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CHAPTER 1

BACKGROUND INFORMATION

1.1 Notation

The used notations will be the same as in [2]. All integrals will be considered in the sense of Lebesgue. We consider the space $L^2(\mathbb{R})$ equipped with its standard inner product defined by, for all $f, g \in L^2(\mathbb{R})$,

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\, dx,$$

which induces the standard L^2 norm, i.e., for all $f \in L^2(\mathbb{R})$,

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}} |f(x)|^2\, dx.$$

The Fourier transform is defined, for all $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, by

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\, dx,$$

and its inverse by

$$f(x) = (\mathcal{F}^{-1}\hat{f})(x) = \check{\hat{f}}(x) = \int_{\mathbb{R}} \hat{f}(\xi)e^{2\pi i\xi x}\, d\xi.$$

The Fourier transform is classically extended to $L^2(\mathbb{R})$ as a unitary operator. Let us denote, for any arbitrary function $f \in L^2(\mathbb{R}^n)$, the following operators: translations: $T_y f(x) = f(x - y)$, where $y \in \mathbb{R}^n$; dilations: $D_a f(x) = |\det a|^{-1/2} f(a^{-1}x)$, where $a \in \mathrm{GL}_n(\mathbb{R})$; modulations: $E_\nu f(x) = e^{2\pi i\nu\cdot x} f(x)$, where $\nu \in \mathbb{R}^n$. It is straightforward to check that these operators fulfill the following properties:

$$\begin{aligned} \mathcal{F}(E_\nu f) &= T_\nu \hat{f} \quad , \quad \mathcal{F}^{-1}(T_y \hat{f}) = E_y f, \\ \mathcal{F}(T_y f) &= E_{-y} \hat{f} \quad , \quad \mathcal{F}^{-1}(E_\nu \hat{f}) = T_{-\nu} f, \\ \mathcal{F}(D_a f) &= D_{a^{-1}} \hat{f} \quad , \quad \mathcal{F}^{-1}(D_a \hat{f}) = D_{a^{-1}} f. \end{aligned}$$

When used explicitly, the Lebesgue measure will be denoted μ. Additionally, we will use the following notation for projection:

$$\mathrm{proj}_u(v) = \frac{\langle v, u \rangle}{\langle u, u \rangle} u.$$

1.2 Definitions

Definitions 1.2 - 1.4 are taken directly from [3]

Definition 1.1. *A **frame** for an inner product space V is a collection $\{v_i\}_{i\in\mathcal{S}}$ such that there are constants $A, B > 0$ such that for every $u \in V$,*

$$A\|u\|^2 \leq \sum_{i\in\mathcal{S}} |\langle u, v_i\rangle|^2 \leq B\|u\|^2.$$

*A **tight frame** is satisfied if there exists a constant $B > 0$ such that for every $u \in V$,*

$$\frac{1}{B}\sum_{i\in\mathcal{S}} |\langle u, v_i\rangle|^2 = \|u\|^2.$$

*A **Parseval frame** is a tight frame of bound $B = 1$.*

The strategy used in this paper to achieve an orthonormal basis relies on constructing a Parseval frame where each vector v_i is of unit norm. In that case,

$$1 = \sum_{i\in\mathcal{S}} |\langle v_i, v_k\rangle|^2 = \|v_k\|^4 + \sum_{i\in\mathcal{S}\setminus\{k\}} |\langle v_i, v_k\rangle|^2$$
$$\implies \langle v_i, v_k\rangle = 0 \quad \text{for } i \neq k$$

Definition 1.2. *Given a partition Ω of the frequency domain, let $\psi \in L^2(\mathbb{R})$ be a function such that its Fourier transform $\hat{\psi}$ satisfies the following two properties:*

1. *$\hat{\psi}$ is localized around the zero frequency,*
2. *There exists a subset $E \subseteq \operatorname{supp}\hat{\psi}$ and $0 \leq \delta < 1$, such that*

$$\int_E |\hat{\psi}(\xi)|^2\, d\xi = (1-\delta)\|\hat{\psi}\|^2_{L^2}.$$

This property guarantees that ψ is mostly supported by E.

Definition 1.3. *An **empirical wavelet system** generated by ψ is the collection*

$$\{\psi_{n,b} \mid n \in \mathcal{N},\ b \in \mathbb{R}\},$$

which can be defined either in the frequency domain as

$$\forall\xi \in \mathbb{R}, \quad \hat{\psi}_{n,b}(\xi) = E_{-b}T_{\omega_n}D_{a_n}\psi_b(\xi) = e^{-2\pi i b\xi} \cdot |a_n|^{-1/2} \cdot \hat{\psi}_b\left(\frac{\xi-\omega_n}{a_n}\right), \tag{1.1}$$

or in the time domain as

$$\forall t \in \mathbb{R}, \quad \psi_{n,b}(t) = T_bE_{\omega_n}D_{1/a_n}\psi(t) = e^{2\pi i\omega_n(t-b)} \cdot |a_n|^{1/2} \cdot \psi\big(a_n(t-b)\big), \tag{1.2}$$

where ω_n is the center of the Fourier support Ω_n, and $a_n \in \mathbb{R}\setminus\{0\}$ is a scaling factor whose choice depends on Ω_n and the prototype ψ_b.

The notations Ω_n, ω_n used in this definition will be explained in the next section. We follow the practices of [3], and also adopt the short-hand,

$$\psi_n = E_{\omega_n} D_{1/a_n}\psi \quad \text{and} \quad \hat{\psi}_n = T_{\omega_n} D_{a_n}\hat{\psi}_b.$$

Thus, the full family is obtained via translation:

$$\psi_{n,b} = T_b\psi_n \quad \text{and} \quad \hat{\psi}_{n,b} = E_{-b}\hat{\psi}_n.$$

Definition 1.4. *Let $b = kb_n$, where $k \in \mathbb{Z}$, and $\{b_n\}_{n\in\mathcal{N}} \subset \mathbb{R} \setminus \{0\}$. Then a* ***discrete empirical wavelet system*** *is the family of functions, for all $n \in \mathcal{N}$, $k \in \mathbb{Z}$, either defined in the Fourier domain by*

$$\forall \xi \in \mathbb{R}, \quad \psi_{n,k}(\xi) = E_{-b_n k} T_{\omega_n} D_{a_n} \psi_b(\xi) = e^{-2\pi i b_n k \xi} \cdot |a_n|^{-1/2} \cdot \hat{\psi}\left(\frac{\xi - \omega_n}{a_n}\right),$$

or in the time domain by

$$\forall t \in \mathbb{R}, \quad \psi_{n,k}(t) = T_{b_n k} E_{\omega_n} D_{1/a_n}\psi(t) = e^{2\pi i \omega_n (t - b_n k)} \cdot |a_n|^{1/2} \cdot \psi\big(a_n(t - b_n k)\big).$$

The corresponding discrete empirical wavelet transform is then given by

$$(E_\psi f)(n,k) = \langle f, E_{-b_n k}\psi_n\rangle = \langle f, T_{b_n k}\hat{\psi}_n\rangle.$$

Finally, a superscript is used to remind the reader of the wavelet kernel being used. We will use "LP" for Littlewood-Paley, "M" for Meyer, and "S" for Shannon.

