23/11/2021

UNIT-2

SET THEORY

SET : Set is a well defined collection of distinct objects

* The object in a Set are called its Elements or its members.
* Elements in Set must be distinct
* Set represented by Capital letter [A-Z]
* Elements of Set represent by a to z, natural, integers
* Symbol of Set { }

Ex: $A = \{1,2,3,4\}$

$B = \{Red, Blue, Black\}$

$C = \{a,b,c,d\}$

Types of Sets:

1. Finite Set : The Set A is said to be finite set if it contain finite no. of diff elements $A = \{1,2,3\}$
2. Infinite Set : Set A is infinite Set if it contain infinite no of diff elements. $A = \{1,2,3 \ldots\}$
3. Single term Set If Set contain only one element $A = \{x \; 2 < x < 4\} = \{3\}$
4. Null Set : If set contain no element. It is Empty Set. Denoted by ϕ
5. Equality of Sets : Set A, B are said to be equal if every element of A is an element of B and also element of A is of element of B

Equality of two Sets A, B $\rightarrow$ A = B

Equivalent Set :

If element of one set can be put into ~~1&1~~ one to one corresponding with elements of ~~one two~~ other sets. Then two sets is called Equivalent Set.

$A = \{1,2,3,4\}$ $B = \{a,b,c,d\}$

$A \sim B$

Equivalent Set is denoted by $A \sim B$

7. Sub Set

let A, B are two non-empty sets. The Set A is subset of B if every element of A is an element of B.

$A = \{1,2,3\}$ $B = \{1,2,3,4,5\}$

$A \subseteq B$

8 Proper Set

Set A is proper subset of B (or) A is properly contained in B if and only if

1) Every element of A is also an element is B i.e $A \subseteq B$.

2) There is atleast one element in B which is not in A i.e $A \neq B$.

$A = \{1,2,3\}$ $B = \{1,2,3,4,5\}$

9. Power Set :

If S is any set then family of all subsets of S is called the power set of S.

$S = \{a,b,c\}$

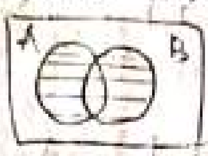
$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\} \Rightarrow 2^3 \rightarrow 2^n$

Operation on Sets

1. Union on Set

let A, B be any two non empty Sets Then union of A and B is the set of all element either A or B or in both A & B

* Denoted by $A \cup B$

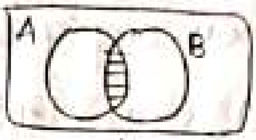


$A \cup B$

2. InterSection on Set

let A, B are two Sets, Intersection of two Sets is the element which present in both Sets

* Denoted by $A \cap B$



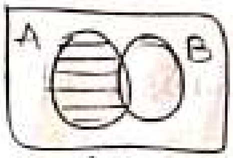
$A \cap B$

3. Difference of sets :

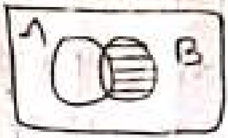
If A, B are two Sets, Difference of Sets is A-B i.e

$A-B = \{x / x \in A \ \& \ x \notin B\}$

$B-A = \{x / x \in B \ \& \ x \notin A\}$



$A-B$



$B-A$

4. Complement Set :

let A be any Set then complement of Set A is set of all elements in Universal set U but not in A.

* Denoted by A^c (or) A' (or) $\bar{A}$

$A^c = \{x \mid x \in U \ \& \ x \notin A\}$

$\bar{A}$ (or) A^c (or) A'

1. let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 3, 5\}$

$B = \{2, 4, 6, 8\}$

$C = \{2, 5, 10\}$

Verify 1) $(A \cap B)^c = A^c \cup B^c$

2) $(A \cup B)^c = A^c \cap B^c$

3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

4) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

1. $(A \cap B)^c = A^c \cup B^c$

$(A \cap B) \Rightarrow \{\phi$

$(A \cap B)^c = \{U\} = \{1,2,3,4,5,6,7,8,9,10\}$

$\cancel{A^c} = \{\cancel{1,2,3,4},5,6,7,8,9,10\}$

$\cancel{B^c = \{}$

$A^c = \{2,4,6,7,8,9,10\}$

$B^c = \{1,3,5,7,9,10\}$

$A^c \cup B^c = \{1,2,3,4,5,6,7,8,9,10\}$

LHS = RHS

2) $(A \cup B)^c = A^c \cap B^c$

$A \cup B = \{1,2,3,4,5,6,8\}$

$(A \cup B)^c = \{7,9,10\}$ $\{\cancel{7,10}\}$

$A^c = \{2,4,6,7,8,9,10\}$

$B^c = \{1,3,5,7,9,10\}$

$A^c \cap B^c = \{7,9,10\}$

$\therefore$ LHS = RHS

3) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$B \cup C = \{2,4,5,6,8,10\}$

$A \cap (B \cup C) = \{\cancel{2,4,6},\{5\}$

$A \cap B = \{\ \}$

$A \cap C = \{5\}$

$(A \cap B) \cup (A \cap C) = \{5\}$

LHS = RHS

Cartesian Product:

let A, B are two sets cartesian product of A, B is $A \times B$

$A \times B = \{(a,b) / a \in A, b \in B\}$

$A = \{1,2,3\}$

$B = \{4,5\}$

$A \times B = \{(1,4)(1,5)(2,4)(2,5)(3,4)(3,5)\}$

Note:

If set A has M elements, B has N elements then

$A \times B = \cancel{M \times N}\ MN$

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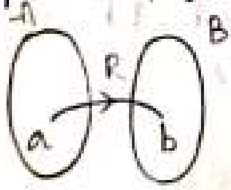
* **Relation:**

let A, B are two sets and subset of $A \times B$ is called a Binary Relation or Relation from A to B

Note:

* If $R \subseteq A \times B$ then R is relation from $A \times B$
* $A = B$, then $R \in A$
* If order pair $(a,b) \in R$ can written as "aRb"



Domain of R

It denoted as dom R

dom R $= \{a / a \in A\ (a,b) \in R$ for some $b \in B\}$

range of R is denoted rang R

Rang R $= \{b / b \in B\ (a,b) \in R$ for some $a \in A\}$

$A = \{1,2,3\}$

$B = \{a,b\}$

$A \times B = \{(1,a)(1,b)(2,a)(2,b)(3,a)(3,b)\}$

$R = \{(1,a)(2,b)\}$ — $R = \{(1,a)(2,a)\}$

dom $R = \{1,2\}$ — dom $R = \{1,2\}$

Rang $R = \{a,b\}$ — Rang $R = \{a\}$

Types of Relations [Properties]

1. Reflexive Relation

A Relation R is said to be Reflexive relation on set A if $a_i R a_i \ \forall a_i \in A$ i.e order pair $\boxed{(a_i, a_i) \in R \ \forall a_i \in A}$

2. Symmetric Relation:

A relation R is said to Symmetric on set A if $aRb \Rightarrow bRa \ \forall a,b \in A$ i.e $(a,b) \in R \Rightarrow (b,a) \in R$

3. Transitive Relation

A relation R is said to transitive on set A if $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$, i.e $aRb, bRc \Rightarrow aRc$
$\forall a,b,c \in A$

Ex: $A = \{1,2,3\}$

$A \times A = \{(1,2)(1,3)(2,1)(2,2)(2,3)(3,1)(3,2)(3,3)(1,1)\}$

$R = \{(1,1)(2,2)(3,3)(1,2)(2,1)(3,1)(1,3)\}$

Reflexive: Clearly $(1,1) \in R$
$(2,2) \in R$
$(3,3) \in R \ \forall 1,2,3 \in A$
R is reflexive

Symmetric: $(a,b) \in R \Rightarrow (b,a) \in R$
then R is Symmetric

Transitive: $(1,1) \to (1,2)(1,3)$
$(1,2) \in R$
$(1,3) \in R$

$(2,2) \to (2,1)$
$(2,1) \in R$

$(3,3) \to (3,1)$
$(3,1) \in R$

$(1,2) \to (2,2)(2,1)$
$(1,2) \in R$
$(1,1) \in R$

$(2,1) \to (1,1)(1,2)$
$(2,1) \in R$
$(2,2) \in R$

$(3,1) \to (1,1)(1,2)$
$(3,1) \in R$
$(3,2) \notin R$

$\therefore$ R is not Transitive

2. let $A = \{a,b,c\}$
$R = \{(a,a)(a,b)(a,c)(b,b)(c,c)(b,c)(c,a)\}$

$A = \{a,b,c\}$
Reflexive: $(a,a)(b,b)(c,c) \in R \ \forall a,b,c \in A$ $\therefore$ R is reflexive on A.

Symmetric: $(a,b) \in R \Rightarrow (b,a) \in R$

If $(a,b) \in R$ then $(b,a) \notin R$ then it is not Symmetric

Transitive: $(a,a) \to (a,b)(a,c)$
$(a,b) \in R$
$(a,c) \in R$

$(b,b) \to (b,c)$
$(b,c) \in R$

$(c,c) \to (c,a)$
$(c,a) \in R$

$(a,b) \to (b,b)(b,c)$
$(a,b) \in R$
$(a,c) \in R$

$(b,c) \to (c,c)(c,a)$
$(b,c) \in R$
$(b,a) \notin R$

$(a,c) \to (c,c)(c,a)$
$(a,c) \in R$
$(a,a) \in R$

$\therefore$ It is not Transitive

Equivalence Relation

Relation R is said to be equivalence relation on set A if R is reflexive, R is Symmetric, R is transitive.

1. Let $A = \{1,2,3,4\}$ and Relation R defined on A is
$R = \{(1,1)(1,2)(2,1)(2,2)(3,4)(4,3)(3,3)(4,4)\}$ verify that R is an equivalence relation on A.

To show that R is equivalence relation we need to show R is
~~AXA=~~
reflexive, R is symmetric, R is transitive

Reflexive $= \{(1,1)(2,2)(3,3)(4,4)\}\ \forall \{1,2,3,4 \in A$

$\therefore$ R is reflexive

Symmetric : if $(a,b) \in R \Rightarrow (b,a) \in R\ \forall\ a,b \in A$

$\{(1,1)(2,2)(1,2)(2,1)(4,3)(3,4)(3,3)(4,4)\}$

It is Symmetric

Transitive : $(a,b) \in R\ (b,c) \in R \Rightarrow (a,c) \in R$

$(1,1) \rightarrow (1,2)$
$(1,2) \in R$

$(1,2) \rightarrow (2,1)(2,2)$
$(1,1) \in R$
$(1,2) \in R$

$(2,1) \rightarrow (1,1)(1,2)$
$(2,1) \in R$
$(2,2) \in R$

$(3,3) \rightarrow (3,4)$
$(3,4) \in R$

$(2,2) \rightarrow (2,1)$
$(2,1) \in R$

$(4,4) \rightarrow (4,3)$
$(4,3) \in R$

$(3,4) \rightarrow (4,3)(4,4)$
$(3,3) \in R$
$(3,4) \in R$

$\therefore$ It is Transitive

$\therefore$ It is Equivalence Relation

$(4,3) \rightarrow (3,3)(3,4)$
$(4,3)$
$(4,4) \in R$

5. If $A = \{1, 2, 3\}$

$R = \{(1,1)\ (2,2)\ (1,2)\ (1,3)$
$(2,1)\ (2,3)\}$

Sol $A = \{1, 2, 3\}$

$R = \{(1,1)\ (2,2)\ (1,2)\ (1,3)$
$(2,1)\ (2,3)\}$

Reflexive :- $(1,1) \in R$
$(2,2) \in R$
$(3,3) \notin R \quad \forall \{1,2,3\} \in A$

$\therefore$ R is not Reflexive.

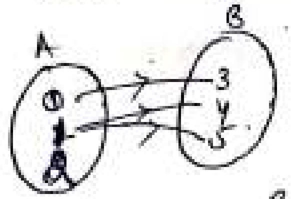
$\therefore$ R is not an equivalence relation

Representations of Relation.
by venn diagrams

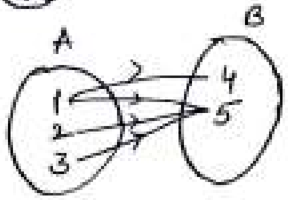
Let A, B are two sets & $A = \{0, 1, 2\}$
$B = \{3, 4, 5\}$

and $R = \{(1,3)\ (2,4)\ (2,5)\}$.

Then venn diagram is.



Eg:-



$\therefore R = \{(1,4)\ (1,5)\ (2,5)\ (3,5)\}$

Representation of relation by matrix

If A, B are two sets and R is a relation from A to B.

If $a_i R a_j = 1$ if $a_i \in A$ & $a_j \in B$.

$a_i \not R a_j = 0$.

Eg:- $\{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 1 \end{array}$$

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Eg :- $M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 0 & 1 & 1 \\ b & 0 & 1 & 0 & 0 \\ c & 1 & 0 & 1 & 0 \\ d & 1 & 1 & 1 & 0 \end{array}$

$R = \{ (a,a)\ (a,c)\ (a,d)\ (b,b)\ (c,a)\ (c,c)\ \&\ (d,a)\ (d,b)\ (d,c) \}$.

Graphical representation of a Relation

Let R be the relation on a set A. denote the each & every element with dots which are called vertices (or) nodes.

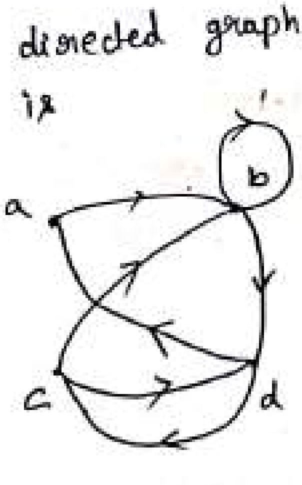
Draw an arrow called an edge from vertex x to y, if there is a relation b/w x & y.

The representation of R is called a direct graph.

Eg :- $A = \{a, b, c, d\}$.

$R = \{ (a,b)\ (b,b)\ (b,d),\ (c,b)\ (c,d)\ (d,a)\ (d,c) \}$ defined on A.

The directed graph of this relation is



1. Let $A = \{1,2,3,4,6\}$ & R be the relation on A, defined a R b iff a is multiple of b. Represent the relation R as a matrix & draw its directed graph and verify that Relation is an equivalence relation or not.

Sol $A = \{1,2,3,4,6\}$

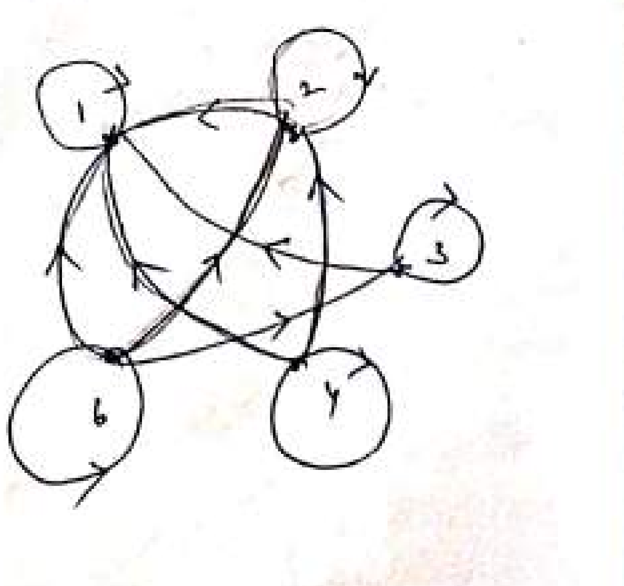
$R = \{ (a,b) \mid a \text{ is multiple of } b \}$

~~[illegible]~~

$R = \{ (1,1)\ (2,1)\ (2,2)\ (3,1)\ (3,3)\ (4,1)\ (4,2)\ (4,4)\ (6,1)\ (6,2)\ (6,3)\ (6,6) \}$

matrix $M_R = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 6 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 & 1 & 0 \\ 6 & 1 & 1 & 1 & 0 & 1 \end{array}$

Directed graph



$R = \{(1,1)\ (2,1)\ (2,2)\ (3,1)\ (3,3)\ (4,1)\ (4,2)\ (4,4)\ (6,1)\ (6,2)\ (6,3)\ (6,6)\}$

Reflexive:- $(1,1) \in R$ & $(2,2) \in R$ $(3,3) \in R$ $(4,4) \in R$ $(6,6) \in R$.

