Neuroscience, Learning, Memory, Cognition Course

Mohammad Nourbakhsh marvast
Student id: 401200482

October 2023

1 Fast Fourier Transform

Fast Fourier Transform (FFT) is an efficient implementation of Discrete Fourier transform (DFT). They are algorithms based on Fourier analysis that can map signals or waves from their usual domain (time) to the frequency domain.

1.1 Fourier Transform

The Fourier transform of a function $f\colon \mathbb{R} \to \mathbb{R}$ is defined as:

$$F(s) := \int_{-\infty}^{\infty} f(x) e^{-i2\pi xs} dx$$

If $F(s)$ is the Fourier transform of $f(x)$, then $f(-x)$ is the Fourier transform of $F(s)$ or equivalently,

$$f(x) = \int_{-\infty}^{\infty} F(s) e^{i2\pi xs} ds$$

A function $f : \mathbb{R} \to \mathbb{R}$ has Fourier transform if

1. $\int_{-\infty}^{\infty} |f(x)| dx$ exists.
2. Discontinues points of $f(x)$ are finite.

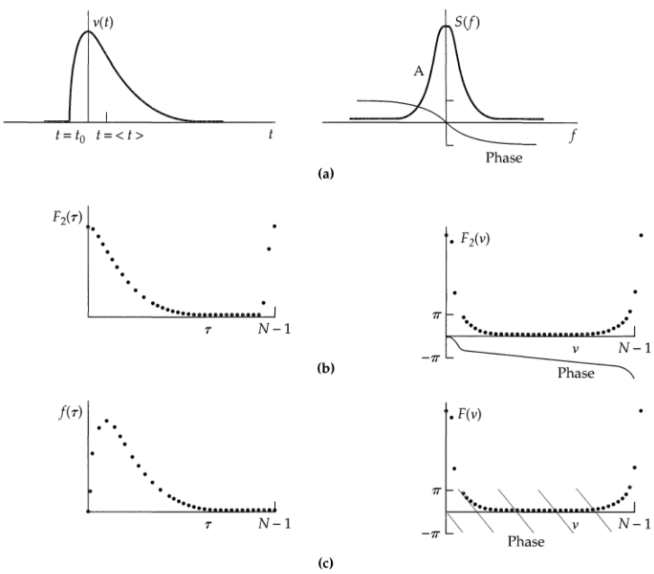
1.2 Discrete Fourier transform

Let us have a N-sample of the function f, i.e. we have values of f in just $N \in \mathbb{N}$ points $\{0, \ldots, N-1\}$. The discrete Fourier transform of f is given by:

$$F(v) = \frac{1}{N} \sum_{k=0}^{N-1} f(k) e^{-i2\pi(\frac{v}{N})k}$$

and likewise the continuous version,

$$f(k) = \sum_{v=1}^{N} F(v) e^{i2\pi(\frac{v}{N})k}$$



(a) A function and its Fourier transform; (b) and (c) two ways of representing the function by N samples and the corresponding discrete Fourier transforms, are shown by modulus (dots) and phase (small dots).

1.3 Fast Fourier Transform

We can write the discrete Fourier transform formula as a matrix product:

$$\begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & W & \dots & W^{N-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & W^{N-1} & \dots & W^{(N-1)^2} \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix}$$

where $W = e^{-\frac{i2\pi}{N}}$ is the Nth root of unity.

By noticing to $W^N = 1$, the order of this algorithm would be $\frac{N}{\log_2 N}$.

2 White Gaussian Noise

2.1 Intuitive Definition

White Noise: White noise is a traditional term in signal processing to refer to stochastic processes made of independent random variables.
A white noise contains all the frequency components in equal proportion. The reason for this name is that the spectrum of these stochastic processes (signals) is flat, i.e., all frequencies have the same magnitude. It just so happens that this is what the spectrum of white light looks like.

