12

Mathematical Preliminaries for Transforms, Subbands, and Wavelets

12.1 Overview

In this chapter we will review some of the mathematical background necessary for the study of transforms, subbands, and wavelets. The topics include Fourier series, Fourier transforms, and their discrete counterparts. We will also look at sampling and briefly review some linear system concepts.

12.2 Introduction

The roots of many of the techniques we will study can be found in the mathematical literature. Therefore, in order to understand the techniques, we will need some mathematical background. Our approach in general will be to introduce the mathematical tools just prior to when they are needed. However, there is a certain amount of background that is required for most of what we will be looking at. In this chapter we will present only that material that is a common background to all the techniques we will be studying. Our approach will be rather utilitarian; more sophisticated coverage of these topics can be found in [175]. We will be introducing a rather large number of concepts, many of which depend on each other. In order to make it easier for you to find a particular concept, we will identify the paragraph in which the concept is first introduced.

We will begin our coverage with a brief introduction to the concept of vector spaces, and in particular the concept of the inner product. We will use these concepts in our description of Fourier series and Fourier transforms. Next is a brief overview of linear systems, then a look at the issues involved in sampling a function. Finally, we will revisit the Fourier concepts in the context of sampled functions and provide a brief introduction to Z-transforms. Throughout, we will try to get a physical feel for the various concepts.

12.3 Vector Spaces

The techniques we will be using to obtain compression will involve manipulations and decompositions of (sampled) functions of time. In order to do this we need some sort of mathematical framework. This framework is provided through the concept of vector spaces.

We are very familiar with vectors in two- or three-dimensional space. An example of a vector in two-dimensional space is shown in Figure 12.1. This vector can be represented in a number of different ways: we can represent it in terms of its magnitude and direction, or we can represent it as a weighted sum of the unit vectors in the x and y directions, or we can represent it as an array whose components are the coefficients of the unit vectors. Thus, the vector $\mathbf{v}$ in Figure 12.1 has a magnitude of 5 and an angle of 36.86 degrees,

$$\mathbf{v} = 4u_x + 3u_y$$

and

$$\mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

We can view the second representation as a decomposition of V into simpler building blocks, namely, the *basis vectors*. The nice thing about this is that any vector in two dimensions can be decomposed in exactly the same way. Given a particular vector $\mathbf{A}$ and a

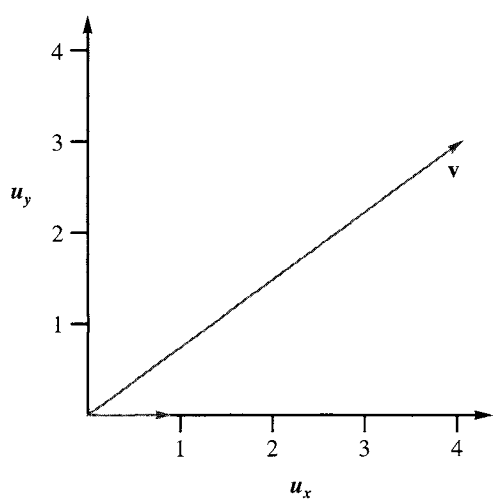

FIGURE 12.1 **A vector.**

basis set (more on this later), decomposition means finding the coefficients with which to weight the unit vectors of the basis set. In our simple example it is easy to see what these coefficients should be. However, we will encounter situations where it is not a trivial task to find the coefficients that constitute the decomposition of the vector. We therefore need some machinery to extract these coefficients. The particular machinery we will use here is called the *dot product* or the *inner product.*

12.3.1 Dot or Inner Product

Given two vectors **a** and **b** such that

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

the inner product between **a** and **b** is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

Two vectors are said to be *orthogonal* if their inner product is zero. A set of vectors is said to be orthogonal if each vector in the set is orthogonal to every other vector in the set. The inner product between a vector and a unit vector from an orthogonal basis set will give us the coefficient corresponding to that unit vector. It is easy to see that this is indeed so. We can write u_x and u_y as

$$u_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad u_y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

These are obviously orthogonal. Therefore, the coefficient a_1 can be obtained by

$$\mathbf{a} \cdot u_x = a_1 \times 1 + a_2 \times 0 = a_1$$

and the coefficient of u_y can be obtained by

$$\mathbf{a} \cdot u_y = a_1 \times 0 + a_2 \times 1 = a_2.$$

The inner product between two vectors is in some sense a measure of how "similar" they are, but we have to be a bit careful in how we define "similarity." For example, consider the vectors in Figure 12.2. The vector **a** is closer to u_x than to u_y. Therefore $\mathbf{a} \cdot u_x$ will be greater than $\mathbf{a} \cdot u_y$. The reverse is true for **b**.

12.3.2 Vector Space

In order to handle not just two- or three-dimensional vectors but general sequences and functions of interest to us, we need to generalize these concepts. Let us begin with a more general definition of vectors and the concept of a vector space.

> A *vector space* consists of a set of elements called vectors that have the operations of vector addition and scalar multiplication defined on them. Furthermore, the results of these operations are also elements of the vector space.

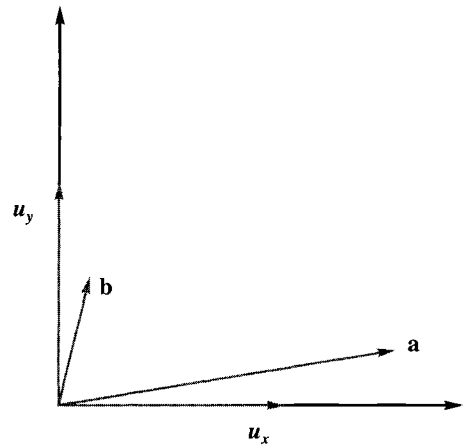

FIGURE 12.2 **Example of different vectors.**

By *vector addition* of two vectors, we mean the vector obtained by the pointwise addition of the components of the two vectors. For example, given two vectors **a** and **b**:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{12.1}$$

the vector addition of these two vectors is given as

$$\mathbf{a}+\mathbf{b} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{bmatrix}. \tag{12.2}$$

By *scalar multiplication*, we mean the multiplication of a vector with a real or complex number. For this set of elements to be a vector space it has to satisfy certain axioms.

Suppose V is a vector space; $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are vectors; and α and β are scalars. Then the following axioms are satisfied:

1. $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ (commutativity).
2. $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ and $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ (associativity).
3. There exists an element θ in V such that $\mathbf{x}+\theta = \mathbf{x}$ for all $\mathbf{x}$ in V. θ is called the additive identity.
4. $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$, and $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$ (distributivity).
5. $1 \cdot \mathbf{x} = \mathbf{x}$, and $0 \cdot \mathbf{x} = \theta$.
6. For every $\mathbf{x}$ in V, there exists a $(-\mathbf{x})$ such that $\mathbf{x}+(-\mathbf{x}) = \theta$.

A simple example of a vector space is the set of real numbers. In this set zero is the additive identity. We can easily verify that the set of real numbers with the standard operations of addition and multiplication obey the axioms stated above. See if you can verify that the set of real numbers is a vector space. One of the advantages of this exercise is to emphasize the fact that a vector is more than a line with an arrow at its end.

Example 12.3.1:

Another example of a vector space that is of more practical interest to us is the set of all functions $f(t)$ with finite energy. That is,

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty. \tag{12.3}$$

Let's see if this set constitutes a vector space. If we define additions as pointwise addition and scalar multiplication in the usual manner, the set of functions $f(t)$ obviously satisfies axioms 1, 2, and 4.

- If $f(t)$ and $g(t)$ are functions with finite energy, and α is a scalar, then the functions $f(t)+g(t)$ and $\alpha f(t)$ also have finite energy.
- If $f(t)$ and $g(t)$ are functions with finite energy, then $f(t)+g(t)=g(t)+f(t)$ (axiom 1).
- If $f(t)$, $g(t)$, and $h(t)$ are functions with finite energy, and α and β are scalars, then $(f(t)+g(t))+h(t)=f(t)+(g(t)+h(t))$ and $(\alpha\beta)f(t)=\alpha(\beta f(t))$ (axiom 2).
- If $f(t)$, $g(t)$, and $h(t)$ are functions with finite energy, and α is a scalar, then $\alpha(f(t)+g(t))=\alpha f(t)+\alpha g(t)$ and $(\alpha+\beta)f(t)=\alpha f(t)+\beta f(t)$ (axiom 4).

Let us define the additive identity function $\theta(t)$ as the function that is identically zero for all t. This function satisfies the requirement of finite energy, and we can see that axioms 3 and 5 are also satisfied. Finally, if a function $f(t)$ has finite energy, then from Equation (12.3), the function $-f(t)$ also has finite energy, and axiom 6 is satisfied. Therefore, the set of all functions with finite energy constitutes a vector space. This space is denoted by $L_2(f)$, or simply L_2. ♦

12.3.3 Subspace

A *subspace* S of a vector space V is a subset of V whose members satisfy all the axioms of the vector space and has the additional property that if $\mathbf{x}$ and $\mathbf{y}$ are in S, and α is a scalar, then $\mathbf{x}+\mathbf{y}$ and $\alpha\mathbf{x}$ are also in S.

Example 12.3.2:

Consider the set S of continuous bounded functions on the interval [0, 1]. Then S is a subspace of the vector space L_2. ♦

12.3.4 Basis

One way we can generate a subspace is by taking linear combinations of a set of vectors. If this set of vectors is *linearly independent*, then the set is called a *basis* for the subspace.

> A set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$ is said to be linearly independent if no vector of the set can be written as a linear combination of the other vectors in the set.