$\therefore$ R is reflexive on A.

symmetric:- $(1,1) \Rightarrow (1,1) \in R$

$(2,1) \in R \Rightarrow (1,2) \notin R$.

$\therefore$ R is not symmetric.

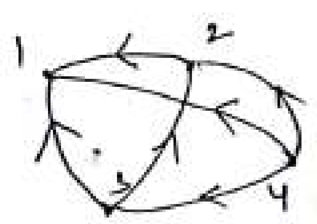
$\therefore$ R is not an equivalence relation.

2. let $X = \{1,2,3,4\}$ & $R = \{(x,y) / x > y\}$. draw the directed graph of R.

Sol) $X = \{1,2,3,4\}$

$R = \{(x,y) \mid x > y\}$

$R = \{(2,1)\ (3,1)\ (3,2)\ (4,1)\ (4,2)\ (4,3)\}$.



3. Let $X = \{1,2,3,4\}$ &
$R = \{(1,1)\ (1,4)\ (4,1)\ (4,4)\ (2,2)\ (2,3)\ (3,2)\ (3,3)\}$
write the matrix of R & sketch its graph. and verify R is equivalence or not.

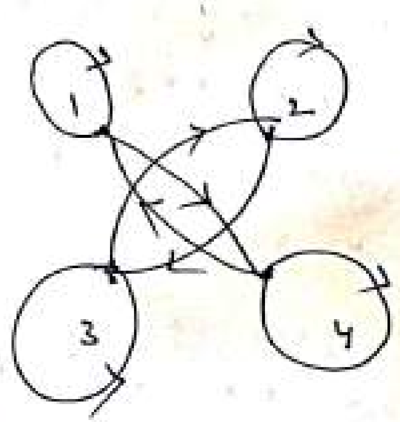
Sol:- $X = \{1,2,3,4\}$

$R = \{(1,1)\ (1,4)\ (4,1)\ (4,4)\ (2,2)\ (2,3)\ (3,2)\ (3,3)\}$

$\therefore$

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

directed graph



Reflexive:- clearly $(a_i, a_i) \in R \ \forall a_i \in X$

$\therefore$ R is reflexive on A.

symmetric: clearly

for every $(a,b) \in R \Rightarrow (b,a) \in R$

$\therefore$ R is symmetric on A.

Transitive :-

$(1,1) — (1,4)$
$(1,4) \in R$

$(1,4) — (4,1)\ (4,4)$
$(1,1) \in R$
$(1,4) \in R$

(4,1) — (1,1) (1,4)
$\underbrace{}_{(4,1)\in R}$ $\underbrace{}_{(4,4)\in R}$

(4,4) — (4,1) ⇒ (4,4) ∈ R

(2,2) — (2,3)
(2,3) ∈ R

(2,3) — (3,2) (3,3)
(2,2) ∈ R . (2,3) ∈ R .

(3,2) → (2,2) (2,3)
(3,2) ∈ R (3,3) ∈ R

(3,3) → (3,2)
(3,2) ∈ R

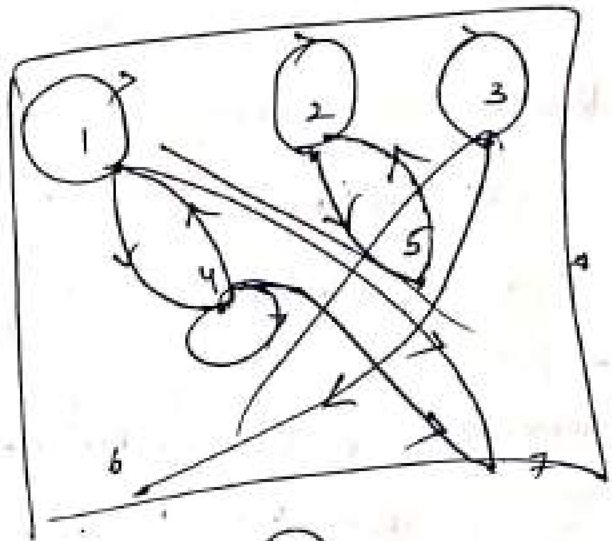
∴ R is transitive

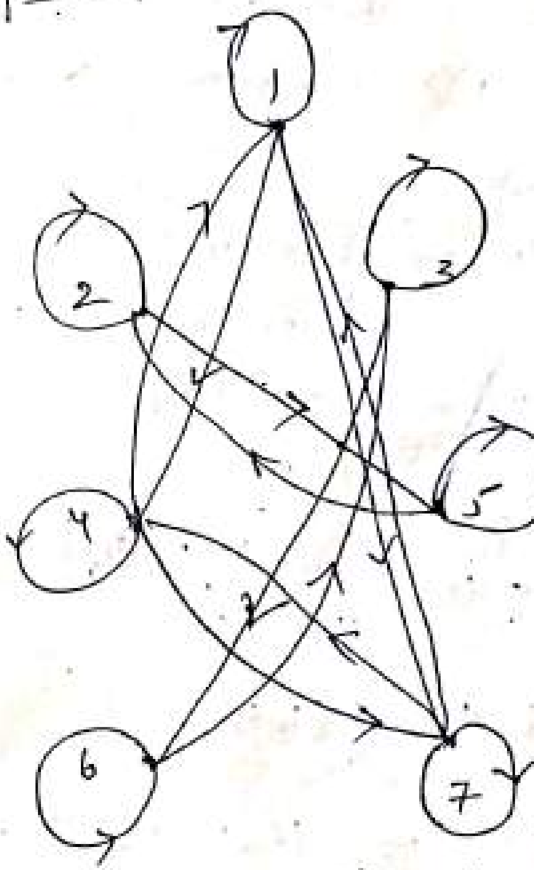
∴ R is an equivalence relation.

4. let X = {1, 2, 3, 4, 5, 6, 7} &
R = {(x,y) / x-y is divisible by 3}.
write M_R & directed graph.
& verify R is an equivalence relation.

Sol) X = {1, 2, 3, 4, 5, 6, 7}.
R = {(x,y) | x-y is divisible by 3}.
R = {(1,1) (1,4) (1,7) (2,2) (2,5)
(3,3) (3,6) (4,1) (4,4) (4,7)
(5,2) (5,5) (6,3) (6,6)
(7,1) (7,4) (7,7)}.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$





$R = \{ (1,1) (1,4) (1,7) (2,2) (2,5)$
$(3,3) (3,6) (4,1) (4,4) (4,7)$
$(5,2) (5,5) (6,3) (6,6)$
$(7,1) (7,4) (7,7) \}.$

Reflexive :- for every $a_i \in A$.
$(a_i, a_i) \in R$. i.e $(1,1) (2,2) (3,3) (4,4) (5,5) (6,6) (7,7) \in R$
$\therefore$ R is Reflexive

Symmetric :- for every $(a,b) \in R$
then $(b,a) \in R$.
$\therefore$ R is symmetric.

Transitive :-

$(1,1) — (1,4) (1,7)$
$(1,4) \in R$
$(1,7) \in R$ $\in R$

$(1,4) — (4,1) (4,4)$
$(1,1) \in R$
$(1,4) \in R$

$(1,7) — (7,1) (7,4) (7,7)$
$(1,1)$ $(1,4)$ $(1,7)$

$(2,2) — (2,5)$.
$(2,5)$

$(2,5) — (5,2) (5,5)$
$(2,2)$ $(2,5) \in R$.

$(3,3) — (3,6)$
$(3,6) \in R$.

$(3,6) — (6,3) (6,6)$

$(4,1) — (1,1) (1,4) (1,7)$
$(4,1) \in R$
$(4,4) \in R$
$(4,7) \in R$

$(4,4) — (4,1) (4,7)$

$(5,2) — R (2,2) (2,5)$

$(5,5) — (5,2)$

$(6,3) — (3,3) (3,6)$
$(6,3) \in R$
$(6,6) \in R$.

$(6,6) — (6,3)$
$(6,3) \in R$.

$(7,1) — (1,1) (1,4) (1,7)$
$(7,1) \in R$ $(7,4) \in R$
$(7,7) \in R$.

$(7,4) — (4,1) (4,4) (4,7)$

$(7,7) — (7,1) (7,4) (7,7)$

$\therefore$ R is Transitive.
$\therefore$ R is an equivalence relation.

5. Let $A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$ on this set define the relation R by $(x,y) \in R$ iff $x-y$ is a multiple of 5. verify that R is an equivalence relation

Sol) $A = \{1,2,3, ---12\}$

$R = \{(x,y) \mid x-y \text{ is multiple of } 5\}$

$R = \{ (1,1)\ (1,6)\ (1,11)\ (2,2)\ (2,7)$
$(2,12)\ (3,3)\ (3,8)$
$(4,4)\ (4,9)\ (5,5)\ (5,10)$
$(6,1)\ (6,6)\ (6,11)\ (7,2)\ (7,7)$
$(8,3)\ (8,8)\ (9,4)\ (9,9)$
$(10,5)\ (10,10)\ (11,6)\ (11,11)$
$(12,7)\ (12,12) \}$.

Reflexive: $(1,1)\ (2,2)\ (3,3)\ (4,4)$
$(5,5)\ (6,6)\ (7,7)\ (8,8)\ (9,9)$
$(10,10)\ (11,11)\ (12,12) \in R$.

for $1,2 \ldots 12 \in A$.

$\therefore$ R is reflexive.

Symmetric: for every $(a,b) \in R$
$\Rightarrow (b,a) \in R$.

$\therefore$ R is symmetric

Transitive: R is transitive.

$\therefore$ R is an equivalence relation.

5. Let $A = \{1,2,3,4\}$ & R_1, R_2 defined on A is

$R_1 = \{ (1,1)\ (2,1)\ (2,2)\ (3,3)\ (4,4)$
$(4,3) \}$

$R_2 = \{ (1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,1)$
$(3,3)\ (1,3)\ (4,1)\ (4,4) \}$

verify that R_1, R_2 are not equivalence relation.

6. The matrix relation R on the set $A = \{1,2,3\}$ given by

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

S.T R is an equivalence relation

Sol) $R = \{ (1,1)\ (2,2)\ (2,3)$
$(3,2)\ (3,3) \}$.

Reflexive:- $(1,1)\ (2,2)\ (3,3) \in R$
for $1,2,3 \in A$

$\therefore$ R is Reflexive

Symmetric:- for every $(a,b) \in R \Rightarrow (b,a) \in R$.

$\therefore$ R is symmetric

Transitive: $(1,1)$ — There is no pairs

$(2,2)$ — $(2,3)$
$(2,3) \in R$

$(2,3)$ — $(3,2)\ (3,3)$

$(3,2)$ — $(2,2)\ (2,3)$

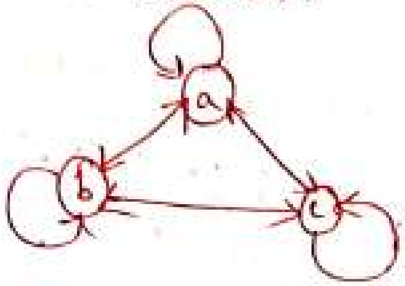
$(3,3)$ — $(3,2)$
$(3,3) \in R$

$\therefore$ R is Transitive

$\therefore$ R is an equivalence relation

7. A relation R on set {a, b, c} is represented by the digraph given below s.t R is an equivalence relation.

Sol



Sol:- $A = \{a, b, c\}$

$R = \{(a,a)\ (a,b)\ (a,c)\ (b,b)\ (b,a)\ (b,c)\ (c,c)\ (c,a)\ (c,b)\}$

~~clearly~~ it is Reflexive

" sym

" Tran

$\therefore$ R is an Equivalence relation.

V.Imp

[redacted]

of ~~relation~~ [redacted]

Some more problems on equivalence relation

1. on the set Z, a relation R is defined by aRb iff a-b is divided by m i.e $m | a-b$ verify that given relation is an equivalence relation or not

Sol:- Given relation is

$R = \{(a,b) \in R \mid \frac{a-b}{m}\ (\text{or})\ m | a-b\}$.

since Z is infinite set.

Reflexive

for any integer $m \in Z$.

w.k.t $\frac{0}{m}$ (or) $m | 0$

$\Rightarrow \frac{a-a}{m}$ or $m | a-a$

$\Rightarrow aRa \Rightarrow (a,a) \in R$.

$\therefore (a,a) \in R \ \forall a \in Z$.

$\therefore$ R is Reflexive.

Symmetric

let $(a,b) \in R$

i.e $\frac{a-b}{m}$ (or) $m | a-b$

$-\frac{(b-a)}{m}$ or $m | -(b-a)$ $\left(\frac{6}{2}\ (\text{or})\ \frac{-6}{2}\right.$

$\Rightarrow bRa$.

$\Rightarrow (b,a) \in R$.

$\therefore \forall (a,b) \in R$ then $(b,a) \in R$

$\therefore$ R is symmetric

Transitive

let $(a,b) \in R$ & $(b,c) \in R$

i.e $\frac{a-b}{m}$ & $\frac{b-c}{m}$ (or) $m | a-b$ & $m | b-c$

$\Rightarrow \frac{a-b+b-c}{m} = \frac{a-c}{m}$ $\left(\frac{6}{2}, \frac{8}{2} \Rightarrow \frac{14}{2}\right.$

$\Rightarrow aRc$

$\therefore (a,b) \in R$ & $(b,c) \in R$

$\Rightarrow (a,c) \in R$

$\therefore$ R is transitive.

Asymmetric Relation :-

A Relation R is said to be Asymmetric on A, if

$aRb \Rightarrow b \not R a$ for $(a,b) \in R$.

Eg:- $A = \{1,2,3,4\}$

$R = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,1)$
$(1,3)\ (2,4)\}$

R is not symmetric

since $(1,3) \in R \Rightarrow (3,1) \notin R$

$\therefore$ It is Asymmetric relation on set A.

Antisymmetric Relation :-

A Relation R is said to be Antisymmetric relation on set A

if $(a,b) \in R$ & $(b,a) \in R$ then $a=b$ $\forall (a,b) \in R$.

Eg:-

$A = \{1,2,3,4\}$.

$R_1 = \{(1,1)\ (2,2)\ (3,3)\ (1,3)\}$

$\therefore$ R is antisymmetric relation on set A. ($\because$ $1=1$, $2=2$, $3=3$) $(1,3) \in R$ but $(3,1) \notin R$.

$R_2 = \{(1,1)\ (2,2)\ (3,1)\ (1,3)\}$

$1=1$, $2=2$ but $(3,1) \in R$ & $(1,3) \in R$ $\Rightarrow 1 \neq 3$)

Operations on Relations

Union :-

R_1, R_2 are two relations from set A to set B

then union of R_1 & R_2 denoted by $R_1 \cup R_2$.

if $(a,b) \in R_1$ (or) $(a,b) \in R_2$.

$\Rightarrow (a,b) \in R_1 \cup R_2$.

Intersection

R_1, R_2 are two relations from set A to set B.

then intersection of R_1 & R_2 denoted by $R_1 \cap R_2$

if $(a,b) \in R_1$ & $(a,b) \in R_2$

$\Rightarrow (a,b) \in R_1 \cap R_2$.

Complement of a Relation

Given relation R from set A to set B, if $(a,b) \in \bar{R}$ (or) R'

if and only if $(a,b) \notin R$

Difference.

$(a,b) \in R-S \Leftrightarrow (a,b) \in R$ & $(a,b) \notin S$

(or)

if $(a,b) \in R$ & $(a,b) \notin S$

then $(a,b) \in R-S$.

$(b,b) \to (b,a)(b,c)$

$(b,a) \in R$

$(b,c) \in R$

$(b,a) \to (a,a)\ (a,b)\ (a,c)$

$(b,a) \in R$ $\quad (b,b) \in R$ $\quad (b,c) \in R$.

$(b,c) \to (c,c)\ (c,a)\ (c,b)$

$(b,c) \in R$ $\quad (b,a) \in R$ $\quad (b,b) \in R$.

$(c,c) \to (c,a)\ (c,b)$

$(c,a) \in R$ $\quad (c,b) \in R$

$(c,a) \to (a,a)\ (a,b)\ (a,c)$

$(c,a) \in R$ $\quad (c,b) \in R$ $\quad (c,c) \in R$.

