Assignment 9

Copying Model

Tiago de Paula Alves (187679)
tiagodepalves@gmail.com

May 26, 2022

Use the rate equation approach to show that the directed copying model (Section 5.9 of the book) leads to a scale-free network with incoming degree exponent $\gamma_{\text{in}} = (2-p)/(1-p)$, where p is the probability involved in the model.

1 Discrete Rate Equation

We can see the preferential attachment showing up in the probability $\Pi(k)$ that a specific node v with $\deg_{\text{in}}(v) = k$ will be connected to the new node:

$$\begin{aligned}\Pi(k) &= p \cdot \mathbf{P}\left[v \text{ is chosen as the } \textit{Target}\right] + (1-p) \cdot \mathbf{P}\left[v \text{ is chosen as a } \textit{Copied Connection}\right] \\ &= p \cdot \frac{1}{N} + (1-p) \cdot \frac{\deg_{\text{in}}(v)}{L} \\ &= \frac{p}{N} + \frac{1-p}{L} k\end{aligned}$$

With this probability, we can analyze the graph generated by the directed version of the copying model.

1.1 Growth at Degree k

For simplicity, we assume an initial model of a single loop, with $N = L = 1$. At each step, we add a new node and a new link, resulting in $N = L = t$, for any time step t. Let $N(k,t) = N \cdot p_{k,t}$ be the expected number of vertices with degree k at time step t. Therefore, the expected number of new links to degree k nodes is given by

$$\begin{aligned}R(k,t) &= \Pi(k) \cdot N(k,t) \\ &= \left(\frac{p}{N} + \frac{1-p}{L} \cdot k\right) \cdot p_{k,t} N \\ &= \left(\frac{p + (1-p)\,k}{t}\right) p_{k,t} \cdot t \\ &= (p + k - kp)\, p_{k,t}\end{aligned}$$

We can now model the growth of nodes of degree k with the time-dependent rate equation:

$$\begin{aligned}N(k,t+1) &= N(k,t) + R(k-1,t) - R(k,t) \\ (N+1)p_{k,t+1} &= Np_{k,t} + (p+k-1-(k-1)p)p_{k-1,t} - (p+k-kp)p_{k,t} \\ Np_{k,t+1} + p_{k,t+1} &= Np_{k,t} + (2p+k-1-kp)p_{k-1,t} - (p+k-kp)p_{k,t}\end{aligned}$$

1.2 Stable Rate Equation

Finally, we can assume that $p_{k,t}$ will eventually stabilize as $p_k = \lim_{t\to\infty} p_{k,t}$. Therefore, when $t \to \infty$, the rate equation becomes

$$\begin{aligned}Np_k + p_k &= Np_k + (2p+k-1-kp)p_{k-1} - (p+k-kp)p_k \\ 0 &= -p_k + 2pp_{k-1} + kp_{k-1} - p_{k-1} - kpp_{k-1} - pp_k - kp_k + kpp_k\end{aligned}$$

Which can be rewritten as

$$
\begin{aligned}
p_k - 2pp_{k-1} + p_{k-1} + pp_k &= kpp_k - kpp_{k-1} - kp_k + kp_{k-1} \\
p_k - 2pp_{k-1} + p_{k-1} + pp_k &= kp(p_k - p_{k-1}) - k(p_k - p_{k-1}) \\
p_k - pp_{k-1} + p_{k-1} &= (kp - k)(p_k - p_{k-1}) - pp_k + pp_{k-1} \\
p_k - pp_{k-1} + p_{k-1} &= (kp - k)(p_k - p_{k-1}) - p(p_k - p_{k-1}) \\
-pp_{k-1} + 2p_{k-1} &= (kp - k - p)(p_k - p_{k-1}) - (p_k - p_{k-1}) \\
(2 - p)p_{k-1} &= (kp - k - p - 1)(p_k - p_{k-1}) \\
(p - 2)p_{k-1} &= (k + p + 1 - kp)(p_k - p_{k-1}) \qquad (1)
\end{aligned}
$$

2 Continuum Approximation

For a large k, we can assume p_k is continuous, such that $p_k \approx p_{k-1}$ and

$$\frac{\mathrm{d}p_k}{\mathrm{d}k} \approx \frac{p_k - p_{k-1}}{k - (k-1)} = p_k - p_{k-1}$$

Therefore, equation (1) can be approximated by the following differential equation

$$
\begin{aligned}
(p - 2)p_{k-1} \approx (p - 2)p_k &= (k + p + 1 - kp)\frac{\mathrm{d}p_k}{\mathrm{d}k} \\
\frac{p - 2}{k + p + 1 - kp} &= \frac{1}{p_k}\frac{\mathrm{d}p_k}{\mathrm{d}k} \\
-(2 - p)\int \frac{1}{(1 - p)k + p + 1}\,\mathrm{d}k &= \int \frac{1}{p_k}\frac{\mathrm{d}p_k}{\mathrm{d}k}\,\mathrm{d}k \\
-\frac{2 - p}{1 - p}\ln(k + p + 1 - kp) + C_2 &= \ln p_k + C_1
\end{aligned}
$$

So,

$$
\begin{aligned}
\ln p_k &= -\frac{2 - p}{1 - p}\ln(k + p + 1 - kp) + C_3 \\
p_k &= C_4 \cdot (k + p + 1 - kp)^{-\frac{2-p}{1-p}}
\end{aligned}
$$

2.1 The Degree Exponent

For $k \geq 1$, we have

$$(1 - p)k \leq k + p + 1 - kp \leq k + 1 \leq 2k$$

And

$$C_4 \cdot 2^{-\frac{2-p}{1-p}} \cdot k^{-\frac{2-p}{1-p}} \leq p_k \leq C_4 \cdot (1 - p)^{-\frac{2-p}{1-p}} \cdot k^{-\frac{2-p}{1-p}}$$

Therefore,

$$p_k \sim k^{-\frac{2-p}{1-p}} = k^{-\gamma_{\text{in}}}$$

For $\gamma_{\text{in}} = \frac{2-p}{1-p}$, as proposed.