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Unit - 3

: Elementary Combinatorics :

BASICS COUNT: There are two types of basic rules of counting (1) Sum rule (2) Product rule.

SUM RULE:

If an event can occur in m ways and another event can occur in n ways and if these two events cannot occur simultaneously then one of the two events can occur in m+n ways.

Ex: If there are 14 boys and 12 girls in a class. Find the no of ways of selecting one student as CR

Sol: Selection of one student either from 14 boys or from 12 girls as a CR, clearly the selection of 1 boy in 14 ways or selection of 1 girl in 12 ways.

∴ By sum rule, number of ways of selecting a student CR either from boys or girls is 14+12 = 26 ways.

PRODUCT RULE:

If an event can occur in m ways and second event can occur in n ways and if the number of ways the second event can occur doesn't depend upon the occurance of first event then the two events can

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Ex: If there are 14 boys and 12 girls in a class. Find the no of ways of selecting one student as CR

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∴ By sum rule, number of ways of selecting a student CR either from boys or girls is $14+12 = 26$ ways.

PRODUCT RULE:

If an event can occur in m ways and second event can occur in n ways and if the number of ways the second event can occur doesn't depend upon the occurance of first event then the two events can occur simultaneously in mn ways.

Ex: Three persons enter into a car where there are 5 seats. In how many way can they take up their seats

Sol: Given no of persons = 3

No of seats = 5

First person can sit in any of the five seats so number of ways for first person is 5 ways Now remaining seats are 4 second person can sit in one of the 4 seats. So no of ways for 2nd person is 4 ways Now the remaining seats are 3. 3rd person can sit in one of the 3 seats. So no of ways for third person is 3 ways

∴ By product rule, the no of ways in which all the three persons can takeup their seats = 5×4×3 = 60 ways

In a set theory concept the two rules can be written as:

(i) SUM RULE: If A, B are disjoint sets then $|A+B| = |A| + |B|$

(ii) PRODUCT RULE: If A×B is the cartesian products of the sets A, B then $|A \times B| - |A| \cdot |B|$

(iii) FACTORIAL NOTATION: The product of first n natural numbers is denoted by n!

i.e $n! = n(n-1)(n-2) \ldots 3.2.1$

Also $n! = n(n-1)!$

Ex: $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

clearly from the above definition, n factorial is recursive.

Ex: Find n if $(n+1)! = 12(n-1)!$

$$\frac{(n+1)!}{(n-1)!} = 12$$

$$\frac{(n+1)n\cancel{(n-1)!}}{\cancel{(n-1)!}} = 12$$

$$n^2+n-12=0$$

$$n=3 \text{ or } n=-4$$

$n^2+n-12=0$
$n^2+4n-3n-12=0$
$n(4+n)-3(n+4)=0$
$(4+n)(n-3)=0$
$n=3, n=-4$

PERMUTATIONS:

From a given finite set of elements, selecting some or all of them and arranging them in a sequential order is called a permutation. i.e Permutation is an arrangement of finite set of objects in a particular order. The number of permutations of n distinct objects taken r at a time is given by

$$^{n}P_r = \frac{n!}{(n-r)!}$$

The number of permutations of n dissimilar things taken r at a time = no. of ways of filling r blank places arranged in a row by n dissimilar things. The above formula is used for without repitition.

Ex: The permutations formed by two objects at a time from {A, B, C} are AB, BA, BC, CB, AC, CA i.e $^3P_2 = \frac{3!}{1!} = 6$

Ex 2: For the set {A, B, C} if taken three at a time then we get $^3P_3$ namely ABC, ACB, BAC, BCA, CAB, CBA

i.e $^3P_3 = \frac{3!}{0!} = 6$

PERMUTATIONS WITH REPETITIONS:

Out of n objects in a set, p objects are exactly alike of first kind, q objects are exactly alike of second kind, r objects are exactly alike of third kind and remaining objects all are distinct then the number of permutations of n objects taken all at a time is:

$$\frac{n!}{p!\,q!\,r!}$$

Ex: How many ways are there to arrange the nine letter word "Allahabad".

Sol: Given Total letters in above word = 9

Clearly the word has 4 'a's, 2 l's. Therefore number of ways of arranging 9 letters in the word:

$$\frac{9!}{4!\,2!} = \frac{9\times 8\times 7\times 6\times 5\times \cancel{4}\times \cancel{3}\times \cancel{2}\times 1}{\cancel{4}\times \cancel{3}\times \cancel{2}\times 1\times \cancel{2}\times 1}$$

$= 7560$ ways.

The number of permutations of n dissimilar things taken r at a time when repetition of this is allowed is n^r ways

Ex: Find the number of different telephone numbers formed by taking 3 digits from 1, 2, 3, 4, 5.

Sol: To form a telephone number and with repetitions by taking 3 digits will have $5^3 = 125$ ways.

COMBINATIONS

Any selection which can be made by taking sum or all of objects at a time out of given number of objects is called combination.

The no of combinations of n distinct objects taken r at a time is given by

$$nC_r = \frac{n!}{r!(n-r)!}$$

NOTE:

(i) $nC_0 = rC_n = 1$

(ii) $nC_r = nC_{n-r}$

(iii) $nP_r = nC_r (r!)$

COMBINATIONS WITH REPITITIONS:

When repitions are allowed, no of arrangements of n distinct objects taken r at a time is n^r

Ex: Prove that $nC_r + nC_{r-1} = (n+1)C_r$

LHS: $nC_r + nC_{r-1}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{(n+1)}{r(n-r+1)}\right]$$

$$= \frac{n!(n+1)}{r(r-1)!(n+1-r)(n-r)!}$$

$$= \frac{(n+1)!}{r!(n+1-r)!}$$

$$= {}^{(n+1)}C_r$$

Ex: Prove that ${}^{n}C_r = {}^{(n-1)}C_{r-1} + {}^{(n-1)}C_r$

(or)

$$C(n,r) = C(n-1, r-1) + C(n-1, r)$$

Sol RHS $\frac{(n-1)!}{(r-1)!(n-1-r+1)!} + \frac{(n-1)!}{r!(n-1-r)!}$

$$\frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!}$$

$$\frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{1}{(n-r)} + \frac{1}{r}\right]$$

$$\frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{r+n-r}{r(n-r)}\right]$$

$$\frac{(n-1)!}{(r-1)!(n-r-1)!}\left[\frac{n}{r(n-r)}\right]$$

$$\frac{n(n-1)!}{r(r-1)!\,(n-r)(n-r-1)!}$$

$$= \frac{n!}{r!\,(n-r)!} = nC_r$$

27/12/21 Q) Consider the set {a, b, c, d}. In how many ways can we select two of ~~ar~~ these letters (repetition is not allowed) when :

(i) order matters

(ii) order doesnt matter.