A direct consequence of this definition is the following theorem:

Theorem *A set of vectors* $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N\}$ *is linearly independent if and only if the expression* $\sum_{i=1}^{N} \alpha_i \mathbf{x}_i = \theta$ *implies that* $\alpha_i = 0$ *for all* $i = 1, 2, \ldots, N$.

Proof The proof of this theorem can be found in most books on linear algebra [175]. □

The set of vectors formed by all possible linear combinations of vectors from a linearly independent set $\mathbf{X}$ forms a vector space (Problem 1). The set $\mathbf{X}$ is said to be the *basis* for this vector space. The basis set contains the smallest number of linearly independent vectors required to represent each element of the vector space. More than one set can be the basis for a given space.

Example 12.3.3:

Consider the vector space consisting of vectors $[ab]^T$, where a and b are real numbers. Then the set

$$\mathbf{X} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

forms a basis for this space, as does the set

$$\mathbf{X} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

In fact, any two vectors that are not scalar multiples of each other form a basis for this space. ♦

The number of basis vectors required to generate the space is called the *dimension* of the vector space. In the previous example the dimension of the vector space is two. The dimension of the space of all continuous functions on the interval [0, 1] is infinity.

Given a particular basis, we can find a representation with respect to this basis for any vector in the space.

Example 12.3.4:

If $\mathbf{a} = [34]^T$, then

$$\mathbf{a} = 3\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix}$$

and

$$\mathbf{a} = 4\begin{bmatrix}1\\1\end{bmatrix} + (-1)\begin{bmatrix}1\\0\end{bmatrix}$$

so the representation of $\mathbf{a}$ with respect to the first basis set is (3, 4), and the representation of $\mathbf{a}$ with respect to the second basis set is (4, -1). ♦

In the beginning of this section we had described a mathematical machinery for finding the components of a vector that involved taking the dot product or inner product of the vector to be decomposed with basis vectors. In order to use the same machinery in more abstract vector spaces we need to generalize the notion of inner product.

12.3.5 Inner Product—Formal Definition

An inner product between two vectors $\mathbf{x}$ and $\mathbf{y}$, denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$, associates a scalar value with each pair of vectors. The inner product satisfies the following axioms:

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$, where * denotes complex conjugate.
2. $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$.
3. $\langle \alpha\mathbf{x}, \mathbf{y} \rangle = \alpha\langle \mathbf{x}, \mathbf{y} \rangle$.
4. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, with equality if and only if $\mathbf{x} = \theta$. The quantity $\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ denoted by $\|\mathbf{x}\|$ is called the *norm* of $\mathbf{x}$ and is analogous to our usual concept of distance.

12.3.6 Orthogonal and Orthonormal Sets

As in the case of Euclidean space, two vectors are said to be *orthogonal* if their inner product is zero. If we select our basis set to be orthogonal (that is, each vector is orthogonal to every other vector in the set) and further require that the norm of each vector be one (that is, the basis vectors are unit vectors), such a basis set is called an *orthonormal basis set*. Given an orthonormal basis, it is easy to find the representation of any vector in the space in terms of the basis vectors using the inner product. Suppose we have a vector space S_N with an orthonormal basis set $\{\mathbf{x}_i\}_{i=1}^N$. Given a vector $\mathbf{y}$ in the space S_N, by definition of the basis set we can write $\mathbf{y}$ as a linear combination of the vectors $\mathbf{x}_i$:

$$\mathbf{y} = \sum_{i=1}^{N} \alpha_i \mathbf{x}_i.$$

To find the coefficient α_k, we find the inner product of both sides of this equation with $\mathbf{x}_k$:

$$\langle \mathbf{y}, \mathbf{x}_k \rangle = \sum_{i=1}^{N} \alpha_i \langle \mathbf{x}_i, \mathbf{x}_k \rangle.$$

Because of orthonormality,

$$\langle \mathbf{x}_i, \mathbf{x}_k \rangle = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

and

$$\langle \mathbf{y}, \mathbf{x}_k \rangle = \alpha_k.$$

By repeating this with each $\mathbf{x}_i$, we can get all the coefficients α_i. Note that in order to use this machinery, the basis set has to be orthonormal.

We now have sufficient information in hand to begin looking at some of the well-known techniques for representing functions of time. This was somewhat of a crash course in vector spaces, and you might, with some justification, be feeling somewhat dazed. Basically, the important ideas that we would like you to remember are the following:

- Vectors are not simply points in two- or three-dimensional space. In fact, functions of time can be viewed as elements in a vector space.
- Collections of vectors that satisfy certain axioms make up a vector space.
- All members of a vector space can be represented as linear, or weighted, combinations of the basis vectors (keep in mind that you can have many different basis sets for the same space). If the basis vectors have unit magnitude and are orthogonal, they are known as an *orthonormal basis set.*
- If a basis set is orthonormal, the weights, or coefficients, can be obtained by taking the inner product of the vector with the corresponding basis vector.

In the next section we use these concepts to show how we can represent periodic functions as linear combinations of sines and cosines.

12.4 Fourier Series

The representation of periodic functions in terms of a series of sines and cosines was discovered by Jean Baptiste Joseph Fourier. Although he came up with this idea in order to help him solve equations describing heat diffusion, this work has since become indispensable in the analysis and design of systems. The work was awarded the grand prize for mathematics in 1812 and has been called one of the most revolutionary contributions of the last century. A very readable account of the life of Fourier and the impact of his discovery can be found in [176].

Fourier showed that any periodic function, no matter how awkward looking, could be represented as the sum of smooth, well-behaved sines and cosines. Given a periodic function $f(t)$ with period T,

$$f(t) = f(t + nT) \qquad n = \pm 1, \pm 2, \ldots$$

we can write $f(t)$ as

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nw_0 t + \sum_{n=1}^{\infty} b_n \sin nw_0 t, \qquad w_0 = \frac{2\pi}{T}. \tag{12.4}$$

This form is called the *trigonometric Fourier series representation* of $f(t)$.

A more useful form of the Fourier series representation from our point of view is the exponential form of the Fourier series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jnw_0 t}. \tag{12.5}$$

We can easily move between the exponential and trigonometric representations by using Euler's identity

$$e^{j\phi} = \cos\phi + j \sin\phi$$

where $j = \sqrt{-1}$.

In the terminology of the previous section, all periodic functions with period T form a vector space. The complex exponential functions $\{e^{jn\omega_0 t}\}$ constitute a basis for this space. The parameters $\{c_n\}_{n=-\infty}^{\infty}$ are the representations of a given function $f(t)$ with respect to this basis set. Therefore, by using different values of $\{c_n\}_{n=-\infty}^{\infty}$, we can build different periodic functions. If we wanted to inform somebody what a particular periodic function looked like, we could send the values of $\{c_n\}_{n=-\infty}^{\infty}$ and they could synthesize the function.

We would like to see if this basis set is orthonormal. If it is, we want to be able to obtain the coefficients that make up the Fourier representation using the approach described in the previous section. In order to do all this, we need a definition of the inner product on this vector space. If $f(t)$ and $g(t)$ are elements of this vector space, the inner product is defined as

$$\langle f(t), g(t) \rangle = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) g(t)^* dt \tag{12.6}$$

where t_0 is an arbitrary constant and * denotes complex conjugate. For convenience we will take t_0 to be zero.

Using this inner product definition, let us check to see if the basis set is orthonormal.

$$\langle e^{jn\omega_0 t}, e^{jm\omega_0 t} \rangle = \frac{1}{T} \int_0^T e^{jn\omega_0 t} e^{-jm\omega_0 t} dt \tag{12.7}$$

$$= \frac{1}{T} \int_0^T e^{j(n-m)\omega_0 t} dt \tag{12.8}$$

When $n = m$, Equation (12.7) becomes the norm of the basis vector, which is clearly one. When $n \neq m$, let us define $k = n - m$. Then

$$\langle e^{jn\omega_0 t}, e^{jm\omega_0 t}\rangle = \frac{1}{T}\int_0^T e^{jk\omega_0 t} dt \tag{12.9}$$

$$= \frac{1}{jk\omega_0}(e^{jk\omega_0 T} - 1) \tag{12.10}$$

$$= \frac{1}{jk\omega_0}(e^{jk2\pi} - 1) \tag{12.11}$$

$$= 0 \tag{12.12}$$

where we have used the facts that $\omega_0 = \frac{2\pi}{T}$ and

$$e^{jk2\pi} = \cos(2k\pi) + j\sin(2k\pi) = 1.$$

Thus, the basis set is orthonormal.

Using this fact, we can find the coefficient c_n by taking the inner product of $f(t)$ with the basis vector $e^{jn\omega_0 t}$:

$$c_n = \langle f(t), e^{jn\omega_0 t}\rangle = \frac{1}{T}\int_0^T f(t) e^{jn\omega_0 t}\, dt. \tag{12.13}$$

What do we gain from obtaining the Fourier representation $\{c_n\}_{n=-\infty}^{\infty}$ of a function $f(t)$? Before we answer this question, let us examine the context in which we generally use Fourier analysis. We start with some signal generated by a source. If we wish to look at how this signal changes its amplitude over a period of time (or space), we represent it as a function of time $f(t)$ (or a function of space $f(x)$). Thus, $f(t)$ (or $f(x)$) is a representation of the signal that brings out how this signal varies in time (or space). The sequence $\{c_n\}_{n=-\infty}^{\infty}$ is a different representation of the same signal. However, this representation brings out a different aspect of the signal. The basis functions are sinusoids that differ from each other in how fast they fluctuate in a given time interval. The basis vector $e^{2j\omega_0 t}$ fluctuates twice as fast as the basis vector $e^{j\omega_0 t}$. The coefficients of the basis vectors $\{c_n\}_{n=-\infty}^{\infty}$ give us a measure of the different amounts of fluctuation present in the signal. Fluctuation of this sort is usually measured in terms of frequency. A frequency of 1 Hz denotes the completion of one period in one second, a frequency of 2 Hz denotes the completion of two cycles in one second, and so on. Thus, the coefficients $\{c_n\}_{n=-\infty}^{\infty}$ provide us with a frequency profile of the signal: how much of the signal changes at the rate of $\frac{\omega_0}{2\pi}$ Hz, how much of the signal changes at the rate of $\frac{2\omega_0}{2\pi}$ Hz, and so on. This information cannot be obtained by looking at the time representation $f(t)$. On the other hand, the use of the $\{c_n\}_{n=-\infty}^{\infty}$ representation tells us little about how the signal changes with time. Each representation emphasizes a different aspect of the signal. The ability to view the same signal in different ways helps us to better understand the nature of the signal, and thus develop tools for manipulation of the signal. Later, when we talk about wavelets, we will look at representations that provide information about both the time profile and the frequency profile of the signal.

