19 February 1996

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 211 (1996) 199–203

On the construction of one-dimensional iterative maps from the invariant density: the dynamical route to the beta distribution

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Received 12 July 1995; revised manuscript received 30 November 1995; accepted for publication 8 December 1995
Communicated by C.R. Doering

Abstract

Using the Frobenius–Perron functional equation we construct a one-dimensional iterative map resulting from a given invariant density. As a specific example we focus on the symmetric beta distribution and obtain a class of maps with a broad range of universal properties. The analytical behaviour as well as the bifurcation routes of these maps are studied in some detail.

One-dimensional iterative maps are an important tool to study the characteristic features of the transition from order to chaos as they typically occur in more complicated dynamical systems of either physical, chemical or biological origin. Of particular relevance are the maps of the interval [0, 1] which possess a single maximum and bifurcation route to chaos. This route and its universal scaling laws have been studied in detail in the literature [1] and are now well understood. The underlying fundamental (Feigenbaum) constants occurring in the corresponding scaling relations as well as the fractal dimensions of the Feigenbaum attractors have been calculated for certain classes of simple maps [2,3]. In contrast to our detailed knowledge on the transition from the ordered phase to the chaotic bands the chaotic regime and the geometric properties of the corresponding strange attractors are much less understood. One of the challenging questions in the chaotic regime is the construction of the invariant measure (density) characteristic for strange attractors. The knowledge of this measure is the key for the understanding of the statistical properties of the corresponding dynamical systems [4]. It allows also the calculation of relevant experimental observables like, for example, time correlations and their power spectra [5]. Several investigations have been performed in order to elucidate the connection between the invariant density and its underlying map [6]. In particular, a number of interesting general properties relating maps to their invariant measure (density) have been established [7].

On the other hand, there exist many interesting physical phenomena whose statistical properties are known but whose underlying dynamical mechanisms leading to this statistical behaviour are completely unknown. Examples are systems described by the beta distribution,

$$P_{p,q}(x)=\frac{x^{p-1}(1-x)^{q-1}}{B_0(p,q)}\quad (p,\, q>0) \tag{1}$$

SSDI 0375-9601(95)00971-X

where $B_0(p, q)$ is the beta function. The distribution (1) possesses a wide range of applications since it leads in special cases to several frequently used distributions like, for example, the binomial one. One could therefore ask for a dynamical system which is for the case of fully developed chaotic behaviour described by the invariant density given in Eq. (1). For the special case $p=\frac{1}{2}$, $q=\frac{1}{2}$ at least one solution, i.e. dynamical system, belonging to the density (1) is well-known: it is the extensively studied logistic map [1]. The distribution (1) is in general asymmetric in the interval [0, 1]. In the following investigation we restrict ourselves to the symmetric case $p=q$. We will show that a rather general class of one-dimensional maps can be derived from this symmetric case.

The purpose of the present paper is as follows. We will construct one-dimensional iterative maps belonging to a given class of invariant densities. The key link between the density in the ergodic limit and the map is hereby provided via the Frobenius–Perron equation. As a specific guiding example we will use the densities given in Eq. (1) for $p=q$ which are of broad interest (see above). Furthermore we will study certain aspects of these maps and derive some analytical properties which provide us with the order of their maxima and consequently the universal constants of their bifurcation routes.

The invariant ergodic density $\rho(x)$ of a one-dimensional map $y=f(x)$ in the interval $[a, b]$ obeys the Frobenius–Perron equation

$$\rho(x)=\int_a^b \mathrm{d}y\ \delta(x-f(y))\,\rho(y), \tag{2}$$

which can be presented in the following differential form,

$$\rho(f(x))=\sum_{x\in f^{-1}(y)} \frac{\rho(x)}{|\mathrm{d}f(x)/\mathrm{d}x|}. \tag{3}$$