$(c,b) \to (b,b)\ (b,a)\ (b,c)$

$(c,b) \in R$ $\quad (c,a) \in R$ $\quad (c,c) \in R$.

$\therefore$ R is transitive.

$\therefore$ R is an equivalence relation.

Some more problems on equivalance relation.

1. on the set Z, a relation R is defined by aRb iff a-b is divided by 'm' i.e $m/a-b$. verify that the given relation is an equivalance relation.

Sol:- Given relation is

$aRb \Leftrightarrow m/a-b$.

i.e $R = \{(a,b) \in R \mid m/a-b\}$.

i.e $R = \{(a,b) \in R \mid a-b$ is ... by $m\}$

> Note:- In this problem, given set is integers (z), so, it has infinite number of elements, it can't be possible to construct the set. so, we follow this method for this type of problem.

(i) Reflexive:-

for any integer, $m \in z^+$

then m divides '0' ('0' is divided by m i.e $\frac{0}{m}$)

$\Rightarrow m|0$

for $a \in z^+$, $\quad m|a-a$ $\quad$ (i.e $\frac{a-a}{m}$)

$\Rightarrow aRa$. $\left[\because (a,b) \in R \Rightarrow m|a-b \text{ i.e } aRb\right]$

$\therefore$ 'R' is Reflexive.

(ii) Symmetric.

for $m \in z$, aRb

$\Rightarrow m|a-b$ (By the defin of R. (or $\frac{a-b}{m}$))

$\Rightarrow m|-(b-a)$ $\quad$ (eg $2|6 = 2|-6$)

$\Rightarrow m|b-a$

$\Rightarrow bRa$

$\therefore$ for $m \in z$, $aRb \Rightarrow bRa$.

$\therefore$ R is symmetric.

(iii) for $m \in z$, let aRb & bRc

$aRb \Rightarrow m|a-b$

$bRc \Rightarrow m|b-c$

then $m|a-\cancel{b}+\cancel{b}-c$

$\Rightarrow m|a-c$

$\Rightarrow aRc$

[eg: $2|8$ & $2|6$ then $2|8+6$ → eg $3|9$ & $3|15$ then $3|24$]

∴ for $m \in z$, ~~~~ aRb, bRc

then aRc.

∴ R is transitive.

∴ R is satisfying Reflexive, symmetric and transitive.

∴ R is an equivalance relation on z.

imp

2. On the set of natural numbers $N \times N$ $(a,b) R (c,d)$ iff $a+d = b+c$.

Then s.t is an equivalance relation.

sol ~~(i) Reflexive~~

Given definition (or) Relation is

$\boxed{(a,b) R (c,d) \text{ iff } a+d = b+c}$

(i) Reflexive :- for $(a,b) \in N \times N$

we know that $a+b = b+a$

$\Rightarrow (a,b) R (a,b)$ [∵ from the above condition]

∴ R is Reflexive.

(ii) symmetric :- for $a, b, c, d \in N$.

consider, $(a,b) R (c,d)$

$\Rightarrow a+d = b+c$

$\Rightarrow b+c = a+d$

$\Rightarrow c+b = d+a$

$\Rightarrow c+b = a+d$

$\Rightarrow (c,d) R (a,b)$

∴ $(a,b) R (c,d) \Rightarrow (c,d) R (a,b)$

∴ R is symmetric.

(iii) Transitive :-

let $a, b, c, d, e, f \in N$.

and let $(a,b) (c,d) (e,f) \in N \times N$

consider,

$(a,b) R (c,d)$ & $(c,d) R (e,f)$

$\Rightarrow a+d = b+c$ & $c+f = d+e$

$\Rightarrow$
$$\begin{aligned} a+d &= b+c \\ c+f &= d+e \end{aligned}$$

Add $a+\cancel{d}+\cancel{c}+f = b+\cancel{c}+\cancel{d}+e$

$\Rightarrow a+f = b+e$

$\Rightarrow a+f = e+b$

$(a,b) R (e,f)$.

∴ if $(a,b) R (c,d)$ & $(c,d) R (e,f)$

$\Rightarrow (a,b) R (e,f)$.

∴ R is transitive.

∴ R is Reflexive, symmetric and transitive.

∴ R is an equivalance relation.

3. If 'z' is the set of all integers the relation 'R' on 'z' defined by aRb iff $a \le b$ verify that equivalance relation (or) not.

sol Given that 'z' is the set of all integers.

and $\boxed{aRb \Leftrightarrow a \le b}$

(i) Reflexive :- for $a \in Z$,

we know that $a \le a$

$\therefore aRa$.

$\therefore$ for $a \in Z$ then $(a,a) \in R$.

$\therefore$ R is Reflexive.

(ii) Symmetric :- for $a, b \in Z$

consider $aRb \Rightarrow a \le b$ ($\because$ By the condition)

$\Rightarrow b \not\le a$ $\left(\begin{matrix} 2<3 \\ \text{but } 3 \not< 2 \end{matrix}\right)$

$\therefore$ 'R' is not symmetric.

$\therefore$ R is not an equivalence relation on 'Z'.

4. On the set of all integers Z, the relation R is defined by $(a,b) \in R$ iff $a^2 - b^2$ is an even integer. Show that R is an equivalance relation.

Sol:- Given 'Z' is the set of all integers.

$\boxed{aRb \Leftrightarrow a^2 - b^2 \text{ is an even integer}}$

(i) Reflexive :- for $a \in Z$.

$aRa \Leftrightarrow a^2 - a^2 = 0$ is an even integer

$\therefore$ for $a \in Z$ then aRa.

$\therefore$ R is reflexive.

~~(ii) Transitive~~

(ii) Symmetric :- for $a, b \in Z$

consider $aRb \Rightarrow a^2 - b^2$ is an even integer

$\Rightarrow -(b^2 - a^2)$ is also even integer.

$\Rightarrow b^2 - a^2$ is an even integer

$\Rightarrow bRa$ $\left[\begin{matrix} 4^2 - 2^2 \text{ is even integer} \\ 2^2 - 4^2 = -12 \text{ is an even integer.} \end{matrix}\right]$

$\therefore$ for $a, b \in Z$ and if aRb then bRa.

Then R is symmetric.

(iii) Transitive :-

if $a, b, c \in Z$ & aRb, bRc

consider aRb & bRc

$\Rightarrow a^2 - b^2$ is an even integer & $b^2 - c^2$ is an even integer.

$\Rightarrow a^2 - b^2 + b^2 - c^2$ is also an even integer ($\because$ Sum of two even integers is also an even integer)

$\Rightarrow a^2 - c^2$ is an even integer.

$\Rightarrow aRc$

$\therefore aRb$ & $bRc \Rightarrow aRc$.

$\therefore$ R is ~~symmetric~~ transitive.

So, R is reflexive, symmetric, and transitive.

$\therefore$ R is an equivalance relation.

Partial orders :-

→ A relation R on set A is said to be a partial ordering relation (or) partial order on A. if

(i) R is reflexive

(ii) R is antisymmetric

(iii) R is transitive on A.

Partial order set :- A set 'A' with a partial order 'R' defined on it is called a partially ordered set (or) an ordered set (or) a poset and it is denoted by the pair (A, R).

Problems on partial order relation.

1. In the set of integers the relation R is defined by aRb iff a divides b show that it is partially order relation or not.

Sol :- Given that $aRb \Leftrightarrow a|b$.

(i) Reflexive :- let $a \in z$

we know that $a|a$

$\Rightarrow aRa$

$\therefore$ for every $a \in z$ then aRa

$\therefore$ R is Reflexive.

(ii) Symmetric :- for $a, b \in z$

consider $aRb \Leftrightarrow a|b$

$\Leftrightarrow b \nmid a$

$\therefore$ It is not symmetric

(iii) Transitive :-

for $a, b, c \in z$

let aRb & bRc

$\Rightarrow a|b$ & $b|c$

$\Rightarrow a|c$

$\Rightarrow aRc$.

$\left[\begin{array}{l} 2|6 \text{ \& } 6|12 \\ \Rightarrow 2|12. \end{array} \right.$

for $aRb \ \exists \ bRc$ then aRc.

$\therefore$ R is Transitive.

(iv) Antisymmetric :-

let $a, b \in z$.

aRb & $bRa \Rightarrow a|b$ & $b|a$

$\Rightarrow a = b$

$\therefore$ R is Antisymmetric

$\therefore$ The given relation R is a partial ordering relation.

and the given set is poset.

2. If 'z' is the set of all integers the relation R on 'z' defined by $aRb \Leftrightarrow a \leq b$.

Sol: Given 'z' is set of all integers

$\boxed{aRb \Leftrightarrow a \leq b}$.

(i) Reflexive :-

for $a \in z$

we know that $a \leq a$

$\Rightarrow aRa$

$\therefore$ for $a \in z$ then aRa

$\therefore$ 'R' is Reflexive.

Operations on Relations

Union :-

R_1, R_2 are two relations from set A to set B
then union of R_1 & R_2 denoted by $R_1 \cup R_2$.

If $(a,b) \in R_1$ (or) $(a,b) \in R_2$.
$\Rightarrow (a,b) \in R_1 \cup R_2$.

Intersection

R_1, R_2 are two relations from set A to set B.
then intersection of R_1 & R_2 denoted by $R_1 \cap R_2$

If $(a,b) \in R_1$ & $(a,b) \in R_2$
$\Rightarrow (a,b) \in R_1 \cap R_2$.

Complement of a Relation

Given relation R from set A to set B, if $(a,b) \in \bar{R}$ (or) R' if and only if $(a,b) \notin R$

Difference :

$(a,b) \in R-S \Leftrightarrow (a,b) \in R$ & $(a,b) \notin S$
(or)
If $(a,b) \in R$ & $(a,b) \notin S$
then $(a,b) \in R-S$.

Composition of Relations

Let R be the relation from set A to B and S be the relation from B to C.

Then the composite relation of R and 'S' is denoted by ROS (or) RS and it is defined by

$$\boxed{ROS = \{ (a, c) \mid (a,b) \in R \ \& \ (b,c) \in S \} \text{ where } a \in A \ \& \ b \in B \ \& \ c \in C.}$$

1. Let R, S be two relations on $A = \{1,2,3\}$ &

$R = \{ (1,1) (1,2) (2,3) (3,1) (3,3) \}$

$S = \{ (1,2) (1,3) (2,1) (3,3) \}$.

then find

$R \cup S$, $R \cap S$, $R - S$, R^c, $R \circ S$

S^2 (or) $S \circ S$, $R^2 = R \circ R$, $S \circ R$

$S - R$.

Sol) $A = \{ 1,2,3 \}$

$R = \{ (1,1) (1,2) (2,3) (3,1) (3,3) \}$

$S = \{ (1,2) (1,3) (2,1) (3,3) \}$

$R \cup S = \{ (1,1) (1,2) (2,3) (3,1) (3,3)$
$(1,3) (2,1) \}$

$R \cap S = \{ (1,2) (3,3) \}$

$R^c = \{ (1,3) (2,1) (2,2) (3,2) \}$

$R - S = \{ (1,1) (2,3) (3,1) \}$

$R \circ S = \{ (1,2) (1,1) (2,3) (3,2) (3,3) \}$
$(1,3)$.

$S \circ S = \{ (1,1) (1,3) (2,2) (2,3) (3,3) \}$

$R \circ R = \{ (1,1) (1,2) (1,3) (2,1) (2,3)$
$(3,1) (3,2) (3,3) \}$.

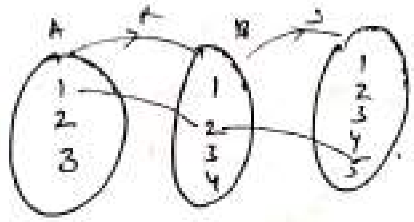
$S \circ R = \{ (1,3) (1,1) (2,1) (2,2) (3,3) \}$
$(3,1) \}$.

$S - R = \{(1,3), (2,1)\}$.

2. Let $A = \{1,2,3\}$
$B = \{1,2,3,4\}$
$C = \{1,2,3,4,5\}$.
be sets & relation
$R = \{(1,2)\ (3,4)\ (2,2)\}$ from A to B.
$S = \{(1,3)\ (2,5)\ (3,1)\ (4,2)\}$ from B to C.
Then find: ROS, SOR, $RO(SOR)$, $(ROS)OR$, $R^{\sim}$, $S^{\sim}$, R^3.

Sol $R = \{(1,2)\ (3,4)\ (2,2)\}$

$S = \{(1,3)\ (2,5)\ (3,1)\ (4,2)\}$.



$ROS = \{(1,5).(3,2)\ (2,5)\}$.

$SOR = \{(1,4)\ (3,2)\ (4,2)\}$.

$RO(SOR) = \{(3,2)\}$

$(ROS)OR = \{(3,2)\}$

~~ROR = {(1,2)}~~
~~R^3 = ROROR = {...}~~

$ROR = \{(1,2)(2,2)\}$ & $SOS = \{(1,1)\ (3,3)\ (4,5)\}$.

$RO\,ROR = \{(1,2), (2,2)\}$.

$SOSOS = \{(1,3)\ (3,1)\}$

3. Let $A = \{1,2,3\}$ & $B = \{1,2,3,4\}$.
the relations R & S from A to B are represented by the following matrices. Determine the representati[on] $R \cup S$, $R \cap S$, R^c, S^c and their matrices & ~~digraphs~~.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Sol
$$M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array} \quad M_S = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$R = \{(1,1)\ (1,3)\ (2,4)\ (3,1)\ (3,2)\ (3,3)\}$

$S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,4)\ (3,2)\ (3,4)\}$.

$R^c = \{(1,2)\ (2,1)\ (2,3)\ (1,4)\ (3,4)\ (2,2)\}$

$S^c = \{(2,1)\ (2,2)\ (2,3)\ (3,1)\ (3,3)\}$.

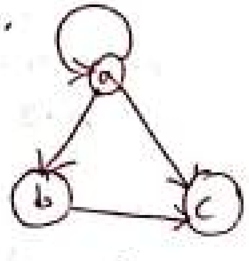
$R \cup S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,4)\ (3,1)\ (3,2)\ (3,3)\ (3,4)\}$

$R \cap S = \{(1,1)\ (1,3)\ (2,4)\ (3,2)\}$.

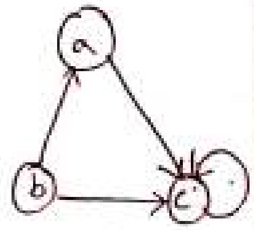
$$M_{R^c} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad M_{S^c} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$M_{R \cup S} = [\quad]$ $M_{R \cap S} = [\quad]$

4. The digraphs of two relations R & S on the set $A = \{a, b, c\}$ are given below. Draw the digraphs of $\bar{R}$, $R \cup S$, $R \cap S$,



(i) R



(ii) S

Sol) from the graphs.

$R = \{(a,a)\ (a,b)\ (a,c)\ (b,c)\}$.

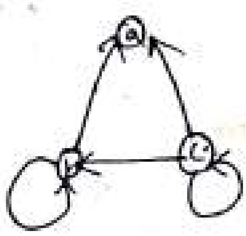
$S = \{(a,c)\ (b,a)\ (b,c)\ (c,c)\}$.

$\bar{R} = \{(b,a)\ (b,b)\ (c,a)\ (c,b)\ (c,c)\}$.

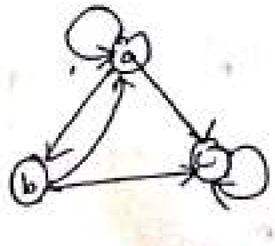
$R \cup S = \{(a,a)\ (a,b)\ (a,c)\ (b,a)\ (b,c)\ (c,c)\}$

$R \cap S = \{(b,c),\ (a,c)\}$.

