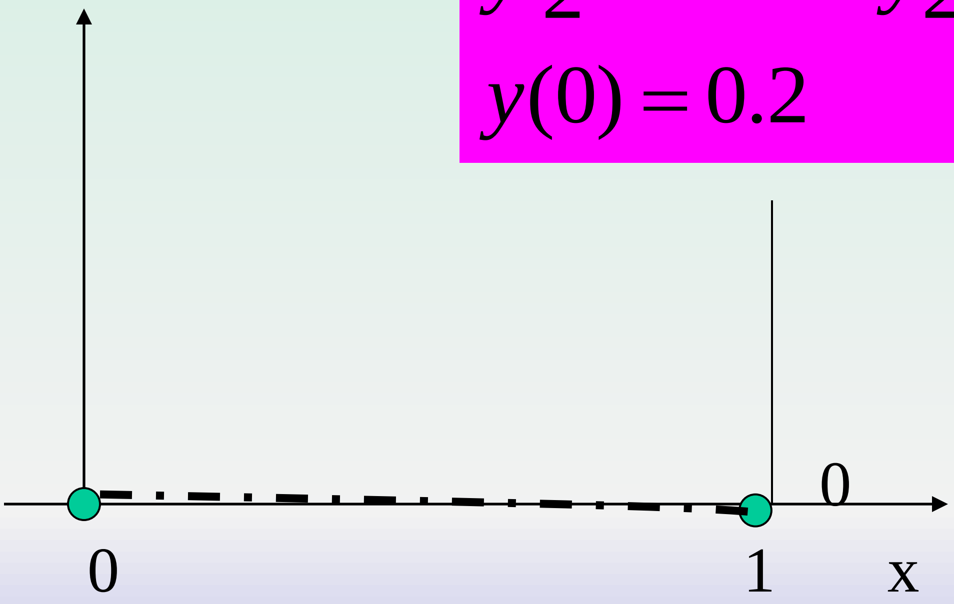
$$y(0) = 0, \quad y(1) = 2$$

$$y'_2 = x^2 - 2y_2 - y_1$$

$$y(0) = 0.2$$

$$Guess\#2$$

$$\dot{y}(0) = 1$$



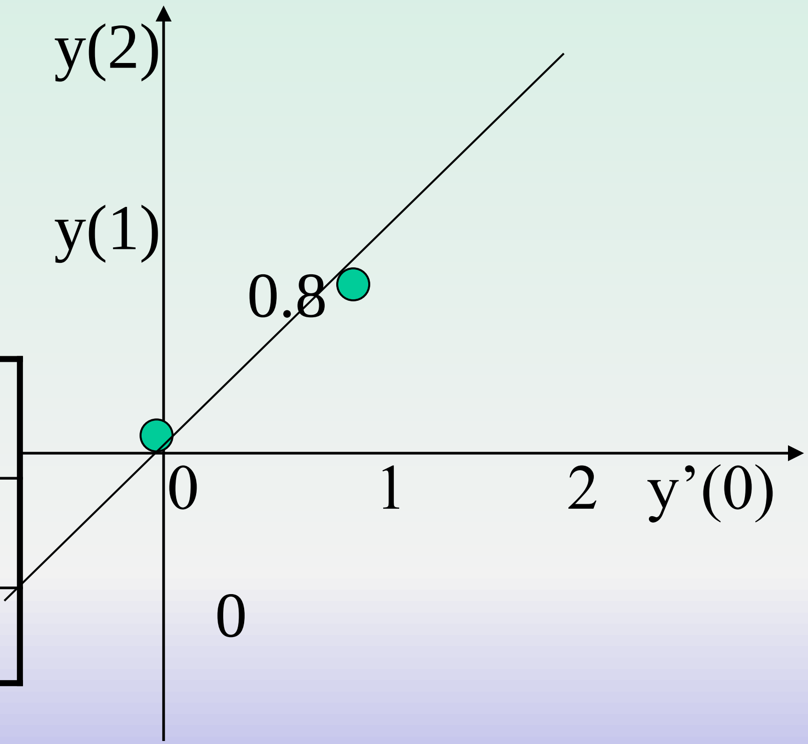
Example 0

Interpolation for Guess # 3

$$y'_2 = x^2 - 2y_2 - y_1$$
$$y(0) = 0.2$$

$$y'' + 2y' + y = x^2$$
$$y(0) = 0.2, \quad y(1) = 0.8$$

Guess	$\dot{y}(0)$	y(1)
1	0	0.8
2	1	0.0



$$y'(0) = 1.6$$

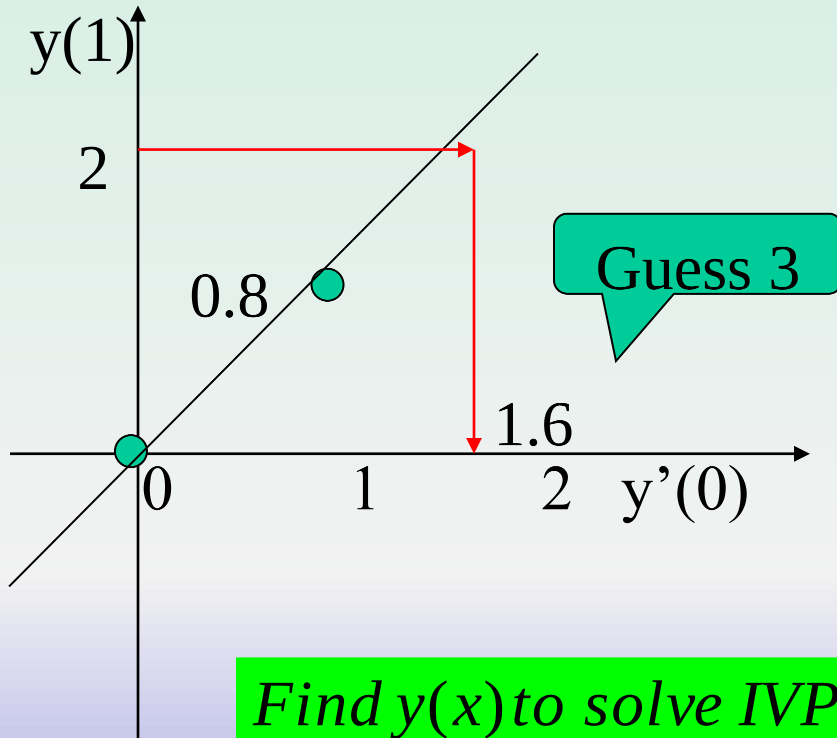