We assume now that our dynamical system described by the map $f(x)$ possesses in the fully developed chaotic regime an invariant density $\rho(x)$ given by the equation

$$\rho(x)=\frac{2^{2\gamma-1}B_0(\frac{1}{2}, 1-\gamma)}{x^\gamma(1-x)^\gamma}, \tag{4}$$

where γ is an arbitrary real number smaller than unity. The constant nominator of the fraction on the r.h.s. of Eq. (4) ensures the normalization of the density. Our aim is to derive the map belonging to the class of densities in Eq. (4) by using Eq. (3). In order to proceed with our calculation we restrict ourselves to maps which are symmetric with respect to $x=\frac{1}{2}$, i.e. we are dealing with the doubly symmetric case (both the density as well as the map are symmetric). Eq. (3) then simplifies to

$$\rho(y)=2\frac{\rho(x)}{|\mathrm{d}y/\mathrm{d}x|}. \tag{5}$$

This equation can be integrated in the regions of monotonic behaviour of the density. The symmetry around $x=\frac{1}{2}$ and the uniqueness of the maximum yield the fact that $y=f(x)$ has its maximum at $x=\frac{1}{2}$. We can therefore write

$$\int_0^y \rho(w)\ \mathrm{d}w=2\int_0^x \rho(u)\ \mathrm{d}u \quad \text{for } x\leqslant\tfrac{1}{2}. \tag{6}$$

Inserting the density (4) results in the equation

$$B(1-\gamma, 1-\gamma, y)=2B(1-\gamma, 1-\gamma, x) \quad \text{for } x\leqslant\tfrac{1}{2} \tag{7}$$

and

$$B(1-\gamma, 1-\gamma, y)=2B(1-\gamma, 1-\gamma, 1-x) \quad \text{for } 1\geqslant x>\tfrac{1}{2}, \tag{8}$$

where B is the incomplete beta function related to Eq. (1) by its defining equation

$$B(p, q, x)=\int_0^x t^{p-1}(1-t)^{q-1}\ \mathrm{d}t. \tag{9}$$

Expressions (7) and (8) can be combined in the form

$$B(1-\gamma, 1-\gamma, y) -2B\left(1-\gamma, 1-\gamma, \tfrac{1}{2}-|x-\tfrac{1}{2}|\right)=0. \tag{10}$$

This is an implicit defining equation for the maps $y=f(x)$ on the interval [0, 1] which possess the invariant densities (4). This family of maps will in the following be called the symmetric beta maps (SBM). Eq. (10) represents a special case of the general relationship between ergodic measures and their underlying maps which has been investigated in detail in Refs. [7].

The existence as well as uniqueness of the solution $y = f(x)$ to Eq. (10) can be argued as follows. Since the maps are symmetric with respect to $x = \frac{1}{2}$ we need to consider only the interval $[0, \frac{1}{2}]$. According to Eqs. (4), (5) $y = f(x)$ is differentiable and in particular continuous in this interval and possesses a maximum at $x = \frac{1}{2}$ with the value $y = 1$, i.e. $B(1 - \gamma, 1 - \gamma, 1) = 2B(1 - \gamma, 1 - \gamma, \frac{1}{2})$. The images of the points $x = 0, \frac{1}{2}$ are therefore $y = 0, 1$, respectively. Since the derivative of the incomplete beta function B is positive on the considered interval $[0, \frac{1}{2}]$, the incomplete beta function is monotonically increasing and in particular invertible. This means we can, formally, write $y = B^{-1} \circ [2B(1 - \gamma, 1 - \gamma, x)]$ which yields that also $y = f(x)$ is monotonically increasing and, as already mentioned, continuous. In total we, therefore, arrive at the conclusion that Eq. (10) possesses for each value of γ a unique solution.

Next let us briefly address the question how the family of maps $y = f(x)$ can be obtained from Eq. (10). Since the maps are given implicitly we determine the image for a given value of x by a simple numerical algorithm: the problem corresponds for given x to a root searching procedure with respect to y in Eq. (10). Our method of choice for finding the roots was a combined bisection and Newtonian method.

In order to extend the definition of the SBM to the nonergodic regime we use the scaling $y \to y/r$ with $0 < r \leqslant 1$. The final form of the SBM is therefore given by

$$B(1 - \gamma, 1 - \gamma, y/r) - 2B\left(1 - \gamma, 1 - \gamma, \tfrac{1}{2} - |x - \tfrac{1}{2}|\right) = 0. \tag{11}$$