The Fourier series provides us with a frequency representation of *periodic* signals. However, many of the signals we will be dealing with are not periodic. Fortunately, the Fourier series concepts can be extended to nonperiodic signals.

12.5 Fourier Transform

Consider the function $f(t)$ shown in Figure 12.3. Let us define a function $f_P(t)$ as

$$f_P(t) = \sum_{n=-\infty}^{\infty} f(t-nT) \tag{12.14}$$

where $T > t_1$. This function, which is obviously periodic ($f_P(t+T) = f_P(t)$), is called the *periodic extension* of the function $f(t)$. Because the function $f_P(t)$ is periodic, we can define a Fourier series expansion for it:

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_P(t)e^{-jn\omega_0 t}dt \tag{12.15}$$

$$f_P(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}. \tag{12.16}$$

Define

$$C(n, T) = c_n T$$

and

$$\Delta\omega = \omega_0,$$

and let us slightly rewrite the Fourier series equations:

$$C(n, T) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f_P(t)e^{-jn\Delta\omega t}dt \tag{12.17}$$

$$f_P(t) = \sum_{n=-\infty}^{\infty} \frac{C(n, T)}{T} e^{jn\Delta\omega t}. \tag{12.18}$$

We can recover $f(t)$ from $f_P(t)$ by taking the limit of $f_P(t)$ as T goes to infinity. Because $\Delta\omega = \omega_0 = \frac{2\pi}{T}$, this is the same as taking the limit as $\Delta\omega$ goes to zero. As $\Delta\omega$ goes to zero, $n\Delta\omega$ goes to a continuous variable ω. Therefore,

$$\lim_{T\to\infty\Delta\omega\to 0} \int_{-\frac{T}{2}}^{\frac{T}{2}} f_P(t)e^{-jn\Delta\omega t}dt = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt. \tag{12.19}$$

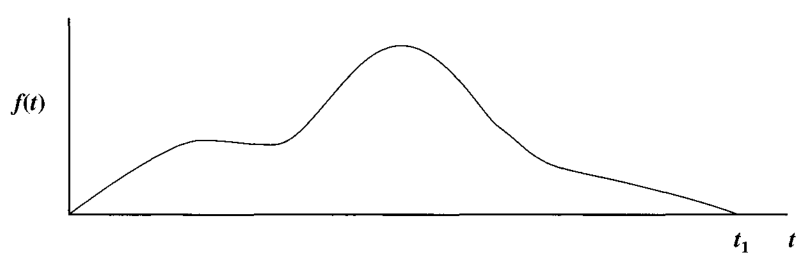

FIGURE 12. 3 **A function of time.**

From the right-hand side, we can see that the resulting function is a function only of ω. We call this function the Fourier transform of $f(t)$, and we will denote it by $F(\omega)$. To recover $f(t)$ from $F(w)$, we apply the same limits to Equation (12.18):

$$f(t) \lim_{T\to\infty} f_P(t) = \lim_{T\to\infty \Delta\omega\to 0} \sum_{n=-\infty}^{\infty} C(n, T)\frac{\Delta\omega}{2\pi} e^{jn\Delta\omega t} \tag{12.20}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\, d\omega. \tag{12.21}$$

The equation

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt \tag{12.22}$$

is generally called the *Fourier transform*. The function $F(\omega)$ tells us how the signal fluctuates at different frequencies. The equation

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)e^{j\omega t}\, d\omega \tag{12.23}$$

is called the *inverse Fourier transform*, and it shows us how we can construct a signal using components that fluctuate at different frequencies. We will denote the operation of the Fourier transform by the symbol $\mathcal{F}$. Thus, in the preceding, $F(\omega) = \mathcal{F}[f(t)]$.

There are several important properties of the Fourier transform, three of which will be of particular use to us. We state them here and leave the proof to the problems (Problems 2, 3, and 4).

12.5.1 Parseval's Theorem

The Fourier transform is an energy-preserving transform; that is, the total energy when we look at the time representation of the signal is the same as the total energy when we look at the frequency representation of the signal. This makes sense because the total energy is a physical property of the signal and should not change when we look at it using different representations. Mathematically, this is stated as

$$\int_{-\infty}^{\infty} |f(t)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega. \tag{12.24}$$

The $\frac{1}{2\pi}$ factor is a result of using units of radians (ω) for frequency instead of Hertz (f). If we substitute $\omega = 2\pi f$ in Equation (12.24), the 2π factor will go away. This property applies to any vector space representation obtained using an orthonormal basis set.

12.5.2 Modulation Property

If $f(t)$ has the Fourier transform $F(\omega)$, then the Fourier transform of $f(t)e^{j\omega_0 t}$ is $F(w - w_0)$. That is, multiplication with a complex exponential in the time domain corresponds to a shift in the frequency domain. As a sinusoid can be written as a sum of complex exponentials, multiplication of $f(t)$ by a sinusoid will also correspond to shifts of $F(\omega)$. For example,

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Therefore,

$$\mathcal{F}[f(t)\cos(\omega_0 t)] = \frac{1}{2}\left(F(\omega - \omega_0) + F(\omega + \omega_0)\right).$$

12.5.3 Convolution Theorem

When we examine the relationships between the input and output of linear systems, we will encounter integrals of the following forms:

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t - \tau) d\tau$$

or

$$f(t) = \int_{-\infty}^{\infty} f_1(t - \tau) f_2(\tau) d\tau.$$

These are called convolution integrals. The convolution operation is often denoted as

$$f(t) = f_1(t) \otimes f_2(t).$$

The convolution theorem states that if $F(\omega) = \mathcal{F}[f(t)] = \mathcal{F}[f_1(t) \otimes f_2(t)]$, $F_1(\omega) = \mathcal{F}[f_1(t)]$, and $F_2(\omega) = \mathcal{F}[f_2(t)]$, then

$$F(\omega) = F_1(\omega) F_2(\omega).$$

We can also go in the other direction. If

$$F(\omega) = F_1(\omega) \otimes F_2(\omega) = \int F_1(\sigma) F_2(\omega - \sigma) d\sigma$$

then

$$f(t) = f_1(t) f_2(t).$$

As mentioned earlier, this property of the Fourier transform is important because the convolution integral relates the input and output of linear systems, which brings us to one of the major reasons for the popularity of the Fourier transform. We have claimed that the Fourier series and Fourier transform provide us with an alternative frequency profile of a signal. Although sinusoids are not the only basis set that can provide us with a frequency profile, they do, however, have an important property that helps us study linear systems, which we describe in the next section.

12.6 Linear Systems

A linear system is a system that has the following two properties:

- **Homogeneity:** Suppose we have a linear system L with input $f(t)$ and output $g(t)$:

  $$g(t) = L[f(t)].$$

  If we have two inputs, $f_1(t)$ and $f_2(t)$, with corresponding outputs, $g_1(t)$ and $g_2(t)$, then the output of the sum of the two inputs is simply the sum of the two outputs:

  $$L[f_1(t) + f_2(t)] = g_1(t) + g_2(t).$$

- **Scaling:** Given a linear system L with input $f(t)$ and output $g(t)$, if we multiply the input with a scalar α, then the output will be multiplied by the same scalar:

  $$L[\alpha f(t)] = \alpha L[f(t)] = \alpha g(t).$$

The two properties together are referred to as *superposition.*

12.6.1 Time Invariance

Of specific interest to us are linear systems that are *time invariant.* A time-invariant system has the property that the shape of the response of this system does not depend on the time at which the input was applied. If the response of a linear system L to an input $f(t)$ is $g(t)$,

$$L[f(t)] = g(t),$$

and we delay the input by some interval t_0, then if L is a time-invariant system, the output will be $g(t)$ delayed by the same amount:

$$L[f(t - t_0)] = g(t - t_0). \tag{12.25}$$

12.6.2 Transfer Function

Linear time-invariant systems have a very interesting (and useful) response when the input is a sinusoid. If the input to a linear system is a sinusoid of a certain frequency ω_0, then the output is also a sinusoid of the same frequency that has been scaled and delayed; that is,

$$L[\cos(\omega_0 t)] = \alpha \cos(\omega_0 (t - t_d))$$

or in terms of the complex exponential

$$L[e^{j\omega_0 t}] = \alpha e^{j\omega_0 (t - t_d)}.$$

Thus, given a linear system, we can characterize its response to sinusoids of a particular frequency by a pair of parameters, the gain α and the delay t_d. In general, we use the phase $\phi = \omega_0 t_d$ in place of the delay. The parameters α and ϕ will generally be a function of the frequency, so in order to characterize the system for all frequencies, we will need a pair of functions $\alpha(\omega)$ and $\phi(\omega)$. As the Fourier transform allows us to express the signal as coefficients of sinusoids, given an input $f(t)$, all we need to do is, for each frequency ω, multiply the Fourier transform of $f(t)$ with some $\alpha(\omega)e^{j\phi(\omega)}$, where $\alpha(\omega)$ and $\phi(\omega)$ are the gain and phase terms of the linear system for that particular frequency.

