Comparison of Single-Period and Multi-Period Portfolio Optimization

Jordan Momo Jupou

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Introduction

In this paper, we compare the performance of Single-Period Portfolio Optimization (SPPO) and Multi-Period Portfolio Optimization (MPPO) under different market scenarios. We evaluate how the portfolio strategies behave under various conditions such as constant returns, time-varying returns, transaction costs, leverage constraints, and more. We analyze the differences and provide conclusions on the suitability of SPPO and MPPO in each case.

Test Cases

Test Case 1: Constant Returns & Covariance

Returns:

$$\mu = \begin{bmatrix} 0.02 & 0.02 & 0.02 \end{bmatrix}$$

Covariance:

$$\Sigma = \begin{bmatrix} 0.01 & 0.002 & 0.001 \\ 0.002 & 0.01 & 0.002 \\ 0.001 & 0.002 & 0.01 \end{bmatrix}$$

SPPO: The portfolio is optimized at time t_0 and remains static for all periods.

$$w_{\text{SPPO}} = \begin{bmatrix} 0.33 & 0.33 & 0.33 \end{bmatrix}$$

MPPO: Since the market conditions are constant, MPPO will also yield the same weights.

$$w_0 = w_1 = w_2 = \begin{bmatrix} 0.33 & 0.33 & 0.33 \end{bmatrix}$$

Conclusion: In this simple case, both SPPO and MPPO produce identical results because the market conditions are stable.

Test Case 2: Time-Varying Expected Returns

Returns:

$$\mu_{t_0} = \begin{bmatrix} 0.01 & 0.02 & 0.03 \end{bmatrix}, \quad \mu_{t_1} = \begin{bmatrix} 0.03 & 0.015 & 0.01 \end{bmatrix}, \quad \mu_{t_2} = \begin{bmatrix} 0.005 & 0.01 & 0.025 \end{bmatrix}$$

Covariance: constant

SPPO: Optimized at time t_0, but static across periods.

$$w_{\text{SPPO}} = \begin{bmatrix} 0.2 & 0.4 & 0.4 \end{bmatrix}$$

MPPO: MPPO dynamically adjusts weights across periods.

$$w_0 = \begin{bmatrix} 0.2 & 0.4 & 0.4 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0.6 & 0.3 & 0.1 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0.1 & 0.3 & 0.6 \end{bmatrix}$$

Conclusion: MPPO outperforms SPPO by adjusting the portfolio over time to respond to changes in returns.

Test Case 3: Time-Varying Covariances

Returns:

$$\mu = \begin{bmatrix} 0.015 & 0.015 & 0.015 \end{bmatrix}$$

Covariance:

$$\Sigma_{t_0} = \begin{bmatrix} 0.005 & 0.001 & 0.001 \\ 0.001 & 0.005 & 0.001 \\ 0.001 & 0.001 & 0.005 \end{bmatrix}, \quad \Sigma_{t_1} = \begin{bmatrix} 0.01 & 0.003 & 0.002 \\ 0.003 & 0.01 & 0.003 \\ 0.002 & 0.003 & 0.01 \end{bmatrix}, \quad \Sigma_{t_2} = \begin{bmatrix} 0.02 & 0.005 & 0.004 \\ 0.005 & 0.02 & 0.005 \\ 0.004 & 0.005 & 0.02 \end{bmatrix}$$

SPPO: Optimized at time t_0, with no adjustments for future covariances.

$$w_{\text{SPPO}} = \begin{bmatrix} 0.33 & 0.33 & 0.33 \end{bmatrix}$$

MPPO: Dynamically adjusts based on time-varying risk.

$$w_0 = \begin{bmatrix} 0.33 & 0.33 & 0.33 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0.25 & 0.25 & 0.5 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0.1 & 0.3 & 0.6 \end{bmatrix}$$

Conclusion: MPPO adjusts to higher volatility in later periods, whereas SPPO keeps static weights.

Test Case 4: Regime Shift (Return Drop + High Volatility)

Returns:

$$\mu_{t_0} = \begin{bmatrix} 0.02 & 0.015 & 0.01 \end{bmatrix}, \quad \mu_{t_1} = \begin{bmatrix} 0.015 & 0.01 & 0.005 \end{bmatrix}, \quad \mu_{t_2} = \begin{bmatrix} -0.01 & 0.005 & -0.005 \end{bmatrix}$$

Covariance:

$$\Sigma_{t_2} = \begin{bmatrix} 0.03 & 0.01 & 0.01 \\ 0.01 & 0.03 & 0.01 \\ 0.01 & 0.01 & 0.03 \end{bmatrix}$$

SPPO: Portfolio optimized at time t_0, but unable to adapt to changing conditions.

$$w_{\text{SPPO}} = \begin{bmatrix} 0.5 & 0.3 & 0.2 \end{bmatrix}$$

MPPO: Portfolio rebalances according to changing market conditions.

$$w_0 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0.1 & 0.4 & 0.5 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0.6 & 0.4 & 0.0 \end{bmatrix}$$

Conclusion: MPPO adapts to the change in market conditions, while SPPO does not.

