Chaos

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Definition

There is NO standard definition for chaos!

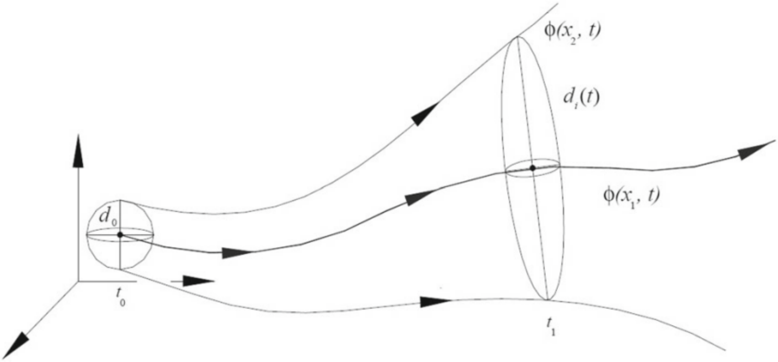
But there are always two elements that at least one of them is essential for any definition:

- Sensitivity to Initial Condition
- Complex Orbit Structure

Sensitivity to Initial Condition

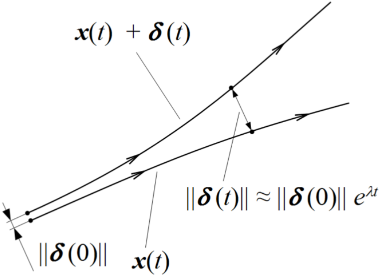
$$\exists \epsilon > 0 : \forall x \in \Lambda : \forall U_x : \exists y \in U_x : \exists t > 0 : ||\phi(t,x) - \phi(t,y)|| > \epsilon$$

Lyapunov Exponents



Lyapunov Exponents

$$\sigma_{x_r}(\delta) := \limsup_{t\to\infty} \frac{1}{t} \ln \frac{||\delta(t)||}{||\delta(t_0)||} = \limsup_{t\to\infty} \frac{1}{t} \ln ||\delta(t)||$$



Lyapunov Exponent

PRODUCTS OF RANDOM MATRICES

BY H. FURSTENBERG AND H. KESTEN[1]

Princeton University

Figure: 1960

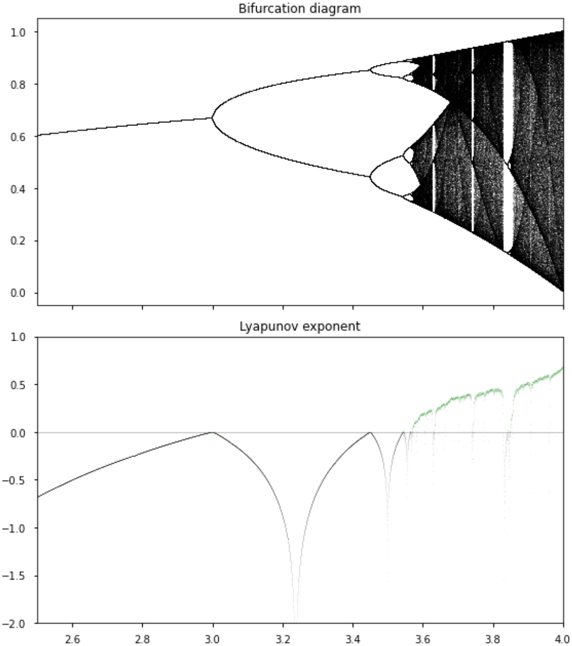
Lyapunov Exponent

УДК 517.863

МУЛЬТИПЛИКАТИВНАЯ ЭРГОДИЧЕСКАЯ ТЕОРЕМА. ХАРАКТЕРИСТИЧЕСКИЕ ПОКАЗАТЕЛИ ЛЯПУНОВА ДИНАМИЧЕСКИХ СИСТЕМ

В. И. Оселедец

Figure: Multiplicative ergodic theorem: Characteristic Lyapunov exponents of dynamical systems - 1968


Bifurcation diagram
1.0
0.8
0.6
0.4
0.2
0.0
Lyapunov exponent
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0

Lyapunov Exponent

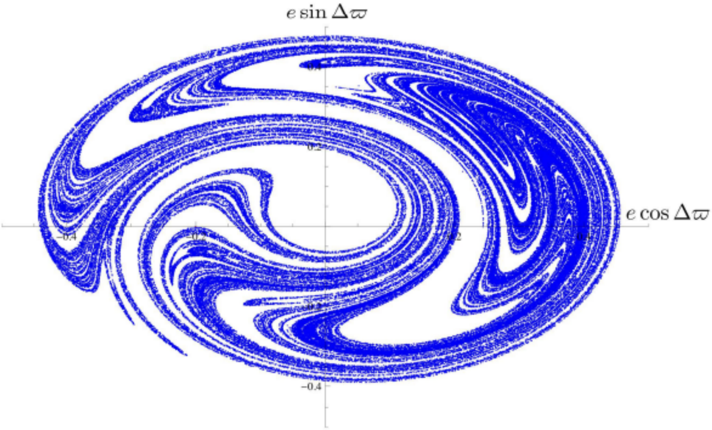
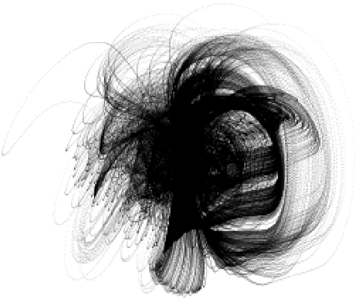