White Gaussian Noise (WGN): A stochastic process $X(t)$ is said to be WGN if $X(t)$ is normally distributed for each t and values $X(t_1)$ and $X(t_2)$ are independent for $t_1 \neq t_2$.
This implies that the power spectral density of this process is uniform over the entire frequency domain.

2.2 Mathematical Explanation

A WGN process $W(t)$ is a Gaussian Process with mean $\mu(t) = 0$, for all t and variance σ^2 with Delta function autocorrelation $R_W(t_1, t_2) = \sigma^2\delta(t_1 - t_2)$.
Now define the discrete time process $W_h(n)$ as the integral of $W(t)$ over the interval $[nh, (n+1)h)$, i.e.,

$$W_h(n) = \int_{nh}^{(n+1)h} W(t)dt$$

The mean value of $W_h(n)$ is $\mu_{W_h(n)} = 0$ and its autocorrelation function $R_{W_h(n1,n2)} = \sigma^2 h$ when $n_1 = n_2$ and $R_{W_h(n1,n2)} = 0$ elsewhere.

3 Cross Spectrum

3.1 Definition

Cross Spectrum or Cross Power Spectral Density is a type of covariance measurement between two signals. It is the Fourier transform of the cross-correlation function.
It offers insights into the correspondence between the power of one signal at different frequencies and the power of another signal at those same frequencies.

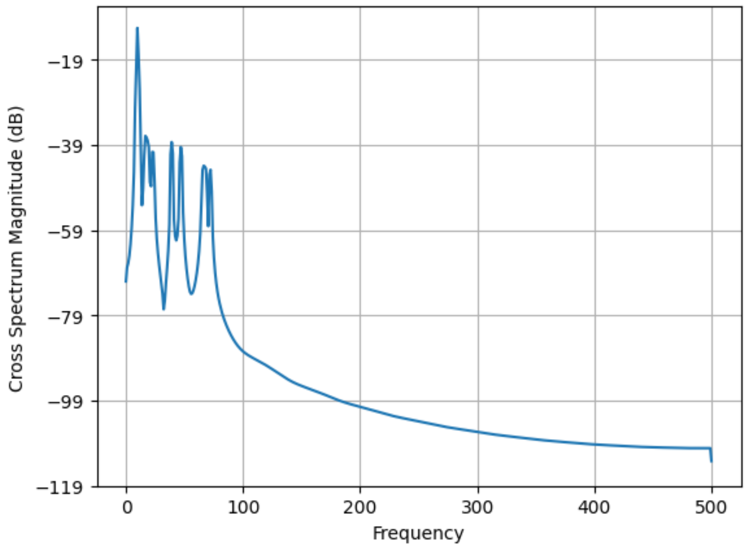
A cross-correlation function that states the relation of two signals $x(t)$, $y(t)$, Mathematically, is defined as

$$R_{xy}(t) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} x(s)y(t-s)ds$$

Cross-spectrum is the Fourier transform of $R_{xy}(t)$ and provides a statement of how common activity between two processes is distributed across frequency.

3.2 Calculation

In the exercises, we had the following cross-spectrum:



it indicates that the two signals are not correlated at any frequency domain. In each frequency, the strength of the relation between two signals is depicted.

3.3 np.random.seed

By using np.random.seed we initialize the starting point of the random number generator. without using np.random.seed, by every execution of our program, we generate different random numbers; By using np.random.seed, each time we have the same random number sequences.

4 The Leaky Integrate-and-Fire (LIF) model

Now, let's implement one of the simplest mathematical models of a neuron: the leaky integrate-and-fire (LIF) model. The basic idea of LIF neuron was proposed in 1907 by Louis Édouard Lapicque, long before we understood the electrophysiology of a neuron.
The subthreshold membrane potential dynamics of a LIF neuron is described by

$$C_m \frac{dV}{dt} = -g_L(V - E_L) + I$$

Dividing both sides of the above equation by g_L gives

$$\tau_m \frac{dV}{dt} = -(V - E_L) + \frac{I}{g_L}$$

4.1 Parameters

- C_m: The membrane capacitance.
- E_L: Resting potential of the neuron (-70 mV).
- I: The initial current flowing into the neuron from other neurons or external sources.
- τ_m: The membrane time constant.
- g_L: The leak conductance, How much easier can ions flow into the neuron or flow out.

