# B

# A Brief Review of Matrix Concepts

In this appendix we will look at some of the basic concepts of matrix algebra. Our intent is simply to familiarize you with some basic matrix operations that we will need in our study of compression. Matrices are very useful for representing linear systems of equations, and matrix theory is a powerful tool for the study of linear operators. In our study of compression techniques we will use matrices both in the solution of systems of equations and in our study of linear transforms.

## B.1 A Matrix

A collection of real or complex elements arranged in $M$ rows and $N$ columns is called a matrix of order $M \times N$

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0N-1} \\ a_{10} & a_{11} & \cdots & a_{1N-1} \\ \vdots & \vdots & & \vdots \\ a_{(M-1)0} & a_{(M-1)1} & \cdots & a_{M-1N-1} \end{bmatrix} \tag{B.1}$$

where the first subscript denotes the row that an element belongs to and the second subscript denotes the column. For example, the element $a_{02}$ belongs in row 0 and column 2, and the element $a_{32}$ belongs in row 3 and column 2. The generic $ij$th element of a matrix $\mathbf{A}$ is sometimes represented as $[\mathbf{A}]_{ij}$. If the number of rows is equal to the number of columns ($N = M$), then the matrix is called a *square matrix*. A special square matrix that we will be using is the *identity matrix* $\mathbf{I}$, in which the elements on the diagonal of the matrix are 1 and all other elements are 0:

$$[\mathbf{I}]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases} \tag{B.2}$$

If a matrix consists of a single column ($N = 1$), it is called a *column matrix* or *vector* of dimension $M$. If it consists of a single row ($M = 1$), it is called a *row matrix* or *vector* of dimension $N$.

The *transpose* $\mathbf{A}^T$ of a matrix $\mathbf{A}$ is the $N \times M$ matrix obtained by writing the rows of the matrix as columns and the columns as rows:

$$\mathbf{A}^T = \begin{bmatrix} a_{00} & a_{10} & \cdots & a_{(M-1)0} \\ a_{01} & a_{11} & \cdots & a_{(M-1)1} \\ \vdots & \vdots & & \vdots \\ a_{0(N-1)} & a_{1(N-1)} & \cdots & a_{M-1N-1} \end{bmatrix} \tag{B.3}$$

The transpose of a column matrix is a row matrix and vice versa.

Two matrices $\mathbf{A}$ and $\mathbf{B}$ are said to be equal if they are of the same order and their corresponding elements are equal; that is,

$$\mathbf{A} = \mathbf{B} \quad \Leftrightarrow \quad a_{ij} = b_{ij}, i = 0, 1, \ldots M-1; j = 0, 1, \ldots N-1. \tag{B.4}$$

## B.2 Matrix Operations

You can add, subtract, and multiply matrices, but since matrices come in all shapes and sizes, there are some restrictions as to what operations you can perform with what kind of matrices. In order to add or subtract two matrices, their dimensions have to be identical—same number of rows and same number of columns. In order to multiply two matrices, the order in which they are multiplied is important. In general $\mathbf{A} \times \mathbf{B}$ is not equal to $\mathbf{B} \times \mathbf{A}$. Multiplication is only defined for the case where the number of columns of the first matrix is equal to the number of rows of the second matrix. The reasons for these restrictions will become apparent when we look at how the operations are defined.

When we add two matrices, the resultant matrix consists of elements that are the sum of the corresponding entries in the matrices being added. Let us add two matrices $\mathbf{A}$ and $\mathbf{B}$ where

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \end{bmatrix}$$

The sum of the two matrices, $\mathbf{C}$, is given by

$$\mathbf{C} = \begin{bmatrix} c_{00} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} = \begin{bmatrix} a_{00}+b_{00} & a_{01}+b_{01} & a_{02}+b_{02} \\ a_{10}+b_{10} & a_{11}+b_{11} & a_{12}+b_{12} \end{bmatrix} \tag{B.5}$$

Notice that each element of the resulting matrix $\mathbf{C}$ is the sum of corresponding elements of the matrices $\mathbf{A}$ and $\mathbf{B}$. In order for the two matrices to have corresponding elements, the dimension of the two matrices has to be the same. Therefore, addition is only defined for matrices with identical dimensions (i.e., same number of rows and same number of columns).

Subtraction is defined in a similar manner. The elements of the difference matrix are made up of term-by-term subtraction of the matrices being subtracted.