1.3 Partitions in the Fourier Domain

Empirical wavelet systems are built from a partitioning of the Fourier domain Ω. The Fourier domain can be divided by boundaries ν_n, for $n \in \mathcal{N} \setminus \{0\} \subset \mathbb{Z}$ with $\mathcal{N} = \{n_l, \ldots, n_r\}$ and $n_l, n_r \in \mathbb{Z} \cup \{-\infty, +\infty\}$, such that $\nu_n \leq 0$ if $n \leq 0$ and $\nu_n > 0$ if $n > 0$. As shown in [4], these boundaries can be obtained by way of scale-space representation. A partition $\{\Omega_n\}_{n\in\mathcal{N}}$ of the Fourier domain is a set of successive disjoint open intervals defined as

$$\Omega_n = \begin{cases} (\nu_{n-1}, \nu_n) & \text{if } n < 0, \\ (\nu_{-1}, \nu_1) & \text{if } n = 0, \\ (\nu_n, \nu_{n+1}) & \text{if } n > 0, \end{cases}$$

satisfying $\Omega = \bigcup_{n\in\mathcal{N}} \Omega_n$.

Define $\mathcal{N}_{\text{comp}} = \{n \in \mathcal{N} : \Omega_n \text{ is bounded}\}$. In the case that $\mathcal{N}_{\text{comp}} \neq \mathcal{N}$, the partition contains sets of the form $(-\infty, \nu_{n_l+1})$ and $(\nu_{n_r-1}, +\infty)$, called the left and right rays, respectively. These are discussed more extensively in [1].

While outside the focus of this thesis, for a real-valued function, the magnitude of the Fourier spectrum is even, and therefore one can consider intervals $\{\Omega_n\}_{n\in\mathcal{N}}$ built from symmetric boundaries $\nu_{-n} = \nu_n$.

1.4 Foundations of Empirical Wavelet Frames

Empirical wavelets are built on the intervals Ω_n from a wavelet kernel $\psi \in L^2(\mathbb{R})$ whose Fourier transform $\widehat{\psi}$ is localized in frequency and mostly supported on a connected open subset Λ.

Let γ_n be a continuous bijection on $\mathbb{R}$ such that $\Lambda = \gamma_n(\Omega_n)$ if Ω_n is bounded, and $\Lambda \subsetneq \gamma_n(\Omega_n)$ otherwise. The discrete empirical wavelet system, denoted $\{\psi_n\}_{n\in\mathcal{N}}$, corresponding to the partition $\{\Omega_n\}_{n\in\mathcal{N}}$ is defined, for all $\xi \in \mathbb{R}$, by

$$\widehat{\psi}_n(\xi) = \frac{1}{\sqrt{a_n(\xi)}}\,\widehat{\psi} \circ \gamma_n(\xi),$$

where $a_n(\xi) > 0$ is a dilation factor that can be used for normalization. For our constructions, the mapping γ_n will be a piecewise affine function on $\mathbb{R}$. In this case the normalizing coefficient is defined as in [5]

$$a_n = \frac{1}{|\gamma_n'|},$$

assuming that γ_n is piecewise differentiable, to preserve that

$$\int_{\Omega_n} |\widehat{\psi}_n(\xi)|^2 \, d\xi = \int_E |\widehat{\psi}(\xi)|^2 \, d\xi.$$

The following result is used extensively and can be found in [3],

Theorem 1.5. *Let*

$$L^2_{comp}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) \mid \operatorname{supp} \widehat{f} \subseteq \Gamma_{comp}\},$$

and $\Gamma_{comp} = \bigcup_{n\in\mathcal{N}_{comp}} \overline{\Omega_n}$. *The system* $\{T_{b_n k}\psi_n\}_{(n,\mathcal{K})\in\mathcal{N}_{comp}\times\mathbb{Z}}$ *is a Parseval frame for* $L^2_{comp}(\mathbb{R})$ *if and only if, for almost every* $\xi \in \Gamma_{comp}$,

$$\sum_{n\in\mathcal{N}_\alpha} \frac{1}{|b_n|}\,\widehat{\psi}_n(\xi)\,\overline{\widehat{\psi}_n(\xi+\alpha)} = \delta_{\alpha,0}, \quad \text{for every } \alpha \in \mathcal{K},$$

where

$$\mathcal{K} = \bigcup_{n\in\mathcal{N}_{comp}} b_n^{-1}\mathbb{Z}, \quad \mathcal{N}_\alpha = \{n \in \mathcal{N}_{comp} \mid b_n\alpha \in \mathbb{Z}\},$$

and $\delta_{\alpha,0}$ *denotes the Kronecker delta function on* $\mathbb{R}$, *i.e.,* $\delta_{\alpha,0} = 1$ *if* $\alpha = 0$, *and* $\delta_{\alpha,0} = 0$ *otherwise.*

In practice, results from **Theorem 1.5** must be split into two cases: i. global and ii. local. The global case occurs when there exists a pair b_n, b_l such that $b_n\alpha, b_l\alpha \in \mathbb{Z}$. The local case occurs when there is no such pair. In section 2.1, a proof is provided for either case for the Meyer system. In later sections, we only consider the local case.

CHAPTER 2

EXPLICIT CONSTRUCTION OF DISCRETE EMPIRICAL WAVELET FRAMES

2.1 Meyer Wavelet System

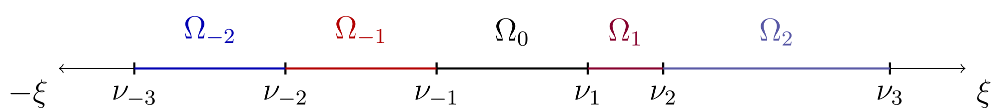
Definition 2.1. *Let the 1D Meyer wavelet, compactly supported by* $[-1,1]$, *be defined for* $\xi \in \mathbb{R}$*:*

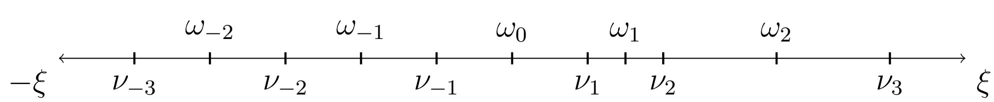
$$\hat{\psi}^M(\xi) = e^{i\frac{2\pi}{3}(\xi+1)} \begin{cases} \sin\left(\frac{\pi}{2}\beta(\xi+1)\right) & \textit{if } -1 \leq \xi \leq 0, \\ \cos\left(\frac{\pi}{2}\beta(\xi)\right) & \textit{if } 0 \leq \xi \leq 1, \\ 0 & \textit{otherwise}, \end{cases}$$

where $\beta(x)$ is a continuous function on $[0,1]$ such that $\beta(0)=0$, $\beta(1)=1$, and $\beta(x)+\beta(1-x)=1$ for every $x \in [0,1]$. We define γ_n such that it satisfies

$$\gamma_n : [\omega_{n-1}, \omega_{n+1}] \mapsto [-1,1],$$

where $\omega_n = (\nu_n + \nu_{n-1})/2$ i.e. the center of the interval Ω_n. The following diagrams are given for a visual reference.





