

Recurrence RelationsGenerating functions of sequences

A function $a: \mathbb{Z}^+ \rightarrow \mathbb{R}$ is called sequence of real numbers.

We use $A = \{a_n\}_{n=0}^{\infty}$ to denote such sequences.

Example $A = \{3^n\}_{n=0}^{\infty}$ is a sequence

$$A = \{1, 3, 9, 27, 81, \dots, 3^n, \dots\}$$

Generating functions

Defn Let $A = \{a_n\}_{n=0}^{\infty} = \{a_0, a_1, \dots, a_n, \dots\}$

be the sequence. Then its generating function is defined to be $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

Example

The generating function of the sequence $\{3^n\}_{n=0}^{\infty}$

is $A(x) = 1 + 3x + 3^2 x^2 + 3^3 x^3 + \dots + 3^n x^n + \dots$

Example

Find the generating function for the following sequence $\{(-1)^{n-1}\}_{n=1}^{\infty}$

Solution

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✓ Sequence of $\{(-1)^{n-1} \binom{n-1}{n}\}$ is

$$\{1, -2, 3, -4, \dots\}$$

∴ generating function for the above sequence

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^{n-1} \binom{n-1}{n} x^n$$

$$A(x) = 1 - 2x + 3x^2 - 4x^3 + \dots$$

closed form expressions for generating function

Example

Find the generating function for $a_n = 3^n$, $n \geq 0$
in closed form

Solution

The generating function for the given

$$\text{sequence is } A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 3^n x^n$$

$$A(x) = 1 + 3x + 3^2 x^2 + 3^3 x^3 + \dots$$

which is a ^{infinite} geometric series with common ratio $3x$.

So by using sum of infinite terms of geometric series we can write $\left(\frac{a}{1-r}\right)$

$$A(x) = \frac{1}{1-3x}$$

(3)

Example
Find the sequences generated by the following functions

i) $(2+x)^3$

$$= 2^3 \left(1 + \frac{x}{2}\right)^3 = 8 \left({}^3C_0 1^3 \left(\frac{x}{2}\right)^0 + {}^3C_1 1^2 \frac{x}{2} + {}^3C_2 1 \left(\frac{x}{2}\right)^2 + {}^3C_3 \left(\frac{x}{2}\right)^3 \right)$$

$$= 8 \left(1 + \frac{3}{2}x + \frac{3}{4}x^2 + \frac{1}{8}x^3 \right)$$

The sequence generated by $(2+x)^3$ is

$$8, 12, 6, 1, 0, 0, 0, \dots$$

Example

Find the sequence generated by the following function $2x^2(1-x)^{-1}$

$$= 2x^2 (1 + x + x^2 + x^3 + \dots)$$

$$= 2x^2 + 2x^3 + 2x^4 + \dots$$

$$= 0 + 0x + 2x^2 + 2x^3 + \dots$$

The required sequence is

$$0, 0, 2, 2, 2, \dots$$

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Problem

Find the closed form expression for the
generating function of the sequence
0, 1, 2, 3, ...

Solution

generating function is

$$0 + 1x + 2x^2 + 3x^3 + \dots$$

$$x(1 + 2x + 3x^2 + \dots) = x(1-x)^{-2}$$

$$\underline{\underline{\frac{x}{(1-x)^2}}}$$

Problem

Find the closed form expression for the
✓ generating function of the sequence

$$1^2, 2^2, 3^2, \dots$$

Soln

we have we have see

$$1 + 2(2x) + 2x^2 + 3x^3 + \dots = \frac{x}{(1-x)^2} \quad \text{--- (1)}$$

$$\text{diff } 1 + 2(2x) + 3(3x^2) = \frac{d}{dx} \left(\frac{x}{(1-x)^2} \right)$$

$$= \frac{(1-x)^2(1) + x(2(1-x))}{(1-x)^4}$$

$$= \frac{1-x+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$$

Example

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Let 'n' be a +ve integer and

 $a_k = C(n, k)$ for $k = 0, 1, 2, \dots, n$. Find the

generating function for the sequence

 (a_0, a_1, \dots, a_n) Solution

The generating function for the given

sequence is $\sum_{k=0}^n a_k x^k$

$$A(x) = nC_0 + nC_1 x + nC_2 x^2 + \dots + nC_n x^n$$

$$= \underline{\underline{(1+x)^n}}$$

~~calculating coefficient of generating function~~ExampleFind the generating function for the sequence $(1, 1, 1, 1, 1, 1)$ in ~~the~~ closed formSolutionGenerating function of $(1, 1, 1, 1, 1, 1)$ is

$$A(x) = 1 + 1x + 1x^2 + 1x^3 + 1x^4 + 1x^5$$

$$= \frac{x^6 - 1}{x - 1}$$

$$\left[\frac{\text{By P.D. } (x^6 - 1)}{x - 1} \right]$$

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Table of generating functions

S.No	sequence	generating function
1	$(1+x)^k$	$C(k, n)$
2	1	$\frac{1}{1-x}$
3	a^n	$\frac{1}{1-ax}$
4	$(-1)^n$	$\frac{1}{1+x}$
5	$(-1)^n a^n = (-a)^n$	$\frac{1}{1+ax}$ $\frac{1}{1+ax}$
6	$C(k-1+n, n)$ k is a fixed +ve integer	$\frac{1}{(1-x)^k}$
7	$C(k-1+n, n) a^n$	$\frac{1}{(1-ax)^k}$
8	$C(k-1+n, n) (-a)^n$	$\frac{1}{(1+ax)^k}$
9	$n+1$	$\frac{1}{(1-x)^2}$
10	n	$\frac{x}{(1-x)^2}$
11	n^2	$\frac{x(1+x)}{(1-x)^3}$

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Calculating coefficient of generating function

Example

Find the coefficient of x^{27} in

$$(x^4 + x^5 + x^6)^5$$

Solution we have $(x^4 + x^5 + x^6)^5$

$$= (x^4)^5 (1 + x + x^2)^5$$

$$= x^{20} \left[(1-x)^{-1} \right]^5$$

$$= x^{20} (1-x)^{-5}$$

$$= x^{20} \sum_{r=0}^{\infty} C(5+r-1, r) x^r$$

$$= x^{20} \sum_{r=0}^{\infty} C(4+r, r) x^r$$

$$= \sum_{r=0}^{\infty} C(4+r, r) x^{20+r}$$

Take $20+r=27$ then $r=7$

\therefore The coefficient of x^{27} is $C(4+7, 7)$

$$= C(11, 7) = \frac{11!}{7!4!}$$

$$= \underline{\underline{330}}$$

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ExampleFind the coefficient of x^{20} in

$$(x + x^2 + 2x^3 + x^4)(x^2 + x^3 + x^4 + \dots)^5$$

Solution

$$\text{We have } (x + x^2 + 2x^3 + x^4)(x^2 + x^3 + x^4 + \dots)^5$$

$$= x(1 + x + 2x^2 + x^3)(x^2)^5(1 + x + x^2 + \dots)^5$$

$$= x^{11}(1 + x + 2x^2 + x^3)[(1 - x)^{-1}]^5$$

$$= x^{11}(1 + x + 2x^2 + x^3)(1 - x)^{-5}$$

$$= x^{11}(1 + x + 2x^2 + x^3) \sum_{r=0}^{\infty} C(4+r, r) x^r$$

$$\therefore \text{Coefficient of } x^{20} \text{ in } (x + x^2 + 2x^3 + x^4)(x^2 + x^3 + x^4 + \dots)^5$$

$$\text{is } C(4+9, 9) + C(4+8, 8) + 2C(4+7, 7) + C(4+6, 6)$$

$$= C(13, 9) + C(12, 8) + 2C(11, 7) + C(10, 6)$$

ExampleFind the coefficient of x^{12} in

$$\frac{x^2}{(1-x)^{10}}$$

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Solution

$$\frac{x^2}{(1-x)^{10}} = x^2 (1-x)^{-10}$$

$$= x^2 \sum_{r=0}^{\infty} C(10+r-1, r)$$

$$= x^2 \sum_{r=0}^{\infty} C(10+r-1, r) x^r$$

$$= x^2 \sum_{r=0}^{\infty} C(9+r, r) x^r$$

∴ The

∴ The coefficient of x^{12} is $C(19, 10)$ (Here $r=10$)Example #Find the coefficient of x^{12} in

$$\frac{x^2 - 3x}{(1-x)^4}$$

Solution

$$\frac{x^2 - 3x}{(1-x)^4} = (x^2 - 3x) (1-x)^{-4}$$

$$= (x^2 - 3x) (1-x)^{-4}$$

$$= (x^2 - 3x) \sum_{r=0}^{\infty} C(4+r-1, r) x^r$$

$$(x^2 - 3x) \sum_{r=0}^{\infty} C(3+r, r) x^r$$

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The coefficient of x^{12} is

$$C(13, 10) - 3C(14, 11)$$

|

Here $r = 10$

|

Here $r = 11$

(11)

Find the generating function that determines the number of non-negative integer solutions of the

$$\text{equation } x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

under the constraints $0 \leq x_1 \leq 3$, $0 \leq x_2 \leq 4$, $2 \leq x_3$

$2 \leq x_4 \leq 5$, x_5 is odd with $x_5 \leq 7$

solution

$$f_1(x) = x^0 + x^1 + x^2 + x^3$$

$$f_2(x) = x^0 + x^1 + x^2 + x^3 + x^4$$

$$f_3(x) = x^2 + x^3 + x^4 + x^5 + x^6$$

$$f_4(x) = x^2 + x^3 + x^4 + x^5$$

$$f_5(x) = x^1 + x^3 + x^5 + x^7 + x^9$$

Sum of n term of
a geometric
series

$$S_n = \frac{a(1-r^n)}{1-r}$$

Hence the generating function for the number of non-negative integer solutions of the given equation under the given constraints is

$$f(x) = f_1(x) f_2(x) f_3(x) f_4(x) f_5(x)$$

$$\frac{1-x^4}{1-x} \cdot \frac{1-x^5}{1-x} \cdot x^2 \cdot \frac{1-x^5}{1-x} \cdot x^2 \cdot \frac{1-x^4}{1-x} \cdot x(1+x^2+x^4+x^6+x^8)$$

$$x^5 (1-x^4)^2 (1-x^5)^2 (1-x)^{-4} \frac{(1-(x^2)^5)}{1-x^2}$$

$$\underline{\underline{x^5 (1-x^4)^2 (1-x^5)^2 (1-x)^{-5} (1+x)^{-1} (1-x^{10})}}$$

2. Build a generating function for $a_r =$ the number of integral solutions to the equation $e_1 + e_2 + e_3 = r$ if $0 < e_i$ for each i .

Solution

$0 < e_i$ means $1 \leq e_i$

$$\begin{aligned} \text{Generating function is } & (x + x^2 + x^3 + \dots)^3 \\ & x^3 (1 + x + x^2 + \dots)^3 \\ & x^3 \left[(1-x)^{-1} \right]^3 \\ & \underline{\underline{x^3 [1-x]^{-3}}} \end{aligned}$$

3. Find a generating function for $a_r =$ the number of ways the sum $|r|$ can be obtained when (1) 2 distinguishable dice are tossed (2) 2 distinguishable dice are tossed and the first shows an even number and the second shows an odd number

Solution

i) We want the number of integral solutions

$$e_1 + e_2 = r \quad \text{where } 1 \leq e_i \leq 6$$

Then a_r is the coefficient of x^r in the generating function $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$

ii) We want the number of integral solutions

$$\text{to } e_1 + e_2 = r \quad 1 \leq e_1 \leq 6$$

and e_1 is even while $1 \leq e_2 \leq 6$ and e_2 is odd

Then a_r is the coefficient of x^r in the generating function $(x^2 + x^4 + x^6)(x + x^2 + x^3 + x^4 + x^5 + x^6)$

Problem

In how many ways can we distribute 24 pens to 4 children so that each child gets at least 3 pens but not more than eight

Solution

The given problem is equivalent to the problem of finding the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 24$ with $3 \leq x_i \leq 8$ for each x_i . Keeping the constraints on x_i in mind, let us consider the

functions $f_i(x) = x^3 + x^4 + x^5 + x^6 + x^7 + x^8$
 $i = 1, 2, 3, 4$

Therefore, the generating function for the problem is

$$f(x) = f_1(x) f_2(x) f_3(x) f_4(x)$$

$$(x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4$$

The required number is the coefficient of x^{24} in this function

Now we find that $f(x) = x^{12} (1 + x + x^2 + x^3 + x^4 + x^5)^4$

$$= x^{12} \left(\frac{1-x^6}{1-x} \right)^4$$

$$= x^{12} (1-x^6)^4 (1-x)^{-4}$$

$(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$

$$x^{12} \sum_{r=0}^{\infty} \binom{4+r-1}{r} (-x^6)^r \sum_{s=0}^{\infty} \binom{4+s-1}{s} x^s$$

$$x^{12} \sum_{r=0}^{\infty} (-1)^r \binom{4+r-1}{r} x^{6r} \cdot \sum_{s=0}^{\infty} \binom{3+s}{s} x^s$$

from this we find the coefficient of x^{24} in $f(x)$

is $\sum_{r=0, s=12} \binom{4}{r} \binom{5}{s} = \binom{4}{0} \binom{5}{12} + \binom{4}{1} \binom{5}{11} + \binom{4}{2} \binom{5}{10} + \binom{4}{3} \binom{5}{9} + \binom{4}{4} \binom{5}{8}$

$$= 455 + 366 + 76 + 125 = 125$$

Recurrence relations

Defn A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely a_0, a_1, \dots, a_{n-1} , for all integers $n \geq 1$.

