UNIT 4 QUANTUM MECHANICS

LECTURE 3

Revision

Need of quantum mechanics

To overcome the limitation of classical mechanics

Classical mechanics failed to explain….

` 1) Stability of atom

2) Spectral distribution of black body radiation

Planck's quantum hypothesis

3) Origin of discrete spectra of atoms

4) Photoelectric effect

particle nature of light by Einstein

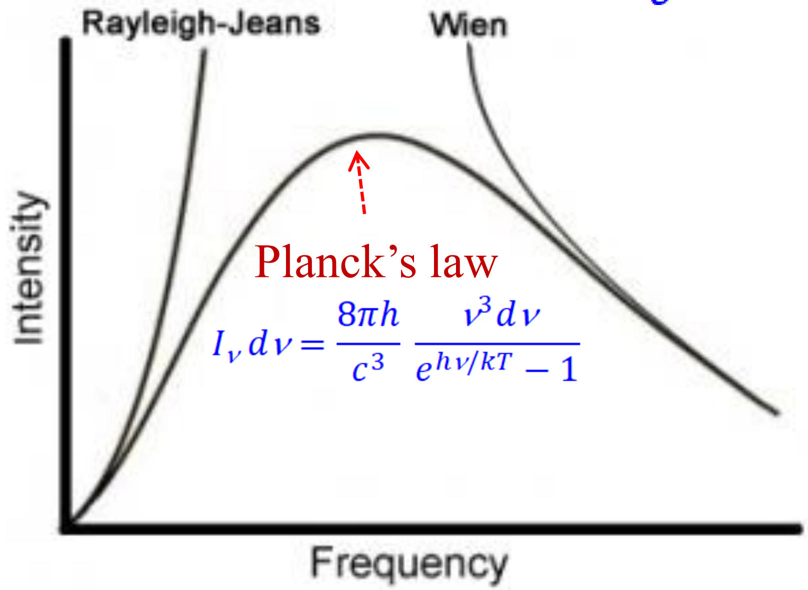
5) Compton effect

6) Raman effect

Black body radiation and Planck's hypothesis

$$I_\nu \, d\nu = \frac{8\pi \nu^2}{c^3} kT \, d\nu$$

$$I_\nu \, d\nu = \frac{A \nu^3}{c^4} e^{-B\nu/T} \, d\nu$$

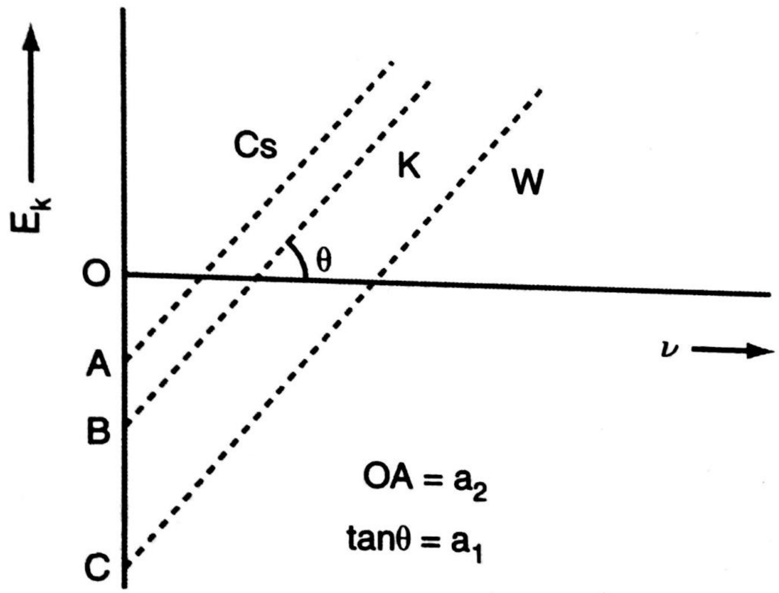


- ✓ Rayleigh-Jeans can be deduced from Planck's law for low frequency (large wavelength) and high temperature
- ✓ Similarly Wien's law can be deduced from Planck's law for high frequency (low wavelength) and low temperature

This discovery was a pioneering insight of modern physics and is of fundamental importance to quantum theory.