Sol Given set {a, b, c, d}

(i) order matters and repitition is not allowed

$n = 4$, $r = 2$

$\therefore 4P_2 = P(4,2) = \frac{4!}{2!} = \frac{4\times3\times\cancel{2!}}{\cancel{2!}} = 12$

ab	ba	ca	da
ac	bc	cb	db
ad	bd	cd	dc

(ii) if order doesnt matter and repition is not allowed then the number of ways of selecting two letters out of 4 is

$$4C_2 = C(4,2) = \frac{4!}{2!\,2!} = \frac{4\times3\times2!}{2!\,2!}$$

$$= \frac{\overset{2}{\cancel{4}}\times3}{2\times1}$$

$$= 6$$

ab	bc
ac	bd
ad	cd

8) In how many ways can the letters of the word "COMPUTER" can be arranged. How many of them begin with C and with R

(ii) How many of them donot begin with C but end with R

Sol Given word COMPUTER

Total letters in the word is 8

∴ All the letters can be arranged in 8! ways

i.e $P(8,8) = 8!$

(i) From the problem, C occupies first place and R occupies last place. Then the remaining letters are OMPUTE which can be arranged in 6! ways

i.e $6! = 720$ ways

(ii) From the problem, first place is not filled with C but it ends with R. where R is fixed then remaining letters are COMPUTE but C is not arranged in first place so first place is filled with letters OMPUTE in 6 ways. In the second place, again 6 letters will be available including C. Consequently third, 4th, 5th, 6th, 7th places can be filled up by 5, 4, 3, 2, 1 ways.

∴ By product rule, the required number of arrangements is $6 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 4320$ ways

Q) How many four digit numbers are there with distinct digits

Sol We know that distinct digits are 0,1,2,3,4,5,6,7, 8,9. Therefore total number of arrangements of 10 digits taken 4 at a time is $10P_4$. But these numbers also include the numbers which has 0 at thousands place. Because such numbers are not four digit numbers so keeping 0 at the thousandth place then the three places can be filled up remaining 9 digits in $9P_3$ ways.

Hence the total number of four digits numbers is $10P_4 - 9P_3$

$$= \frac{10!}{6!} - \frac{9!}{6!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times \cancel{6!}}{\cancel{6!}} - \frac{9 \times 8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times 3!}{\cancel{6} \times \cancel{5} \times \cancel{4} \times 3!}$$

$$= (10 \times 9 \times 8 \times 7) - (9 \times 8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4})$$

$$= 5040 - 504$$

$$= 4536 \text{ ways}$$

Q) How many five digit numbers can be formed from the digits 1,2,3,4,5 using the digit once? How many of them are even.

Sol Total no of ways of 5 digit numbers formed by 5 digits 1,2,3,4,5 is $5P_5$ $(5!) = 120$ ways

For the number to be even it must have either 4 or 2 at its units place.

If 2 is in the units place, we have the remaining digits 1,3,4,5 for remaining 4 places. These 4 places can be arranged in 4! (24 ways)

Similarly if 4 is at the units place then the remaining 4 places can be arranged again in 4! ways

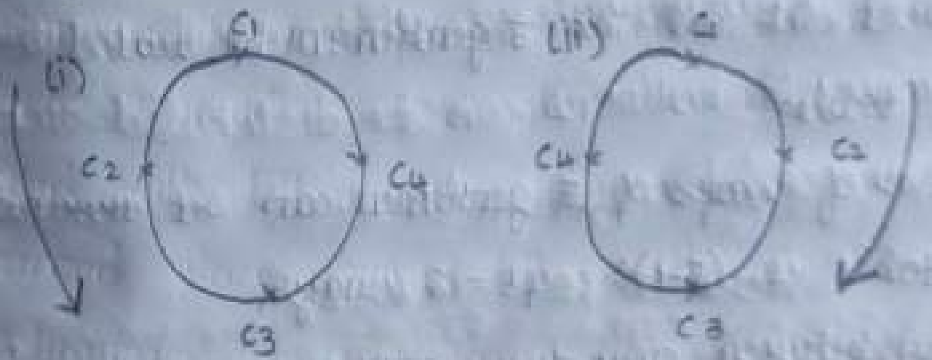
Therefore by sum rule. Total no of even numbers are 24+24 = 48

CIRCULAR PERMUTATION :

An circular permutation is an arrangements of object around a circle or with a simple closed curve.

In circular permutation clockwise and anticlockwise arrangements of objects are possible and both are distinguishable

The circular permutation can be shown in the following figure :



In the first permutation C_1, C_2, C_3, C_4 in the anti clockwise direction and in the second permutation C_1, C_2, C_3, C_4 which is in clockwise. When distinction is made between clockwise & anticlockwise arrangements circular permutation of n objects taken all n at a time is $(n-1)!$

If anticlockwise & clockwise arrangements are not distint (Arrangements of beads in a necklace or arrangement of flowers in a garland etc) Then the number of circular permutation of n distinct things is $(n-1)!/2$

Q) In how many different ways can 5 gentleman and 5 ladies can sit around a table if

(i) There is no restriction

(ii) No two ladies sit together.

Sol) Given arrangement of 5 gentlemen and 5 ladies around a table

Total persons $= 5+5 = 10$

The number of different ways of sitting 10 persons around a table is $(10-1)! = 9!$ ways.

(ii) First let all the 5 gentlemen be seated around a table.

$\therefore$ No of ways of 5 gentlemen can sit around a table is $(5-1)! = 4! = 24$ ways

Now between any two men, a women can be seated. Hence all 5 ladies can be seated in 5 Intermediate places at $5!$ ways.

Hence the required no of ways of 5 women can sit around a table if no two ladies sit together is $5! - 4!$ ways $= 120 - 24 = 96$ ways

Q) Find the number of ways in which 7 different beads can be arranged to form a necklace

Sol Fixing the position of one bead, the remaining beads can be arranged in 6! ways but this is a ring permutation so the required no. of arrangements will be $\frac{6!}{2} = 360$

Q) A man has 7 relatives four of them are ladies and 3 are gentlemen. His wife has again 7 relatives in which 3 of them are ladies, 4 were gentlemen. In how many way can they invite for a dinner party of 3 ladies and 3 gentlemen so that, 3 from man's relatives and 3 from wife's relatives.

Sol Given Man's relatives : 4 ladies & 3 men

Wife's relatives : 3 ladies & 4 men

We have to find the number of ways inviting to the party of 3 ladies and 3 men in such a way that 3 members from wife's relatives and husbands relatives

Clearly they can invite in 4 possible ways

(1) Three ladies from husband side + Three gents from wife's side

$\therefore$ No of ways $= 4C_3 \times 4C_3 = 16$

(2) Three gents from husband side + Three ladies from wife side

$\therefore$ No of ways $= 3C_3 \times 3C_3 = 1$

(iii) Two ladies and one gent from husband side + one lady, two gents from wife side

$\therefore$ No of ways: $4C_2 \times 3C_1 \times 3C_1 \times 4C_2 = 324$

(iv) one lady and two gents from husband side two ladies and one gent from wife side

$\therefore$ No. of ways $= 4C_1 \times 3C_2 \times 3C_2 \times 4C_1 = 144$

$\therefore$ Total no of ways $= 16 + 1 + 324 + 144$

$= 485$ ways

Q) Formula : The no of unorderd choices of r from n objects with repititions is $C(n+r-1, r)$ (or) $C(n+r-1, n-1)$

Q) In how many ways can 12 balloons be distributed at a birthday party among 10 children.