Eg: $a_n - 3a_{n-1} + 2a_{n-2} = 0$ — Homogeneous recurrence relation

Eg: $a_n - 5a_{n-1} + 6a_{n-2} = n^2 + 1$ — inhomogeneous recurrence relation

The Fibonacci Recurrence relation

The recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ with initial conditions $F_0 = F_1 = 1$, is known as Fibonacci recurrence relation.

Solutions of recurrence relations

A sequence $\{a_n\}_{n=0}^{\infty}$ is said to be a solution of a recurrence relation if each value a_n (i.e., a_0, a_1, \dots, a_{n-1}) satisfies the recurrence relation.

Ex $\{a_n\}_{n=0}^{\infty}$ where $a_n = 2^n$ is the solution of the recurrence relation $a_n = 2a_{n-1}$, $n \geq 1$.

Methods of solving recurrence relationsSubstitution method

1. Solve the recurrence relation $a_n = a_{n-1} + f(n)$ for $n \geq 1$ by substitution method.

Solution

$$\text{Here } a_1 = a_0 + f(1)$$

$$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$$

$$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$$

$$\vdots$$

$$a_n = a_{n-1} + f(n) = a_0 + f(1) + f(2) + \dots + f(n)$$

$$= a_0 + \sum_{k=1}^n f(k)$$

9. solve the recurrence relation

$$a_n = a_{n-1} + n^2, \text{ where } a_0 = 7 \text{ by substitution}$$

method

Solution

given

$$a_0 = 7$$

$$a_1 = a_0 + 1^2 = 7 + 1^2$$

$$a_2 = a_1 + 2^2 = 7 + 1^2 + 2^2$$

$$a_3 = a_2 + 3^2 = 7 + 1^2 + 2^2 + 3^2$$

$$\vdots$$

$$a_n = 7 + (1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$= 7 + \frac{n(n+1)(2n+1)}{6}$$

$$=$$

3. solve

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$$a_n = a_{n-1} + 3^n \text{ where } a_0 = 1$$

solution

$$a_0 = 1$$

$$a_1 = a_0 + 3^1 = 1 + 3$$

$$a_2 = a_1 + 3^2 = 1 + 3 + 3^2$$

$$a_3 = a_2 + 3^3 = 1 + 3 + 3^2 + 3^3$$

⋮

$$a_n = 1 + 3 + 3^2 + \dots + 3^n$$

$$a_n = \frac{3^{n+1} - 1}{3 - 1} = \frac{3^{n+1} - 1}{2} \text{ is the solution}$$

$$\left(S_n = a \left(\frac{r^{n+1} - 1}{r - 1} \right) \right)$$

4. If a_n is a solution of $a_{n+1} = k a_n$ for $n \geq 0$

and $a_3 = \frac{153}{49}$ and $a_5 = \frac{1377}{2401}$ find the value of k .

solution

The general solution is $a_{n+1} = k a_n$ is

$$\text{That is } a_n = k^n a_0 ; n \geq 1$$

$$\text{Hence } a_3 = k^3 a_0 \Rightarrow k^3 a_0 = \frac{153}{49}$$

$$a_5 = k^5 a_0 \Rightarrow k^5 a_0 = \frac{1377}{2401}$$

$$\frac{k^5 a_0}{k^3 a_0} = \frac{1377}{2401} \times \frac{49}{153} = \frac{9}{49}$$

$$k^2 = \frac{9}{49} \Rightarrow k = \pm \frac{3}{7}$$

Method - 2

Method of Generating functions

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

1. $\sum_{n=k}^{\infty} a_n x^n = A(x) - a_0 - a_1 x - \dots - a_{k-1} x^{k-1}$

2. $\sum_{n=k}^{\infty} a_{n-1} x^n = x \left[A(x) - a_0 - a_1 x - \dots - a_{k-2} x^{k-2} \right]$

3. $\sum_{n=k}^{\infty} a_{n-2} x^n = x^2 \left[A(x) - a_0 - a_1 x - \dots - a_{k-3} x^{k-3} \right]$

4. $\sum_{n=k}^{\infty} a_{n-3} x^n = x^3 \left[A(x) - a_0 - a_1 x - \dots - a_{k-4} x^{k-4} \right]$

5. $\sum_{n=k}^{\infty} a_{n-k} x^n = x^k \left[A(x) \right]$

Where $A(x)$ is called a generating function for a given recurrence relation.

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Q. 1) Solve the recurrence relation using generating function

$$a_n - 7a_{n-1} + 10a_{n-2} = 0 \text{ for } n \geq 2$$

$$a_0 = 10, a_1 = 41$$

Solution

Step 1. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

Step 2. Consider the given recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$. Next multiply each term in the

recurrence relation by x^n and sum from 2 to ∞

$$\sum_{n=2}^{\infty} a_n x^n - 7 \sum_{n=2}^{\infty} a_{n-1} x^n + 10 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

Step 3. Replace each infinite sum by an expression from the equivalent expressions:

$$[A(x) - a_0 - a_1 x] - 7x[A(x) - a_0] + 10x^2[A(x)] = 0$$

Step 4. Simplify

$$A(x)(1 - 7x + 10x^2) = a_0 + a_1 x - 7a_0 x$$

$$A(x) = \frac{a_0 + a_1 x - 7a_0 x}{1 - 7x + 10x^2} = \frac{a_0 + x(a_1 - 7a_0)}{(1-2x)(1-5x)}$$

Given $a_0 = 10$
 $a_1 = 41$

$$= \frac{10 + x(41 - 70)}{(1-2x)(1-5x)}$$

step 5. Decompose $A(x)$ as a sum of partial fractions: (20)

$$A(x) = \frac{10 - 29x}{(1-2x)(1-5x)}$$

using:

$$\frac{10 - 29x}{(1-2x)(1-5x)} = \frac{A}{1-2x} + \frac{B}{1-5x}$$

$$10 - 29x = A(1-5x) + B(1-2x) \quad \text{--- (1)}$$

put $x = \frac{1}{5}$ in (1)

$$10 - 29\left(\frac{1}{5}\right) = A(0) + B\left(1 - 2\left(\frac{1}{5}\right)\right)$$

$$\frac{50 - 29}{5} = \frac{3}{5} B$$

$$21 = 3B$$

$$B = \underline{\underline{7}}$$

put $x = \frac{1}{2}$ in (1)

$$10 - 29\left(\frac{1}{2}\right) = A\left(1 - \frac{5}{2}\right) + B(0)$$

$$-\frac{9}{2} = -\frac{3}{2} A$$

$$9 = 3A \quad A = \underline{\underline{3}}$$

step 6 Express $A(x)$ as a sum of series

$$\begin{aligned} A(x) &= \frac{3}{1-2x} + \frac{7}{1-5x} \\ &= 3 \sum_{n=0}^{\infty} 2^n x^n + 7 \sum_{n=0}^{\infty} 5^n x^n \end{aligned}$$

(step 1) Express a_n as the coefficient of x^n in $A(x)$ and in the sum of the other series:

$$a_n = (3)2^n + (1)5^n$$

(Q2) using generating functions solve the recurrence relation $a_n - 4a_{n-1} + 3a_{n-2} = 0, n \geq 2$ with initial conditions $a_0 = 2$ and $a_1 = 4$

Solution

step 1. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

step 2 consider the given recurrence relation

$$a_n - 4a_{n-1} + 3a_{n-2} = 0 \quad \text{--- (1)}$$

Multiply each term in (1) by x^n and sum from 2 to ∞ , we get

$$\sum_{n=2}^{\infty} a_n x^n - 4 \sum_{n=2}^{\infty} a_{n-1} x^n + 3 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

step 3 Replace each infinite sum by an expression from the equivalent expressions

$$[A(x) - a_0 - a_1 x] - 4x[A(x) - a_0] + 3x^2[A(x)] = 0$$

step 4 simplify

$$A(x)[1 - 4x + 3x^2] = a_0 + a_1 x - 4a_0 x$$

$$A(x) = \frac{a_0 + a_1 x - 4a_0 x}{1 - 4x + 3x^2} \quad (22)$$

Given $a_0 = 2, a_1 = 4$

$$\therefore A(x) = \frac{2 + 4x - 8x}{1 - 4x + 3x^2} = \frac{2 - 4x}{(1-x)(1-3x)}$$

(Step 5) Decompose $A(x)$ as a sum of partial fractions

$$A(x) = \frac{1}{1-x} + \frac{1}{1-3x}$$

Step 6 Express $A(x)$ as sum of infinite series

$$\begin{aligned} A(x) &= (1-x)^{-1} + (1-3x)^{-1} \\ &= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (3x)^n \end{aligned}$$

Step 7 ~~Ex~~ Express a_n as the coefficient of x^n

$$\underline{\underline{a_n = 1^n + 3^n}}$$

Q1

Solve $a_n - 3a_{n-1} - 2 = 0$, $n \geq 1$ with $a_0 = 1$

by generating functions

Solution

Given recurrence relation is

$$a_n - 3a_{n-1} - 2 = 0 \quad (1)$$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function

Multiply each term in (1) by x^n and sum from 1 to ∞ , we get-

$$\sum_{n=1}^{\infty} a_n x^n - 3 \sum_{n=1}^{\infty} a_{n-1} x^n - 2 \sum_{n=1}^{\infty} x^n$$

Replace the infinite sums by equivalent expressions

we get-

$$[A(x) - a_0] - 3x[A(x)] - 2x \sum_{n=1}^{\infty} x^{n-1}$$

$$\Rightarrow A(x) [1 - 3x] = a_0 + 2x(1-x)^{-1}$$

Given $a_0 = 1$

$$A(x)(1-3x) = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}$$

$$A(x) = \frac{1+x}{(1-x)(1-3x)} = \frac{-1}{1-x} + \frac{2}{1-3x}$$
$$= -(1-x)^{-1} + 2(1-3x)^{-1}$$

$$= - \sum_{n=0}^{\infty} x^n + 2 \sum_{n=0}^{\infty} (3x)^n \quad (24)$$

$$\therefore \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(-1) + 2 \cdot 3^n \right] x^n$$

$$a_n = \underline{-1 + 2(3)^n} \quad \text{required solution}$$

Q2) Solve the recurrence relation

$$a_n - 6a_{n-1} = 0 \quad \text{for } n \geq 1 \quad \text{and } a_0 = 1$$

by using generating functions

Solution

Given recurrence relation is

$$a_n - 6a_{n-1} = 0 \quad \text{--- (1)}$$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function

Multiply each term in (1) by x^n and sum from 1 to ∞

$$\text{we get } \sum_{n=1}^{\infty} a_n x^n - 6 \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

Replace the infinite sum by equivalent expressions we get-

$$\begin{aligned} [A(x) - a_0] - 6x[A(x)] &= 0 \\ A(x)[1 - 6x] &= a_0 \end{aligned}$$

given $a_0 = 1$

(25)

$$A(x) = \frac{1}{1-6x} = (1-6x)^{-1}$$

$$= \sum_{n=0}^{\infty} (6x)^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 6^n x^n$$

$\therefore a_n = 6^n$ which is the required solution

(Q3) solve the recurrence relation
 $a_n - 9a_{n-1} + 20a_{n-2} = 0$ for $n \geq 2$

given that $a_0 = -3$, $a_1 = -10$.

$$\text{Ans } \underline{\underline{2 \cdot 5^n - 5 \cdot 4^n}}$$

Method of characteristic roots (2)

Defn

$$\text{Let } a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = 0,$$

$$n \geq k, c_k \neq 0 \quad \text{--- (1)}$$

be a linear homogeneous recurrence relation of degree k . Then the equation $t^k + c_1 t^{k-1} + c_2 t^{k-2} + \dots + c_k = 0$ --- (2)

is called the characteristic equation of the given recurrence relation (1).

Degree of (2) is k , therefore it has ' k ' roots

Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be the roots of the equation (2)

and the roots $\alpha_1, \alpha_2, \dots, \alpha_k$ are called characteristic roots.