This equation represents an implicit definition of one-parameter families of maps with r as a parameter. If r varies from 0 to 1 the iteratives of the maps defined by Eq. (11) run through a bifurcation route which finally ends up in the fully developed chaotic and ergodic case for $r = 1$. We remark that the SBM includes a special cases the logistic map for $\gamma = \frac{1}{2}$ and the tent map for $\gamma = 0$. We will show in the following that the class of maps defined through Eq. (11) is actually very rich and includes in particular maps with a maximum arbitrary high order and also nondifferentiable maxima at the point $x = \frac{1}{2}$. In Fig. 1 we show the SBM for $r = 1$ and for six different values of γ. Let us now study some analytical properties of this map. For any value of r we have the properties $y(0) = 0$ and $y(1) = 0$ and the value at the maximum is $y(\frac{1}{2}) = r$. The most interesting feature of the SBM is its analytical behaviour around the maximum. The derivative of the map in Eq. (11) is given by

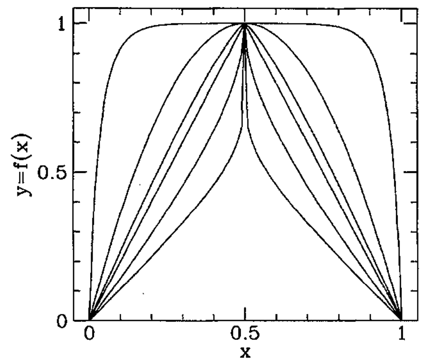

Fig. 1. The symmetric beta map (SBM) defined by Eq. (10) on the interval [0, 1] is illustrated for six different values of the parameter γ. From the bottom to the top curve we have $\gamma = -10.0, -1.0, 0.0, 0.2, 0.5, 0.8$.

$$\frac{dy}{dx} = \frac{2y^{\gamma}(r-y)^{\gamma}}{x^{\gamma}(1-x)^{\gamma} r^{2\gamma-1}} \times \left[\Theta\left(\tfrac{1}{2} - x\right) - \Theta\left(x - \tfrac{1}{2}\right)\right]. \tag{12}$$

Near the maximum $x = \frac{1}{2}$, $y = r$ we can write $y = r - \epsilon$, $\epsilon \ll r$ and $|x - \frac{1}{2}| = \delta$, $\delta \ll \frac{1}{2}$ and we arrive at

$$\frac{dy}{dx} \propto |x - \tfrac{1}{2}|^{\gamma/(1-\gamma)}\left[\Theta\left(\tfrac{1}{2} - x\right) - \Theta\left(x - \tfrac{1}{2}\right)\right]. \tag{13}$$

From Eq. (13) we conclude that for $0 < \gamma < 1$ the first derivative of the map is continuous and zero at $x = \frac{1}{2}$. For $\gamma = 0$ (tent map) it is for $x < \frac{1}{2}$ and $x > \frac{1}{2}$ constant and discontinuous at $x = \frac{1}{2}$ while for $\gamma < 0$ it is one sided defined and singular at $x = \frac{1}{2}$. It is a tedious but straightforward task to calculate the behaviour of the nth derivative of the SBM near the maximum. We obtain

$$\frac{d^n y}{dx^n} \propto |x - \tfrac{1}{2}|^{(n\gamma - n + 1)/(1-\gamma)} \times \left[\Theta\left(\tfrac{1}{2} - x\right) - \Theta\left(x - \tfrac{1}{2}\right)\right]^n. \tag{14}$$

From Eq. (14) we conclude that $\gamma_n = (n-1)/n$ are "critical" points in the following sense. For $1 > \gamma > (n-1)/n$ the first n derivatives of the SBM vanish while for $0 < \gamma < (n-1)/n$ the nth derivative is singular. At the critical points $\gamma = \gamma_n$ the nth derivative is constant. For $n = 2k$ $(k = 1, 2, \dots)$ Eq. (14) leads to a continuous maximum of order $2k$. From the above we conclude that the maximum structure of the SBM is similar to that of the map $g(x) = 1 - |1 - 2x|^{2z}$ where $z = 1/2(1-\gamma)$ which has in some detail been studied in the literature [2]. The invariant densities of the latter class of maps $g(x)$ can, however, only be determined numerically. We mention that the SBM fulfills the condition $f'(0) = 2^{1/(1-\gamma)}$ which was in Ref. [7] shown to be necessary for a symmetric map to possess a symmetric invariant density. In particular the members of the family of symmetric beta maps are all smoothly conjugate to each other.

