THEORY OF DISTRIBUTION

SSTA021

DETAILS

Lecturers	: Mr MAJA TF
	Dr D MAPOSA
Office Number	: 2027 & 2004C
	: Mathematics Building, 2nd floor
Phone Number	: 015 268 2867
Email	: tshepo.maja@ul.ac.za
	: daniel.maposa@ul.ac.za

Specific Learning Outcomes

- State Axioms and Laws of probability.
- State and prove Law of Total Probability and the Bayes' Theorem.
- Distinguish between a Discrete and a Continuous Random Variable.
- Understand and apply the special Probability Mass Functions and Probability Density Functions (Special PDFs and PMFs).
- State and prove the Means, Variances and Moment Generating Functions of the Special PMFs and PDFs.
- Calculate Univariate, Bivariate, Marginal and Conditional probabilities.

CONT'S

- Calculate Expectations and Variances of given functions of the jointly distributed Random Variables.
- Determine the Characteristic Functions of Random Variables.
- Determine the Covariance and Correlation of several Random Variables.
- Determine the distributions of random functions and random variables using the following techniques: Moment Generating Function technique, Cumulative Distribution technique and the Transformation technique.
- Understand the basic techniques and procedures of Mathematical Statistics.

MODULE ASSESSMENT

A module mark will be obtained from continuous assessment based on assignments, quizzes and tests. The contribution of assessments towards a module mark is as follows:

- ❖ Average of assignments : 15%
- ❖ Average of quizzes : 15%
- ❖ Average of tests : 70%

TEST DATES

- TEST 1 : MARCH 2020
- TEST 2 : APRIL 2020

Take note of the following:

- In this course, students are encouraged to attend all the lectures and all tutorials. If there will be no lecture or tutorial, students will know in advance.
- The exact dates for tests, quizees and assignments will be given in class.
- Students must always bring along this document whenever they attend all lectures and all tutorials.
- During lectures, students are not expected to recopy definitions, theorems, concepts or any materials that are readily available in this document.

CONT'S

- However, this document serves only as a brief summary of the course; students are therefore expected to copy formal notes, proofs of theorems and possible solutions to examples and exercises that will be done in class.
- The tutorial questions will be arranged separately and forwarded to students.
- Not only the examples and exercises included in this document will be used for assessment purposes. Students are therefore advised to reinforce this module by practicing several possible examples and exercises from the textbooks, e.t.c
- Unless amended, students should only consult during the allocated consultation slots and no students will consult TWO days before the test.

CHAPTER 1: INTRODUCTORY PROBABILITY

- This unit introduces the basic concepts of probability. It outlines rules and techniques for assigning probabilities to events.
- If the probability of an event is high, one would expect that it would occur rather than not occur. If the probability of rain is 95%, it is more likely that it would rain than not rain.
- Probability principles are the foundation for the probability distribution, the concept of mathematical expectation and the binomial and Poisson distribution

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WHAT IS PROBABILITY

- **Probability** is a way of expressing the likely occurrence of a particular event as a number between 0 and 1.
  - In order to determine any probability, you first need to obtain data. We can obtain data through an experimentation, observation or experience.
- which may be assigned to an event in order to indicate the likelihood of that particular event.
  - Note: Probability Theory is useful for solving the problems involving uncertainty.

DEFINITIONS

- **Experiment** is any situation whose outcomes cannot be predicted with certainty.
  - ➢Examples of an experiment include rolling a die, flipping a coin.
  - ➢By an outcome or simple event we mean any result of the experiment. For example, the experiment of rolling a die yields six outcomes, namely, the outcomes 1,2,3,4,5, and 6.
- An experiment whose outcome is uncertain before it is performed is called a **Random Experiment**.

Cont's: Sample Space and Events

- **Sample Space (S)** is a collection of all possible outcomes of a random experiment.
- **Element\Member of a sample space** is each outcome in a sample space.
- **Event** is a subset of the sample space or population.
- **Simple Event** It is an event that results in exactly one outcome.

CONT'S

- **Compound Event** It is an event that results in more than one outcome.
- **Rare\Impossible Event** It is an event which cannot occur or event whose probability or likelihood of occurrence is zero.
- **Sure\Certain Event** It is an event which contains all the elements or event whose probability or likelihood of occurrence is 1 or 100%.

SET THEORY

Intersection of Events $\boldsymbol{A}$ and $\boldsymbol{B}$

The **intersection** of two events A *and* B, denoted by the symbol $\boldsymbol{A \cap B}$, is the event containing all elements that are common to A *and* B

Mutually Exclusive/Disjoint Events

Two events A *and* B are **mutually exclusive**, or **disjoint**, if $\boldsymbol{A} \cap \boldsymbol{B} = \emptyset$, that is, if A and B have no elements in common.

Union of Events $\boldsymbol{A}$ and $\boldsymbol{B}$

The **union** of the two events A and B, denoted by the symbol $\boldsymbol{A} \cup \boldsymbol{B}$, is the event containing all the elements that belong to A *and* B or both.

Complement of an Event

The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol A', $\overline{A}$.

Venn Diagrams

Venn Diagram is a tool used to portray graphically the concepts of union, intersection, complement and disjoint sets.

