The Fundamental Theorem of Dynamical Systems

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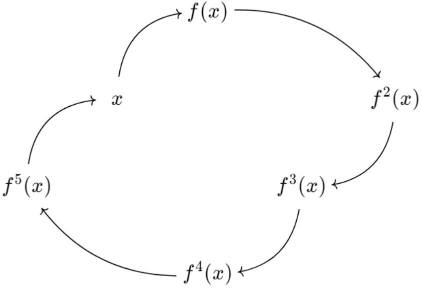
Recurrence Behaviours of Dynamics - Fixed Points and Periodic Points

In all slides consider (X, d) as a compact metric space and a homeomorphism $f : X \to X$ as the dynamic.

- Fixed Points: $F.P.(X, f) := \{x \in X : f(x) = x\}$
- Periodic Points: $P.P.(X, f) := \{x \in X : \exists n \in \mathbb{N} \text{ s.t. } f^n(x) = x\}$



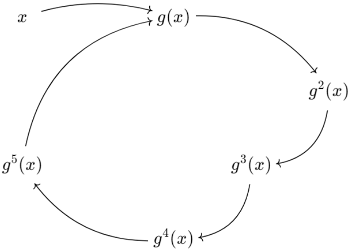
Recurrence Behaviours of Dynamics - Recurrent Points

Definition. The ω-limit set of $x \in X$ is the set of all limit points of orb(x).

$$\omega(x) := \{\bar{x} \in X : \exists \{n_i\}_{i\in\mathbb{N}} \text{ s.t. } \lim_{i\to\infty} f^{n_i} = \bar{x}\}$$

- Recurrent Points: $x \in X$ is a recurrent point if it's a limit point of it's own orbit.

$$R.P.(X, f) := \{x \in X : x \in \omega(x)\}$$



Consider Irrational rotation on S^1:

$$\forall x \in S^1 : \omega(x) = S^1 \implies R.P.(S^1, R_\alpha) = S^1$$

Recurrence Behaviours of Dynamics - Non-Wandering Points

- Non-Wandering Points: $x \in X$ is a non-wandering point if every open neighbourhood of that come back to itself.

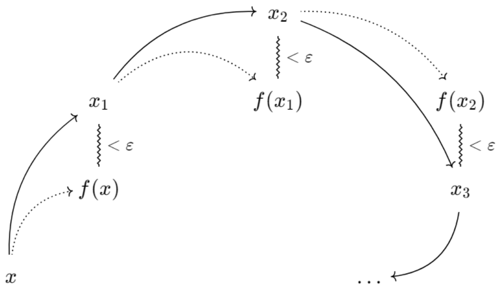
$$N.W.P.(X, f) := \{x \in X : \forall U_x \ni x, \exists n \in \mathbb{N}, f^n(U_x) \cap U_x \neq \emptyset\}$$

Consider The doubling map on S^1: consider $x = \frac{1}{2}$. x is not a recurrent point since $f^n(x) = 0$ for all $n \geq 1$. But every open interval $I_x \ni x$ goes to an interval $f(I_x)$ with length $2\times$ (length(I_x)).Thus for some $n \in \mathbb{N}$, $f^n(I_x)$ cover the whole S^1. Therefore x is a non-wandering point.

Recurrence Behaviours of Dynamics - Chain Recurrent Points

Definition. We called $\{x_n\}_{n\in\mathbb{N}} \subset X$ an ε-pseudo orbit of $x = x_0$ if

$$\forall n \in \mathbb{N} : d(x_n, f(x_{n-1})) < \varepsilon$$

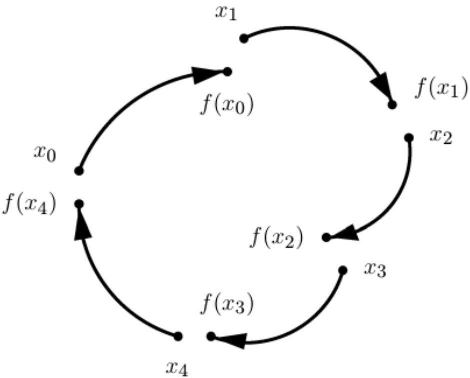


We say there is an ε-pseudo orbit from x to y if there is an ε-pseudo orbit of x passing through y. If for every ε it's true, we write $x \curvearrowright y$

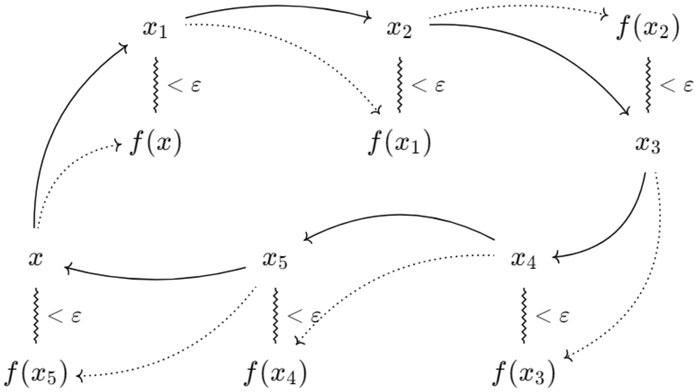
Recurrence Behaviours of Dynamics - Chain Recurrent Points

- Chain Recurrent Points: $x \in X$ is a chain recurrent point if there exists ε-chain from x to itself for all $\varepsilon > 0$.

