Branch and Bound: General Method, FIFO Branch and Bound, LC Brach and Bound, Applications:0/1 knapsack Problem, Travelling Salesperson Problem.

Introduction to Branch and Bound Technique

- The Branch and Bound Technique is a problem solving strategy, which is most commonly used in optimization problems, where the goal is to minimize a certain value.
- The optimized solution is obtained by means of a state space tree (A state space tree is a tree where the solution is constructed by adding elements one by one, starting from the root. Note that the root contains no element).
- This method is best when used for combinatorial problems with exponential time complexity, since it provides a more efficient solution to such problems.

The Algorithm: Branch and Bound Technique

- In this technique, the first step is to create a function *U* (which represents an upper bound to the value that node and its children shall achieve), that we intend to minimize.
- We call this function the objective function.
- Note that the branch and bound technique can also be used for maximization problems, since multiplying the objective function by -1 coverts the problem to a minimization problem.
- Let this function have an initial maximum value, according to the conditions of the given problem. Also, let U_0 be the initial value of *U*.
- We also calculate a cost function *C* which will give us the exact value that the particular node shall achieve.
- The next question is the order in which the tree is to be searched. For this, there exist multiple types of branch and bound, which we shall discuss in detail later.
- For now, let us assume that there is a set *S* consisting of subsets of the given set of values in the order in which they need to be searched.

The algorithm of the branch and bound method for this problem will be as follows:

For each subset *s* in S, do the following:

1. Calculate the value of $U(s)$.
2. If $U(s)$ U_0, then $U_0 = U(s)$.

In the end, the the subset *s* for which the current value of U_0 is obtained will be the best solution to the given problem, and the value of the cost function at that node will give us the solution. Here, after each level, the value of U_0 tells us that there shall be a node with cost less than that value.

Types of Solutions:

For a branch and bound problem, there are two ways in which the solution can be represented:

- **Variable size solution**: This solution provides the subset of the given set that gives the optimized solution for the given problem. For example, if the we are to select a combination of elements from {A, B, C, D, E} that optimizes the given problem, and it is found that A, B, and E together give the best solution, then the solution will be {A, B, E}.
- **Fixed size solution**: This solution is a sequence of 0s and 1s, with the digit at the i^{th} position denoting whether the i^{th} element should be included or not. Hence, for the earlier example, the solution will be given by {1, 1, 0, 0, 1}.

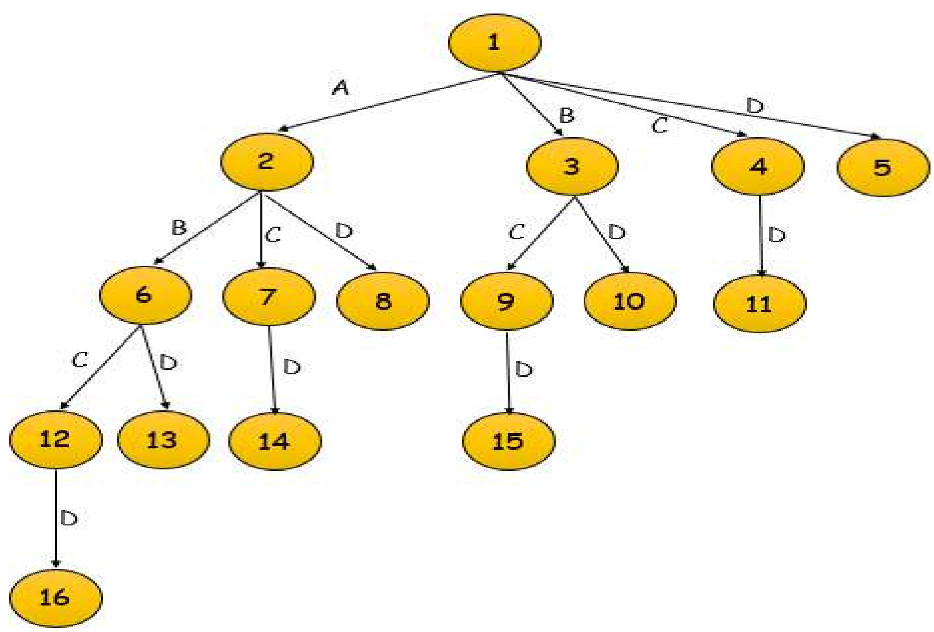
Types of Branch and Bound:

There are multiple types of the Branch and Bound method, based on the order in which the state space tree is to be searched. We will be using the variable solution method to denote the solutions in these methods.

1. FIFO Branch and Bound

The First-In-First-Out approach to the branch and bound problem follows the **queue** approach in creating the state-space tree. Here, breadth first search is performed, i.e., the elements at a particular level are all searched, and then the elements of the next level are searched, starting from the first child of the first node in the previous level.

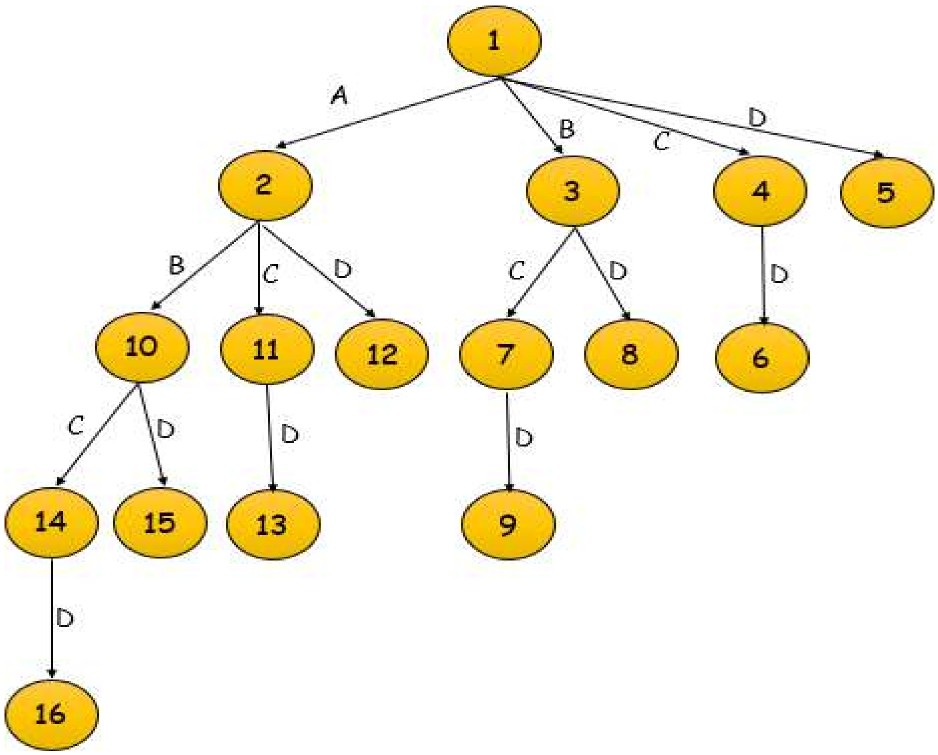
For a given set {A, B, C, D}, the state space tree will be constructed as follows:



Here, note that the number assigned to the node signifies the order in which the tree shall be constructed. The element next to the set denotes the next element to be added to the subset. Note that if an element is getting added, it is assumed here that all elements in the set preceeding that element are not added. For example, in node 4, D is getting added. This implies that elements A, B and C are not added.

2. LIFO Branch and Bound

The Last-In-First-Out approach to this problem follows the **stack** approach in creating the state space tree. Here, when nodes get added to the state space tree, think of them as getting added to a stack. When all nodes of a level are added, we pop the topmost element from the stack and then explore it. Hence, the state space tree for the same example {A, B, C, D} will be as follows:



Here, one can see that the main difference lies in the order in which the nodes have been explored.

3. Least Cost-Branch and Bound

- This method of exploration uses the cost function in order to explore the state space tree.
- Although the previous two methods calculate the cost function at each node, this is not used as a criterion for further exploration.
- In this method, after the children of a particular node have been explored, the next node to be explored would be that node out of the unexplored nodes which has the least cost.
- For example, in the previous example, after reaching node 5, the next node to be explored would be that which has the least cost among nodes 2, 3, 4, 5.

Why use Branch and Bound?

- The Branch and Bound method is preferred over other similar methods such as backtracking when it comes to optimization problems.
- Here, the cost and the objective function help in finding branches that need not be explored.
- Suppose the cost of a particular node has been determined. If this value is greater than that of U_0, this means that there is no way this node or its children shall give a solution. Hence, we can kill this node and not explore its further branches.
- This method helps us rule out cases not worth exploring, and is therefore more efficient.

Problems that can be solved using Branch and Bound

The Branch and Bound method can be used for solving most combinatorial problems. Some of these problems are given below:

1. ***Job Sequencing***: Suppose there is a set of N jobs and a set of N workers. Each worker takes a specific time to complete each of the N jobs. The job sequencing problem deals with finding that order of the workers, which minimizes the time taken to complete the job.
2. ***0/1 Knapsack problem***: Given a set of weights of N objects and a sack which can carry a maximum of W units. The problem deals with finding that subset of items such that the maximum possible weight is carried in the sack. Here, one cannot take part of an object, i.e., one can either take an object or not take it. This problem can also be solved using the backtracking, brute force and the dynamic programming approach.
3. ***Traveling Salesman Problem***: Here, we are given a set of N cities, and the cost of traveling between all pairs of cities. The problem is to find a path such that one starts from a given node, visits all cities exactly once, and returns back to the starting city.

Applications:

1. **Travelling Salesperson Problem.(LCBB)**
2. **0/1 knapsack Problem.(LCBB,FIFOBB ,LIFOBB)**

1. Travelling Salesperson Problem using LCBB:

- Travelling Salesman Problem (TSP) is an interesting problem. Problem is defined as “given n cities and distance between each pair of cities, find out the path which visits each city exactly once and come back to starting city, with the constraint of minimizing the travelling distance.”
- TSP has many practical applications. It is used in network design, and transportation route design.
- The objective is to minimize the distance. We can start tour from any random city and visit other cities in any order. With n cities, n! Different permutations are possible.
- Exploring all paths using brute force attacks may not be useful in real life applications.

LCBB using Static State Space Tree for Travelling Salesman Problem

- Branch and bound is an effective way to find better, if not best, solution in quick time by pruning some of the unnecessary branches of search tree.
- It works as follow :

Consider directed weighted graph G = (V, E, W), where node represents cities and weighted directed edges represents direction and distance between two cities.

1. Initially, graph is represented by cost matrix C, where

C_{ij} = cost of edge, if there is a direct path from city i to city j
C_{ij} = ∞, if there is no direct path from city i to city j.

2. Convert cost matrix to reduced matrix by subtracting minimum values from appropriate rows and columns, such that each row and column contains at least one zero entry.

3. Find cost of reduced matrix. Cost is given by summation of subtracted amount from the cost matrix to convert it in to reduce matrix.

4. Prepare state space tree for the reduce matrix

5. Find least cost valued node A (i.e. E-node), by computing reduced cost node matrix with every remaining node.

6. If <i, j> edge is to be included, then do following :

(a) Set all values in row i and all values in column j of A to ∞
(b) Set A[j, 1] = ∞
(c) Reduce A again, except rows and columns having all ∞ entries.

7. Compute the cost of newly created reduced matrix as,

Cost = L + Cost(i, j) + r

Where, L is cost of original reduced cost matrix and r is A[i, j].