A review of set notation

Description	Notation	Extra Information
The universal set	S	Includes all elements
Subset	$A \subset B$	Not the same as $B \subset A$
Empty (null) set	$\emptyset$	$\emptyset$ is a set, not an element
Union of two sets	$A \cup B$	Two shaded circles
Intersection of sets	$A \cap B$	Circles must overlap
Complement of set	$\overline{A}$	Careful: $\overline{A} \cup \overline{B} \neq \overline{A \cup B}$
Disjoint (Mutually exclusive)	$A \cap B = \emptyset$	Two circles do not overlap

LAWS IN SET THEORY

De Morgan's law (a) De Morgan's law (b)	$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$
Associative law (a) Associative law (b)	$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$
Distributive law (a) Distributive law (b)	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Commutative law (a) Commutative law (b)	$A \cup B = B \cup A$ $A \cap B = B \cap A$

EXAMPLE 1

1. Use Venn diagrams to verify the distributive laws.
2. Use Venn diagrams to verify DeMorgan’s second law.

EXAMPLE 2

An experiment involves tossing a pair of dice, one green and one red, and recording the numbers that come up. If x equals the outcome on the green die and y the outcome on the red die, describe the sample space S

a) by listing the elements (x, y);

b) list the elements corresponding to the event A that the sum is greater than 8;

c) list the elements corresponding to the event B that a 2 occurs on either die;

d) list the elements corresponding to the event C that number greater than 4 comes up on the green die;

CONT'S

e) list the elements corresponding to the event $A \cap C$;

f) list the elements corresponding to the event $A \cap B$;

g) list the elements corresponding to the event $B \cap C$;

h) construct a Venn diagram to illustrate the intersections and unions of the events A, B, and C.

.

$$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),$$
$$(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),$$
$$(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),$$
$$(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),$$
$$(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),$$
$$(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}.$$

EXAMPLE 3

If $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $A = \{0, 2, 4, 6, 8\}, B = \{1, 3, 5, 7, 9\}$, $C = \{2, 3, 4, 5\}$, and $D = \{1, 6, 7\}$, list the elements of the sets corresponding to the following events:

CONT'S

$a. A \cup C$

b. $A \cap B$

$c. \overline{C}$

$d. (\overline{C} \cap D) \cup B$

$e. A \cap C \cap \overline{D}$

PROBABILITY

Definition

Probability is a real-valued set function P that assigns, to each event A in the sample space S, a number $P(A) = \frac{n(A)}{n(S)}$, called the probability of the event A, such that the following properties are satisfied:

a. $0 \leq P(A) \leq 1;$

b. $P(\emptyset) = 0$

c. $P(S) = 1;$

d. if $A_1, A_2, A_3, \ldots$ are events and $A_i \cap A_j = \emptyset$, $i \neq j$, then $P(A_1 \cup A_2 \cup \cdot \cdot \cdot \cup A_k) = P(A_1) + P(A_2) + \cdot \cdot \cdot + P(A_k)$ for each positive integer k, and $P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$ for an infinite, but countable, number of events.

Axioms of Probability

- **Axiom 1:** The probability of an event A is nonnegative, i.e.$P(A) \geq 0$

- **Axiom 2:** The probability of a sure event is 1, i.e. $P(S) = 1$.

- **Axiom 3:** The probability of a union of disjoint or mutually exclusive events is equal to the sum of the probabilities of those events, i.e. if $A_1, A_2, \ldots.$ are disjoint, then $\bigcap_{i=1}^{\infty} A_i = \emptyset$ so that

$$P[\bigcup_{i=1}^{\infty} A_i] = \sum_{i=1}^{\infty} P(A_i)$$

$A \subseteq \not B$

Key laws of a probability

Law 1: If $A = \emptyset$ (rare or impossible event), then $P(A) = P(\emptyset) = 0$.

Law2: If $A_i, i = 1, \dots.., n$ are disjoint events, then $P[\cup_{i=1}^{n} A_i] = \sum_{i=1}^{n} P(A_i)$

Law 3: For any event A, $\overline{A} = 1 - P(A)$.

Law 4: For any event A, $0 \leq P(A) \leq 1$..

Law 5: If $A\ is\ subset\ of\ B$, then $P(A) \leq P(B)$.

Law 6: For any two events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Conditional Probability

The **conditional probability** of an event *A*, given that event *B* has occurred, is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Provided that $P(B) > 0$.

EXAMPLE

The probability that a regularly scheduled flight departs on time is $P(D) = 0.83$; the probability that it arrives on time is $P(A) = 0.82$; and the probability that it departs and arrives on time is $P(D \cap A) = 0.78$. Find the probability that a plane

a. arrives on time, given that it departed on time

b. departed on time, given that it has arrived on time.

EXAMPLE

A study of the posttreatment behavior of a large number of drug abusers suggests that the likelihood of conviction within a two-year period after treatment may depend upon the offenders education. The proportions of the total number of cases falling in four education–conviction categories are shown in the following table:

CONT'S

Status within 2 years after treatment			
Education	**Convicted**	**Not convicted**	**Total**
10 years or more	0.10	0.30	0.40
9 years or less	0.27	0.33	0.60
Total	0.37	0.63	1.00

Suppose that a single offender is selected from the treatment program. Define the events:

A: The offender has 10 or more years of education.

B: The offender is convicted within two years after completion of treatment.

Find the following:

a. $P(A)$.

b. $P(B)$.

c. $P(A \cap B)$.

d. $P(A \cup B)$.

e. $P(\overline{A})$.

f. $P(\overline{A \cup B})$.

g. $P(\overline{A \cap B})$.

h. $P(A|B)$.

i. $P(B|A)$.

Independent Events

Two events *A* and *B* are **independent** if and only if

$$P(A \cap B) = P(A) \times P(B) \; or \; P(B|A) \; = \; P(B) \; or \; P(A|B)$$
$$= \; P(A)$$

Assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

EXAMPLE

- Let A and B be independent events with $P(A) = 1/4$ and $P(B) = 2/3$. Compute

a. $P(A \cap B)$

b. $P(A \cap \overline{B})$

c. $P(\overline{A} \cap \overline{B})$

d. $P(A \cup B)$

e. $P(\overline{A} \cap B)$.

EXAMPLE

Suppose that a foreman must select one worker from a pool of four available workers (numbered 1, 2, 3, and 4) for a special job. He selects the worker by mixing the four names and randomly selecting one. Let A denote the event that worker 1 or 2 is selected, let B denote the event that worker 1 or 3 is selected, and let C denote the event that worker 1 is selected. Are A and B independent? Are A and C independent?

