Historical Mathematical & Philosophical Events

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Benacerraf's identification problem

I:

$$\begin{aligned}
&0 = \varnothing \\
&1 = \{\varnothing\} \\
&2 = \{\{\varnothing\}\} \\
&3 = \{\{\{\varnothing\}\}\} \\
&\vdots
\end{aligned}$$

II:

$$\begin{aligned}
&0 = \varnothing \\
&1 = \{\varnothing\} \\
&2 = \{\varnothing, \{\varnothing\}\} \\
&3 = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\} \\
&\vdots
\end{aligned}$$

The simple question that Benacerraf asks is:

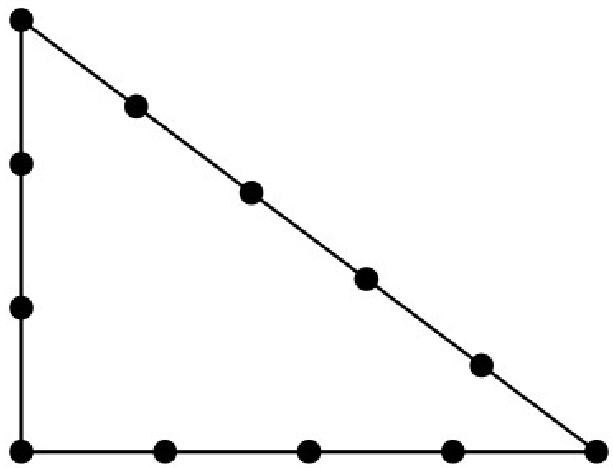
Which of these consists solely of true identity statements: I or II?

It seems very difficult to answer this question. It is not hard to see how a successor function and addition and multiplication operations can be defined on the number-candidates of I and on the number-candidates of II so that all the arithmetical statements that we take to be true come out true. Indeed, if this is done in the natural way, then we arrive at *isomorphic* structures (in the set-theoretic sense of the word), and isomorphic structures make the same sentences true (they are *elementarily equivalent*). It is only when we ask extra-arithmetical questions, such as '1∈3?' that the two accounts of the natural numbers yield diverging answers. So it is impossible that both accounts are correct. If both accounts were correct, then the transitivity of identity would yield a purely set theoretic falsehood.

Summing up, we arrive at the following situation. On the one hand, there appear to be no reasons why one account is superior to the other. On the other hand, the accounts cannot both be correct. This predicament is sometimes called labeled Benacerraf's *identification problem*.

The proper conclusion to draw from this conundrum appears to be that neither account I nor account II is correct. Since similar considerations would emerge from comparing other reasonable-looking attempts to reduce natural numbers to sets, it appears that natural numbers are not sets after all. It is clear, moreover, that a similar argument can be formulated for the rational numbers, the real numbers… Benacerraf concludes that they, too, are not sets at all.

Pythagorean Theorem:



Right angle by rope stretching

At first sight, arithmetic and geometry seem to be completely unrelated realms. Arithmetic is based on counting, the epitome of a discrete (or digital) process. The facts of arithmetic can be clearly understood as outcomes of certain counting processes, and one does not expect them to have any meaning beyond this. Geometry, on the other hand, involves continuous rather than discrete objects, such as lines, curves, and surfaces. Continuous objects cannot be built from simple elements by discrete processes, and one expects to see geometrical facts rather than arrive at them by calculation.

The Pythagorean theorem was the first hint of a hidden, deeper relationship between arithmetic and geometry, and it has continued to hold a key position between these two realms throughout the history of mathematics.

Infinity

Perhaps the most interesting—and most modern—feature of Greek mathematics is its treatment of infinity.

Using this method, Euclid found that the volume of a tetrahedron equals 1/3 of its base area times its height, and Archimedes found the area of a parabolic segment. Both of them relied on an infinite process that is fundamental to many calculations of area and volume: the **summation of an infinite geometric series**.