Next let us study the bifurcation routes of the SBM for different values of γ. We are particularly interested in the continuous parameter interval $0 < \gamma < 1$ which includes a wide range of the orders of the corresponding maxima of the SBM. In order to investigate the universal scaling laws of the bifurcation routes of the SBM for different γ we have to continuously change the parameter r for each value of γ in Eq. (10) from 0 to 1. At subsequent critical values of the parameter r, i.e. for $r = r_i$ with $i = 1, \dots, \infty$, the stable periodic orbits of period 2^i become unstable. Simultaneously a new stable periodic orbit of period 2^{i+1} is created by a period doubling bifurcation. Two universal Feigenbaum constants characterize the transition from order to chaos via the period doubling bifurcation route: the limiting ratio

$$\lim_{i\to\infty} \frac{r_{i+1} - r_i}{r_i - r_{i-1}} = \frac{1}{\delta}, \quad \lim_{i\to\infty} \frac{d_{i+1}}{d_i} = \alpha.$$

The value of α determines the asymptotic scaling ratio of two subsequent distances of the fixpoints closest to the fixpoint $\frac{1}{2}$ for superstable orbits. Scanning through the period doubling bifurcation route we obtain for each value of γ a pair of universal constants (δ, α).

In Table 1 we provide the critical values r_i for the first few bifurcation points and the distances d_i of the closest fixpoints to $\frac{1}{2}$ for the first few superstable orbits of the SBM as well as the map $g(x) = 1 - |1 - 2x|^{1/(1-\gamma)}$. The chosen value of γ is 0.2. The two classes of maps have in common that they possess a wide range of possible orders of the corresponding maxima. The sequences α_i and δ_i converge, as expected, against the universal constants α and δ, respectively, which are uniquely determined by the order of the maximum of the underlying map.

Finally we want to make a comment on the ergodic limit of the map, i.e. for fully developed chaos at $r = 1$. For the SBM the invariant density is, of course, given by Eq. (4). In addition to this most important property there is a general statement which can be applied to all symmetric maps of the interval with a maximum of a certain order $2z$. The invariant densities of these maps possess all the same singularity structures at $x \to 0^+$ and, due to symmetry, also at $x \to 1^-$. These singularities of the density are for $x \to 0$ of the form $\rho(x) \propto x^{-(1-1/2z)}$ (see Ref. [8]).

Let us conclude. We use a general applicable

Table 1
The values r_i for the scaling parameter r, the distances d_i of the nearest fixpoint to $x = \frac{1}{2}$ as well as the iterations of the universal constants α_i and δ_i are given from the 2^2 to the 2^9 period doubling bifurcation of the symmetric beta map (SBM) as well as the map $g(x) = 1 - |1 - 2x|^{1/(1-\gamma)}$. The value of the parameter γ is 0.2

r_i		d_i		α_i		δ_i	
SBM	$g(x)$	SBM	$g(x)$	SBM	$g(x)$	SBM	$g(x)$
0.68788	0.70351	1.280×10^{-2}	1.397×10^{-2}	2.62437	2.27537	3.19696	3.67370
0.70794	0.72353	2.599×10^{-3}	2.815×10^{-3}	13.1754	13.3029	3.17647	3.24376
0.71415	0.72969	5.394×10^{-4}	5.829×10^{-4}	4.95390	4.99802	3.23392	3.24991
0.71606	0.73158	1.127×10^{-4}	1.217×10^{-4}	4.82649	4.83927	3.25189	3.25640
0.71664	0.73216	2.359×10^{-5}	2.547×10^{-5}	4.78926	4.79308	3.25726	3.25860
0.71682	0.73234	4.941×10^{-6}	5.334×10^{-6}	4.77797	4.77913	3.25889	3.25930
0.71688	0.73239	1.035×10^{-6}	1.117×10^{-6}	4.77452	4.77487	3.25939	3.25951
0.71689	0.73241	2.168×10^{-7}	2.341×10^{-7}	4.77346	4.77357	3.25956	3.25958

method for the construction of one-dimensional iterative maps (SBM) by solving the Frobenius–Perron equation for a given invariant density. As a specific example we investigated the symmetric beta distribution. The construction of these maps is an important step towards the simulation of the dynamics of systems whose statistical properties are described by a given distribution. Although the construction of such a map is, due to its implicit character, based on a numerical approach, its implementation is straightforward and can, therefore, be applied to a broad class of invariant ergodic measures (densities). It is a challenging open question to construct a map for the asymmetric beta distribution. However, to accomplish this a further understanding of the general properties of invariant densities would be helpful.

Financial support by the Deutsche Forschungsgemeinschaft (P.S.) as well as the European Community (F.K.D.) are gratefully acknowledged.

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