We could have generalized matrix addition and matrix subtraction from our knowledge of addition and subtraction of numbers. Multiplication of matrices is another kettle of fish entirely. It is easiest to describe matrix multiplication with an example. Suppose we have two different matrices **A** and **B** where

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \\ b_{20} & b_{21} \end{bmatrix} \tag{B.6}$$

The product is given by

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} = \begin{bmatrix} a_{00}b_{00} + a_{01}b_{10} + a_{02}b_{20} & a_{00}b_{01} + a_{01}b_{11} + a_{02}b_{21} \\ a_{10}b_{00} + a_{11}b_{10} + a_{12}b_{20} & a_{10}b_{01} + a_{11}b_{11} + a_{12}b_{21} \end{bmatrix}$$

You can see that the $i, j$ element of the product is obtained by adding term by term the product of elements in the $i$th row of the first matrix with those of the $j$th column of the second matrix. Thus, the element $c_{10}$ in the matrix **C** is obtained by summing the term-by-term products of row 1 of the first matrix **A** with column 0 of the matrix **B**. We can also see that the resulting matrix will have as many rows as the matrix to the left and as many columns as the matrix to the right.

What happens if we reverse the order of the multiplication? By the rules above we will end up with a matrix with three rows and three columns.

$$\begin{bmatrix} b_{00}a_{00} + b_{01}a_{10} & b_{00}a_{01} + +b_{01}a_{11} & b_{00}a_{02} + b_{01}a_{12} \\ b_{10}a_{00} + b_{11}a_{10} & b_{10}a_{01} + +b_{11}a_{11} & b_{10}a_{02} + b_{11}a_{12} \\ b_{20}a_{00} + b_{21}a_{10} & b_{20}a_{01} + +b_{21}a_{11} & b_{20}a_{02} + b_{21}a_{12} \end{bmatrix}$$

The elements of the two product matrices are different as are the dimensions.

As we can see, multiplication between matrices follows some rather different rules than multiplication between real numbers. The sizes have to match up—the number of columns of the first matrix has to be equal to the number of rows of the second matrix, and the order of multiplication is important. Because of the latter fact we often talk about premultiplying or postmultiplying. Premultiplying **B** by **A** results in the product **AB**, while postmultiplying **B** by **A** results in the product **BA**.

We have three of the four elementary operations. What about the fourth elementary operation, division? The easiest way to present division in matrices is to look at the formal definition of division when we are talking about real numbers. In the real number system, for every number $a$ different from zero, there exists an inverse, denoted by $1/a$ or $a^{-1}$, such that the product of $a$ with its inverse is one. When we talk about a number $b$ divided by a number $a$, this is the same as the *multiplication* of $b$ with the inverse of $a$. Therefore, we could define division by a matrix as the multiplication with the inverse of the matrix. **A/B** would be given by $\mathbf{AB}^{-1}$. Once we have the definition of an inverse of a matrix, the rules of multiplication apply.

So how do we define the inverse of a matrix? Following the definition for real numbers, in order to define the inverse of a matrix we need to have the matrix counterpart of 1. In matrices this counterpart is called the *identity matrix*. The identity matrix is a square matrix with diagonal elements being 1 and off-diagonal elements being 0. For example, a $3 \times 3$ identity matrix is given by

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{B.7}$$

The identity matrix behaves like the number one in the matrix world. If we multiply any matrix with the identity matrix (of appropriate dimension), we get the original matrix back. Given a square matrix $\mathbf{A}$, we define its inverse, $\mathbf{A}^{-1}$, as the matrix that when premultiplied or postmultiplied by $\mathbf{A}$ results in the identity matrix. For example, consider the matrix

$$\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \tag{B.8}$$

The inverse matrix is given by

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -2 \\ -0.5 & 1.5 \end{bmatrix} \tag{B.9}$$

To check that this is indeed the inverse matrix, let us multiply them:

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -0.5 & 1.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{B.10}$$

and

$$\begin{bmatrix} 1 & -2 \\ -0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{B.11}$$

If $\mathbf{A}$ is a vector of dimension $M$, we can define two specific kinds of products. If $\mathbf{A}$ is a column matrix, then the *inner product* or *dot product* is defined as

$$\mathbf{A}^T\mathbf{A} = \sum_{i=0}^{M-1} a_{i0}^2 \tag{B.12}$$

and the *outer product* or *cross product* is defined as

$$\mathbf{AA}^T = \begin{bmatrix} a_{00}a_{00} & a_{00}a_{10} & \cdots & a_{00}a_{(M-1)0} \\ a_{10}a_{00} & a_{10}a_{10} & \cdots & a_{10}a_{(M-1)0} \\ \vdots & \vdots & & \vdots \\ a_{(M-1)0}a_{00} & a_{\{(M-1)1\}}a_{10} & \cdots & a_{(M-1)0}a_{(M-1)0} \end{bmatrix} \tag{B.13}$$

Notice that the inner product results in a scalar, while the outer product results in a matrix.