Example 0

Interpolation for Guess # 3

$$y'' + 2y' + y = x^2$$

$$y(0) = 0.2, \quad y(1) = 0.8$$

Guess	$\dot{y}(0)$	y(1)
1	0	0.8
2	1	0



Find $y(x)$ *to solve IVP*

$$\ddot{y} + 2\dot{y} + y = x^2$$

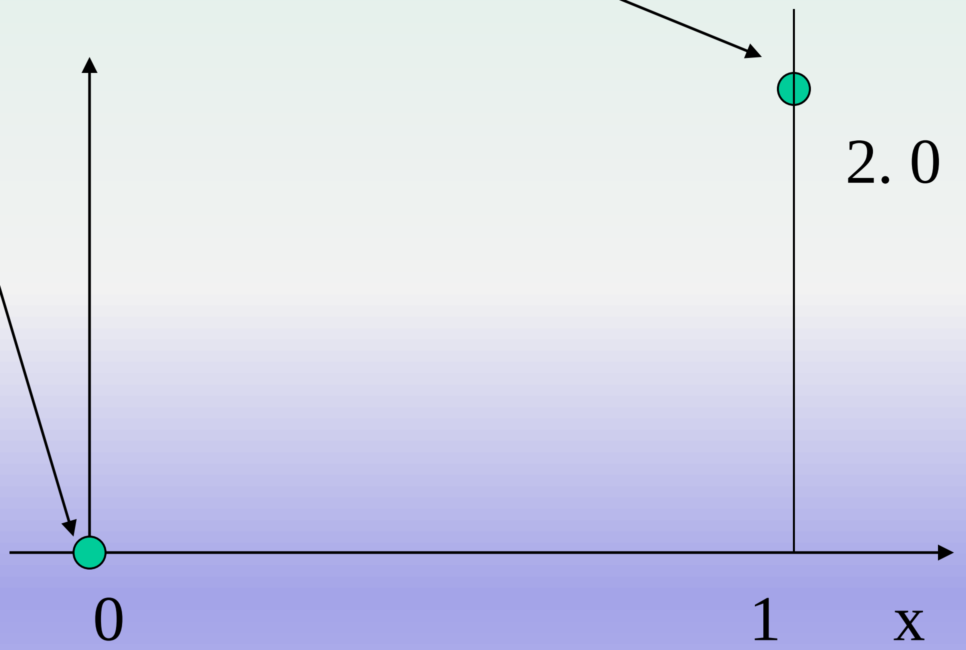
$$y(0) = 0.2, \quad y'(0) = 1.6$$

Example 1

Original BVP

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

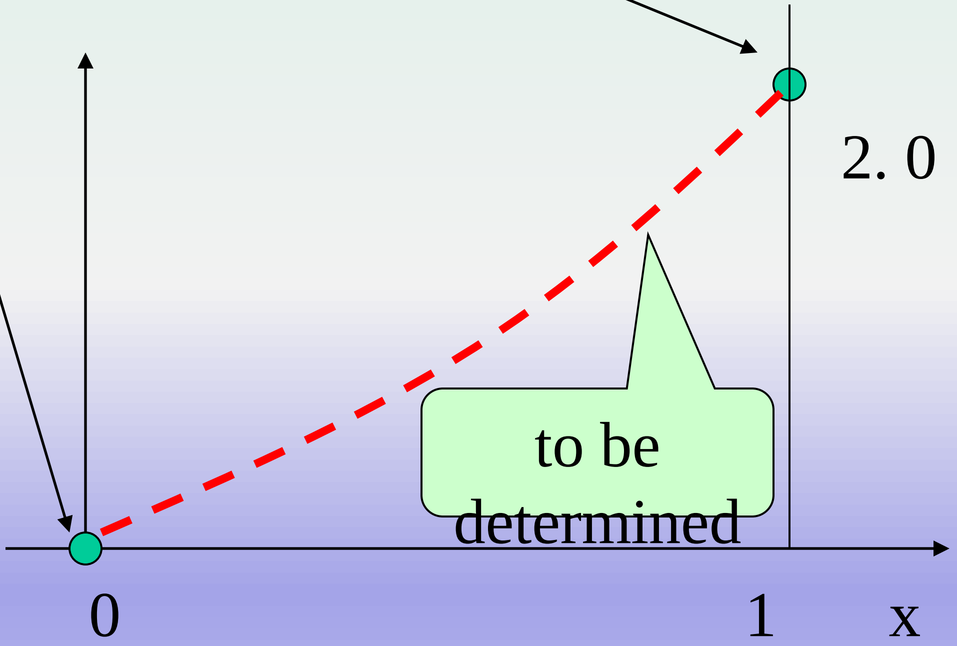


Example 1

Original BVP

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$



Example 1

Step1: Convert to a System of First Order ODEs

$$y'' - 4y' + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

Convert to a system of first order Equations

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ 4(y_1 - x) \end{bmatrix}, \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ ? \end{bmatrix}$$