EXAMPLE

1. If $P(A) = 0.8, P(B) = 0.5,$ and $P(A \cup B) = 0.9,$ are A and B independent events? Why or why not?

2. Let $P(A) = 0.3$ and $P(B) = 0.6$.

   a. Find $P(A \cup B)$ when A and B are independent.

   b. Find $P(A|B)$ when A and B are mutually exclusive.

Theorem

If A and B are independent events, then the following pairs of events are also independent:

$a.$ A and $\overline{B}$

$b.$ $\overline{A}$ and B

$c.$ $\overline{A}$ and $\overline{B}$

The law of Total Probability

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0, \forall_i = 1,2, \ldots, k,)$, then for any event A in S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i) P(A|B_i)$$

EXAMPLE

A company buys microchips from three suppliers—I, II, and III. Supplier I has a record of providing microchips that contain 10% defectives; Supplier II has a defective rate of 5%; and Supplier III has a defective rate of 2%. Suppose 20%, 35%, and 45% of the current supply came from Suppliers I, II, and III, respectively. If a microchip is selected at random from this supply, what is the probability that it is defective?

EXAMPLE

In a certain assembly plant, three machines, B_1, B_2 and B_3, make 30%, 45%, and 25%, respectively, of the products. It is known from past experience that 2%, 3%, and 2% of the products made by each machine, respectively, are defective. Now, suppose that a finished product is randomly selected. What is the probability that it is defective?

Bayes Theorem

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0, \forall_i = 1,2, \ldots, k,)$, then for any event A in S such that $P(A) \neq 0$,

$$P(B_r|A) = \frac{P(B_r \cap A)}{\sum_{i=1}^{k} P(B_i \cap A)} = \frac{P(B_r)P(A|\, B_r)}{\sum_{i=1}^{k} P(B_i)P(A|B_i)}$$

For $r = 1,2, \ldots, k$

EXAMPLE

Consider again the information of a selected microchip. If a randomly selected microchip is defective, what is the probability that it came from Supplier B_{II} ?

Example

With reference to machine problem, if a product was chosen randomly and found to be defective, what is the probability that it was made by machine B_3?

EXAMPLE

A manufacturing firm employs three analytical plans for the design and development of a particular product. For cost reasons, all three are used at varying times. In fact, plans 1, 2, and 3 are used for 30%, 20%, and 50% of the products, respectively. The defect rate is different for the three procedures as follows:

$$\boldsymbol{P(D|P1) = 0.01, P(D|P2) = 0.03, P(D|P3) = 0.02},$$

where $P(D|Pj)$ is the probability of a defective product, given plan j. If a random product was observed and found to be defective, which plan was most likely used and thus responsible?

Methods of Enumeration: Counting Techniques

In this section, we develop counting techniques that are useful in determining the number of outcomes associated with the events of certain random experiments.

Definition: Functional Notation

If , the number $n!$, reads n factorial is defined by

$$n! = n(n-1)\ldots(2)(1)$$

We define $0! = 1$. That is, we say that zero positions can be filled with zero objects in one way.

Definition: Permutation (Ordered Selection)

Permutation *is an ordered arrangement of* r *distinct objects.*

❖ The number of ways of ordering n distinct objects taken *r* at a time will be designated by the symbol P_r^n

Theorem

The number of permutations of n distinct objects taken r at a time is given by

$$nP_r = \frac{n!}{(n-r)!}, r = 0,1, \dots, n$$

Example

A small company has 12 account managers. Three potential customers have been identified and each customer has quite different needs. The company's director decides to send an account manager to visit each of the potential customers and considers the customers' needs in making his selection. How many ways are there for him to assign three different account managers to make the contacts?

Example

The number of possible four-letter code words, selecting from the 26 letters in the alphabet, in which all four letters are different

EXAMPLE

The names of 3 employees are to be randomly drawn, without replacement, from a bowl containing the names of 30 employees of a small company. The person whose name is drawn first receives $100, and the individuals whose names are drawn second and third receive $50 and $25, respectively. How many sample points are associated with this experiment?

Definition: Combinations (Unordered Selection)

A combination of n distinct objects taken r at a time is an unordered selection of r distinct objects from a total of n distinct objects.

Theorem

The number of combinations of n distinct objects taken r at a time is given by

$$nC_r = \frac{n!}{r!\,(n-r)!}\,, r = 0,1,\ldots,n$$

EXAMPLE

Most states conduct lotteries as a means of raising revenue. In Florida's lottery, a player selects six numbers from 1 to 53. For each drawing, balls numbered from 1 to 53 are placed in a hopper. Six balls are drawn from the hopper at random and without replacement. To win the jackpot, all six of the player's numbers must match those drawn in any order. How many winning numbers are possible?

EXAMPLE

The number of possible 5-card hands (in 5-card poker) drawn from a deck of 52 playing cards is.

EXAMPLE

Find the number of ways of selecting two applicants out of five.

Example

If A and B are independent events. Then $\overline{A}$ and B are independent. Similarly A and $\overline{B}$ are independent.

EXAMPLE

If events A, B and C are independent, show that:

$$P[(A \cap B) \cup C] = P(A)P(B)P(\overline{C}) + P(C)$$

Exercise

If A and B are independent events, show that $\overline{A}$ and $\overline{B}$ are also independent.

CHAPTER 2: RANDOM VARIABLES

Contents

Definition: Random Variable

Random variable is a function that associates a real number with each element in the sample space.

- We shall use a capital letter, say X, to denote a random variable and its corresponding small letter, x in this case, for one of its values.

Discrete sample space

Discrete sample space is a sample space that contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers.

Continuous sample space

Continuous sample space is a sample space that contains an infinite number of possibilities equal to the number of points on a line segment.

Discrete random variable

A random variable is called a **discrete random variable** if its set of possible outcomes is countable.