Sol This is an unordered selection with repetitions. Given 12 balloons can be distributed among 10 children

Here $n = 10$ $r = \cancel{10}\ 12$

Here no. of selections will be $C(n+r-1, r)$

$C(10+12-1, 12)$

$C(21, 12)$

$C(21, 9)$

If we want to ensure that every child

gets atleast one balloon. Then we must give one balloon to each child. Then distribution of remaining 2 balloons which can be done if $n = 10$ $r = 2$ is

$$C(10+2-1, 2)$$

$$C(11, 2)$$

$$^{11}C_2 = \frac{11!}{9! \times 2!} = \frac{11 \times \overset{5}{\cancel{10}} \times 9!}{9! \times \cancel{2}} = 55 \text{ ways}$$

BINOMIAL THEOREM

Statement: For any real numbers x, y and any integer $n \geq 0$ such that $(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots + {}^nC_n x^0 y^n$

$$= \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$$

(or)

$$= \sum_{r=0}^{n} C(n,r) x^{n-r} y^r \quad \text{—} \; ①$$

An expression consisting of two terms is known as binomial expression. The right hand side of the expression in eq ① are called binomial expansions. The expansion of $(x+y)^n$ contains $n+1$ terms.

- In the expansion $(x+y)^n$, the sum of powers of x and y in each term is equal to n
- In the expansion, the coefficients ${}^nC_0, {}^nC_1, {}^nC_2 \ldots {}^nC_n$ are called binomial coefficients and these are also denoted by $C(n,0)$, $C(n,1)$ etc
- In the expansion $(r+1)$th term is called general term and it is denoted by T_{r+1}

$T_{r+1} = nC_r\, x^{n-r} y^r$

Here $nC_r = c(n,r) = \binom{n}{r}$ is the coefficient of $(r+1)$th term

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* The quantity $nC_r = \frac{n!}{r!\,(n-r)!}$ is known as binomial coefficient

* The symbol nC_r has two meanings (i) combinatorial meaning (ii) Algebraic meaning.

* The first case means that number of ways of choosing r objects from n distinct objects

→ second case means that algebraic meaning

$$nC_r = \frac{n!}{r!\,(n-r)!}$$

* An identity which results from some counting process is known as combinatorial identity

<u>Some identities involving binomial coefficients are:</u>

(1) $C_0 + C_1 + C_2 + \dots + C_n = 2^n$

(2) $C_1 + C_3 + \dots = C_2 + C_4 + \dots = 2^{n-1}$

(3) $C_r = C_{n-r}$ (or) $nC_r = nC_{n-r}$

(4) $C(n,r) \cdot C(r,k) = C(n,k) \cdot C(n-k, r-k)$,

This is called Newton's Identity $\boxed{n \geq r \geq k \geq 0 \text{ are Integers}}$

(5) $C(n+1, r) = C(n,r) + C(n, r-1)$

This is called pascal's identity.

(6) $C(n+m, r) = C(n,0) \cdot C(m,r) + C(n,1) \cdot C(m, r-1)$.

$+ C(n,r) \cdot C(m,0)$

This is called Vandermonde's identity

MULTINOMIAL THEOREM:

Statements: For any positive integers n, k

$$(x_1 + x_2 + \dots + x_k)^n = \sum \binom{n}{n_1, n_2, \dots k} \cdot x_1^{n_1} \cdot x_2^{n_2} \dots x_k^{n_k}$$

$$n_1 + n_2 + \dots + n_k = n$$

i.e The summation is taken over k-tuples of non negative integers $n_1, n_2, \dots, n_k$ such that $n_1 + n_2 + \dots + n_k = n$

Here $\sum \binom{n}{n_1, n_2, \dots, n_k} = \dfrac{n!}{n_1!\, n_2! \dots n_k!}$ is called as multinomial coefficient

Multinomial coefficient denotes the number of distinguishable arrangements of n objects in which n_1 objects of type 1, n_2 objects of type 2, ..., n_k objects of type k.

The general term in the above expansion of $(x_1 + x_2 + \dots + x_k)^n$ is $\dfrac{n!}{n_1!\, n_2! \dots n_k!} \times x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3} \dots x_k^{n_k}$

where $n_1, n_2, n_3, \dots, n_k$ are non negative integers which are not exceeding n

The multinomial expansion i.e $(x_1 + x_2 + \dots + x_k)^n$ has $\binom{n+k-1}{n} = (n+k-1)C_n$ terms

No. of distinct terms in the multinomial expansion = No of non negative integer solutions of the equation

$x_1 + x_2 + \dots + x_k = n$

Here the number of terms $= C(n+k-1, n) =$

$(n+k-1)C_n$

$C(n+r-1, r) = C(n+r-1, n-1)$ represents no of combination of n distinct objects taken r at a time with repititions allowed

i.e $x_1 + x_2 + \dots + x_n = r$.

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19) Find the cofficient of x^9y^3 in the expansion of $(2x-3y)^{12}$

Sol From binomial expansion $(x+y)^n = \sum_{r=0}^{n} nC_r x^{n-r} y^r$

$$(2x-3y)^{12} = \sum_{r=0}^{n} 12C_r (2x)^{12-r}(-3y)^r \quad \text{--- (1)}$$

In eq (1) put $r=3$ to obtain the coefficient of x^9y^3

$$\therefore 12C_3 (2x)^{12-3}(-3y)^3$$

$$= \frac{12!}{3!\,9!}(2x)^9 \cdot (-3)^3 y^3$$

$$= \frac{12\times 10\times 11\times \cancel{9!}}{3!\,\cancel{9!}} 2^9 (-3)^3 x^9y^3$$

$$= \frac{\overset{4}{\cancel{12}}\times \overset{5}{\cancel{10}}\times 11}{\cancel{3}\times\cancel{2}\times 1} \cdot 512 \cdot (-27) x^9y^3$$

$$= (220)\cdot(512)\cdot(-27)x^9y^3$$

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20) (i) Compute $\binom{7}{2,3,2} = \frac{7!}{2!\,3!\,2!} = \frac{7\times 6\times 5\times 4\times 3!}{\cancel{3!}\times 2!\times 2!}$

$= 210$

(ii) $\binom{8}{4,2,2,0} = \dfrac{8!}{4!\,2!\,2!\,0!} = \dfrac{\overset{4}{\cancel{8}} \times 7 \times \overset{3}{\cancel{6}} \times 5 \times \cancel{4!}}{\cancel{4!}\,\cancel{2!}\,\cancel{2!}\,.\,1}$