8. If all nodes are not visited then go to step 4.

Reduction procedure is described below :

Raw Reduction:

Matrix M is called reduced matrix if each of its row and column has at least one zero entry or entire row or entire column has ∞ value. Let M represents the distance matrix of 5 cities. M can be reduced as follow:

$M_{RowRed} = \{M_{ij} - \min \{M_{ij} \mid 1 \leq j \leq n, \text{ and } M_{ij} < \infty\}\}$
Consider the following distance matrix:

M =

∞	20	30	10	11
15	∞	16	4	2
3	5	∞	2	4
19	6	18	∞	3
16	4	7	16	∞

Find the minimum element from each row and subtract it from each cell of matrix.

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$$M = \begin{array}{|c|c|c|c|c|} \hline \infty & 20 & 30 & 10 & 11 \\ \hline 15 & \infty & 16 & 4 & 2 \\ \hline 3 & 5 & \infty & 2 & 4 \\ \hline 19 & 6 & 18 & \infty & 3 \\ \hline 16 & 4 & 7 & 16 & \infty \\ \hline \end{array} \begin{array}{c} \rightarrow 10 \\ \rightarrow 2 \\ \rightarrow 2 \\ \rightarrow 3 \\ \rightarrow 4 \end{array}$$

Reduced matrix would be:

$$M_{RowRed} = \begin{array}{|c|c|c|c|c|} \hline \infty & 10 & 20 & 0 & 1 \\ \hline 13 & \infty & 14 & 2 & 0 \\ \hline 1 & 3 & \infty & 0 & 2 \\ \hline 16 & 3 & 15 & \infty & 0 \\ \hline 12 & 0 & 3 & 12 & \infty \\ \hline \end{array}$$

Row reduction cost is the summation of all the values subtracted from each rows:

Row reduction cost (M) = 10 + 2 + 2 + 3 + 4 = 21

Column reduction:

Matrix M_{RowRed} is row reduced but not the column reduced. Matrix is called column reduced if each of its column has at least one zero entry or all ∞ entries.
$M_{ColRed} = \{M_{ji} - \min \{M_{ji} \mid 1 \le j \le n, \text{ and } M_{ji} < \infty \}\}$

To reduced above matrix, we will find the minimum element from each column and subtract it from each cell of matrix.

$$M_{RowRed} = \begin{array}{|c|c|c|c|c|} \hline \infty & 10 & 20 & 0 & 1 \\ \hline 13 & \infty & 14 & 2 & 0 \\ \hline 1 & 3 & \infty & 0 & 2 \\ \hline 16 & 3 & 15 & \infty & 0 \\ \hline 12 & 0 & 3 & 12 & \infty \\ \hline \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 0 & 3 & 0 & 0 \end{array}$$

Column reduced matrix M_{ColRed} would be:

$$M_{ColRed} = \begin{array}{|c|c|c|c|c|} \hline \infty & 10 & 17 & 0 & 1 \\ \hline 12 & \infty & 11 & 2 & 0 \\ \hline 0 & 3 & \infty & 0 & 2 \\ \hline 15 & 3 & 12 & \infty & 0 \\ \hline 11 & 0 & 0 & 12 & \infty \\ \hline \end{array}$$

Each row and column of M_{ColRed} has at least one zero entry, so this matrix is reduced matrix.

Column reduction cost (M) = 1 + 0 + 3 + 0 + 0 = 4

State space tree for 5 city problem is depicted in Fig. 6.6.1. Number within circle indicates the order in which the node is generated, and number of edge indicates the city being visited.

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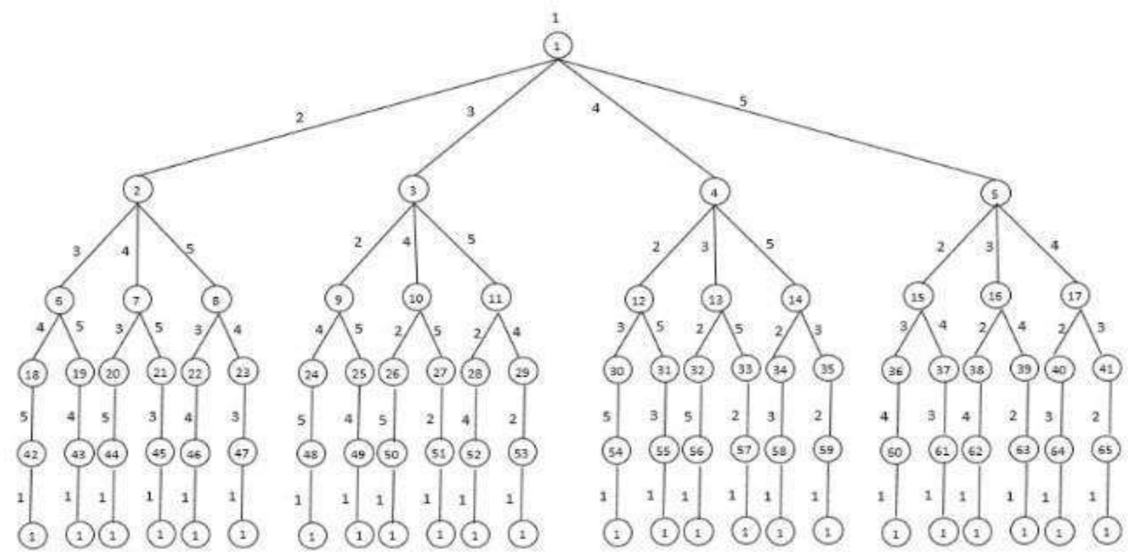


Example

Example: Find the solution of following travelling salesman problem using branch and bound method.

Cost Matrix =

∞	20	30	10	11
15	∞	16	4	2
3	5	∞	2	4
19	6	18	∞	3
16	4	7	16	∞

Solution:

I. The procedure for dynamic reduction is as follow:

II. Draw state space tree with optimal reduction cost at root node.

III. Derive cost of path from node i to j by setting all entries in i^{th} row and j^{th} column as ∞.

Set M[j][i] = ∞

- Cost of corresponding node N for path i to j is summation of optimal cost + reduction cost + M[j][i]
- After exploring all nodes at level i, set node with minimum cost as E node and repeat the procedure until all nodes are visited.
- Given matrix is not reduced. In order to find reduced matrix of it, we will first find the row reduced matrix followed by column reduced matrix if needed. We can find row reduced matrix by subtracting minimum element of each row from each element of corresponding row. Procedure is described below:
- Reduce above cost matrix by subtracting minimum value from each row and column.

$$
\begin{array}{|c|c|c|c|c|}
\hline
\infty & 20 & 30 & 10 & 11 \\ \hline
15 & \infty & 16 & 4 & 2 \\ \hline
3 & 5 & \infty & 2 & 4 \\ \hline
19 & 6 & 18 & \infty & 3 \\ \hline
16 & 4 & 7 & 16 & \infty \\ \hline
\end{array}
\begin{array}{c}
\rightarrow 10 \\ \rightarrow 2 \\ \rightarrow 2 \\ \rightarrow 3 \\ \rightarrow 4
\end{array}
\Rightarrow
\begin{array}{ccccc}
\begin{array}{|c|c|c|c|c|}
\hline
\infty & 10 & 20 & 0 & 1 \\ \hline
13 & \infty & 14 & 2 & 0 \\ \hline
1 & 3 & \infty & 0 & 2 \\ \hline
16 & 3 & 15 & \infty & 0 \\ \hline
12 & 0 & 3 & 12 & \infty \\ \hline
\end{array} \\
\downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow \\
1\ \ \ 0\ \ \ 3\ \ \ 0\ \ \ 0
\end{array}
= M_1'
$$

M_1' is not reduced matrix. Reduce it subtracting minimum value from corresponding column. Doing this we get,

$$
\begin{array}{|c|c|c|c|c|}
\hline
\infty & 10 & 17 & 0 & 1 \\ \hline
12 & \infty & 11 & 2 & 0 \\ \hline
0 & 3 & \infty & 0 & 2 \\ \hline
15 & 3 & 12 & \infty & 0 \\ \hline
11 & 0 & 0 & 12 & \infty \\ \hline
\end{array}
= M_1
$$

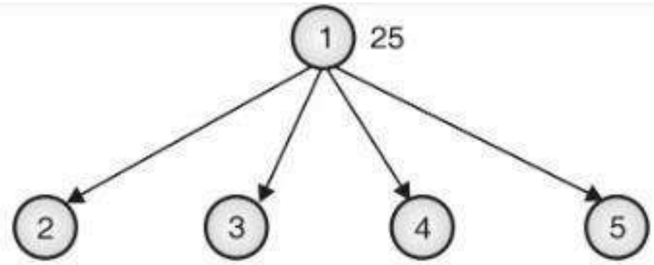
Cost of M_1 = C(1)

= Row reduction cost + Column reduction cost

= (10 + 2 + 2 + 3 + 4) + (1 + 3) = 25

This means all tours in graph has length at least 25. This is the optimal cost of the path.

State space tree



Let us find cost of edge from node 1 to 2, 3, 4, 5.

Select edge 1-2:

Set M_1 [1] [] = M_1 [] [2] = ∞

Set M_1 [2] [1] = ∞

Reduce the resultant matrix if required.

∞	∞	∞	∞	∞	→ x
∞	∞	11	2	0	→ 0
0	∞	∞	0	2	→ 0
15	∞	12	∞	0	→ 0
11	∞	0	12	∞	→ 0
↓	↓	↓	↓	↓	
0	x	0	0	0	

$= M_2$

M_2 is already reduced.

Cost of node 2 :

C(2) = C(1) + Reduction cost + M_1 [1] [2]

= 25 + 0 + 10 = 35

Select edge 1-3

Set M_1 [1][ ] = M_1 [ ] [3] = ∞

Set M_1 [3][1] = ∞

Reduce the resultant matrix if required.

$$M_1 \Rightarrow \begin{array}{|c|c|c|c|c|}\hline \infty & \infty & \infty & \infty & \infty \\ \hline 12 & \infty & 11 & 2 & 0 \\ \hline 0 & \infty & \infty & 0 & 2 \\ \hline 15 & \infty & 12 & \infty & 0 \\ \hline 11 & \infty & 0 & 12 & \infty \\ \hline \end{array} \begin{array}{c} \rightarrow x \\ \rightarrow 0 \\ \rightarrow 0 \\ \rightarrow 0 \\ \rightarrow 0 \end{array} \Rightarrow \begin{array}{|c|c|c|c|c|}\hline \infty & \infty & \infty & \infty & \infty \\ \hline 1 & \infty & \infty & 2 & 0 \\ \hline \infty & 3 & \infty & 0 & 2 \\ \hline 4 & 3 & \infty & \infty & 0 \\ \hline 0 & 0 & \infty & 12 & \infty \\ \hline \end{array} = M_3$$

$$\begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 11 & 0 & x & 0 & 0 \end{array}$$

Select edge 1-4:

Set M_1 [1][ ] = M_1[ ][4] = ∞

Set M_1 [4][1] = ∞

Reduce resultant matrix if required.

$$M_1 \Rightarrow \begin{array}{|c|c|c|c|c|}\hline \infty & \infty & \infty & \infty & \infty \\ \hline 12 & \infty & 11 & \infty & 0 \\ \hline 0 & 3 & \infty & \infty & 2 \\ \hline \infty & 3 & 12 & \infty & 0 \\ \hline 11 & 0 & 0 & \infty & \infty \\ \hline \end{array} \begin{array}{c} \rightarrow x \\ \rightarrow 0 \\ \rightarrow 0 \\ \rightarrow 0 \\ \rightarrow 0 \end{array} = M_4$$

$$\begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & x & 0 \end{array}$$

Matrix M_4 is already reduced.

