Unit - 4

Recurrence Relations

Generating functions of sequences

A function $a\ Z^+ \to R$ is called sequence of real numbers

We use $A = \{a_n\}_{n=0}^{\infty}$ to denote such sequences.

Example $A = \{3^n\}_{n=0}^{\infty}$ is a sequence

$$A = \{1, 3, 9, 27, 81 \cdots 3^n \cdots\}$$

Generating functions

Defn Let $A = \{a_n\}_{n=0}^{\infty} = (a_0, a_1, \cdots a_n, \cdots)$ be the sequence. Then its generating function is defined to be $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 \cdots$

Example

The generating function of the sequence $\{3^n\}_{n=0}^{\infty}$ is $A(x) = 1 + 3x + 3^2x^2 + 3^3x^3 \cdots + 3^n x^n + \cdots$

Example

Find the generating function for the following sequence $\left\{\frac{(-1)^{n-1}}{n}\right\}$

Solution ③

Sequence of $\{(-1)^{n-1}(n)\}$ is

$\{1, -2, 3, -4 \ldots\ldots\}$

$\therefore$ Generating function for the above sequence is

$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (-1)^{n-1} n \, x^n$

$A(x) = 1 - 2x + 3x^2 \cancel{+} - 4x^3 + \ldots\ldots$

closed form expressions for Generating functi

Example

Find the generating function for $a_n = 3^n, n\geq$

in closed form

Solution

The generating function for the given sequence is $A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 3^n x^n$

$A(x) = 1 + 3x + 3^2 x^2 + 3^3 x^3 \ldots$

which is a infinite geometric series with common ratio $3x$.

So by using sum of infinite terms of Geometric series we can write $\left(\frac{a}{1-r}\right)$

$$A(x) = \frac{1}{1-3x}$$

Example

Find the sequences generated by the following functions

i) $(2+x)^3$

$$= 2^3\left(1+\frac{x}{2}\right)^3 = 8\left(3C_0\, 1^3\left(\frac{x}{2}\right)^0 + 3C_1(1)^2\frac{x}{2} + 3C_2(1)^1\left(\frac{x}{2}\right)^2 + 3C_3\left(\frac{x}{2}\right)^3\right)$$

$$= 8\left(1+\frac{3}{2}x+\frac{3}{4}x^2+\frac{1}{8}x^3\right)$$

The sequence generated by $(2+x)^3$ is

8, 12, 6, 1, 0, 0, 0 - - - -

Example

Find the sequence generated by the following function $2x^2(1-x)^{-1}$

$$= 2x^2(1+x+x^2+x^3 - \cdots)$$

$$= 2x^2+2x^3+2x^4+ \cdots$$

$$= 0+0x+2x^2+2x^3 - -$$

The required sequence is

0, 0, 2, 2, 2 . . . .)

Problem

Find the closed form expression ~~for~~ for the generating function of the sequence

0, 1, 2, 3 ...

solution

Generating function is

$$0 + 1\cdot x + 2x^2 + 3x^3 \dots$$

$$x(1 + 2x + 3x^2 \dots) = x(1-x)^{-2}$$

$$\frac{x}{(1-x)^2}$$

Problem Find the closed form expression for the generating function of the sequence

$1^2, 2^2, 3^2 \dots$

Soln

~~we have~~ we have see

$$~~1+2(2x~~ x + 2x^2 + 3x^3 \dots \frac{x}{(1-x)^2} \quad \text{--- (1)}$$

diff $1 + 2(2x) + 3(3x^2) = \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right)$

$$= \frac{(1-x)^2(1) + x(2(1-x)}{(1-x)^4}$$

$$= \frac{1-x+2x}{(1-x)^3} = \frac{1+x}{(1-x)^3}$$

Example

let 'n' be a +ve integer and

$a_k = C(n, k)$ for $k = 0, 1, 2 \ldots n$. Find the generating function for the sequence

$(a_0, a_1 \ldots a_n)$

solution

The generating function for the given sequence is $\sum_{k=0}^{n} a_k x^k$

$A(x) = nC_0 + nC_1 x + nC_2 x^2 \ldots + nC_n x^n$

$= (1+x)^n$

~~calculating coefficient of generating function~~

Example

Find the generating function for the sequence $(1, 1, 1, 1, 1, 1)$ in ~~the~~ closed form