$= 420$

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23) Determine the coefficient of

(i) xyz^2 in the expansion of $(2x-y-z)^4$

(ii) $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$

Sol (i) We know that from multinomial theorem:

General term in the expansion of

$(x_1+x_2+\dots+x_k)^n$ is $\dfrac{n!}{n_1!\,n_2!\dots n_k!}\,x_1^{n_1}\cdot x_2^{n_2}\dots x_k^{n_k}$

(i) General term in expansion of $(2x-y-z)^4$ is

$\binom{4}{n_1,n_2,n_3}\cdot(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}$ — ①

Put $n_1=1$, $n_2=1$, $n_3=2$ in eq ① to obtain the coefficient of xyz^2

$\therefore \binom{4}{1,1,2}(2x)^1(-y)^1(-z)^2$

$= \dfrac{4!}{1!\,1!\,2!}(-2xyz^2)$

$= \dfrac{4\times3\times\cancel{2!}}{\cancel{2!}}$

$= 12\,(-2xyz^2)$

$= -24xyz^2$

$\therefore$ Coefficient of xyz^2 in expansion of $(2x-y-z)^4 = -24$

(ii) General term in the expansion of $(a+2b-3c+2d+5)^{16}$

is $\binom{16}{n_1,n_2,n_3,n_4,n_5}\cdot(a)^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}$ — ①

Since $n_1+n_2+n_3+n_4+n_5=16$

put $n_1=2$, $n_2=3$, $n_3=2$, $n_4=5$, $n_5=?$ in ①

$$2+3+2+5+n_5=16$$

$$12+n_5=16$$

$$\boxed{n_5=4}$$

∴ coefficient of $a^2b^3c^2d^5$ in expansion general term is

$$\binom{16}{2,3,2,5,4} a^2(2b)^3(-3c)^2(2d)^5(5)^4$$

$$= \frac{16!}{2!\,3!\,2!\,5!\,4!} a^2 2^3 b^3 3^2 c^2 2^5 d^5 5^4$$

$$= \frac{16!\cdot 8\cdot 9\cdot 2^5\cdot 5^4}{2!\,3!\,2!\,5!\,4!} a^2b^3c^2d^5$$

Q) The Find the no of non negative Integer solutions of the equation $x_1+x_2+x_3+x_4+x_5=8$

Sol Given equation $x_1+x_2+x_3+x_4+x_5=8$ — ①

∴ The no of non negative Integer solutions of eq ① is $c(n+r-1, n) = c(8+5-1, 8)$

$$= c(12, 8)$$

$$= \frac{12!}{8!\,4!}$$

$$= \frac{12\times 11\times 10\times 9\times 8!}{8!\times 4\times 3\times 2}$$

$$= 495$$

Q7) Find the number of non negative integer solutions of the inequality $x_1+x_2+x_3+x_4+x_5+x_6<10$

Sol $x_1+x_2+x_3+x_4+x_5+x_6<10$ — ①

It is possible when

$$x_1+x_2+\dots+x_6=9-x_7<10 \quad ②$$

so that x_7 is a nonnegative integer

$x_7=10,11,12,\dots$ all are negative integers

$x_7=1,2,3,4,\dots$

∴ to find number of non negative integer solutions of ① it is enough to find the number of non negative integer solutions of eq ②

Eq ② can be written as

$$x_1+x_2+\dots+x_6+x_7=9$$

∴ no. of terms =

$$n=9,\ r=7$$

$$c(n+r-1,n)$$

$$c(9+7-1,9)$$

$$c(15,9)$$

$$15C_9=\frac{15!}{9!\times6!}=\frac{15\times14\times13\times12\times11\times10\times9!}{9!\times6\times5\times4\times3\times2}$$

$$=7\times13\times11\times5$$

$$=5005$$

Q) Find the no. of positive integer solutions of the equation $x_1+x_2+x_3=17$ where x_1, x_2, x_3 are non negative integers with $x_1 \geq 1$ $x_2 \geq 1$ $x_3 \geq 1$

<u>Sol</u> Put $x_1-1=y_1 \Rightarrow x_1=y_1+1$

$x_2-1=y_2 \Rightarrow x_2=y_2+1$

$x_3-1=y_3 \Rightarrow x_3=y_3+1$

$(y_1+1+y_2+1+y_3+1)=17$

$y_1+y_2+y_3+3=17$

$y_1+y_2+y_3=14$

$C(14+3-1, 14) = C(16,14)$

$= \dfrac{16!}{14!\,2!}$

$= \dfrac{\overset{8}{\cancel{16}} \times 15 \times \cancel{14!}}{\cancel{14!} \cdot \cancel{2!}}$

$= 8 \times 15$

$= 120$

21/01/2022

PRINCIPLE OF INCLUSION - EXCLUSION

Consider a finite set 'S' consisting of 'n' number of elements then the number n is called cardinality of 'S' or size of 'S' or order of 'S' and it is denoted by $n(S)$ or $|S|$ or $O(S)$

Example : If $A = \{1, 2, 6\}$, $B = \{a, b, c, d\}$ then $O(A) = 3$, $O(B) = 4$

NOTE 1) We know that $|\phi| = 0$

2) For any non empty finite set S $|S| \geq 1$

3) For any two finite sets A, B if $A \subseteq B$ then $|A| \leq |B|$

4) If A is a subset of A, finite univeral set U then the no. of elements in the complement of A denoted by $\bar{A}$ is given by

$$|\bar{A}| = |U| - |A|$$

If $A \cap B \neq \phi$ then the sum rule $|A \cup B| = |A| + |B|$ doesn't hold.

Ex : If $A = \{a, b, c\}$, $B = \{c, d, e, f\}$ then $|A| = 3$, $|B| = 4$ and $|A \cup B| = 6$ Now take $|A| + |B| = 3 + 4 = 7$

clearly $|A \cup B| \neq |A| + |B|$ (since $A \cap B \neq \phi$)

$\therefore$ General sum rule for any two sets A, B if $A \cap B \neq \phi$ then $|A \cup B| = |A| + |B| - |A \cap B|$

The above formula canbe extended for three sets A, B, C such that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

The above formulae is called principle of inclusion - exclusion.

If U is a finite universal set and A, B are subsets of U then order of $|\bar{A} \cap \bar{B}| = |\overline{A \cup B}| = |U| - |A \cup B|$

$$= |U| - [|A| + |B| - |A \cap B|]$$

$$= |U| - |A| - |B| + |A \cap B| \quad \text{--- (1)}$$

If $A \cap B = \phi$ then eq (1) becomes

$$|\bar{A} \cap \bar{B}| = |U| - |A| - |B|$$

This is called principle of disjunctive counting.