This pair of functions $\alpha(\omega)$ and $\phi(\omega)$ constitute the *transfer function* of the linear time-invariant system $H(\omega)$:

$$H(\omega) = |H(\omega)|\, e^{j\phi(\omega)}$$

where $|H(\omega)| = \alpha(\omega)$.

Because of the specific way in which a linear system responds to a sinusoidal input, given a linear system with transfer function $H(\omega)$, input $f(t)$, and output $g(t)$, the Fourier transforms of the input and output $F(\omega)$ and $G(\omega)$ are related by

$$G(w) = H(\omega)F(\omega).$$

Using the convolution theorem, $f(t)$ and $g(t)$ are related by

$$g(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$$

or

$$g(t) = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau$$

where $H(\omega)$ is the Fourier transform of $h(t)$.

12.6.3 Impulse Response

To see what $h(t)$ is, let us look at the input-output relationship of a linear time-invariant system from a different point of view. Let us suppose we have a linear system L with input $f(t)$. We can obtain a staircase approximation $f_S(t)$ to the function $f(t)$, as shown in Figure 12.4:

$$f_S(t) = \sum f(n\Delta t)\text{rect}\left(\frac{t-n\Delta t}{\Delta t}\right) \tag{12.26}$$

where

$$\text{rect}\left(\frac{t}{T}\right) = \begin{cases} 1 & |t| < \frac{T}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{12.27}$$

The response of the linear system can be written as

$$L[f_S(t)] = L\left[\sum f(n\Delta t)\text{rect}\left(\frac{t-n\Delta t}{\Delta t}\right)\right] \tag{12.28}$$

$$= L\left[\sum f(n\Delta t)\frac{\text{rect}\left(\frac{t-n\Delta t}{\Delta t}\right)}{\Delta t}\Delta t\right]. \tag{12.29}$$

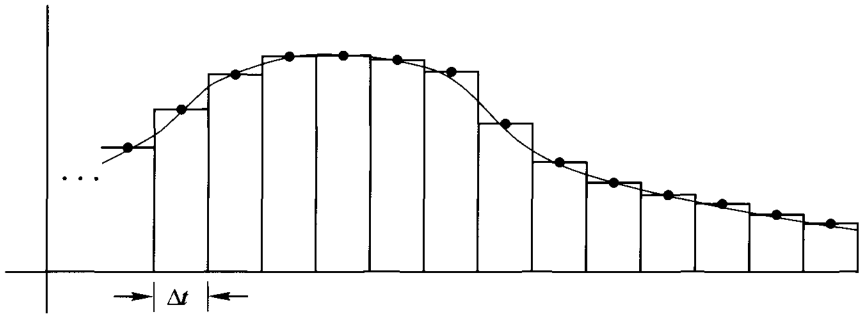

FIGURE 12.4 A function of time.

For a given value of Δt, we can use the superposition property of linear systems to obtain

$$L[f_S(t)] = \sum f(n\Delta t) L\left[\frac{\text{rect}(\frac{t-n\Delta t}{\Delta t})}{\Delta t}\right]\Delta t. \tag{12.30}$$

If we now take the limit as Δt goes to zero in this equation, on the left-hand side $f_S(t)$ will go to $f(t)$. To see what happens on the right-hand side of the equation, first let's look at the effect of this limit on the function $\text{rect}(\frac{t}{\Delta t})/\Delta t$. As Δt goes to zero, this function becomes narrower and taller. However, at all times the integral of this function is equal to one. The limit of this function as Δt goes to zero is called the *Dirac delta function*, or *impulse function*, and is denoted by $\delta(t)$:

$$\lim_{\Delta t \to 0} \frac{\text{rect}(\frac{t-n\Delta t}{\Delta t})}{\Delta t} = \delta(t). \tag{12.31}$$

Therefore,

$$L[f(t)] = \lim_{\Delta t \to 0} L[f_S(t)] = \int f(\tau) L[\delta(t-\tau)] d\tau. \tag{12.32}$$

Denote the response of the system L to an impulse, or the *impulse response*, by $h(t)$:

$$h(t) = L[\delta(t)]. \tag{12.33}$$

Then, if the system is also time invariant,

$$L[f(t)] = \int f(\tau) h(t-\tau) d\tau. \tag{12.34}$$

Using the convolution theorem, we can see that the Fourier transform of the impulse response $h(t)$ is the transfer function $H(\omega)$.

The Dirac delta function is an interesting function. In fact, it is not clear that it is a function at all. It has an integral that is clearly one, but at the only point where it is not zero, it is undefined! One property of the delta function that makes it very useful is the *sifting* property:

$$\int_{t_1}^{t_2} f(t)\delta(t-t_0)dt = \begin{cases} f(t_0) & t_1 \le t_0 \le t_2 \\ 0 & \text{otherwise.} \end{cases} \tag{12.35}$$

12.6.4 Filter

The linear systems of most interest to us will be systems that permit certain frequency components of the signal to pass through, while attenuating all other components of the signal. Such systems are called *filters*. If the filter allows only frequency components below a certain frequency W Hz to pass through, the filter is called a *low-pass filter*. The transfer function of an ideal low-pass filter is given by

$$H(\omega) = \begin{cases} e^{-j\alpha\omega} & |\omega| < 2\pi W \\ 0 & \text{otherwise.} \end{cases} \tag{12.36}$$

This filter is said to have a *bandwidth* of W Hz. The magnitude of this filter is shown in Figure 12.5. A low-pass filter will produce a smoothed version of the signal by blocking higher-frequency components that correspond to fast variations in the signal.

A filter that attenuates the frequency components below a certain frequency W and allows the frequency components above this frequency to pass through is called a *high-pass filter*. A high-pass filter will remove slowly changing trends from the signal. Finally, a signal that lets through a range of frequencies between two specified frequencies, say, W_1 and W_2, is called a *band-pass filter*. The bandwidth of this filter is said to be $W_2 - W_1$ Hz. The magnitude of the transfer functions of an ideal high-pass filter and an ideal band-pass filter with bandwidth W are shown in Figure 12.6. In all the ideal filter characteristics, there is a sharp transition between the *passband* of the filter (the range of frequencies that are not attenuated) and the *stopband* of the filter (those frequency intervals where the signal is completely attenuated). Real filters do not have such sharp transitions, or *cutoffs*.

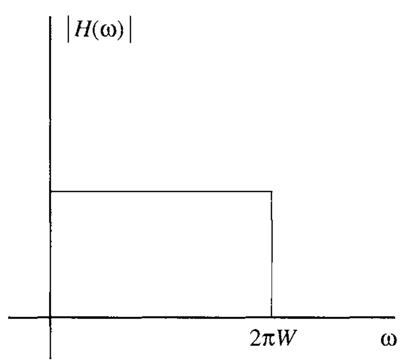

FIGURE 12.5 Magnitude of the transfer function of an ideal low-pass filter.

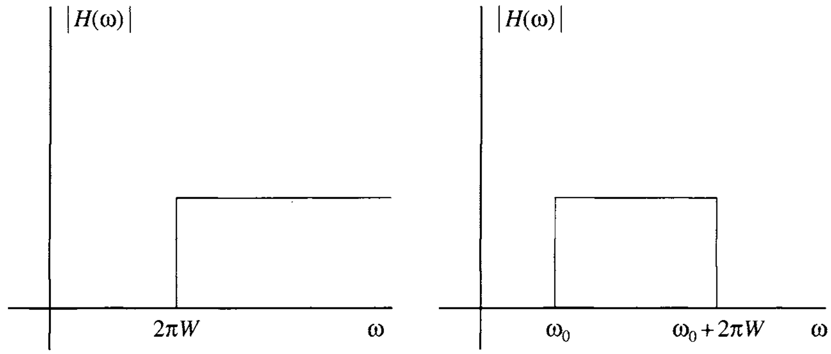

FIGURE 12.6 **Magnitudes of the transfer functions of ideal high-pass (left) and ideal band-pass (right) filters.**

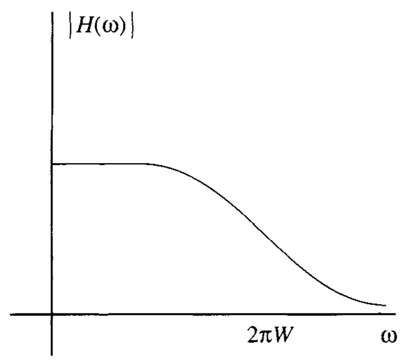

FIGURE 12.7 **Magnitude of the transfer functions of a realistic low-pass filter.**

The magnitude characteristics of a more realistic low-pass filter are shown in Figure 12.7. Notice the more gentle rolloff. But when the cutoff between stopband and passband is not sharp, how do we define the bandwidth? There are several different ways of defining the bandwidth. The most common way is to define the frequency at which the magnitude of the transfer function is $1/\sqrt{2}$ of its maximum value (or the magnitude squared is 1/2 of its maximum value) as the cutoff frequency.

12.7 Sampling

In 1928 Harry Nyquist at Bell Laboratories showed that if we have a signal whose Fourier transform is zero above some frequency W Hz, it can be accurately represented using $2W$ equally spaced samples per second. This very important result, known as the *sampling theorem*, is at the heart of our ability to transmit analog waveforms such as speech and video using digital means. There are several ways to prove this result. We will use the results presented in the previous section to do so.