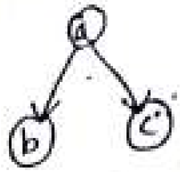
digraph of $\bar{R}$ =



digraph of $R \cup S$ =



digraph $R \cap S$ =



5. let $A = \{1, 2, 3\}$ &

$R = \{(1,1)\ (1,2)\ (2,3)\ (3,1)\}$

$S = \{(2,1)\ (3,1)\ (3,2)\ (3,3)\}$

compute $\bar{R}$, $R \cap S$, $R \cup S$, R^c:

6. let $A = \{a, b, c, d\}$

$R = \{(a,a)\ (a,c)\ (b,c)\ (c,a)\ (d,b)\ (d,d)\}$.

$S = \{(a,b)\ (b,c)\ (c,a)\ (c,b)\ (d,c)\}$

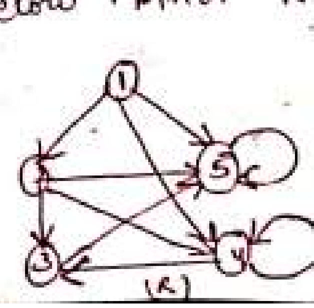
find $M_{R \cap S}$, $M_{R \cup S}$, M_{R^c}, $M_{\bar{S}}$

7. let $A = \{1, 2, 3\}$ & R & S be relations on A, whose matrices are given below find the matrices of $\bar{R}$, R^c, $R \cap S$ & $R \cup S$.

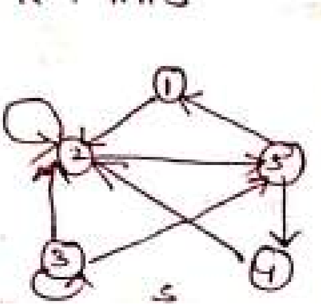
$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

8. let $A = \{1, 2, 3, 4, 5\}$

R & S be relations on A whose corresponding digraphs are as given below. find $\bar{R}$, R^c, $R \cap S$



(R)



S

8. let $A = \{1,2,3,4\}$ $B = \{w,x,y,z\}$
and $C = \{5,6,7\}$

let R_1 be the relation from A to B

defined $R_1 = \{(1,x)\ (2,x)\ (3,y)\ (3,z)\}$

& $R_2 = \{(w,5)\ (x,6)\}$ from B to C

$R_3 = \{(w,5)\ (w,6)\}$ from B to C.

find $R_1 o R_2$ & $R_1 o R_3$.

Sol)

$$R_1 o R_2 = \{(1,6)\ (2,6)\}$$

$$R_1 o R_3 = \{\ \}\ \text{or}\ \phi.$$

9. let $A = \{1,2,3,4\}$

$R = \{(1,1)\ (1,2)\ (2,3)\ (3,4)\}$.

$S = \{(3,1)\ (4,4)\ (2,4)\ (1,4)\}$.

be the relations on A.

Determine ROS, SOR, R^2, S^2

Sol)

$$ROS = \{(1,4)\ (1,1)\ (3,4)\}.$$

$$SOR = \{(3,1)\ (3,2)\}$$

$$R^2 = \{(1,1)\ (1,2)\ (1,3)\ (2,4)\}$$

$$S^2 = \{(3,4)\ (2,4)\ (1,4)\ (4,4)\}.$$

10. If $A = \{1,2,3,4\}$ &

$R = \{(1,2)\ (1,3)\ (2,4)\ (4,4)\}$

$S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,3)\ (3,4)\}$

find ROS, SOR, R^2, S^2 write their matrix

Equivalence class :- let R be ~~the~~ an equivalence relation on set A, $x \in A$. The equivalence class of 'a' is given by $[a]_R = \{ x \in A \mid (a,x) \in R \}$. Here 'a' is called generator of equivalence class $[a]_R$.

Some set $A = \{e_1, e_2, e_3, \ldots e_n\}$
Some R (eq)
$[e_1]_R = \{ x \in A \mid (x, e) \in R \}$.

Eg :- ~~Sam~~ $S = \{1,2,3\}$
$R = \{ (1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,1) \}$

Find equivalence classes $[1], [2], [3]$.
(should be in second memb)

$[1]_R = \{ \cancel{(1,1)\ (2,1)} \} = \{1, 2\}$

$[2]_R = \{ \cancel{(2,2)\ (1,2)} \} = \{2, 1\}$

$[3]_R = \{ \cancel{(3,3)} \} = \{3\}$

(channel ? Kalpit Kamal Jain)

Eg :- 2) . $X = \{a, b, c, d, e\}$
$R = \{ (a,a)\ (b,b)\ (c,c)\ (d,d)\ (e,e)\ (a,b)\ (b,a)\ (b,e)\ (e,b)\ (a,e)\ (e,a)\ (c,d)\ (d,c) \}$

find $[a], [b], [c], [d], [e]$.

Sol
$[a] = \{a, b, e\}$ $\quad$ $[d] = \{d, c\}$
$[b] = \{b, a, e\}$ $\quad$ $[e] = \{e, b, a\}$.
$[c] = \{c, d\}$

$\therefore$ The partitions of X are $\{ (a,b,e), (c,d) \}$.

Quotient set :- The collection of all equivalence class of A determined by the equivalence relation R on A of by $\frac{A}{R}$ and is defined as $\frac{A}{R} = \{ [a] \mid a \in A \}$

'S' is a non empty set

$S_1, S_2, S_3 --- S_m$ are subsets of 'S'

$S_1, S_2, S_3 -- S_m$ are said to be partitions of 's'

if 1. $S_i \neq \phi$. (any subset, that subset $\neq \phi$)

2. $S_i \cap S_j = \phi$. for $i \neq j$

3. $\bigcup_{i=1}^{m} S_i = S$.

Partition of a set

let 's' be a non empty set, $S_1, S_2, -- S_m$ are subsets of S, The collection of subsets S_i is a partition of S. if and only if

(i) $S_i \neq \phi$ for each i .

(ii) $S_i \cap S_j = \phi$ for $i \neq j$

(iii) $\bigcup_{i=1}^{m} S_i = S$ when $\cup_{i=1}^{m}$.

represents the union of the subsets S_i for all and $S_1, S_2 -- S_m$ are called the Blocks of partition .

. let $A = \{a, b, c, d\}$ & $R = \{(a,a)\ (a,b)\ (b,a)\ (b,b)\ (c,d)\ (d,c)$
$(c,c)\ (d,d)\}$
be an equivalence relation on R find $\frac{A}{R}$.

Sol $A = \{a, b, c, d\}$

$[a]_R = \{a, b\}$ $\quad$ $[c]_R = \{d, c\}$

$[b]_R = \{a, b\}$ $\quad$ $[d]_R = \{c, d\}$

∴ The partitions of A are $\{\{a, b\}, \{c, d\}\}$

Hence $\frac{A}{R} = \{[a], [c]\}$

let A be the equivalence relation on the set A
$A = \{1, 2, 3, 4, 5, 6\}$ where $R = \{(1,1)\ (1,5)\ (2,2)\ (2,3)$
$(2,6)\ (3,2)\ (3,3)\ (3,6)\ (4,4)\ (5,1)\ (5,5)\ (6,2)\ (6,3)\ (6,6)\}$
find the partition of A induced by R i.e equivalence
class of R.

$A = \{1, 2, 3, 4, 5, 6\}$

$[1]_R = \{1, 5\}$ $\quad$ $[3]_R = \{2, 3, 6\}$ $\quad$ $[5]_R = \{1, 5\}$

$[2]_R = \{2, 3, 6\}$ $\quad$ $[4]_R = \{4\}$ $\quad$ $[6]_R = \{2, 3, 6\}$.

The partitions are $\{1, 5\}, \{2, 3, 6\}, \{4\}$

$\frac{A}{R} = \{[1], [2], [4]\}$.

Hasse Diagrams (v.imp)

→ partial ordering (≤) on a set 'P' can be represented by means of a diagram known as Hasse diagram of (P, ≤).

$(P, \leq)$ → poset — represent p.or.so.

→ In Hasse diagram, each element is represented by a small circle or a dot.

→ In Hasse diagram, we represent the vertices by dots or small circles, we don't put arrows on edges and we don't draw self-loops at vertices.

→ In digraph of partial order, there is an edge from vertex A to vertex B & there is an edge from vertex B to vertex C.

as such we need not exhibit an edge from A to C explicitly.
It will done automatically.

Hasse diagram (or) poset diagram

A partial ordering $\leq$ on a set 'P' can be represented by means of diagram known as Hasse diagram (or) poset diagram.

1. Draw the Hasse diagram representing the positive divisors of 36. ~~and verify it~~

Sol: The set of all positive divisors of 36 is

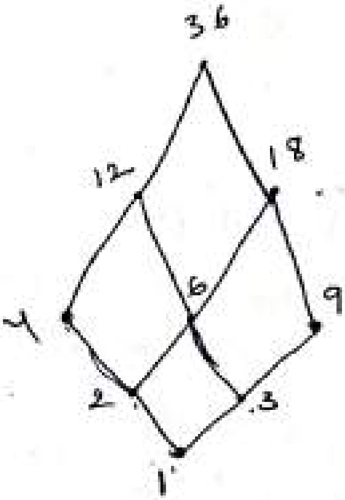
$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

The relation R of divisibility ($aRb \iff a$ divides b) is a partial ordered on this set.

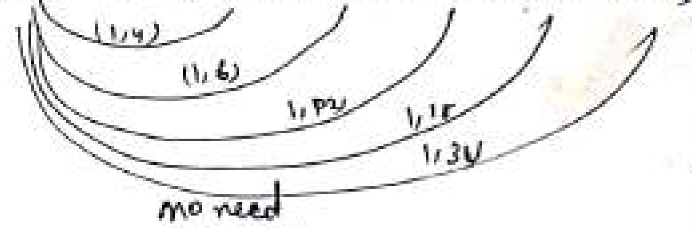
The Hasse diagram for this partial ordered is

$R = \{ \cancel{(1,1)}\ \underline{(1,2)}\ \underline{(1,3)}\ \cancel{(1,4)}\ (1,6)$
$(1,9)\ (1,12)\ (1,18)\ (1,36)$
$\cancel{(2,2)}\ \cancel{(2,4)}\ \cancel{(3,6)}\ (2,12)\ (2,18)\ (2,36)$
$\cancel{(3,3)}\ (3,6)\ (3,9)\ (3,12)\ (3,18)\ (3,36)$
$\cancel{(4,4)}\ (4,12)\ (4,36)\ \cancel{(6,6)}\ (6,12)\ (6,18)\ (6,36)$
$\cancel{(9,9)}\ (9,18)\ (9,36)\ \cancel{(12,12)}\ (12,36)$
$\cancel{(18,18)}\ (18,36)\ \cancel{(36,36)} \}$ ~~No need~~

It should not be self loops



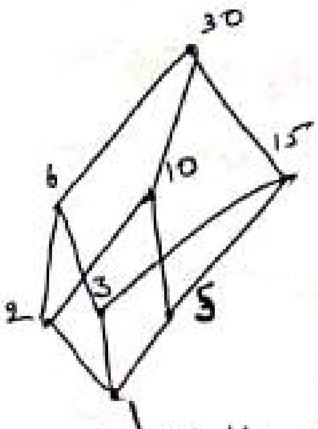
(1,2) → (2,4) (2,6) (2,12) (2,18) (2,36)



no need

2. Draw the Hasse diagram for the poset & determine whether the poset is totally ordered or not?

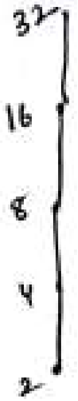
$A = \{1, 2, 3, 5, 6, 10, 15, 30\}$



This is not totally ordered.

(∵ It is not chain | di 2

(ii) $A = \{2, 4, 8, 16, 32\}$.



Totally orded

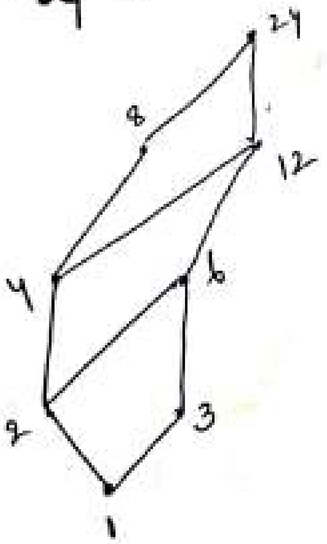
Let $(P, \leq)$ be a poset. If every pair of elements of A are comparable then $(P, \leq)$ is called totally ordered set (or) a chain (or) simply ordered set.

It is totally ordered.

3. Draw the Hasse diagram for D_{24}

sol $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$.

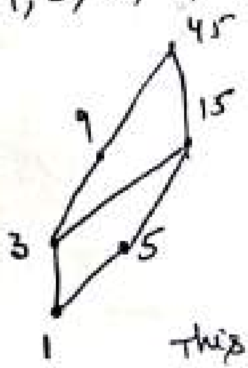
$\{(1,1)(2,2)(3,3)(4,4)(6,6)(8,8)(12,12)(24,24)$
$(1,2)(1,3)(1,4)(1,6)(1,8)(1,12)(1,24)$
$(2,4)(2,6)(2,8)(2,12)(2,24)$
$(3,6)(3,12)(3,24)$
$(4,8)(4,12)(4,24)$
$(6,12)(6,24)(8,24)(12,24)\}$



This is not totally ordered set.

4. D_{45}

sol $D_{45} = \{1, 3, 5, 9, 15, 45\}$.

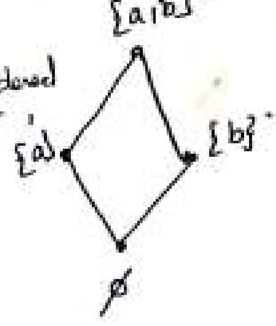


This is not totally ordered set.

Hasse diagram for sets

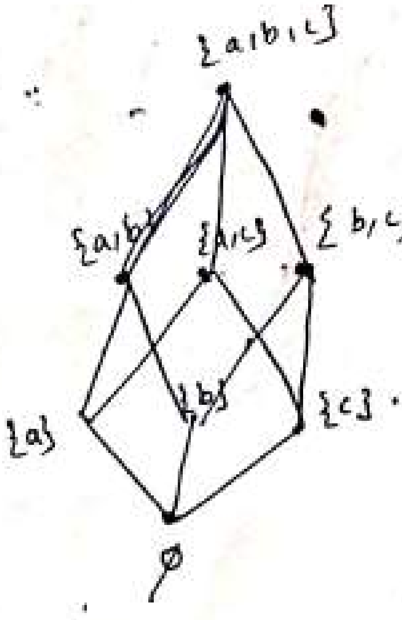
1. Set $A = \{a, b\}$.

sol $P(S) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$



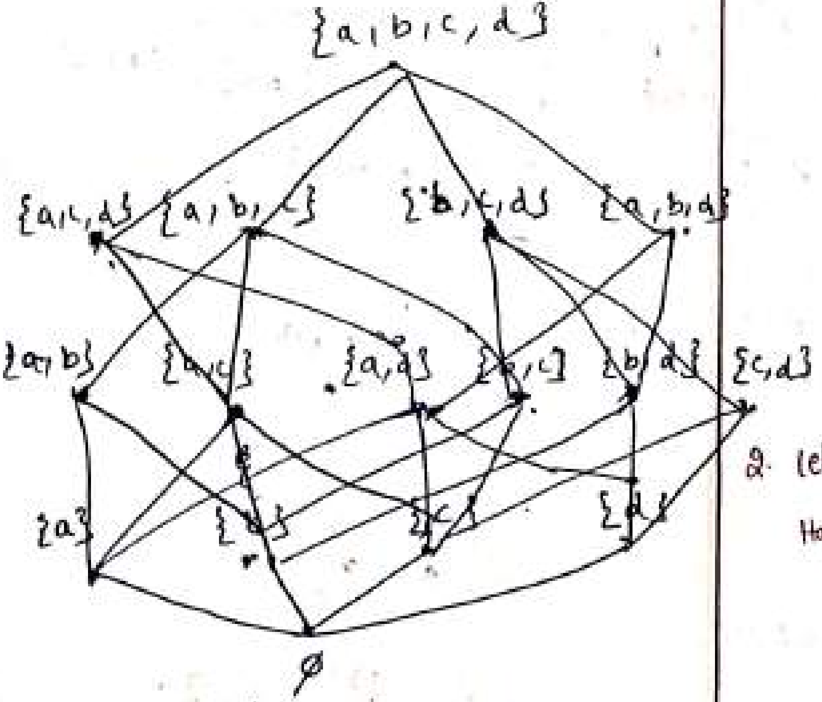
2 Set $A = \{a, b, c\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.