There are two types of characteristic roots

1. If the characteristic equation of a linear homogeneous recurrence relation of degree k has ' k ' distinct roots say $\alpha_1, \alpha_2, \dots, \alpha_k$ then

$$a_n = c_1 \alpha_1^n + c_2 \alpha_2^n + \dots + c_k \alpha_k^n$$

where c_1, c_2, \dots, c_k are constants, is the general solution of the given recurrence relation

2. If the characteristic equation of a linear homogeneous recurrence relation of degree k has a root ' λ ' repeated k times then

$$a_n = (D_1 + D_2 n + D_3 n^2 \dots D_k n^{k-1}) \lambda^n$$

where D_1, D_2, \dots, D_k are constants, is the general solution of the given recurrence relation

Example 1 solve $a_n - 3a_{n-1} + 2a_{n-2} = 0$ for $n \geq 2$

Solution

The characteristic equation is

$$t^2 - 3t + 2 = 0$$

$$(t-1)(t-2) = 0$$

$$t = 1, 2$$

The general solution is $a_n = C_1 1^n + C_2 2^n$

Example 2

Solve $a_n - 3a_{n-1} -$

$$9a_{n-2} = 0$$

Solution

The characteristic equation is

$$t^2 - 6t + 9 = 0$$

$$(t-3)^2 = 0$$

The general solution is $a_n = (D_1 + D_2 n) 3^n$

Example

Write the general form of the solution to

$$a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$$

Solution

characteristic equation is

$$t^3 - 3t^2 + 3t - 1 = 0$$

$$(t-1)^3 = 0$$

$$t = 1, 1, 1$$

The general solution is

$$a_n = (D_1 + D_2 n + D_3 n^2) 1^n$$

Example

solve the recurrence relation

$$a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0 \text{ for } n \geq 3$$

with the initial conditions $a_0 = 1, a_1 = 4, a_2 = 8$

Solution

The characteristic equation is

$$t^3 - 7t^2 + 16t - 12 = 0$$

$$f(t) = f(2) = 0$$

$(t-2)$ is a factor of $f(t)$

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$f(t) = (t-2)(t^2 - 5t + 6)$$

$$(t-2)^3(t-3)$$

$$t = 2, 2, 3 //$$

The general solution is (29)

$$a_n = D_1 2^n + D_2 n 2^n$$

$$a_n = (D_1 + D_2 n) 2^n + D_3 3^n \text{ --- (1)}$$

Put $n=0$ in (1)

$$a_0 = D_1 + D_3 \text{ --- (2)}$$

$$\text{Put } n=1 \quad a_1 = (D_1 + D_2) 2 + D_3 (3) \text{ --- (3)}$$

Put $n=2$

$$a_2 = (D_1 + 2D_2) 2^2 + D_3 (3^2)$$

$$= (D_1 + 2D_2) 4 + 9D_3 \text{ --- (4)}$$

Solve (2), (3) & (4) we get

$$D_1 = 5, \quad D_2 = 3 \quad D_3 = -4$$

Hence the unique solution of the recurrence relation is

$$a_n = (5) 2^n + 3(n 2^n) - 4(3^n)$$

Example

Find the characteristic polynomial, characteristic equation for the homogeneous recurrence relations whose general solution has the form

$$1) a_n = B_1 + n B_2$$

(30)

$$2) a_n = B_1 + n B_2 + n^2 B_3$$

$$3) a_n = B_1 2^n + B_2 3^n$$

$$4) a_n = B_1 2^n + B_2 n 2^n$$

Solution

1) Given general solution is

$$a_n = B_1 + n B_2$$

$$= B_1 (1)^n + n B_2 (1)^n$$

characteristic polynomial is $(t-1)^2$

characteristic equation is $(t-1)^2 = 0$

2) Given general solution is

$$a_n = B_1 + n B_2 + n^2 B_3$$

$$= B_1 1^n + n B_2 1^n + n^2 B_3 1^n$$

characteristic polynomial is $(t-1)^3$

characteristic equation is $(t-1)^3 = 0$

3) Given general solution is $a_n = B_1 2^n + B_2 3^n$

characteristic polynomial is $(t-2)(t-3) = t^2 - 5t + 6$

characteristic equation is $t^2 - 5t + 6 = 0$

4) Given general solution is

$$a_n = B_1 2^n + B_2 n 2^n$$

(31)
characteristic polynomial is $(t-2)^2$
characteristic equation is $(t-2)^2 = 0$

Example

solve the Fibonacci relation

$$a_n = a_{n-1} + a_{n-2} \text{ with } a_0 = 0, a_1 = 1 \text{ as}$$

initial conditions

Solution The characteristic equation is

$$a_n - a_{n-1} - a_{n-2} = 0$$

$$t^2 - t - 1 = 0$$

$$t = \frac{1 \pm \sqrt{5}}{2}$$

The complete solution is

$$a_n = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n \quad \text{--- (1)}$$

put $n=0$ in (1)

$$a_0 = c_1 + c_2 \quad \text{--- (2)}$$

put $n=1$ in (1)

$$a_1 = c_1 \left(\frac{1+\sqrt{5}}{2} \right) + c_2 \left(\frac{1-\sqrt{5}}{2} \right)$$

$$1 = c_1 \left(\frac{1+\sqrt{5}}{2} \right) + c_2 \left(\frac{1-\sqrt{5}}{2} \right) \quad \text{--- (3)}$$

Solving (2) and (3) $c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}}$

(32)

Complete solution of the Fibonacci relation is given by

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

Example

solve $a_n = 2(a_{n-1} - a_{n-2})$ for $n \geq 2$

with $a_0 = 1$ and $a_1 = 2$

solution

The characteristic equation is

$$a_n - 2a_{n-1} + 2a_{n-2} = 0$$

$$t^2 - 2t + 2 = 0$$

$$t = 1 \pm i$$

The general solution is

$$a_n = r^n (c_1 \cos n\theta + c_2 \sin n\theta)$$

where c_1, c_2 are arbitrary constants

$$r = |1 \pm i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$a_n = (\sqrt{2})^n \left[c_1 \cos \frac{n\pi}{4} + c_2 \sin \frac{n\pi}{4} \right] \quad \text{--- (1)}$$

(33)

put $n=0$

$$a_0 = (\sqrt{2})^0 [c_1 \cos 0 + c_2 \sin 0]$$

$$\underline{1 = c_1}$$

put $n=1$

$$a_1 = (\sqrt{2})^1 \left[c_1 \cos \frac{\pi}{4} + c_2 \sin \frac{\pi}{4} \right]$$

$$2 = \sqrt{2} \left[1 \left(\frac{1}{\sqrt{2}} \right) + c_2 \left(\frac{1}{\sqrt{2}} \right) \right]$$

$$2 = 1 + c_2 \Rightarrow c_2 = 1$$

The general solution is

$$a_n = (\sqrt{2})^n \left[\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4} \right] \quad n \geq 2$$

(34)

Solution of inhomogeneous linear recurrence

Relations

$$a_0 + C_1 a_{n-1} + \dots + C_k a_{n-k} = f(n) \text{ for } n \geq k$$

where $C_k \neq 0$ and $f(n)$ is a specified function of n

~~For~~ Characteristic Roots Method

The solution of a linear inhomogeneous recurrence relation with constant coefficients is the sum of the two parts. There are homogeneous solution and particular solution.

A solution, which satisfies the recurrence relation when the right-hand side of the ~~equation~~ ^{relation} is set to zero is called homogeneous solution.

A solution which satisfies the recurrence relation with $f(n)$ on the right-hand side is called particular solution. The homogeneous solution is denoted by $a_n^{(h)}$ and the particular solution is denoted by $a_n^{(p)}$.

Follow the same procedure as in solving homogeneous recurrence relations for determining the homogeneous solution.

~~There is~~ To determine the particular solution use the following rules based on the nature of $f(n)$.

Rule 1 If $f(n)$ is of the form of a polynomial of degree m in n (ii) $b_0 + b_1 n + b_2 n^2 + \dots + b_{m-1} n^{m-1} + b_m n^m$

Then particular solution will be of the form:

$$Q_0 + Q_1 n + Q_2 n^2 \dots + Q_{m-1} n^{m-1} + Q_m n^m$$

provided one is not a characteristic root of the recurrence relation

Rule 2

If $f(n)$ is of the form

$$(b_0 + b_1 n + b_2 n^2 + \dots + b_{m-1} n^{m-1} + b_m n^m) a^n$$

then particular solution is of the form

$$(Q_0 + Q_1 n + Q_2 n^2 \dots + Q_{m-1} n^{m-1} + Q_m n^m) a^n$$

where 'a' is not a characteristic root of the recurrence relation

Rule 3

If 'a' is the characteristic root of the multiplicity (r-1) when $f(n)$ is of the form

$$(b_0 + b_1 n + b_2 n^2 \dots + b_{m-1} n^{m-1} + b_m n^m) a^n, \text{ then}$$

particular solution is of the form

$$n^{r-1} (Q_0 + Q_1 n + Q_2 n^2 \dots + Q_{m-1} n^{m-1} + Q_m n^m) a^n$$

→ General solution = Homogeneous solution + particular solution

$$a_n = a_n^{(h)} + a_n^{(p)}$$

(36)
Q) solve the recurrence relation

$$a_n - 9a_{n-1} + 20a_{n-2} = 1$$

Solution

Given recurrence relation is

$$a_n - 9a_{n-1} + 20a_{n-2} = 1 \quad \text{--- (1)}$$

Consider homogeneous relation

$$a_n - 9a_{n-1} + 20a_{n-2} = 0$$

The characteristic equation is $t^2 - 9t + 20 = 0$
 $t = 4, 5$

$$a_n^{(h)} = C_1 4^n + C_2 5^n, \text{ Here one is not a}$$

characteristic root-

Since the right side $f(n) = 1$ is a constant, by rule 1, the particular solution will also be a constant, say Q .

Substituting Q in (1) we get-

$$Q - 9Q + 20Q = 1$$

$$Q = \frac{1}{12}$$

$$a_n^{(p)} = \frac{1}{12}$$

(37)

Hence the general solution

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = C_1 4^n + C_2 5^n + \frac{1}{12}$$

Q Solve $a_n - a_{n-1} - 6a_{n-2} = -30$ with $a_0 = 20$,
 $a_1 = -5$ — (1)

Solution

consider homogeneous relation $a_n - a_{n-1} - 6a_{n-2} = 0$

The characteristic equation $t^2 - t - 6 = 0$

$$(t-3)(t+2) = 0 \quad t = -2, 3$$

$a_n^{(h)} = C_1 (-2)^n + C_2 (3)^n$, Here is not a characteristic root

since the right side $f(n) = -30$ is a constant, by rule 1,

the particular solution will also be a constant - say Q .

~~Put $a_n = Q$~~ substituting Q in (1)

$$Q - Q - 6Q = -30$$

$$Q = 5$$

$$a_n^{(p)} = 5$$

General solution $a_n = a_n^{(h)} + a_n^{(p)}$

$$= C_1 (-2)^n + C_2 (3)^n + 5$$

— (2)

put $n=0$ in (2)

$$a_0 = C_1(-2)^0 + C_2(3)^0 + 5$$

$$20 = C_1 + C_2 + 5$$

$$C_1 + C_2 = 15 \quad \text{--- (3)}$$

put $n=1$ in (2)

$$a_1 = C_1(-2)^1 + C_2(3)^1 + 5$$

$$-5 = -2C_1 + 3C_2 + 5$$

$$-2C_1 + 3C_2 = -10 \quad \text{--- (4)}$$

solving (3) and (4) $C_1 = 11, C_2 = 4$

Complete solution is $a_n = 11(-2)^n + 4(3)^n + 5$

Problem

$$\text{solve } a_n - 7a_{n-1} + 10a_{n-2} = 4^n$$

Solution

Given recurrence relation is

$$a_n - 7a_{n-1} + 10a_{n-2} = 4^n \quad \text{--- (1)}$$

Homogeneous relation is $a_n - 7a_{n-1} + 10a_{n-2} = 0$

The characteristic equation is $t^2 - 7t + 10 = 0$

$$(t-2)(t-5) = 0$$

$$t = 2, 5$$

$$a_n(h) = C_1 2^n + C_2 5^n$$

(39)

Since 4 is not a characteristic root-

by Rule 2, the particular solution is $Q_0 4^n$

Substituting in ①, we get-

$$Q_0 4^n - 7Q_0 4^{n-1} + 10Q_0 4^{n-2} = 4^n$$

$$4^{n-2} Q_0 [4^2 - 7 \times 4 + 10] = 4^n$$

$$-2Q_0 4^{n-2} = 4^n$$

$$Q_0 = -8$$

$$a_n^{(p)} = -8 4^n$$

Hence general solution is $a_n = a_n^{(h)} + a_n^{(p)}$
 $C_1 2^n + C_2 5^n + (-8) 4^n$

Problem

Solve the recurrence relation

$$a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n \quad \text{with } a_0 = 1, a_1 = 2$$

—①

Solution

Given recurrence relation is

$$a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n \quad \text{--- i}$$

Homogeneous relation is $a_n - 7a_{n-1} + 10a_{n-2} = 0$

The characteristic equation is $t^2 - 7t + 10 = 0$

$$(t-2)(t-5) = 0 \quad t = 2, 5$$

$$a_n^{(h)} = C_1 2^n + C_2 5^n$$

By Rule 1, the particular ⁽⁴⁰⁾solution is of the form $Q_0 + Q_1 n$

substituting this solution in (1), we get-

$$(Q_0 + Q_1 n) - 7(Q_0 + Q_1(n-1)) + 10(Q_0 + Q_1(n-2)) = 6 + 8n$$
$$\Rightarrow (4Q_0 - 13Q_1) + 4Q_1 n = 6 + 8n$$

By comparing like power coefficients on both sides, we get-

$$4Q_0 - 13Q_1 = 6$$

$$4Q_1 = 8$$

$$Q_1 = 2, Q_0 = 8$$

$$a_n^{(p)} = 8 + 2n$$

Then general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = C_1 2^n + C_2 5^n + 8 + 2n \quad \text{--- (2)}$$

put $n=0$ in (2) we get-

$$a_0 = C_1 + C_2 + 8$$

$$\Rightarrow 1 = C_1 + C_2 + 8$$

$$C_1 + C_2 = -7$$

put $n=1$ in (2) we get-

$$a_1 = 2C_1 + 5C_2 + 10$$

$$\Rightarrow 2 = 2C_1 + 5C_2 + 10$$

$$2C_1 + 5C_2 = -8$$

By solving these two equations, we get-

$$C_1 = -9, C_2 = 2$$

Hence complete solution is $a_n = -9(2)^n + 2(5)^n + 8 + 2n$

Problem

(41)

Solve the recurrence relation

$$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$$

Solution

Given recurrence relation is

$$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1 \quad \text{--- (1)}$$

The homogeneous relation is $a_n + 5a_{n-1} + 6a_{n-2} = 0$

The characteristic equation is $t^2 + 5t + 6 = 0$

$$(t+2)(t+3) = 0$$

$$t = -2, -3$$

$$\therefore a_n^{(h)} = C_1 (-2)^n + C_2 (-3)^n \quad \text{Here '1' is not a}$$

characteristic root.