12.7.1 Ideal Sampling—Frequency Domain View

Let us suppose we have a function $f(t)$ with Fourier transform $F(\omega)$, shown in Figure 12.8, which is zero for ω greater than $2\pi W$. Define the periodic extension of $F(\omega)$ as

$$F_P(\omega) = \sum_{n=-\infty}^{\infty} F(\omega - n\sigma_0), \qquad \sigma_0 = 4\pi W. \tag{12.37}$$

The periodic extension is shown in Figure 12.9. As $F_P(\omega)$ is periodic, we can express it in terms of a Fourier series expansion:

$$F_P(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{1}{2W}\omega}. \tag{12.38}$$

The coefficients of the expansion $\{c_n\}_{n=-\infty}^{\infty}$ are then given by

$$c_n = \frac{1}{4\pi W}\int_{-2\pi W}^{2\pi W} F_P(\omega)e^{-jn\frac{1}{2W}\omega}d\omega. \tag{12.39}$$

However, in the interval $(-2\pi W, 2\pi W)$, $F(\omega)$ is identical to $F_P(\omega)$; therefore,

$$c_n = \frac{1}{4\pi W}\int_{-2\pi W}^{2\pi W} F(\omega)e^{-jn\frac{1}{2W}\omega}d\omega. \tag{12.40}$$

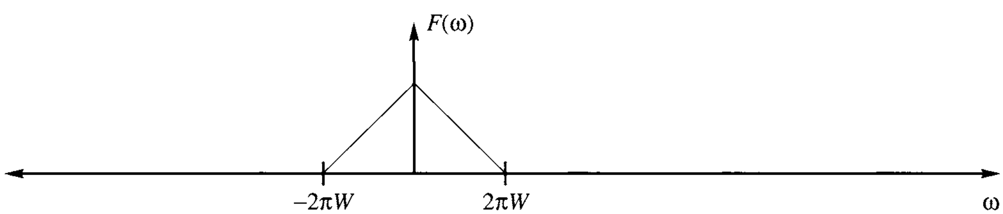

FIGURE 12.8 **A function $F(\omega)$.**

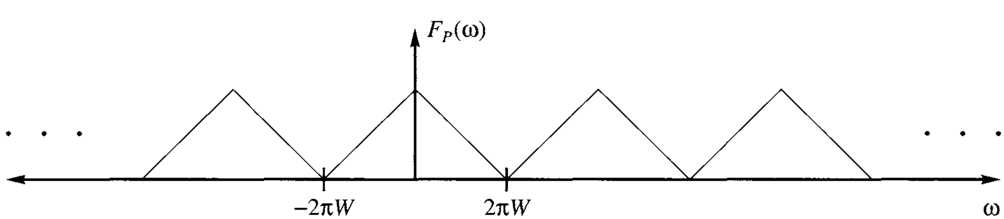

FIGURE 12.9 **The periodic extension $F_P(\omega)$.**

The function $F(\omega)$ is zero outside the interval $(-2\pi W, 2\pi W)$, so we can extend the limits to infinity without changing the result:

$$c_n = \frac{1}{2W}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-jn\frac{1}{2W}\omega}d\omega\right]. \tag{12.41}$$

The expression in brackets is simply the inverse Fourier transform evaluated at $t = \frac{n}{2W}$; therefore,

$$c_n = \frac{1}{2W}f\left(\frac{n}{2W}\right). \tag{12.42}$$

Knowing $\{c_n\}_{n=-\infty}^{\infty}$ and the value of W, we can reconstruct $F_P(\omega)$. Because $F_P(\omega)$ and $F(\omega)$ are identical in the interval $(-2\pi W, 2\pi W)$, therefore knowing $\{c_n\}_{n=-\infty}^{\infty}$, we can also reconstruct $F(\omega)$ in this interval. But $\{c_n\}_{n=-\infty}^{\infty}$ are simply the samples of $f(t)$ every $\frac{1}{2W}$ seconds, and $F(\omega)$ is zero outside this interval. Therefore, given the samples of a function $f(t)$ obtained at a rate of $2W$ samples per second, we should be able to exactly reconstruct the function $f(t)$.

Let us see how we can do this:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-j\omega t}d\omega \tag{12.43}$$

$$= \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} F(\omega)e^{-j\omega t}d\omega \tag{12.44}$$

$$= \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} F_P(\omega)e^{-j\omega t}d\omega \tag{12.45}$$

$$= \frac{1}{2\pi}\int_{-2\pi W}^{2\pi W} \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{1}{2W}\omega}e^{-j\omega t}d\omega \tag{12.46}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} c_n \int_{-2\pi W}^{2\pi W} e^{jw(t-\frac{n}{2W})}d\omega. \tag{12.47}$$

Evaluating the integral and substituting for c_n from Equation (12.42), we obtain

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right)\text{ Sinc }\left[2W\left(t-\frac{n}{2W}\right)\right] \tag{12.48}$$

where

$$\text{Sinc}[x] = \frac{\sin(\pi x)}{\pi x}. \tag{12.49}$$

Thus, given samples of $f(t)$ taken every $\frac{1}{2W}$ seconds, or, in other words, samples of $f(t)$ obtained at a rate of $2W$ samples per second, we can reconstruct $f(t)$ by interpolating between the samples using the Sinc function.

12.7.2 Ideal Sampling—Time Domain View

Let us look at this process from a slightly different point of view, starting with the sampling operation. Mathematically, we can represent the sampling operation by multiplying the function $f(t)$ with a train of impulses to obtain the sampled function $f_S(t)$:

$$f_S(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT), \qquad T < \frac{1}{2W}. \tag{12.50}$$

To obtain the Fourier transform of the sampled function, we use the convolution theorem:

$$\mathcal{F}\left[f(t) \sum_{n=-\infty}^{\infty} \delta(t-nT) \right] = \mathcal{F}[f(t)] \otimes \mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t-nT) \right]. \tag{12.51}$$

Let us denote the Fourier transform of $f(t)$ by $F(\omega)$. The Fourier transform of a train of impulses in the time domain is a train of impulses in the frequency domain (Problem 5):

$$\mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t-nT) \right] = \sigma_0 \sum_{n=-\infty}^{\infty} \delta(w - n\sigma_0) \qquad \sigma_0 = \frac{2\pi}{T}. \tag{12.52}$$

Thus, the Fourier transform of $f_S(t)$ is

$$F_S(\omega) = F(\omega) \otimes \sum_{n=-\infty}^{\infty} \delta(w - n\sigma_0) \tag{12.53}$$

$$= \sum_{n=-\infty}^{\infty} F(\omega) \otimes \delta(w - n\sigma_0) \tag{12.54}$$

$$= \sum_{n=-\infty}^{\infty} F(\omega - n\sigma_0) \tag{12.55}$$

where the last equality is due to the sifting property of the delta function.

Pictorially, for $F(\omega)$ as shown in Figure 12.8, $F_S(\omega)$ is shown in Figure 12.10. Note that if T is less than $\frac{1}{2W}$, σ_0 is greater than $4\pi W$, and as long as σ_0 is greater than $4\pi W$, we can recover $F(\omega)$ by passing $F_S(\omega)$ through an ideal low-pass filter with bandwidth W Hz ($2\pi W$ radians).

What happens if we do sample at a rate less than $2W$ samples per second (that is, σ_0 is less than $4\pi W$)? Again we can see the results most easily in a pictorial fashion. The result

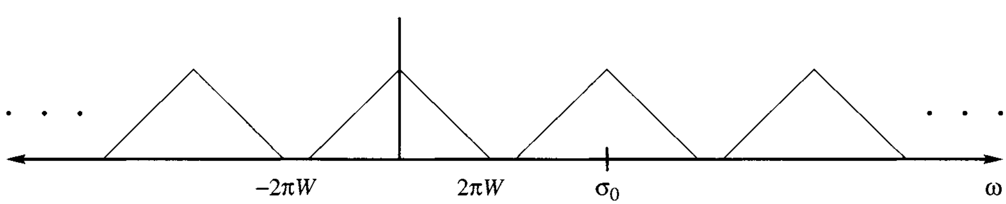

FIGURE 12.10 **Fourier transform of the sampled function.**

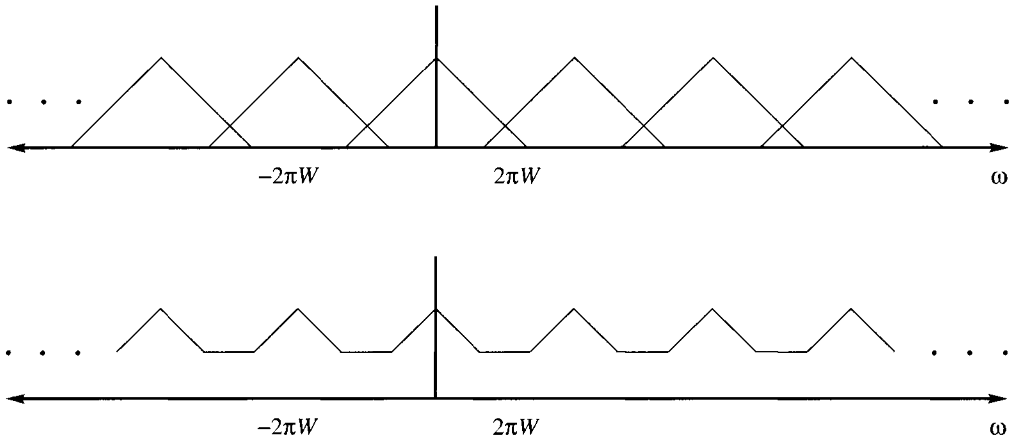

FIGURE 12.11 **Effect of sampling at a rate less than 2*W* samples per second.**

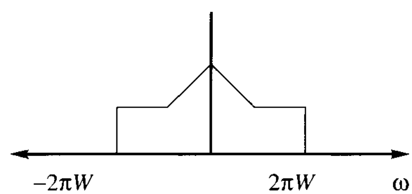

FIGURE 12.12 **Aliased reconstruction.**

for σ_0 equal to $3\pi W$ is shown in Figure 12.11. Filtering this signal through an ideal low-pass filter, we get the distorted signal shown in Figure 12.12. Therefore, if σ_0 is less than $4\pi W$, we can no longer recover the signal $f(t)$ from its samples. This distortion is known as *aliasing*. In order to prevent aliasing, it is useful to filter the signal prior to sampling using a low-pass filter with a bandwidth less than half the sampling frequency.

