DEPT. MATEMÀTIQUES	FIB	UPC	Year 2019-2020 Q1
Geometric Tools for Computer Graphics		**Final exam**	January 7, 2020

Duration: 2 hours.

Publication of final grades: Monday, January 13, through the Racó.

Revision: Anyone wishing to revise the grades obtained should: *i)* send the professor an e-mail by Monday, January 13, at 23:59; and *ii)* come to the following room on Tuesday, January 14, at 10:00: Omega Building, 4th floor, office 432 (North Campus).

1. **(5 points)** Let C_0 be the cylinder $x^2 + y^2 = 4$.
   (a) Obtain a parametrization of C_0.
   (b) Rotate C_0 about the axis Ox by $\pi/4$, and obtain a parametrization of the resulting cylinder C.
   (c) Intersect C with the plane $z = 1$, and parameterize the resulting curve, Γ.
   (d) Obtain the implicit equation of Γ. What kind of curve is it?
   (e) Describe a sequence of basic 2-dimensional affinities within the plane $z = 1$, transforming Γ into a circle centered at point $(1, 3, 1)$ with radius 5.

2. **(5 points)** Let Γ_0 be a spiral curve located in $z \geq 0$ on the cone $x^2 + y^2 = z^2$. The spiral orthogonally projects onto the plane $z = 0$ in an Archimedean spiral.
   (a) Obtain a parametrization of Γ_0. Justify your answer.
   (b) Compute the orthogonal projection onto the plane $z = 0$ of the vector tangent to Γ_0 at its start.
   (c) Consider a copy Γ of Γ_0 in the following position:
      i. Spiral Γ starts at point $A = (2, 3, 0)$.
      ii. The axis of Γ (i.e., the axis of the cone it belongs to) is a halfline ℓ through A, forming a $60°$ angle with the plane $z = 0$.
      iii. Halfline ℓ entirely lies in the octant $x \geq 0$, $y \geq 0$, $z \geq 0$, and orthogonally projects onto $z = 0$ in a line forming a $45°$ angle with Ox^+.

      Obtain a parametrization of Γ. Consider several methods to solve this problem (as many as you can).
   (d) Explain which of the methods you prefer in this case, and why.

SOLUTION

1. (a) **[2 points]** A possible parametrization of C_0 is $(2\cos t, 2\sin t, s)$, with $t \in [0, 2\pi]$ and $s \in \mathbb{R}$.

   (b) **[2 points]** $C = R^x_{\pi/4}(C_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2\cos t \\ 2\sin t \\ s \end{pmatrix} = \begin{pmatrix} 2\cos t \\ \sqrt{2}\sin t - \frac{s}{\sqrt{2}} \\ \sqrt{2}\sin t + \frac{s}{\sqrt{2}} \end{pmatrix}$, with $t \in [0, 2\pi]$ and $s \in \mathbb{R}$.

   (c) **[2 points]** The fact that $1 = z = \sqrt{2}\sin t + \frac{s}{\sqrt{2}}$ implies that $\frac{s}{\sqrt{2}} = 1 - \sqrt{2}\sin t$. Therefore, $y = \sqrt{2}\sin t - \frac{s}{\sqrt{2}} = \sqrt{2}\sin t - (1 - \sqrt{2}\sin t) = 2\sqrt{2}\sin t - 1$. A possible parametrization of Γ is then $(2\cos t, 2\sqrt{2}\sin t, 1)$, with $t \in [0, 2\pi]$.

   (d) **[2 points]** The implicit equation of Γ is $z = 1$, $\frac{x^2}{4} + \frac{(y+1)^2}{8} = 1$, since $\frac{x}{2} = \cos t$, $\frac{y+1}{2\sqrt{2}} = \sin t$ and $\cos^2 t + \sin^2 t = 1$. Therefore, Γ is an ellipse in the plane $z = 1$, with center $(0, -1, 1)$ and axes 2 and $2\sqrt{2}$.

   (e) **[2 points]** Within the plane $z = 1$, a possible sequence of 2-dimensional affinities transforming Γ into the desired circle is as follows:

   1st Translate Γ by vector $(0, 1)$ within the plane $z = 1$. After that, the curve stays in the plane and is still an ellipse, but its center is now located in the origin of the plane $z = 1$.

   2nd Scale by factor $\frac{5}{2}$ and $\frac{5}{2\sqrt{2}}$, with scaling center at the origin of the plane $z = 1$. This transforms the ellipse into a circle of radius 5, still in the plane $z = 1$ and centered at the origin of the plane $z = 1$.

   3rd Translate by vector $(1, 3)$ within the plane $z = 1$, to center the circle in the appropriate point, without leaving the plane $z = 1$.

