Question Bank for 23CST401 Discrete Mathematics

TKM College of Engineering, Kollam

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Course Code: 23CST401 **Course Name**: Discrete Mathematics
Year of Introduction: 2023 **Credits**: 3

1 Module I: Mathematical Logic

1.1 Part A: Short-Answer Questions (2 marks each)

1. Show the implication $(P \land Q) \Rightarrow (P \to Q)$ without constructing the truth table.
2. Write the negation of the statement: "If I drive, then I will not walk."
3. Show that the statements RVM, ¬RVS, ¬M, ¬S cannot exist simultaneously without using a truth table.
4. Represent the statement "Not every city in Canada is clean" in symbolic form.
5. Define a tautology and provide an example.
6. State the duality law and illustrate with an example.
7. Write the conjunctive normal form (CNF) for $(P \lor Q) \land (\neg P \lor R)$.
8. List two rules of inference used in propositional logic.
9. Express the statement "Some students are intelligent but not hardworking" using predicate logic.
10. Define a biconditional statement and give its truth table.
11. Write the disjunctive normal form (DNF) for $(\neg P \land Q) \lor (P \land R)$.
12. Define a contradiction and provide an example.

1.2 Part B: Long-Answer Questions (8 marks each)

1. (a) Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$. (4 marks)
   (b) Show that from $(\exists x)(F(x) \wedge S(x)) \to (y)(M(y) \to W(y))$ and $(\exists y)(M(y) \wedge \neg W(y))$, the conclusion $(x)(F(x) \to \neg S(x))$ follows. (4 marks)
2. (a) Show that $(x)(P(x) \vee Q(x)) \to ((x)P(x) \vee (\exists x)Q(x))$ using the indirect method of proof. (4 marks)
   (b) Discuss the indirect method of proof and show that the premises: (i) If Jack misses many classes, then he fails high school; (ii) If Jack fails high school, then he is uneducated; (iii) If Jack reads a lot, then he is not uneducated; (iv) Jack misses many classes and reads a lot, are inconsistent. (4 marks)
3. Prove that the argument $(P \to Q) \wedge (Q \to R) \wedge P \to R$ is valid using rules of inference.
4. Construct the principal disjunctive normal form (PDNF) for $(P \to Q) \wedge (Q \vee \neg R)$ and verify it using a truth table.
5. Prove that $(\neg P \vee Q) \wedge (P \vee R) \to (Q \vee R)$ is a tautology using the theory of inference.
6. Show that the premises $(x)(P(x) \to Q(x))$ and $(\exists x)P(x)$ imply $(\exists x)Q(x)$ using predicate calculus.
7. Prove that $(P \wedge (P \to Q)) \to Q$ is a tautology using logical equivalences.
8. Convert $(P \wedge \neg Q) \vee (Q \wedge R)$ into its principal conjunctive normal form (PCNF).
9. Show that the statement "All men are mortal, and Socrates is a man, therefore Socrates is mortal" is valid using predicate logic.
10. Discuss the role of quantifiers in predicate calculus and show that $\neg(\forall x)P(x) \equiv (\exists x)\neg P(x)$.
11. Prove that $(P \vee Q) \wedge (\neg P \vee R) \to (Q \vee R)$ is a tautology using a truth table.
12. Show that the premises $(\exists x)(P(x) \wedge \neg Q(x))$ and $(x)(P(x) \to R(x))$ imply $(\exists x)(R(x) \wedge \neg Q(x))$.

2 Module II: Algebraic Structures

2.1 Part A: Short-Answer Questions (2 marks each)

1. Define a monoid and provide an example.
2. Show that for a group $(A, \cdot)$, $(ab)^{-1} = b^{-1}a^{-1}$.
3. Define a permutation group and give an example using S_3.
4. State Lagrange's Theorem for finite groups.
5. Define a subgroup and give an example using integers under addition.

6. Define a homomorphism between two groups and provide an example.
7. Define a coset in a group and illustrate with an example.
8. Give an example of a permutation in S_4 and find its inverse.
9. Explain the concept of a submonoid with an example.
10. Define the order of an element in a permutation group and give an example.
11. Give an example of a group that is not abelian.
12. Write the composition of the permutations $(1\ 2\ 3)$ and $(2\ 3\ 4)$ in S_4.