We then define

$$\gamma_n(\xi) = \begin{cases} \dfrac{\xi - \omega_n}{\omega_n - \omega_{n-1}} & \text{if } \xi \leq \omega_n, \\ \dfrac{\xi - \omega_n}{\omega_{n+1} - \omega_n} & \text{if } \xi \geq \omega_n, \end{cases}$$

To increase smoothness, the function $\beta(x)$ chosen is

$$\beta(x) = x^4(35 - 84x + 70x^2 - 20x^3).$$

Under this construction we have that

$$\|\hat{\psi}^M \circ \gamma_n\|^2 = (\omega_n - \omega_{n-1}) \int_{-1}^{0} |\hat{\psi}^M(\xi)|^2 d\xi + (\omega_{n+1} - \omega_n) \int_{0}^{1} |\hat{\psi}^M(\xi)|^2 d\xi.$$

Note that

$$\begin{aligned} 1 &= \int_0^1 \cos^2\left(\frac{\pi}{2}\beta(\xi)\right) + \int_0^1 \sin^2\left(\frac{\pi}{2}\beta(\xi)\right) \\ \implies 1 &= \int_0^1 \cos^2\left(\frac{\pi}{2}\beta(\xi)\right) + \int_0^1 \sin^2\left(\frac{\pi}{2}(1-\beta(1-\xi))\right) \qquad (1) \\ \implies 1 &= 2\int_0^1 \cos^2\left(\frac{\pi}{2}\beta(\xi)\right). \end{aligned}$$

This implies

$$\|\hat{\psi}^M \circ \gamma_n\|^2_{L^2(\mathbb{R})} = \frac{\omega_{n+1} - \omega_{n-1}}{2},$$

so we define

$$a_n(\xi) = \frac{\omega_{n+1} - \omega_{n-1}}{2}.$$

Hence, in the Fourier domain, the empirical Meyer wavelets, compactly supported by $[\omega_{n-1}, \omega_{n+1}]$, read, for every $n \in \mathcal{N}_{\text{comp}}$ and $\xi \in \mathbb{R}$,

$$\hat{\psi}_n^M(\xi) = \frac{1}{\sqrt{a_n^M(\xi)}}\, \hat{\psi}^M(\gamma_n(\xi)).$$

In cases where Γ_{comp} is compact (or, equivalently, where $|\mathcal{N}_{comp}|$ is finite), then $\Gamma_{comp} = [\nu_m, \nu_l]$, where

$$\nu_m = \min_n \nu_n,\ \ \nu_l = \max_n \nu_n.$$

In the Meyer system,

$$\bigcup_{n \in \mathcal{N}_{comp}} \operatorname{supp} \hat{\psi}_n^M = [\omega_{m+1}, \omega_l].$$

As such, one must add two additional boundaries ν_{m-1} and ν_{l+1}. Let

$$\mathcal{N}^* = \mathcal{N}_{comp} \cup \{m-1, l+1\},\ \ \mathcal{K} = \bigcup_{n \in \mathcal{N}^*} b_n^{-1}\mathbb{Z},\ \ \mathcal{N}_\alpha^* = \{n \in \mathcal{N}^* : b_n\alpha \in \mathbb{Z}\}.$$

Then

$$\Gamma_{comp} \subset \bigcup_{n \in \mathcal{N}^*} \operatorname{supp} \hat{\psi}_n^M.$$

This is important for bounding the Meyer frame in the global case. For the local case, we let $\Gamma_{comp} = \mathbb{R}$ for simplicity.

Theorem 2.2. *Let $\Gamma_{comp} = \mathbb{R}$. If $a_n^M : \xi \mapsto \omega_{n+1} - \omega_{n-1}$, then the system $\{T_{b_n k}\psi_n^M\}_{n\in\mathcal{N}_{comp}, k\in\mathbb{Z}}$ is a Parseval frame on $L^2_{comp}(\mathbb{R})$ in the local case, for*

$$b_n = \frac{1}{\omega_{n+1} - \omega_{n-1}}$$

Proof. Let $\alpha \in \mathcal{K} \setminus \{0\}$ and $l \in \mathcal{N}_\alpha$. There exists $j \in \mathbb{Z} \setminus \{0\}$ such that

$$j = b_l \alpha.$$

Then

$$\alpha = b_l^{-1} j = (\omega_{l+1} - \omega_{l-1}) j,$$

and it follows that

$$|\alpha| \geq \omega_{l+1} - \omega_{l-1}.$$

Then, if $\xi \in [\omega_{l-1}, \omega_{l+1}]$, we have that $\xi + \alpha \notin (\omega_{l-1}, \omega_{l+1})$, and

$$\hat{\psi}_l^M(\xi)\overline{\hat{\psi}_l^M(\xi+\alpha)} = 0.$$

Let $\alpha = 0$. For every $\xi \in \Gamma_{comp}$, there exists $n \in \mathcal{N}$ such that $\xi \in [\omega_n, \omega_{n+1}]$ and we have

$$\begin{aligned}
&\sum_{n\in\mathcal{N}^\alpha} \frac{1}{|b_n|}\hat{\psi}_n^M(\xi)\overline{\hat{\psi}_n^M(\xi+\alpha)} = |\hat{\psi}_j^M(\xi)|^2 + |\hat{\psi}_{j+1}^M(\xi)|^2 \\
&= \cos^2\left(\frac{\pi}{2}\beta\left(\frac{\xi - \omega_{j+1}}{\omega_{j+1} - \omega_j}\right)\right) + \sin^2\left(\frac{\pi}{2}\beta\left(\frac{\xi - \omega_j}{\omega_{j+1} - \omega_j} + 1\right)\right) \\
&= \cos^2\left(\frac{\pi}{2}\beta\left(\frac{\xi - \omega_{j+1}}{\omega_{j+1} - \omega_j}\right)\right) + \sin^2\left(\frac{\pi}{2}\beta\left(\frac{\xi - \omega_{j+1}}{\omega_{j+1} - \omega_j}\right)\right) \\
&= 1.
\end{aligned} \tag{2}$$

Hence, for every $\alpha \in \mathcal{K}$ and $\xi \in \Gamma_{comp}$,

$$\sum_{l\in\mathcal{N}_\alpha} \frac{1}{|b_l|}\hat{\psi}_l^M(\xi)\overline{\hat{\psi}_l^M(\xi+\alpha)} = \delta_{\alpha,0},$$

which completes the proof. It is worth observing that here, $\|\widehat{\psi}_n\| = \sqrt{2}$. □

For the global case, one is forced to consider the bound over all such pairs of b_m, b_l. As we will see in the following proof, it is then important that there exists

$$\min_{n\in\mathcal{N}^*} |\omega_{n+1} - \omega_{n-1}|, \ \max_{n\in\mathcal{N}^*} |\omega_{n+1} - \omega_{n-1}|.$$

For this reason, we restrict that Γ_{comp} be compact. There are different conditions that that can also satisfy the existence of such terms.

Theorem 2.3. *Let Γ_{comp} be compact. The system $\{T_{b_n k}\psi_n^M\}_{(n,k)\in\mathcal{N}^*\times\mathbb{Z}}$ is a tight frame of bound 2/C on $L^2_{comp}(\mathbb{R})$ in the global case for*

$$b_n = \frac{C}{\omega_{n+1}-\omega_{n-1}}, \quad \textit{with} \quad 0 < C \leq \frac{\min_{n\in\mathcal{N}^*}|\omega_{n+1}-\omega_{n-1}|}{\max_{n\in\mathcal{N}^*}|\omega_{n+1}-\omega_{n-1}|}$$

Proof. Let $\alpha \in \mathcal{K}$, with $\alpha \neq 0$. Let $b_j\alpha = k$ for $k \in \mathbb{Z}$. If we have that $\exists l \in \mathcal{N}_\alpha$ such that $b_l\alpha = m$. Then

$$|\alpha| = |b_j^{-1}k| = \left|\frac{\omega_{j+1}-\omega_{j-1}}{C}\right||k|,$$

and

$$|k| = |b_j\alpha| = |b_j b_l^{-1} m| = \frac{|\omega_{l+1}-\omega_{l-1}|}{|\omega_{j+1}-\omega_{j-1}|}|m| \geq C|m|.$$