GENERAL INCLUSION - EXCLUSION PRINCIPLE (For n sets)

Let $A_1, A_2, \ldots, A_n$ be any n finite sets then

$$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| = \Sigma|A_i| - \Sigma|A_i \cap A_j| + \Sigma|A_i \cap A_j \cap A_k|$$
$$+ \ldots (-1)^{n-1} \Sigma|A_i \cap A_j \cap \ldots \cap A_n|$$

(Q) Find the number of integers between 1 and 250 that are divisible by any of the integers 2, 3, 7

Sol Let A_1 be the set of all integers between 1 to 250 that are divisible by 2.

Similarly A_2, A_3 be the set of integers between 1, 250 which are divisible by 3, 7 respectively.

then $|A_1| = \left[\frac{250}{2}\right] = 125$

$$|A_2| = \left[\frac{250}{3}\right] = 83$$

$|A_3| = \left[\frac{250}{7}\right] = 35$

where $[x]$ denotes the largest integer not exceeding x.

$|A_1 \cap A_2|$ = no of integers b/w 1 and 250 which are divisible by both 2 and 3

$$= \left[\frac{250}{2\times 3}\right] = \left[\frac{250}{6}\right] = 41$$

Similarly $|A_2 \cap A_3| = \left[\frac{250}{3\times 7}\right] = \left[\frac{250}{21}\right] = 11$

$$|A_3 \cap A_1| = \left[\frac{250}{7\times 2}\right] = \left[\frac{250}{14}\right] = 17$$

$$|A_1 \cap A_2 \cap A_3| = \left[\frac{250}{2\times 3\times 7}\right] = \left[\frac{250}{42}\right] = 5$$

From sum rule, we have

$$|A_1 \cup A_2 \cup A_3| = 125 + 83 + 35 - (41 + 11 + 17) + 5$$

$$= 179$$

∴ No of integers b/w 1 and 250 that are divisible by any of the integers 2, 3, 7 are 179

5/1/22 9) Out of 30 students in a hostel, 15 study history, 8 study economics, 6 study geography. It is known that 3 students study all these subjects. Show that seven or more students study none of these subjects

<u>Sol</u>: Given $|U| = 30$ where U is the set of all students in the hostel.

Let A_1, A_2, A_3 be the set of all students who study history, economics, geography respectively.

Given $|A_1| = 15$ $|A_2| = 8$ $|A_3| = 6$

$$|A_1 \cap A_2 \cap A_3| = 3$$

Now we have to find number of students who donot study none of the three subjects

i.e $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = ?$

WKT $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |\overline{A_1 \cup A_2 \cup A_3}|$

$$= |U| - |A_1 \cup A_2 \cup A_3|$$

$$= |U| - [|A_1| + |A_2| + |A_3| - \Sigma|A_i \cap A_j| + |A_1 \cap A_2 \cap A_3|] \quad \text{--- ①}$$

Let $\Sigma|A_i| = |A_1| + |A_2| + |A_3|$

$$= 15 + 8 + 6 = 29$$

$$|A_1 \cap A_2 \cap A_3| = 3$$

$$\Sigma|A_i \cap A_j| = S_1$$

From ① $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = 30 - 29 + S_1 - 3$

$$= S_1 - 2$$

We know that $A_1 \cap A_2 \cap A_3$ is a subset of $|A_i \cap A_j|$

$\therefore$ Each $|A_i \cap A_j|$ which are three in number is $\geq$ $|A_1 \cap A_2 \cap A_3|$

$\therefore S_1 = \Sigma|A_i \cap A_j| \geq 3|A_1 \cap A_2 \cap A_3|$

$$= 3(3)$$

$$= 9$$

$\therefore S_1 \geq 9$ —②

substituting ② in above equation

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = S_1 - 2$$
$$\geq 9 - 2$$
$$\geq 7$$

$$\therefore |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| \geq 7$$

PIGEONHOLE PRINCIPLE

If 'n' objects are put into 'm' containers with $n > m$ then atleast one container must contain more than one object.

Ex: If 8 people were chosen, atleast two of them will have born on the same day of a week.

NOTE: To apply the pigeon hole principle, we must decide which objects represents the pigeons and which objects represents the pigeon holes.

GENERALIZATION OF THE PIGEONHOLE PRINCIPLE:

If 'k' pigeons are placed into 'n' pigeonholes, then one of the pigeon hole must contain atleast integer part of $\left[\frac{k-1}{n}\right]+1$ pigeons.

Here $[x]$ denotes the greatest integer $\leq x$

Pb) Prove that if any 30 people are selected, we may choose a subset of 5, so that all 5 were born on the sameday of the week

Sol Assign each person to the day of the week on which he/she was born. Here 30 pigeons are being assigned to 7 pigeon holes.

∴ By generalization of pigeon hole principle, $k=30$ and $n=7$ we get

$$\left[\frac{30-1}{7}\right]+1$$

$$=\left[\frac{29}{7}\right]+1$$

$$=4+1$$

$$=5$$

Hence 5 of the people have born on the same day of the week.

Unit -3: Elementary combinatorics

* permutation:

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

* combination:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Eg: {A,B,C}

3 letters 2

$$^{3}P_{2} = \frac{3!}{1!} = 6 \text{ ways}$$

∴ AB, BA, BC, CA, AC, CB

→ combinatorics deals with the arrangement of objects according to some pattern and counting no. of ways which it can be done.

Basic rules of counting:

→ In many situations of computational work we employ two basic rules of counting namely:

a) Sum rule b) product rule

* Sum rule:

- If an event can occur in m ways and the other event can occur in n ways, if these 2 events cannot occur simultaneously then one of the 2 events can occur in (n+m) ways.

If there are 14 boys and 12 girls in a class. Then find the no. of ways of selecting

Sol: To select one student either from 14 boys or 12 girls as a CR, clearly the selection of one boy in 14 ways or selection of one girls in 12 ways hence from sum rule the no. of ways of selecting from boys/girls in 14+12 = 26 ways.

* product rule

- If an event can occur in m ways and 2nd event way can occur in n ways. If the number of ways 2nd event can occur doesn't depend on occurrence of 1st event. Then 2 events can occur simultaneously in (m × n) ways

Eg: Three persons enter into a car where there are 5 seats in how many ways can they take up three seats.

Sol: Given no. of persons = 3

no of seats = 5

- First person can sit in any of 5 seats.
- So no of ways of 1st person is 5 ways.
- now remaining seats are 4
- 2nd person can sit in one 4 seats
- no. of ways for 2nd person is 4 ways
- remaining = 3
- 3rd person can sit in one of 3 seats
- no of ways = 3
- remaining = 1

Therefore by product rule,

number of ways in which all 3 person can take seats is $5 \times 4 \times 3 = 60$ ways.