   In other words, as seen in 3 dimensions, if Γ' is the desired circle, we can obtain it from Γ as follows:

$$\Gamma' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{5}{2} & 0 & 0 & 0 \\ 0 & \frac{5}{2\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2\cos t \\ 2\sqrt{2}\sin t - 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + 5\cos t \\ 3 + 5\sin t \\ 1 \\ 1 \end{pmatrix}, \text{ with } t \in [0, 2\pi].$$

2. (a) **[2 points]** The projection of Γ_0 is an Archimedian spiral, that can be parametrized as $(x(t), y(t), 0) = (at\cos t, at\sin t, 0)$, with $t \in [0, +\infty)$, where a is a fix positive constant. Since Γ_0 must belong to the cone $x^2 + y^2 = z^2$, we need that $z^2(t) = x^2(t) + y^2(t) = a^2t^2\cos^2 t + a^2t^2\sin^2 t = a^2t^2$. Since we also have the requirement that $z(t) \geq 0$, we obtain that $z(t) = at$. All together, a possible parametrization of Γ_0 is $(at\cos t, at\sin t, at)$, with $t \in [0, +\infty)$. Notice that a can be any fix positive real number.

   (b) **[2 points]** The vector tangent to Γ_0 at an arbitrary point $\Gamma_0(t)$ is

$$\Gamma_0'(t) = (x'(t), y'(t), z'(t)) = (a\cos t - at\sin t, a\sin t + at\cos t, a).$$

   At its start, it is $\Gamma_0'(0) = (a\cos 0 - a0\sin 0, a\sin 0 + a0\cos 0, a) = (a, 0, a)$. Its orthogonal projection onto the plane $z = 0$ is vector $(a, 0, 0)$.

   (c) **[4 points]**

   Method 1: changing coordinates. By condition i, the new origin must be $W = A = (2, 3, 0)$. On the other hand, the axis of the new cone is ℓ and the old one is $Oz+$. Therefore, we need $e_3' = (x, y, z)$ to have the same direction as ℓ. Due to condition iii, $x = y$ and $x, z \geq 0$. Due to condition ii, we need to choose $z = \sin 60° = \frac{\sqrt{3}}{2}$ and $x\sqrt{2} = \sqrt{x^2 + x^2} = \sqrt{x^2 + y^2} = \cos 60° = \frac{1}{2}$. Therefore, $e_3' = (\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2})$, which is a unit vector. Since the problem has no further restrictions, we can choose $e_1' = \frac{e_3' \wedge e_3}{\|e_3' \wedge e_3\|} = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0)$. And then $e_2' = e_3' \wedge e_1' = (\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2})$. All together, a possible parametrization of Γ is:

$$\Gamma = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 2 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 3 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} at\cos t \\ at\sin t \\ at \\ 1 \end{pmatrix}, \text{with } t \in \mathbb{R}^+.$$

   Method 2: concatenating basic affinities. A possible option is to first rotate Γ_0 about axis Ox angle $\alpha = -(90° - 30°) = -30°$, then rotate it about axis Oz angle $\beta = -45°$, and finally translate it by vector $W = A = (2, 3, 0)$. In other words:

$$\begin{aligned} \Gamma &= \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} at\cos t \\ at\sin t \\ at \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 2 \\ \frac{-1}{\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 3 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} at\cos t \\ at\sin t \\ at \\ 1 \end{pmatrix}, \text{with } t \in \mathbb{R}^+. \end{aligned}$$

   Notice that the result of multiplying the three matrices is the same matrix obtained by the first method.

Method 3: A unique rotation followed by a translation. Naturally, the translation will be of vector $W = A = (2, 3, 0)$. As for the rotation, due to condition *iii*, the axis of the cone needs to end up in the plane $x = y$. Therefore, we can appropriately orient the cone by a unique rotation of angle $\alpha = (90^\circ - 30^\circ) = 30^\circ$ using as axis the line through the origin that is perpendicular to the plane $x = y$, orienting the rotation from the current axis towards the new one. In order to do that, we consider vectors $v_1 = (0, 0, 1)$, which is the direction of the current axis, and $v_2 = (1, 1, 0)$. Both lie in the plane $x = y$. Compute the cross product $u = v_1 \wedge v_2 = (-1, 1, 0)$. Vector u is the direction of the rotation axis we seek. Therefore, $\Gamma = T_W \circ R^{0,u}_{30^\circ}(\Gamma_0)$. This can be achieved by:

Method 3.1. Foley Van-Damm In this case, this method is simple, since the rotation axis already belongs to the plane $z = 0$. Therefore,

$$R^{0,u}_{30^\circ} = R^z_{45^\circ} \circ R^y_{30^\circ} \circ R^z_{-45^\circ}.$$

In fact, if $\alpha = \text{angle}(u, e_2)$, we have:

$$\begin{aligned}
&\cos\alpha = \tfrac{u \cdot e_2}{\|u\|\|e_2\|} = \tfrac{(-1,1,0)\cdot(0,1,0)}{\sqrt{2}} = \tfrac{1}{\sqrt{2}},\\
&\text{sign}(\alpha) = \text{sign}(\det(u, e_2)) = \text{sign}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1,\\
&\sin\alpha = \text{sign}(\alpha)\sqrt{1 - \cos^2\alpha} = -\tfrac{1}{\sqrt{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Gamma &= T_W \circ R^{0,u}_{30^\circ}(\Gamma_0)\\
&= T_W \circ R^z_{45^\circ} \circ R^y_{30^\circ} \circ R^z_{-45^\circ}(\Gamma_0)\\
&= \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\\
&\quad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} at\cos t \\ at\sin t \\ at \\ 1 \end{pmatrix}\\
&= \begin{pmatrix} \frac{1}{2} + \frac{\sqrt{3}}{4} & -\frac{1}{2} + \frac{\sqrt{3}}{4} & \frac{1}{2\sqrt{2}} & 2 \\ -\frac{1}{2} + \frac{\sqrt{3}}{4} & \frac{1}{2} + \frac{\sqrt{3}}{/4} & \frac{1}{2\sqrt{2}} & 3 \\ -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \sqrt{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} at\cos t \\ at\sin t \\ at \\ 1 \end{pmatrix}, \text{with } t \in \mathbb{R}^+.
\end{aligned}$$