Then we have that

$$|\alpha| \geq (\omega_{j+1}-\omega_{j-1})|m| \geq (\omega_{j+1}-\omega_{j-1}).$$

Hence if $\xi \in [\omega_{j-1}, \omega_{j+1}]$, we have that $\hat{\psi}_j^M(\xi)\overline{\hat{\psi}_j^M(\xi+\alpha)} = 0$. If $\alpha = 0$, then for every $\xi \in \Gamma_{comp}$, we have that there is a $j \in \mathcal{N}^*$ such that $\xi \in [\omega_j, \omega_{j+1}]$, we have that

$$\frac{C}{2}\sum_{n\in\mathcal{N}_\alpha^*}\frac{1}{|b_n|}\hat{\psi}_n^M(\xi)\overline{\hat{\psi}_n^M(\xi+\alpha)} = \left|\psi^M \circ \gamma_n(\xi)\right|^2 + \left|\psi^M \circ \gamma_{n+1}(\xi)\right|^2$$
$$= 1,$$

as shown in (2). Hence, we've achieved the desired result. □

2.2 Littlewood-Paley Wavelet System

The Littlewood-Paley wavelet is compactly supported on $[-1, 1]$ in the Fourier domain, defined by:

$$\hat{\psi}^{LP}(\xi) = \begin{cases} \sin(\frac{\pi}{2}\beta(2\xi+2)) & \text{if } -1 < \xi < -1/2 \\ 1 & \text{if } -1/2 \leq \xi \leq 1/2 \\ \cos(\frac{\pi}{2}\beta(2\xi-1)) & \text{if } 1/2 < \xi \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

In this case we introduce transition constants τ_n to satisfy a frame over the partition $\mathcal{N}$. We require that $\tau_n > 0$, with $\tau_n + \tau_{n+1} < |\Omega_n|$. As we will see later, this is chosen to satisfy the partition of unity property. Here we have that

$$\begin{aligned} \gamma_n(\xi) &: [\nu_n - \tau_n, \nu_n + \tau_n] \mapsto \left[-1, -\frac{1}{2}\right] \\ &: [\nu_n + \tau_n, \nu_{n+1} - \tau_{n+1},] \mapsto \left[-\frac{1}{2}, \frac{1}{2}\right] \\ &: [\nu_{n+1} - \tau_{n+1}, \nu_{n+1} + \tau_{n+1}] \mapsto \left[\frac{1}{2}, 1\right] \end{aligned}$$

via

$$\gamma_n(\xi) = \begin{cases} \dfrac{\xi - \nu_n - 3\tau_n}{4\tau_n} & \text{if } \xi \leq \nu_n + \tau_n \\ \dfrac{\xi - \nu_n - \tau_n}{\nu_{n+1} - \tau_{n+1} - \nu_n - \tau_n} - \dfrac{1}{2} & \text{if } \nu_n + \tau_n \leq \xi \leq \nu_{n+1} - \tau_{n+1} \\ \dfrac{\xi - \nu_{n+1} + 3\tau_{n+1}}{4\tau_{n+1}} & \text{if } \nu_{n+1} - \tau_{n+1} \leq \xi \end{cases}$$

To compute the normalizing coefficient a_n, we have the following:

$$\|\hat{\psi}_n^{LP}(\xi)\|^2 = \int_{\nu_n - \tau_n}^{\nu_n + \tau_n} |\hat{\psi}_n^{LP}(\xi)|^2 \,\mathrm{d}\xi + \int_{\nu_n + \tau_n}^{\nu_{n+1} - \tau_{n+1}} |\hat{\psi}_n^{LP}(\xi)|^2 \,\mathrm{d}\xi + \int_{\nu_{n+1} - \tau_{n+1}}^{\nu_{n+1} + \tau_{n+1}} |\hat{\psi}_n^{LP}(\xi)|^2 \,\mathrm{d}\xi,$$

where

$$\gamma_n'(\xi) = \begin{cases} \dfrac{1}{4\tau_n} & \text{if } \xi \leq \nu_n + \tau_n \\ \dfrac{1}{\nu_{n+1} - \tau_{n+1} - \nu_n - \tau_n} & \text{if } \nu_n + \tau_n \leq \xi \leq \nu_{n+1} - \tau_{n+1} \\ \dfrac{1}{4\tau_{n+1}} & \text{if } \nu_{n+1} - \tau_{n+1} \leq \xi \end{cases}$$

By applying similar reasoning to what's used in (1), it can be shown that

$$\int_{-1}^{-\frac{1}{2}} |\hat{\psi}^{LP}(\xi)|^2 \, \mathrm{d}\xi = \frac{1}{4},$$
$$\int_{\frac{1}{2}}^{1} |\hat{\psi}^{LP}(\xi)|^2 \, \mathrm{d}\xi = \frac{1}{4}.$$

We define $a_n^{LP}(\xi) = \sqrt{|\Omega_n|}$. One can verify that

$$\begin{aligned}\|\hat{\psi}_n^{LP}(\xi)\|^2 &= \int_{\nu_n-\tau_n}^{\nu_n+\tau_n} |\hat{\psi}_n^{LP}(\xi)|^2 \, \mathrm{d}\xi + \int_{\nu_n+\tau_n}^{\nu_{n+1}-\tau_{n+1}} |\hat{\psi}_n^{LP}(\xi)|^2 \, \mathrm{d}\xi + \int_{\nu_{n+1}-\tau_{n+1}}^{\nu_{n+1}+\tau_{n+1}} |\hat{\psi}_n^{LP}(\xi)|^2 \, \mathrm{d}\xi \\ &= \frac{1}{|\Omega_n|}\left(4\tau_n\left(\frac{1}{4}\right) + |\Omega_n| - \tau_n - \tau_{n+1} + 4\tau_{n+1}\left(\frac{1}{4}\right)\right) \\ &= 1.\end{aligned}$$

Then, in the Fourier domain, the empirical Littlewood-Paley wavelets, compactly supported by $[\nu_n - \tau_n, \nu_{n+1} + \tau_{n+1}]$, read, for every $n \in \mathcal{N}_{\text{comp}}$ and $\xi \in \mathbb{R}$,

$$\hat{\psi}_n^{LP}(\xi) = \frac{1}{\sqrt{a_n^{LP}(\xi)}} \left(\hat{\psi}^{LP} \circ \gamma_n\right)(\xi).$$

Note that in this case, if Γ_{comp} is compact, then

$$\bigcup_{n\in\mathcal{N}_{comp}} \operatorname{supp} \hat{\psi}_n^{LP} = [\nu_m - \tau_m, \nu_l + \tau_l]$$

where

$$\nu_m = \min_n \nu_n, \;\; \nu_l = \max_n \nu_n.$$

However, additional boundaries ν_{m-1}, ν_{l+1} must again be placed. Without the additional boundaries, one can observe that on the intervals $[\nu_m, \nu_m + \tau_m], [\nu_l - \tau_l, \nu_l]$, the condition that

$$\sum_{n\in\mathcal{N}_\alpha} \frac{1}{|b_n|} |\hat{\psi}_n^{LP}(\xi)|^2 = 1$$

cannot be met. We now seek to show that our system defines a frame in the local case.