Note :- In set theory concept the above two rules can be written as:

Sum rule: If A, B are two disjoint sets then $|A+B| = |A| + |B|$

product rule: If $A \times B$ cartesian product of A, B then order of

$$|A \times B| = |A| \cdot |B|$$

- factorial notation:
  - The product of first n natural numbers is denoted by $n!$
  - i.e $n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$

  Eg $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

  $= 720$

  - $n!$ can be written as $n(n-1)!$

Eg: Find n if $(n+1)! = 12(n-1)!$.

$$\Rightarrow \frac{(n+1)!}{(n-1)!} = 12$$

$$\Rightarrow \frac{(n+1)\,n\,(n-1)!}{(n-1)!} = 12$$

$$\Rightarrow n^2 + n - 12 = 0$$

$$n = 3, -4$$

permutation: From a given finite set of elements selecting some or all of them and arranging in a sequential order is called permutation. That is, permutation is an arrangement of finite set of objects in a particular order. The no. of permutations of n distinct objects taken (^nP_r) r at a time is given by,

$$^nP_r = \frac{n!}{(n-r)!}$$

→ No of permutations of n dissimilar things taken r at a time:

- No. of ways of filling r blank places arranged in a row by n dissimilar things

Eg: permutation formed by 2 objects at a time from set of A, B, C is given by AB, BA, AC, CA, BC, CB.

i.e, $^3P_2 = \frac{3!}{1} = 6$ ways.

② permutation formed by 3 obj at a time from set of A, B, C is given by,

$$^3P_3 = \frac{3!}{0!} = 6 \text{ ways}$$

* permutation with repetitions: Out of n objects in a set p objects are exactly alike of 1st kind, q obj are exactly alike 2nd kind, r alike 3rd kind and rest are dissimilar.

number of ways of arranging: $\frac{n!}{p!\,q!\,r!}$

Eg: How many ways are there to arrange 9 letter word, "ALLAHABAD"

A - 4
L - 2
H - 1
D - 1
B - 1

$$= \frac{9!}{4! \times 2!} = \frac{9\times8\times7\times6\times5\times4!}{4!\times2!}$$

$= 72\times105 = 7560$

2) Find the no. of diff telephone num- formed by taking 3 digits from 1,2,3

Sol To form a telephone no., with repetitions by taking 3 digits from 1,2,3 in $3^3 = 27$ ways.

3) If the number of different telephone numbers formed by 3 digits from 1,2,3,4 is.

$4^3 = 64$

4) Combinations: Any selection which can be made by taking some or all object at a time and out of given no. of object is called combination.

The no of combination of n distinct obj taken r at a time is given by

$$^nC_r = \frac{n!}{r!(n-r)!}$$

order doesn't matter

selecting 2 from {A,B,C}

$$^nC_r = {}^3C_2 = \frac{3!}{2!\times1!} \qquad [AB, BC, AC]$$

= 3 ways

Note:

1) $^nC_0 = {}^nC_n = 1$

2) $^nC_r = {}^nC_{n-r}$

3) $^nC_1 = n = {}^nC_{n-1}$

Combinations with repetitions:

- with repetitions, no. of arrangments of n distinct obj taken r at a time is $^{(n+r-1)}C_r$

Results.

① Prove that $^nC_r + {}^nC_{r-1} = {}^{(n+1)}C_r$

LHS = $^nC_r + {}^nC_{r-1}$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$\frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n-r+1+r}{r(n-r+1)}\right]$$

$$\frac{n!\,(n+1)}{(r-1)!\,r\,(n-r)!\,(n-r+1)} = \frac{(n+1)!}{r!\,(n+1-r)!}$$

or

$= {}^{(n+1)}C_r$ => RHS

How many 4 digit numbers are there with distinct digits?

Solution

- We know that the distinct digit are 0,1,...,9 these are total no. of arrangements of 10 digits taken four at a time is ${}^{10}P_4$ but, these no. also includes the numbers which has 0 at thousands place since such num. are not 4 digit numbers
- Hence keeping 0 at thousands place then the 3 places can be filled up with the 9 digits in ${}^{9}P_3$ ways.
- Thus the total no. of 4 digit numbers is ${}^{10}P_4 - {}^{9}P_3$

$${}^{10}P_4 = \frac{10!}{(10-4)!} \qquad {}^{9}P_3 = \frac{9!}{(9-3)!}$$

$= 10\times9\times8\times7 \qquad = 9\times8\times7$

$= 5040 \qquad = 504$

$\Rightarrow 4536$

Q) How many 5 digit nos can be formed using digits 1,2,3,4,5 without repitition. How many of them are even?

Sol: Total no. of ways of 5 digit no.s formed using 5 digits 1,2,3,4,5 is ${}^{5}P_5 = 5!$ ways (120 ways)

- It must have either 4 or 2 at the unit place, if 2 is in unit's place then remaining digits are 1,3,4,5 for the remaining 4 places in 4! ways
- similarly if 4 is at unit place then the remaining 4 places arranged in 4! ways

Therefore by sum rule, total ways = 24 + 24 = 48 ways *circular permutation: If n distinct objects are arranged along the circumference of circle, then the no. of circular permutations of n objects taken all at a time is $(n-1)!$

- If we impose condition that no 2 objects be adjacent in any 2 arrangements then the required permutations are $\frac{(n-1)!}{2}$ (clockwise or anticlockwise)

Q) How many diff. ways can 5 gentleman and 5 women can sit in a round table if,

i) There is no restriction

ii) No 2 ladies sit together

Sol) given arrangment for 5 men and 5 women in round table is $(n-1)!$

i) $\Rightarrow (10-1)!$

$\Rightarrow 9!$

$\Rightarrow 362880$

ii) First let of all 5 men can be seated around the table.

no of ways of 5 gentlemen can sit around table is $(5-1)!$

$= 4!$

$= 24$

now b/w any 2 men a women can be seated. Hence all 5 ladies at 5 intermediate ways at 5! ways

hence required no of ways is

Q) A man has 7 relatives 4 of them are ladies and 3 of them are gentleman. His wife has again 7 other relatives, (3 girls, 4 boys) In how many ways can they sit at dinner that 3 from man's relatives and 3 from girls relatives.

Sol) Man side: 4 ladies 3men

women side: 3 ladies 4 men

dinner: 3 ladies 3men

Man	Women	
3 ladies	3 men	$= 4C_3 \times 4C_3$
3 men	3 ladies	$= 3C_3 \times 3C_3$
2L, 1m	2m, 1l	$= 4C_2 \times 3C_1 \times 3C_1 \times 4C_2$
1l, 2m	2l, 1m	$= 4C_1 \times 3C_2 \times 3C_2 \times 4C_1$

*when repetitions are allowed, the no. of arrangment of N distinct objects (r) (similar object) are taken at a time given as: $^{(n+r-1)}C_r$

- When repetions are allowed, the no. of arrangments for n-similar objects taken r (distinct) at a time is given by: $^{(n+r-1)}C_{r-1}$

n → similar objects r → distinct objects

Q) The numbers of 3 combinations of 5 distinct objects with unlimited repetitions.

now $n = 35$

$r = 3$

$^{(n+r-1)}C_r = 7C_3 = 35$

Q) Enumerate no. of ways of placing 20 indistinguishable balls into 5 boxes where each box is non empty.