Continuous random variable

Continuous random variable is when a random variable can take on values on a continuous scale.

DISCRETE PROBABILITY DISTRIBUTIONS

DEFINITION: PROBABILITY MASS FUNCTION

The set of ordered pairs $(x, P(X = x))$ is a probability function, probability mass function, or probability distribution of the discrete $rv\, X$, if for each possible outcome x , the following conditions are fulfilled:

$a.\ P(X = x) \geq 0$

$b.\ \sum_{\forall x} P(X = x) = 1$

$c.\ P(X = x) = P_i, \forall_i$

EXAMPLE

A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Definition: Cumulative Distribution Function (CDF)

The **cumulative distribution function** $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)\,, for - \infty < x < \infty.$$

where $f(t)$ is the value of the probability distribution of X at t is called the distribution function $(df\ )$ or the Cumulative Distribution Function of X.

EXAMPLE

If the probability of a random variable X with space $R_X = \{1, 2, 3, \ldots, 12\}$ is given by

$$f(x) = k\,(2x - 1),$$

then, what is the value of the constant k?

EXAMPLE

If the probability density function of the random variable X is given by

$$\frac{1}{144}(2x - 1) for\ x = 1, 2, 3, \ldots, 12$$

then find the cumulative distribution function of X.

CONTINUOUS PROBABILITY DISTRIBUTIONS

Definition: Probability Density Function ($\boldsymbol{PDF}$)

The function $f(x)$ is a Probability Density Function (PDF) for the continuous $rv\ X$, defined over the set of real numbers R, if the following three conditions are fulfilled:

a. $f(x) \geq 0, \forall_x \epsilon R$

b. $\int_{-\infty}^{\infty} f(x)dx = 1$

c. $P(a < X < b) = \int_a^b f(x)dx$

EXAMPLE

Suppose that the error in the reaction temperature, in degrees Celsius, for a controlled laboratory experiment is a continuous $rv\ X$, having the pdf

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & otherwise \end{cases}$$

a. Verify that $f(x)$ is a legitimate pdf.

b. Find $P(0 < x < 1)$

EXERCISE

Given $f\ (y) = cy^2, 0 \leq y \leq 2$, and $f\ (y) = 0$ elsewhere,

a. Find the value of *c* for which $f\ (y)$ is a valid density function.

b. Find $P(1 \leq Y \leq 2)$

c. find $P(1 < Y < 2)$.

Definition: CDF, CONTINUOS $\boldsymbol{rv}\ \boldsymbol{X}$

The $CDF, F(x)$ of a continuous $rv\ X$ with $PDF\ f(x)$ is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt\ , -\infty < x < \infty$$

EXAMPLE

For the density function of previous example, find

$F(x)$

THEOREM

The values of $F(x)$ of the distribution function of a continuous rv satisfy the following conditions:

$a.\ P(a < X < b) = F(b) - F(a)$

$b.\ f(x) = \frac{dF(x)}{dx}$

EXAMPLE

For the pdf of previous example, find $F(x)$ and use it to evaluate

$$P(0 < X < 1)$$

EXAMPLE

The Department of Energy (DOE) puts projects out on bid and generally estimates what a reasonable bid should be. Call the estimate *b*. The DOE has determined that the density function of the winning (low) bid is

$$f(y) = \begin{cases} \frac{5}{8b}, \frac{2}{5} b \leq y \leq 2b \\ 0, elsewhere \end{cases},$$

Find *F*(*y*) and use it to determine the probability that the winning bid is less than

the DOE's preliminary estimate *b*.

MEAN OF A RANDOM VARIABLE: DISCRET AND CONTINUOUS

Let X be a random variable with probability distribution $f(x)$. The mean, or expected value, of X is

$\mu = E(X) = \sum_x xf(x)$, If X is discrete,

and

$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$, If X is continuous.

VARIANCE OF A RANDOM VARIABLE: DISCRET AND CONTINUOUS

Let X be a random variable with probability distribution $f(x)$ and mean μ. The variance of X is

$$\sigma^2 = [(X - \mu)^2] = \sum_x (x - \mu)^2 f(x)$$

If X is discrete, and

$$\sigma^2 = [(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

If X is continuous.

EXAMPLE

Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday. The probability distribution for company A is

x	1	2	3
$f(x)$	0.3	0.4	0.3

and that for company B is

EXAMPLE CON'T

x	**0**	**1**	**2**	**3**	**4**
$f(x)$	**0.2**	**0.1**	**0.3**	**0.3**	**0.1**

Show that the variance of the probability distribution for company B is greater than that for company A.

Alternative Variance Formula

An alternative and preferred formula for finding σ^2, which often simplifies the calculations, is stated in the following theorem.

Theorem

The variance of a random variable X is

$$\boldsymbol{\sigma^2 = E\left(X^2\right) - \mu^2.}$$

PROVE

EXAMPLE

Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. The following is the probability distribution of X.

$\boldsymbol{x}$	**0**	**1**	**2**	**3**
$\boldsymbol{f(x)}$	**0.51**	**0.38**	**0.10**	**0.01**

Using alternative variance formula, calculate σ^2.

EXAMPLE

The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density

$$f(x) = \begin{cases} 2(x-1), 1 < x < 2 \\ 1, \qquad\qquad elsewhere \end{cases}$$

Find the mean and variance of X.

THEOREM

Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$\boldsymbol{\mu_{g(X)} = E[g(x)] = \sum_x g(x)f(x)}$, If X is discrete, and

$\boldsymbol{\mu_{g(X)} = E[g(x)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx}$, If X is continuous.

EXAMPLE

Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

EXAMPLE

Let X be a random variable with density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & elsewhere \end{cases}$$

Find the expected value of $g(X) = 4X + 3$.