2.2 Part B: Long-Answer Questions (8 marks each)

1. State and prove Lagrange's Theorem for finite groups.
2. (a) Prove that the set $\mathbb{Q} \setminus \{1\}$ of rational numbers other than 1 forms an abelian group with respect to the operation $*$ defined by $a * b = a + b - ab$. (4 marks)
   (b) Show that the direct product of two groups is a group. (4 marks)
3. (a) For a group $(A, *)$, show that $(A, *)$ is abelian if and only if $a^2 * b^2 = (a * b)^2$ for all $a, b \in A$. (4 marks)
   (b) Prove that the intersection of two subgroups of a group is a subgroup. (4 marks)
4. Prove that the symmetric group S_3 is a group under composition of permutations and find all its subgroups.
5. Prove that the set of positive integers under multiplication forms a semigroup but not a group.
6. Show that the set of 2×2 matrices with real entries under matrix addition forms a group.
7. Prove that if $\phi : G \to H$ is a group homomorphism, then the kernel of ϕ is a subgroup of G.
8. Let G be a group and H a subgroup. Prove that the number of left cosets of H in G equals the number of right cosets.
9. Prove that the permutation group S_4 has 24 elements and identify a subgroup of order 4.
10. Prove that the image of a monoid homomorphism is a submonoid.
11. Show that the alternating group A_3 is a subgroup of S_3 and determine its order.
12. Prove that the set of all permutations in S_n that fix a specific element forms a subgroup isomorphic to S_{n-1}.

3 Module III: Sets, Relations, Posets, and Lattice

3.1 Part A: Short-Answer Questions (2 marks each)

1. Show that the divisibility relation '$|$' is a partial ordering on $\mathbb{Z}^+$.
2. Define a bounded lattice and give an example.
3. Define an equivalence relation and provide an example.
4. Draw the Hasse diagram for the set $\{1, 2, 4\}$ with the divisibility relation.
5. Define the least upper bound (lub) and greatest lower bound (glb) in a poset.
6. Give an example of an irreflexive relation on the set $\mathbb{Z}$.
7. Define a complete lattice and provide an example.
8. Define a dual lattice and illustrate with an example.
9. Define an equivalence class and give an example.
10. Define a distributive lattice and provide an example.
11. Show that the relation "less than" on $\mathbb{R}$ is irreflexive.
12. Give an example of an irreflexive relation that is also antisymmetric.

3.2 Part B: Long-Answer Questions (8 marks each)

1. For $A = \{a, b, c\}$, let $P(A)$ be its power set and '$\leq$' the subset relation. Draw the Hasse diagram of $(P(A), \leq)$.
2. Let $A = \{1, 2, 3, \ldots, 12\}$. Let R be the equivalence relation on $A \times A$ defined by $(a, b)R(c, d)$ if and only if $a + d = b + c$. Prove that R is an equivalence relation and find the equivalence class of $(2, 5)$.
3. (a) Let R and S be two relations on a set A. If R and S are symmetric, prove that $R \cap S$ is symmetric. (4 marks)

   (b) Prove that a relation R on a set A is an equivalence relation if and only if it is reflexive, symmetric, and transitive. (4 marks)
4. Prove that the relation on $\mathbb{Z}$ defined by xRy if $x - y$ is divisible by 3 is an equivalence relation and find its equivalence classes.
5. Draw the Hasse diagram for the poset $(\{1, 2, 3, 6\}, |)$ and identify its maximal and minimal elements.
6. Prove that the power set of a set under the subset relation forms a complete lattice.
7. Prove that the lattice of subsets of a set is distributive.
8. For $A = \{1, 2, 3\}$, define a relation R on A that is a partial order but not a total order.

9. Prove that in a lattice, the greatest lower bound and least upper bound are unique if they exist.
10. Discuss the application of equivalence relations in database systems with an example.
11. Prove that the relation R on $\mathbb{Z}$ defined by xRy if $x < y$ is irreflexive and transitive.
12. Show that an irreflexive and antisymmetric relation on a set A can be extended to a partial order on A.