Theorem 2.4. *Let $\Gamma_{comp} = \mathbb{R}$. The system $\{T_{b_n k}\psi_n^M\}_{(n,k)\in\mathcal{N}_{comp}\times\mathbb{Z}}$ is a tight frame of bound 1/C on $L^2_{comp}(\mathbb{R})$ in the local case for*

$$b_n = \frac{C}{|\Omega_n + \tau_n + \tau_{n+1}|} \quad , \quad 0 < C \leq 1$$

Proof. Let $\alpha \in K$, with $\alpha \neq 0$. Let $b_j \alpha = k$ for $k \in \mathbb{Z}$. Then

$$|\alpha| = |b_j^{-1} k| = \frac{|\Omega_j + \tau_j + \tau_{j+1}|}{C} |k|$$

and therefore

$$|\alpha| \geq |\Omega_j + \tau_j + \tau_{j+1}|$$

Hence, if $\xi \in [\nu_n - \tau_n, \nu_{n+1} + \tau_{n+1}]$, we have that $\xi + \alpha \notin [v_n - \tau_n, v_{n+1} + \tau_{n+1}]$. Further, for every $\xi \in \Gamma_{comp}$ there is a $k \in \mathcal{N}$ such that $\xi \in [\nu_k - \tau_k, \nu_{k+1} + \tau_{k+1}]$. In particular, if $\xi \in [\nu_{k+1} - \tau_{k+1}, \nu_{k+1} + \tau_{k+1}]$, then

$$\begin{aligned} C \sum_{n \in \mathcal{N}_\alpha} \frac{1}{|b_n|} \hat{\psi}_n^{LP}(\xi) \overline{\hat{\psi}_n^{LP}(\xi + \alpha)} &= |\hat{\psi_k^{LP}}(\xi)|^2 + |\hat{\psi}_{k+1}^{LP}(\xi)|^2 \\ &= |\hat{\psi}^{LP} \circ \gamma_k(\xi)|^2 + |\hat{\psi}^{LP} \circ \gamma_{k+1}(\xi)|^2, \end{aligned}$$

the full expression of which is,

$$\begin{aligned} &= \cos^2\left(\frac{\pi}{2}\beta\left(\frac{2(\xi - v_{k+1} + 3\tau_{k+1})}{4\tau_{k+1}} - 1\right)\right) + \sin^2\left(\frac{\pi}{2}\beta\left(\frac{2(\xi - v_{k+1} - 3\tau_{k+1})}{4\tau_{k+1}} + 2\right)\right) \\ &= \cos^2\left(\frac{\pi}{2}\beta\left(\frac{2(\xi - v_{k+1} + \tau_{k+1})}{4\tau_{k+1}}\right)\right) + \sin^2\left(\frac{\pi}{2}\beta\left(\frac{2(\xi - v_{k+1} + \tau_{k+1})}{4\tau_{k+1}}\right)\right) \\ &= 1 \end{aligned}$$

as desired. $\square$

2.3 Shannon Wavelet System

Define the Shannon Wavelet kernel as

$$\hat{\psi}^S(\xi) = \begin{cases} e^{i\pi(\xi + \frac{3}{2})} & \text{if } -\frac{1}{2} < \xi < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

We take a natural choice of γ_n, for which

$$\gamma_n : [\nu_n, \nu_{n+1}] \mapsto [-1/2, 1/2]$$

via

$$\begin{aligned} \gamma_n(\xi) &= \frac{\xi}{|\Omega_n|} + \frac{1}{2} - \frac{\nu_{n+1}}{|\Omega_n|} \\ &= \frac{2\xi - (\nu_{n+1} + \nu_n)}{2|\Omega_n|} \\ &= \frac{\xi - \omega_n}{|\Omega_n|} \end{aligned}$$

We then define the empirical Shannon wavelet system as

$$\hat{\psi}_n^S(\xi) = \frac{1}{\sqrt{|\Omega_n|}} \hat{\psi}^S \circ \gamma_n(\xi)$$

We now have one more construction for the local case.

Theorem 2.5. *The system* $\{T_{b_n,k}\hat{\psi}_n^S\}$ *is a tight frame on* $L^2_{comp}(\mathbb{R})$ *of bound* $1/C$ *for*

$$b_n = \frac{C}{|\Omega_n|} \quad , \quad 0 < C \leq 1$$

Proof. Let $\alpha \in \mathcal{K}$, with $\alpha \neq 0$. Let $b_j\alpha = k$ for $k \in \mathbb{Z}$. Then

$$|\alpha| = |b_j^{-1}k| = \left|\frac{|\Omega_j|}{C}\right| |k|$$

and therefore

$$|\alpha| \geq |\Omega_j|$$

Hence if $\xi \in [\nu_n, \nu_{n+1}]$, we have that $\xi + \alpha \notin (\nu_n, \nu_{n+1})$, and therefore for every $\xi \in \mathbb{R}$ we have that $\hat{\psi}^S(\xi)\overline{\hat{\psi}^S(\xi+\alpha)} = 0$. Further, we also have that for every $\xi \in \Gamma_{comp}$, there exists an $n \in \mathcal{N}$ such that $\xi \in [\nu_n, \nu_{n+1}]$, and

$$C \sum_{n \in \mathcal{N}_\alpha} \frac{1}{|b_n|} \hat{\psi}_n^S(\xi)\overline{\hat{\psi}_n^S(\xi)} = \|\psi^S\|^2 = 1$$

which completes the proof. □

One may note that for $C = 1$, we indeed have that $\|\hat{\psi}_n^S\| = 1$. Hence, for $C = 1$, the system $\{T_{b_n,k}\hat{\psi}_n^S\}$ is an orthonormal basis for $L^2_{comp}(\mathbb{R})$.

CHAPTER 3
NUMERICAL RESULTS

Conditions to create an orthonormal basis from the Littlewood-Paley system were studied quite closely but remained unclear. This lead us to question what properties one could expect from an orthonormalized Littlewood-Paley system. Thankfully, there is at least one result which can be used to force this in the finite case. Allowing that Γ_{comp} be compact, we may apply Gram-Schmidt to the family $\{\hat{\psi}_n^{LP}\}_{n\in\mathcal{N}_{comp}}$. First, note that the family $\{\hat{\psi}_n^{LP}\}_{n\in\mathcal{N}_{comp}}$ is linearly independent, since

$$\langle\hat{\psi}_i^{LP},\hat{\psi}_j^{LP}\rangle = 0 \;\text{ if } j \neq i \pm 1$$

and for neighboring pairs of vectors, $\hat{\psi}_i^{LP},\hat{\psi}_{i+1}^{LP}$, we have that

$$\text{supp}\{\hat{\psi}_i^{LP}\} \neq \text{supp}\{\hat{\psi}_{i+1}^{LP}\}$$

Hence, there are no scalars $c_1, c_2 \in \mathbb{R}\setminus\{0\}$, such that

$$c_1\hat{\psi}_i^{LP} + c_2\hat{\psi}_{i+1}^{LP} = 0$$

We begin by fixing $\tau_n = \tau$, with $2\tau < \min_n|\Omega_n|$. Since we will compute the normalized system, here we set $a_n = 1$, where

$$\frac{1}{\sqrt{a_n}}\hat{\psi}_n^{LP}(\xi) = \hat{\psi}^{LP}\circ\gamma_n(\xi)$$