Photoelectric effect

Discovered by Hertz but explained by Einstein



Kinetic energy of the Ejected Electron E_k

$$E_k = h\nu - h\nu_0$$

Incident photon energy

Work function

This effect says about the emission/ejection of electrons from the metal surface upon irradiation with light waves

- **W**e knew the wave nature of light or electromagnetic radiation with the help of Maxwell’s equation- electromagnetic theory
- **T**hen assumed particle nature of light with the Planck’s black body radiation
- **W**ith Einstein’s photoelectric equation we experimentally proved particle nature of light

✓ *Importance of quantum mechanics and quantum/particle nature of light*

Wave nature of particles?? A mathematical relation connecting wavelength (λ) to momentum (p) De Broglie!

Dual nature of radiation

In the case of radiation (Plank's theory), we know Energy, $E = h\nu$

Now will go to Einstein special theory of relativity and that famous equation

$$E = mc^2$$

De Broglie hypothesized that the two energies would be equal

$$mc^2 = h\nu = \frac{hc}{\lambda} \Longrightarrow mc = \frac{h}{\lambda}$$

But *mc* is nothing but the momentum of photon, $p = \frac{h}{\lambda}$

.. by mixing Einstein's famous matter-energy relation with Planck's famous quantum oscillator theory.. Wavelength of the wave is related to the momentum of its particle through the Planck's constant ..

Dual nature of MATTER

If a wave can be so, then why not a particle?

de Broglie extended matter concept of radiation and applied to particles as well..

Because real particles do not travel at the speed of light, De Broglie used velocity (v) for the speed of light (c).

$$E = mv^2 = h\nu \quad \Rightarrow \quad mv = \frac{h}{\lambda}$$

$$\lambda = \frac{h}{mv} = \frac{h}{p}$$

λ is the de Broglie wavelength of the matter wave of the particle moving with velocity v and momentum p

The **de Broglie wavelength** is the **wavelength**, λ, associated with a massive particle and is related to its momentum, p, through the Planck constant, h: In other words, you can say that matter also behaves like waves.

1. If particle is accelerated through the kinetic energy

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE_k}}$$

2. If a charged particle having charge (q) is accelerated through electrostatic potential V

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mqV}}$$

3. If the particle having mass (m) is accelerated by means of thermal energy

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{3mkT}}$$

4. If the particle having rest mass (m_0) is moving with a velocity(v) comparable to the speed of light (c)

$$\lambda = \frac{h}{p} = \frac{h\sqrt{1 - (v/c)^2}}{m_0 v}$$

Rayleigh-Jeans law is deduced from Planck's radiation formula under the condition of

$$I_\nu d\nu = \frac{8\pi\nu^2}{c^3} kT\, d\nu$$

a) **High frequency and low temperature**
b) **Low frequency and high temperature**
c) **High frequency and high temperature**
d) **Low frequency and low temperature**

$$I_\nu d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}$$

Ans: B

Which of the following phenomena show the particle nature of light?

a) **Photoelectric effect**
b) **Interference**
c) **Diffraction**
d) **Polarization**

Ans: A

Einstein's photoelectric equation relate the kinetic energy of the ejected electron to the incident photon energy and -------

a) Work function
b) Wave function
c) Both a and b
d) None of these

Ans: A

Saturated photoelectric current depends

a) on the energy of the incident radiation
b) on the intensity of the incident radiation
c) Both a and b
d) None of the above

Ans: B

Velocities associated with de Broglie wave

We will see what is…

i. Phase or wave velocity **(v_p)**
ii. Group velocity **(v_g)**.. Wave packet
iii. Particle velocity **(v)**

Analogy: city marathon runners

- Initially it would appear that all of them are running at the same speed. As time passes, group spreads out (disperses)
- Because each runner in the group is running with different speed.
- If you think of phase velocity to be like the speed of an individual runner, then the group velocity is the speed of the entire group as a whole.

Phase or Wave Velocity (v_p) for EM wave

A wave travelling in the +x direction is given by

$$y = a\ \sin(\omega t - kx) \longrightarrow 1$$

Where **a** is the amplitude, ω $(=2\pi\nu)$ is the angular frequency and k $(=2\pi/\lambda)$ is the propagation constant

By definition the ratio of the angular frequency to the propagation constant is the phase velocity, $\mathbf{v_p}$

$$v_p = \frac{\omega}{k} \dashrightarrow 1a$$

Now we will see why $\mathbf{v_p}$ it is called wave velocity also ?