$$\mathcal{R}(X, f) = \{x \in X : x \curvearrowright x\}$$



Recurrence Behaviours of Dynamics - Chain Recurrent Points



Recurrence Behaviours of Dynamics - Chain Recurrent Points

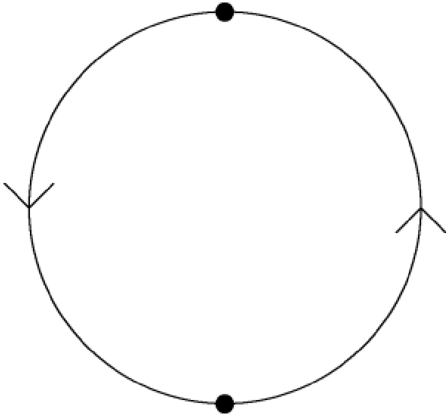


Figure: $f(\theta) = \theta + \alpha \cos^2 \theta$

Recurrence Behaviours of Dynamics

Conclusion:

$$F.P.(X,f) \subset P.P.(X,f) \subset R.P.(X,f) \subset N.W.P.(X,f) \subset \mathcal{R}(X,f)$$

Fundamental Theorem of Dynamical Systems

We want to see whether we can decompose the space X to the chain recurrent classes like what we did for finite dynamics using grand-orbits.

Notice that $\mathcal{R}(X, f)$ is a closed invariant set!

Fundamental Theorem of Dynamical Systems

Let define a relation:

$$x \sim y \longleftrightarrow (x \curvearrowright y \text{ and } y \curvearrowright x)$$

it's obvious that this is not an equivalence relation on X since necessarily we do not have $x \sim x$ for all $x \in X$.

Thus, let's work on $\mathcal{R}(X, f)$!

$\sim$ is an equivalence relation on $\mathcal{R}(X, f)$

Fundamental Theorem of Dynamical Systems

$\sim$ is an equivalence relation on $\mathcal{R}(X, f)$

Each equivalence classes are somehow like **Periodic-Orbits**.

Look at the induced dynamic of f in the space $\mathcal{R}(X, f)/\sim$:

Every point is a fixed point!

Complete Lyapunov Function

Definition. A complete Lyapunov function for the system (X, f) is a continuous function $g : X \to \mathbb{R}$ satisfying:

(i) if $x \notin \mathcal{R}(X, f)$, then $g\big(f(x)\big) < g(x)$

(ii) if $x, y \in \mathcal{R}(X, f)$, then $g(x) = g(y)$ if and only if $x \sim y$.

(iii) $g(\mathcal{R}(X, f))$ is a compact nowhere dense subset of $\mathbb{R}$.

The History

It's as simple as ABC :)

where ABC stand for Anosov, Bowen, and Conley!

Attractors & Repellers

A set $A \subset X$ is an attractor for f if

(i) A is a nonempty compact invariant set.

(ii) there exists a neighborhood U of A such that $f(\bar{U}) \subset U$ and $\bigcap_{n\geq 0} f^n(\bar{U}) = A$. U is called an isolating neighbourhood for A.

Notice that $V := X \setminus \bar{U}$ is an isolating neighbourhood of $A^* := \bigcap_{n\geq 0} f^{-n}(\bar{V})$ and A^* is an attractor for (X, f^{-1}). We call A^* a repeller dual to A.

Attractors & Repellers

Lemma. There are at most countable number of attractors and repellers.

Lemma. Let $\{A_n\}_{n\in\mathbb{N}}$ be the attractors of (X, f). Then

$$\mathcal{R}(X, f) = \bigcap_{n\in\mathbb{N}} \Big(A_n \cup A_n^*\Big)$$

Lemma. if $x, y \in \mathcal{R}(X, f)$, then $x \sim y$ if and only if there is no attractor A such that $x \in A$ and $y \in A^*$ or $x \in A^*$ and $y \in A$

The Conley's Theorem

Lemma. There is a continuous function $g : X \to [0,1]$ such that $g^{-1}(0) = A$, $g^{-1}(1) = A^*$ and g is strictly decreasing on orbits of points in $X \setminus (A \cup A^*)$.

The Conley's Fundamental Theorem of Dynamical systems

Theorem. There is a Complete Lyapunov Function for the system (X, f).

How Big the Recurrent Part is?

Poincare Recurrence Theorem Suppose μ is a finite Borel measure on X and $f : X \to X$ is a measure preserving transformation. If $E \subset X$ is measurable and $\mathcal{N}$ is the subset of E given by

$$\mathcal{N} := \{x \in E : \#(\mathsf{orb}(x) \cap E) < \infty\}$$

Then $\mathcal{N}$ is measurable and $\mu(\mathcal{N}) = 0$.

Corollaries of Poincare Recurrence Theorem

Corollary. Under the assumptions of Poincare Recurrence Theorem, the set of not-recurrence points are of measure zero.

Corollary. If you could find a finite f-invariant Borel measure μ such that $\mu(U) > 0$ for all non-empty open sets $U \subset X$, then recurrent points are dense in X.

Corollary. If you could find a finite f-invariant Borel measure μ such that $\mu(U) > 0$ for all non-empty open sets $U \subset X$, then if X is connected, for every $x, y \in X$, $x \curvearrowright y$.

References

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Thank you!