THEOREM

Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$\boldsymbol{\sigma^2_{g(X)} = E\left\{\left[g(X) - \mu_{g(X)}\right]^2\right\} = \sum_x \left[g(x) - \mu_{g(X)}\right]^2 f(x)}$, If X is discrete, and

$\boldsymbol{\sigma^2_{g(X)} = E\left\{\left[g(X) - \mu_{g(X)}\right]^2\right\} = \int_{-\infty}^{\infty} \left[g(x) - \mu_{g(X)}\right]^2 f(x)\, dx}$, If X is continuous

EXAMPLE

Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution

$\boldsymbol{x}$	**0**	**1**	**2**	**3**
$\boldsymbol{f(x)}$	$\boldsymbol{\frac{1}{4}}$	$\boldsymbol{\frac{1}{8}}$	$\boldsymbol{\frac{1}{2}}$	$\boldsymbol{\frac{1}{8}}$

Let X be a random variable having the density function

EXAMPLE

Let X be a random variable having the density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & elsewhere \end{cases}$$

Find the variance of the random variable $g(X) = 4X + 3$.

MEANS AND VARIANCES OF LINEAR COMBINATIONS OF RANDOM VARIABLES

THEOREM

For any random variable X and constants a and b,

1. $\boldsymbol{E(aX + b) = aE(X) + b}$.

2. $\boldsymbol{V(aX + b) = a^2V(X)}$.

Example

1.Prove the $E(aX + b) = aE(X) + b$ and $V(aX +$

EXAMPLE

Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X = x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

EXAMPLE

Let X be a random variable having the density function

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0, & elsewhere \end{cases}$$

Find the variance of the random variable $g(X) = 4X + 3$.

THEOREM

If the space R_X of the random variable X is given by
R_X =$\{x_1 < x_2 < x_3 < \cdot\cdot\cdot < x_n\}$, then

$$f(x_1) = F(x_1)$$
$$f(x_2) = F(x_2) - F(x_1)$$
$$f(x_3) = F(x_3) - F(x_2)$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$
$$f(x_n) = F(x_n) - F(x_{n-1}).$$

This Theorem tells us how to find the probability density function (pdf) given the cumulative distribution function (cdf).

THEOREM

$$\mathrm{P}(X \leq x_i) = F(x_i)$$

$$\mathrm{P}(X = x_i) = F(x_i) - F(x_i{}^-)$$

$$\mathrm{P}(X \geq x_i) = 1 - F(x_i{}^-)$$

$$\mathrm{P}(X < x_i) = F(x_i{}^-)$$

EXAMPLE: CDF DISCRETE CASE

Find the probability density function of the random variable X whose cumulative distribution function is **GIVEN IN THE NEXT SLIDE**

Also, find $(a) P(X \leq 3)$.

CDF

$$F(x) = \begin{cases} 0 & if\ x < -1 \\ 0.25\ if & -1 \leq x < 1 \\ 0.50\ if & 1 \leq x < 3 \\ 0.75\ if & 3 \leq x < 5 \\ 1\quad if & x \geq 5 \end{cases}$$

CHAPTER 3

SPECIAL PROBABILITY DISTRIBUTIONS

DISCRETE PROBABILITY DISTRIBUTION FUNCTIONS (PMFs)

- Discrete Uniform Distribution
- Bernoulli Distribution
- Binomial Distribution
- Poisson Distribution
- Geometric Distribution
- Hypergeometric Distribution
- Negative Binomial (Pascal) Distribution

Bernoulli Distribution

Strictly speaking, the Bernoulli process must possess the following properties:

1. The experiment consists of repeated trials.
2. Each trial results in an outcome that may be classified as a success or a failure.
3. The probability of success, denoted by p, remains constant from trial to trial.
4. The repeated trials are independent.

DEFINITION:BERNOULLI DISTRIBUTION

Let P be the probability of success in a Bernoulli experiment and let $X = 1$ if the experiment results in a success and $X = 0$ otherwise. Then X has a Bernoulli distribution with a probability mass function that is given as follows:

$$P(X = x) = p^x q^{1-x}, x = 0,1$$

With $E(X) = p$ and $\sigma^2 = pq$

PROVE

1. MEAN
2. VARIANCE

Binomial distribution

A random variable X has a Binomial Distribution and it is referred to as a Binomial random variable if and only if its probability distribution is given by

$$P(X = x) = \binom{n}{x} p^x q^{n-x}, x = 0,1, \dots n$$

Where n is fixed, p is constant through out the trials and trials are independent

EXAMPLE

The probability that a certain kind of component will survive a given shock test is 75%. Find the probability that exactly 2 of the next 4 components will survive a shock test. Determine the mean, the variance and the standard deviation of the components that will survive a shock.

THEOREM

The mean and variance of the binomial distribution $b(x;\, n, p)$ are

$$\mu = np$$

and

$$\sigma^2 = npq.$$

EXERCISE

The probability that a patient recovers from a rare blood disease is 0.4. If 15 people are known to have contracted this disease, what is the probability that

a. at least 10 survive
b. from 3 to 8 survive
c. exactly 5 survive?

POISSON DISTRIBUTION

A random variable X has a Poisson Distribution if and only if its probability mass function is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0,1,2,\ldots$$

where λ is the average number of outcomes per unit time

NOTE

a. X ~ number of events in a specified time or space. Its range is 0, 1, 2,..

b. Number of events in one interval or space is independent of the number of events in any other disjoint interval or space.

c. The average number of events occurring in a given interval or space is proportional to the length or size of the region in question.

EXAMPLE

The average number of radioactive particles passing through a counter during 1 millisecond in a laboratory experiment is 4. What is the probability that at least 6 particles enter the counter in a given millisecond?

EXAMPLE

A life insurance salesman sells on the average 3 life insurance policies per week. Use Poisson distribution to calculate the probability that in a given week he will sell

a. some policies

b. 2 or more policies but less than 5 policies.

c. Assuming that there are 5 working days per week, what is the probability that in a given day he will sell one policy?