In equation 1 $(\omega t\text{-}kx)$ is called the phase of the wave motion. And is a constant for plane wave

$$\omega t - kx = constant$$

$$\frac{d}{dt}(\omega t - kx) = 0$$

$$\omega - k\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = \frac{\omega}{k}$$

But dx/dt is the velocity of the wave.. And same as equation 1**. so phase velocity is nothing but the wave velocity**

$$v_p = \frac{\omega}{k} = \frac{2\pi\nu}{2\pi/\lambda} = \nu\lambda = c$$

For an electromagnetic wave in vacuum.

Phase or Wave Velocity (v_p) for Matter wave

We have $v_p = \nu\lambda;\ \mathrm{E} = \mathrm{h}\nu;\ or\ \ \nu = E/h$ ---------------------> 2

According to de Broglie $\lambda = \frac{\mathrm{h}}{\mathrm{p}} = \frac{\mathrm{h}}{\mathrm{mv}}$ ---------------------> 3

From 2 and 3, phase velocity for the de Broglie wave

$$v_p = \nu\lambda = \frac{\mathrm{E}}{\mathrm{h}} \times \frac{\mathrm{h}}{\mathrm{mv}} = \frac{\mathrm{mc}^2}{\mathrm{mv}} \qquad \therefore v_p = \frac{\mathrm{c}^2}{\mathrm{v}} \qquad \text{--------------------->}\ 4$$

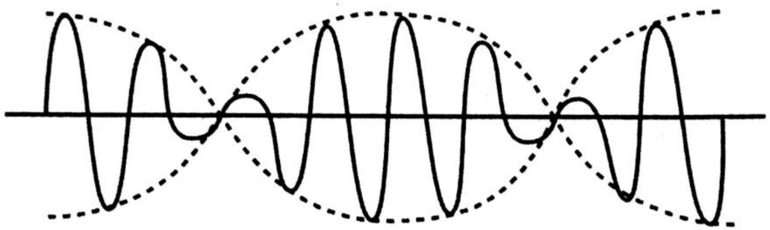
Since v << c, eqn.(4) implies that phase velocity of de Broglie wave of the particle moving with velocity ‘v’ is greater than c, the speed of light!!

Group Velocity (v_g) of De Broglie wave

V_g, introduced to overcome the difficulty of $v_p > c$ of matter wave: Here each moving particle is associated with a group of wave or wave packet rather than a single wave.

$$y_1 = a \sin(\omega_1 t - k_1 x)$$

$$y_2 = a \sin(\omega_2 t - k_2 x)$$



$$y = y_1 + y_2 = a\,[\sin(\omega_1 t - k_1 x) + \sin(\omega_2 t - k_2 x)]$$

$$\sin A + \sin B = 2\,Sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$y = 2a\,sin\left[\frac{(\omega_1 + \omega_2)t}{2} - \frac{(k_1 + k_2)x}{2}\right] cos\left[\frac{(\omega_1 - \omega_2)t}{2} - \frac{(k_1 - k_2)x}{2}\right]$$

$$\omega = \frac{\omega_1 + \omega_2}{2} \qquad k = \frac{k_1 + k_2}{2} \qquad \Delta\omega = \omega_1 - \omega_2$$

Group Velocity (v_g) of De Broglie wave

$$\therefore y = 2a\cos\left[\frac{\Delta\omega t}{2} - \frac{\Delta k x}{2}\right]\sin(\omega t - kx) \dashrightarrow \quad 5$$

Eqn.5 has two parts,

(1) A wave with angular frequency ω, propagation constant k and has the velocity v_p , given by

$$v_p = \frac{\omega}{k}$$

And is the phase velocity

(2) Another wave with angular frequency $\Delta\omega/2$, propagation constant $\Delta k/2$ and has the velocity v_g , given by

$$v_g = \frac{\Delta\omega}{\Delta k}$$

And is the group velocity.. Velocity of the wave packet.. Envelop showed by the dotted lines in the figure