3. set $A = \{a, b, c, d\}$ $2^4 = 16$

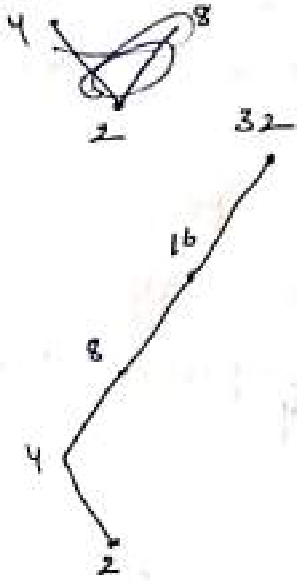
$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\},$
$\{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}$
$\{b,d\}, \{c,d\}$ $\{a,b,c\}$
$\{a,c,d\}, \{a,b,d\}$
$\{b,c,d\}, \{a,b,c,d\}\}$



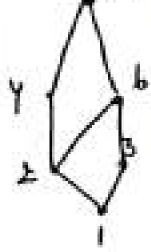
Exercise

1. Let $A = \{2, 4, 8, 16, 32\}$ Draw the Hasse diagram of $(A, |)$.

Sol

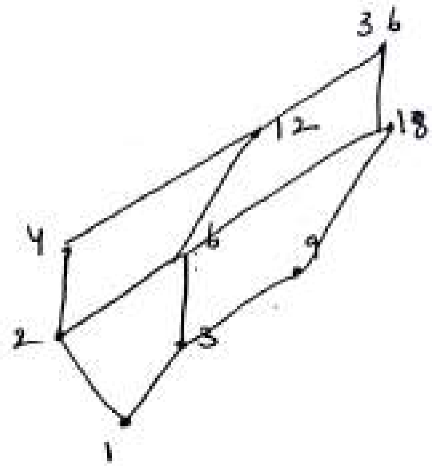


2. Let $A = \{1, 2, 3, 4, 6, 12\}$ Draw the Hasse diagram of $(A, |)$.

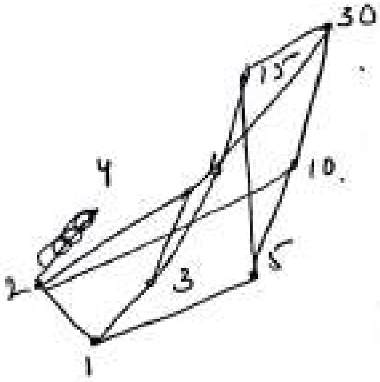


3. Draw the H.d representing the positive divisors of 36.

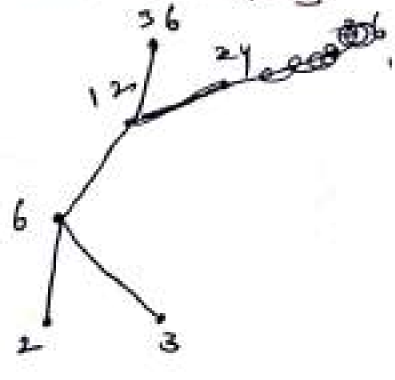
Sol) $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$



4. $A = \{1, 2, 3, 4, 5, 6, 10, 15, 30\}$

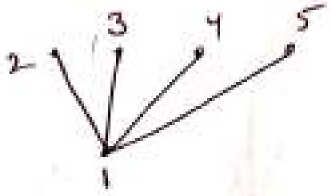


5. $A = \{2, 3, 6, 12, 24, 36\}$



6. $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$

6. Find the matrix of the partial order whose Hasse diagram is



Sol:- $R = \{(1,2)(1,3)\ (1,4)(1,5)$
$(1,1)(2,2)(3,3)(4,4)(5,5)\}$

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

<u>maximal element</u> & <u>minimal element</u>

→ let $(P, \leq)$ be a poset,
$A \leq P$, An element $a \in A$ is said to be minimal element of A if $\exists$ no 'x' in $A \ni x < a$.

→ let $(P, \leq)$ be a poset,
$A \leq P$, An element $a \in A$ is said to be maximal element of A if $\exists$ no x in $A \ni x > a$.

1. let $D_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$ and relation $|$ be dividing partial ordering on D_{24} then Draw the Hasse diagram of $(D_{24}, |)$ and also find the followi

(i) all lower bounds of 8, 12

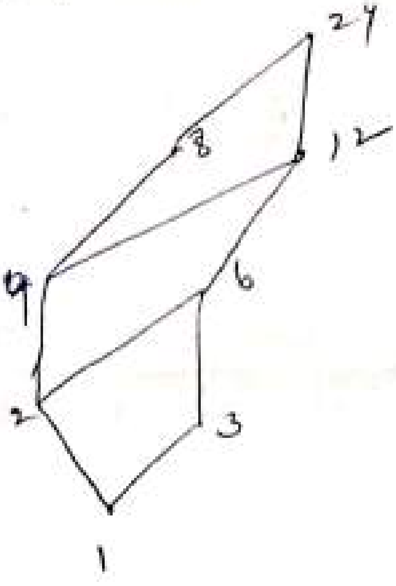
(ii) all upper " of 8, 12

(iii) g.l.b of 8, 12

(iv) l.u.b of 8, 12

(v) greatest & least element of this poset - exist

Sol



1 is all (
$1 \leq x \,\forall$
{1,2

(i) lower bounds of 8, 12 are 1, 2, 4.

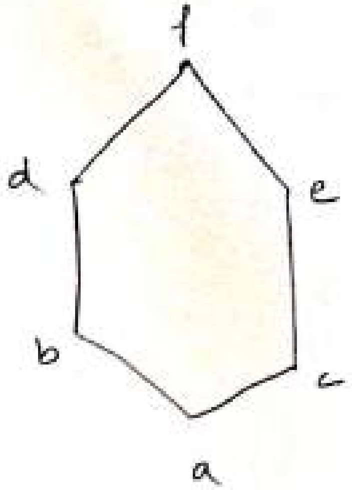
(ii) upper bond 8, 12 ~~are~~ is 24.

(iii) g.l.b of 8, 12 ~~are~~ is, 4.

(iv) l.u.b of 8, 12 is 24

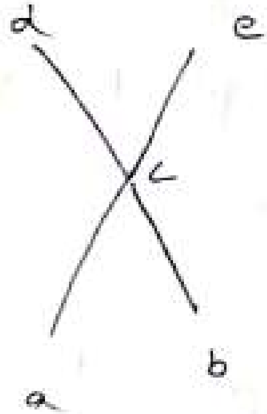
(v) G. elem = 24 & least ele 1.

Determine the greatest & least elements of following. (2)



(i)

(ii)



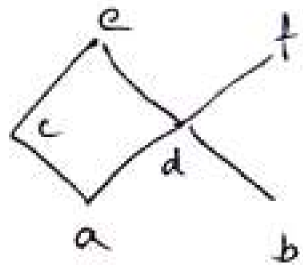
(iii)

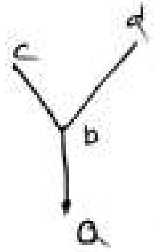
i) Greatest element is f an least elem a.

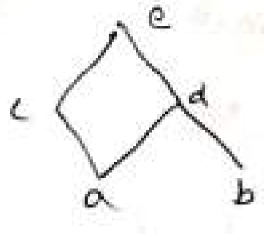
ii) " " e, no least element.

iii) No Greatest no least.

Determine maximal element & minimal el.







i) maxial eleme e, f & minimal a, b.

(ii) max c, d " a

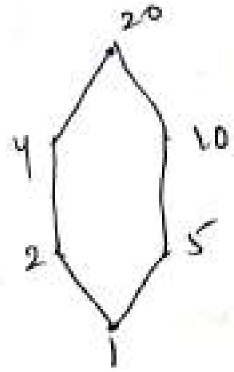
(ii) max e " a, b.

4. Draw a poset diagram & find minimal & maxi[mal], greatest, least.

(i) $(D_{20}, |)$ (ii) $(D_{30}, |)$ (iii) $(A, |)$ whe $A = \{2, 3, 4, 6, 8, 2\ldots$

Sol (iv) $(A, |)$ $A = \{2, 3, 4, 6, 12, 1\ldots$ $24, 36\}$

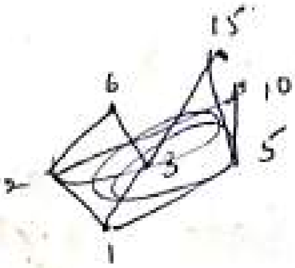
i-Sol): $D_{20} = \{1, 2, 4, 5, 10, 20\}$.

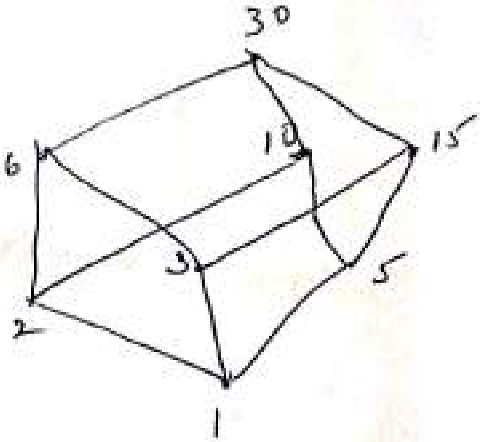


Maxim = 20 , greate elm = 20

Minim = 1 least elm = 1

(ii)-Sol: $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

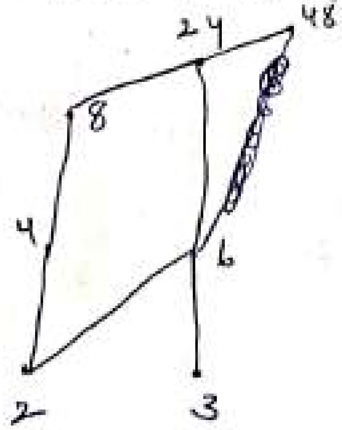




max = 30 great = 30.

minim = 1 least = 1

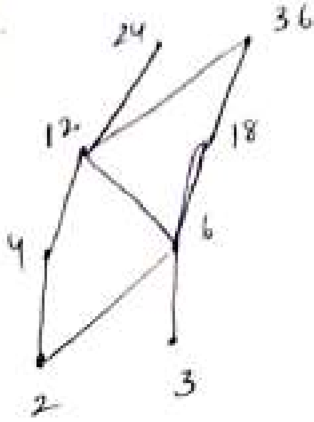
(iii) - sol $A = \{2, 3, 4, 6, 8, 24, 48\}$



Maxi = 48 , grea = 48

minim = 2, 3 , least = does not exi

$A = \{2, 3, 4, 6, 12, 18, 24, 36\}$



minimal = 2, 3 [exp. — does not ex

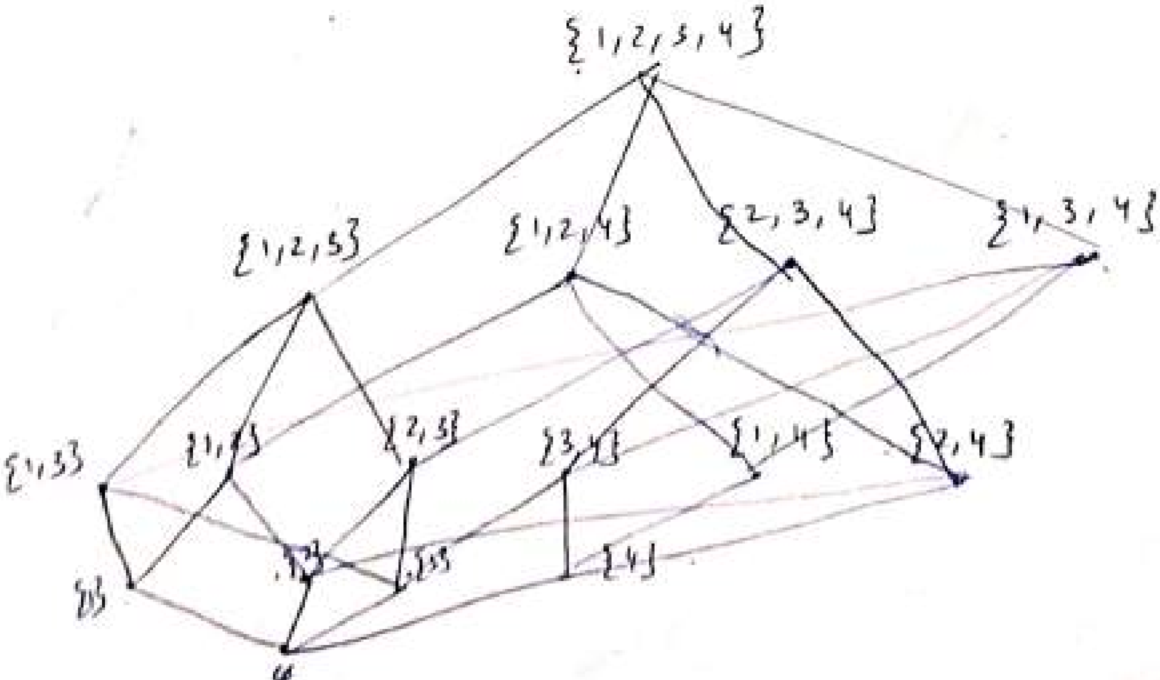
max = 24, 36 great " "

*) consider $(P(S), \subseteq)$ when $S = \{1,2,3,4\}$ for each of

find infimum & supremum of B

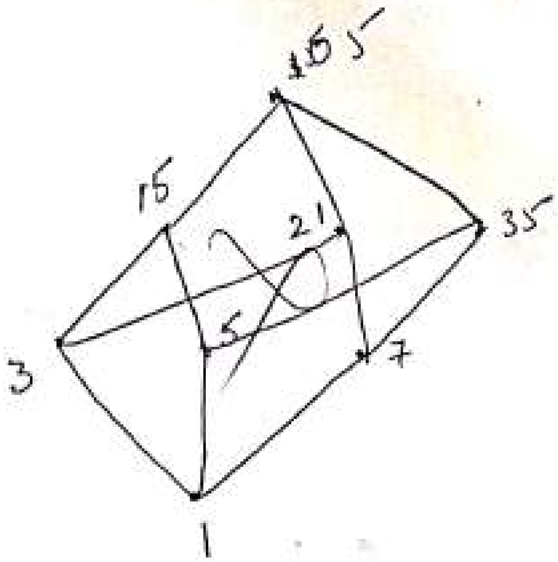
(i) $B = \{\{1\}, \{2\}, \{3\}, \{1,2\}\}$

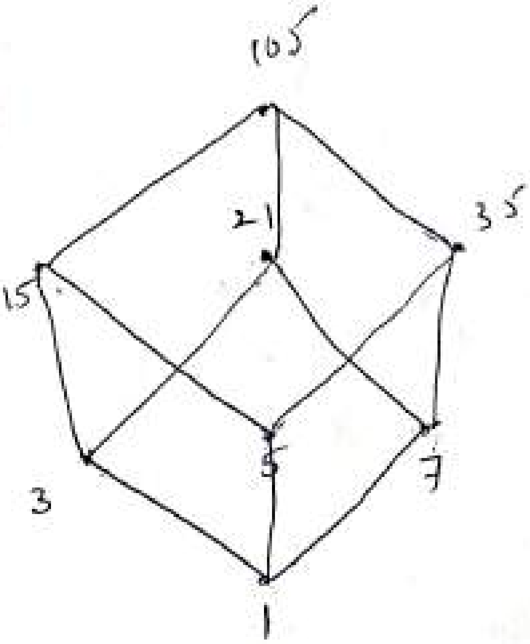
So) $P(S) = \{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{1,3\}, \{1,4\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{1,3,4\}, \{1,2,3,4\}\}$



3. Show that $(D_{105}, |)$ is a lattice.

Sol $D_{105} = \{1, 3, 5, 7, 15, 21, 35, 105\}$





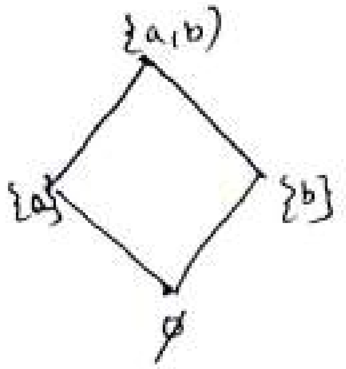
l.u.b	1	3	5	7	15	21	35	105
1	1	3	5	7	15	21	35	105
3	3	3	15	21	15	21	105	105
5	5	15	5	35	15	105	35	105
7	7	21	35	7	105	21	35	105
15	15	15	15	105	15	105	105	105
21	21	21	105	21	105	21	105	105
35	35	105	35	35	105	105	35	105
105	105	105	105	105	105	105	105	105

g.l.b	1	3	5	7	15	21	35	105
1	1	1	1	1	1	1	1	1
3	1	1	1	1	3	3	1	3
5	1	1	5	1	5	1	1	5
7	1	1	1	7	1	7	7	7
15	1	3	5	1	15	3	5	15
21	1	3	1	7	3	21	7	21
35	1	1	5	7	5	7	35	35
105	1	3	5	7	15	21	35	105

$\therefore$ It is a lattice.