We shall see that the Greeks' rejection of irrational numbers was just part of a general rejection of infinite processes. In fact, until the late 19th century most mathematicians were reluctant to accept infinity as more than "potential."
For example, the natural numbers 1, 2, 3, ..., can be accepted as a potential infinity—generated from 1 by the process of adding 1— without accepting that there is a completed totality {1, 2, 3, ...}. The same applies to any sequence x1, x2, x3, ... (of rational numbers, say), where xn+1 is obtained from xn by a definite rule.

And yet a beguiling possibility arises when xn tends to a limit x. If x is something we already accept—for geometric reasons, say—then it is very tempting to view x as somehow the "completion" of the sequence x1, x2, x3, … . It seems that the Greeks were afraid to draw such conclusions. According to tradition, they were frightened off by the paradoxes of Zeno, around 450 BCE.

We know of Zeno's arguments only through Aristotle, who quotes them in his Physics in order to refute them, and it is not clear what Zeno himself wished to achieve. Was there, for example, a tendency toward speculation about infinity that he disapproved of? His arguments are so extreme they could almost be parodies of loose arguments about infinity he heard among his contemporaries. Consider his first paradox, the dichotomy:

> There is no motion because that which is moved must arrive at the middle (of its course) before it arrives at the end.
>
> Aristotle, *Physics*, Book VI, Ch. 9

The full argument presumably is that before getting anywhere one must first get half way, and before that a quarter of the way, and before that one eighth of the way, ad infinitum. The completion of this infinite sequence of steps no longer seems impossible to most mathematicians, since it represents nothing more than an infinite set of points within a finite interval. It must have frightened the Greeks though, because in all their proofs they were very careful to avoid completed infinities and limits.

The first mathematical processes we would recognize as infinite were probably devised by the Pythagoreans, for example, the recurrence relations

$$x_{n+1} = x_n + 2y_n,$$
$$y_{n+1} = x_n + y_n$$

for generating integer solutions of the equations $x^2 - 2y^2 = \pm 1$. We saw in Section 3.4 why it is likely that these relations arose from an attempt to understand $\sqrt{2}$, and it is easy for us to see that $x_n/y_n \to \sqrt{2}$ as $n \to \infty$.

concept of zero[1]

The zero was born in India:
It was the Indian sages who first drew a symbol to represent zero, a digit that does not appear in Greek writings or among Roman numerals.

Understanding the concept of zero (and negative numbers) is not the same as having a symbolic notation. The Roman number system has no such symbols. The first recorded use of a symbol for zero is said to be by Brahmagupta in 628 CE.

1. **Numbers**
   - $\mathbb{N}$ *counting numbers,* $\mathbb{Q}$ *rationals,* $\mathbb{P}$ *primes (6th century BCE)*
   - $\mathbb{Z}$ *common integers,* $\mathbb{I}$ *irrationals (5th century BCE)*
   - *zero* $\in \mathbb{Z}$ *(7th century CE)*
2. **Geometry** *(e.g., lines, circles, spheres, toroids, other shapes)*
   - *Composition of polynomials (Descartes, Fermat),*
   - *Euclid's geometry and algebra* $\Rightarrow$ *analytic geometry (17th century CE)*
   - *Fundamental theorem of algebra (18th cetury CE)*
3. **Infinity** *($\infty \rightarrow$ sets)*
   - *Taylor series, functions, calculus (Newton, Leibniz) (17th and 18th century CE)*
   - $\mathbb{R}$ *real,* $\mathbb{C}$ *complex (19th century CE)*
   - *Set theory (20th century CE)*

Three streams that follow from the Pythagorean theorem: numbers, geometry, and infinity.

1 For more Mathematically & Philosophically information see the paper Absence Perception and the Philosophy of Zero

Other Examples

- No Free Lunch Theorem[2]
- What is Probability

References

Benacerraf's identification problem
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An Invitation to Mathematical Physics and Its History, Jont Allen
Probability Branden Fitelson, Alan Hájek, and Ned Hall
No Free Lunch Theorem

2 https://www.geeksforgeeks.org/what-is-no-free-lunch-theorem/