Cost of node 4:

C(4) = C(1) + Reduction cost + M_1 [1] [4]

= 25 + 0 + 0 = 25

Select edge 1-5:

Set M_1 [1] [ ] = M_1 [ ] [5] = ∞

Set M_1 [5] [1] = ∞

Reduce the resultant matrix if required.

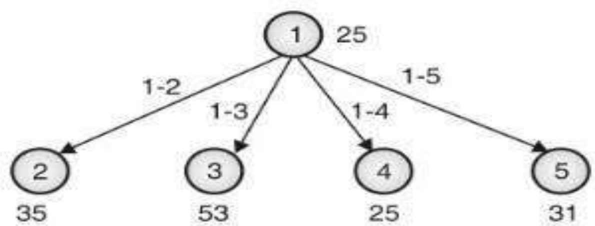
$$M_1 \Rightarrow \begin{array}{|c|c|c|c|c|}\hline \infty & \infty & \infty & \infty & \infty \\ \hline 12 & \infty & 11 & 2 & \infty \\ \hline 0 & 3 & \infty & 0 & \infty \\ \hline 15 & 3 & 12 & \infty & \infty \\ \hline \infty & 0 & 0 & 12 & \infty \\ \hline \end{array} \begin{array}{l} \to x \\ \to 2 \\ \to 0 \\ \to 3 \\ \to 0 \end{array} \Rightarrow \begin{array}{c} \begin{array}{|c|c|c|c|c|}\hline \infty & \infty & \infty & \infty & \infty \\ \hline 10 & \infty & 9 & 0 & \infty \\ \hline 0 & 3 & \infty & 0 & \infty \\ \hline 12 & 0 & 9 & \infty & \infty \\ \hline \infty & 0 & 0 & 12 & \infty \\ \hline \end{array} \\ \begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & 0 & x \end{array} \end{array} = M_5$$

Cost of node 5:

C(5) = C(1) + reduction cost + M_1 [1] [5]

= 25 + 5 + 1 = 31

State space diagram:



Node 4 has minimum cost for path 1-4. We can go to vertex 2, 3 or 5. Let's explore all three nodes.

Select path 1-4-2 : (Add edge 4-2)

Set M_4 [1] [] = M_4 [4] [] = M_4 [] [2] = ∞

Set M_4 [2] [1] = ∞

Reduce resultant matrix if required.

$$M_4 \Rightarrow \begin{array}{c} \begin{array}{|c|c|c|c|c|}\hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & 11 & \infty & 0 \\ \hline 0 & \infty & \infty & \infty & 2 \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 11 & \infty & 0 & \infty & \infty \\ \hline \end{array} \\ \begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & x & 0 \end{array} \end{array} \begin{array}{l} \to x \\ \to 0 \\ \to 0 \\ \to x \\ \to 0 \end{array} = M_6$$

Matrix M_6 is already reduced.
Cost of node 6:

$C(6) = C(4) + \text{Reduction cost} + M_4 [4] [2]$
$= 25 + 0 + 3 = 28$

Select edge 4-3 (Path 1-4-3):

Set $M_4 [1] [\ ] = M_4 [4] [\ ] = M_4 [\ ] [3] = \infty$
Set $M [3][1] = \infty$

Reduce the resultant matrix if required.

$$M_4 \Rightarrow \begin{array}{|c|c|c|c|c|} \hline \infty & \infty & \infty & \infty & \infty \\ \hline 12 & \infty & \infty & \infty & 0 \\ \hline \infty & 3 & \infty & \infty & 2 \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 11 & 0 & \infty & \infty & \infty \\ \hline \end{array} \begin{array}{l} \rightarrow x \\ \rightarrow 0 \\ \rightarrow 2 \\ \rightarrow \infty \\ \rightarrow 0 \end{array} \Rightarrow \begin{array}{c} \begin{array}{|c|c|c|c|c|} \hline \infty & \infty & \infty & \infty & \infty \\ \hline 12 & \infty & \infty & \infty & 0 \\ \hline \infty & 1 & \infty & \infty & 0 \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 11 & 0 & \infty & \infty & \infty \\ \hline \end{array} \\ \begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 11 & 0 & x & x & 0 \end{array} \end{array} = M_7'$$

M_7' is not reduced. Reduce it by subtracting 11 from column 1.

$$\therefore M_7' \Rightarrow \begin{array}{|c|c|c|c|c|} \hline \infty & \infty & \infty & \infty & \infty \\ \hline 1 & \infty & \infty & \infty & 0 \\ \hline \infty & 1 & \infty & \infty & 2 \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 0 & 0 & \infty & \infty & \infty \\ \hline \end{array} = M_7$$

Cost of node 7:

$C(7) = C(4) + \text{Reduction cost} + M_4 [4] [3]$

$= 25 + 2 + 11 + 12 = 50$

Select edge 4-5 (Path 1-4-5):

$$M_4 \Rightarrow \begin{array}{|c|c|c|c|c|}\hline \infty & \infty & \infty & \infty & \infty \\ \hline 12 & \infty & 11 & \infty & \infty \\ \hline 0 & 3 & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & 0 & 0 & \infty & \infty \\ \hline \end{array} \begin{array}{l} \rightarrow x \\ \rightarrow 11 \\ \rightarrow 0 \\ \rightarrow x \\ \rightarrow 0 \end{array} \Rightarrow \begin{array}{ccccc} \begin{array}{|c|c|c|c|c|}\hline \infty & \infty & \infty & \infty & \infty \\ \hline 1 & \infty & 0 & \infty & \infty \\ \hline 0 & 3 & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & 0 & 0 & \infty & \infty \\ \hline \end{array} \\ \downarrow \;\; \downarrow \;\; \downarrow \;\; \downarrow \;\; \downarrow \\ 0 \;\; 0 \;\; 0 \;\; x \;\; x \end{array} = M_8$$

Matrix M_8 is reduced.

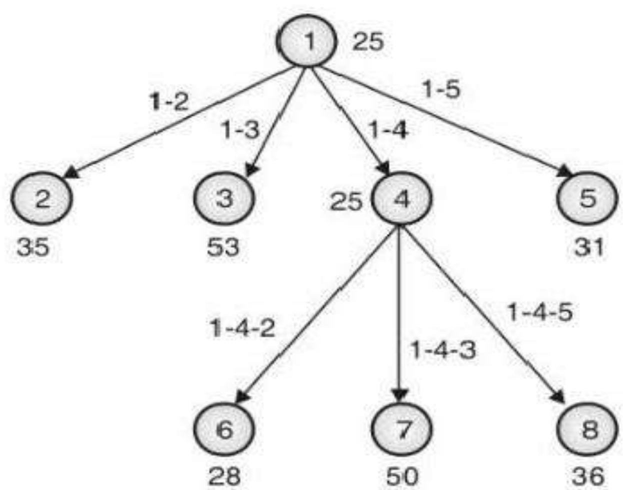
Cost of node 8:

$C(8) = C(4) + \text{Reduction cost} + M_4[4][5]$

$= 25 + 11 + 0 = 36$

State space tree

Path 1-4-2 leads to minimum cost. Let's find the cost for two possible paths.



Add edge 2-3 (Path 1-4-2-3):

Set $M_6[1][\,] = M_6[4][\,] = M_6[2][\,]$

$= M_6[\,][3] = \infty$

Set $M_6[3][1] = \infty$

Reduce resultant matrix if required.

$$M_6 \Rightarrow \begin{array}{|c|c|c|c|c|} \hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 0 & \infty & \infty & \infty & 2 \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 11 & \infty & \infty & \infty & \infty \\ \hline \end{array} \begin{array}{l} \rightarrow x \\ \rightarrow x \\ \rightarrow 0 \\ \rightarrow x \\ \rightarrow 11 \end{array} \Rightarrow \begin{array}{c} \begin{array}{|c|c|c|c|c|} \hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 0 & \infty & \infty & \infty & 2 \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 0 & \infty & \infty & \infty & \infty \\ \hline \end{array} \\ \begin{array}{ccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & x & x & x & 2 \end{array} \end{array} = M_9'$$

$$\therefore M_9' \Rightarrow \begin{array}{|c|c|c|c|c|} \hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 0 & \infty & \infty & \infty & 0 \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 0 & \infty & \infty & \infty & \infty \\ \hline \end{array} = M_9$$

Cost of node 9:

C(9) = C(6) + Reduction cost + M_6 [2][3]

= 28 + 11 + 2 + 11 = 52

Add edge 2-5 (Path 1-4-2-5):

Set M_6 [1][ ] = M_6 [4][ ] = M_6 [2][ ] = M_6 [ ][5] = ∞

Set M_6 [5][1] = ∞

Reduce resultant matrix if required.

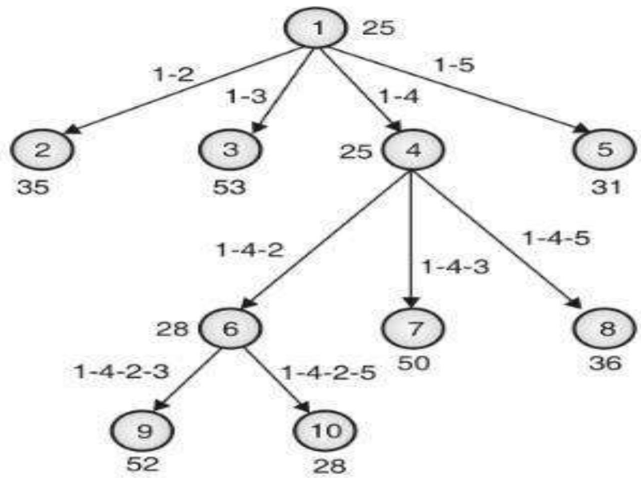
$$\therefore M_6 \Rightarrow \begin{array}{|c|c|c|c|c|} \hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline 0 & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & 0 & \infty & \infty \\ \hline \end{array} = M_{10}$$

Cost of node 10:

C(10) = C(6) + Reduction cost + M_6 [2][5]

= 28 + 0 + 0 = 28

State space tree



Add edge 5-3 (Path 1-4-2-5-3):

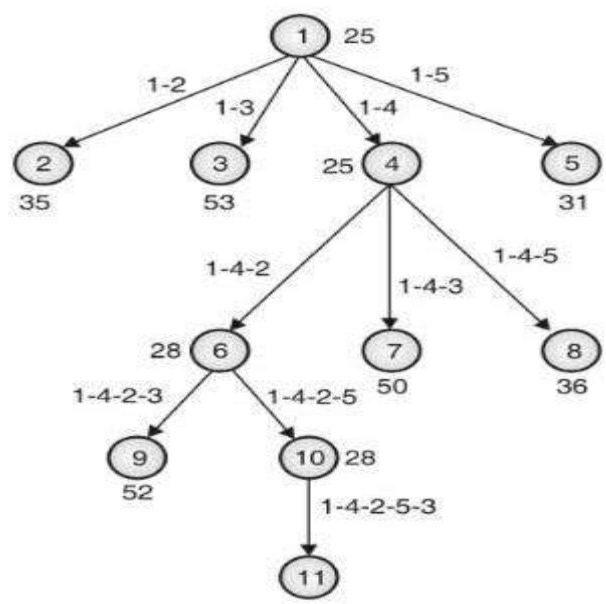
$$\therefore\ M_{10} \Rightarrow \begin{array}{|c|c|c|c|c|}\hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline \infty & \infty & \infty & \infty & \infty \\ \hline \end{array} = M_{11}$$

Cost of node 11:

$C(11) = C(10) + \text{Reduction cost} + M_{10}[5][3]$

$= 28 + 0 + 0 = 28$

State space tree:



2. 0/1 knapsack Problem (LCBB,FIFOBB ,LIFOBB):

Knapsack Problem using Branch and Bound is discussed in this article. LC and FIFO, both variants are described with example.