① S.T $(P(A), \subseteq)$ is a lattice when $A = \{a, b\}$

Sol $P(A) = \{ \phi, \{a\}, \{b\}, \{a, b\} \}$



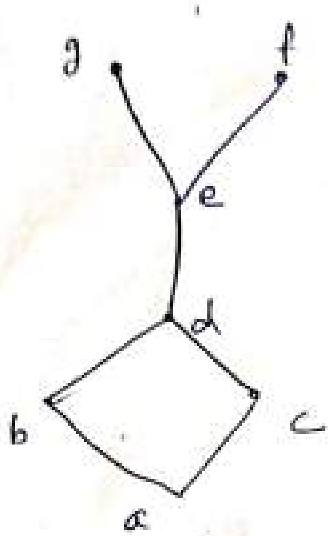
g.l.b	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$
ϕ	ϕ	ϕ	ϕ	ϕ
$\{a\}$	ϕ	$\{a\}$	ϕ	$\{a\}$
$\{b\}$	ϕ	ϕ	$\{b\}$	$\{b\}$
$\{a,b\}$	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$

l.u.b	ϕ	$\{a\}$	$\{b\}$	$\{a,b\}$
ϕ	$\{\phi\}$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$

$\therefore$ It is a lattice.

① which of the following poset represent lattice

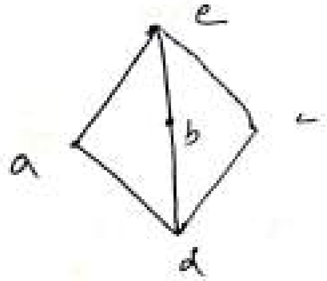
(i)



there is no least upper bound of g, f.

∴ It is not lattice.

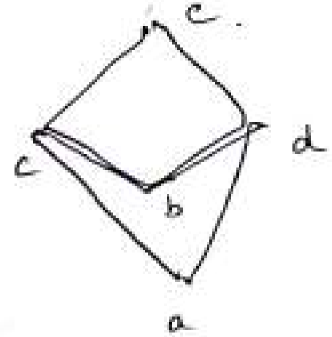
(ii)



g.l.b	a	b	c	d	e
a	a	d	d	d	a
b	d	o	d	d	d
c	d	d	c	d	c
d	d	d	d	d	d
e	d	d	c	d	e

∴ It is a lattice.

(iii)

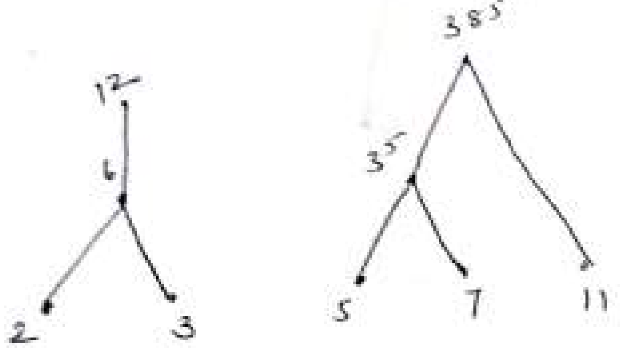


the g.l.b c, d does not exists.

It is not lattice.

① $A = \{2, 3, 5, 6, 7, 11, 12, 35, 385\}$

represented by the following Hasse diagram

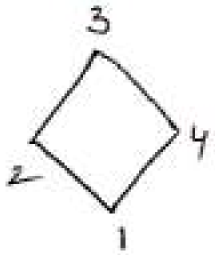


Here there is no ~~6~~ joining 12 and ~~385~~ .

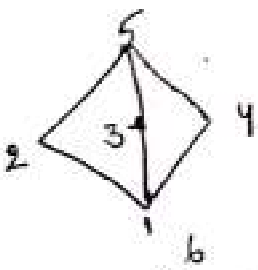
i.e They are not comparable.

∴ It is not lattice.

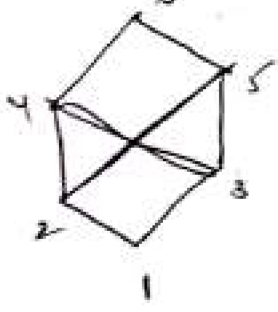
Which of the following diagrams represent lattice

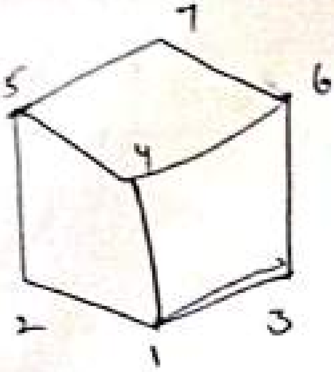
1. 

— It is a lattice

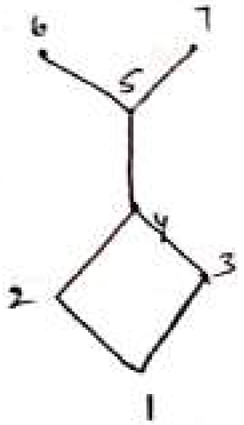
2. 

— It is a lattice

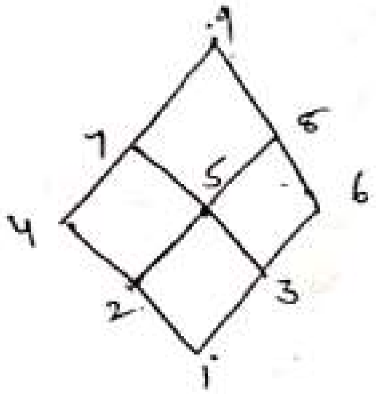
3. 

(d) 

Lattice

(e) 

for (2,3) ~~the~~ LUB does not exist.

(f) 

Lattice

Transitive closure

let X be any finite set and R be the relation on X. The relation $R^+ = R \cup R^2 \cup R^3 \cup \cdots$ in X is called the transitive closure of R in X.

Eg:-

$R_1 = \{(a,b), (a,c), (c,b)\}$.

$R_2 = \{(a,b), (b,c), (c,a)\}$

$R_3 = \{(a,b), (b,c), (c,c)\}$

$R_4 = \{(a,b), (b,a), (c,c)\}$

$R_1^2 = R_1 o R_1 = \{(a,b)\}$.

$R_1^3 = R_1 o R_1 o R_1 = \phi$.

$R_1^4 = \phi$.

$R_2^2 = R_2 o R_2 = \{(a,c)\ (b,a)\ (c,b)\}$

$R_2^3 = \{(a,a)\ (b,b)\ (c,c)\}$

$R_2^4 = \{(a,b)\ (b,c)\ (c,a)\}$

$R_3^2 = \{(a,c)\ (b,c)\ (c,c)\}$

$R_3^3 = \{(a,c)\ (b,c)\ (c,c)\}$

$R_1^+ = R_1 \cup R_1^2 \cup R_1^3 \cdots = R_1$

$R_2^+ = R_2 \cup R_2^2 \cup R_2^3 \cdots =$

$= \{(a,b)\ (b,c), (c,a)\ (a,c)\ (b,a)\ (c$
$(a,a)\ (b,b)\ (c,c)\}$

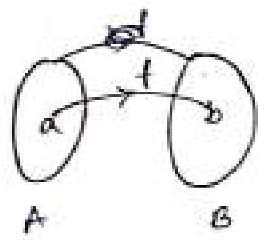
$R_3^+ =$

Functions

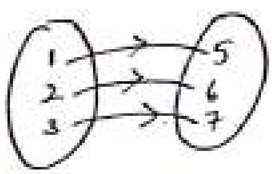
Let A, B be two non empty sets.
a relation f from A to B is called a function, if for every $a \in A$ there is a unique $b \in B$ such that $(a,b) \in f$.
Then we write $f(a) = b$.

this can be written as
$f : A \to B$.

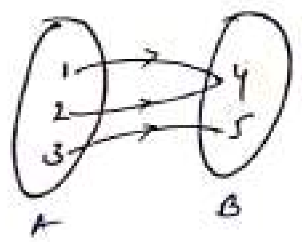
→ Here b is called the image and 'a' is called the preimage.



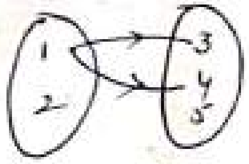
Eg :-



Is a function.
$f(1) = 5,\ f(2) = 6,\ f(3) = 7$.

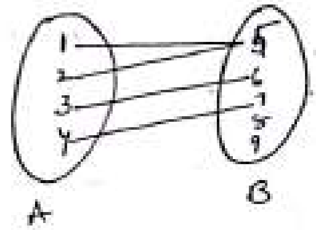


$f(1) = 4,\ f(2) = 4,\ f(3) = 5$
Is a function.



f is mapping from 1 to 3 & 1 to 4.

this is not a function.



domain $A = \{1,2,3,4\}$.
co-domain $B = \{5,6,7,8,9\}$.
$f(A) = \{5,6,7\}$.

Range:- If $f : A \to B$ is a function.
The elements of co-domain B which has pre image in 'A' is called Range & it's denoted by
$f(A)$ i.e $f(A) = \{f(x) \mid x \in A\}$ = Range

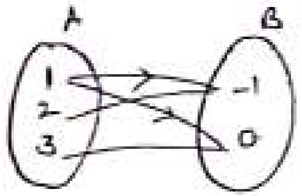
Note:- functions are called mappings or transformations or correspondence.

Def:- If domain & codomain of a function 'f' are both the same set. then function is called operator (or) transformation on the set.
i.e $f : A \to A$ is called operator & transformation on A.

1. let $A = \{1, 2, 3\}$ & $B = \{-1, 0\}$ & R be a relation from A to B defined by
$R = \{(1,-1), (1,0) (2,-1) (3,0)\}$
is R a function from A to B.

Sol
$A = \{1, 2, 3\}$
$B = \{-1, 0\}$
$R = \{(1,-1) (1,0) (2,-1) (3,0)\}$.

The pictorial representation of R is



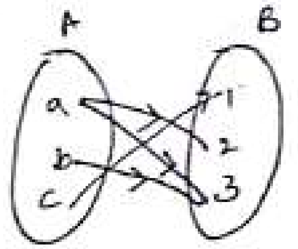
from the above, $1 \in A$ is mapping to two different element.
i.e 1 is not uniquly mapping.
i.e $f(1) = -1$, & $f(1) = 0$

$\therefore$ R is not a function from A to B?

2. State whether or not each of the relations given below defines a function of $A = \{a, b, c\}$ into $B = \{1, 2, 3\}$

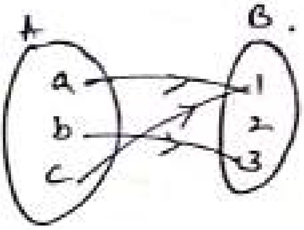
(i) $f = \{(a,2) (a,3) (b,3) (c,1)\}$
(ii) $f = \{(a,1) (b,3) (c,1)\}$
(iii) $f = \{(a,2) (b,3)\}$
(iv) $f = \{(a,1) (b,2) (c,3)\}$
(v) $f = \{(a,1) (b,1) (c,1)\}$
(vi) $f = \{(a,1) (a,2) (b,3) (c,3)\}$
(vii) $f = \{(a,1) (a,2) (b,3) (c,2)\}$

Sol
(i)



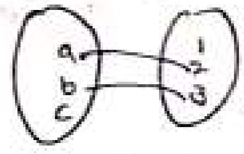
f is not function since 'a' has not unique image in B.

(ii)



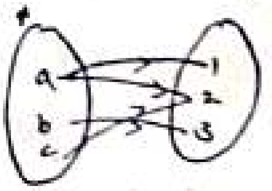
f is a function, since every element has unique image in B.

(iii)



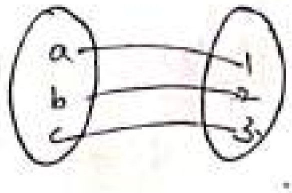
f is not a function since 'c' has no image in B.

(VP)



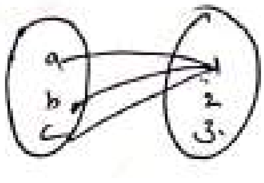
f is not function.

(iv)



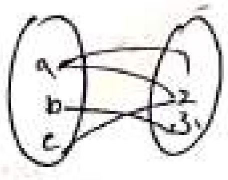
f is function

(v)



∴ f is a function

(vi)



f is not function

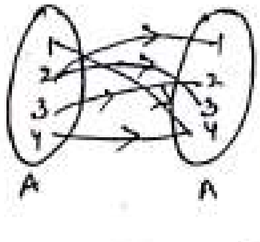
3. Let $A = \{1, 2, 3, 4\}$ determine whether (or) not the following relations on A are functions.

(i) $f = \{(2,3)\ (1,4)\ (2,1)\ (3,2)\ (4,4)\}$

(ii) $g = \{(3,1)\ (4,2)\ (2,1)\}$

(iii) $h = \{(2,1)\ (3,4)\ (1,4)\ (4,4)\}$

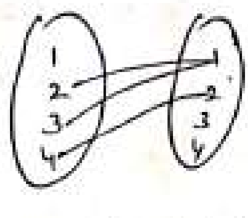
(i)



f is not function since 2 in A has not unique map to B

i.e $f(2) = 1$ & $f(2) = 3$ not unique map

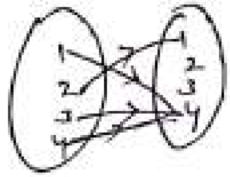
(ii)



g is not function.

since 1 has no image in B.

(iii).



∴ h is a function.

$f(1) = 4 \quad f(2) = 1 \quad f(3) = 4$

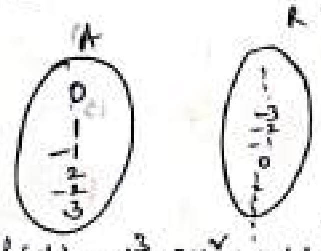
$f(4) = 4$

4. Let $A = \{0, \pm 1, \pm 2, 3\}$ consider the function $f : A \to R$ defined by

$f(x) = x^3 - 2x^2 + 3x + 1$.

for $x \in A$ find the range of A

Sol) Range $f(A) = \{f(x) \mid x \in A\}$.



$f(x) = x^3 - 2x^2 + 3x + 1$

$0 \in A, \ f(0) = 1 \in R$

$1 \in A, \ f(1) = 1 - 2 + 3 + 1 = 3 \in R$

$-1 \in A, \ f(-1) = -1 - 2 - 3 + 1 = -5 \in R$

$2 \in A, \ f(2) = \cancel{8} - 8 + 6 + 1 = 7 \in R$

$-2 \in A, \ f(-2) = -8 - 8 - 3(2) + 1$

$= -\cancel{16} - 21 \in R$

$3 \in A, \ f(3) = 19 \in R$

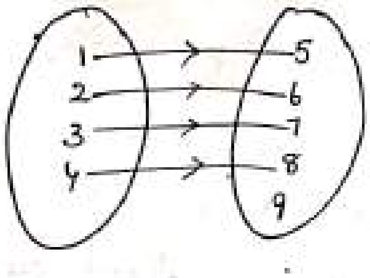
one-one function (or) Injective

A mapping $f : A \to B$ is called one-one function, if different elements of A is mapped to different elements of B.