The problem will be solved using RK2 with $h = 0.01$

for different values of $y_2(0)$ until we have $y(1) = 2$

Example 1

Guess # 1

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ 4(y_1 - x) \end{bmatrix}$$

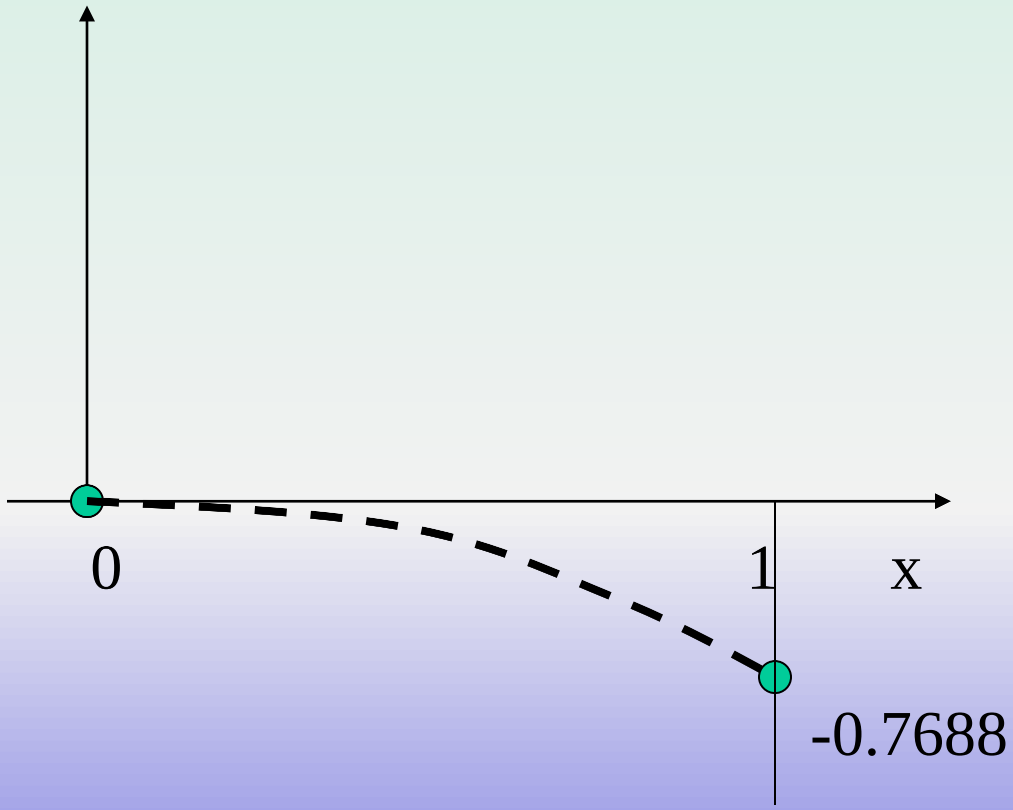
$$y(0) = 0$$

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

$$Guess\#1$$

$$\dot{y}(0) = 0$$



Example 1

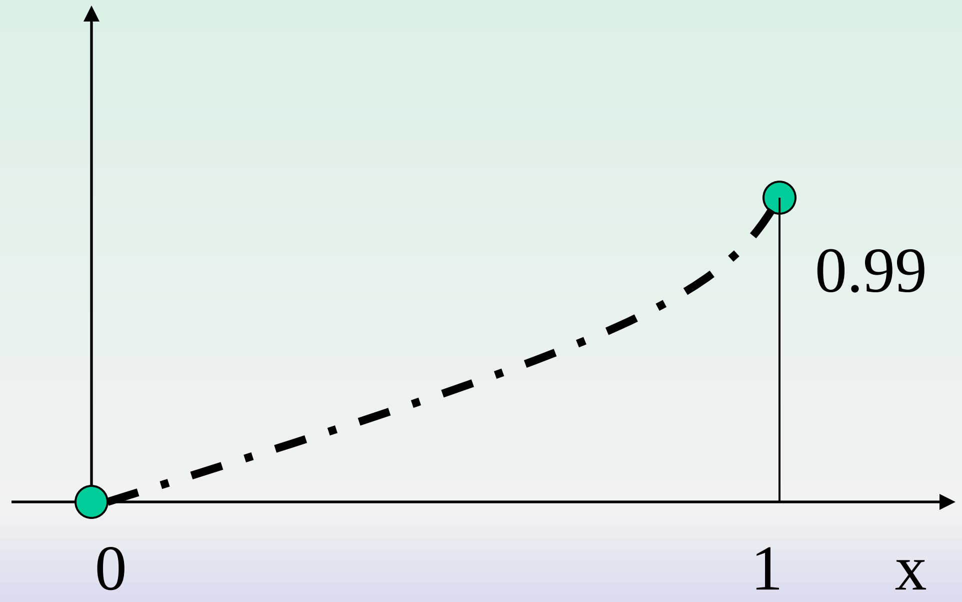
Guess # 2

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

$$Guess \#2$$

$$\dot{y}(0) = 1$$



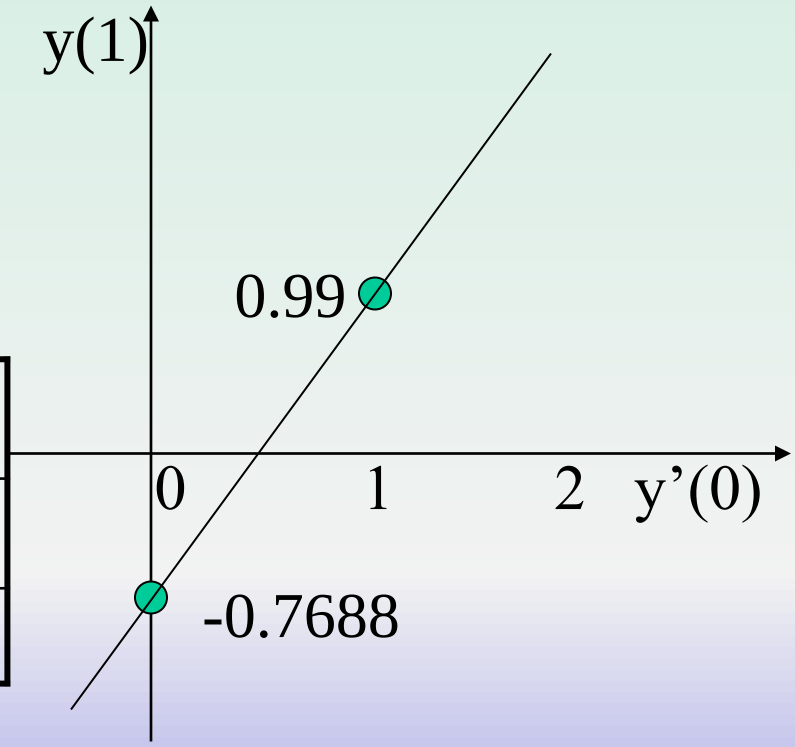
Example 1