Notice that in this case (as in the following ones), the result is not identical to that of the previous methods. The reason is that the problem has one degree of freedom: given one solution (like the one from the previous two methods), any rotation about line ℓ is also a valid solution.

Method 3.2. Compact formula The compact formula states that, if $\|u\| = 1$, then $R^{0,u}_{\alpha}(x) = (x \cdot u)u + (x - (x \cdot u)u)\cos\alpha + (u \wedge x)\sin\alpha$.
In our case, $u = \frac{1}{\sqrt{2}}(-1, 1, 0)$, $x = (at\cos t, at\sin t, at)$, and $\alpha = 30°$.
Therefore:

$$\begin{array}{rcl}
\Gamma & = & (x \cdot u)u + (x - (x \cdot u)u)\frac{\sqrt{3}}{2} + (u \wedge x)\frac{1}{2} + (2, 3, 0) \\
& = & \left\{ \begin{array}{l} x(t) = 2 + \frac{1}{4}(\sqrt{2} - (-6 + \sqrt{3})at\cos t + (-2 + \sqrt{3})at\sin t) \\ y(t) = 3 + \frac{1}{4}(\sqrt{2} + (-2 + \sqrt{3})at\cos t - (-6 + \sqrt{3})at\sin t) \\ z(t) = 1 - \frac{at(\cos t + \sin t)}{2\sqrt{2}}) \end{array} \right. \\
& & \text{with } t \in \mathbb{R}^+.
\end{array}$$

Method 3.3. Quaternions if $\|u\| = 1$, then $R^{0,u}_{\alpha}(x) = Q_{\frac{\alpha}{2}} X Q^*_{\frac{\alpha}{2}}$, where

$$\begin{aligned}
X &= (0, x_1, x_2, x_3), \\
Q_{\frac{\alpha}{2}} &= (\cos\tfrac{\alpha}{2}, u_1\sin\tfrac{\alpha}{2}, u_2\sin\tfrac{\alpha}{2}, u_3\sin\tfrac{\alpha}{2}), \\
Q^*_{\frac{\alpha}{2}} &= (\cos\tfrac{\alpha}{2}, -u_1\sin\tfrac{\alpha}{2}, -u_2\sin\tfrac{\alpha}{2}, -u_3\sin\tfrac{\alpha}{2}).
\end{aligned}$$

In our case, $u = \frac{1}{\sqrt{2}}(-1, 1, 0)$, $x = (at\cos t, at\sin t, at)$, and $\frac{\alpha}{2} = 15°$.
Therefore, our quaternions are

$$\begin{aligned}
Q_{15°} &= (\tfrac{1+\sqrt{3}}{2\sqrt{2}}, \tfrac{1}{4}(1 - \sqrt{3}), \tfrac{1}{4}(-1 + \sqrt{3}), 0), \\
X &= (0, at\cos t, at\sin t, at), \\
A &= (0, 2, 3, 0).
\end{aligned}$$

We obtain:

$$\begin{array}{rcl}
\Gamma & = & R^{0,u}_{\alpha}(\Gamma_0) \\
& = & Q_{15°} X Q^*_{15°} + A \\
& = & \begin{pmatrix} 0 \\ 2 + \frac{1}{4}at(\sqrt{2} + (2 + \sqrt{3})\cos t + (-2 + \sqrt{3})\sin t) \\ 3 + \frac{1}{4}at(\sqrt{2} + (-2 + \sqrt{3})\cos t + (2 + \sqrt{3})\sin t) \\ \frac{1}{4}at(2\sqrt{3} - \sqrt{2}\cos t - \sqrt{2}\sin t)) \end{pmatrix}, \text{ with } t \in \mathbb{R}^+.
\end{array}$$

Observation: Since methods 3.2 and 3.3 are just a straightforward application of two formulae, it was unnecessary to carry out the calculations. It was enough to indicate which values were involved and which computations needed to be done.

(d) **[2 points]** Among all the methods, the conditions of the problem make method 2 to be he simplest one, followed by method 3.1. The three variants of method 3 require the same preprocessing (i.e., computing the direction u of the rotation axis), but then evaluating cos and sin of 15° and multiplying the quaternions require more involved calculations that are difficult to do by hand without errors. Finally, although method 1 requires a bit of thinking in order to find the appropriate e_3, after that it is simple and results in a single matrix that is the easy to apply to the curve.