By rule 1, The particular solution is form

$$Q_0 + Q_1 n + Q_2 n^2$$

Substituting in (1)

$$(Q_0 + Q_1 n + Q_2 n^2) + 5(Q_0 + Q_1 (n-1) + Q_2 (n-1)^2) +$$

$$6(Q_0 + Q_1 (n-2) + Q_2 (n-2)^2) = 3n^2 - 2n + 1$$

Simplifies

$$(12Q_0 - 17Q_1 + 29Q_2) + (2Q_1 - 34Q_2)n + 12Q_2 n^2 = 3n^2 - 2n + 1$$

Comparing like power coefficients on both sides we get

$$12Q_2 = 3$$

(42)

$$12Q_1 - 34Q_2 = -2$$

$$12Q_0 - 17Q_1 + 29Q_2 = 1$$

By solving we get $Q_1 = \frac{13}{24}$, $Q_2 = \frac{1}{4}$, $Q_0 = \frac{71}{288}$

$$\therefore Q_n^{(p)} = \frac{71}{288} + \frac{13}{24}n + \frac{1}{4}n^2$$

Hence complete solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$= C_1(-2)^n + C_2(-3)^n + \frac{71}{288} + \frac{13}{24}n + \frac{1}{4}n^2$$

Q Solve the recurrence relation

$$a_n + a_{n-1} = 3n \cdot 2^n$$

Solution

Given relation is $a_n + a_{n-1} = 3n \cdot 2^n$

Homogeneous relation is $a_n + a_{n-1} = 0$

The characteristic equation $t + 1 = 0$
 $t = -1$

$$a_n^{(h)} = C_1 (-1)^n$$

Since 2 is not a characteristic root, by Rule 2, the particular solution is $(Q_0 + Q_1 n) 2^n$.

Substituting this solution in (1), we obtain

$$(Q_0 + Q_1 n) 2^n + (Q_0 + Q_1 (n-1)) 2^{n-1} = 3n 2^n$$

which simplifies to $(\frac{3}{2} Q_0 - \frac{1}{2} Q_1) 2^n + \frac{3}{2} Q_1 n 2^n = 3n 2^n$

Comparing both sides we get-

$$\frac{3}{2} Q_1 = 3 \quad \frac{3}{2} Q_0 - \frac{1}{2} Q_1 = 0$$

$$Q_0 = \frac{2}{3}, \quad Q_1 = 2$$

$$\Rightarrow a_n^{(p)} = \left(\frac{2}{3} + 2n\right) 2^n$$

Hence complete solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$= C_1 (-1)^n + \left(\frac{2}{3} + 2n\right) 2^n$$

(9) Solve $a_r - 10a_{r-1} + 25a_{r-2} = 2^r$
solution

Given recurrence relation is

$$a_r - 10a_{r-1} + 25a_{r-2} = 2^r \quad (1)$$

Homogeneous solution is relation is

$$a_r - 10a_{r-1} + 25a_{r-2} = 0$$

The characteristic equation is $t^2 - 10t + 25 = 0$

$$(t-5)^2 = 0 \quad t = 5, 5$$

$$a_r^{(h)} = (C_1 + C_2 r) 5^r$$

44
Hence Since 2 is not characteristic root,
by rule 2, the particular solution of the form

$$a_r^{(p)} = Q 2^r$$

substituting this solution in (1), we get -

$$Q 2^r - 10Q 2^{r-1} + 25Q 2^{r-2} = 2^r$$

$$\Rightarrow 2^r \left[Q - \frac{10Q}{2} + \frac{25Q}{4} \right] = 2^r$$

$$4Q - \frac{20Q}{4} + \frac{25Q}{4} = 1$$

$$9Q = 4 \Rightarrow Q = \frac{4}{9}$$

$$a_r^{(p)} = \frac{4}{9} 2^r$$

Hence general solution is $a_r = a_r^{(h)} + a_r^{(p)}$

$$= (C_1 + C_2 r) 5^r + \frac{4}{9} 2^r$$

Problem.

(45)

Find the general solution of

$$a_n - 3a_{n-1} - 4a_{n-2} = 4^n$$

Solution.

Given recurrence relation is

$$a_n - 3a_{n-1} - 4a_{n-2} = 4^n$$

Homogeneous relation $a_n - 3a_{n-1} - 4a_{n-2} = 0$

The characteristic equation is $t^2 - 3t - 4 = 0$

$$(t+1)(t-4) = 0 \Rightarrow t = -1, 4$$

$$(h) \quad a_n = C_1 (-1)^n + C_2 (4)^n$$

Since 4 is a characteristic root, by Rule 3, the multiplicity 1, particular solution is $Q_n 4^n$

Substituting this solution in (1), we get-

$$Q_n 4^n - 3Q(n-1) 4^{n-1} - 4Q(n-2) 4^{n-2} = 4^n$$

$$Q_n 4^n - 3Q_n 4^{n-1} + 3Q 4^{n-1} - 4Q_n 4^{n-2} + 8Q 4^{n-2} = 4^n$$

$$(16Q_n - 12Q_n + 12Q - 4Q_n + 8Q) 4^{n-2} = 16 \cdot 4^{n-2}$$

$$20Q = 16$$

$$\Rightarrow Q = \frac{16}{20} = \frac{4}{5}$$

$$(p) \quad a_n = \frac{4}{5} n 4^n$$

The general solution is $a_n = a_n^{(h)} + a_n^{(p)}$
 $= C_1 (-1)^n + C_2 (4)^n + \frac{4}{5} n 4^n$

(46)

Problem

$$\text{Solve } a_n - 6a_{n-1} + 8a_{n-2} = n4^n$$

$$\text{where } a_0 = 8, a_1 = 22$$

Solution

Given recurrence relation is

$$a_n - 6a_{n-1} + 8a_{n-2} = n4^n \quad \text{--- (1)}$$

Homogeneous relation is $a_n - 6a_{n-1} + 8a_{n-2} =$

The characteristic equation is $t^2 - 6t + 8 = 0$

$$(t-2)(t-4) = 0$$

$$\Rightarrow t = 2, 4$$

$$a_n^{(h)} = C_1 (2)^n + C_2 (4)^n$$

Since 4 is the characteristic root with multiplicity 1, by rule 3, the particular solution is of the form

$$n(Q_0 + Q_1 n)4^n$$

Substituting this solution in (1), we get-

$$n(Q_0 + Q_1 n)4^n - 6[(n-1)(Q_0 + Q_1(n-1)) \cdot 4^{n-1}] +$$

$$8[(n-2)(Q_0 + Q_1(n-2)) \cdot 4^{n-2}] = n4^n \quad \text{--- (2)}$$

The equation (2) holds for all values of n and in

particular when $n=0$ $Q_0 + Q_1 = 0$ --- (3)

putting $n=1$ in (2), we get $Q_0 + 3Q_1 = 2$ — (4)

By solving (3), (4) we get

$$Q_0 = -1 \quad Q_1 = 1$$

$$a_n^{(p)} = n(-1+n)4^n + n(n-1)4^n$$

The general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = C_1 2^n + C_2 4^n + n(n-1)4^n$$

The initial conditions $a_0 = 8, a_1 = 22$

$$\Rightarrow C_1 = 5 \quad C_2 = 3$$

Hence complete solution is

$$a_n = 5 \cdot 2^n + 3 \cdot 4^n + n(n-1)4^n$$

=====

Unit-4: Recurrence Relations

Numeric function:

- A function whose domain is the set of whole numbers and range is called a numeric function.

Eg: $3^0, 3^1, 3^2, \dots, 3^n, 3 \dots$

Generating function: Let $\{a_0, a_1, a_2, \dots, a_n, \dots\}$ be a numeric function for 'a' then the infinite series $G(x)$ is said to be a generating function which is denoted by " $a_0x^0 + a_1x^1 + a_2x^2 + \dots$ ".

$G(x) = \sum_{n=0}^{\infty} a_n x^n$ is called a generating function.

Ex. for numeric function $\{3^n\}$ the generating function is given by,

$$\begin{aligned} G(x) &= 3^0x^0 + 3^1x^1 + 3^2x^2 + \dots \\ &= \sum_{n=0}^{\infty} 3^n x^n \end{aligned}$$

② $(1, 1, 1, 1, 1, 1, 1)$

$$\begin{aligned} G.F &= 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3 + 1 \cdot 2^4 + 1 \cdot 2^5 + 1 \cdot 2^6 \dots \\ &= 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 \dots \text{ in G.F} \end{aligned}$$

$$a = 1, r = 2$$

$$= \frac{a(2^n - 1)}{2 - 1} \quad \text{if } n > 1$$

$$= \frac{1(2^n - 1)}{2 - 1} \quad \text{if } (n > 1)$$

3) Write the generating function of $a_n = 2^n, n \geq 0$.

$$\begin{aligned} G(x) &= 2^0x^0 + 2^1x^1 + 2^2x^2 + 2^3x^3 + \dots \\ &= 2^0 + 2x + 4x^2 + 8x^3 + \dots \\ &= 2x + 4x^2 + 8x^3 + \dots \\ &= (1 - 2x)^{-1} \\ &= \frac{1}{1 - 2x} \end{aligned}$$

4) Write generating function $(1, 2, 3, 4, \dots)$

$$\begin{aligned} G(x) &= 1 \cdot x^0 + 2 \cdot x^1 + 3 \cdot x^2 + 4 \cdot x^3 + \dots \\ &= 4x^2 + 3x^3 + 4x^4 + \dots \\ &= (1 - x)^{-2} \\ &= \frac{1}{(1 - x)^2} \end{aligned}$$

5) Write G.F for $(1, -2, 3, -4)$

$$\begin{aligned} G(x) &= 1x^0 - 2x + 3x^2 - 4x^3 + \dots \\ &= 1 - 2x + 3x^2 - 4x^3 + \dots \\ &= (1 + x)^{-2} \end{aligned}$$

6) Find the G.F of the seq $A = \{a_n\}_{n=0}^{\infty}$ where $a_n = \begin{cases} 1 & \text{if } 0 \leq n \leq 2 \\ 3 & \text{if } 3 \leq n \leq 6 \\ 0 & \text{if } n \geq 6 \end{cases}$

$$G(x) = 1x^0 + 1x^1 + 1x^2 + 3x^3 + 3x^4 + 3x^5 + 0x^6 + 0x^7 + \dots$$

$$= 1 + x + x^2 + 3x^3 + 3x^4 + 3x^5$$

Closed form expressions of generating functions:

- To express generating function in closed form in the closed form expression for the numeric geometric series, which is expression of the form:

$$a + ar + ar^2 + \dots$$

Note: It may be finite or infinite series

Q) G.F = ?

$$a_n = 3^n, \quad n \geq 0 \quad \text{In closed form}$$

$$G(x) = 3x^0 + 3x^1 + 3x^2 + 3x^3 + \dots$$

$$= 3(x^0 + x^1 + x^2 + x^3 + \dots)$$

$$= 3(1-x)$$

2) Find the G.F in closed form of Fibonacci sequence $\{f_n\}$ defined by

$$f_n = f_{n-1} + f_{n-2}$$

Consider the generating function $f(x) = f_0 + f_1x + f_2x^2 + \dots$

$$= \sum_{n=0}^{\infty} f_n x^n \quad \text{--- (1)}$$

We have the Fibonacci series $f_{n-1} + f_{n-2} = f_n$

Multiply with x^n on both sides

$$f_n x^n = f_{n-1} x^n + f_{n-2} x^n$$

Taking sum on all $n=2 \rightarrow \infty$ on both sides

$$= \sum_{n=2}^{\infty} f_n x^n = \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n \quad \text{--- (2)}$$

$$= \sum_{n=2}^{\infty} f_n x^n = f_2 x^2 + f_3 x^3 + f_4 x^4 + \dots + f$$

$$= (f_0 + f_1x + f_2x^2 + f_3x^3 + \dots) - f_0 - f_1x$$

$$\sum_{n=2}^{\infty} f_n x^n = G(x) - f_0 - f_1x \quad \text{--- (3)} \quad [\text{from (1)}]$$

$$\text{Now } \sum_{n=2}^{\infty} f_{n-1} x^n = f_1 x^2 + f_2 x^3 + f_3 x^4 + \dots$$

$$= x(f_1x + f_2x^2 + f_3x^3 + \dots)$$

add and sub f_0

$$= x(G(x) - f_0) \quad \text{--- (4)}$$

$$\text{Now } \sum_{n=2}^{\infty} f_{n-2} x^n = f_0 x^2 + f_1 x^3 + f_2 x^4 + \dots +$$

$$= x^2(G(x) - f_0) \quad \text{--- (5)}$$

Sub (3) (4) (5) in (2)

$$G(x) - f_0 - f_1x = x(G(x) - f_0) + x^2(G(x) - f_0)$$

$$G(x) - f_0 - f_1x = x(G(x) - 2f_0 + x^2 G(x))$$

$$G(x)[1 - x - x^2] = -xf_0 + f_0 + f_1x$$

$$\text{Since } f_0 = 0, f_1 = 1$$

$$= -0 + 0 + 2$$

$$G(x)(1 - 2 - x^2) = 2$$

$$\therefore G(x) = \frac{x}{1 - 2 - x^2}, \text{ is the req. G.F for the } \{f_n\}$$

