Quantum Teleportation

Quantum teleportation is a mechanism for transferring a quantum state from one qubit to another qubit. It is the fundamental algorithm underlying quantum communication and quantum key distribution.

It is called *teleportation* because the state of the first qubit is lost due to measurement, and then reconstructed in the second qubit. In other words, the quantum state is *moved* but not *copied*. Quantum teleportation can be performed even if the two qubits are separated by great distances. However, reconstruction requires the transmission of two classical bits, so it does not permit faster-than-light communication.

The procedure is as follows:

1. Prepare two ancillary qubits as an EPR pair.
2. Perform a Bell measurement of the first and second qubits. After measurement, these qubits have states $|b_0\rangle$ and $|b_1\rangle$, where b_0 and b_1 are classical bits.
3. If $b_1 = 1$ then apply a Pauli-X gate to the third qubit.
4. If $b_0 = 1$ then apply a Pauli-Z gate to the third qubit.

At the end of this process, the state of the third qubit is equal to the original state of the first qubit. The circuit diagram is shown in Figure 1. In this diagram, the quantum state $|\psi\rangle$ is transferred from the first qubit, labeled q_0, to the third qubit, labeled q_2.

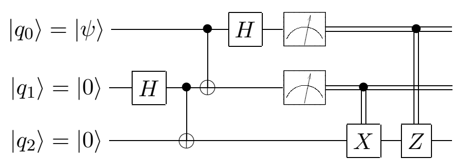

Figure 1: Circuit diagram for quantum teleportation.

Alegbraic verification

Let $|\psi\rangle = a\,|0\rangle + b\,|1\rangle$ be the initial state of q_0. The initial state of the register is

$$|\psi\rangle \otimes |0\rangle \otimes |0\rangle = a\,|000\rangle + b\,|100\rangle\,. \tag{1}$$

We prepare the two ancillary qubits q_1 and q_2 as an EPR pair by applying a Hadamard gate to q_1, then applying a CNOT gate to q_2 controlled by q_1. The result is

$$|\psi\rangle \otimes \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(a\,|000\rangle + a\,|011\rangle + b\,|100\rangle + b\,|111\rangle\right). \tag{2}$$

The Bell measurement can be split into three steps:

1. Apply a CNOT gate to q_1 controlled by q_0.
2. Apply a Hadamard gate to q_0.
3. Measure q_0 and q_1, yielding two classical bits b_0 and b_1.

Applying the CNOT gate yields

$$\frac{1}{\sqrt{2}}\left(a\left|000\right\rangle + a\left|011\right\rangle + b\left|110\right\rangle + b\left|101\right\rangle\right). \tag{3}$$

The Hadamard gate maps $|0\rangle$ to $(|0\rangle + |1\rangle)/\sqrt{2}$ and $|1\rangle$ to $(|0\rangle - |1\rangle)/\sqrt{2}$. So applying a Hadamard gate to q_0 yields

$$\frac{1}{2}\left(a\left|000\right\rangle + a\left|100\right\rangle + a\left|011\right\rangle + a\left|111\right\rangle + b\left|010\right\rangle - b\left|110\right\rangle + b\left|001\right\rangle - b\left|101\right\rangle\right). \tag{4}$$

We perform a measurement of the first two qubits (q_0 and q_1), yielding two classical bits (b_0 and b_1). There are four possible outcomes, as shown in the following table.

b_0	b_1	new state
0	0	$a\left\|000\right\rangle + b\left\|001\right\rangle$
0	1	$a\left\|011\right\rangle + b\left\|010\right\rangle$
1	0	$a\left\|100\right\rangle - b\left\|101\right\rangle$
1	1	$a\left\|111\right\rangle - b\left\|110\right\rangle$

Recall that the Pauli-X gate maps $a\left|0\right\rangle + b\left|1\right\rangle$ to $b\left|0\right\rangle + a\left|1\right\rangle$, and the Pauli-Z gate maps $a\left|0\right\rangle + b\left|1\right\rangle$ to $a\left|0\right\rangle - b\left|1\right\rangle$.

If $(b_0, b_1) = (0, 0)$ then we do nothing.

$$\begin{aligned} a\left|000\right\rangle + b\left|001\right\rangle &= \left|00\right\rangle \otimes \left(a\left|0\right\rangle + b\left|1\right\rangle\right) && (5)\\ &= \left|00\right\rangle \otimes \left|\psi\right\rangle && (6) \end{aligned}$$

If $(b_0, b_1) = (0, 1)$ then we apply a Pauli-X gate to q_2.

$$\begin{aligned} a\left|011\right\rangle + b\left|010\right\rangle &\mapsto a\left|010\right\rangle + b\left|011\right\rangle && (7)\\ &= \left|01\right\rangle \otimes \left|\psi\right\rangle && (8) \end{aligned}$$

If $(b_0, b_1) = (1, 0)$ then we apply a Pauli-Z gate to q_2.

$$\begin{aligned} a\left|100\right\rangle - b\left|101\right\rangle &\mapsto a\left|100\right\rangle + b\left|101\right\rangle && (9)\\ &= \left|10\right\rangle \otimes \left|\psi\right\rangle && (10) \end{aligned}$$

If $(b_0, b_1) = (1, 1)$ then we apply a Pauli-X gate followed by a Pauli-Z gate to q_2.

$$\begin{aligned} a\left|111\right\rangle - b\left|110\right\rangle &\mapsto a\left|110\right\rangle - b\left|111\right\rangle && (11)\\ &\mapsto a\left|110\right\rangle + b\left|111\right\rangle && (12)\\ &= \left|11\right\rangle \otimes \left|\psi\right\rangle && (13) \end{aligned}$$