- As discussed earlier, the goal of knapsack problem is to maximize sum_{i}^{} p_i x_i*sumipi xi* given the constraints sum_{i}^{} w_i x_i le M*sumiwi xileM*, where M is the size of the knapsack. A maximization problem can be converted to a minimization problem by negating the value of the objective function.
- The modified knapsack problem is stated as,

minimize -sum_{i}^{} p_i x_i−*sumipi xi* subjected to sum_{i}^{} w_i x_i le M*sumiwi xileM*,
Where, x_i in {0, 1}, 1 le i le n*xiin*0,1,1*leilen*

- Node satisfying the constraint sum_{i}^{} w_i x_i le M*sumiwi xileM* in state space tree is called the answer state, the remaining nodes are infeasible.
- For the minimum cost answer node, we need to define hat{c}(x) = -sum_{i}^{} p_i x_i*hatc*(*x*)=−*sumipi xi* for every answer node x_i.
- Cost for infeasible leaf node would be ∞.
- For all non-leaf nodes, cost function is recursively defined as

  c(x) = min{ c(LChild(x)), c(RChild(x)) }
- For every node x, hat{c}(x) le c(x) le u(x)*hatc*(*x*)*lec*(*x*)*leu*(*x*).
- Algorithm for knapsack problem using branch and bound is described below :
- For any node N, upper bound for feasible left child remains N. But upper bound for its right child needs to be calculated.

i. 0/1 Knapsack using LCBB:

LC branch and bound solution for knapsack problem is derived as follows:

1. Derive state space tree.
2. Compute lower bound hat{c}(.)*hatc*(.) and upper bound u(.)*u*(.) for each node in state space tree.
3. If lower bound is greater than upper bound than kill that node.
4. Else select node with minimum lower bound as E-node.
5. Repeat step 3 and 4 until all nodes are examined.
6. The node with minimum lower bound value hat{c}(.)*hatc*(.) is the answer node. Trace the path from leaf to root in the backward direction to find the solution tuple.

Variation:

The bounding function is a heuristic computation. For the same problem, there may be different bounding functions. Apart from the above-discussed bounding function, another very popular bounding function for knapsack is,

$$ub = v + (W - w) * (v_{i+1} / w_{i+1})$$

where,

v is value/profit associated with selected items from the first i items.

W is the capacity of the knapsack.

w is the weight of selected items from first i items

Example:

Solve the following instance of knapsack using LCBB for knapsack capacity M = 15.

i	**P_i**	**W_i**
1	10	2
2	10	4
3	12	6
4	18	9

Solution:

Let us compute u(1) and $hatc(1)$. If we include first three item then $sumw_i leM$, but if we include 4[th] item, it exceeds knapsack capacity. So,

u(1) $= -sump_i$ such that $sumw_i leM$

= − (p_1 + p_2 + p_3)

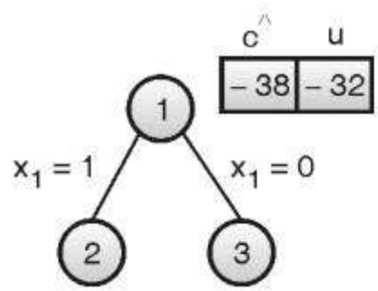
= -(10 + 10 + 12) = − 32

$hatc(1) = u(1) - fracM - Weight;of;selected;itemsWeight;of;remaining;items * Profit;of;remaining;items$

That gives,

$hatc(1) = -32 - frac15 - (2+4+6)9 * 18$ = -38

State Space Tree:



Node 2 : Inclusion of item 1 at node 1

Inclusion of item 1 is compulsory. So we will end up with same items in previous step.

u(2) = − (p_1 + p_2 + p_3) = − (10 + 10 + 12) = − 32

$hatc(2) = u(2) - fracM - Weight;of;selected;itemsWeight;of;remaining;items * Profit;of;remaining;items$ $hatc(2) = -32 - frac15 - (2+4+6)9 * 18$ = -38

Node 2 is inserted in list of live nodes.

Node 3 : Exclusion of item 1 at node 1

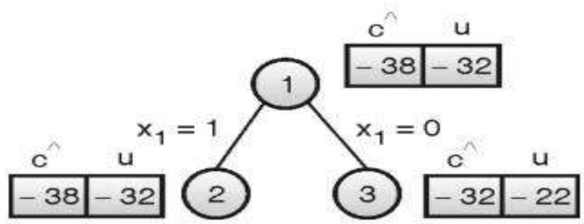
We are excluding item 1 and including 2 and 3. Item 4 cannot be accommodated in knapsack along with items 2 and 3. So,

$u(3) = -(p_2 + p_3) = -(10 + 12) = -22$

$hatc(3) = u(3) - fracM - Weight; of; selected; itemsWeight; of; remaining; items * Profit; of; remaining; items\ hatc(3) = -22 - frac15 - (4+6)9 * 18$ = -32

Node 3 is inserted in the list of live nodes.

State-space tree:



At level 1, node 2 has minimum $hatc(.)$, so it becomes E-node.

Node 4 : Inclusion of item 2 at node 2

Item 1 is already added at node 2, and we must have to include item 2 at this node. After adding items 1 and 2, the knapsack can accommodate item 3 but not 4.

$u(4) = -(p_1 + p_2 + p_3) = -(10 + 10 + 12) = -32$

$hatc(4) = u(4) - fracM - Weight; of; selected; itemsWeight; of; remaining; items * Profit; of; remaining; items\ hatc(4) = -32 - frac15 - (2+4+6)9 * 18$ = -38

Node 5: Exclusion of item 2 at node 2

At node 5, item 1 is already added, we must have to skip item 2. Only item 3 can be accommodated in knapsack after inserting item 1. $u(5) = -(p_1 + p_3) = -(10 + 12) = -22$

$hatc(5) = u(5) - fracM - Weight; of; selected; itemsWeight; of; remaining; items * Profit; of; remaining; items\ hatc(5) = -22 - frac15 - (2+6)9 * 18$ = -36

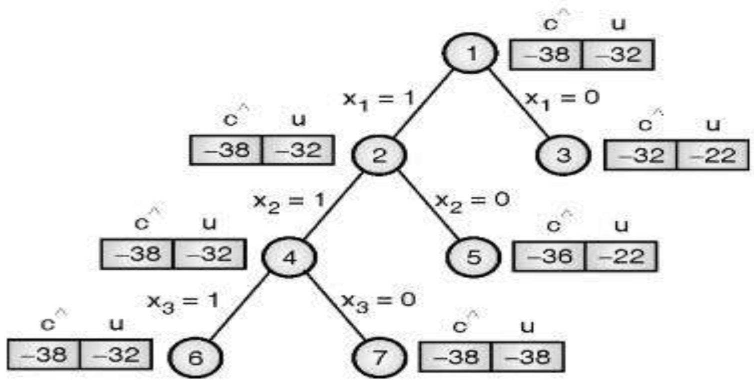
Node 7 : Exclusion of item 2 at node 2

On excluding item 3, we can accommodate item 1, 2 and 4 in knapsack.

u(7) = -(10 + 10 + 18) = − 38

$hatc(7) = u(7) -$
$fracM - Weight; of; selected; itemsWeight; of; remaining; items *$
$Profit; of; remaining; items\ hatc(7) = -38 - frac15 - (2+4+9)6 * 12$ =
-38

State Space Tree:

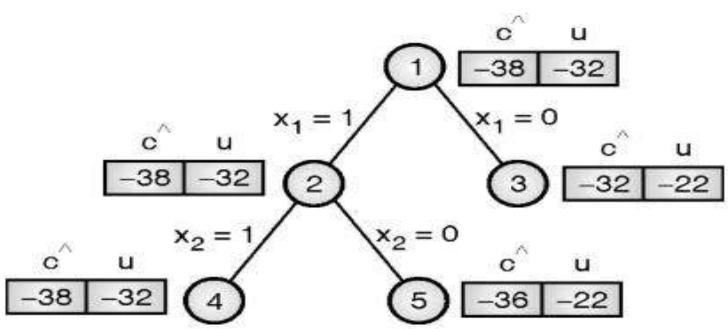


When there is tie, we can select any node. Let's make node 7 E-node.

Node 8 : Inclusion of item 4 at node 7

u(8) = -(10 + 10 + 18) = − 38

State Space Tree:



At level 2, node 4 has minimum $hatc(.)$, so it becomes E-node.

Node 6 : Inclusion of item 3 at node 4

At node 4, item 1 and 2 are already added, we have to add item 3. After including item 1, 2 and 3, item 4 cannot be accommodated. u(6) = − (10 + 10 + 12) = − 32

$hatc(6) = u(6) -$
$fracM - Weight; of; selected; itemsWeight; of; remaining; items *$
$Profit; of; remaining; items\ hatc(6) = -32 -$
$frac15 - (2+4+6)9 * 18$ = -38

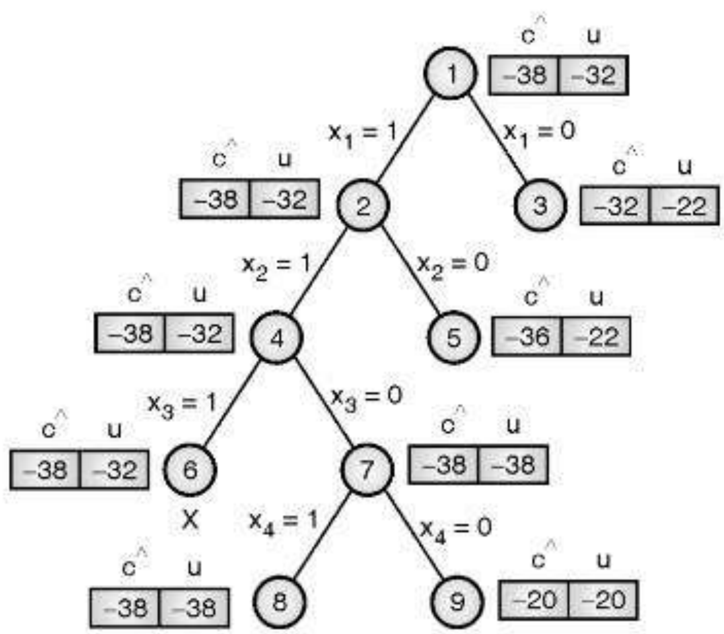
$hatc(8) = u(8) -$
$fracM - Weight; of; selected; itemsWeight; of; remaining; items *$
$Profit; of; remaining; items\ hatc(8) = -38 - frac15 - (2 + 4 + 9)6 * 12 =$
-38

Node 9 : Exclusion of item 4 at node 7

This node excludes both the items, 3 and 4. We can add only item 1 and 2.

u(9) = -(10 + 10) = − 20

$hatc(9) = u(9) -$
$fracM - Weight; of; selected; itemsWeight; of; remaining; items *$
$Profit; of; remaining; items\ hatc(9) = -20 - frac15 - (2 + 4)0 * 0 =$ -20



- At last level, node 8 has minimum hat{c}(.)*hatc*(.), so node 8 would be the answer node. By tracing from node 8 to root, we get item 4, 2 and 1 in the knapsack.