1) Find the sequence generated by the following functions:

$$\text{i) } (2 + x^2)^3$$

$$\text{ii) } 2x^2(1 - x)^{-1}$$

$$\text{iii) } 3x^3 + e^{2x}$$

$$\underline{\text{Sol}} = 2^3 \left(1 + \frac{x}{2}\right)^3$$

$$= 8 \left[3C_0 + 3C_1 \frac{x}{2} + 3C_2 \left(\frac{x}{2}\right)^2 + 3C_3 \left(\frac{x}{2}\right)^3 \right]$$

$$= 8 \left[1 + \frac{3}{2}x + \frac{3}{4}x^2 + 1 \cdot \frac{x^3}{8} \right]$$

$$= 8 + 12x + 6x^2 + x^3$$

\therefore The seq. generated by $(2 + x^2)^3$ is 8, 12, 6, 1, 0, 0, 0, ...

$$\text{ii) } 2x^2(1 - x)^{-1}$$

$$= 2x^2(1 + x + x^2 + x^3 + \dots)$$

$$= 2x^2 + 2x^3 + 2x^4 + 2x^5 + \dots$$

$$\Rightarrow 0, 0, 2, 2, 2, \dots$$

$$\text{iii) } 3x^3 + 1 + \frac{2x}{1!} + \frac{4x^2}{2!} + \dots$$

$$= 1 + 2x + 2x^2 + \left(\frac{4}{2} + 3\right)x^3 + \dots$$

$$= 1 + 2x + 2x^2 + \frac{13}{2}x^3 + \frac{2}{3}x^4 + \dots$$

$$\text{Seq: } 1, 2, 2, \frac{13}{2}, \frac{2}{3}, \dots$$

1) Sequence to general function:

$$\text{a) } 0, 1, 2, 3, \dots$$

$$= 0x^0 + 1 \cdot x + 2x^2 + 3x^3 + \dots$$

$$= x(1 + 2x + 3x^2 + \dots)$$

$$= x(1 - x)^{-2}$$

$$\text{b) } 1^2, 2^2, 3^2, \dots$$

$$= 1^2x^0 + 2^2x + 3^2x^2 + \dots$$

$$= 1 + 4x + 9x^2 + \dots$$

$$=$$

$$\frac{x}{(1 - x)^2}$$

$$= \frac{(1 - x)^4 (1 - x)(2 + 1)}{(1 - x)^4}$$

$$= \frac{(1 - x)^4 + (2x - 2x^2)}{(1 - x)^4}$$

Extended binomial theorem:

Let x be real no. with $|x| < 1$ and n be a real no., then

$$(1+x)^n = \sum_{r=0}^{\infty} nC_r x^r$$

$$\textcircled{1} -^n C_r = (-1)^r (n+1-r) C_r$$

$$\textcircled{2} (1+x)^{-n} = \sum_{r=0}^{\infty} (-n)C_r x^r = \sum_{r=0}^{\infty} (-1)^r (n+1-r) C_r x^r$$

$$\textcircled{3} (1-x)^{-n} = \sum_{r=0}^{\infty} (n+1-r) C_r x^r$$

Calculating coeff of General function:

Q) Find the coeff of x^{32} in ~~$(1+x^3+x^9)^{10}$~~ $(1+x^3+x^9)^{10}$

We want the coeff solutions of

$$e_1 + e_2 + \dots + e_{10} = 32$$

$$\begin{array}{l} e_1 = 0, 5, 9 \rightarrow \text{Three 9's} \rightarrow 27 \\ \text{One } 5 \rightarrow 5 \\ \text{Six } 0's \rightarrow 0 \\ \hline 32 \end{array}$$

$$\Rightarrow \frac{10!}{3! 1! 6!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{3! \times 1! \times 6!} = \frac{10 \times 9 \times 8 \times 7}{3 \times 2 \times 1} = 15 \times 56 = 840$$

$$\begin{array}{r} 56 \\ \times 15 \\ \hline 280 \\ 560 \\ \hline 840 \end{array}$$

Q) Find coeff of x^{27} in $(x^4 + x^5 + x^6 + \dots)^5$

Given: $(x^4 + x^5 + x^6 + \dots)^5$

$$= (x^4)^5 (1 + x + x^2 + \dots)$$

$$= x^{20} [1-x]^{-5}$$

$$= x^{20} \sum_{r=0}^{\infty} (n+r-1) C_r x^r \quad (n=5)$$

$$= \sum_{r=0}^{\infty} (4+r) C_r x^{20+r}$$

1) To get coeff of x^{27} put $20+r=27$
 $r=7$

$$= (4+7) C_7 = {}^{11}C_7$$

Q) x^{20} in $(1+x^2+2x^3+x^4)(x^2+x^3+x^4+\dots)^5$

$$\Rightarrow x(1+x+2x^2+x^3) x^{10} (1-x)^{-5}$$

$$\Rightarrow x^{11} (1+x+2x^2+x^3) \sum_{r=0}^{\infty} (5+r-1) C_r x^r$$

$$\Rightarrow (1+x+2x^2+x^3) \sum_{r=0}^{\infty} (4+r) C_r x^{r+1}$$

$$\begin{array}{cccc} r+1=20 & r+1=20 & r+1=20 & r+1=20 \\ r=9 & r=8 & r=7 & r=6 \end{array}$$

$$\Rightarrow (4+9)C_9 + (4+8)C_8 + (4+7)C_7 + (4+6)C_6$$

$$= 13C_9 + 12C_8 + 11C_7 + 10C_6$$

$$= 2080$$

Extended binomial theorem:

Let x be real no. with $|x| < 1$ and n be a real no., then

$$(1+x)^n = \sum_{r=0}^{\infty} {}^nC_r x^r$$

$$\textcircled{1} {}^{-n}C_r = (-1)^r (n+r-1)C_r$$

$$\textcircled{2} (1+x)^{-n} = \sum_{r=0}^{\infty} (-n)C_r x^r = \sum_{r=0}^{\infty} (-1)^r (n+r-1)C_r \cdot x^r$$

$$\textcircled{3} (1-x)^{-n} = \sum_{r=0}^{\infty} (n+r-1)C_r \cdot x^r$$

Calculating coeff of General function:

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$$e_1 = 0, 5, 9 \rightarrow \text{Three 9's} \rightarrow 27$$

$$\text{One 5} \rightarrow 5$$

$$\text{Six 0's} \rightarrow 0$$

$$\underline{32}$$

$$\Rightarrow \frac{10!}{3! 1! 6!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{3! \times 1! \times 6!}$$

$$= \frac{5 \times 3 \times 10 \times 8 \times 7}{3 \times 2 \times 1} = 15 \times 56 = 840$$

$$\begin{array}{r} 56 \\ \times 15 \\ \hline 280 \\ 560 \\ \hline 840 \end{array}$$

Q) Find coeff of x^{27} in $(x^4+x^5+x^6+\dots)^5$

$$\text{Given, } (x^4+x^5+x^6+\dots)^5$$

$$= (x^4)^5 (1+x+x^2+\dots)^5$$

$$= x^{20} [1-x]^{-5}$$

$$= x^{20} \sum_{r=0}^{\infty} (n+r-1)C_r \cdot x^r \quad (n=5)$$

$$= \sum_{r=0}^{\infty} (4+r)C_r x^{20+r}$$

$$\textcircled{1} \text{ To get coeff of } x^{27} \text{ put } 20+r=27$$

$$r=7$$

$$= (4+7)C_7 = {}^{11}C_7$$

Q) x^{20} in $(x+x^2+2x^3+x^4)(x^2+x^3+x^4+\dots)^5$

$$\Rightarrow x(1+x+2x^2+x^3)2^{10}(1-x)^{-5}$$

$$\Rightarrow x^{11}(1+x+2x^2+x^3) \sum_{r=0}^{\infty} (5+r-1)C_r x^r$$

$$\Rightarrow (1+x+2x^2+x^3) \sum_{r=0}^{\infty} (4+r)C_r x^{r+1}$$

$$r+11=20 \quad r+12=20 \quad r+13=20 \quad r+14=20$$

$$r=9$$

$$r=8$$

$$r=7$$

$$r=6$$

$$\Rightarrow (4+9)C_9 + (4+8)C_8 + (4+7)C_7 + (4+6)C_6$$

$$= 13C_9 + 12C_8 + 2({}^{11}C_7) + 10C_6$$

$$= 2080$$

in case of non negatives e^i can take values $0, 1, 2, \dots$

$$\exists f_i(x) = x^0 + x^1 + x^2 + \dots$$

The GF for non neg is $f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x)$

$$\Rightarrow (1+x+x^2+\dots)(1+x+x^2+\dots)(1+x+x^2+\dots)(1+x+x^2+\dots)$$

$$\Rightarrow (1+x+x^2+\dots)^4 = ((1-x^{-1})^{-1})^4 = (1-x)^{-4}$$

$$\sum_{r=0}^{\infty} (4+r-1) C_r x^r$$

$$\Rightarrow \sum_{r=0}^{\infty} (3+r) C_r x^r$$

$$\text{put } x=25$$

\Rightarrow no. of non neg int sol is coeff of x^{25} in the expansion

$$\Rightarrow 25 C_{25}$$

$$\Rightarrow 3276$$

ii) $f_i(x) = x^1 + x^2 + x^3 + \dots$ for $i=1, 2, 3, 4$

$$f(x) = f_1 f_2 f_3 f_4$$

$$= (x+x^2+\dots)(x+x^2+\dots)(x+x^2+\dots)(x+x^2+\dots)$$

$$= (x+x^2+\dots)^4$$

$$= x^4 (1+x+\dots)^4$$

$$= x^4 (1-x)^{-4}$$

$$= x^4 \sum_{r=0}^{\infty} (4+r-1) C_r x^r$$

$$= \sum_{r=0}^{\infty} (3+r) C_r \cdot x^{4+r}$$

$$\text{put } x+4=25$$

$$x=21$$

$$(3+21) C_{21} = 2021$$

Generally recursive definition can be used to solve counting problems

\rightarrow sequence $\{a_n\}$ is an eqn that captures $\{a_n\}$ in terms of one or more of the previous terms, of the sequence $\{a_n\}$ namely

$$a_0, a_1, a_2, \dots, a_{n-1}, \forall n \geq 1$$

\rightarrow also reference as difference equation

$$i) a_n - a_{n-1} = n \Rightarrow a_n = n + n a_{n-1}$$

$$ii) a_n - 3a_{n-1} + 2a_{n-2} = 0$$

$$iii) a_n - 5a_{n-1} + 6a_{n-2} = n^2 + 1$$

$$iv) a_n - a_{n-1} = -1 \text{ not linear}$$

$\left. \begin{array}{l} \rightarrow \\ \text{linear} \end{array} \right\} \begin{array}{l} \rightarrow \text{deg 2} \\ \rightarrow \text{deg 2} \end{array}$

Linear recurrence relation:

- Let n, k be non neg int, a recurrence relation is of form

$$c_0(n)a_n + c_1(n)a_{n-1} + c_2(n)a_{n-2} + \dots + c_k(n)a_{n-k} = f(n) \text{ for } n \geq k$$

where $c_0(n), c_1(n), \dots, c_k(n)$ are function of n

• If $c_0(n), c_k(n)$ are not 0 then $k \rightarrow$ degree of the above RR

• If $c_0(n), \dots, c_k(n)$ are constants then (1) is called a linear LR with constant coeff

• If $f(x)$

A sequence $\{a_n\}_{n=0}^{\infty}$ is said to be a solution of RR if each value a_n satisfies the recurrence relation.

methods to solve recurrence relations

- method 1: Substitution
- method 2: Generating function
- method 3: Characteristic root method

Ex 1

(i) Solve the RR:

$$a_n = a_{n-1} + n^2 \text{ where } a_0 = 7$$

$$\text{If } n=1, a_1 = a_0 + 1^2 = 7 + 1^2$$

$$\text{If } n=2, a_2 = a_1 + 4 = 7 + (1^2 + 2^2)$$

$$\text{If } n=3, a_3 = a_2 + 3^2 = 7 + (1^2 + 2^2 + 3^2)$$

\vdots

$$a_n = a_{n-1} + n^2 = 7 + (1^2 + 2^2 + \dots + n^2)$$

$$= 7 + \frac{n(n+1)(2n+1)}{6}$$

\therefore required sol of RR

(ii) $a_n = a_{n-1} + \frac{1}{n(n+1)}$ where $a_0 = 1$

$$n=1 \Rightarrow a_1 = a_0 + \frac{1}{1 \cdot 2} = 1 + \frac{1}{2}$$

$$n=2 \Rightarrow a_2 = a_1 + \frac{1}{2 \cdot 3} = 1 + \frac{1}{2} + \frac{1}{6} = 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3}$$

$$a_n = 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)}$$

(iii) If a_n is sol of RR $a_{n+1} = k(a_n)$ for $n \geq 0$ and $a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{2401}$

find 'k'.

put $n=3$

$$a_{3+1} = a_4 = \frac{153}{49}$$

$$a_{3+1} = a_4 = k(a_3)$$

$$= a_4 = k\left(\frac{153}{49}\right)$$

$$a_5 = k(a_4) = k^2\left(\frac{153}{49}\right)$$

$$\text{given } a_5 = \frac{1377}{2401}$$

$$\Rightarrow k^2\left(\frac{153}{49}\right) = \frac{1377}{2401}$$

$$\Rightarrow k^2 = \frac{1377}{2401} \times \frac{49}{153}$$

$$k^2 = \frac{01}{49} \Rightarrow k = \pm \frac{1}{7}$$

Method-2: Using generating functions

(i) $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$ where $a_0 = 10$, $a_1 = 4$

Sol let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ — (1)