Group Velocity (v_g) of De Broglie wave

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{\partial\omega}{\partial k} = \frac{\partial(2\pi\nu)}{\partial\,(2\pi/\lambda)}$$

$$= \frac{\partial(\nu)}{\partial\,(1/\lambda)} = -\lambda^2 \frac{\partial \nu}{\partial \lambda}$$

So group velocity is given by

$$\therefore\ v_g = -\frac{\lambda^2}{2\pi}\frac{\partial\omega}{\partial\lambda}$$

Now we will see relation between v_p and v_g

Relation between v_p and v_g

$$v_p = \frac{\omega}{k} \qquad v_g = -\frac{\lambda^2}{2\pi}\frac{\partial \omega}{\partial \lambda}$$

But $$k = \frac{2\pi}{\lambda} \Rightarrow dk = -\frac{2\pi}{\lambda^2}d\lambda$$

so $$\frac{k}{dk} = -\frac{\lambda}{d\lambda}$$

$$v_g = \frac{d\omega}{dk} = \frac{d(v_p k)}{dk}$$

$$v_g = v_p + k\frac{dv_p}{dk}$$

$$v_g = v_p + \left(-\frac{\lambda}{d\lambda}\right)dv_p \Rightarrow v_g = v_p - \lambda\frac{dv_p}{d\lambda} \dashrightarrow 6$$

That is group velocity is less than the phase velocity in a dispersive medium where v_p is a function of k or λ.. And for a non-dispervive medium v_p is independent of k or λ, equation 6 gives

$$v_g = v_p \quad \text{because} \quad \frac{dv_p}{d\lambda} = 0$$

Relation between v_g and particle velocity (v) of De Broglie wave

Consider a material particle of rest mass m_0. Let its mass be m when moving with a velocity v. then it energy is given by

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

We know that

$$\omega = 2\pi\nu = \frac{2\pi E}{h} = \frac{2\pi mc^2}{h}$$

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi mv}{h}$$

Substitute the value of m from the above into the last two equation of ω and k

Relation between v_g and particle velocity (v) of De Broglie wave

$$\omega = \frac{2\pi m_0 c^2}{h\sqrt{1 - v^2/c^2}} \qquad k = \frac{2\pi m_0 v}{h\sqrt{1 - v^2/c^2}}$$

and differentiation with respect to the velocity of the particle v

$$\frac{d\omega}{dv} = \frac{2\pi m_0 v}{h\,(1 - v^2/c^2)^{3/2}} \qquad \frac{dk}{dv} = \frac{2\pi m_0}{h\,(1 - v^2/c^2)^{3/2}}$$

But Vg is defined as

$$v_g = \frac{d\omega}{dk} = \frac{d\omega/dv}{dk/dv}$$

$$v_g = v$$

Phase,wave group & particle velocities

Phase velocity (v_p) of the wave is larger than the group velocity (v_g) of the waves?

It depends on the nature of the medium.

1) $v_p = v_g$ for non-dispersive medium- velocity not depend on wave length. Examples sound waves in air and electromagnetic waves in vacuum.

2) $v_p > v_g$ for normal-dispersive medium- electromagnetic radiation in medium where refractive index depends on the wavelength and hence velocity of EM changes in the medium.

3) $v_p < v_g$ for anomalous-dispersive medium..This we see in matter-wave cases

For non-dispersive medium phase velocity (v_p) is independent of the wavelength of the wave and hence group velocity v_g is

a) $v_g > v_p$
b) $v_g < v_p$
c) $v_g = v_p$
d) none of the above

Ans: C

For dispersive medium phase velocity (v_p) is dependent of the wavelength of the wave and hence group velocity v_g is

a) $v_g > v_p$
b) $v_g < v_p$
c) $v_g = v_p$
d) none of the above

Ans: B

Matter-wave is associated with moving particle. In that case, the particle velocity is equal to the group velocity. True or False?

(a) True
(b) False

Ans: A

UNIT 4-Quantum Mechanics

Text Book: ENGINEERING PHYSICS by HITENDRA K MALIK AND A K SINGH, MCGRAW HILL EDUCATION, 1st Edition, (2009)

References:

- ENGINEERING PHYSICS by B K PANDEY AND S CHATURVEDI, CENGAGE LEARNING, 1st Edition, (2009).
- ENGINEERING PHYSICS by D K BHATTACHARYA, POONAM TONDON OXFORD UNIVERSITY PRESS.