Solution

Generating function of $(1, 1, 1, 1, 1, 1)$ is

$A(x) = 1 + 1x + 1x^2 + 1x^3 + 1x^4 + 1x^5$

$= \dfrac{x^6 - 1}{x - 1}$

$\boxed{\text{By G.P} \quad \dfrac{a(r^n - 1)}{r - 1}}$

Table of Generating functions

S.NO	sequence	generating function
1.	$(1+x)^k$	$C(k,n)$
2.	1	$\frac{1}{1-x}$
3	a^n	$\frac{1}{1-ax}$
4	$(-1)^n$	$\frac{1}{1+x}$
5.	$(-1)^n a^n = (-a)^n$	$\frac{1}{1+ax}$
6.	$C(k-1+n, n)$ k is a fixed +ve integer	$\frac{1}{(1-x)^k}$
7.	$C(k-1+n, n)\, a^n$	$\frac{1}{(1-ax)^k}$
8	$C(k-1+n, n)(-a)^n$	$\frac{1}{(1+ax)^k}$
9.	$n+1$	$\frac{1}{(1-x)^2}$
10	n	$\frac{x}{(1-x)^2}$
11	n^2	$\frac{x(1+x)}{(1-x)^3}$

Calculating coefficient of Generating function

Example

Find the coefficient of x^{27} in

$(x^4+x^5+x^6 \ldots)^5$

Solution: we have $(x^4+x^5+x^6 \ldots)^5$

$= (x^4)^5 (1+x+x^2 \ldots)^5$

$= x^{20} \left[(1-x)^{-1}\right]^5$

$= x^{20} (1-x)^{-5}$

$= x^{20} \sum_{r=0}^{\infty} C(5+r-1, r) x^r$

$= x^{20} \sum_{r=0}^{\infty} C(4+r, r) x^r$

$= \sum_{r=0}^{\infty} C(4+r, r) x^{20+r}$

Take $20+r = 27$ then $r = 7$

$\therefore$ The coefficient of x^{27} is $C(4+7, 7)$

$C(11,7) = \frac{11!}{7!4!}$

$= \underline{\underline{330}}$

Example

Find the coefficient of x^{20} in

$(x+x^2+2x^3+x^4)(x^2+x^3+x^4\ldots)^5$

Solution

We have $(x+x^2+2x^3+x^4)(x^2+x^3+x^4\ldots)^5$

$$= x(1+x+2x^2+x^3)(x^2)^5(1+x+x^2\ldots)^5$$

$$= x^{11}(1+x+2x^2+x^3)\left[(1-x)^{-1}\right]^5$$

$$= x^{11}(1+x+2x^2+x^3)(1-x)^{-5}$$

$$= x^{11}(1+x+2x^2+x^3)\sum_{r=0}^{\infty} C(4+r,r)\,x^r$$

$\therefore$ Coefficient of x^{20} in $(x+x^2+2x^3+x^4)(x^2+x^3+x^4+\ldots)^5$

is $C(4+9,9)+C(4+8,8)+2\,C(4+7,7)+C(4+6,6)$

$$= C(13,9)+C(12,8)+2C(11,7)+C(10,6)$$

Example

Find the coefficient of x^{12} in

$$\frac{x^2}{(1-x)^{10}}$$

Solution

$$\frac{x^2}{(1-x)^{10}} = x^2(1-x)^{-10}$$

$$= x^2 \sum_{r=0}^{\infty} C\left(\cancel{-10+r-1,r} \atop r+9,r\right)$$

$$= x^2 \sum_{r=0}^{\infty} C(10+r-1, r)\, x^r$$

$$= x^2 \sum_{r=0}^{\infty} C(9+r, r)\, x^r$$

∴ ~~The~~

∴ The coefficient of x^{12} is $C(19,10)$

(Here $r=10$

Example ~~7~~

Find the coefficient of x^{12} in

$$\frac{x^2-3x}{(1-x)^4}$$

Solution

$$\frac{x^2-3x}{(1-x)^4} = (x^2-3x)(1-x)^{-4}$$

$$=(x^2-3x)(1-x)^{-4}$$

$$=(x^2-3x)\sum_{r=0}^{\infty} C(4+r-1, r)\, x^r$$

$$(x^2-3x)\sum_{r=0}^{\infty} C(3+r, r)\, x^r$$

• The coefficient of x^{12} is

$$\underbrace{C(13,10)}_{\text{Here } r=10} - \underbrace{3C(14,11)}_{\text{Here } r=11}$$