Given R.R,

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

mul each term with $\textcircled{2} x^n$ and sum overall $n = 2$ to ∞

$$\textcircled{2} \text{ becomes } \sum_{n=2}^{\infty} a_n x^n - 7 \sum_{n=2}^{\infty} a_{n-1} x^n + 10 \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 10x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$\Rightarrow \left(\sum_{n=0}^{\infty} a_n x^n - a_2 - a_1 x \right) - 7x \left(\sum_{n=0}^{\infty} a_n x^n - a_0 \right) + 10x^2 \left(\sum_{n=0}^{\infty} a_n x^n \right)$$

$$\Rightarrow A(x) - a_2 - a_1 x - 7x A(x) + 7x a_0 + 10x^2 A(x) = 0$$

here $a_0 = 10$ and $a_1 = 41$

$$\Rightarrow A(x) - 7x A(x) + 10x^2 A(x) - a_2 - a_1 x + 7x a_0 = 0$$

$$A(x) [1 - 7x + 10x^2] = 10 - 41x + 7x(10) = 0$$

$$(10x^2 - 7x + 1) A(x) = 10 - 29x = 0$$

$$A(x) = \frac{10 - 29x}{10x^2 - 7x + 1}$$

$$\Rightarrow 10x^2 - 7x + 1$$

$$\Rightarrow 10x^2 - 5x - 2x + 1$$

$$= 5x(2x-1) - 1(2x-1)$$

$$= (2x-1)(5x-1)$$

$$= \frac{10 - 29x}{(2x-1)(5x-1)} = \frac{A}{2x-1} + \frac{B}{5x-1}$$

$$10 - 29x = A(5x-1) + B(2x-1)$$

$$\text{put } x = \frac{1}{5} \text{ to get } B, B = -7$$

$$\therefore A(x) = \frac{-3}{2x-1} - \frac{7}{5x-1}$$

$$= \sum_{n=0}^{\infty} a_n x^n = \frac{3}{1-2x} + \frac{7}{1-5x}$$

$$= 3 \sum_{n=0}^{\infty} (2x)^n + 7 \sum_{n=0}^{\infty} (5x)^n$$

$$= \sum_{n=0}^{\infty} 3(2^n) x^n + \sum_{n=0}^{\infty} 7(5^n) x^n = \sum_{n=0}^{\infty} (3 \cdot 2^n + 7 \cdot 5^n) x^n$$

$$\therefore a_n = 3 \cdot 2^n + 7 \cdot 5^n$$

Q) using GF method, solve $a_n - 4a_{n-1} + 3a_{n-2} = 0$, $n \geq 2$ with condition $a_0 = 2$ and $a_1 = 4$

$$\text{Sol } A(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$a_n - 4a_{n-1} + 3a_{n-2} = 0 \quad \text{--- (1)}$$

\therefore mul x^n obs

$$\sum a_n x^n - 4 \sum a_{n-1} x^n + 3 \sum a_{n-2} x^n = 0$$

$$= A_2 - 4A_2[A(2) - a_0] + 3A_2^2[A(2) - a_0 - a_{12}] = 0$$

$$= A(2) - 4A_2(A(2)) + 3A_2^2(A(2)) - 4A_2a_0 + 3A_2^2a_0$$

...

$$Q) \text{ Solve R.R } a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$$

★ Method 2: Characteristic roots method

$$\text{Let } a_n + c_1a_{n-1} + c_2a_{n-2} + \dots + c_ka_{n-k} = a_0 \quad \text{--- (1)}$$

when $c_k \neq 0$ is a linear homogeneous RE of degree k ,

the characteristic eqn of (1) is of the form.

$$t^k + c_1t^{k-1} + \dots + c_k = 0$$

Degree of equation (2) is k so it has k roots, $\alpha_1, \alpha_2, \dots, \alpha_k$

these are called the characteristic roots.

Two cases may arise:

Case i) If $\alpha_1, \alpha_2, \dots, \alpha_k$ are distinct roots then the gen sol of R.R is

$$a_n = C_1\alpha_1^n + C_2\alpha_2^n + \dots + C_k\alpha_k^n$$

where C_1, C_2, \dots, C_k are constants

Case ii) If root α is repeated k then gen sol of RE is,

$$a_n = (C_1 + C_2n + C_3n^2 + C_4n^3 + \dots + C_kn^{k-1})\alpha^n$$

where C_1, C_2, \dots, C_k constants.

$$\textcircled{1} \text{ Solve R.R: } a_n - 3a_{n-1} + 2a_{n-2} = 0$$

Sol The characteristic eqn of following R.R is.

$$t^2 - 3t + 2 = 0$$

$$t^2 - 2t - t + 2 = 0$$

$$t(t-2) - 1(t-2) = 0$$

$$(t-1)(t-2) = 0$$

\therefore roots are real and distinct

General solution: $C_1 2^n + C_2 1^n$

$$\textcircled{2} \text{ Solve: } a_n - 6a_{n-1} + 9a_{n-2} = 0$$

$$t^2 - 6t + 9 = 0$$

$$t^2 - 3t - 3t + 9 = 0$$

$$t(t-3) - 3(t-3) = 0$$

$$(t-3)(t-3) = 0$$

\therefore roots are real and same

General Solution: $(C_1 + C_2n)3^n$

$\textcircled{3}$ Find the characteristic equation given homogeneous R.R takes form:

$$\text{i) } a_n = \beta_1 + n\beta_2$$

$$\text{ii) } a_n = \beta_1 2^n + \beta_2 3^n$$

} β_1, β_2 are constants

i) roots are $t=1,1$

\therefore char eq. is $(t-1)^2 = 0$

$$t^2 - 2t + 1 = 0$$

ii) $a_n = B_1 2^n + B_2 3^n$

roots are $t=2,3$

$$\Rightarrow (t-2)(t-3) = 0$$

$$\Rightarrow t^2 - 5t + 6 = 0$$

5) $a_n = a_{n-1} + a_{n-2}$

with $a_0 = 0, a_1 = 1$

Given R.R $a_n - a_{n-1} - a_{n-2} = 0$

char eq. $t^2 - t - 1 = 0$

$$t = \frac{1 \pm \sqrt{5}}{2}$$

$$\Rightarrow t = \frac{1 + \sqrt{5}}{2}$$

$$\Rightarrow t = \frac{1 - \sqrt{5}}{2}$$

GS of Fibonacci relations is

$$a_n = c_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n \quad \text{--- (1)}$$

given initial conditions are,

$$a_0 = 0, a_1 = 1$$

put $n=0$ in (1)

$$a_0 = c_1 \left(\frac{1 + \sqrt{5}}{2} \right)^0 + c_2 \left(\frac{1 - \sqrt{5}}{2} \right)^0 = 0$$

$$c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \quad \text{--- (2)}$$

put $n=1$ in (1)

$$a_1 = c_1 \left(\frac{1 + \sqrt{5}}{2} \right) + c_2 \left(\frac{1 - \sqrt{5}}{2} \right) = 1$$

$$\Rightarrow c_1 (1 + \sqrt{5}) + c_1 (1 - \sqrt{5}) = 2 \quad (\text{from (2)})$$

$$\Rightarrow c_1 + \sqrt{5} c_1 - c_1 + \sqrt{5} c_1 = 2$$

$$2\sqrt{5} c_1 = 2$$

$$c_1 = \frac{1}{\sqrt{5}} \Rightarrow c_2 = -\frac{1}{\sqrt{5}} \quad (\because c_2 = -c_1)$$

\therefore GS of RR

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

* Solution in-homogeneous linear recurrence relation:

• general form is of form: $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = f(n)$

(for $n \geq k$)

where $f(n)$ is some specified function in terms of n

To solve IHR we have 2 methods:

- characteristic root method
- Generating function

- The solution of a linear in-homogeneous recurrence relation with constant coeff is sum of two parts
- one is homogeneous solution and 2nd one is particular solution
- A sol, which satisfies RR when RHS is 0 is called homogeneous solution i.e. a sol which satisfies RR in of form

$$a_n = a_n^{(h)} + a_n^{(p)}$$

procedure:

① depends on nature of particular sol

Rule 1: If $f(n)$ is a prototype polynomial of degree m in n it can be expressed as

$$(b_0 + b_1n + b_2n^2 + \dots + b_mn^m)$$

then PS can be expressed as

$$(c_0 + c_1n + c_2n^2 + \dots + c_mn^m)$$

Where ' 1 ' is not a characteristic root of a recurrence relation

Rule 2: If $f(n) = (b_0 + b_1n + b_2n^2 + \dots + b_mn^m)$ then the

P.S

Rule 3: If ' α ' is the characteristic root with multiplicity ' s ' when $f(n)$ is in form:

$$f(n) = (b_0 + b_1n + b_2n^2 + b_3n^3 + \dots + b_mn^m)$$

$$PS \text{ is in form: } n^s (c_0 + c_1n + c_2n^2 + c_3n^3 + \dots + c_mn^m) \alpha^n$$

Problems:

$$\textcircled{1} \text{ Solve the recurrence relation: } a_n - a_{n-1} - 6a_{n-2} = -30 \quad \textcircled{1}$$

$$\text{with } a_0 = 20, a_1 = 5$$

$$\text{Sol } a_n = a_n^{(h)} + a_n^{(p)}$$

$h \rightarrow$ homogeneous

$p \rightarrow$ particular integral

To find $a_n^{(h)}$:

$$\text{Here RR from } \textcircled{1}, a_n - a_{n-1} - 6a_{n-2} = 0 \quad \textcircled{2}$$

$$\text{Char eq of } \textcircled{2} \text{ is } t^2 - t - 6 = 0$$

$$t^2 - 3t + 2t - 6 = 0$$

$$(t-3)(t+2) = 0$$

$$t = 3, -2$$

To find $a_n^{(p)}$:

Since RHS of $\textcircled{1}$ is:

$$f(n) = -30$$

Let $a = -30$ using rule ①

$$a_n - a_{n-1} = a_{n-2} = a$$

from ①

$$a - 6a - 6a = -30$$

$$-6a = -30$$

$$a = 5$$

$$a_n^{(p)} = 5$$

Hence the G.S is

$$a_n = C_1(3)^n + C_2(-2)^n + 5 \quad \text{--- (3)}$$

To find C_1, C_2

Given $a_0 = 20$ and $a_1 = -5$

$$\text{from (2)} \quad a_n = C_1 3^n + C_2 (-2)^n + 5$$

put $n=0$

$$a_0 = C_1 + C_2 + 5 = 20$$

$$= C_1 + C_2 = 15 \quad \text{--- (4)}$$

put $n=1$

$$a_1 = C_1(3) + C_2(-2) + 5 = -5$$

$$3C_1 - 2C_2 = -10 \quad \text{--- (5)}$$

$$a_n = C_1 2^n + C_2 5^n$$

$$\text{Q) Solve } a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n \quad \text{--- (1)}$$

$$\text{G.S } \Rightarrow a_n = a_n^{(h)} + a_n^{(p)}$$

To find $a_n^{(h)}$:

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

$$t^2 - 7t + 10 = 0$$

$$t^2 - 5t - 2t + 10 = 0$$

$$t(t-5) - 2(t-5) = 0$$

$$(t-5)(t-2) = 0$$

$$a_n^{(h)} = C_1 5^n + C_2 2^n$$

$\therefore 1$ is not a char. root

$$\Rightarrow 4a_0 + 4a_1 - 13a_2 = 6 + 8n$$

\therefore compare with r.h.s

$$= (4a_0 - 13a_2) + n(4a_1) = 6 + 8n$$

$$= 4a_0 - 13a_2 = 6 \quad \text{--- (6)}$$

$$(4a_1) \propto 8n \quad \text{--- (7)}$$

$$= a_1 = 2 \quad \text{--- put in (6)}$$

$$\Rightarrow 4a_0 - 26 = 6$$

$$\Rightarrow a_0 = \frac{32}{4}$$

To find $a_n^{(p)}$:

$$\text{Given } y(n) = 6 + 8n$$

by rule 1,

$$\text{p.s. is of form } a_0 + a_1 n$$

$$\text{i.e., } a_n = a_0 + a_1 n$$

$$a_{n-1} = a_0 + a_1(n-1)$$

$$a_{n-2} = a_0 + a_1(n-2)$$

$$= (a_0 + a_1 n) - 7(a_0 + a_1(n-1)) + 10(a_0 + a_1(n-2))$$

$$= 6 + 8n$$

$$= a_0 + a_1 n - 7a_0 + 7a_1(n-1) + 10a_0 + 10a_1(n-2)$$

$$= 6 + 8n$$

$$+ a_1 n$$

$$= a_0 - 7a_0 + 10a_0 - 7a_1 + 7a_1 + 10a_1 n - 20a_1$$

$$= 6 + 8n$$

$$\text{G.S: } a_n = C_1 5^n + C_2 2^n + (8 + 2n)$$

③ Solve $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$

$$a_n^{(h)} = a_n - 7a_{n-1} + 10a_{n-2} = 0$$

$$= t^2 - 7t + 10 = 0$$

$$= t^2 - 5t - 2t + 10 = 0$$

$$= t(t-5) - 2(t-5) = 0$$

$$= (t-2)(t-5)$$

$$\Rightarrow C_1 5^n + C_2 2^n$$

$$f(n) = 4^n$$

particular solution for $a_n = 4^n$

$$a_n = 4^n$$

$$a_{n-1} = 4^{n-1}$$

$$a_{n-2} = 4^{n-2}$$

Sub in ③

$$= 4^n - 7 \cdot 4^{n-1} + 10 \cdot 4^{n-2} = 4^n$$

$$= 4^n - 7 \cdot 4^{n-1} + 10 \cdot 4^{n-2} = 4^n$$

$$= 4^{n-2} (16 - 7(4) + 10) = 4^n$$

$$= 0(16 - 28 + 10) = \frac{4^n}{4^2 \cdot 4^2}$$

$$= 0 = \frac{16}{-2} = -8$$

$$= a_n^{(p)} = 4^n = (-8)(4)^n$$

$$C_5 = a_n = C_1(2)^n + C_2(5)^n - 8(4)^n$$

④ Solve $a_n + a_{n-1} = (3n) 2^n$

$$a_n^{(h)} = t + 1 = 0$$

$$t = -1$$

$$f(n) = (3n) 2^n$$

\therefore Solution is of form

$$a_n \neq 1/3n/2^n$$

$$d/dt \neq 1/3(4)^n - 1/3(2)^n$$

$$= (3n)/2^n + 1/3(4)^n - 1/3(2)^n$$

$$= a_n = (C_0 + C_1 n) 2^n$$

$$a_{n-1} = (C_0 + C_1(n-1)) 2^{n-1}$$

$$\left. \begin{aligned} &= \frac{2}{2} (C_0 + C_1 n) + \frac{1}{2} (C_0 + C_1(n-1)) = (3n)(2^n) 2 \\ &= 2(C_0 + C_1 n) + (C_0 + C_1(n-1)) = 2(3n) \\ &= C_0 + C_0 + nC_1 + nC_1 - C_1 = 6n \\ &= 2C_0 + 2nC_1 - C_1 \end{aligned} \right\} \times$$