4 Module IV: Counting Theory

4.1 Part A: Short-Answer Questions (2 marks each)

1. Explain the Pigeonhole Principle.
2. In how many ways can the letters of the word ALLAHABAD be arranged?
3. How many arrangements are there for the letters in MASSASAUGA with all 4 A's together?
4. Find the coefficient of x^9y^3 in the expansion of $(x+y)^{12}$.
5. Define derangements and provide an example.
6. State the Principle of Inclusion-Exclusion for three sets.
7. How many ways can 5 distinct books be arranged on a shelf?
8. Find the number of ways to arrange the letters in "BOOK".
9. State the Binomial Theorem and give an example.
10. Explain the Rule of Product with an example.
11. How many ways can 3 identical red balls be distributed into 4 distinct boxes?
12. Find the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 10$.

4.2 Part B: Long-Answer Questions (8 marks each)

1. Find the number of integers between 1 and 1000 inclusive that are not divisible by 5, 6, or 8.
2. (a) Explain the Binomial Theorem. Find the coefficient of x^9y^3 in the expansion of $(x+y)^{12}$, $(x+2y)^{12}$, and $(2x-3y)^{12}$. (4 marks)
   (b) How many 5-digit numbers can be formed from the digits 1, 2, 3, 4, 5 without repetition? (i) How many are even? (ii) How many are even and greater than 30,000? (4 marks)
3. (a) At a party with 8 guests, each brings a gift and receives another gift, but not their own. How many ways can the gifts be distributed? (4 marks)

(b) Six exam papers, two mathematical, are scheduled one per day. How many orders are possible if: (i) The two mathematical papers are consecutive? (ii) They are not consecutive? (4 marks)

4. Prove the Pigeonhole Principle and show that selecting any five numbers from 1 to 8 ensures at least two sum to 9.
5. How many ways can 7 distinct people be seated in a row if 3 specific people must sit together?
6. Find the number of ways to distribute 5 identical balls into 3 distinct boxes with no box empty.
7. Using the Principle of Inclusion-Exclusion, find the number of integers from 1 to 100 divisible by 2, 3, or 5.
8. Calculate the number of derangements of the set $\{1, 2, 3, 4\}$.
9. Find the coefficient of x^5 in the expansion of $(2x-3)^8$ using the Binomial Theorem.
10. Discuss the application of the Pigeonhole Principle in computer science, such as in hash functions.
11. How many ways can 4 identical apples and 3 identical oranges be distributed into 5 distinct baskets?
12. Find the number of ways to select 6 items from 3 types of items (e.g., apples, oranges, bananas) with repetition allowed.

5 Module V: Fundamentals of Graphs

5.1 Part A: Short-Answer Questions (2 marks each)

1. Define a planar graph and give an example.
2. Define the chromatic number of a graph and illustrate with a bipartite graph.
3. Define an Euler graph and provide an example.
4. Define a cut set in a graph and explain with an example.
5. State the Four Color Theorem.
6. Define vertex connectivity and edge connectivity in a graph.
7. Define a bipartite graph and give an example.
8. Define a Hamiltonian circuit and provide an example.
9. Define a matching in a graph and give an example.
10. Define a vertex cover in a graph and provide an example.
11. Define graph isomorphism and give an example of two isomorphic graphs.
12. Define an edge cover in a graph and give an example.

5.2 Part B: Long-Answer Questions (8 marks each)

1. Determine if simple graphs with the degree sequences (a) $2, 3, 3, 3, 3, 3, 4, 5$, (b) $1, 3, 3, 4, 5, 6, 6$, and (c) $1, 2, 3, 3, 4, 5, 6$ are possible. If yes, draw the graphs.
2. Find all cut sets of a complete graph K_4 and determine its edge connectivity.
3. Write a detailed explanation of applications of graph coloring in computer science.
4. Prove that a connected graph with all vertices of even degree is Eulerian.
5. Prove that the complete graph K_3 is planar but K_5 is not.
6. Show that a graph is bipartite if and only if it contains no odd-length cycles.
7. Prove that the chromatic number of a bipartite graph is at most 2.
8. Discuss the application of Euler graphs in network routing with an example.
9. For a graph with 5 vertices and degree sequence $2, 2, 3, 3, 4$, determine if it has a Hamiltonian circuit.
10. Explain the Four Color Theorem and its significance in graph theory.
11. Find a maximum matching and a minimum vertex cover in a bipartite graph with partite sets $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ and edges $\{(u_1, v_1), (u_1, v_2), (u_2, v_2), (u_2, v_3), (u_3, v_3)\}$.
12. Prove that in any graph, the size of a maximum matching is at most the size of a minimum vertex cover.