In order to find the inverse of a matrix, we need the concepts of determinant and cofactor. Associated with each square matrix is a scalar value called the *determinant* of the matrix. The determinant of a matrix $\mathbf{A}$ is denoted as $|\mathbf{A}|$. To see how to obtain the determinant of an $N \times N$ matrix, we start with a $2 \times 2$ matrix. The determinant of a $2 \times 2$ matrix is given as

$$|\mathbf{A}| = \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{vmatrix} = a_{00}a_{11} - a_{01}a_{10}. \tag{B.14}$$

Finding the determinant of a $2 \times 2$ matrix is easy. To explain how to get the determinants of larger matrices, we need to define some terms.

The *minor* of an element $a_{ij}$ of an $N \times N$ matrix is defined to be the determinant of the $N-1 \times N-1$ matrix obtained by deleting the row and column containing $a_{ij}$. For example, if $\mathbf{A}$ is a $4 \times 4$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{B.15}$$

then the minor of the element $a_{12}$, denoted by $M_{12}$, is the determinant

$$M_{12} = \begin{vmatrix} a_{00} & a_{01} & a_{03} \\ a_{20} & a_{21} & a_{23} \\ a_{30} & a_{31} & a_{33} \end{vmatrix} \tag{B.16}$$

The cofactor of $a_{ij}$ denoted by $\mathbf{A}_{ij}$ is given by

$$\mathbf{A}_{ij} = (-1)^{i+j} M_{ij}. \tag{B.17}$$

Armed with these definitions we can write an expression for the determinant of an $N \times N$ matrix as

$$|\mathbf{A}| = \sum_{i=0}^{N-1} a_{ij}\mathbf{A}_{ij} \tag{B.18}$$

or

$$|\mathbf{A}| = \sum_{j=0}^{N-1} a_{ij}\mathbf{A}_{ij} \tag{B.19}$$

where the $a_{ij}$ are taken from a single row or a single column. If the matrix has a particular row or column that has a large number of zeros in it, we would need fewer computations if we picked that particular row or column.

Equations (B.18) and (B.19) express the determinant of an $N \times N$ matrix in terms of determinants of $N-1 \times N-1$ matrices. We can express each of the $N-1 \times N-1$ determinants in terms of $N-2 \times N-2$ determinants, continuing in this fashion until we have everything expressed in terms of $2 \times 2$ determinants, which can be evaluated using (B.14).

Now that we know how to compute a determinant, we need one more definition before we can define the inverse of a matrix. The *adjoint* of a matrix $\mathbf{A}$, denoted by $(\mathbf{A})$, is a matrix whose $ij$th element is the cofactor $\mathbf{A}_{ji}$. The inverse of a matrix $\mathbf{A}$, denoted by $\mathbf{A}^{-1}$, is given by

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}(\mathbf{A}). \tag{B.20}$$

Notice that for the inverse to exist the determinant has to be nonzero. If the determinant for a matrix is zero, the matrix is said to be singular. The method we have described here works well with small matrices; however, it is highly inefficient if $N$ becomes greater than 4. There are a number of efficient methods for inverting matrices; see the books in the Further Reading section for details.

Corresponding to a square matrix $\mathbf{A}$ of size $N \times N$ are $N$ scalar values called the *eigenvalues* of $\mathbf{A}$. The eigenvalues are the $N$ solutions of the equation $|\lambda\mathbf{I} - \mathbf{A}| = 0$. This equation is called the *characteristic equation.*

## Example B.2.1:

Let us find the eigenvalues of the matrix

$$\begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$$

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{A}| &= 0 \\ \left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix} \right| &= 0 \\ (\lambda - 4)(\lambda - 1) - 10 &= 0 \\ \lambda_1 = -1 \qquad \lambda_2 &= 6 \end{aligned} \tag{B.21}$$

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The eigenvectors $V_k$ of an $N \times N$ matrix are the $N$ vectors of dimension $N$ that satisfy the equation

$$\mathbf{A}V_k = \lambda_k V_k. \tag{B.22}$$

### Further Reading

1. The subject of matrices is covered at an introductory level in a number of textbooks. A good one is *Advanced Engineering Mathematics*, by E. Kreyszig [129].

2. Numerical methods for manipulating matrices (and a good deal more) are presented in *Numerical Recipes in C*, by W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.P. Flannery [178].