Once we have the samples of a signal, sometimes the actual times they were sampled at are not important. In these situations we can normalize the sampling frequency to unity. This means that the highest frequency component in the signal is at 0.5 Hz, or π radians. Thus, when dealing with sampled signals, we will often talk about frequency ranges of $-\pi$ to π.

12.8 Discrete Fourier Transform

The procedures that we gave for obtaining the Fourier series and transform were based on the assumption that the signal we were examining could be represented as a continuous function of time. However, for the applications that we will be interested in, we will primarily be dealing with samples of a signal. To obtain the Fourier transform of nonperiodic signals, we started from the Fourier series and modified it to take into account the nonperiodic nature of the signal. To obtain the discrete Fourier transform (DFT), we again start from the Fourier series. We begin with the Fourier series representation of a sampled function, the discrete Fourier series.

Recall that the Fourier series coefficients of a periodic function $f(t)$ with period T is given by

$$c_k = \frac{1}{T}\int_0^T f(t)e^{jkw_0t}dt. \tag{12.56}$$

Suppose instead of a continuous function, we have a function sampled N times during each period T. We can obtain the coefficients of the Fourier series representation of this sampled function as

$$F_k = \frac{1}{T}\int_0^T f(t)\sum_{n=0}^{N-1}\delta\left(t-\frac{n}{N}T\right)e^{jkw_0t}dt \tag{12.57}$$

$$= \frac{1}{T}\sum_{n=0}^{N-1} f\left(\frac{n}{N}T\right)e^{j\frac{2\pi kn}{N}} \tag{12.58}$$

where we have used the fact that $w_0 = \frac{2\pi}{T}$, and we have replaced c_k by F_k. Taking $T = 1$ for convenience and defining

$$f_n = f\left(\frac{n}{N}\right),$$

we get the coefficients for the discrete Fourier series (DFS) representation:

$$F_k = \sum_{n=0}^{N-1} f_n e^{j\frac{2\pi kn}{N}}. \tag{12.59}$$

Notice that the sequence of coefficients $\{F_k\}$ is periodic with period N.

The Fourier series representation was given by

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jn\omega_0 t}. \tag{12.60}$$

Evaluating this for $t = \frac{n}{N}T$, we get

$$f_n = f(\frac{n}{N}T) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kn}{N}}. \tag{12.61}$$

Let us write this in a slightly different form:

$$f_n = \sum_{k=0}^{N-1}\sum_{l=-\infty}^{\infty} c_{k+lN}e^{j\frac{2\pi n(k+lN)}{N}} \tag{12.62}$$

but

$$e^{j\frac{2\pi n(k+lN)}{N}} = e^{j\frac{2\pi kn}{N}}e^{j2\pi nl} \tag{12.63}$$

$$= e^{j\frac{2\pi kn}{N}}. \tag{12.64}$$

Therefore,

$$f_n = \sum_{k=0}^{N-1} e^{j\frac{2\pi kn}{N}} \sum_{l=-\infty}^{\infty} c_{k+lN}. \tag{12.65}$$

Define

$$\bar{c}_k = \sum_{l=-\infty}^{\infty} c_{k+lN}. \tag{12.66}$$

Clearly, $\bar{c}_k$ is periodic with period N. In fact, we can show that $\bar{c}_k = \frac{1}{N}F_k$ and

$$f_n = \frac{1}{N}\sum_{k=0}^{N-1} F_k e^{j\frac{2\pi kn}{N}}. \tag{12.67}$$

Obtaining the discrete Fourier transform from the discrete Fourier series is simply a matter of interpretation. We are generally interested in the discrete Fourier transform of a finite-length sequence. If we assume that the finite-length sequence is one period of a periodic sequence, then we can use the DFS equations to represent this sequence. The only difference is that the expressions are only valid for one "period" of the "periodic" sequence.

The DFT is an especially powerful tool because of the existence of a fast algorithm, called appropriately the *fast Fourier transform* (FFT), that can be used to compute it.

12.9 Z-Transform

In the previous section we saw how to extend the Fourier series to use with sampled functions. We can also do the same with the Fourier transform. Recall that the Fourier transform was given by the equation

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt. \tag{12.68}$$

Replacing $f(t)$ with its sampled version, we get

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)e^{-j\omega t}\,dt \tag{12.69}$$

$$= \sum_{n=-\infty}^{\infty} f_n e^{-j\omega nT} \tag{12.70}$$

where $f_n = f(nT)$. This is called the discrete time Fourier transform. The Z-transform of the sequence $\{f_n\}$ is a generalization of the discrete time Fourier transform and is given by

$$F(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n} \tag{12.71}$$

where

$$z = e^{\sigma T + jwT}. \tag{12.72}$$

Notice that if we let σ equal zero, we get the original expression for the Fourier transform of a discrete time sequence. We denote the Z-transform of a sequence by

$$F(z) = \mathcal{Z}[f_n].$$

We can express this another way. Notice that the magnitude of z is given by

$$|z| = e^{\sigma T}.$$

Thus, when σ equals zero, the magnitude of z is one. Because z is a complex number, the magnitude of z is equal to one on the unit circle in the complex plane. Therefore, we can say that the Fourier transform of a sequence can be obtained by evaluating the Z-transform of the sequence on the unit circle. Notice that the Fourier transform thus obtained will be periodic, which we expect because we are dealing with a sampled function. Further, if we assume T to be one, ω varies from $-\pi$ to π, which corresponds to a frequency range of -0.5 to 0.5 Hz. This makes sense because, by the sampling theorem, if the sampling rate is one sample per second, the highest frequency component that can be recovered is 0.5 Hz.

For the Z-transform to exist—in other words, for the power series to converge—we need to have

$$\sum_{n=-\infty}^{\infty} |f_n z^{-n}| < \infty.$$

Whether this inequality holds will depend on the sequence itself and the value of z. The values of z for which the series converges are called the *region of convergence* of the Z-transform. From our earlier discussion, we can see that for the Fourier transform of the sequence to exist, the region of convergence should include the unit circle. Let us look at a simple example.

Example 12.9.1:

Given the sequence

$$f_n = a^n u[n]$$

where $u[n]$ is the unit step function

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{12.73}$$

the Z-transform is given by

$$F(z) = \sum_{n=0}^{\infty} a^n z^{-n} \tag{12.74}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n. \tag{12.75}$$

This is simply the sum of a geometric series. As we confront this kind of sum quite often, let us briefly digress and obtain the formula for the sum of a geometric series.

Suppose we have a sum

$$S_{mn} = \sum_{k=m}^{n} x^k = x^m + x^{m+1} + \cdots + x^n \tag{12.76}$$

then

$$xS_{mn} = x^{m+1} + x^{m+2} + \cdots + x^{n+1}. \tag{12.77}$$

Subtracting Equation (12.77) from Equation (12.76), we get

$$(1-x)S_{mn} = x^m - x^{n+1}$$

and

$$S_{mn} = \frac{x^m - x^{n+1}}{1-x}.$$

If the upper limit of the sum is infinity, we take the limit as n goes to infinity. This limit exists only when $|x| < 1$.

Using this formula, we get the Z-transform of the $\{f_n\}$ sequence as

$$F(z) = \frac{1}{1-az^{-1}}, \qquad |az^{-1}| < 1 \tag{12.78}$$

$$= \frac{z}{z-a}, \qquad |z| > |a|. \tag{12.79}$$

♦

In this example the region of convergence is the region $|z| > a$. For the Fourier transform to exist, we need to include the unit circle in the region of convergence. In order for this to happen, a has to be less than one.

Using this example, we can get some other Z-transforms that will be useful to us.

Example 12.9.2:

In the previous example we found that

$$\sum_{n=0}^{\infty} a^n z^{-n} = \frac{z}{z-a}, \qquad |z| > |a|. \tag{12.80}$$

If we take the derivative of both sides of the equation with respect to a, we get

$$\sum_{n=0}^{\infty} na^{n-1} z^{-n} = \frac{z}{(z-a)^2}, \qquad |z| > |a|. \tag{12.81}$$

Thus,

$$\mathcal{Z}[na^{n-1}u[n]] = \frac{z}{(z-a)^2}, \qquad |z| > |a|.$$

If we differentiate Equation (12.80) m times, we get

$$\sum_{n=0}^{\infty} n(n-1)\cdots(n-m+1)a^{n-m} = \frac{m!z}{(z-a)^{m+1}}.$$

In other words,

$$\mathcal{Z}\left[\binom{n}{m}a^{n-m}u[n]\right] = \frac{z}{(z-a)^{m+1}}. \tag{12.82}$$

♦

In these examples the Z-transform is a ratio of polynomials in z. For sequences of interest to us, this will generally be the case, and the Z-transform will be of the form

$$F(z) = \frac{N(z)}{D(z)}.$$

The values of z for which $F(z)$ is zero are called the *zeros* of $F(z)$; the values for which $F(z)$ is infinity are called the *poles* of $F(z)$. For finite values of z, the poles will occur at the roots of the polynomial $D(z)$.