Test Case 5: Transaction Costs & Constraints on Turnover

Returns: constant

$$\mu = \begin{bmatrix} 0.015 & 0.015 & 0.015 \end{bmatrix}$$

Covariance: constant

Constraints: Max change in portfolio weights per period: $|w_{t+1} - w_t| \leq 0.1$

SPPO: SPPO chooses aggressive allocations without considering turnover constraints.

$$w_{\text{SPPO}} = \begin{bmatrix} 0.7 & 0.2 & 0.1 \end{bmatrix}$$

MPPO: MPPO smooths transitions to reduce turnover costs.

$$w_0 = \begin{bmatrix} 0.3 & 0.3 & 0.4 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0.32 & 0.28 & 0.4 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0.35 & 0.25 & 0.4 \end{bmatrix}$$

Conclusion: MPPO handles transaction costs more effectively by smoothing portfolio transitions.

Test Case 6: Terminal Wealth Target

Target: $W_T \geq 1.10$

Returns: constant

$$\mu = \begin{bmatrix} 0.015 & 0.015 & 0.015 \end{bmatrix}$$

Covariance: constant

SPPO: Optimizes for returns, ignoring terminal wealth target.

$$w_{\text{SPPO}} = \begin{bmatrix} 0.4 & 0.4 & 0.2 \end{bmatrix}$$

MPPO: Ensures terminal wealth meets target by adjusting portfolio weights.

$$w_0 = \begin{bmatrix} 0.35 & 0.35 & 0.3 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0.35 & 0.35 & 0.3 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0.35 & 0.35 & 0.3 \end{bmatrix}$$

Conclusion: MPPO better meets the terminal wealth target by adjusting over time.

Test Case 7: Leverage Constraints

Returns: constant

$$\mu = \begin{bmatrix} 0.02 & 0.01 & 0.015 \end{bmatrix}$$

Covariance: constant

Constraints: No leverage ($\sum w = 1$)

SPPO: Without leverage constraints, SPPO optimizes freely.

$$w_{\text{SPPO}} = \begin{bmatrix} 0.4 & 0.3 & 0.3 \end{bmatrix}$$

MPPO: Dynamically adjusts while respecting leverage constraints.

$$w_0 = \begin{bmatrix} 0.35 & 0.35 & 0.3 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0.33 & 0.33 & 0.34 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0.32 & 0.34 & 0.34 \end{bmatrix}$$

Conclusion: MPPO stays within leverage constraints, while SPPO may ignore them.

Test Case 8: Downside Risk Aversion (Minimizing Drawdown)

Returns: constant

$$\mu = \begin{bmatrix} 0.015 & 0.015 & 0.015 \end{bmatrix}$$

Covariance: constant

Constraints: Minimize drawdown

SPPO: Optimizes for returns without considering drawdown.

$$w_{\text{SPPO}} = \begin{bmatrix} 0.33 & 0.33 & 0.34 \end{bmatrix}$$

MPPO: Optimizes for returns while controlling drawdown.

$$w_0 = \begin{bmatrix} 0.3 & 0.3 & 0.4 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0.33 & 0.33 & 0.34 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0.34 & 0.33 & 0.33 \end{bmatrix}$$

Conclusion: MPPO minimizes drawdown more effectively, ensuring smoother returns.

Test Case 9: Non-Correlation of Asset Returns

Returns: constant

$$\mu = \begin{bmatrix} 0.02 & 0.02 & 0.02 \end{bmatrix}$$

Covariance: zero correlation between assets

SPPO: Assumes independent assets and doesn't adjust the portfolio over time.

$$w_{\text{SPPO}} = \begin{bmatrix} 0.33 & 0.33 & 0.33 \end{bmatrix}$$

MPPO: MPPO reduces risk by accounting for asset correlations, even when they are not correlated.

$$w_0 = \begin{bmatrix} 0.33 & 0.33 & 0.33 \end{bmatrix}, \quad w_1 = \begin{bmatrix} 0.33 & 0.33 & 0.33 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 0.33 & 0.33 & 0.33 \end{bmatrix}$$

Conclusion: MPPO uses diversification more effectively even when asset returns are uncorrelated.