Solution vector X = {x_1, x_2, x_4} and profit = $p_1 + p_2 + p_4$ = 38.

Java Program For 0/1 kanpsack using LCBB: **knapsack_LCBB.java**

```
import java.util.*;
class Item
{

   // Stores the weight     // of items
   float weight;

   // Stores the values     // of items
   int value;

   // Stores the index     // of items
   int idx;
   public Item(){}
   public Item(int value, float weight,  int idx)
   {
      this.value = value;
      this.weight = weight;
      this.idx = idx;
   }
}

class Node
{
   // Upper Bound: Best case     // (Fractional Knapsack)

   float ub;

   // Lower Bound: Worst case // (0/1)

     float lb;

   // Level of the node in     // the decision tree
   int level;

   // Stores if the current // item is selected or not

     boolean flag;
```

```
    // Total Value: Stores the // sum of the values of the    // items included

    float tv;

    // Total Weight: Stores the sum of the weights of included items
    float tw;
    public Node()  {}
    public Node(Node cpy)
    {
        this.tv = cpy.tv;
        this.tw = cpy.tw;
        this.ub = cpy.ub;
        this.lb = cpy.lb;
        this.level = cpy.level;
        this.flag = cpy.flag;
    }
}

// Comparator to sort based on lower bound

class sortByC implements Comparator<Node>
 {
    public int compare(Node a, Node b)
    {
        boolean temp = a.lb > b.lb;
        return temp ? 1 : -1;
    }
}

class sortByRatio implements Comparator<Item>
 {
    public int compare(Item a, Item b)
    {
        boolean temp = (float)a.value / a.weight > (float)b.value / b.weight ;
        return temp ? -1 : 1;
    }
```

```
}

class knapsack_LCBB
{

    private static int size;
    private static float capacity;

    // Function to calculate upper bound (includes fractional part of the items)

  static float upperBound(float tv, float tw,  int idx, Item arr[])
    {
        float value = tv;
        float weight = tw;
        for (int i = idx; i < size; i++)
          {
            if (weight + arr[i].weight
                <= capacity) {
                weight += arr[i].weight;
                value -= arr[i].value;
            }
            else {
                value -= (float)(capacity
                                - weight)
                        / arr[i].weight
                        * arr[i].value;
                break;
            }
        }
        return value;
    }
```

```
// Calculate lower bound (doesn't include fractional part of items)

 static float lowerBound(float tv, float tw, int idx, Item arr[])
   {
      float value = tv;
      float weight = tw;
      for (int i = idx; i < size; i++) {
         if (weight + arr[i].weight
            <= capacity) {
            weight += arr[i].weight;
            value -= arr[i].value;
         }
         else {
            break;
         }
      }
      return value;
   }

   static void assign(Node a, float ub, float lb, int level, boolean flag, float tv, float tw)
   {
      a.ub = ub;
      a.lb = lb;
      a.level = level;
      a.flag = flag;
      a.tv = tv;
      a.tw = tw;
   }

   public static void solve(Item arr[])
   {
      // Sort the items based on the profit/weight ratio
      Arrays.sort(arr, new sortByRatio());

      Node current, left, right;
      current = new Node();
      left = new Node();
```

```
    right = new Node();

    // min_lb -> Minimum lower bound of all the nodes explored

    // final_lb -> Minimum lower bound of all the paths that reached the final level

 float minLB = 0, finalLB  = Integer.MAX_VALUE;
    current.tv = current.tw = current.ub= current.lb = 0;
    current.level = 0;
    current.flag = false;

    // Priority queue to store elements based on lower bounds
    PriorityQueue<Node> pq= new PriorityQueue<Node>( new sortByC());

    // Insert a dummy node
    pq.add(current);

    // curr_path -> Boolean array to store at every index if the element is included or not

 // final_path -> Boolean array to store the result of selection array when it reached the
//last level

    boolean currPath[] = new boolean[size];
    boolean finalPath[] = new boolean[size];

    while (!pq.isEmpty())
      {
       current = pq.poll();
       if (current.ub > minLB || current.ub >= finalLB)
            {
     // if the current node's best case/ value is not optimal than minLB,  then there is no
//reason to explore that node. Including finalLB eliminates all those paths whose best
//values is equal to  the finalLB.
          continue;
       }
```

```
        if  (current.level != 0)
            currPath[current.level - 1] = current.flag;

        if (current.level == size)
        {
            if (current.lb < finalLB)
                  {
                // Reached last level
                for (int i = 0; i < size; i++)
                    finalPath[arr[i].idx]    = currPath[i];
                finalLB = current.lb;
            }
            continue;
        }

        int level = current.level;

        // right node -> Excludes current item/ Hence, cp, cw will obtain the value
        // of that of parent

  assign(right, upperBound(current.tv,current.tw, level + 1, arr), lowerBound(current.tv,
current.tw, level + 1, arr),  level + 1, false,  current.tv, current.tw);

        if (current.tw + arr[current.level].weight  <= capacity)
       {

            // left node -> includes current item c and lb should be calculated
            // including the current item.
left.ub = upperBound(current.tv - arr[level].value, current.tw + arr[level].weight,
                level + 1, arr);
left.lb = lowerBound( current.tv - arr[level].value, current.tw  + arr[level].weight,
                level + 1, arr);
assign(left, left.ub, left.lb, level + 1, true, current.tv - arr[level].value, current.tw
                    + arr[level].weight);
        }

        // If the left node cannot/ be inserted
```

```
        else {

            // Stop the left node from
            // getting added to the
            // priority queue
            left.ub = left.lb = 1;
        }

        // Update minLB
        minLB = Math.min(minLB, left.lb);
        minLB = Math.min(minLB, right.lb);

        if (minLB >= left.ub)
            pq.add(new Node(left));
        if (minLB >= right.ub)
            pq.add(new Node(right));
    }
    System.out.println("Items taken"+ "into the knapsack are");
    for (int i = 0; i < size; i++) {
        if (finalPath[i])
            System.out.print("1 ");
        else
            System.out.print("0 ");
    }
    System.out.println("\nMaximum profit" + " is " + (-finalLB));
}

// Driver code
public static void main(String args[])
{
    size = 4;
    capacity = 15;

    Item arr[] = new Item[size];
    arr[0] = new Item(10, 2, 0);
    arr[1] = new Item(10, 4, 1);
    arr[2] = new Item(12, 6, 2);
```

```
        arr[3] = new Item(18, 9, 3);

        solve(arr);
    }
}
```

output=

Items taken into the knapsack are :
1 1 0 1
Maximum profit is : 38

ii. 0/1 Knapsack using FIFOBB

- In FIFO branch and bound approach, variable tuple size state space tree is drawn. For each node N, cost function hat{c}(.)*hatc*(.) and upper bound u(.) is computed similarly to the previous approach. In LC search, E node is selected from two child of current node.
- In FIFO branch and bound approach, both the children of siblings are inserted in list and most promising node is selected as new E node.
- Let us consider the same example :

Example: Solve following instance of knapsack using FIFO BB.

i	Pi	Wi
1	10	2
2	10	4
3	12	6
4	18	9

Solution:

Let us compute u(1) and $hatc(1)$.

u(1) = $sump_i$ such that $sumw_i leM$

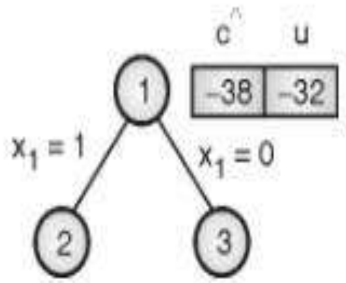
= − (10 + 10 + 12) = − 32

Upper = u(1) = − 32

If we include the first three items then $sumw_i leM$, but if we include 4th item, it exceeds the knapsack capacity.

$hatc(1) = u(1) - fracM - Weight; of; selected; itemsWeight; of; remaining; items * Profit; of; remaining; items\ hatc(1) = -32 - frac15 - (2+4+6)9 * 18$ = -38

State space tree:



Node 2 : Inclusion of item 1 at node 1

u(2) = -(10 + 10 + 12) = – 32

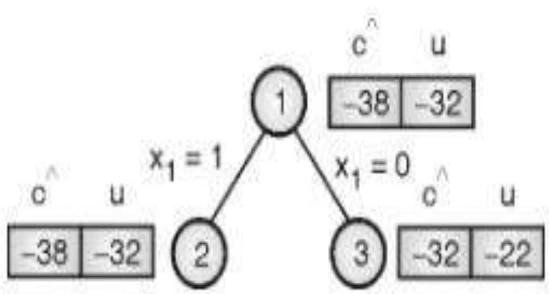
$hatc(2) = -32 - frac15 - (2 + 4 + 6)9 * 18$ = -38

Node 3 : Exclusion of item 1 at node 1

We are excluding item 1, including 2 and 3. Item 4 cannot be accommodated in a knapsack along with 2 and 3.

u(3) = -(10 + 12) = – 22

$hatc(3) = -22 - frac15 - (4 + 6)9 * 18$ = -32



In LC approach, node 2 would be selected as E-Node as it has minimum (x). But in FIFO approach, all child of node 2 and 3 are expanded and the most promising child becomes E-node.

