Exploring Shor's Algorithm

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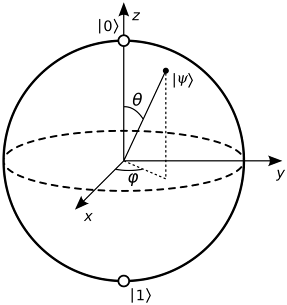
Outline

- Review of quantum information theory
- Timeline of integer factorization
- Shor's algorithm
- Quantum order finding
    - Quantum Fourier transform
    - Quantum phase estimation
    - Continued fractions

Qubits

1. A qubit (quantum bit) is the basic unit of quantum information
2. While a classical bit can only be $|0\rangle$ or $|1\rangle$, a qubit can exist in a superposition of $|0\rangle$ and $|1\rangle$
3. When a qubit is measured, the outcome is either $|0\rangle$ or $|1\rangle$



Qubit measurement

1. If we have a qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers, then:
2. The probability of measuring $|0\rangle$ is $|\alpha|^2$
3. The probability of measuring $|1\rangle$ is $|\beta|^2$
4. $|\alpha|^2 + |\beta|^2 = 1$ because probabilities must add to 1
5. After measurement, the qubit is in the state that was observed

n-qubit systems

1. A quantum system with n qubits has 2^n basis states.
2. Each basis state corresponds to a possible classical configuration of the qubits, represented as a binary string of length n.

Examples:

1. A 1-qubit system has 2 basis states: $|0\rangle$ and $|1\rangle$
2. A 2-qubit system has 4 basis states: $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$
3. A 3-qubit system has 8 basis states: $|000\rangle$ through $|111\rangle$

We often write these states using decimal notation: $|0\rangle, |1\rangle, |2\rangle, |3\rangle$, and so on.

Quantum superposition

1. A quantum state is a superposition of basis states
2. Each basis state has an associated complex number called its amplitude
3. The sum of their squared magnitudes equals 1

When we measure a quantum state:

1. The result is always one of the basis states
2. The probability of measuring a particular basis state $|s\rangle$ is $|\alpha|^2$, where α is that state's amplitude
3. The measurement disturbs the quantum state, collapsing it to the observed basis state

History of integer factorization

- Euclid (c. 300 BC) - Unique factorization, GCD algorithm
- Fermat (1643) - Fermat Factorization Method.
  $M = a^2 - b^2 = (a - b)(a + b)$
- Euler (1763) - Euler's totient function
- Gauss (1801) - Modular arithmetic, congruences
- Kraitchik (1920s) - Congruence of squares method:
  If M divides $a^2 - b^2$ but not $a \pm b$, then $\gcd(a - b, M)$ and $\gcd(a + b, M)$ are non-trivial factors of M

Recent history of integer factorization

- Miller (1976) - Reduction of factorization to order finding
- Rivest, Shamir, Adleman (1977) - RSA encryption
- Dixon (1981) - Quadratic sieve algorithm (up to 100 digits)
- Shor (1994) - Quantum algorithm for integer factorization
- Pollard (1998) - Number field sieve algorithm (over 100 digits)
- Boudot, et al (2020) - RSA-250 factored using GNFS

RSA encryption

RSA encryption is used to secure communications over the internet.

1. Compute $M = p \cdot q$, where p and q are large primes.
2. Compute the totient: $\phi(M) = (p-1)(q-1)$.
3. Choose a public exponent e such that $1 < e < \phi(M)$ and $\gcd(e, \phi(M)) = 1$.
4. Compute the private exponent d such that $(d \cdot e) \bmod \phi(M) = 1$.
5. The public key is (M, e) and the private key is d.
6. To encrypt a message m, compute $c = m^e \bmod M$.
7. To decrypt a ciphertext c, compute $m = c^d \bmod M$.
8. The security of RSA relies on the difficulty of factoring large numbers.

Multiplicative order

Assume that $\gcd(a, M) = 1$. If we compute successive powers

$$1, a, a^2, a^3, \ldots$$

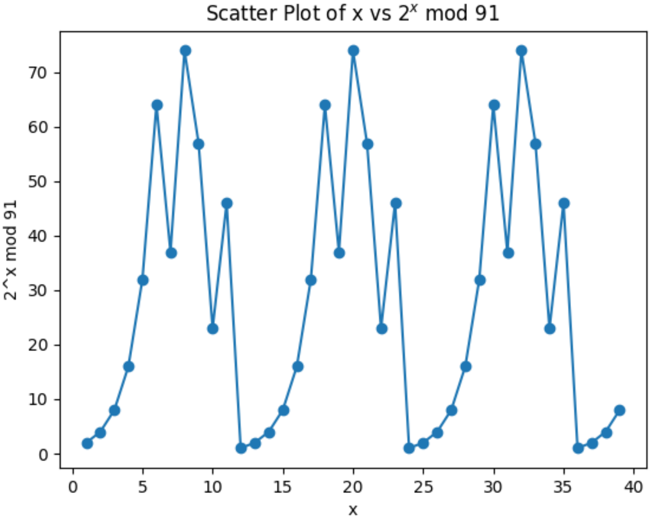
we will eventually see a repetition.

The order of a mod M is the smallest positive integer r such that $a^r \equiv 1 \pmod{M}$.

Example: Compute the order of 2 mod 15.

- $2^1 \equiv 2$
- $2^2 \equiv 4$
- $2^3 \equiv 8$
- $2^4 \equiv 1 \pmod{15}$, so $r = 4$.

Example: Powers of 2 mod 91



Outline of Shor's Algorithm

Given a composite integer M, Shor's algorithm finds a non-trivial factor of M with high probability.