Figure: A strange attractor - Lyapunov Exponent $= 1.87 \times 10^{-6}$

Complex Orbit Structure

NOSÉ-HOOVER SYSTEM

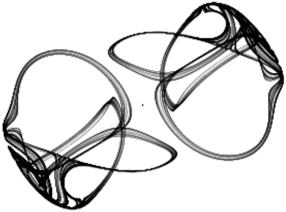


Complex Orbit Structure

- Topologically Transitivity
- Density of Periodic Points
- Topologically Mixing
- Containing a Dense orbit
- Containing Homoclinic/Hetroclinic point
- Containing Smale Horseshoe
- Containing Strange Attractor
- Positivity of Topological/Metric Entropy

Strange Attractor

Thomas System



Topological/Metric Entropy

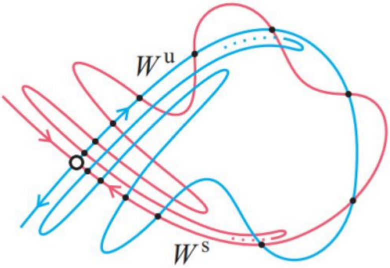
$$\mathcal{U}_M^N := \bigvee_{n=M}^{N} T^{-n}(\mathcal{U})$$

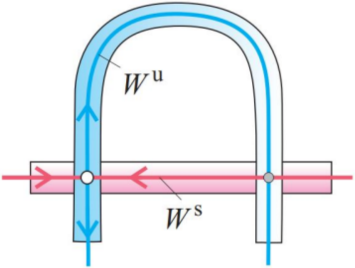
$$h_{top}(\mathcal{U}, T) := \lim_{n\to\infty} \frac{1}{n} \log(\mathcal{N}(\mathcal{U}_0^{n-1})) \tag{1}$$

$$h_{\text{top}}(T) := \sup\{h_{\text{top}}(\mathcal{U}, T) \mid \mathcal{U} \in C_X^\circ\} \tag{2}$$

Stable & Unstable Manifolds

$$W^s(p) = \{v \in \mathbb{R}^{\nvDash} : \lim_{n\to\infty} |f^n(v) - f^n(p)| = 0\}$$
$$W^u(p) = \{v \in \mathbb{R}^{\nvDash} : \lim_{n\to\infty} |f^{-n}(v) - f^{-n}(p)| = 0\}$$





Recipe of Chaos

Stretching & Folding

Recipe of Chaos

Recipe of Chaos

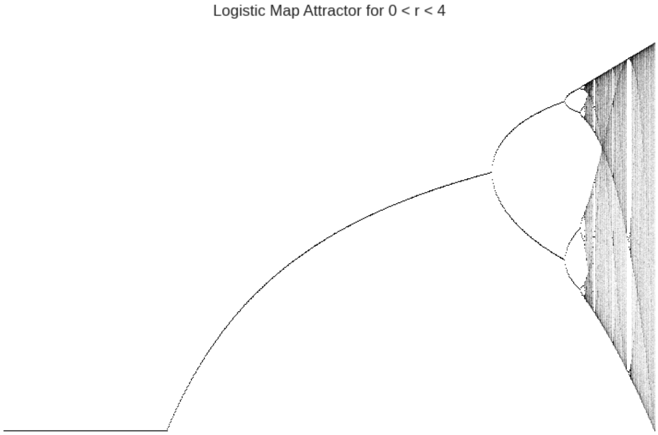
Recipe of Chaos

Recipe of Chaos

Recipe of Chaos

Chaotic Transitions

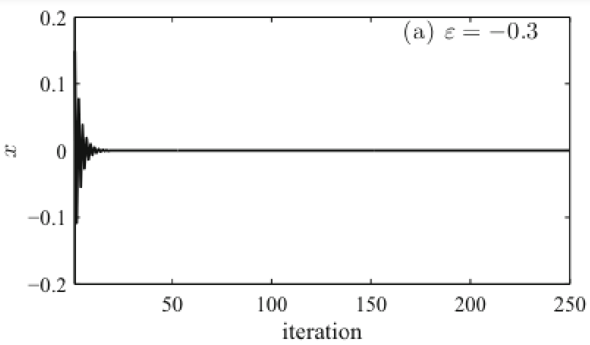
- Cascade of Period Doubling Bifurcations $\longrightarrow$ Chaotic Attractor