The inner product is difficult to evaluate based on choice of β. For simplicity, if $\beta(x) = x$, then

$$\begin{aligned}
\langle\hat{\psi}_n^{LP}(\xi),\hat{\psi}_{n+1}^{LP}(\xi)\rangle &= \int_{v_{n+1}-\tau}^{v_{n+1}+\tau}\cos\left(\frac{\pi}{2}\left(2\gamma_n(\xi)-1\right)\right)\sin\left(\frac{\pi}{2}\left(2\gamma_{n+1}(\xi)+2\right)\right)\mathrm{d}\xi\\
&= \int_{v_{n+1}-\tau}^{v_{n+1}+\tau}\cos\left(\frac{\pi}{2}\left(\frac{\xi-\nu_{n+1}}{2\tau}+\frac{1}{2}\right)\right)\sin\left(\frac{\pi}{2}\left(\frac{\xi-\nu_{n+1}}{2\tau}+\frac{1}{2}\right)\right)\mathrm{d}\xi\\
&= \frac{2\tau}{\pi}\left[\sin^2\left(\frac{\pi}{2}\left(\frac{\xi-\nu_{n+1}}{2\tau}+\frac{1}{2}\right)\right)\right]_{v_{n+1}-\tau}^{v_{n+1}+\tau}\\
&= \frac{2\tau}{\pi}.
\end{aligned}$$

Let $|\mathcal{N}_{comp}|$ be finite, and select m such that

$$m = \min_{n\in\mathcal{N}}\nu_{n+1}.$$

Reindex such that for $i \geq 1$, $\hat{\psi}_i^{LP}(\xi) = \hat{\psi}_{(m-1)+i}^{LP}(\xi)$. Recall that the Gram-Schmidt process on an inner product space, V, takes a finite set of vectors $\{v_1, v_2, ..., v_n\}$ and produces an orthogonal set $\{\tilde{e}_1, \ldots, \tilde{e}_n\}$ via

$$\begin{aligned}
\tilde{e}_1 &= v_1, \\
\tilde{e}_2 &= v_2 - \mathrm{proj}_{u_1}(v_2), \\
&\vdots \\
\tilde{e}_n &= v_n - \sum_{k=1}^{n-1} \mathrm{proj}_{u_k}(v_n).
\end{aligned}$$

Finally we normalize the orthogonal vectors, with

$$\begin{aligned}
e_1 &= \frac{\tilde{e}_1}{\|\tilde{e}_1\|} \\
&\vdots \\
e_n &= \frac{\tilde{e}_n}{\|\tilde{e}_n\|}
\end{aligned}$$

Since $\operatorname{supp} \hat{\psi}_i^{LP} \cap \operatorname{supp} \hat{\psi}_j^{LP} = \emptyset$ for $|i-j| > 1$,

$$\begin{aligned}
\tilde{e}_n^{LP} &= \hat{\psi}_n^{LP} - \sum_{k=1}^{n-1} \langle \hat{\psi}_n^{LP}, e_k^{LP} \rangle e_k^{LP} \\
&= \hat{\psi}_n^{LP} - \langle \hat{\psi}_n^{LP}, e_{n-1}^{LP} \rangle e_{n-1}^{LP}.
\end{aligned}$$

Recalling that

$$\|\hat{\psi}_{k+1}^{LP} - \mathrm{proj}_{e_{k-1}^{LP}}(\hat{\psi}_k^{LP})\|^2 = \|\hat{\psi}_{k+1}^{LP}\|^2 - \frac{|\langle \hat{\psi}_k^{LP}, e_{k-1}^{LP} \rangle|^2}{\|e_{k-1}^{LP}\|^2},$$

and letting $\rho = \frac{2\tau}{\pi}$, then

$$e_1^{LP}(\xi) = \frac{1}{\sqrt{|\Omega_1|}}\hat{\psi}_1^{LP}(\xi),$$

$$e_2^{LP}(\xi) = \frac{1}{\sqrt{|\Omega_2| - \frac{\rho^2}{|\Omega_1|}}}\left(\hat{\psi}_2^{LP}(\xi) - \frac{\rho}{\sqrt{|\Omega_1|}}e_1^{LP}(\xi)\right),$$

$$e_3^{LP}(\xi) = \frac{1}{\sqrt{|\Omega_3| - \cfrac{\rho^2}{|\Omega_2| - \cfrac{\rho^2}{|\Omega_1|}}}}\left(\hat{\psi}_3^{LP}(\xi) - \frac{\rho}{\sqrt{|\Omega_2| - \frac{\rho^2}{|\Omega_1|}}}e_2^{LP}(\xi)\right),$$

$$\vdots$$

$$e_n^{LP}(\xi) = \frac{1}{\sqrt{|\Omega_n| - \cfrac{\rho^2}{|\Omega_{n-1}| - \cfrac{\rho^2}{\ddots - \cfrac{\rho^2}{|\Omega_1|}}}}}\left(\hat{\psi}_n^{LP}(\xi) - \frac{\rho}{\sqrt{|\Omega_{n-1}| - \cfrac{\rho^2}{|\Omega_{n-2}| - \cfrac{\rho^2}{\ddots - \cfrac{\rho^2}{|\Omega_1|}}}}}e_{n-1}^{LP}(\xi)\right).$$

While this achieves an orthonormal set, the Gram-Schmidt process introduces 'leakage,' where

$$\text{supp}(e_1^{LP}) \subset \text{supp}(e_2^{LP}) \subset \ldots \subset \text{supp}(e_n^{LP}).$$

And, on closer inspection, the partition of unity property is not respected.

However, for the usual choice, $\beta(x) = x^4(35 - 84x + 70x^2 - 20x^3)$, the integral must be evaluated numerically. In this case we have that

$$\begin{aligned}
\langle \hat{\psi}_n^{LP}(\xi), \hat{\psi}_{n+1}^{LP}(\xi)\rangle &= \int_{v_{n+1}-\tau}^{v_{n+1}+\tau} \cos\left(\frac{\pi}{2}\beta\left(2\gamma_n(\xi) - 1\right)\right) \sin\left(\frac{\pi}{2}\beta\left(2\gamma_{n+1}(\xi) + 2\right)\right) \mathrm{d}\xi \\
&= 2\tau \int_0^1 \cos\left(\frac{\pi}{2}\beta(\xi)\right) \sin\left(\frac{\pi}{2}\beta(\xi)\right) \mathrm{d}\xi \\
&= 2\tau M
\end{aligned}$$

with $M \approx 0.17886...$ Letting $\rho = 2\tau M$, the previous Gram-Schmidt formulas hold.