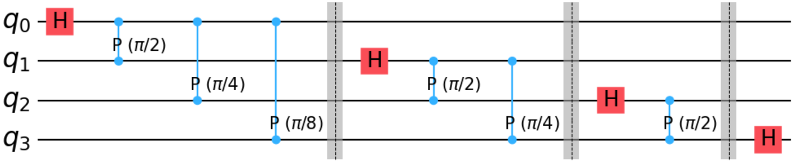
1. Choose a random integer a such that $1 < a < M$.
2. Compute the greatest common divisor of a and M.
   If $\gcd(a, M) > 1$, then we are done.
3. Compute the order r of a mod M. (How???)
4. If r is even, then M divides $a^r - 1 = \left(a^{r/2} - 1\right)\left(a^{r/2} + 1\right)$.
   Compute $x = a^{r/2} \pmod{M}$.
5. If m is odd, or if $x = M - 1$, go back to step 1.
6. Compute $\gcd(x - 1, M)$ and $\gcd(x + 1, M)$.

Quantum Fourier Transform

The quantum Fourier transform (QFT) is defined by

$$|j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle$$

where $N = 2^n$ is the number of basis states in the system.



Matrix form of the QFT

$$\mathrm{QFT}_n = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \dots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \dots & \omega^{(N-1)^2} \end{bmatrix}$$

where $\omega = e^{2\pi i/N}$ and $N = 2^n$.

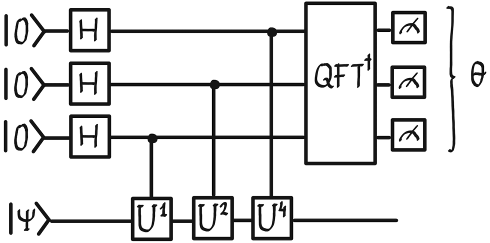
Quantum Phase Estimation

The quantum phase estimation algorithm estimates the phase of an eigenvector of a unitary operator.

Given a unitary operator U and an eigenvector $|\psi\rangle$ such that $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$, the quantum phase estimation algorithm estimates θ.

If $|\psi\rangle$ is not an eigenvector of U, then the algorithm will estimate the phase of a random eigenvector of U.

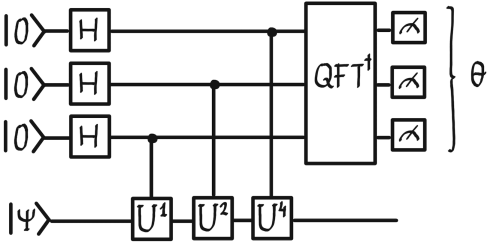
QPE Circuit - Step 1



Step 1: Apply Hadamard gates to the top register, placing it in an equal superposition of all basis states.

$$|0\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle$$

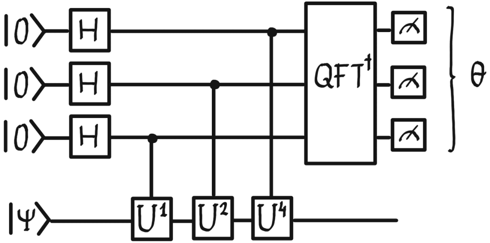
QPE Circuit - Step 2



Step 2: Apply controlled-U^{2^j} gates to the bottom register, controlled on the top register.

$$\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle\,|\psi\rangle \mapsto \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle\,U^k\,|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}e^{2\pi ik\theta}\,|k\rangle\,|\psi\rangle$$

QPE Circuit - Step 2



After applying the controlled-U^{2^j} gates, the system state is

$$\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}|k\rangle \mapsto \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}e^{2\pi ik\theta}|k\rangle$$

QPE Circuit - Step 3

Suppose that $\theta = j/N$ for some integer r. Then the state of the top register after step 2 is

$$\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle$$

But this is the same as applying the QFT to the state $|j\rangle$. Therefore, applying the inverse QFT to the top register will yield the state $|j\rangle$.

If $N\theta$ is not an integer then j/N will be an approximation to θ, with high probability.

Quantum Order Finding

Let M be a number to be factored, and let a be a number such that $1 < a < M - 1$ and $\gcd(a, M) = 1$.

We wish to find the order r of a mod M. We can do this using the quantum phase estimation algorithm.

Let U be a unitary operator defined by $U\,|x\rangle = |ax \bmod M\rangle$ for $x < M$, and $U\,|x\rangle = |x\rangle$ for $x \geq M$.

Note that since $U^r = I$, the eigenvalues of U are $e^{2\pi ij/r}$ for $j = 0, 1, \ldots, r - 1$.

Quantum Order Finding

Apply the QPE algorithm to U and the state $|1\rangle$.
This yields the state $|c\rangle$, where $c/N \approx j/r$ for some integer j.

To compute r, we need to find the best approximation to c/N whose denominator r is less than M. This is done using continued fractions.

Continued fraction example

Suppose that we are factoring $M = 21$ with $a = 2$ and $N = 2^10 = 1024$.
QPE yields the approximation $c/N = 171/1024 = 0.1669921875$.
Compute the continued fraction expansion of $171/1024$.

$$\frac{171}{1024} = [0; 5, 1, 84, 2] = 0 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{84 + \cfrac{1}{2}}}}$$

Continued fraction example

Truncating the continued fraction expansion yields the approximation

$$\frac{j}{r} = [0; 5, 1] = 0 + \cfrac{1}{5 + \cfrac{1}{1}} = \frac{1}{6}$$

Therefore, the order of 2 mod 21 is 6.