THEOREM

Both the mean and the variance of the Poisson distribution $p(x;\ \lambda)$ are λ.

HYPERGEOMETRIC DISTRIBUTION

Then X has a Hypergeometric Distribution with parameters N, K and n. The Hypergeometric random variable X is denoted by $X \sim Hyper(N, K, n)$ and its PMF Is given as follows:

$$P(X = x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}, x = 0,1, \dots, n; x \leq k; n - x \leq N - k$$

NB: Hypergeometric distribution is appropriate for sampling without replacement.

MEAN AND VARIANCE OF HYPERGEOMETRIC

The mean and variance of the hypergeometric distribution are

$$\mu = \frac{nk}{N}$$

And

$$\sigma^2 = \frac{N-n}{N-1} \times n \times \frac{k}{N}(1 - \frac{k}{N})$$

EXAMPLE

Batches (lots) of 40 components each are called acceptable if they contain no more than 3 defectives. The procedure for sampling the batch is to select 5 components at random and to reject the batch if a defective is found.

a) Calculate the probability that exactly one defective will be found in the sample if there are 3 defectives in the entire batch.

b) Calculate the mean and the variance number of the components

EXAMPLE

In a small pond there are 50 fish, 10 of which have been tagged. If a fisherman's catch consists of 7 fish selected at random and without replacement, and X denotes the number of tagged fish, the probability that exactly 2 tagged fish are caught

EXAMPLE

Suppose a box contained 100 microchips, 80 good and 20 defective. The number of defectives in the box is unknown to a purchaser, who decides to select 10 microchips at random without replacement and to consider the microchips in the box acceptable if the 10 items selected include no more than three defectives. The number of defectives selected is denoted by X. What is the probability of the lot being acceptable.

GEOMETRIC DISTRIBUTION

Suppose that independent Bernoulli trials with success probability are performed until a success, i.e. the first success occurs. Let X be a random variable that denote the number of trials required to get the first success. Then X is said to have a Geometric Distribution with probability mass function that is given as follows:

$$P(X = x) = pq^{x-1}, x = 1,2, \dots$$

MEAN AND VARIANCE

The mean and variance of a random variable following geometric distribution are

$$\mu = \frac{1}{p}$$

And

$$\sigma^2 = \frac{1-p}{p^2}$$

EXAMPLE

If the probability is 75% that an applicant for a driver's license will pass the road test on any given try,

a) What is the probability that an applicant will finally pass the test on the fourth try?

b) Determine the mean and the variance of the random variable.

EXAMPLE

For a certain manufacturing process, it is known that, on the average, 1 in every 100 items is defective. What is the probability that the fifth item inspected is the first defective item found?

EXAMPLE

The probability that a student pilot passes the written test for a private pilot's license is 0.7. Find the probability that a given student will pass the test

(a) on the third try;

(b) before the fourth try.

NEGATIVE BINOMIAL

Suppose that independent Bernoulli trials with success probability p are performed until a total of r successes are obtained. Let X be a random variable that denote the number of trials required, then X has a Negative Binomial or Pascal Distribution with parameters r and p. The probability mass function of X is given by

$$P(X = x) = \binom{x-1}{r-1} p^r q^{x-r}, x = r, r+1, \ldots$$

NOTE: Probabilities depend on the number of successes desired

MEAN AND VARIANCE OF NEGATIVE BINOMIAL

$$\mu = \frac{r}{p}$$

And

$$\sigma^2 = \frac{rq}{p^2}$$

EXAMPLE

In an NBA (National Basketball Association) championship series, the team that wins four games out of seven is the winner. Suppose that teams A and B face each other in the championship games and that team A has probability 0.55 of winning a game over team B.

(a) What is the probability that team A will win the series in 6 games?

(b) What is the probability that team A will win the series?

(c) If teams A and B were facing each other in a regional playoff series, which is decided by winning three out of five games, what is the probability that team A would win the series?

CONTINUOUS PROBABILITY DISTRIBUTION FUNCTIONS (PDFs)

Continuous Uniform [Triangular] Distribution

If $a < b$, a random variable X is said to have a Continuous Uniform or Triangular probability distribution on the interval (a, b) if and only if the density function of X is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & for\ a \le x \le b \\ 0, & for\ x < a\ or\ x > b \end{cases}$$

CONT'S

Application: Uniform distribution is applicable whenever the probability is proportional to the interval's length, where a and b are the smallest and largest values that the variable can assume.

The mean and variance of the uniform distribution

The mean and variance of the uniform distribution are

$$\mu = \frac{b + a}{2}$$

And

$$\sigma^2 = \frac{(b - a)^2}{12}$$

EXAMPLE

Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval $[0, 4]$.

(a) What is the probability density function?

(b) What is the probability that any given conference lasts at least 3 hours?

EXAMPLE

A farmer living in western Nebraska has an irrigation system to provide water for crops, primarily corn, on a large farm. Although he has thought about buying a backup pump, he has not done so. If the pump fails, delivery time X for a new pump to arrive is uniformly distributed over the interval from 1 to 4 days. The pump fails. It is a critical time in the growing season in that the yield will be greatly reduced if the crop is not watered within the next 3 days. Assuming that the pump is ordered immediately and that installation time is negligible, what is the probability that the farmer will suffer major yield loss?

EXAMPLE

The daily amount of coffee, in liters, dispensed by a machine located in an airport lobby is a random variable X having a continuous uniform distribution with a = 7 and b = 10. Find the probability that on a given day the amount of coffee dispensed by this machine will be

(a) at most 8.8 liters;

(b) more than 7.4 liters but less than 9.5 liters;

(c) at least 8.5 liters.