Since it is given that 5 boxes are non empty $^{19}C_{15}$

- Place one ball in one box therefore we must

Q) Find the no. of non negative integral solutions of $x_1+x_2+x_3+x_4+x_5=50$

$n=50$ $\quad (n+r-1)_{C_r} = {}^{54}C_{50}$

$r=50$

I

$(n+r-1)_{C_{r-1}}$ $\quad [n=50, r=5]$

$(50+5-1)_{C_4} = {}^{54}C_4$

Q) How many integral solutions are there to $x_1+x_2+x_3+x_4+x_5=20$

where $x_1 \geq 3, x_2 \geq 2, x_3 \geq 4, x_4 \geq 6, x_5 \geq 0$

Sol first distribute 3 objects to x_1, 2 obj to x_2, 4 obj to x_3, 6 obj to x_4 and zero obj to x_5. Then remaining obj are 5 so,

we have to distribute 5 objects to the given 5 unknowns with unlimited repititions.

→ consider $x_1-3=y_1$

$x_2-2=y_2$

$x_3-4=y_3$

$x_4-6=y_4$

$x_5=y_5$

Such that,

$y_1+y_2+y_3+y_4+y_5=5$

here $n=5, r=5$

$(n+r-1)C_r = {}^9C_5$ such solutions exist

Q) Find the number of integral solutions to $x_1+x_2+x_3+x_4=50$

where $x_1 \geq -4, x_2 \geq 7, x_3 \geq -14, x_4 \geq 10$

* Binomial theorem:

$$(x+y)^n = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_n x^0 y^n$$

- Any sum of 2 unlike symbols such as

- The binomial theorem is a formula for powers of a binomial.

*Statement: If n is a small +ve integer then, 1

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$$

*remarks:

→ The expansion of $(x+y)^n$ contains $n+1$ terms.

→ The sum of powers (x,y) in each term is equal to 'n'.

→ The $(r+1)$th term in expansion of $(x+y)^n$ is ${}^nC_r x^{n-r} y^r$.

denoted by T_{r+1}.

→ The integers ${}^nC_0, {}^nC_1, \dots, {}^nC_n$ are called the binomial coefficients of the expansion. These are also denoted by $C_0, C_1, \dots, C_n$

- $C(n,0), C(n,1) \dots C(n,n)$

- $\binom{n}{r}$

→ The expansion of $(y-x)^n = {}^nC_0 y^n - {}^nC_1 y^{n-1}x + {}^nC_2 y^{n-2}x^2 + \cdots + (-1)^n {}^nC_n x^n$

Problems:

Q) Find the middle term of $\left(2x - \frac{1}{3x}\right)^{10}$

Q) $\left(x - \frac{3}{y}\right)^9$. Find the middle term

i) Given $\left(2x - \frac{1}{3x}\right)^{10}$

total terms $= n+1$
$= 11$

mid term $= 6$

$T_{r+1} = {}^nC_r x^{n-r} y^r$

$T_6 = T_{5+1} = {}^nC_5 x^{n-5} y^5$

$= {}^{10}C_5 x^5 y^5$

$= {}^{10}C_5 (2x)^5 \left(-\frac{1}{3x}\right)^5$

$= {}^{10}C_5 (2)^5 \left(\frac{-1}{3}\right)^5$

$= \frac{10!}{5!\,5!} (32) \times \left(\frac{-1}{3^5}\right)$

$= \frac{10\times9\times8\times7\times6\times5!}{120\times5!} \times \frac{32}{3^5}$

$= \frac{10\times8\times7\times6}{120} \times \frac{32}{3^5} = \frac{896}{27}$

ii) $\left(x - \frac{3}{y}\right)^9$

Total no of terms $= n+1 = 9+1 = 10$

middle term $= T_5, T_6$

$T_5 = {}^9C_4 x^{9-4} \left(\frac{-3}{y}\right)^4$

$T_6 = T_{5+1} = {}^9C_5 x^{9-5} \left(-\frac{3}{y}\right)^5$

② Coef. of $x^9 \cdot y^3$, $(2x-3y)^{12}$

Sol $(2x-3y)^{12} = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r$

$= \sum_{r=0}^{12} {}^{12}C_r (2x)^{12-r} y^r$ — ①

to get coef of $x^9 \cdot y^3$ taking $r=3$ in ①

So coef of x^9y^3 is ${}^{12}C_3 (2x)^{12-3} (-3y)^3$

$\Rightarrow -({}^{12}C_3 2^9 \cdot 3^3) x^9 y^3$

$\therefore$ coef of $x^9 \cdot y^3$ is $(-{}^{12}C_3 2^9 3^3)$

Q) Find the term independent of x in expansion: $\left(x^2 + \frac{1}{x}\right)^{12}$.

$T_{r+1} = {}^nC_r (x^2)^{n-r} (x^{-1})^r$

$= {}^nC_r (x)^{2(n-r)-r}$

$= {}^nC_r (x)^{2n-3r}$

$2n - 3r = 0$ // Independent term power → 0

$2(12) - 3r = 0$

$r = \frac{24}{3} = 8$

$\boxed{r=8}$

Q) Find 2 successive terms in the expansion of $(1+x)^{24}$ whose coeff. are in ratio 4:1

Sol Let 2 successive terms be

T_r and T_{r+1}

$T_{r+1} = {}^nC_r\, x^{n-r} y^r$

$= {}^{24}C_r (1)^{24-r} x^r$

$= {}^{24}C_r x^r$

$T_r = {}^nC_{r-1} x^{n-r+1} y^{r-1}$ (put $r = r-1$)

$= {}^nC_{r-1} (1)^{n-r+1} x^{r-1}$

$= {}^nC_{r-1} x^{r-1}$

Given,

$= \frac{{}^nC_{r-1}}{{}^{24}C_r} = \frac{4}{1}$

$= \frac{{}^{24}C_{r-1}}{{}^{24}C_r} = \frac{4}{1}$

$= \frac{24!}{(24-r+1)!(r-1)!} \times \frac{(24-r)!\,r!}{24!} = 4$

$= \frac{(24-r)!\,r(r-1)!}{(24-r+1)!\,(r-1)!} = \frac{(24-r)!\,r}{(24-r+1)(24-r)!} = 4$

$= r = 100 - 4r$ | $\boxed{r=20}$

$= 5r = 100$

Properties of Binomial Theorem:

① PT: $C(n,0) + C(n,1) + C(n,2) + \cdots + C(n,n)$ (or)

$C_0 + C_1 + C_2 + \cdots + C_n$

Sol: consider, $(1+x)^n$

$\Rightarrow$ put $x = 1$

2^n

....

