

Assignment 5

Calculus and Differential Equations

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Solve the differential equation $f' + 2f = x$ with initial condition $f(0) = 0$. Please clarify every step you take.

Analyzing the Equation

First note that the given equation is a *linear differential equation*. That is, given two solutions $f_1(x)$ and $f_2(x)$ to the original equation, their difference $g(x) = f_1(x) - f_2(x)$ follows a *homogeneous differential equation*:

$$\begin{aligned} g' + 2g &= (f_1 - f_2)' + 2(f_1 - f_2) \\ &= f_1' - f_2' + 2f_1 - 2f_2 \\ &= (f_1' + 2f_1) - (f_2' + 2f_2) \\ &= x - x \\ &= 0 \end{aligned} \tag{1}$$

Therefore, we can describe all solutions to the original equation as a combination of a *particular solution* (not restricted to the same initial condition) and a *homogeneous solution*, which is a solution to the equivalent homogeneous equation (1).

Finding the Solutions

Some Particular Solution

Since the inhomogeneous part of the differential equation comes from a polynomial ($p(x) = x$), we assume that we can find a polynomial $f(x) = \sum_{i=0}^n a_i x^i$ of degree n that is a solution to the equation:

$$\begin{aligned} x &= f' + 2f \\ &= \frac{d}{dx} \sum_{i=0}^n a_i x^i + 2 \sum_{i=0}^n a_i x^i \\ &= \sum_{i=0}^n a_i \frac{dx^i}{dx} + \sum_{i=0}^n 2a_i x^i \\ &= a_0 \frac{dx^0}{dx} + \sum_{i=1}^n a_i \cdot i x^{i-1} + \sum_{i=0}^n 2a_i x^i \\ &= 0 + \sum_{i=0}^{n-1} (i+1)a_{i+1} x^i + \sum_{i=0}^{n-1} 2a_i x^i + 2a_n x^n \\ &= 2a_n x^n + \sum_{i=0}^{n-1} ((i+1)a_{i+1} + 2a_i) x^i \end{aligned}$$

Since this must hold for any x , we have the following system:

$$2a_2 + 2a_1 = 1 \quad \text{when } i = 1 \quad (2)$$

$$(i+1)a_{i+1} + 2a_i = 0 \quad \text{for } i \neq 1 \text{ and } i \neq n \quad (3)$$

$$a_n = 0 \quad \text{considering the first term} \quad (4)$$

From equations (3) and (4), we get that $a_i = 0$ for $2 \leq i \leq n$, and from equation (2) we have:

$$a_1 = \frac{1 - 2a_2}{2} = \frac{1}{2}$$

Applying equation (3) with $i = 0$ results in:

$$a_0 = -\frac{1}{2}a_1 = -\frac{1}{4}$$

Therefore, our particular solution is $f(x) = \frac{1}{2}x - \frac{1}{4}$.

Homogeneous Solutions

To reach our homogeneous solution we must solve the homogeneous equation:

$$f' + 2f = 0$$

Using a common technique called *separation of variables*, we get:

$$\begin{aligned} \frac{df}{dx} + 2f &= 0 \\ \frac{df}{dx} &= -2f \\ \frac{1}{f} \frac{df}{dx} &= -2 \\ \int \frac{1}{f} \frac{df}{dx} dx &= \int -2 dx \\ \int \frac{1}{f} df &= -2x + C_1 \\ \ln f + C_2 &= -2x + C_1 \\ f(x) &= e^{-2x+C_1-C_2} = C_3 \cdot e^{-2x} \end{aligned}$$

Therefore, the homogeneous part of the solution is $f(x) = A \exp(-2x)$, for some $A \in \mathbb{R}$.

General Solution

As described earlier, any solution to the original equation can now be achieved by adding some homogeneous solution to our particular solution, resulting in:

$$\begin{aligned} f(x) &= Ae^{-2x} + \frac{1}{2}x - \frac{1}{4} \\ &= \frac{Be^{-2x} + 2x - 1}{4} \end{aligned} \quad (5)$$

For some other $B \in \mathbb{R}$.

Applying the Initial Condition

The last step is to find the specific function $f(x)$ that solves both equations. It is now just a matter of applying our constraints to the general solution on equation (5).

$$\begin{aligned} 0 = f(0) &= \frac{Ae^{-2 \cdot 0} + 2 \cdot 0 - 1}{4} \\ &= \frac{A \cdot 1 - 1}{4} \\ &= A - 1 \end{aligned}$$

That is, $A = 1$ and:

$$f(x) = \frac{e^{-2x} + 2x - 1}{4}$$

Verification

We can also verify that the differential equation holds:

$$\begin{aligned} f' + 2f &= \frac{\frac{d}{dx}e^{-2x} + \frac{d}{dx}2x - \frac{d}{dx}1}{4} + 2 \frac{e^{-2x} + 2x - 1}{4} \\ &= \frac{-2e^{-2x} + 2 - 0 + 2e^{-2x} + 2 \cdot 2x - 2}{4} \\ &= \frac{4x}{4} \\ &= x \end{aligned}$$

And the initial condition:

$$\begin{aligned} f(0) &= \frac{e^{-2 \cdot 0} + 2 \cdot 0 - 1}{4} \\ &= \frac{e^0 - 1}{4} \\ &= \frac{1 - 1}{4} \\ &= 0 \end{aligned}$$