4.2 Euler's Method for Solving ODEs

Let's look at the following ODE:

$$\dot{X} = f(X, t), \quad X \in \mathbb{R}^n$$

with initial condition given as $X(t_0) = X_0 \in \mathbb{R}^n$.
From calculus, we know that:

$$\begin{aligned} X(t) &= X(t_0) + \dot{X}|_{t=t_0}(t - t_0) + O(h^2) \\ &= X_0 + f(X_0, t_0)(t - t_0) \end{aligned}$$

We can approximate $X(t)$ near $t = t_0$ by the above formula. Thus, let's start with the initial guess X_0 and calculate $X_1 = X(t_1)$ for a time t_1 near t_0 by the below formula:

$$X_1 := X_0 + f(X_0, t_0)(t_1 - t_0)$$

Then for a time t_2 near t_1 we can repeat the same process:

$$X_2 := X_1 + f(X_1, t_1)(t2 - t_1)$$

By repeating the same process we have a sequence

$$\{X_n := X_{n-1} + f(X_{n-1}, t_{n-1})(t_n - t_{n-1})\}_{n \in \mathbb{N}}$$

by starting from our initial condition X_0.

Algorithm

1. Define h as a time step.
2. for i in range(1,n):
   (a) $F = f(x_{i-1}, t_{i-1})$
   (b) $t_i := t_{i-1} + h$
   (c) $X_i := X_{i-1} + Fh$

5 The Hodgkin-Huxley model

The Hodgkin–Huxley model, or conductance-based model, is a mathematical model that describes how action potentials in neurons are initiated and propagated. It is a set of nonlinear differential equations that approximates the electrical characteristics of excitable cells such as neurons and muscle cells. It is a continuous-time dynamical system.

Equations of the Hodgkin-Huxley model:

- $C_M \dfrac{dV}{dt} = -g_{Na} n^4 (V - V_{Na}) - g_k m^3 h (V - V_K) - g_l (V - V_l) + I$ (V: Membrane Potential)
- $\dfrac{dh}{dt} = \alpha_h(V)(1-h) - \beta_h(V)h$ (h: Na Inactivation)
- $\dfrac{dm}{dt} = \alpha_m(V)(1-m) - \beta_m(V)m$ (m: Na Activation)
- $\dfrac{dn}{dt} = \alpha_n(V)(1-n) - \beta_n(V)n$ (n: K Activation)

5.1 Parameters

- C_M: Membrane Capacitance, i.e. the ability of the cell membrane to store electrical charge.
- g_{Na}: The value of Na^+ maximal conductance, i.e. the measure of how easily sodium ions can flow through the sodium channels in the cell membrane.
- g_K: The value of K^+ maximal conductance
- g_l: The value of leaking maximal conductance
- V_{Na}: Na^+ equilibrium potential, i.e. the membrane potential at which sodium ions are in electrochemical equilibrium.
- V_K: K^+ equilibrium potential
- V_l: Leaking equilibrium potential
- $\alpha_h(V)$: The rate at which Na^+ channels transition to inactivated states during fast inactivation.
- $\alpha_m(V)$: The rate at which Na^+ channels transition to activated states during slow inactivation.
- $\alpha_n(V)$: The rate at which K^+ channels transition to activated states.
- $\beta_h(V)$: The rate at which Na^+ channels are reactivated during fast inactivation.
- $\beta_m(V)$: The rate at which Na^+ channels are inactivated during slow inactivation.
- $\beta_n(V)$: The rate at which K^+ channels transition to inactivated states.