Node 4 : Inclusion of item 2 at node 2

u(4) = -(10 + 10 + 12) = – 32

$hatc(4) = -32 - frac15 - (2+4+6)9 * 18$ = -38

Node 5 : Exclusion of item 2 at node 2

We are excluding item 1, including 2 and 3. Item 4 cannot be accommodated in a knapsack along with 2 and 3.

u(5) = -(10 + 12) = – 22

$hatc(5) = -22 - frac15 - (4+6)9 * 18$ = -32

Node 6 : Inclusion of item 2 at node 3

u(6) = -(p_2 + p_3) = – (10 + 12) = – 22

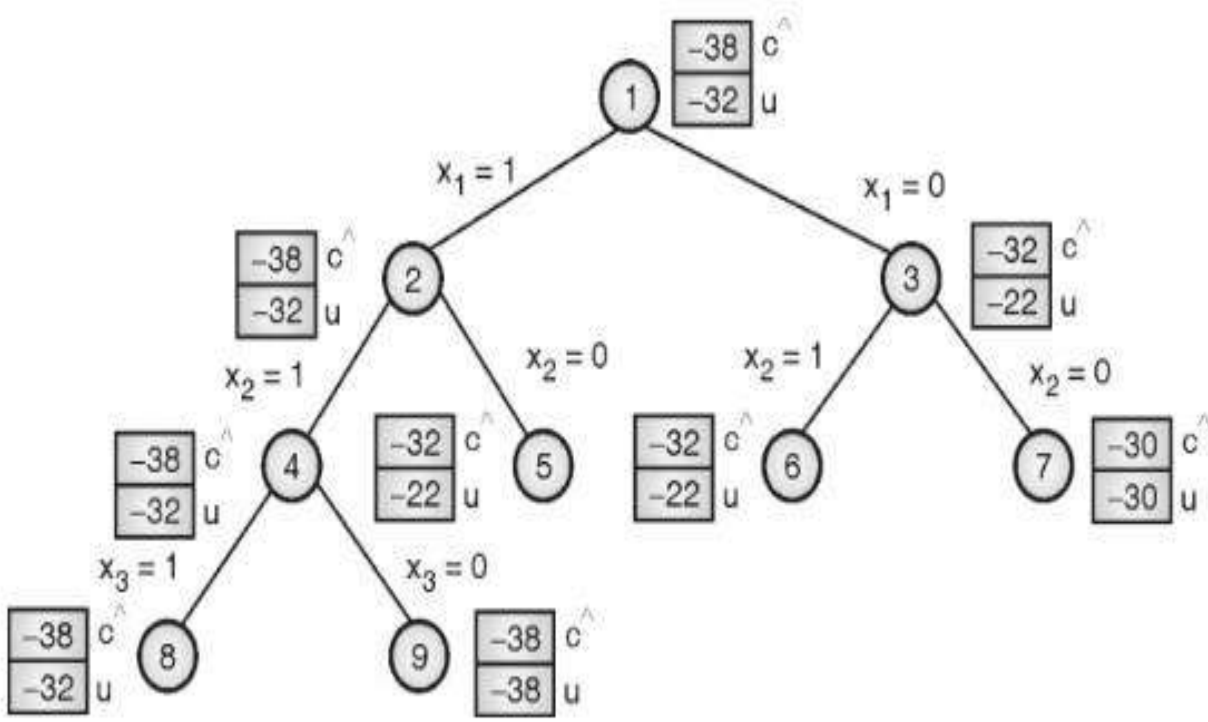
$hatc(6) = -22 - frac15 - (4+6)9 * 18$ = -32

Node 7 : Exclusion of item 2 at node 3

We are excluding item 1, including 2 and 3. Item 4 cannot be accommodated in a knapsack along with 2 and 3.

u(7) = -(p_3 + p_4) = – (12 + 18) = – 30

$hatc(7) = -30 - 0$ = -30



Out of node 4, 5, 6 and 7, $hatc(7)$ > upper, so kill node 7. Remaining 3 nodes are live and added in list. Out of 4, 5 and 6, node 4 has minimum $hatc$ value, so it becomes next E-node.

Node 8 : Inclusion of item 3 at node 4

u(8) = -(10 + 10 + 12) = – 32

$hatc(8) = -32 - frac15 - (2+4+6)9 * 18$ = -38

Node 9 : Exclusion of item 3 at node 4

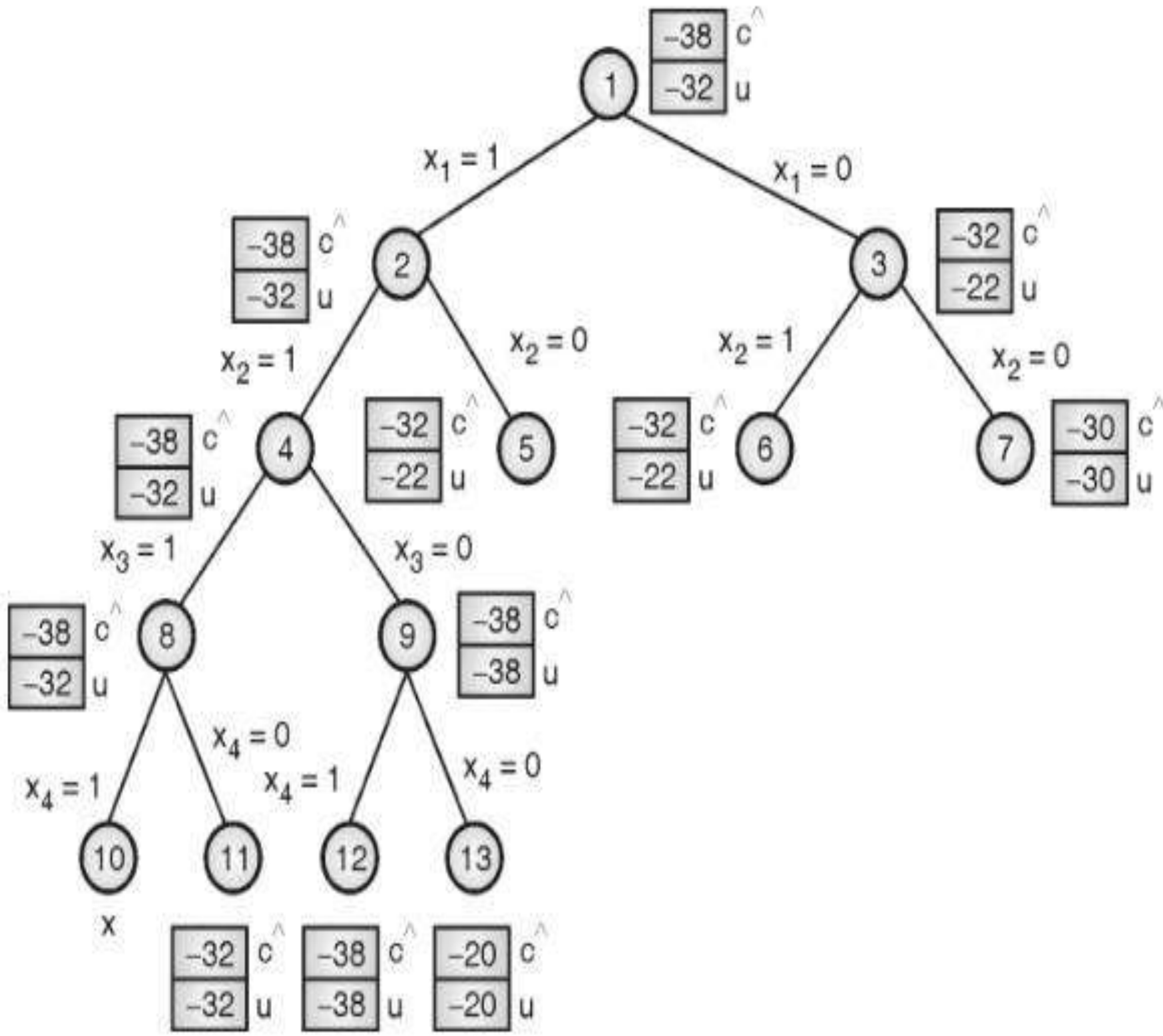
We are excluding item 3, including 1 and 2 and 4.

u(9) = -(p_1 + p_2 + p_4) = – (10 + 10 + 18) = – 38

$hatc(9) = -38 - 0$ = -38

Upper = u(9) = – 38.

$hatc(5)$ > upper and $hatc(6)$ > upper, so kill them. If we continue in this way, final state space tree looks as follow :



Node 12 has minimum cost function value, so it will be the answer node.

Solution vector = $\{x_1, x_2, x_4\}$.

profit = 10 + 10 + 18 = 38

UNIT – V

Deterministic and non-deterministic algorithms

Deterministic: The algorithm in which every operation is uniquely defined is called deterministic algorithm.

Non-Deterministic: The algorithm in which the operations are not uniquely defined but are limited to specific set of possibilities for every operation, such an algorithm is called non-deterministic algorithm.

The non-deterministic algorithms use the following functions:

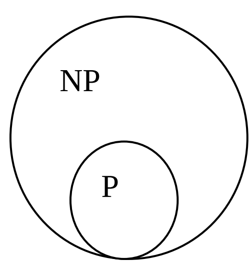
1. Choice: Arbitrarily chooses one of the element from given set.
2. Failure: Indicates an unsuccessful completion
3. Success: Indicates a successful completion

A non-deterministic algorithm terminates unsuccessfully if and only if there exists no set of choices leading to a success signal. Whenever, there is a set of choices that leads to a successful completion, then one such set of choices is selected and the algorithm terminates successfully.

In case the successful completion is not possible, then the complexity is O(1). In case of successful signal completion then the time required is the minimum number of steps needed to reach a successful completion of O(n) where n is the number of inputs.

The problems that are solved in polynomial time are called tractable problems and the problems that require super polynomial time are called non-tractable problems. All deterministic polynomial time algorithms are tractable and the non-deterministic polynomials are intractable.

A nondeterministic algorithm is considered as a superset for deterministic algorithm i.e., the deterministic algorithms are a special case of nondeterministic algorithms.



Example1: A non-deterministic algorithm for searching an element X in the given set of elements A[1:n] in the time complexity O(1).

```
Algorithm Search(A,x)
{
        i<-choice(1:n)
        if(A(i)=X) then
        {
                Write(i);
                Success();
        }
        else
        {
                Write(0);
                Failure();
        }
}
```

Satisfiability Problem:
The satisfiability is a boolean formula that can be constructed using the following literals and operations.

1. A literal is either a variable or its negation of the variable.
2. The literals are connected with operators $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$
3. Parenthesis

The satisfiability problem is to determine whether a Boolean formula is true for some assignment of truth values to the variables. In general, formulas are expressed in Conjunctive Normal Form (CNF).

A Boolean formula is in conjunctive normal form iff it is represented by
$(x_i \vee x_j \vee x_k^1) \wedge (x_i \vee x_j^1 \vee x_k)$

A Boolean formula is in 3CNF if each clause has exactly 3 distinct literals.

Example:

The non-deterministic algorithm that terminates successfully iff a given formula E(x1,x2,x3) is satisfiable.

Reducability:
A problem Q1 can be reduced to Q2 if any instance of Q1 can be easily rephrased as an instance of Q2. If the solution to the problem Q2 provides a solution to the problem Q1, then these are said to be reducable problems.

Let L1 and L2 are the two problems. L1 is reduced to L2 iff there is a way to solve L1 by a deterministic polynomial time algorithm using a deterministic algorithm that solves L2 in polynomial time and is denoted by L1α L2.

If we have a polynomial time algorithm for L2 then we can solve L1 in polynomial time. Two problems L1 and L2 are said to be polynomially equivalent iff L1α L2 and L2 α L1.

Example: Let P1 be the problem of selection and P2 be the problem of sorting. Let the input have n numbers. If the numbers are sorted in array A[ ] the i^{th} smallest element of the input can be obtained as A[i]. Thus P1 reduces to P2 in O(1) time.

Decision Problem:
Any problem for which the answer is either yes or no is called decision problem. The algorithm for decision problem is called decision algorithm.
Example: Max clique problem, sum of subsets problem.

Optimization Problem: Any problem that involves the identification of an optimal value (maximum or minimum) is called optimization problem.
Example: Knapsack problem, travelling salesperson problem.
In decision problem, the output statement is implicit and no explicit statements are permitted.
The output from a decision problem is uniquely defined by the input parameters and algorithm specification.

Many optimization problems can be reduced by decision problems with the property that a decision problem can be solved in polynomial time iff the corresponding optimization problem can be solved in polynomial time. If the decision problem cannot be solved in polynomial time then the optimization problem cannot be solved in polynomial time.

NP HARD AND NP COMPLETE
Polynomial Time algorithms
Problems whose solutions times are bounded by polynomials of small degree are called polynomial time algorithms

Example: Linear search, quick sort, all pairs shortest path etc.