Interpolation for Guess # 3

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

Guess	$\dot{y}(0)$	y(1)
1	0	-0.7688
2	1	0.9900



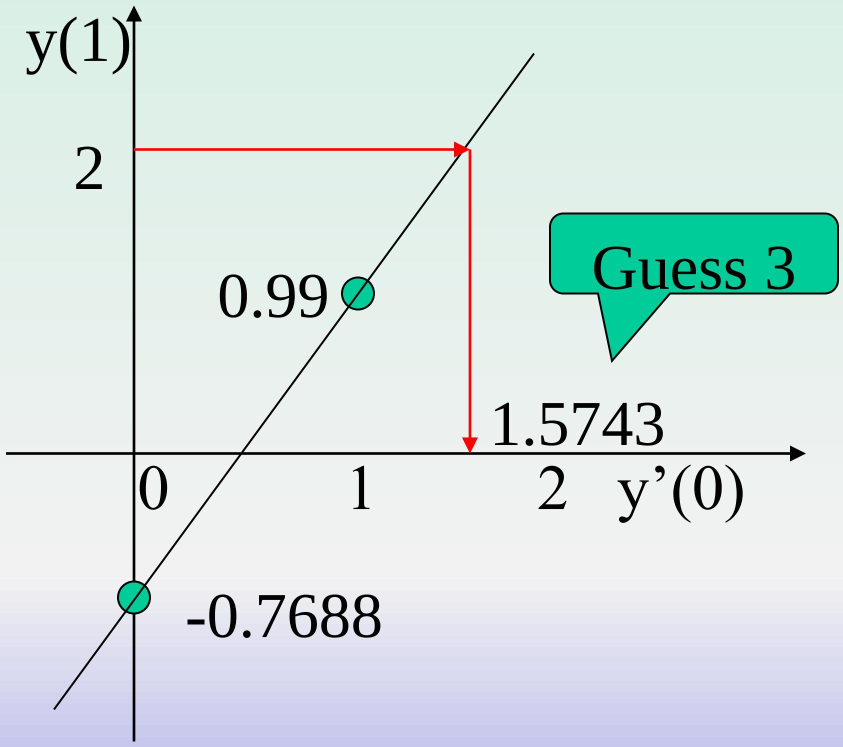
Example 1

Interpolation for Guess # 3

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \;\; y(1) = 2$$

Guess	$\dot{y}(0)$	y(1)
1	0	-0.7688
2	1	0.9900



Example 1

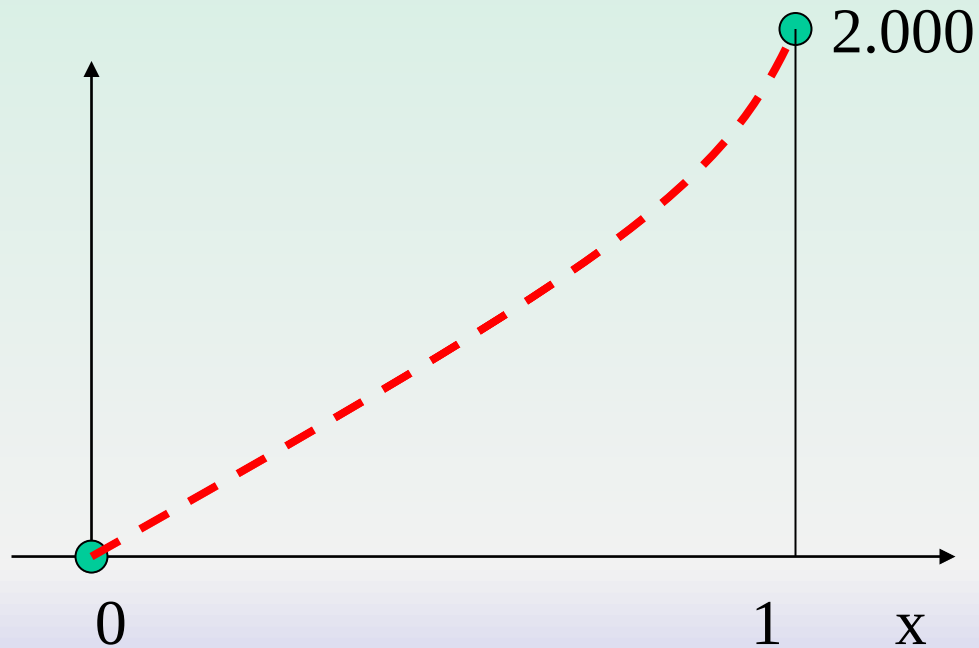
Guess # 3

$$\ddot{y} - 4y + 4x = 0$$

$$y(0) = 0, \quad y(1) = 2$$

$Guess\,\#3$

$$\dot{y}(0) = 1.5743$$



y(1)=2.000

This is the solution to the boundary value problem.

Summary of the Shooting Method

1. Guess the unavailable values for the auxiliary conditions at one point of the independent variable.
2. Solve the initial value problem.
3. Check if the boundary conditions are satisfied, otherwise modify the guess and resolve the problem.
4. Repeat (3) until the boundary conditions are satisfied.

Properties of the Shooting Method

1. Using interpolation to update the guess often results in few iterations before reaching the solution.

2. The method can be cumbersome for high order BVP because of the need to guess the initial condition for more than one variable.


Thank You!