The inverse Z-transform is formally given by the contour integral

$$\frac{1}{2\pi j}\oint_C F(z)z^{n-1}dz$$

where the integral is over the counterclockwise contour C, and C lies in the region of convergence. This integral can be difficult to evaluate directly; therefore, in most cases we use alternative methods for finding the inverse Z-transform.

12.9.1 Tabular Method

The inverse Z-transform has been tabulated for a number of interesting cases (see Table 12.1). If we can write $F(z)$ as a sum of these functions

$$F(z) = \sum \alpha_i F_i(z)$$

TABLE 12.1 Some Z-transform pairs.

$\{f_n\}$	$F(z)$
$a^n u[n]$	$\frac{z}{z-a}$
$nTu[n]$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
$\sin(\alpha nT)$	$\frac{(\sin \alpha nT)z^{-1}}{1-2\cos(\alpha T)z^{-1}+z^{-2}}$
$\cos(\alpha nT)$	$\frac{(\cos \alpha nT)z^{-1}}{1-2\cos(\alpha T)z^{-1}+z^{-2}}$

then the inverse Z-transform is given by

$$f_n = \sum \alpha_i f_{i,n}$$

where $F_i(z) = \mathcal{Z}[\{f_{i,n}\}]$.

Example 12.9.3:

$$F(z) = \frac{z}{z-0.5} + \frac{2z}{z-0.3}$$

From our earlier example we know the inverse Z-transform of $z/(z-a)$. Using that, the inverse Z-transform of $F(z)$ is

$$f_n = 0.5^n u[n] + 2(0.3)^n u[n].$$

♦

12.9.2 Partial Fraction Expansion

In order to use the tabular method, we need to be able to decompose the function of interest to us as a sum of simpler terms. The partial fraction expansion approach does exactly that when the function is a ratio of polynomials in z.

Suppose $F(z)$ can be written as a ratio of polynomials $N(z)$ and $D(z)$. For the moment let us assume that the degree of $D(z)$ is greater than the degree of $N(z)$, and that all the roots of $D(z)$ are distinct (distinct roots are referred to as simple roots); that is,

$$F(z) = \frac{N(z)}{(z-z_1)(z-z_2)\cdots(z-z_L)}. \tag{12.83}$$

Then we can write $F(z)/z$ as

$$\frac{F(z)}{z} = \sum_{i=1}^{L} \frac{A_i}{z-z_i}. \tag{12.84}$$

If we can find the coefficients A_i, then we can write $F(z)$ as

$$F(z) = \sum_{i=1}^{L} \frac{A_i z}{z-z_i}$$

and the inverse Z-transform will be given by

$$f_n = \sum_{i=1}^{L} A_i z_i^n u[n].$$

The question then becomes one of finding the value of the coefficients A_i. This can be simply done as follows: Suppose we want to find the coefficient A_k. Multiply both sides of Equation (12.84) by $(z-z_k)$. Simplifying this we obtain

$$\frac{F(z)(z-z_k)}{z} = \sum_{i=1}^{L} \frac{A_i(z-z_k)}{z-z_i} \tag{12.85}$$

$$= A_k + \sum_{\substack{i=1\\ i\neq k}}^{L} \frac{A_i(z-z_k)}{z-z_i}. \tag{12.86}$$

Evaluating this equation at $z = z_k$, all the terms in the summation go to zero and

$$A_k = \left. \frac{F(z)(z-z_k)}{z} \right|_{z=z_k}. \tag{12.87}$$

Example 12.9.4:

Let us use the partial fraction expansion method to find the inverse Z-transform of

$$F(z) = \frac{6z^2-9z}{z^2-2.5z+1}.$$

Then

$$\frac{F(z)}{z} = \frac{1}{z}\frac{6z^2-9z}{z^2-2.5z+1} \tag{12.88}$$

$$= \frac{6z-9}{(z-0.5)(z-2)}. \tag{12.89}$$

We want to write $F(z)/z$ in the form

$$\frac{F(z)}{z} = \frac{A_1}{z-0.5} + \frac{A_2}{z-2}.$$

Using the approach described above, we obtain

$$A_1 = \left. \frac{(6z-9)(z-0.5)}{(z-0.5)(z-2)} \right|_{z=0.5} \tag{12.90}$$

$$= 4 \tag{12.91}$$

$$A_2 = \left. \frac{(6z-9)(z-2)}{(z-0.5)(z-2)} \right|_{z=2} \tag{12.92}$$

$$= 2. \tag{12.93}$$

Therefore,

$$F(z) = \frac{4z}{z-0.5} + \frac{2z}{z-2}$$

and

$$f_n = [4(0.5)^n + 2(2)^n]u[n]. \qquad \blacklozenge$$

The procedure becomes slightly more complicated when we have repeated roots of $D(z)$. Suppose we have a function

$$F(z) = \frac{N(z)}{(z-z_1)(z-z_2)^2}.$$

The partial fraction expansion of this function is

$$\frac{F(z)}{z} = \frac{A_1}{z-z_1} + \frac{A_2}{z-z_2} + \frac{A_3}{(z-z_2)^2}.$$

The values of A_1 and A_3 can be found as shown previously:

$$A_1 = \left.\frac{F(z)(z-z_1)}{z}\right|_{z=z_1} \tag{12.94}$$

$$A_3 = \left.\frac{F(z)(z-z_2)^2}{z}\right|_{z=z_2}. \tag{12.95}$$

However, we run into problems when we try to evaluate A_2. Let's see what happens when we multiply both sides by $(z-z_2)$:

$$\frac{F(z)(z-z_2)}{z} = \frac{A_1(z-z_2)}{z-z_1} + A_2 + \frac{A_3}{z-z_2}. \tag{12.96}$$

If we now evaluate this equation at $z = z_2$, the third term on the right-hand side becomes undefined. In order to avoid this problem, we first multiply both sides by $(z-z_2)^2$ and take the derivative with respect to z prior to evaluating the equation at $z = z_2$:

$$\frac{F(z)(z-z_2)^2}{z} = \frac{A_1(z-z_2)^2}{z-z_1} + A_2(z-z_2) + A_3. \tag{12.97}$$

Taking the derivative of both sides with respect to z, we get

$$\frac{d}{dz}\frac{F(z)(z-z_2)^2}{z} = \frac{2A_1(z-z_2)(z-z_1) - A_1(z-z_2)^2}{(z-z_1)^2} + A_2. \tag{12.98}$$

If we now evaluate the expression at $z = z_2$, we get

$$A_2 = \left.\frac{d}{dz}\frac{F(z)(z-z_2)^2}{z}\right|_{z=z_2}. \tag{12.99}$$

Generalizing this approach, we can show that if $D(z)$ has a root of order m at some z_k, that portion of the partial fraction expansion can be written as

$$\frac{F(z)}{z} = \frac{A_1}{z - z_k} + \frac{A_2}{(z - z_k)^2} + \cdots + \frac{A_m}{(z - z_k)^m} \tag{12.100}$$

and the lth coefficient can be obtained as

$$A_l = \frac{1}{(m-l)!} \frac{d^{(m-l)}}{dz^{(m-l)}} \frac{F(z)(z - z^k)^m}{z} \bigg|_{z=z_k}. \tag{12.101}$$

Finally, let us drop the requirement that the degree of $D(z)$ be greater or equal to the degree of $N(z)$. When the degree of $N(z)$ is greater than the degree of $D(z)$, we can simply divide $N(z)$ by $D(z)$ to obtain

$$F(z) = \frac{N(z)}{D(z)} = Q(z) + \frac{R(z)}{D(z)} \tag{12.102}$$

where $Q(z)$ is the quotient and $R(z)$ is the remainder of the division operation. Clearly, $R(z)$ will have degree less than $D(z)$.

To see how all this works together, consider the following example.

Example 12.9.5:

Let us find the inverse Z-transform of the function

$$F(z) = \frac{2z^4 + 1}{2z^3 - 5z^2 + 4z - 1}. \tag{12.103}$$

The degree of the numerator is greater than the degree of the denominator, so we divide once to obtain

$$F(z) = z + \frac{5z^3 - 4z^2 + z + 1}{2z^3 - 5z^2 + 4z - 1}. \tag{12.104}$$

The inverse Z-transform of z is δ_{n-1}, where δ_n is the discrete delta function defined as

$$\delta_n = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{12.105}$$

Let us call the remaining ratio of polynomials $F_1(z)$. We find the roots of the denominator of $F_1(z)$ as

$$F_1(z) = \frac{5z^3 - 4z^2 + z + 1}{2(z - 0.5)(z - 1)^2}. \tag{12.106}$$

Then

$$\frac{F_1(z)}{z} = \frac{5z^3-4z^2+z+1}{2z(z-0.5)(z-1)^2} \tag{12.107}$$

$$= \frac{A_1}{z}+\frac{A_2}{z-0.5}+\frac{A_3}{z-1}+\frac{A_4}{(z-1)^2}. \tag{12.108}$$

Then

$$A_1 = \left.\frac{5z^3-4z^2+z+1}{2(z-0.5)(z-1)^2}\right|_{z=0} = -1 \tag{12.109}$$

$$A_2 = \left.\frac{5z^3-4z^2+z+1}{2z(z-1)^2}\right|_{z=0.5} = 4.5 \tag{12.110}$$

$$A_4 = \left.\frac{5z^3-4z^2+z+1}{2z(z-0.5)}\right|_{z=1} = 3. \tag{12.111}$$

To find A_3, we take the derivative with respect to z, then set $z=1$:

$$A_3 = \left.\frac{d}{dz}\left[\frac{5z^3-4z^2+2z+1}{2z(z-0.5)}\right]\right|_{z=1} = -3. \tag{12.112}$$

Therefore,

$$F_1(z) = -1+\frac{4.5z}{z-0.5}-\frac{3z}{z-1}+\frac{3z}{(z-1)^2} \tag{12.113}$$

and

$$f_{1,n} = -\delta_n + 4.5(0.5)^n u[n] - 3u[n] + 3nu[n] \tag{12.114}$$

and

$$f_n = \delta_{n-1} - \delta_n + 4.5(0.5)^n u[n] - (3-3n)u[n]. \tag{12.115}$$

♦

12.9.3 Long Division

If we could write $F(z)$ as a power series, then from the Z-transform expression the coefficients of z^{-n} would be the sequence values f_n.