Non- Polynomial time algorithms

Problems whose solutions times are bounded by non-polynomials are called non-polynomial time algorithms

Examples: Travelling salesman problem, 0/1 knapsack problem etc

It is impossible to develop the algorithms whose time complexity is polynomial for non-polynomial time problems, because the computing times of non-polynomial are greater than polynomial. A problem that can be solved in polynomial time in one model can also be solved in polynomial time.

NP-Hard and NP-Complete Problem:

Let P denote the set of all decision problems solvable by deterministic algorithm in polynomial time. NP denotes set of decision problems solvable by nondeterministic algorithms in polynomial time. Since, deterministic algorithms are a special case of nondeterministic algorithms, $P \subseteq NP$. The nondeterministic polynomial time problems can be classified into two classes. They are

1. NP Hard and
2. NP Complete

NP-Hard: A problem L is NP-Hard iff satisfiability reduces to L i.e., any nondeterministic polynomial time problem is satisfiable and reducable then the problem is said to be NP-Hard.

Example: Halting Problem, Flow shop scheduling problem

NP-Complete: A problem L is NP-Complete iff L is NP-Hard and L belongs to NP (nondeterministic polynomial).

A problem that is NP-Complete has the property that it can be solved in polynomial time iff all other NP-Complete problems can also be solved in polynomial time. (NP=P)

If an NP-hard problem can be solved in polynomial time, then all NP-complete problems can be solved in polynomial time. All NP-Complete problems are NP-hard, but some NP-hard problems are not known to be NP-Complete.

Normally the decision problems are NP-complete but the optimization problems are NP-Hard.

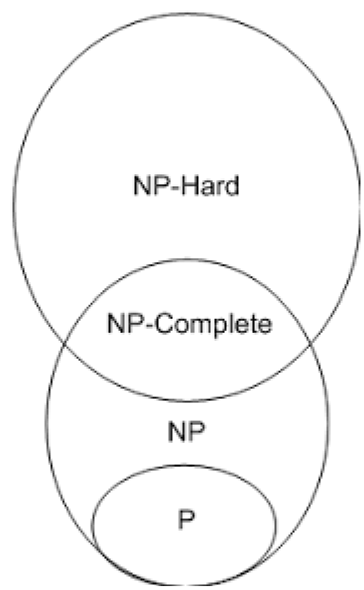
However if problem L1 is a decision problem and L2 is an optimization problem, then it is possible that L1α L2.

Example: Knapsack decision problem can be reduced to knapsack optimization problem.

There are some NP-hard problems that are not NP-Complete.

Relationship between P,NP,NP-hard, NP-Complete

Let P, NP, NP-hard, NP-Complete are the sets of all possible decision problems that are solvable in polynomial time by using deterministic algorithms, non-deterministic algorithms, NP-Hard and NP-complete respectively. Then the relationship between P, NP, NP-hard, NP-Complete can be expressed using Venn diagram as:



Problem conversion

A decision problem D1 can be converted into a decision problem D2 if there is an algorithm which takes as input an arbitrary instance I1 of D1 and delivers as output an instance I2 of D2such that I2 is a positive instance of D2 if and only if I1 is a positive instance of D1. If D1 can be converted into D2, and we have an algorithm which solves D2, then we thereby have an

algorithm which solves D1. To solve an instance I of D1, we first use the conversion algorithm to generate an instance I0 of D2, and then use the algorithm for solving D2 to determine whether or not I0 is a positive instance of D2. If it is, then we know that I is a positive instance of D1, and if it is not, then we know that I is a negative instance of D1. Either way, we have solved D1 for that instance. Moreover, in this case, we can say that the computational complexity of D1 is at most the sum of the computational complexities of D2 and the conversion algorithm. If the conversion algorithm has polynomial complexity, we say that D1 is at most polynomially harder than D2. It means that the amount of computational work we have to do to solve D1, over and above whatever is required to solve D2, is polynomial in the size of the problem instance. In such a case the conversion algorithm provides us with a feasible way of solving D1, given that we know how to solve D2.
Given a problem X, prove it is in NP-Complete.

1. Prove X is in NP.
2. Select problem Y that is known to be in NP-Complete.
3. Define a polynomial time reduction from Y to X.
4. Prove that given an instance of Y, Y has a solution iff X has a solution.

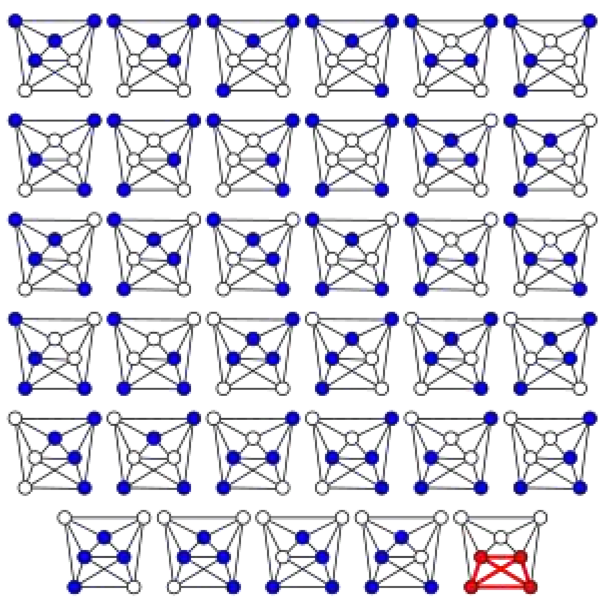
Cook’s theorem

Cook’s Theorem implies that any NP problem is at most polynomially harder than SAT. This means that if we find a way of solving SAT in polynomial time, we will then be in a position to solve any NP problem in polynomial time. This would have huge practical repercussions, since many frequently encountered problems which are so far believed to be intractable are NP. This special property of SAT is called NP-completeness. A decision problem is NP-complete if it has the property that any NP problem can be converted into it in polynomial time. SAT was the first NP-complete problem to be recognized as such (the theory of NP-completeness having come into existence with the proof of Cook’s Theorem), but it is by no means the only one. There are now literally thousands of problems, cropping up in many different areas of computing, which have been proved to be NP-complete.
In order to prove that an NP problem is NP-complete, all that is needed is to show that SAT can be converted into it in polynomial time. The reason for this is that the sequential composition of two polynomial-time algorithms is

itself a polynomial-time algorithm, since the sum of two polynomials is itself a polynomial.
Suppose SAT can be converted to problem D in polynomial time. Now take any NP problem D0. We know we can convert it into SAT in polynomial time, and we know we can convert SAT into D in polynomial time. The result of these two conversions is a polynomial-time conversion of D0 into D. since D0 was an arbitrary NP problem, it follows that D is NP-complete

NP-Hard Graph Problems:
The strategy adopted to show that a problem L2 is NP-hard is:
1. Pick a problem L1, already known to be NP-Hard.
2. Show how to obtain the solution to instance I of L1
3. Conclude from step(2) that L1αL2
4. Conclude from steps (1) and (3) that L2 is NP-hard.

Clique Decision problem:
Let G be a non-directed graph consisting of a set of vertices V and set of edges E. A maximal complete sub graph of a graph G is a clique. The size of the clique is the number of vertices in it. The max clique problem is an optimization problem that has to determine whether graph has a clique of size atleast k for some integer k.

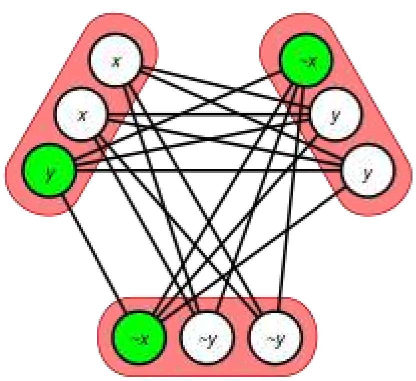


The brute force algorithm finds a 4-clique in this 7-vertex graph (the complement of the 7-vertex path graph) by systematically checking all C(7,4) = 35 4-vertex subgraphs for completeness.

The clique decision problem is NP-complete. This problem was also mentioned in Stephen Cook's paper introducing the theory of NP-complete problems. Because of the hardness of the decision problem, the problem of finding a maximum clique is also NP-hard. If one could solve it, one could also solve the decision problem, by comparing the size of the maximum clique to the size parameter given as input in the decision problem.

Karp's NP-completeness proof is a many-one reduction from the Boolean satisfiability problem. It describes how to translate Boolean formulas in conjunctive normal form (CNF) into equivalent instances of the maximum clique problem Satisfiability, in turn, was proved NP-complete in the Cook–Levin theorem. From a given CNF formula, Karp forms a graph that has a vertex for every pair (v,c), where v is a variable or its negation and c is a clause in the formula that contains v. Two of these vertices are connected by an edge if they represent compatible variable assignments for different clauses. That is, there is an edge from (v,c) to (u,d) whenever $c \neq d$ and u and v are not each other's negations. If k denotes the number of clauses in the CNF formula, then the k-vertex cliques in this graph represent consistent ways of assigning truth values to some of its variables in order to satisfy the formula. Therefore, the formula is satisfiable if and only if a k-vertex clique exists.

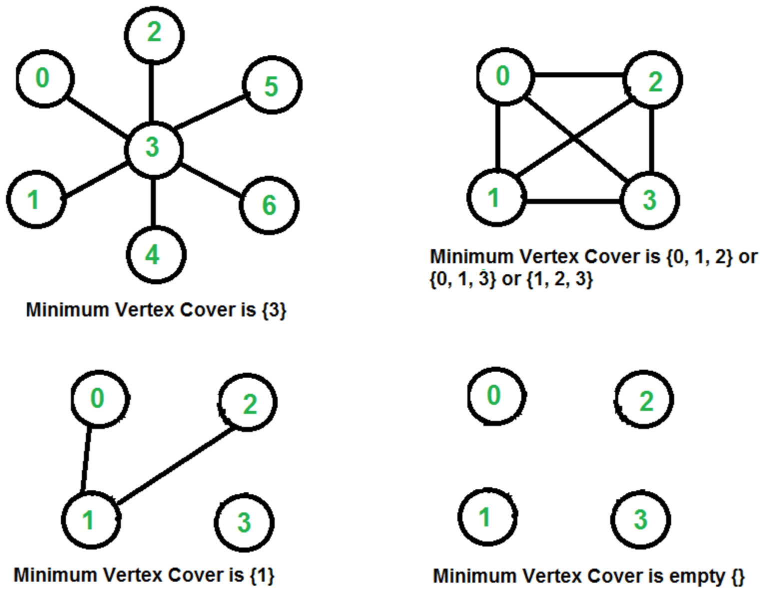
Some NP-complete problems (such as the travelling salesman problem in planar graphs) may be solved in time that is exponential in a sublinear function of the input size parameter n, significantly faster than a brute-force search. However, it is unlikely that such a subexponential time bound is possible for the clique problem in arbitrary graphs, as it would imply similarly subexponential bounds for many other standard NP-complete problems.



The 3-CNF Satisfiability instance $(x \lor x \lor y) \land (\sim x \lor \sim y \lor \sim y) \land (\sim x \lor y \lor y)$ reduced to Clique. The green vertices form a 3-clique and correspond to a satisfying assignment.

Node Cover Problem:

Let G(V,E) be a undirected graph. A set S which is a subset of V is a node cover if all S. The size of node cover is the edges in E are incident to atleast one vertex in S. The size of node cover is the number of vertices in it. The node cover decision problem is to find minimum size vertex cover in the given graph.