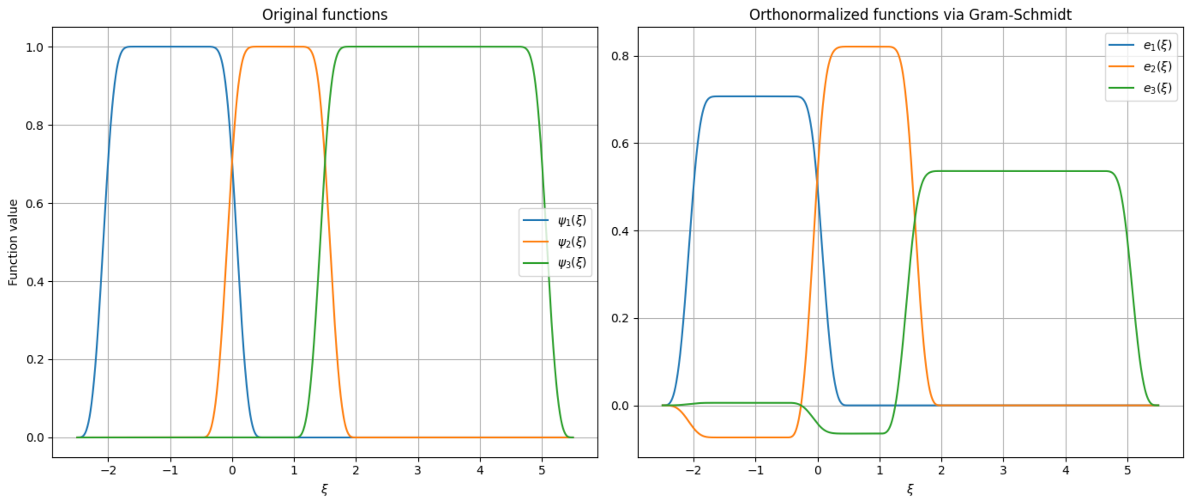


Figure 3.1. Littlewood-Paley system, $\beta(x) = x^4(35 - 84x + 70x^2 - 20x^3)$

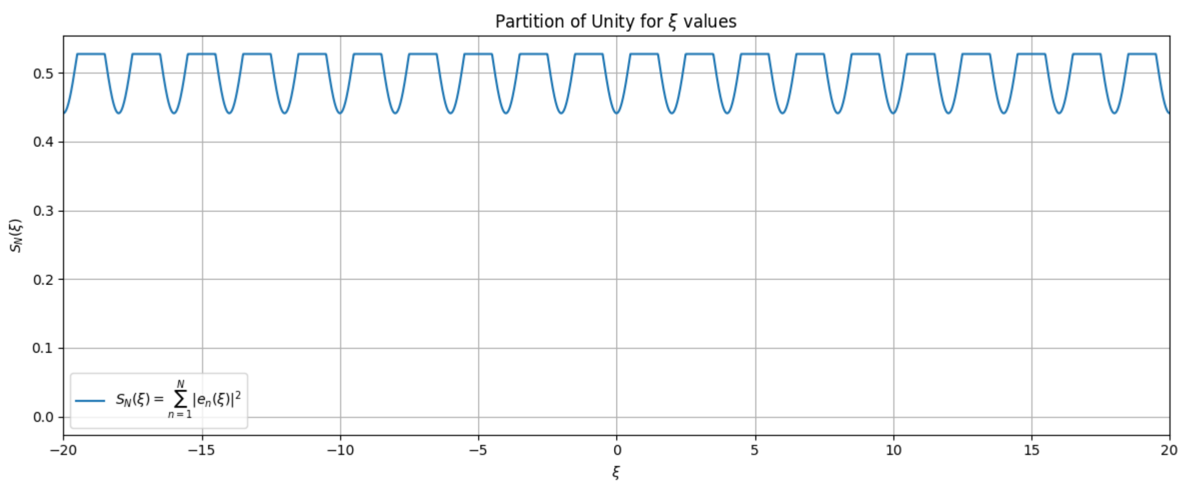


Figure 3.2. LP Partition of Unity, $N = 50$, $\tau = \frac{1}{2}$, $\beta(x) = x$

For the Meyer system, the Gram-Schmidt process also distorts the structure of the supports. Due to having more variation in overlapping supports, we instead give a recursive formula to perform the Gram-Schmidt process. This time, we take

$$\frac{1}{a_n} = \frac{2}{\omega_{n+1} - \omega_{n-1}} \quad , \quad \beta(x) = x^4(35 - 84x + 70x^2 - 20x^3)$$

and re-index as previously given. Then

$$\begin{aligned}
\langle \hat{\psi}_n^M, \hat{\psi}_{n+1}^M \rangle &= \frac{1}{\sqrt{a_n a_{n+1}}} \int_{\omega_n}^{\omega_{n+1}} \cos\left(\frac{\pi}{2}\beta\left(\frac{\xi - \omega_{n+1}}{\omega_{n+1} - \omega_n}\right)\right) \sin\left(\frac{\pi}{2}\beta\left(\frac{\xi - \omega_{n+1}}{\omega_{n+1} - \omega_n}\right)\right) d\xi \\
&= \frac{(\omega_{n+1} - \omega_n)}{\sqrt{a_n a_{n+1}}} \int_{-1}^{0} \cos\left(\frac{\pi}{2}\beta(\xi)\right) \sin\left(\frac{\pi}{2}\beta(\xi)\right) d\xi \\
&= \frac{(\omega_{n+1} - \omega_n)}{\sqrt{a_n a_{n+1}}} \cdot L,
\end{aligned}$$

where

$$L \approx 0.04182\ldots$$

Let

$$\rho_n = \frac{(\omega_n - \omega_{n-1})}{\sqrt{a_n a_{n-1}}} L,$$

and

$$r_n = \langle \psi_n^M, e_{n-1}^M \rangle.$$

Then we have that

$$r_n = \langle \hat{\psi}_n^M, \frac{\hat{\psi}_{n-1}^M - r_{n-1} e_{n-2}^M}{\sqrt{1 - |r_{n-1}|^2}} \rangle.$$

By linearity of the inner product, noting again that

$$\langle \hat{\psi}_i^M, \hat{\psi}_j^M \rangle = 0 \;\text{ if } j \neq i \pm 1,$$

we have that

$$\begin{aligned}
r_n &= \frac{\langle \hat{\psi}_n^M, \hat{\psi}_{n-1}^M \rangle}{\sqrt{1 - |r_{n-1}|^2}} \\
&= \frac{\rho_n}{\sqrt{1 - |r_{n-1}|^2}}.
\end{aligned}$$

Then

$$r_1 = 0,\; r_2 = \rho_2,\; r_n = \frac{\rho_n}{\sqrt{1 - |r_{n-1}|^2}} \quad \text{for} \quad n \geq 3.$$

Thus, $e_1^M = \hat{\psi}_1^M$, and for $n > 1$:

