Assignment 2

Koenigsberg

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Which of the graphs in the attached figure can be drawn without raising your pencil from the paper, and without drawing any line more than once? Why? For the graphs that can be drawn as described, state how many starting points are possible.

1 Image (a)

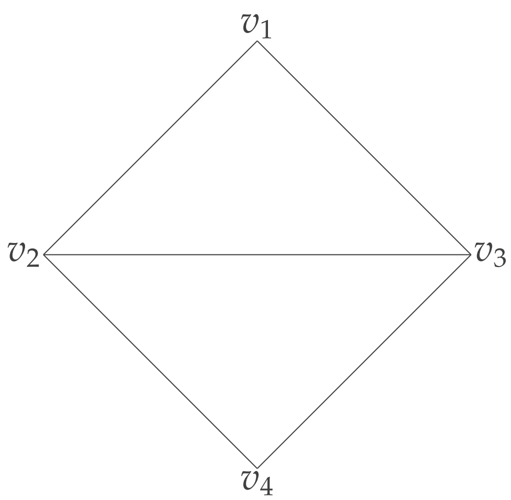

Figure 1: Image a. from question with labelled nodes.

A **trail** is a sequence of *distinct* edges such that any consecutive edges are incident to a common vertex. When the sequence visits every edge of a graph, it is called an **Euler trail**. Following such trail with a pencil would lead us to draw the graph with the specified requirements.

To find if a graph has an Euler trail, however, we need Euler's Theorem, which is stated in terms of a **circuit**, that is, a trail that ends at the same vertex it started. From there we have proposition 2, which won't be proved here, but the main argument behind it is:

If we connect the two vertices with odd degree by an edge $e := xy$, we get an even graph. Since this new graph has an Euler circuit $W := uW_1xeyW_2u$ and the only edge missing is e, the trail $T := xW_1uW_2y$ still is eulerian for the original graph.

Theorem 1 (*Euler's Theorem*)

A connected graph has an Euler circuit if and only if every vertex has even degree.

Proposition 2

A connected graph has an Euler trail if and only if exactly zero or two vertices have odd degree.

1.1 Answer

Finally, by proposition 2, we can see that the graph in figure 1 can be drawn without raising the pencil and without repeating any lines. This is because only v_2 and v_3 have odd degree ($\deg(v_2) = 3 = \deg(v_3)$).

Furthermore, considering the argument for proposition 2, we must start the drawing at one of the two odd vertices, v_2 or v_3.

2 Image (b)

For figure 2, we have nodes v_1, v_2, v_3 and v_4 with degree odd ($\deg(v_i) = 3$ for $i = 1, 2, 3$ and 4), therefore the graph does **not** have an Euler trail and cannot be drawn without raising the pencil or repeating a line.

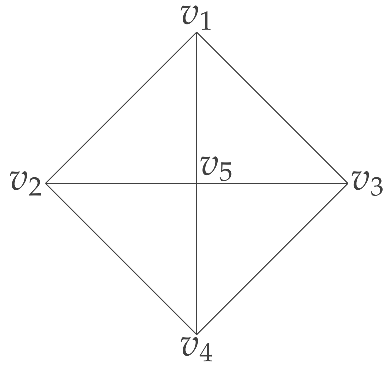

Figure 2: Image `b.` with labelled nodes.

3 Image (c)

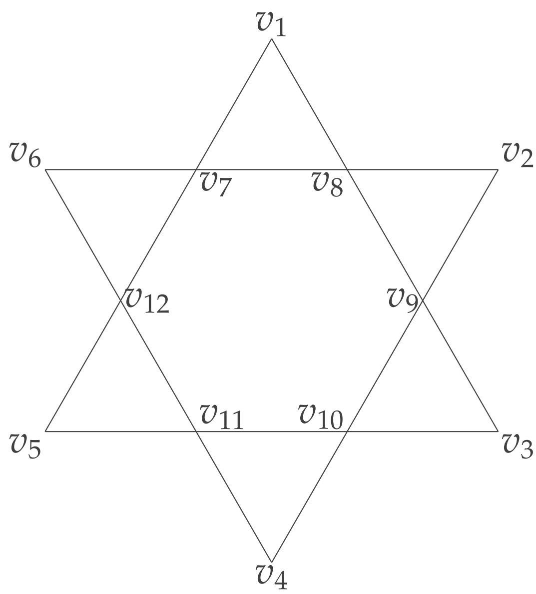

Figure 3: Image `c.` with labelled nodes.

Note that every outer vertex here $(v_1, \ldots, v_6)$ has degree 2 and every inner node $(v_7, \ldots, v_{12})$ has degree 4 therefore, by Euler's Theorem, there is an Euler circuit and the graph can be drawn as described.

Since a circuit is just a collection of cycles, we can start the eulerian trail from any vertex and we will stop at the same vertex. In fact, we can start at any point in the drawing, not only the labelled vertices, by just creating a vertex there with degree two.

4 Image (d)

Table 1: Degrees of each vertex of figure 4.

Vertex	Degree	Is Odd?
v_1	2	—
v_2	2	—
v_3	6	—
v_4	6	—
v_5	3	Yes
v_6	1	Yes

Just like with image (a), we have exactly two odd vertices, v_5 and v_6, in this case. By proposition 2, this proves the graph can be drawn as required. As before, both vertices can be used as the starting point for the drawing.

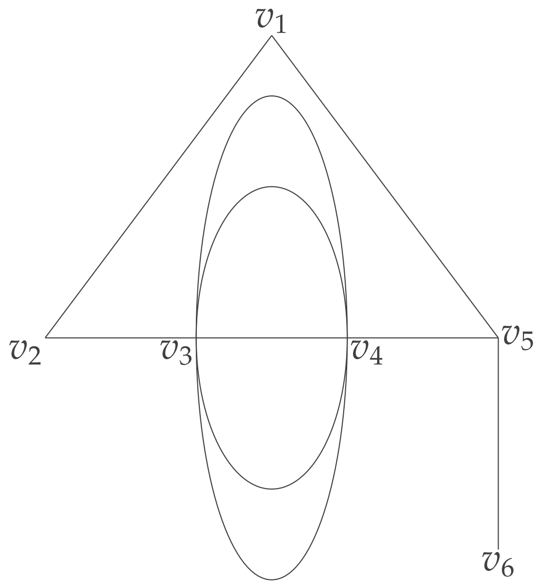

Figure 4: Image `d.` with labelled nodes.