Assignment 5

Calculus and Differential Equations

Tiago de Paula Alves (187679)
tiagodepalves@gmail.com

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Solve the differential equation $f' + 2f = x$ with initial condition $f(0) = 0$. Please clarify every step you take.

Analyzing the Equation

First note that the given equation is a *linear differential equation*. That is, given two solutions $f_1(x)$ and $f_2(x)$ to the original equation, their difference $g(x) = f_1(x) - f_2(x)$ follows a *homogeneous differential equation*:

$$\begin{aligned} g' + 2g &= (f_1 - f_2)' + 2\,(f_1 - f_2) \\ &= f_1' - f_2' + 2f_1 - 2f_2 \\ &= \left(f_1' + 2f_1\right) - \left(f_2' - 2f_2\right) \\ &= x - x \\ &= 0 \end{aligned} \tag{1}$$

Therefore, we can describe all solutions to the original equation as a combination of a *particular solution* (not restricted to the same initial condition) and a *homogeneous solution*, which is a solution to the equivalent homogeneous equation (1).

Finding the Solutions

Some Particular Solution

Since the inhomogeneous part of the differential equation comes from a polynomial ($p(x) = x$), we assume that we can find a polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ of degree n that is a solution to the equation:

$$\begin{aligned} x &= f' + 2f \\ &= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{i=0}^{n} a_i x^i + 2 \sum_{i=0}^{n} a_i x^i \\ &= \sum_{i=0}^{n} a_i \frac{\mathrm{d}x^i}{\mathrm{d}x} + \sum_{i=0}^{n} 2a_i x^i \\ &= a_0 \frac{\mathrm{d}x^0}{\mathrm{d}x} + \sum_{i=1}^{n} a_i \cdot i x^{i-1} + \sum_{i=0}^{n} 2a_i x^i \\ &= 0 + \sum_{i=0}^{n-1} (i+1) a_{i+1} x^i + \sum_{i=0}^{n-1} 2a_i x^i + 2a_n x^n \\ &= a_n x^n + \sum_{i=0}^{n-1} \left((i+1)a_{i+1} + 2a_i\right) x^i \end{aligned}$$

Since this must hold for any x, we have the following system:

$$
\begin{aligned}
2a_2 + 2a_1 &= 1 && \text{when } i = 1 && (2)\\
(i+1)a_{i+1} + 2a_i &= 0 && \text{for } i \neq 1 \text{ and } i \neq n && (3)\\
a_n &= 0 && \text{when } i = n && (4)
\end{aligned}
$$

From equations (3) and (4), we get that $a_i = 0$ for $i \geq 2$, and from equation (2) we have:

$$a_1 = \frac{1 - 2a_2}{2} = \frac{1}{2}$$

Applying equation (3) with $i = 0$ results in:

$$a_0 = -\frac{1}{2}a_1 = -\frac{1}{4}$$

Therefore, our particular solution is $f(x) = \frac{1}{2}x - \frac{1}{4}$.

Homogeneous Solutions

To reach our homogeneous solution we must solve the homogeneous equation:

$$f' + 2f = 0$$

Using a common technique called *separation of variables*, we get:

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x} + 2f &= 0\\
\frac{\mathrm{d}f}{\mathrm{d}x} &= -2f\\
\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x} &= -2\\
\int \frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}x}\,\mathrm{d}x &= \int -2\,\mathrm{d}x\\
\int \frac{1}{f}\,\mathrm{d}f &= -2x + C_1\\
\ln f + C_2 &= -2x + C_1\\
f(x) &= e^{-2x+C_1-C_2} = C_3 \cdot e^{-2x}
\end{aligned}
$$

Therefore, the homogeneous part of the solution is $f(x) = A\exp(-2x)$, for some $A \in \mathbb{R}$.

General Solution

As described earlier, any solution to the original equation can now be achieved by adding some homogeneous solution to the particular solution, resulting in:

$$
\begin{aligned}
f(x) &= Ae^{-2x} + \frac{1}{2}x - \frac{1}{4}\\
&= \frac{Be^{-2x} + 2x - 1}{4} && (5)
\end{aligned}
$$

For some other $B \in \mathbb{R}$.

Applying the Initial Condition

The last step is to find the specific function $f(x)$ that solves both equations. It is now just a matter of applying our constraints to the general solution on equation (5).

$$\begin{aligned} 0 = f(0) &= \frac{Ae^{-2\cdot 0} + 2\cdot 0 - 1}{4} \\ &= \frac{A\cdot 1 - 1}{4} \\ &= A - 1 \end{aligned}$$

That is, $A = 1$ and:

$$f(x) = \frac{e^{-2x} + 2x - 1}{4}$$

Verification

We can also verify that the differential equation holds:

$$\begin{aligned} f' + 2f &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}e^{-2x} + \frac{\mathrm{d}}{\mathrm{d}x}2x - \frac{\mathrm{d}}{\mathrm{d}x}1}{4} + 2\frac{e^{-2x} + 2x - 1}{4} \\ &= \frac{-2e^{-2x} + 2 - 0 + 2e^{-2x} + 2\cdot 2x - 2}{4} \\ &= \frac{4x}{4} \\ &= x \end{aligned}$$

And the initial condition:

$$\begin{aligned} f(0) &= \frac{e^{-2\cdot 0} + 2\cdot 0 - 1}{4} \\ &= \frac{e^0 - 1}{4} \\ &= \frac{1 - 1}{4} \\ &= 0 \end{aligned}$$