② PT $C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + \cdots = 2^{n-1}$

$(1+x)^n = {}^nC_0 + {}^nC_1 x + \cdots + {}^nC_n x^n$

put $x = -1$

$0 = C_0 + C_1(-x) + C_2(-x)^2 + \cdots + C_n(-x)^n$

$0 = C_0 - C_1 x + C_2 x^2 - C_3 x^3 + \cdots + (-1)^n C_n x^n$

$\Rightarrow C_0 + C_1 + C_2 + \cdots = C_1 + C_3 + C_5 + \cdots$

$= \frac{1}{2}[C_0 + C_1 + C_2 + \cdots + C_n]$

$\Rightarrow C_0 + C_2 + \cdots = C_1 + C_3 + \cdots = \frac{1}{2}(2^n)$

$\therefore C_0 + C_2 + \cdots = C_1 + C_3 + \cdots = 2^{n-1}$

Multinomial Theorem:

• The sum of 'r' unlike things say $x_1+x_2+\dots+x_r$ is called a multinomial.

Let n be a +ve integer then $\forall\ x_1, x_2 \dots$ we have

$$(x_1+x_2+x_3+x_4+\dots+x_r)^n = \Sigma \binom{n}{q_1, q_2, q_3, \dots, q_r} x_1^{q_1} \cdot x_2^{q_2} \cdots x_r^{q_r}$$

where $q_1+q_2+q_3+\dots+q_r = n$

and $\binom{n}{q_1\ q_2 \cdots q_r} = \dfrac{n!}{q_1!\,q_2! \cdots q_r!}$

• where the summation extends overall set off non negative integers.

Note: The general term in the expansion of $(x_1+x_2+x_3+\dots+x_r)^n$ is

$$= \frac{n!}{q_1!\,q_2! \cdots q_r!} \times x_1^{q_1} \cdot x_2^{q_2} \cdots x_r^{q_r}$$

• Number of terms in expansion of above is $^{(n+r-1)}C_{r-1}$ (or)

$^{(n+r-1)}C_n$

Problems:

Q) Evaluate $\binom{18}{4\ 5\ 6\ 3}$.

$$\frac{18!}{4! \times 5! \times 6! \times 3!} x_1^4 x_2^5 x_3^6 x_3^3$$

Q) no. of terms in $(x-7y+3z-w)^{25}$

Sol $n=25$ $r=4$ $\Rightarrow$ $^{(25+4-1)}C_3 = {}^{28}C_3$

ii) $(x-7y+3z-w)^{25}$ $[x^5 y^{10} z^5 w^5]$

$\Rightarrow \dfrac{25!}{5!\,10!\,5!\,5!}$

$\Rightarrow \dfrac{25!}{5!\,10!\,5!\,5!} (1)^5 (-7)^{10} (3)^5 (-1)^5$

$\Rightarrow \dfrac{-25!}{5!\,10!\,5!\,5!} (7)^{10} (3)^5$

Q) Find the term which contains x^{11} and y^4 in the expansion of $(2x^3+3$

$$\frac{6!}{2!\,3!} (2)^3 (3)^2$$

Inclusion - exclusion principle

★ If $A \cap B = \phi$ Then $|A \cup B| = |A| + |B|$

★ If $A \cap B \neq \phi$ Then $|A \cup B| = |A| + |B| - |A \cap B|$

• Consider a finite set of elements of n numbers then the no

n is called the cardinality of set (or) size of set (or)

Set

• denoted by $n(S), |S|, O(S)$

Eg: $|A| = 5$ $|B| = 6$
$|A \cap B| = 1$ Then $|A \cup B| = 5+6-1 = 10$

② $A = \{1,2,6\}$

$B = \{a,b,c,d\}$

then $|A| = 3$

$|B| = 4$

$|A \cup B| = |A| + |B| \quad (A \cap B \neq \phi)$

$= 7$

Note:

- for any 2 finite sets A, B

  - $|\phi| = 0$
  - $|A| \geq 1$

If $A \subseteq B$

$|A| \leq |B|$

- comp denoted by $\bar{A}$ (or) A' given by $|A'| = |U| - |A|$

- If $A \cap B \neq \phi$

$|A \cup B| = |A| + |B| - |A \cap B|$

- If $A \cap B = \phi$ then,

$|A \cup B| = |A| + |B|$

- The above formula can be extended for 3 sets A, B, C is

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A|$

$+ |A \cap B \cap C|$

- The above formulas are called the principal of inclusion and exclusion principle.

- If U is a universal sets and A, B are subsets of U then $|\bar{A} \cap \bar{B}| = |\overline{A \cup B}|$

$= U - |A \cup B|$

$= U - [|A| + |B| - |A \cap B|]$ — ①

- If no. of elements in $|A \cap B| = \phi$ Then ① reduces to

$= |\bar{A} \cap \bar{B}| = |U| - [|A| - |B|]$

This formula is called principle of disjunctive counting.

problem:

Q) Find the no. of integers b/w 1 and 250 that are divisible by any of one of the integers 1,2,3,7.

$A = \{\}$ $B = \{\}$ $C = \{\}$

Sol let A, B, C be sets which divisible by 2, 3, 7

$|A| = \left[\frac{250}{2}\right] = 125$ $\qquad |A_1 \cap A_2| = \left[\frac{250}{2 \times 3}\right] = 41$

$|B| = \left[\frac{250}{3}\right] = 83$ $\qquad |A_2 \cap A_3| = \left[\frac{250}{3 \times 7}\right] = 11$

$|C| = \left[\frac{250}{7}\right] = 35$ $\qquad |A_3 \cap A_1| = \left[\frac{250}{7 \times 2}\right] = 17$

$|A_1 \cap A_2 \cap A_3| = \left[\frac{250}{2 \times 3 \times 7}\right] = 5$

$\Rightarrow |A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_3 \cap A_2| - |A_3 - A_1|$

$= 179$

Q) Certain computer sector employees 100 computer programs of these 47 can program in Fortran, 35 in pascal, 23 in both. How many can program neither of programs.

Sol

$|A_1| = 47$

$|A_2| : 33$

$|A_1 \cap A_2| = 23$

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

$= 47 + 35 - 23$

$= 59$

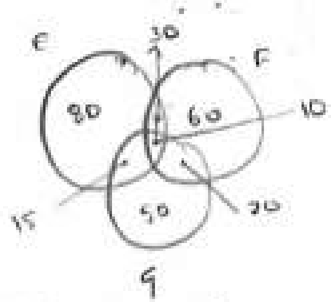
∴ who cannot program $[100 - 59] = 41$

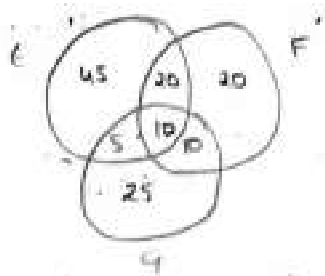
Q) In a language survey of studes it is found that 80 knows eng, 60 → French, 50 → German, 30 → Eng & Fre, 20 → F & Ge, 15 → E & G, 10 → E, F, G.

i) How many know atleast one language

ii) only english

iii) French and one but not both E and G.





Q) Out of 30 students in a hostel 15 study history, 8 academics, 6 study geography. It is known that 3 study all. Show that 7 or more students can study none.