NORMAL DISTRIBUTION

A random variable X is said to have a Normal or Gaussian distribution with parameters μ and σ^2, denoted by $N \sim (\mu, \sigma^2)$ if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, , -\infty < x < \infty, \sigma > 0$$

MEAN AND VARIANCE OF NORMAL

The mean and variance of normal distribution are

$$\mu$$

And

$$\sigma^2$$

EXAMPLE

If X is any random variable with mean μ and variance $\sigma^2 > 0$, then what are the mean and variance of the random variable

$$Y = \frac{X - \mu}{\sigma}?$$

Standard Normal Distribution

The special case of a Normal Distribution where the mean is $\mu = 0$ and the variance is $\sigma^2 = 1$ is called a **Standard Normal Distribution**.

STANDARD NORMAL

A normal random variable is said to be standard normal, if its mean is zero and variance is one. We denote a standard normal random variable X by $X \sim N(0,1)$. The probability density function of standard normal distribution is the following:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \;, -\infty < x < \infty$$

THEOREM

If X has a Normal Distribution with mean μ and the standard deviation σ, then

$$Z = \frac{x-\mu}{\sigma}$$

has the standard normal distribution.

EXAMPLE

1. If $X \sim N(0,1)$, what is the probability of a random variable X less than or equal to -1.72.
2. If $X \sim N(0,1)$, then what is $P(4 \leq X \leq 8)$

EXAMPLE

A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.

EXAMPLE

A research scientist reports that mice will live an average of 40 months when their diets are sharply restricted and then enriched with vitamins and proteins. Assuming that the lifetimes of such mice are normally distributed with a standard deviation of 6.3 months, find the probability that a given mouse will live

(a) more than 32 months;

(b) less than 28 months;

(c) between 37 and 49 months.

GAMMA FUNCTION

The Gamma function is defined by

$$r(\alpha) = \int_{0}^{\infty} x^{\alpha-1} e^{-x}\, dx \quad , \alpha > 0$$

Finding expected values for the exponential distribution is simplified by understanding a certain type of integral called a gamma (r) function.

EXERCISE

1. Show that $r(\alpha + 1) = \alpha r(\alpha)$
2. Show that $r(1) = 1$

Evaluating Gamma function

1. Evaluate $r(7.8)$

2. Note that $r(n+1) = nr(n) = n!$

GAMMA DISTRIBUTION

A random variable X is said to have a Gamma Distribution with parameters α and β, denoted by $Gamma\ (\alpha, \beta)$ if the probability density function is given by

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} & , x > 0, \alpha, \beta > 0 \\ 0 & otherwise \end{cases}$$

Where $\beta = mean\ time\ between\ events$

Note: Gamma distribution deals with time until the occurrence of α Poisson events.

MEAN AND VARIANCE OF GAMMA DISTRIBUTION

THE MEAN AND VARIANCE OF A GAMMA DISTRIBUTION ARE

$$\mu = \alpha\beta$$

AND

$$\sigma^2 = \alpha\beta^2$$

EXAMPLE

Let X have the density function

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\beta^{\alpha}\Gamma(\alpha)} & , x > 0, \alpha, \beta > 0 \\ 0 & otherwise \end{cases}$$

If $\alpha = 4$, what is the mean of $\frac{1}{X^3}$?

EXAMPLE

Suppose that the telephone calls arriving at a particular switch board follow a Poisson process with an average of 5 calls coming per minute.

a) What is the probability that up to a minute will elapse until 2 calls have come in to the switch board?

b) Determine the average number and the standard deviation number of calls.

MOMENT GENERATING FUNCTIONS

The moment generating function of the random variable X is given by $E(e^{tX})$ and is denoted by $M_X(t)$. Hence,

$$M_X(t) = E(e^{tX}) = \begin{cases} \displaystyle\sum_{\forall x} e^{tx} f(x) & if\ X\ is\ discrete \\ \displaystyle\int_{-\infty}^{\infty} e^{tx} f(x) dx & if\ X\ is\ continuous \end{cases}$$

Theorem

Let X be a random variable with MGF $M_X(t)$.Then

$$\frac{d^r M_X(t)}{dt^r}\Big|_{t=0} = \mu_r' = E(X^r)$$

EXAMPLE

- Find the MGF of a binomial distribution and verify that the mean and variance are

$$\mu = np \quad and \ \sigma^2 = npq$$

EXAMPLE

Find the MGF of an exponential distribution and then use it to find the mean and the variance

MULTIVARIATE PROBABILITY DISTRIBUTION

Although the heading of this chapter refers to multivariate distributions, the emphasis is on bivariate distributions. Please concentrate on the bivariate cases and make sure you understand how the move from one to two random variables influences the formulas and calculations. Then, in follow-up modules, it will not be difficult to expand your knowledge to any number of variables.

Bivariate Discrete Random variable

When both X and Y are discrete random variables, we define their joint PMF as follows:

$$P_{XY}(x, y) = P(X = x, Y = y)$$

Properties

The properties of the joint PMF include the following:

1. As a probability, the PMF can neither be negative nor exceed unity, which means that $0 \leq P_{XY}(x, y) \leq 1$.

2. $\sum_x \sum_y p_{XY}(x, y) = 1$

3. $\sum_{x \leq a} \sum_{y \leq b} p_{XY}(x, y) = F_{XY}(a, b)$

Marginal PMFs

The marginal PMFs are obtained as follows:

$$P_X(x) = \sum_y p_{XY}(x, y) = P(X = x)$$

And

$$P_Y(y) = \sum_x p_{XY}(x, y) = P(Y = y)$$

Independence

If X and Y are independent random variables,

$$P_{XY}(x, y) = P_X(x) \times P_Y(y)$$

EXAMPLE

The joint PMF of two random variables X and Y is given by

$$P_{XY}(x, y) = \begin{cases} k(2x + y), & x = 1,2; y = 1,2 \\ 0, & elsewhere \end{cases}$$

Where k is a constant

a. What is the value of k?

b. Find the marginal $PMFs$ of X and Y

c. Are X and Y independent?