Clique decision problem α to node cover decision problem:

Let G be a non-directed graph consisting of set of vertices V and set of edges E.

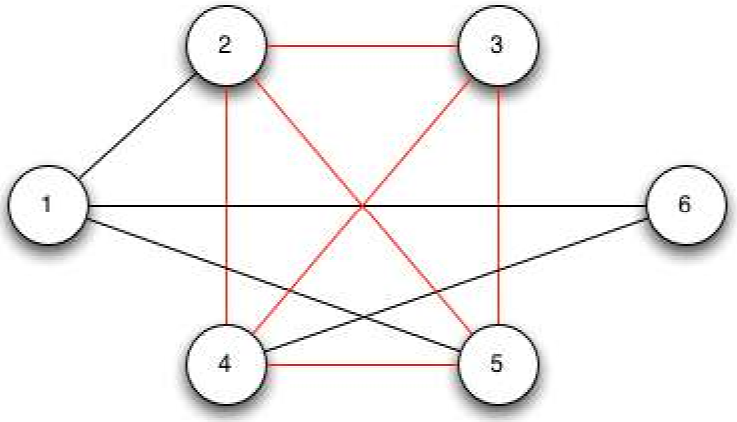
Let G^1 be the complement of a graph G such that G^1 has node cover of size atmost n-k iff G has a clique of size atleast k. The graph G^1 is given by G^1-(V,E^1) where $E^1 = \{(u,v) | u \in V, v \in V \text{ and } (u,v) \notin E\}$

We have to show that G has a clique of size atleast k iff G^1 has a node cover of size atmost n-k.

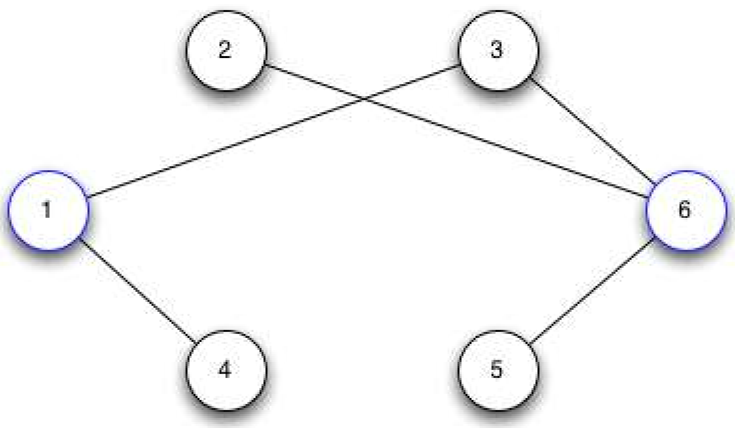
Let C be any clique of G. Since there are no edges in E^1 connecting vertices in C, the remaining n-C vertices in G^1 , must cover all edges in E^1. Similarly if S is node cover of G^1, then V-S must form a complete subgraph in G.

Since G^1 can be obtained from G in polynomial time, clique decision problem can be solved in polynomial deterministic time then we have polynomial deterministic time for node cover decision problem.

ExampleConsider the following graph (which has a clique of size 4 as shown in red)



We first construct the complement graph $\overline{G}$ giving



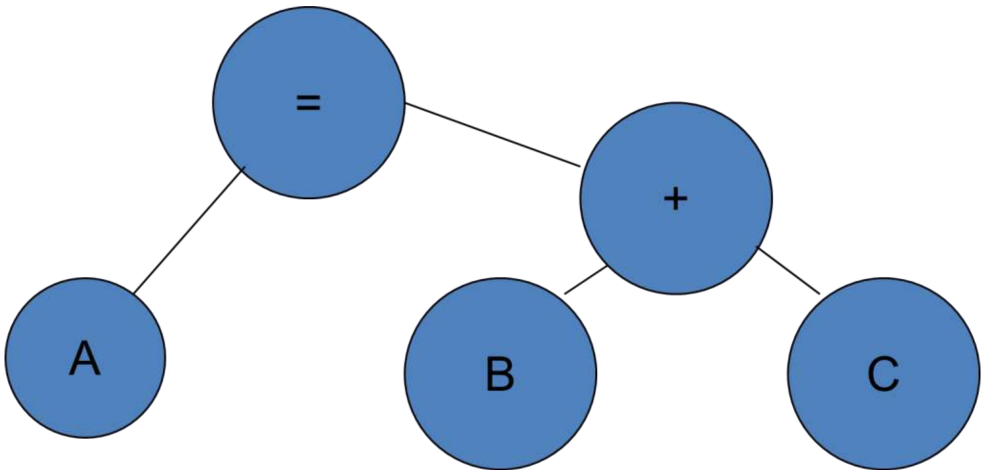
Clearly a vertex cover for $\overline{G}$ is {1,6} (shown in blue) which has size 6 - 4 = 2.

Thus if we could solve vertex cover in polynomial time we could solve clique in polynomial time (and by extension any *NP* problem in polynomial time).

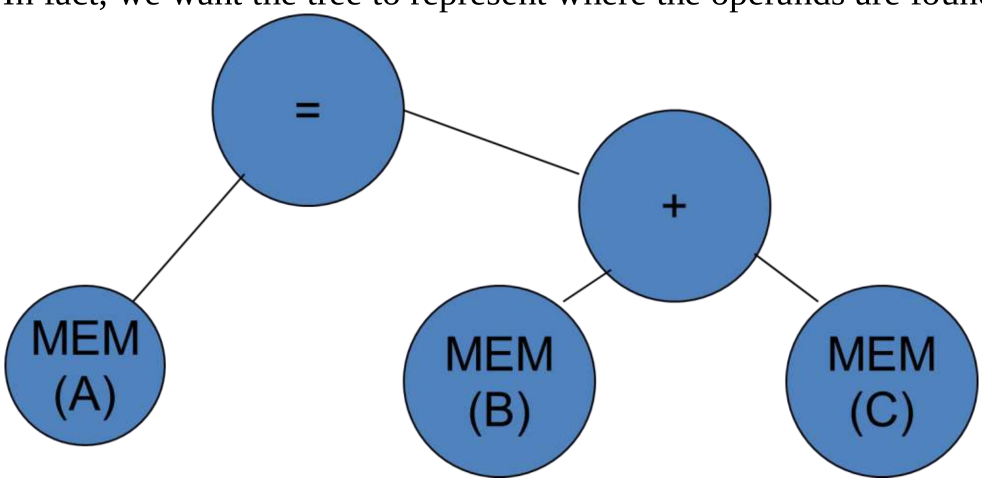
NP-Hard Code Generation Problems: Given an expression tree and a machine architecture, generate a set of instructions that evaluate the tree

- Initially, consider only trees (no common subexpressions)
- Interested in the quality of the program
- Interested in the running time of the algorithm

- Nodes represent
  - Operators (including assignment)
  - Operands (memory, registers, constants)
- No flow of control operations



In fact, we want the tree to represent where the operands are found




```
load   B,r1
load   C,r2
add    r1,r2,r1
store  A,r1
```

- A program is a sequence of instructions
- A program *computes* an expression tree if it transforms the tree according to the desired *goal*
  - Compute the tree into a register
  - Compute the tree into memory

- Compute the tree for its side-effects
    - Condition codes
    - Assignments

Code generation with common subexpressions:

A *directed acyclic graph* (DAG!) is a directed graph that contains no cycles. A rooted tree is a special kind of DAG and a DAG is a special kind of directed graph. For example, a DAG may be used to represent common subexpressions in an optimising compiler.

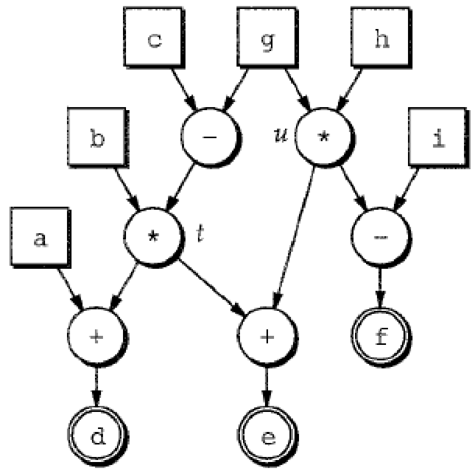
```
          +                          +
        .   .                      .   .
      .       .                  .       .
     *         ()               *<---|     ()
   . .       .   .            . .    |    .   .
  .   .     .     .          .   .   |   .    |
  a   b     f     *          a   b   |   f    |
                 . .                 ^        v
                .   .                |        |
                a   b                |--<----

       Tree                       DAG

expression: a*b+f(a*b)
```

Example of Common Subexpression.

The common subexpression a*b need only be compiled once but its value can be used twice.

```
p = c - g;
t = b * p;
d = a + t;
u = g * h;
e = t + u;
f = u - i;
```

(a)

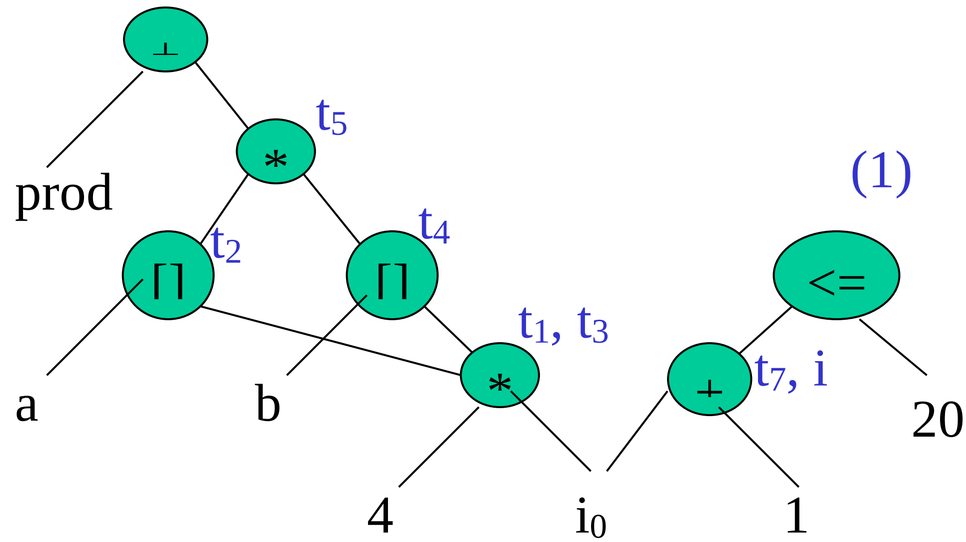


(b)

Fig. 1. Constructing an expression DAG from a basic block: (a) C code for a basic block; (b) the corresponding expression DAG, assuming p, t, and u are not live upon exit of the basic block.

Example of DAG Representation

- **t_1:= 4*i**
- **t_2:= a[t_1]**
- **t_3:= 4*i**
- **t_4:= b[t_3]**
- **t_5:= t_2 * t_4**
- **t_6:= prod + t_5**
- **prod:= t_6**



Corresponding DAG

Practice Questions:

1. What is Decision Problem?
2. What is NP-hard generation problem?
3. State Cooks theorem.
4. Define NP-hard and NP-complete.
5. Write short notes on a) Node covering problem b) NP-hard graph problems c) hard code generation problems