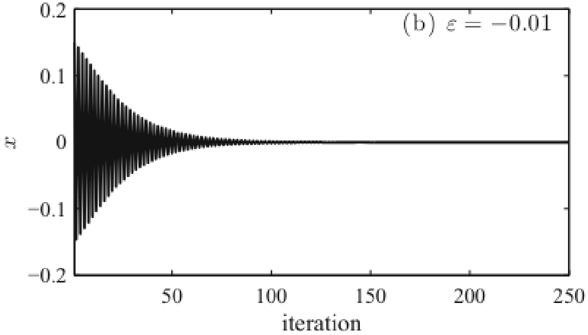
Chaotic Transitions - Intermittency

- Intermittency $\longrightarrow$ Chaotic Attractor

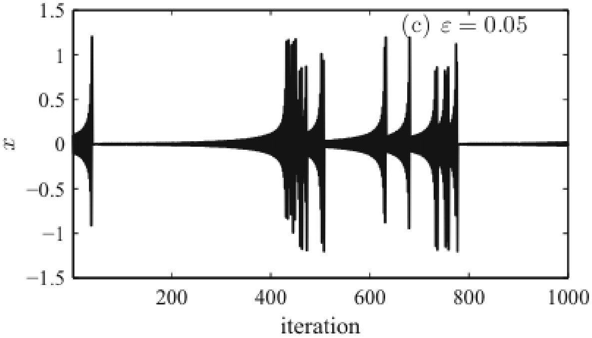
Chaotic Transitions - Intermittency



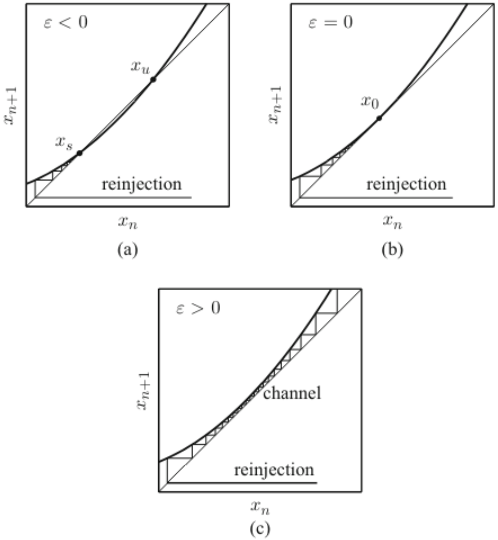
Chaotic Transitions - Intermittency



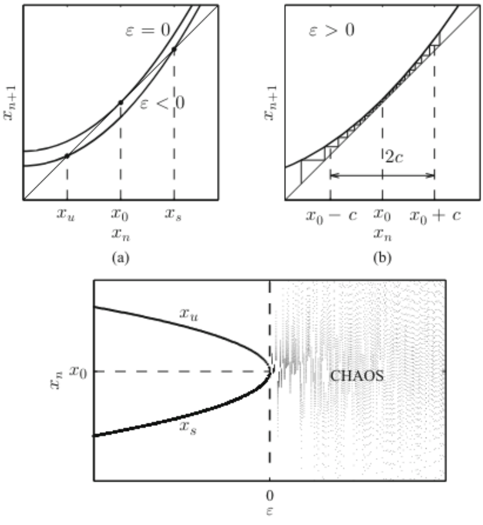
Chaotic Transitions - Intermittency



Chaotic Transitions - Intermittency



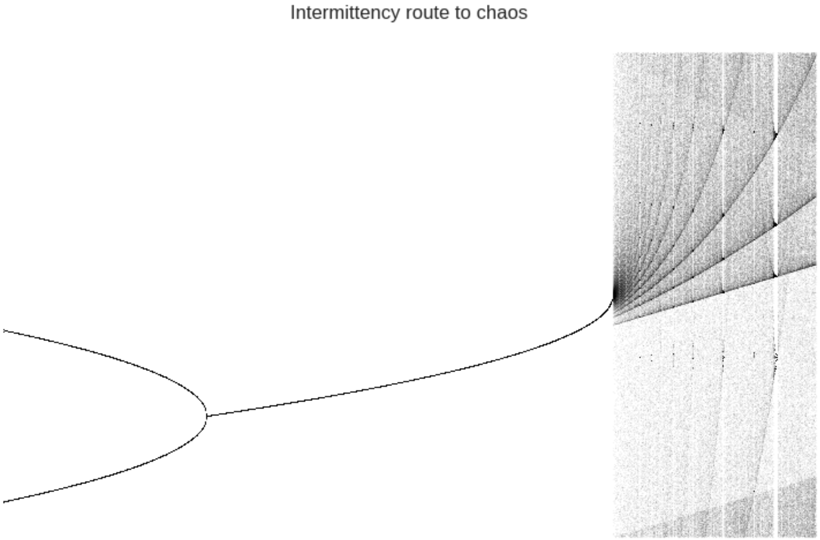
Chaotic Transitions - Intermittency



(a) (b)

(c)

Chaotic Transitions - Intermittency



Chaotic Transitions

Intermittency and Period Doublind Cascade route to chaos

References

- Ott, Edward. "Chaos in dynamical systems." Cambridge university press, 2002.
- Deng, Bo "Notes on Stable and Unstable Manifolds I."
- Artigiani, Mauro "Oseledets' multiplicative ergodic theorem and Lyapunov exponents."

Thank you!