$$e_n^M = \frac{\hat{\psi}_n^M - r_n e_{n-1}^M}{\sqrt{1 - |r_{n-1}|^2}}.$$

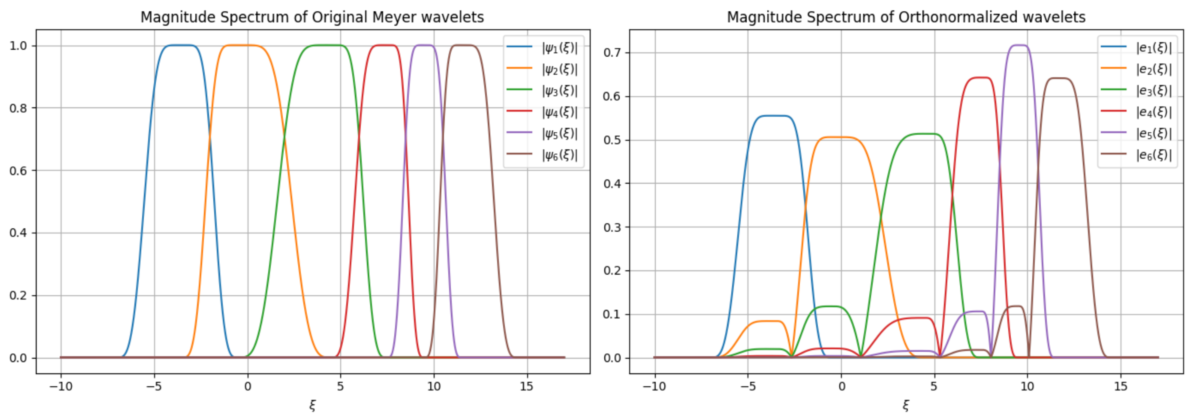


Figure 3.3. Meyer System, $\beta(x) = x^4(35 - 84x + 70x^2 - 20x^3)$

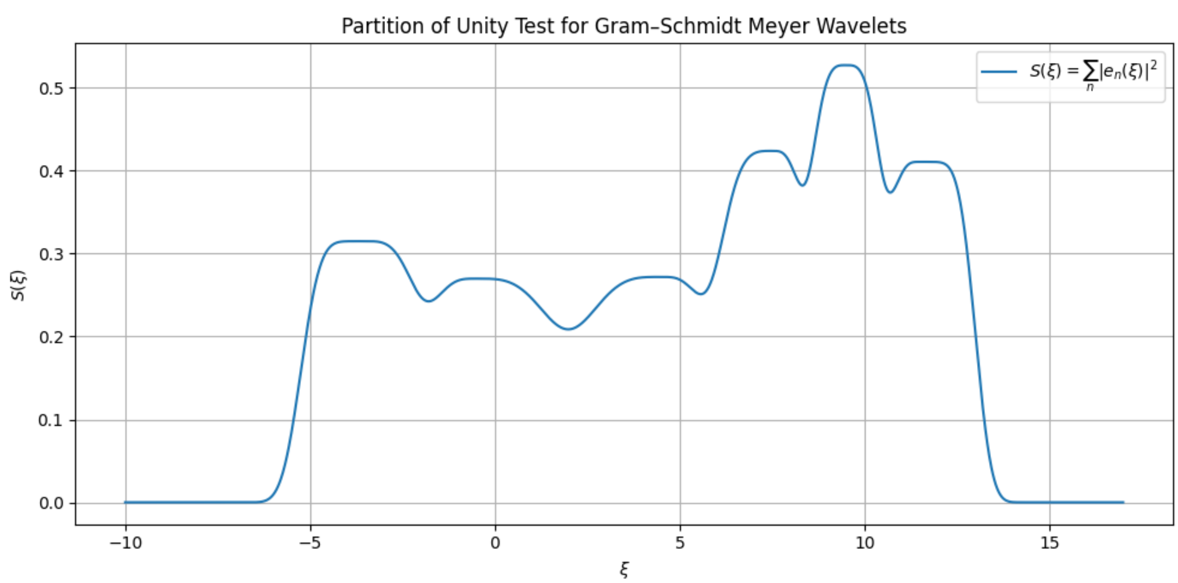


Figure 3.4. Partition of Unity, Meyer, $\beta(x) = x^4(35 - 84x + 70x^2 - 20x^3)$

CHAPTER 4

CLASSIFICATION OF EMPIRICAL ORTHONORMAL WAVELET BASES

In this chapter, we will show that if $\{T_{b_n k}\psi_n\}_{(n,k)\in\mathcal{N}_{comp}\times\mathbb{Z}}$ satisfies an orthonormal basis of $L^2_{\text{comp}}(\mathbb{R})$, then for a.e. $\xi\in\Gamma_{comp}$

$$\frac{1}{|b_n|}|\hat{\psi}_n(\xi)|^2=\chi_{I_n}(\xi),$$

where $I_n=\operatorname{supp}\psi_n$. We will show this by using the following lemma, where our proof strategy is to simply use **Theorem 1.5** in reverse.

Lemma 4.1. *Any discrete empirical wavelet system where*

$$\mu(\operatorname{supp}\hat{\psi}_n\cap\operatorname{supp}\hat{\psi}_{n+1})\neq 0$$

cannot satisfy an orthonormal basis of $L^2_{comp}(\mathbb{R})$.

Proof. Let the system $\{T_{b_n k}\psi_n\}_{(n,k)\in\mathcal{N}_{comp}\times\mathbb{Z}}$ satisfy an orthonormal basis of $L^2_{comp}(\mathbb{R})$. Necessarily the system must be a Parseval frame of $L^2_{comp}(\mathbb{R})$, and satisfy that

$$\|\hat{\psi}_n\|^2=1. \tag{3}$$

Using **Theorem 1.5**, it is required that for every $\alpha\in\mathcal{K}\setminus\{0\}$,

$$\frac{1}{|b_n|}\hat{\psi}_n(\xi)\overline{\hat{\psi}_n(\xi+\alpha)}=0$$

for almost every $\xi\in\Gamma_{comp}$. One can observe that by fixing $l\in\mathcal{N}_{comp}$, for $\alpha=b_l^{-1}\in\mathcal{K}\setminus\{0\}$, we must have that for almost every $\xi\in\Gamma_{comp}$

$$\hat{\psi}_l(\xi)\overline{\hat{\psi}_l(\xi+b_l^{-1})}=0,$$

which is only true if

$$\frac{1}{|b_l|}\geq\mu(\operatorname{supp}\hat{\psi}_l).$$

Let

$$J=\operatorname{supp}\hat{\psi}_n\cup\operatorname{supp}\hat{\psi}_{n+1},\ \ I=\operatorname{supp}\hat{\psi}_n\cap\operatorname{supp}\hat{\psi}_{n+1}.$$

Using (3)

$$\int_J \frac{1}{|b_n|}|\hat{\psi}_n(\xi)|^2 + \frac{1}{|b_{n+1}|}|\hat{\psi}_{n+1}(\xi)|^2 d\xi = \frac{1}{|b_n|}\|\hat{\psi}_n\|^2 + \frac{1}{|b_{n+1}|}\|\hat{\psi}_{n+1}\|^2$$
$$= \frac{1}{|b_n|} + \frac{1}{|b_{n+1}|}$$
$$\geq \mu(\operatorname{supp}\hat{\psi}_n) + \mu(\operatorname{supp}\hat{\psi}_{n+1}).$$

However, on J,

$$\frac{1}{|b_n|}|\hat{\psi}_n(\xi)|^2 + \frac{1}{|b_{n+1}|}|\hat{\psi}_{n+1}(\xi)|^2 \leq 1.$$

Hence we have that

$$\int_J \frac{1}{|b_n|}|\hat{\psi}_n(\xi)|^2 + \frac{1}{|b_{n+1}|}|\hat{\psi}_{n+1}(\xi)|^2 d\xi \leq \mu(J).$$

This implies

$$\frac{1}{|b_n|} + \frac{1}{|b_{n+1}|} \leq \mu(J).$$

Combining both inequalities gives

$$\mu(\operatorname{supp}\hat{\psi}_n) + \mu(\operatorname{supp}\hat{\psi}_{n+1}) \leq \frac{1}{|b_n|} + \frac{1}{|b_{n+1}|} \leq \mu(J)$$
$$\mu(\operatorname{supp}\hat{\psi}_n) + \mu(\operatorname{supp}\hat{\psi}_{n+1}) \leq \frac{1}{|b_n|} + \frac{1}{|b_{n+1}|} \leq \mu(\operatorname{supp}\hat{\psi}_n) + \mu(\operatorname{supp}\hat{\psi}_{n+1}) - \mu(I)$$
$$\implies \mu(I) = 0.$$

$\square$

By **Lemma 4.1**, since any candidate system must have the restriction that

$$\mu(\operatorname{supp}\hat{\psi}_n \cap \operatorname{supp}\hat{\psi}_{n+1}) = 0,$$

in this case, for a.e. $\xi \in \Gamma_{comp}, \exists! l \in \mathcal{N}$ such that $\xi \in \operatorname{supp}\hat{\psi}_l$, and

$$\sum_{n\in\mathcal{N}_\alpha} \frac{1}{|b_n|}|\hat{\psi}_n(\xi)|^2 = \frac{1}{|b_l|}|\hat{\psi}_l(\xi)|^2 = 1.$$

Hence we've shown the result.

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