(or)

Different elements of A has different images in B.

Eg:-



Mathematical way

$f : A \to B$

if $a_1, a_2 \in A \ni a_1 \neq a_2$

$\Rightarrow f(a_1) \neq f(a_2)$.

(or)

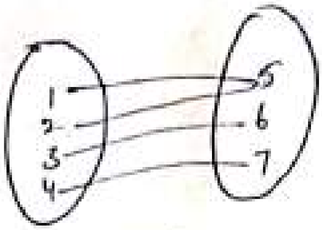
$a_1, a_2 \in A \ni f(a_1) = f(a_2)$

$\Rightarrow a_1 = a_2$

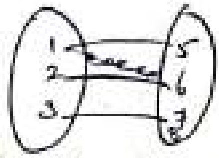
onto function (or) surjective

$f : A \to B$ is said to be onto, function ~~such that~~ if every element of B has pre Image in A.

i.e for every element $b \in B$ there is an element $a \in A$ $\ni f(a) = b$.



on-to.



It is not onto.
only one-one.

Bijection (or) one-one onto

If the function f is both one-one and onto then 'f' is called one-one onto (or) bijection function.

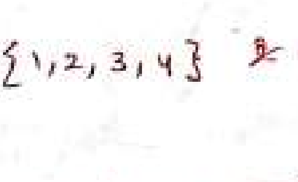
problems on one-one & onto.

1. If $A = \{1, 2, 3, 4\}$ & $B = \{w, x, y, z\}$ and $f = \{(1, w)\ (2, x)\ (3, y)\ (4, z)\}$ then verify 'f' is one-one onto

sol) $A = \{1, 2, 3, 4\}$

$B = \{w, x, y, z\}$

$f = \{(1, w)\ (2, x)\ (3, y)\ (4, z)\}$



one - one

clearly different elements of A are mapped to different elements of B

(or) the elements of A have different images in B.

$\therefore$ It is one-one.

on-to :-

clearly each and every element of B has preimage in A.

$\therefore$ f is on-to

$\therefore$ f is one one on-to.

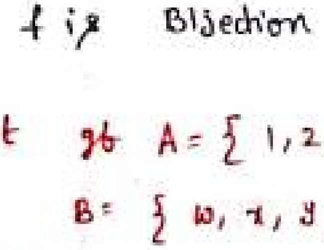
$\therefore$ f is Bijection function.

2. P.T of $A = \{1, 2, 3, 4, 5\}$. $B = \{w, x, y, z\}$.

$f = \{(1,w)\ (2,w)\ (3,x)\ (4,y)\ (5,z)\}$ then f is onto but not one-one.

Sol $A = \{1, 2, 3, 4, 5\}$

$B = \{w, x, y, z\}$



one - one :- 1 & 2 in A are mapping to same element.

$\therefore$ different elements of A are not mapping to different elements of B

$\therefore$ It is not one-one.

on to :- every element in B has preimage.

$\therefore$ It is on-to.

3. $f: R \to R$ is defined by $f(x) = 2x+3 \ \forall \ x \in R$ verify f is Bijection or not.

Sol Given that $f(x) = 2x+3$

f is one-one

let $a, b \in R \ni$

$f(a) = f(b)$

$\Rightarrow 2a+3 = 2b+3$

$\Rightarrow 2a = 2b$

$\Rightarrow a = b$.

$\therefore$ f is one-one (or) f is 1-1

f is onto

for $y \in R$ (codomain) $\exists x \in R$.

$\Rightarrow f(x) = y$.

$\Rightarrow 2x + 3 = y$

$2x = y-3$

$x = \frac{y-3}{2}$

$x = \frac{y-3}{2} \in R$.

$\therefore y \in R \ \exists \ \frac{y-3}{2} \in R \ni$

$f\left(\frac{y-3}{2}\right) = 2\left(\frac{y-3}{2}\right) + 3 = y$.

$\therefore$ f is onto. $\therefore$ f is Bijection.

4. $f: R \to R$ is defined by
$f(x) = 2x+7 \quad \forall\, x \in R$

sol: f is one-one

for $a, b \in R \ni$

$f(a) = f(b)$

$2a+7 = 2b+7$

$2a = 2b$

$a = b$

$\therefore$ f is one-one.

f is onto

let $y \in R \; \exists x \in R \ni f(x) = y$.

$\Rightarrow 2x+7 = y$

$2x = y-7$

$x = \frac{y-7}{2}$.

for $y \in R \; \exists \frac{y-7}{2} \in R \ni f\left(\frac{y-7}{2}\right) = 2\left(\frac{y-7}{2}\right)+7$

$= y$

$\therefore$ ~~...~~

$\therefore$ f is on-to.

$\therefore$ f is Bijection.

5. $f: R \to R$ is defined by
$f(x) = e^x \quad \forall\, x \in R$.

sol:- f is one-one

let $a, b \in R$

$f(a) = f(b)$

$e^a = e^b$

$\Rightarrow a = b$

f is one-one

f is on-to

let $y \in R \; \exists\, x \in R$.

$\Rightarrow f(x) = y$.

$\Rightarrow e^x = y$.

$\Rightarrow x = \log y \notin R$

~~for $y \in R \; \exists \log y \in R \ni f(\log y) = e^{\log y} = y$~~

~~$f(x) = y$~~.

f is not on-to, f^{-1} does not exist.

6. $f: R \to R$ is defined by.
$f(x) = x^3 \quad \forall\, x \in R$.

f is one-one

let $a, b \in R \ni$

$f(a) = f(b)$

$a^3 = b^3$

$\Rightarrow a = b$

f is one-one.

f is onto

let $y \in R \; \exists\, x \in R \ni f(x) = y$.

$\Rightarrow x^3 = y$

$\Rightarrow x = y^{1/3}$.

$f(x) = f(y^{1/3}) = (y^{1/3})^3 = y$.

for $y \in R \; \exists\, y^{1/3} \in R$ (domain) $\ni f(y^{1/3}) = (y^{1/3})^3 = y$

~~$f(x) = y$~~.

f is on-to.

Inverse function

If 'f' is a bijection (i.e f is one-one, onto) from A to B. then the relation f^{-1} mapping from B to A is called inverse mapping of the function 'f' from A to B.

i.e $a \in A \; \exists \; b \in B \ni f(a) = b$

$$\Rightarrow f^{-1}(b) = a$$

2. If $f: Q \to Q$ be defined by $f(x) = 5x+4 \;\; \forall x \in Q$, where Q is the set of rational numbers. Then p.t f is one-one & onto and find f^{-1}

Sol $f: Q \to Q$ defined by

$$f(x) = 5x+4.$$

one-one :- $x_1, x_2 \in Q$.

$$\Rightarrow f(x_1) = f(x_2)$$

$$5x_1 + 4 = 5x_2 + 4$$

$$\Rightarrow 5x_1 = 5x_2$$

$\Rightarrow x_1 = x_2$

f is one-one.

f is on-to

let $y \in \mathbb{Q} \; \exists \; x \in \mathbb{Q}$ (dom) $\ni f(x) = y$

i.e. $5x + 4 = y$

$\Rightarrow x = \frac{y-4}{5} \in \mathbb{Q}$ (domain)

$f(x) = f\left(\frac{y-4}{5}\right) = 5\left(\frac{y-4}{5}\right) + 4$

$= y$

$f(x) = y$

$\therefore$ for any $y \in \mathbb{Q}$ (codomain)

$\exists \; \frac{y-4}{5} \in \mathbb{Q}$ (domain)

$\Rightarrow f\left(\frac{y-4}{5}\right) = y$.

$\therefore f$ is on to.

$\therefore f$ is one-one & on-to

f is Bisection.

then f^{-1} exist.

To find f^{-1}

since $f(x) = y$

$\Rightarrow x = f^{-1}(y)$

$\frac{y-4}{5} = f^{-1}(y)$

$\Rightarrow f^{-1}(y) = \frac{y-4}{5}$

$f^{-1}(x) = \frac{x-4}{5} \quad \forall \; x \in \mathbb{Q}$.

3. $f: R \to R \ni f(x) = \frac{2x+3}{5}$ find $f^{-1}(x)$

Sol f is one-one

let $x_1, x_2 \in R : \ni f(x_1) = f(x_2)$

$\Rightarrow \frac{2x_1+3}{5} = \frac{2x_2+3}{5}$

$\Rightarrow 2x_1+3 = 2x_2+3$

$\Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$.

$\therefore f$ is one-one.

f is on-to

let $y \in R$ (co domain) $\exists x \in R$

$\Rightarrow f(x) = y$

$\Rightarrow \frac{2x+3}{5} = y$

$\Rightarrow 2x+3 = 5y$

$\Rightarrow 2x = 5y-3$

$\Rightarrow x = \frac{5y-3}{2}$

$\therefore$ for $y \in R \; \exists \; \frac{5y-3}{2} \in R$ (domain)

$\Rightarrow f\left(\frac{5y-3}{2}\right) = \frac{\cancel{2}\left(\frac{5y-3}{\cancel{2}}\right)+3}{5} = y$

$\therefore f$ is on-to.

$\therefore f$ is Bijection, f^{-1} exist.

f^{-1} :- $\because f(x) = y$

$\Rightarrow x = f^{-1}(y)$

$\Rightarrow \frac{5y-3}{2} = f^{-1}(y)$ for $y \to x$

$\Rightarrow f^{-1}(x) = \frac{5x-3}{2}$

Composition of functions

$f: A \to B$ & $g: B \to C$ are the functions. The composition of these two functions is defined as the function

$$g\circ f : A \to C$$

with $g\circ f(a) = g[f(a)] \ \forall a \in A$.

1. let $A = \{1,2,3,4\}$ & $B = \{a,b,c\}$ $C = \{w,x,y,z\}$

Eg:- let $A = \{1,2,3,4\}$
$B = \{a,b,c\}$.
$C = \{w,x,y,z\}$

with $f: A \to B$ & $g: B \to C$

given by
$f = \{(1,a)\ (2,a)\ (3,b)\ (4,c)\}$
$g = \{(a,x)\ (b,y)\ (c,z)\}$ find $g\circ f$.

Sol. domain of f

$g\circ f(1) = g[f(1)] = g[a] = x$.

$g\circ f(2) = g[f(2)] = g[a] = x$.

$g\circ f(3) = g[f(3)] = g[b] = y$

$g\circ f(4) = g[f(4)] = g[c] = z$.

$g\circ f = \{(1,x)\ (2,x)\ (3,y)\ (4,z)\}$.

2. consider the function f & g defined by $f(x) = x^3$ & $g(x) = x^2+1$ $\forall x \in R$ find $g\circ f$, $f\circ g$, f^2 & g^2

Sol G.t $f(x) = x^3$
$g(x) = x^2+1$

$$[g\circ f](x) = g[f(x)] = g(x^3)$$
$$= g(x^3) = (x^3)^2+1$$
$$= x^6+1.$$

$$[f\circ g](x) = f[g(x)] = f[x^2+1]$$
$$= (x^2+1)^3$$
$$= (x^2)^3+(1)^3+3x^2\cdot 1^2+3x^2\cdot 1$$
$$= x^6+1+3x^2+3x^2$$
$$= x^6+6x^2+1$$

$$f^2 = f\circ f = f\circ f(x) = f[f(x)]$$
$$= f(x^3) = (x^3)^3 = x^9$$

$$g^2 = g\circ g = g\circ g(x) = g[g(x)]$$
$$= g[x^2+1] = (x^2+1)^2+1$$
$$= x^4+1+2x^2+1$$
$$= x^4+2x^2+2$$

3. let f & g be functions from R to R defined by
$f(x) = ax+b$ & $g(x) = 1-x+x^2$
If $[g\circ f]x = 9x^2-9x+3$
determine a, b.

Given that

$f(x) = ax+b$

$g(x) = 1-x+x^2$

$[g\circ f](x) = 9x^2-9x+3$

$g[f(x)] = 9x^2-9x+3$

$\Rightarrow g[ax+b] = 9x^2-9x+3$

$\Rightarrow 1-(ax+b)+(ax+b)^2 = 9x^2-9x+3$

$\Rightarrow 1-ax-b+a^2x^2+b^2+2abx = 9x^2-9x+3$

$\Rightarrow a^2x^2+x(2ab-a)+(1-b+b^2) = 9x^2-9x+3$

Comparing on b|s.

$a^2=9$ & $2ab-a=-9$

$1-b+b^2=3$

$\Rightarrow \boxed{b=-2}$

$2a(-2)-a=-9$

$-4a-a=-9$

$-5a=-9$

$\boxed{a=\frac{9}{5}}$

$a^2=9 \Rightarrow a=\pm 3$.

$2ab-a=-9$

$1-b+b^2=3$

If $a=3$

$2(3)b-3=-9$

$6b=-6$

$b=-1$

$b=-1$

$\therefore \boxed{a=3 \ \& \ b=-1}$

If $a=-3$

$2(-3)b+3=-9$.

$-6b+3=-9$

$-6b=-12$

$\boxed{b=2}$

$\boxed{\therefore a=-3 \ \& \ b=2}$

are the values.

4. Let $A=B=C=R$ the set of all real numbers &

$f: A\to B$ & $g: B\to C$ be defined by $f(a)=a-1$, $a\in A$

$g(b)=b^2$, $b\in B$

find

(i) $g\circ f(a)$ (ii) $f\circ g(b)$

(iii) $f\circ f(a)$ (iv) $g\circ g(b)$

Sol) G.T $f(a)=a-1$

$g(b)=b^2$

(i) $g\circ f(a) = g[f(a)] = g(a-1) = (a-1)^2$
$= a^2+1-2a$.

(ii) $f\circ g(b) = f[g(b)] = f[b^2] = b^2-1$

(iii) $f\circ f(a) = f[f(a)] = f[a-1]$
$= (a-1)-1 = a-2$

(iv) $g\circ g(b) = g[g(b)] = g[b^2]$
$= (b^2)^2 = b^4$.

5. Let f, g, h be the functions defined by

$f(x) = x+2$, $g(x) = x-2$

$h(x) = 3x$, $\forall x \in R$

find gof, fog, fof, gog, foh, hog, hof.

Sol $f(x) = x+2$

$g(x) = x-2$

$h(x) = 3x$

$gof(x) = g[f(x)] = g[x+2]$

$= (x+2) - 2 = x$.

$fog(x) = f[g(x)] = f[x-2] = x-2+2$

$= x$

$fof(x) = f[f(x)] = f[x+2] = x+2+2$

$= x+4$

$gog(x) = g[g(x)] = g(x-2) = x-2-2$

$= x-4$

$foh(x) = f[h(x)] = f[3x]$

$= 3x+2$

$hog(x) = h[g(x)] = h[x-2]$

$= 3(x-2) = 3x-6$

$hof(x) = h[f(x)] = h[x+2]$

$= 3(x+2) = 3x+6$.

6. The set of all integers and 'c' be the set of all even integers

Let $f: A \to B$ & $g: B \to C$ be defined by $f(a) = a+1$, $a \in A$.

$g(b) = 2b$, $b \in B$

find gof.

Sol $gof(a) = g[f(a)] = g[a+1]$

$= 2(a+1) = 2a+2$

7. Let f & g be functions from R to R defined by

$f(x) = x^2$ & $g(x) = x+5$

P.T $gof \neq fog$.

Sol $gof(x) = g[f(x)]$

$= g[x^2] = x^2+5$

$fog(x) = f[g(x)]$

$= f[x+5]$

$= (x+5)^2 = x^2+25+10x$

$\therefore gof \neq fog$

Algebraic Structures

Binary operations :-

Let A be any non empty set
a mapping $f: A \times A \to A$ is called
a binary operations on A.
i.e $\exists$ a unique $f(a,b) \in A$ $\forall a \in A, b \in A$

we denote a binary operation
by symbol such as
$+, -, *, \div, \cdot, \square, \sqsubset$ etc.

Algebraic Structure

A non empty set 'G' equipped
with one or more binary operations
is called algebraic structure.

Eg :- $(N, +, \cdot)$ is an algebraic
structure.