Example 12.9.6:

Let's find the inverse z-transform of

$$F(z) = \frac{z}{z-a}.$$

Dividing the numerator by the denominator we get the following:

$$
\begin{array}{r|lllll}
 & 1 & + & az^{-1} & + & a^2z^{-2} \quad \cdots \\
\hline
z-a & z & & & & \\
 & z & - & a & & \\
\cline{2-4}
 & & & a & & \\
 & & & a & - & a^2z^{-1} \\
\cline{4-6}
 & & & & & a^2z^{-1}
\end{array}
$$

Thus, the quotient is

$$1+az^{-1}+a^2z^{-2}+\cdots=\sum_{n=0}^{\infty}a^nz^{-n}.$$

We can easily see that the sequence for which $F(z)$ is the Z-transform is

$$f_n = a^n u[n].$$

♦

12.9.4 Z-Transform Properties

Analogous to the continuous linear systems, we can define the transfer function of a discrete linear system as a function of z that relates the Z-transform of the input to the Z-transform of the output. Let $\{f_n\}_{n=-\infty}^{\infty}$ be the input to a discrete linear time-invariant system, and $\{g_n\}_{n=-\infty}^{\infty}$ be the output. If $F(z)$ is the Z-transform of the input sequence, and $G(z)$ is the Z-transform of the output sequence, then these are related to each other by

$$G(z) = H(z)F(z) \tag{12.116}$$

and $H(z)$ is the transfer function of the discrete linear time-invariant system.

If the input sequence $\{f_n\}_{n=-\infty}^{\infty}$ had a Z-transform of one, then $G(z)$ would be equal to $H(z)$. It is an easy matter to find the requisite sequence:

$$F(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n} = 1 \Rightarrow f_n = \begin{cases} 1 & n=0 \\ 0 & \text{otherwise.} \end{cases} \tag{12.117}$$

This particular sequence is called the *discrete delta function*. The response of the system to the discrete delta function is called the impulse response of the system. Obviously, the transfer function $H(z)$ is the Z-transform of the impulse response.

12.9.5 Discrete Convolution

In the continuous time case, the output of the linear time-invariant system was a convolution of the input with the impulse response. Does the analogy hold in the discrete case? We can check this out easily by explicitly writing out the Z-transforms in Equation (12.116). For simplicity let us assume the sequences are all one-sided; that is, they are only nonzero for nonnegative values of the subscript:

$$\sum_{n=0}^{\infty} g_n z^{-n} = \sum_{n=0}^{\infty} h_n z^{-n} \sum_{m=0}^{\infty} f_m z^{-m}. \tag{12.118}$$

Equating like powers of z:

$$\begin{aligned} g_0 &= h_0 f_0 \\ g_1 &= f_0 h_1 + f_1 h_0 \\ g_2 &= f_0 h_2 + f_1 h_1 + f_2 h_0 \\ &\vdots \\ g_n &= \sum_{m=0}^{n} f_m h_{n-m}. \end{aligned}$$

Thus, the output sequence is a result of the discrete convolution of the input sequence with the impulse response.

Most of the discrete linear systems we will be dealing with will be made up of delay elements, and their input-output relations can be written as constant coefficient difference equations. For example, for the system shown in Figure 12.13, the input-output relationship can be written in the form of the following difference equation:

$$g_k = a_0 f_k + a_1 f_{k-1} + a_2 f_{k-2} + b_1 g_{k-1} + b_2 g_{k-2}. \tag{12.119}$$

The transfer function of this system can be easily found by using the *shifting theorem*. The shifting theorem states that if the Z-transform of a sequence $\{f_n\}$ is $F(z)$, then the Z-transform of the sequence shifted by some integer number of samples n_0 is $z^{-n_0}F(z)$.

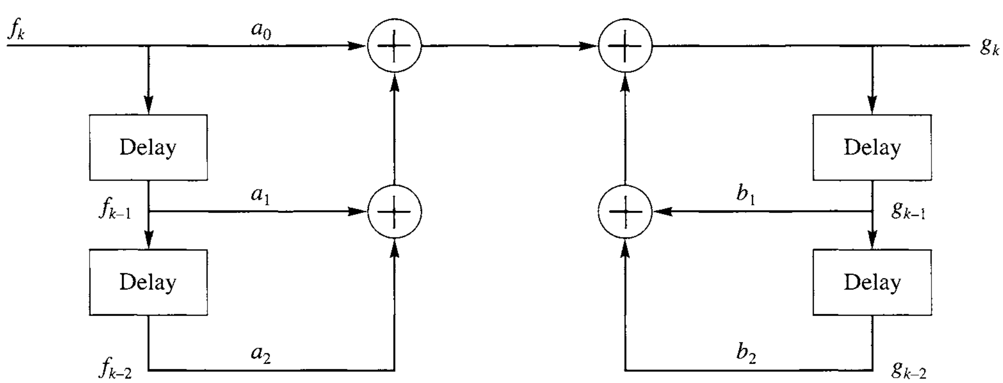

FIGURE 12.13 A discrete system.

The theorem is easy to prove. Suppose we have a sequence $\{f_n\}$ with Z-transform $F(z)$. Let us look at the Z-transform of the sequence $\{f_{n-n_0}\}$:

$$\mathcal{Z}[\{f_{n-n_0}\}] = \sum_{n=-\infty}^{\infty} f_{n-n_0} z^{-n} \tag{12.120}$$

$$= \sum_{m=-\infty}^{\infty} f_m z^{-m-n_0} \tag{12.121}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} f_m z^{-m} \tag{12.122}$$

$$= z^{-n_0} F(z). \tag{12.123}$$

Assuming $G(z)$ is the Z-transform of $\{g_n\}$ and $F(z)$ is the Z-transform of $\{f_n\}$, we can take the Z-transform of both sides of the difference equation (12.119):

$$G(z) = a_0 F(z) + a_1 z^{-1} F(z) + a_2 z^{-2} F(z) + b_1 z^{-1} G(z) + b_2 z^{-2} G(z) \tag{12.124}$$

from which we get the relationship between $G(z)$ and $F(z)$ as

$$G(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 - b_1 z^{-1} - b_2 z^{-2}} F(z). \tag{12.125}$$

By definition the transfer function $H(z)$ is therefore

$$H(z) = \frac{G(z)}{F(z)} \tag{12.126}$$

$$= \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 - b_1 z^{-1} - b_2 z^{-2}}. \tag{12.127}$$

12.10 Summary

In this chapter we have reviewed some of the mathematical tools we will be using throughout the remainder of this book. We started with a review of vector space concepts, followed by a look at a number of ways we can represent a signal, including the Fourier series, the Fourier transform, the discrete Fourier series, the discrete Fourier transform, and the Z-transform. We also looked at the operation of sampling and the conditions necessary for the recovery of the continuous representation of the signal from its samples.

Further Reading

1. There are a large number of books that provide a much more detailed look at the concepts described in this chapter. A nice one is *Signal Processing and Linear Systems*, by B.P. Lathi [177].

2. For a thorough treatment of the fast Fourier transform (FFT), see *Numerical Recipes in C*, by W.H. Press, S.A. Teukolsky, W.T. Vetterling, and B.J. Flannery [178].

12.11 Projects and Problems

1. Let X be a set of N linearly independent vectors, and let V be the collection of vectors obtained using all linear combinations of the vectors in X.

   (a) Show that given any two vectors in V, the sum of these vectors is also an element of V.

   (b) Show that V contains an additive identity.

   (c) Show that for every $\mathbf{x}$ in V, there exists a $(-\mathbf{x})$ in V such that their sum is the additive identity.

2. Prove Parseval's theorem for the Fourier transform.

3. Prove the modulation property of the Fourier transform.

4. Prove the convolution theorem for the Fourier transform.

5. Show that the Fourier transform of a train of impulses in the time domain is a train of impulses in the frequency domain:

$$\mathcal{F}\left[\sum_{n=-\infty}^{\infty} \delta(t-nT)\right] = \sigma_0 \sum_{n=-\infty}^{\infty} \delta(w-n\sigma_0) \qquad \sigma_0 = \frac{2\pi}{T}. \tag{12.128}$$

6. Find the Z-transform for the following sequences:

   (a) $h_n = 2^{-n}u[n]$, where $u[n]$ is the unit step function.

   (b) $h_n = (n^2-n)3^{-n}u[n]$.

   (c) $h_n = (n2^{-n} + (0.6)^n)u[n]$.

7. Given the following input-output relationship:

$$y_n = 0.6y_{n-1} + 0.5x_n + 0.2x_{n-1}$$

   (a) Find the transfer function $H(z)$.

   (b) Find the impulse response $\{h_n\}$.

8. Find the inverse Z-transform of the following:

   (a) $H(z) = \frac{5}{z-2}$.

   (b) $H(z) = \frac{z}{z^2-0.25}$.

   (c) $H(z) = \frac{z}{z-0.5}$.