$$= 2(C_0 + C_1 n) + C_0 + (C_1(n-1)) 2 = 6n$$

$$= 2C_0 + C_0 + 2C_1 n - 2C_1 n - 2C_1 = 6n \quad 2C_0 + C_0 + 2C_1 n + C_1 n - C_1 = 6n$$

$$= 3C_0 +$$

$$= 3C_0 + 3C_1 n - C_1$$

$$= 3C_0 - C_1 = 0 \quad \text{--- ①}$$

$$3C_1 n = 6n \quad \text{--- ②}$$

$$= C_1 = 2 \quad \text{--- ③}$$

$$\left. \begin{aligned} 3C_0 &= 2 \\ C_0 &= \frac{2}{3} \end{aligned} \right\}$$

$$a_n = C_1(-1)^n + \frac{2}{3} + 2n$$

$$Q) a_n - 6a_{n-1} + 8a_{n-2} = n \cdot 4^n$$

$$a_n^{(h)} : t^2 - 6t + 8 = 0$$

$$= t^2 - 2t - 4t + 8 = 0$$

$$= t(t-2) - 4(t-2) = 0$$

$$= (t-4)(t-2) = 0$$

$$= t = 4, 2$$

$$= C_1 4^n + C_2 2^n = C_1 2^n + C_2 4^n$$

\therefore sol is of form [rule 3]

$$n(0_0 + 0_1 n) 4^n$$

$$Q. 4. 7. 11. 15. 19. 23. 27. 31. 35. 39. 43. 47. 51. 55. 59. 63. 67. 71. 75. 79. 83. 87. 91. 95. 99. 103. 107. 111. 115. 119. 123. 127. 131. 135. 139. 143. 147. 151. 155. 159. 163. 167. 171. 175. 179. 183. 187. 191. 195. 199. 203. 207. 211. 215. 219. 223. 227. 231. 235. 239. 243. 247. 251. 255. 259. 263. 267. 271. 275. 279. 283. 287. 291. 295. 299. 303. 307. 311. 315. 319. 323. 327. 331. 335. 339. 343. 347. 351. 355. 359. 363. 367. 371. 375. 379. 383. 387. 391. 395. 399. 403. 407. 411. 415. 419. 423. 427. 431. 435. 439. 443. 447. 451. 455. 459. 463. 467. 471. 475. 479. 483. 487. 491. 495. 499. 503. 507. 511. 515. 519. 523. 527. 531. 535. 539. 543. 547. 551. 555. 559. 563. 567. 571. 575. 579. 583. 587. 591. 595. 599. 603. 607. 611. 615. 619. 623. 627. 631. 635. 639. 643. 647. 651. 655. 659. 663. 667. 671. 675. 679. 683. 687. 691. 695. 699. 703. 707. 711. 715. 719. 723. 727. 731. 735. 739. 743. 747. 751. 755. 759. 763. 767. 771. 775. 779. 783. 787. 791. 795. 799. 803. 807. 811. 815. 819. 823. 827. 831. 835. 839. 843. 847. 851. 855. 859. 863. 867. 871. 875. 879. 883. 887. 891. 895. 899. 903. 907. 911. 915. 919. 923. 927. 931. 935. 939. 943. 947. 951. 955. 959. 963. 967. 971. 975. 979. 983. 987. 991. 995. 999. 1003. 1007. 1011. 1015. 1019. 1023. 1027. 1031. 1035. 1039. 1043. 1047. 1051. 1055. 1059. 1063. 1067. 1071. 1075. 1079. 1083. 1087. 1091. 1095. 1099. 1103. 1107. 1111. 1115. 1119. 1123. 1127. 1131. 1135. 1139. 1143. 1147. 1151. 1155. 1159. 1163. 1167. 1171. 1175. 1179. 1183. 1187. 1191. 1195. 1199. 1203. 1207. 1211. 1215. 1219. 1223. 1227. 1231. 1235. 1239. 1243. 1247. 1251. 1255. 1259. 1263. 1267. 1271. 1275. 1279. 1283. 1287. 1291. 1295. 1299. 1303. 1307. 1311. 1315. 1319. 1323. 1327. 1331. 1335. 1339. 1343. 1347. 1351. 1355. 1359. 1363. 1367. 1371. 1375. 1379. 1383. 1387. 1391. 1395. 1399. 1403. 1407. 1411. 1415. 1419. 1423. 1427. 1431. 1435. 1439. 1443. 1447. 1451. 1455. 1459. 1463. 1467. 1471. 1475. 1479. 1483. 1487. 1491. 1495. 1499. 1503. 1507. 1511. 1515. 1519. 1523. 1527. 1531. 1535. 1539. 1543. 1547. 1551. 1555. 1559. 1563. 1567. 1571. 1575. 1579. 1583. 1587. 1591. 1595. 1599. 1603. 1607. 1611. 1615. 1619. 1623. 1627. 1631. 1635. 1639. 1643. 1647. 1651. 1655. 1659. 1663. 1667. 1671. 1675. 1679. 1683. 1687. 1691. 1695. 1699. 1703. 1707. 1711. 1715. 1719. 1723. 1727. 1731. 1735. 1739. 1743. 1747. 1751. 1755. 1759. 1763. 1767. 1771. 1775. 1779. 1783. 1787. 1791. 1795. 1799. 1803. 1807. 1811. 1815. 1819. 1823. 1827. 1831. 1835. 1839. 1843. 1847. 1851. 1855. 1859. 1863. 1867. 1871. 1875. 1879. 1883. 1887. 1891. 1895. 1899. 1903. 1907. 1911. 1915. 1919. 1923. 1927. 1931. 1935. 1939. 1943. 1947. 1951. 1955. 1959. 1963. 1967. 1971. 1975. 1979. 1983. 1987. 1991. 1995. 1999. 2003. 2007. 2011. 2015. 2019. 2023. 2027. 2031. 2035. 2039. 2043. 2047. 2051. 2055. 2059. 2063. 2067. 2071. 2075. 2079. 2083. 2087. 2091. 2095. 2099. 2103. 2107. 2111. 2115. 2119. 2123. 2127. 2131. 2135. 2139. 2143. 2147. 2151. 2155. 2159. 2163. 2167. 2171. 2175. 2179. 2183. 2187. 2191. 2195. 2199. 2203. 2207. 2211. 2215. 2219. 2223. 2227. 2231. 2235. 2239. 2243. 2247. 2251. 2255. 2259. 2263. 2267. 2271. 2275. 2279. 2283. 2287. 2291. 2295. 2299. 2303. 2307. 2311. 2315. 2319. 2323. 2327. 2331. 2335. 2339. 2343. 2347. 2351. 2355. 2359. 2363. 2367. 2371. 2375. 2379. 2383. 2387. 2391. 2395. 2399. 2403. 2407. 2411. 2415. 2419. 2423. 2427. 2431. 2435. 2439. 2443. 2447. 2451. 2455. 2459. 2463. 2467. 2471. 2475. 2479. 2483. 2487. 2491. 2495. 2499. 2503. 2507. 2511. 2515. 2519. 2523. 2527. 2531. 2535. 2539. 2543. 2547. 2551. 2555. 2559. 2563. 2567. 2571. 2575. 2579. 2583. 2587. 2591. 2595. 2599. 2603. 2607. 2611. 2615. 2619. 2623. 2627. 2631. 2635. 2639. 2643. 2647. 2651. 2655. 2659. 2663. 2667. 2671. 2675. 2679. 2683. 2687. 2691. 2695. 2699. 2703. 2707. 2711. 2715. 2719. 2723. 2727. 2731. 2735. 2739. 2743. 2747. 2751. 2755. 2759. 2763. 2767. 2771. 2775. 2779. 2783. 2787. 2791. 2795. 2799. 2803. 2807. 2811. 2815. 2819. 2823. 2827. 2831. 2835. 2839. 2843. 2847. 2851. 2855. 2859. 2863. 2867. 2871. 2875. 2879. 2883. 2887. 2891. 2895. 2899. 2903. 2907. 2911. 2915. 2919. 2923. 2927. 2931. 2935. 2939. 2943. 2947. 2951. 2955. 2959. 2963. 2967. 2971. 2975. 2979. 2983. 2987. 2991. 2995. 2999. 3003. 3007. 3011. 3015. 3019. 3023. 3027. 3031. 3035. 3039. 3043. 3047. 3051. 3055. 3059. 3063. 3067. 3071. 3075. 3079. 3083. 3087. 3091. 3095. 3099. 3103. 3107. 3111. 3115. 3119. 3123. 3127. 3131. 3135. 3139. 3143. 3147. 3151. 3155. 3159. 3163. 3167. 3171. 3175. 3179. 3183. 3187. 3191. 3195. 3199. 3203. 3207. 3211. 3215. 3219. 3223. 3227. 3231. 3235. 3239. 3243. 3247. 3251. 3255. 3259. 3263. 3267. 3271. 3275. 3279. 3283. 3287. 3291. 3295. 3299. 3303. 3307. 3311. 3315. 3319. 3323. 3327. 3331. 3335. 3339. 3343. 3347. 3351. 3355. 3359. 3363. 3367. 3371. 3375. 3379. 3383. 3387. 3391. 3395. 3399. 3403. 3407. 3411. 3415. 3419. 3423. 3427. 3431. 3435. 3439. 3443. 3447. 3451. 3455. 3459. 3463. 3467. 3471. 3475. 3479. 3483. 3487. 3491. 3495. 3499. 3503. 3507. 3511. 3515. 3519. 3523. 3527. 3531. 3535. 3539. 3543. 3547. 3551. 3555. 3559. 3563. 3567. 3571. 3575. 3579. 3583. 3587. 3591. 3595. 3599. 3603. 3607. 3611. 3615. 3619. 3623. 3627. 3631. 3635. 3639. 3643. 3647. 3651. 3655. 3659. 3663. 3667. 3671. 3675. 3679. 3683. 3687. 3691. 3695. 3699. 3703. 3707. 3711. 3715. 3719. 3723. 3727. 3731. 3735. 3739. 3743. 3747. 3751. 3755. 3759. 3763. 3767. 3771. 3775. 3779. 3783. 3787. 3791. 3795. 3799. 3803. 3807. 3811. 3815. 3819. 3823. 3827. 3831. 3835. 3839. 3843. 3847. 3851. 3855. 3859. 3863. 3867. 3871. 3875. 3879. 3883. 3887. 3891. 3895. 3899. 3903. 3907. 3911. 3915. 3919. 3923. 3927. 3931. 3935. 3939. 3943. 3947. 3951. 3955. 3959. 3963. 3967. 3971. 3975. 3979. 3983. 3987. 3991. 3995. 3999. 4003. 4007. 4011. 4015. 4019. 4023. 4027. 4031. 4035. 4039. 4043. 4047. 4051. 4055. 4059. 4063. 4067. 4071. 4075. 4079. 4083. 4087. 4091. 4095. 4099. 4103. 4107. 4111. 4115. 4119. 4123. 4127. 4131. 4135. 4139. 4143. 4147. 4151. 4155. 4159. 4163. 4167. 4171. 4175. 4179. 4183. 4187. 4191. 4195. 4199. 4203. 4207. 4211. 4215. 4219. 4223. 4227. 4231. 4235. 4239. 4243. 4247. 4251. 4255. 4259. 4263. 4267. 4271. 4275. 4279. 4283. 4287. 4291. 4295. 4299. 4303. 4307. 4311. 4315. 4319. 4323. 4327. 4331. 4335. 4339. 4343. 4347. 4351. 4355. 4359. 4363. 4367. 4371. 4375. 4379. 4383. 4387. 4391. 4395. 4399. 4403. 4407. 4411. 4415. 4419. 4423. 4427. 4431. 4435. 4439. 4443. 4447. 4451. 4455. 4459. 4463. 4467. 4471. 4475. 4479. 4483. 4487. 4491. 4495. 4499. 4503. 4507. 4511. 4515. 4519. 4523. 4527. 4531. 4535. 4539. 4543. 4547. 4551. 4555. 4559. 4563. 4567. 4571. 4575. 4579. 4583. 4587. 4591. 4595. 4599. 4603. 4607. 4611. 4615. 4619. 4623. 4627. 4631. 4635. 4639. 4643. 4647. 4651. 4655. 4659. 4663. 4667. 4671. 4675. 4679. 4683. 4687. 4691. 4695. 4699. 4703. 4707. 4711. 4715. 4719. 4723. 4727. 4731. 4735. 4739. 4743. 4747. 4751. 4755. 4759. 4763. 4767. 4771. 4775. 4779. 4783. 4787. 4791. 4795. 4799. 4803. 4807. 4811. 4815. 4819. 4823. 4827. 4831. 4835. 4839. 4843. 4847. 4851. 4855. 4859. 4863. 4867. 4871. 4875. 4879. 4883. 4887. 4891. 4895. 4899. 4903. 4907. 4911. 4915. 4919. 4923. 4927. 4931. 4935. 4939. 4943. 4947. 4951. 4955. 4959. 4963. 4967. 4971. 4975. 4979. 4983. 4987. 4991. 4995. 4999. 5003. 5007. 5011. 5015. 5019. 5023. 5027. 5031. 5035. 5039. 5043. 5047. 5051. 5055. 5059. 5063. 5067. 5071. 5075. 5079. 5083. 5087. 5091. 5095. 5099. 5103. 5107. 5111. 5115. 5119. 5123. 5127. 5131. 5135. 5139. 5143. 5147. 5151. 5155. 5159. 5163. 5167. 5171. 5175. 5179. 5183. 5187. 5191. 5195. 5199. 5203. 5207. 5211. 5215. 5219. 5223. 5227. 5231. 5235. 5239. 5243. 5247. 5251. 5255. 5259. 5263. 5267. 5271. 5275. 5279. 5283. 5287. 5291. 5295. 5299. 5303. 5307. 5311. 5315. 5319. 5323. 5327. 5331. 5335. 5339. 5343. 5347. 5351. 5355. 5359. 5363. 5367. 5371. 5375. 5379. 5383. 5387. 5391. 5395. 5399. 5403. 5407. 5411. 5415. 5419. 5423. 5427. 5431. 5435. 5439. 5443. 5447. 5451. 5455. 5459. 5463. 5467. 5471. 5475. 5479. 5483. 5487. 5491. 5495. 5499. 5503. 5507. 5511. 5515. 5519. 5523. 5527. 5531. 5535. 5539. 5543. 5547. 5551. 5555. 5559. 5563. 5567. 5571. 5575. 5579. 5583. 5587. 5591. 5595. 5599. 5603. 5607. 5611. 5615. 5619. 5623. 5627. 5631. 5635. 5639. 5643. 5647. 5651. 5655. 5659. 5663. 5667. 5671. 5675. 5679. 5683. 5687. 5691. 5695. 5699. 5703. 5707. 5711. 5715. 5719. 5723. 5727. 5731. 5735. 5739. 5743. 5747. 5751. 5755. 5759. 5763. 5767. 5771. 5775. 5779. 5783. 5787. 5791. 5795. 5799. 5803. 5807. 5811. 5815. 5819. 5823. 5827. 5831. 5835. 5839. 5843. 5847. 5851. 5855. 5859. 5863. 5867. 5871. 5875. 5879. 5883. 5887. 5891. 5895. 5899. 5903. 5907. 5911. 5915. 5919. 5923. 5927. 5931. 5935. 5939. 5943. 5947. 5951. 5955. 5959. 5963. 5967. 5971. 5975. 5979. 5983. 5987. 5991. 5995. 5999. 6003. 6007. 6011. 6015. 6019. 6023. 6027. 6031. 6035. 6039. 6043. 6047. 6051. 6055. 6059. 6063. 6067. 6071. 6075. 6079. 6083. 6087. 6091. 6095. 6099. 6103. 6107. 6111. 6115. 6119. 6123. 6127. 6131. 6135. 6139. 6143. 6147. 6151. 6155. 6159. 6163. 6167. 6171. 6175. 6179. 6183. 6187. 6191. 6195. 6199. 6203. 6207. 6211. 6215. 6219. 6223. 6227. 6231. 6235. 6239. 6243. 6247. 6251. 6255. 6259. 6263. 6267. 6271. 6275. 6279. 6283. 6287. 6291. 6295. 6299. 6303. 6307. 6311. 6315. 6319. 6323. 6327. 6331. 6335. 6339. 6343. 6347. 6351. 6355. 6359. 6363. 6367. 6371. 6375. 6379. 6383. 6387. 6391. 6395. 6399. 6403. 6407. 6411. 6415. 6419. 6423. 6427. 6431. 6435. 6439. 6443. 6447. 6451. 6455. 6459. 6463. 6467. 6471. 6475. 6479. 6483. 6487. 6491. 6495. 6499. 6503. 6507. 6511. 6515. 6519. 6523. 6527. 6531. 6535. 6539. 6543. 6547. 6551. 6555. 6559. 6563. 6567. 6571. 6575. 6579. 6583. 6587. 6591. 6595. 6599. 6603. 6607. 6611. 6615. 6619. 6623. 6627. 6631. 6635. 6639. 6643. 6647. 6651. 6655. 6659. 6663. 6667. 6671. 6675. 6679. 6683. 6687. 6691. 6695. 6699. 6703. 6707. 6711. 6715. 6719. 6723. 6727. 6731. 6735. 6739. 6743. 6747. 6751. 6755. 6759. 6763. 6767. 6771. 6775. 6779. 6783. 6787. 6791. 6795. 6799. 6803. 6807. 6811. 6815. 6819. 6823. 6827. 6831. 6835. 6839. 6843. 6847. 6851. 6855. 6859. 6863. 6867. 6871. 6875. 6879. 6883. 6887. 6891. 6895. 6899. 6903. 6907. 6911. 6915. 6919. 6923. 6927. 6931. 6935. 6939. 6943. 6947. 6951. 6955. 6959. 6963. 6967. 6971. 6975. 6979. 6983. 6987. 6991. 6995. 6999. 7003. 7007. 7011. 7015. 7019. 7023. 7027. 7031. 7035. 7039. 7043. 7047. 7051. 7055. 7059. 7063. 7067. 7071. 7075. 7079. 7083. 7087. 7091. 7095. 7099. 7103. 7107. 7111. 7115. 7119. 7123. 7127. 7131. 7135. 7139. 7143. 7147. 7151. 7155. 7159. 7163. 7167. 7171. 7175. 7179. 7183. 7187. 7191. 7195. 7199. 7203. 7207. 7211. 7215. 7219. 7223. 7227. 7231. 7235. 7239. 7243. 7247. 7251. 7255. 7259. 7263. 7267. 7271. 7275. 7279. 7283. 7287. 7291. 7295. 7299. 7303. 7307. 7311. 7315. 7319. 7323. 7327. 7331. 7335. 7339. 7343. 7347. 7351. 7355. 7359. 7363. 7367. 7371. 7375. 7379. 7383. 7387. 7391. 7395. 7399. 7403. 7407. 7411. 7415. 7419. 7423. 7427. 7431. 7435. 7439. 7443. 7447. 7451. 7455. 7459. 7463. 7467. 7471. 7475. 7479. 7483. 7487. 7491. 7495. 7499. 7503. 7507. 7511. 7515. 7519. 7523. 7527. 7531. 7535. 7539. 7543. 7547. 7551. 7555. 7559. 7563. 7567. 7571. 7575. 7579. 7583. 7587. 7591. 7595. 7599. 7603. 7607. 7611. 7615. 7619. 7623. 7627. 7631. 7635. 7639. 7643. 7647. 7651. 7655. 7659. 7663. 7667. 7671. 7675. 7679. 7683. 7687. 7691. 7695. 7699. 7703. 7707. 7711. 7715. 7719. 7723. 7727. 7731. 7735. 7739. 7743. 7747. 7751. 7755. 7759. 7763. 7767. 7771. 7775. 7779. 7783. 7787. 7791. 7795. 7799. 7803. 7807. 7811. 7815. 7819. 7823. 7827. 7831. 7835. 7839. 7843. 7847. 7851. 7855. 7859. 7863. 7867. 7871. 7875. 7879. 7883. 7887. 7891. 7895. 7899. 7903. 7907. 7911. 7915. 7919. 7923. 7927. 7931. 7935. 7939. 7943. 7947. 7951. 7955. 7959. 7963. 7967. 7971. 7975. 7979. 7983. 7987. 7991. 7995. 7999. 8003. 8007. 8011. 8$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 3 \sum_{n=1}^{\infty} (n-1) x^n$$