EXAMPLE

Let X and Y be a discrete random variables with joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{2}(x+y) & if\ x = 1,2; y = 1,2,3 \\ 0, & otherwise \end{cases}$$

What is the marginal probability density function of X and Y?

Discrete Conditional Distributions

The conditional Distribution of X given $\mathrm{Y} = y$ is

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Which is defined for all real values of X. Similarly, for all real values of Y, the conditional distribution of Y given $\mathrm{X} = x$ is

$$P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)}$$

EXAMPLE

The joint PMF of two random variables X and Y is given by

$$P_{XY}(x,y) = \begin{cases} k(2x+y), & x = 1,2; y = 1,2 \\ 0, & elsewhere \end{cases}$$

Where k is a constant

a) What is the conditional PMF of Y given X

b) What is the conditional PMF of X given Y

Generating joint PMFs using Extended Hyp Geo

The hypergeometric distribution can be generalized to apply in cases where there are more than two types of outcomes of interest. Suppose that a collection consists of a finite number of items N and that there are $k\ + 1$ different types; $K_1$of type 1, K_2 of type 2, and so on. Select n items at random without replacement, and let X_i be the number of items of type i that are selected. The vector $X\ =\ (X_1, X_2, \dots, X_m)$ has an extended hypergeometric distribution and a joint pdf of the form

Cont's

$$f(x_1, x_2, \dots, x_m) = \frac{\binom{K_1}{x_1}\binom{K_2}{x_2}\dots\binom{K_m}{x_m}\binom{K_{m+1}}{x_{m+1}}}{\binom{N}{n}}$$

For all $0 \le x_i \le K_i$, where $K_{m+1} = N - \sum_{i=1}^{k} K_i$ and

$x_{m+1} = n - \sum_{i=1}^{k} x_i$

For Bivariate, the above formula can be simplified as:

$$f(x_1, x_2) = \frac{\binom{K_1}{x_1}\binom{K_2}{x_2}\binom{N-K_1-K_2}{n-x_1-x_2}}{\binom{N}{n}}$$

EXAMPLE

From a group of three Republicans, two Democrats, and one independent, a committee of two people is to be randomly selected. Let Y_1 denote the number of Republicans and Y_2 denote the number of Democrats on the committee. Find the joint probability function of Y_1 and Y_2 and then find the marginal probability function of Y_1.

Example

Find the conditional distribution of Y_1 given that $Y_2 = 1$. That is, given that one of the two people on the committee is a Democrat, find the conditional distribution for the number of Republicans selected for the committee.

Bivariate Continuous Random Variables

If both X and Y are continuous random variables, their joint PDF is given by

$$f_{XY}(x, y) = \frac{d}{d_x d_y} F_{XY}(x, y)$$

CDF

The joint PDF satisfies the following condition:

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u, v) dv du$$

PROPERTIES

The joint PDF also has the following properties:

1. For all x and y, $f_{XY}(x, y) \geq 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dxdy = 1$
3. $f_{XY}(x, y)$ is continuous for all except possibly finitely values of x or of y
4. $P(x_1 < X < x_2, y_1 < Y < y_2) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{XY}(x, y) dxdy$

Marginal PDFs

The marginal PDFs are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$$

And

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx$$

Independence

If X and Y are independent random variables, then

$$f_{XY}(x, y) = f_X(x) \times f_Y(y)$$

EXAMPLE

X and Y are two continuous random variables whose joint PDF is given by

$$f_{XY}(x, y) = \begin{cases} e^{-(x+y)}, & 0 \leq x < \infty, 0 \leq y < \infty \\ 0, & elsewhere \end{cases}$$

Are X and Y independent

Example

Let Y_1 and Y_2 have a joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, y_1 > 0, y_1 > 0 \\ 0, \qquad elsewhere \end{cases}$$

a) What is $P(Y_1$<1, Y_2>5)

b) What is $P(Y_1 + Y_2$ <3)

EXPECTED VALUES OF FUNCTIONS OF RANDOM VARIABLES

DISCRETE CASE

Suppose that the discrete random variables (X, Y) have a joint probability function given by $p(x, y)$. If $g(X, Y)$ is any real-valued function of (X, Y), then the expected value of $g(X, Y)$ is

$$E[g(X,Y)] = \sum_{x}\sum_{y} g(x,y)p(x,y)$$

Theorem

If X and Y are independent means μ_X and μ_Y, respectively, then

$$E(XY) = E(X)E(Y)$$

Further, if X and Y are independent , g is a function of X alone and h is a function of Y alone, then

$$E[g(X)h(Y)] = E[g(X)]E[h(y)]$$

Continuous Case

The sum is over all values of (x, y) for which p(x, y) > 0. If $(X\ , Y\ )$ are continuous random variables with probability density function $f\ (x, y)$, then

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy$$

Covariance

- If X has a mean μ_X and Y has a mean of μ_Y , then

- $Cov(X,Y) = E(XY) - \mu_X\mu_Y$

CORRELATION

The correlation between two random variables X and Y is given by

$$\rho = \frac{cov(X, Y)}{\sqrt{V(X)V(Y)}}$$

EXAMPLE

The joint pdf of X and Y is given by

$$f(x,y) = \begin{cases} \frac{1}{2}, 0 \le x \le y, 0 \le y \le 2 \\ 0, elsewhere \end{cases}$$

Find the $Cov(X,Y)$

Statement for the previous example

A soft-drink machine has a random supply Y at the beginning of a given day and dispenses a random amount X during the day (with measurements in gallons). The machine is not resupplied during the day. It has been observed that X and Y have joint density

Example

The joint PDF of the random variables X and Y is defined as follows

$$f_{XY}(x, y) = \begin{cases} 25e^{-5y}, 0 < x < 0.2, y > 0 \\ 0, \qquad elsewhere \end{cases}$$

a) What is the covariance of X and Y?
b) What is the conditional mean $E[X|Y = y]$