The properties of Binary operations

closure :- G is a non empty set
and '$*$' is a binary operation on G,
if $\forall a, b \in G \Rightarrow a * b \in G$.
Then G is closed

Associative G is a non empty set
and $*$ is a binary operation
on G, if $\forall a, b, c \in G$
$a * (b * c) = (a * b) * c$
$\therefore$ '$*$' is Associative.

Identity :-
let G be non empty set & $*$ is a
binary operation on G. &
$e \in G$ & $a * e = e * a = a$ $\forall a \in G$
Then 'e' is called an identity

Eg :- ① w.r.t addition
$a + 0 = 0 + a = a$ $\forall a \in G$.
'0' is an identity w.r.t addition

② w.r.t multiplication
$1 \times a = a \times 1 = a$ $\forall a \in G$
'1' is an identity w.r.t '$\times$'

Inverse :-
let 'G' be a non empty set &
'$*$' is a binary operation on G
and 'e' is an identity in G, then
$b \in G$ is said to be
inverse of 'a' if $a * b = b * a = e$.
$\forall a, b \in G$.

Commutative :-
let 'G' be a non empty set
and '$*$' is a binary operation on 'G'
if $a * b = b * a$ $\forall a, b \in G$
Then '$*$' is called commutative.

Distributive properties :-

for any $a, b, c \in G$

$$a * (b \circ c) = (a * b) \circ (a * c)$$

$$(b \circ c) * a = (b * a) \circ (c * a)$$

Cancellation property

for any $a, b, c \in G$

$$a * b = a * c \Rightarrow b = c \text{ (left canc)}$$

$$b * a = c * a \Rightarrow b = c \text{ (Right canc)}$$

Groupoid or Quasi Group

Let G be a non empty set and '$*$' is a binary operation on G, if 1. G is closed

then G is called Groupoid.

Eg:- $(N, +)$ is a Groupoid

since $1, 3 \in N$

$1 + 3 = 4 \in N$, N is closed.

$\therefore (N, +)$ is a Groupoid

~~But~~

$(N, -)$

$1, 2 \in N$

$1 - 2 = -1 \notin N$

$\therefore (N, -)$ is not a Groupoid.

Semi group

Let G be a non empty set, and '$*$' is a binary operation on G,

if ① G is closed.

② '$*$' is Associative on G.

then 'G' is called semi group.

Monoide

Let G be a non empty set & '$*$' is a binary operation on G.

if 1. G is closed.

2. $*$ is Associative

3. G has an Identity.

$\therefore G$ is called monoid.

Group

Let G be a non empty set. & '$*$' is a binary operation on G, then $(G, *)$ is said to be a group if

1. G is closed
2. '$*$' is associative on 'G'
3. G has an Identity
4. Every element in G is invertible.

Some standard groups

1. $(Z, +)$, $(Q, +)$ $(R, +)$ $(C, +)$ $\rightarrow$ abelian group $(e = 0)$ $(N, +)$ is not a group Identity does not exist
2. $(Q^*, \cdot)$ $(R^*, \cdot)$ $(C^* \cdot)$ $\longrightarrow$ abelian grou $(e = 1)$

Abelian group

Let G be a non empty set, '*' is a binary operation on G then

(G, *) is said to be abelian group if

1. G is closed
2. '*' is associative
3. G has an identity.
4. Every element in 'G' is invertible.
5. '*' is Commutative.

1. S.T $G = \{1, -1, i, -i\}$ is an abelian group w.r.t '·'.

Sol)

·	1	-1	i	-i
1	$\boxed{1}$	-1	i	-i
-1	-1	$\boxed{1}$	-i	i
i	i	-i	-1	$\boxed{1}$
-i	-i	i	$\boxed{1}$	-1

closure:- clearly G is closed.
for a, b ∈ G then a.b ∈ G.

Associative:-

closure:- since all the entries of in the composition table are the elements of G.

∴ G is closed w.r.t '·'

Associative

The elements of G are complex and the multiplication of complex is associative.

∴ Associative property satisfies in G w.r.t '·'.

Identity:-

from the composition table the row headed by '1' just coinsides with top row of the composition table.

that is $1(1) = 1$
$1(-1) = -1$
$1(i) = i$
$1(-i) = -i$

$(e * a = a * e = a$

∴ '1' is the identity.

Inverse:-

w.r.t identity element is its own inverse.

Inverse of 1 is 1
" -1 is -1
" i is -i
" -i is i.

Commutative.

from the composition table the entries in the first, second third & fourth rows of the table coincide with the corresponding entries in the first, second, third column.

therefore we have $a \cdot b = b \cdot a \; \forall \; a, b \in G$.
∴ '·' is commutative.

	-1	0	1

Tr

3. $G = \{1, \omega, \omega^2\}$. is an abelian w.r.t.

	1	ω	ω^2
1	$\boxed{1}$	ω	ω^2
ω	ω	ω^2	$\boxed{1}$
ω^2	ω^2	$\boxed{1}$	ω

closure, asso $\xrightarrow{\text{complex}}$, identr, inve.

4. s.t $G = \{0, 1, 2, 3, 4, 5\}$ is an abelian group w.r.t $\oplus_6$.

	0	1	2	3	4	5
0	$\boxed{0}$	1	2	3	4	5
1	1	2	3	4	5	$\boxed{0}$
2	2	3	4	5	$\boxed{0}$	1
3	3	4	5	$\boxed{0}$	1	1
4	4	5	$\boxed{0}$	1	2	3
5	5	$\boxed{0}$	1	2	3	4

closure

since all the entries in the composition table are in G.

$\therefore$ G is closed w.r.t $+_6$.

Associative: since elements of 'G' are integers, $+_6$ is associative in G

Identity:

The row headed by '0' is coinciding with top row of the table is

$0 \oplus_6 0 = 0$

$0 \oplus_6 1 = 1$

$0 \oplus_6 2 = 2$

$0 \oplus_6 3 = 3$

$0 \oplus_6 4 = 4$

$0 \oplus_6 5 = 5$ $\therefore$ '0' is the identity.

Inverse

from the table inverse of

0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1

commutative:-

The corresponding rows & columns are identical.

$\oplus_6$ is commutative.

$\therefore$ $(G, +_6)$ is an abelian group

3. In Q^+, an operation 'o' is defined by $aob = \frac{ab}{3}$ $\forall a,b \in G$ is an abelian group.

So) closure :-

for $a,b \in Q^+$, $ab \in Q^+$

$\Rightarrow \frac{ab}{3} \in Q^+$

$\Rightarrow aob \in Q^+$

$\therefore$ for $a,b \in Q^+ \Rightarrow aob \in Q^+$

$\therefore$ Q^+ is closed.

Associative

let $a, b, c \in Q^+$

$ao(boc) = ao\left(\frac{bc}{3}\right) = \frac{abc}{9}$

$(aob)oc = \frac{ab}{3}oc = \frac{abc}{9}$

$\therefore ao(boc) = (aob)oc$

$\therefore$ for $a,b,c \in Q^+$ then

$ao(boc) = (aob)oc$

$\therefore$ 'o' is an associative.

Identity for $a \in Q^+ \exists e \in Q^+ \ni$

$aoe = a$.

$\Rightarrow \frac{ae}{3} = a \Rightarrow \boxed{e=3}$ ~~xake~~.

$\Rightarrow ae = 3a$

$\Rightarrow ae - 3a = 0$

$\Rightarrow a(e-3) = 0$

$a \neq 0, e-3=0$

$\boxed{e=3}$

$\therefore$ $e = 3 \in Q^+$ is an identity in Q^+

Inverse

for $a \in Q^+$, $\exists b \in Q^+ \ni$

~~$aob = boa = e$~~

$aob = e$

$\Rightarrow \frac{ab}{3} = 3 \Rightarrow ab = 9$

$a = \frac{9}{b}$

(or) $b = \frac{9}{a}$

$\therefore b = \frac{9}{a} \in Q^+$ is the inverse of 'a'

Commutative

for $a,b \in Q^+$

$aob = \frac{ab}{3} = \frac{ba}{3} = boa$

$\therefore aob = boa$ $\forall a,b \in Q^+$.

$\therefore$ 'o' is commutative.

$\therefore$ (Q^+, o) is an abelian group

4. In Z, an operation 'o' is defined by $aob = a+b-2$ for $a,b \in Z$ is an abelian group.

So) closure :-

for $a,b \in Z$, $ab \in Z \Rightarrow a+b \in Z$.

$\Rightarrow a+b-2 \in Z$

$\Rightarrow aob \in Z$.

$\therefore aob \in Z$ $\forall a,b \in Z$.

$\therefore$ Z is closure.

for $a, b, c \in Z$

$$a\circ(b\circ c) = a\circ(b+c-2)$$
$$= a+(b+c-2)-2$$
$$= a+b+c-4 \in Z$$

$$(a\circ b)\circ c = (a+b-2)\circ c$$
$$= a+b-2+c-2$$
$$= a+b+c-4 \in Z.$$

$\therefore a\circ(b\circ c) = (a\circ b)\circ c$

$\therefore$ 'o' is an Associative.

Identity:-

for $a \in Z \ \exists\ e \in Z \ni$

$a\circ e = a.$

$a+e-2 = a$

$\cancel{a}+e-2-\cancel{a} = 0$

$\boxed{e=2} \in Z.$

Inverse:-

for ~~$a\circ b \in Z$~~ $a \in Z \ \exists\ b \in Z$

$\Rightarrow a\circ b = e$

$\Rightarrow a+b-2 = e$

$\Rightarrow a+b-2 = 2$

$\Rightarrow a+b = 4$

$\Rightarrow b = 4-a \in Z.$

$\therefore$ $4-a$ is the inverse of 'a'

Commutative:- for $a, b \in Z$

$a\circ b = a+b-2$
$= b+a-2$
$= b\circ a$

$\therefore$ 'o' is commutative.

$\therefore (Z, \circ)$ is an abelian group.

5. In Z, an operation 'o' is defined by $a\circ b = a+b-ab$ for $a, b \in Z$ is an abelian group.

Sol closure.

let $a, b \in Z \Rightarrow ab \in Z$
$\Rightarrow a+b \in Z$
$\Rightarrow a+b-ab \in Z$
$\Rightarrow a\circ b \in Z.$

$\therefore$ for $a, b \in Z \Rightarrow a\circ b \in Z.$

$\therefore$ 'o' is closed.

Associative

for $a, b, c \in Z.$

$\Rightarrow$ ~~$a\circ b = a+b-ab.$~~

$a\circ(b\circ c) = a\circ(b+c-bc)$
$= a+b+c-bc-a(b+c-bc)$
$= a+b+c-bc-ab-ac+abc$
$= a+b+c-ab-ac-bc+abc$

$(a\circ b)\circ c = (a+b-ab)\circ c$

$= a+b-ab+c-(a+b-ab)c$

$= a+b-ab+c-ac-bc+abc$

$= a+b+c-ab-ac-bc+abc$

$\therefore a\circ(b\circ c) = (a\circ b)\circ c$

'o' is Associative.

Identity

for $a \in Z\ \exists\ e \in Z \ni$

$a \circ e = a$

$\Rightarrow a + e - ae = a$

$\Rightarrow \cancel{a} + e - ae - \cancel{a} = 0$

$\Rightarrow e(1-a) = 0.$

$e = 0 \in Z$

$\therefore$

Inverse

for $a \in Z\ \exists\ b \in Z$

$\ni a \circ b = e$

$\Rightarrow a + b - ab = e = 0\ (\because e = 0$

$\Rightarrow a + b = ab \Rightarrow b \neq r$

$\Rightarrow \cancel{a = b \Rightarrow a \neq b}$ $a + b(1-a) = 0$ $b = \frac{-a}{1-a}$

$\Rightarrow$ Inverse does not exist.

$\therefore$ Z, satisfies.

closure, associative,

Identity

$\therefore$ Z is a monoid under '$\circ$'.

6. In R, an operation '$*$' is defined by $a * b = a + b - ab$ for $a, b \in R$ is an abelian group.

closure

Let $a, b \in R \Rightarrow a * b = a + b - ab \in R$.

for $a, b \in R \Rightarrow a * b \in R$

$\therefore$ 'R' is closed.

Associative

$a * (b * c) = a * (b + c - bc)$

$= a + b + c - bc - a(b + c - bc)$

$= a + b + c - bc - ab - ac + abc$

$= a + b + c - ab - bc - ac + abc$

$(a * b) * c = (a + b - ab) * c$

$= a + b - ab + c - c(a + b - ab)$

$= a + b + c - ab - bc + abc$

$\therefore a * (b * c) = (a * b) * c$

$\therefore$

Identity for $a \in R\ \exists\ e \in R \ni$

$a * e = a \Rightarrow a + e - ae = a.$

$a + e(1-a) = a.$

$e(1-a) = 0$

$\boxed{e = 0} \in R.$

Inverse

for $a \in Z \; \exists \; b \in Z \ni$

$$a * b = e$$

$$\Rightarrow a + b - ab = e$$

$$a + b(1-a) = e$$

$$a + b(1-a) = 0$$

$$b(1-a) = -a$$

$$b = \frac{-a}{1-a}$$

$\frac{-a}{1-a}$ is the inverse of b

$\frac{a}{a-1}$ is the inverse of b.

Commutative

$$a * b = a + b - ab$$

$$= b + a - ba$$

$$= b + a - ba$$

$\therefore *$ is commutative

$\therefore (G, *)$ is an abelian group

Sub groups

Let $< G, * >$ be a group and H be a non empty subset of G. then $< H, * >$ is called a subgroup of G if $< H, * >$ itself a group.

Eg:- under addition, the set of all even integers forms a subgroup of the group of all integers.

Eg:- (i) $G = \{1, -1, i, -i\}$

$H = \{1, -1\}$ is a Group

$\therefore$ H is a subgroup.

(ii) $(Z, +)$ is a subgroup of group $(Q, +)$

(iii) $(N, +)$ is not a subgroup of group $(Z, +)$

since '0' identity does not exist in N.

→ Problem :-

→ Let $G = \{2, 4, 6, 8\}$ and binary operation multiplic. modula 10.

i) Show that G is a group.

ii) let $H = \{6, 4\}$. Show that H is a subgroup of G.

Given :- $G = \{2, 4, 6, 8\}$.

10) 32 (

$\odot_{10}$	2	4	6	8
2	2	4	2	6
4	2	6	4	~~8~~ 2
6	2	4	6	8
8	6	2	8	4

10) 12 (1
10

10) 16 (

10) 24 (2
20

10) 48 (4

10) 32 (3

10) 48 (

i) All the elements in the group $\in G$.
Closure property satisfies

ii) Associ. property satisfies

iii) 6 is ~~subset~~ identity elem. ~~of~~ in $\odot_{10}$ & $6 \in G$

iv) Inverse exists.

$\therefore (G, \odot_{10})$ is a group.

$H = \{6, 4\}$.

$\odot_{10}$	4	6
4	6	4
6	4	6

i) ~~all~~ closure prop. satisfies

ii) Ass. prop. satisfies

iii) 6 is identity elem. in $\odot_{10}$ & $6 \in H$.

iv) $4 \odot_{10} \underline{4} = 6$

$6 \odot_{10} \underline{6} = 6$

$\therefore$ Inverse exists

$\therefore$ H is a group.

$\therefore$ H is subset of G.

$\therefore$ H is a subgroup of G.

→ Theorem :-

Show that intersection of two subgroups is also a subgroup.

Proof :-

let H_1 & H_2 be any 2 subgroups

To show that $H_1 \cap H_2$ is a subgroup, it is enough to show that $ab^{-1} \in H_1 \cap H_2$ $\forall a, b \in H_1 \cap H_2$

let $a, b \in H_1 \cap H_2$

$\Rightarrow a, b \in H_1$ & $a, b \in H_2$

$\Rightarrow b^{-1} \in H_1$ & $b^{-1} \in H_2$ (since H_1, H_2 are subgroups)

$\Rightarrow a, b^{-1} \in H_1 \Rightarrow ab^{-1} \in H_1$ (")

Also $a, b^{-1} \in H_1 \Rightarrow ab^{-1} \in H_2$ (")

$\cancel{a, b^{-1} \in H_2 \Rightarrow ab^{-1}}$.

$ab^{-1} \in H_1 \cap H_2$ $\forall a, b \in H_1 \cap H_2$.

$\therefore H_1 \cap H_2$ is a subgroup.