$$\Rightarrow (A(x) - a_0) - x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$$

$$\Rightarrow A(x) - 2 - x A(x)$$

$$\Rightarrow 3 \sum_{n=1}^{\infty} (n-1) x^n = \frac{x^2}{(1-x)^2}$$

$$\therefore A(x)(1-x) - 2 = \frac{3x^2}{(1-x)^2}$$

$$A(x)(1-x) = 2 + \frac{3x^2}{(1-x)^2}$$

$$A(x) = \frac{2}{(1-x)} + \frac{3x^2}{(1-x)^3}$$

$$\sum a_n x^n = 2 \sum_{n=0}^{\infty} x^n + 3 \sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n$$

$$\therefore a_n = 2 + \frac{3}{2} (n(n-1)) \text{ is the required solution.}$$

$$\textcircled{1} a_n - 5a_{n-1} + 6a_{n-2} = n(n-1) \text{ for } n \geq 2 \quad (a_0=1, a_1=3)$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 5 \sum_{n=2}^{\infty} a_{n-1} x^n + 6 \sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2}^{\infty} n(n-1) x^n$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 5x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 6x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = \sum_{n=2}^{\infty} n(n-1) x^n$$

$$\Rightarrow (A(x) - a_0) - 5x(A(x) - a_0) + 6x^2 A(x) = \sum_{n=2}^{\infty} n(n-1) x^n$$

$$\Rightarrow A(x) [1 - 5x + 6x^2] - a_0 + a_1 x + 5xa_0 = \sum_{n=2}^{\infty} n(n-1) x^n$$

$$\Rightarrow A(x) [1 - 5x + 6x^2] - 1 - 5x + 5x = \frac{2x^2}{(1-x)^3}$$

$$\Rightarrow A(x) [1 - 5x + 6x^2] = \frac{2x^2 - (1-x)^3}{(1-x)^3}$$

$$\Rightarrow A(x) (6x^2 - 2x - 3x + 1) = \frac{2x^2 - (1-x)^3}{(1-x)^3}$$

$$(3x(2x-1) - 1(2x-1)) = \frac{2x^2 - (1-x)^3}{(1-x)^3}$$

$$\Rightarrow A(x) = \frac{2x^2}{(1-x)^3(3x-1)} - \frac{1}{(3x-1)(2x-1)}$$

$$\Rightarrow A(x) = \frac{1-3x+5x^2-x^3}{(1-x)^3(1-2x)(1-3x)}$$

$$\Rightarrow A(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{1-2x} + \frac{E}{1-3x}$$

On simplification ,

$$A = \frac{39}{12} \quad B = \frac{3}{2} \quad C = 1 \quad D = -10 \quad E = \frac{21}{4}$$

$$\therefore A(x) = \frac{39}{12} (1-x)^{-1} + \frac{3}{2} (1-x)^{-2} + (1-x)^{-3} - 10(1-2x)^{-1} + \frac{21}{4} (1-3x)^{-1}$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{39}{12} \sum_{n=0}^{\infty} x^n + \frac{3}{2} \sum_{n=0}^{\infty} (n+1) x^n + \sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^n$$

Comparing the coeff of x^n on both sides $-10 \sum_{n=0}^{\infty} 2^n x^n + \frac{21}{4} \sum_{n=0}^{\infty} 3^n x^n$

$$a_n = \frac{39}{12} + \frac{3}{2} (n+1) + \frac{(n+2)(n+1)}{2} - 10(2)^n + \frac{21}{4} (3)^n$$

Solving non linear Recurrence relations

Q) Solve $a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0$ for $n \geq 0$

Given $a_0 = 4, a_1 = 13$

sol) Let $b_n = a_n^2$

Eq ① can be written as

$$a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0 \text{ --- ①}$$

$$\Rightarrow b_{n+2} - 5b_{n+1} + 4b_n = 0 \text{ for } n \geq 0$$

$$\Rightarrow b_n - 5b_{n-1} + 4b_{n-2} = 0 \text{ for } n \geq 2 \text{ put } n = n-2$$

\Rightarrow By characteristic root method

$$t^2 - 5t + 4 = 0$$

$$t^2 - 4t - t + 4 = 0$$

$$t(t-4) - 1(t-4) = 0$$

$$t = 1, 4$$

Sol of ② is $b_n = c_1 4^n + c_2 1^n$ --- ③

To find C_1, C_2

Given $a_0 = 4$

$$\Rightarrow b_0 = a_0^2 = 4^2 = 16$$

$$a_1 = 13 \quad b_1 = a_1^2 = 13^2 = 169$$

Put $n=0$ in (3)

$$b_0 = C_1 4^0 + C_2 1^0 = 16$$

$$C_1 + C_2 = 16$$

Put $n=1$ in (3) - (4)

$$b_1 = C_1 4 + C_2 = 169$$

$$4C_1 + C_2 = 169 \quad \text{--- (5)}$$

Solving (4) and (5)

$$C_1 = 51 \quad C_2 = -35$$

\therefore GS sol of eq (1) is

$$b_n = a_n^2 = 51(4)^n - 35(1)^n$$

$$a_n = \sqrt{51(4)^n - 35(1)^n}$$