Find the generating function that determines the number of non-negative integer solutions of the equations $e_1 + e_2 + e_3 + e_4 + e_5 = 20$ under the constraints $0 \le e_1 \le 3$, $0 \le e_2 \le 4$, $2 \le e_3 \le 6$, $2 \le e_4 \le 5$, e_5 is odd with $e_5 \le 9$

solution

$$f_1(x) = x^0 + x^1 + x^2 + x^3$$

$$f_2(x) = x^0 + x^1 + x^2 + x^3 + x^4$$

$$f_3(x) = x^2 + x^3 + x^4 + x^5 + x^6$$

$$f_4(x) = x^2 + x^3 + x^4 + x^5$$

$$f_5(x) = x^1 + x^3 + x^5 + x^7 + x^9$$

Sum of n terms of a Geometric series

$$= \frac{a(1-r^n)}{1-r}$$

Hence the generating function for the number of non-negative integer solutions of the given equation under the given constraints is

$$f(x) = f_1(x) f_2(x) f_3(x) f_4(x) f_5(x)$$

$$\frac{1-x^4}{1-x} \frac{1-x^5}{1-x} x^2 \frac{1-x^5}{1-x} x^2 \frac{1-x^4}{1-x} x(1+x^2+x^4+x^6+x^8)$$

$$x^5 (1-x^4)^2 (1-x^5)^2 (1-x)^{-4} \frac{(1-(x^2)^5)}{1-x^2}$$

$$x^5(1-x^4)^2(1-x^5)^2(1-x)^{-5}(1+x)^{-1}(1-x^{10})$$

2. Build a generating function for a_r = the number of integral solutions to the equation $e_1+e_2+e_3=r$ if $0<e_i$ for each i

Solution

$0<e_i$ means $1\le e_i$

Generating function is $(x+x^2+x^3 \dots)^3$

$$x^3(1+x+x^2 \dots)^3$$

$$x^3\left[(1-x)^{-1}\right]^3$$

$$\underline{x^3\left[1-x\right]^{-3}}$$

3. Find a generating function for a_r = the number of ways the sum 'r' can be obtained when (1) 2 distinguishable dice are tossed and (2) 2 distinguishable dice are tossed and the first shows an even number and the second shows an odd number

Solution

i) We want the number of integral solutions $e_1 + e_2 = r$ where $1 \leq e_i \leq 6$

Then a_r is the coefficient of x^r in the generating function $(x + x^2 + x^3 + x^4 + x^5 + x^6)^2$

ii) We want the number of integral solutions to $e_1 + e_2 = r$ $1 \leq e_1 \leq 6$ and e_1 is even while $1 \leq e_2 \leq 6$ and e_2 is odd

Then a_r is the coefficient of x^r in the generating function $(x^2 + x^4 + x^6)(x + x^3 + x^5)$

Problem

In how many ways can we distribute 24 pens to 4 children so that each child gets at least 3 pens but not more than eight

solution

The given problem is equivalent to the problem of finding the number of integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 24$ with $3 \leq x_i \leq 8$ for each x_i, Keeping the constraints on x_i in mind, let us consider the function $f_i(x) = x^3 + x^4 + x^5 + x^6 + x^7 + x^8$

$i = 1, 2, 3, 4$

There fore, the generating function for the problem is

$$f(x) = f_1(x) f_2(x) f_3(x) f_4(x)$$

$$(x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^4$$

The required number is the coefficient of x^{24} in this function

Now we find that $f(x) = x^{12}(1 + x + x^2 + x^3 + x^4 + x^5)^4$

$$x^{12}\left(\frac{1-x^6}{1-x}\right)^4$$

$(1-x)^{-n} = \sum_{r=0}^{\infty} C(n+r-1, r)\, x^r$

$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$

$$= x^{12}(1-x^6)^4(1-x)^{-4}$$

$$x^{12} \sum_{r=0}^{4} C(4, r)(-x^6)^r \sum_{s=0}^{\infty} C(4+s-1, s)\, x^s$$

$$x^{12}\left[\sum_{r=0}^{4} (-1)^r C(4, r)\, x^{6r} \cdot \sum_{s=0}^{\infty} C(3+s, s)\, x^s\right.$$

from this we find the coefficient of x^{24} in $f(x)$

$r=0, s=12$

is $C(4,0)\, C(15,12) - C(4,1)\, C(9,6) +$

$C(4,2)\, C(3,0)$

$= 455 - 336 + 6 = \underline{\underline{125}}$

Recurrence relations

Defn A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the ~~form~~ previous terms of the sequence, namely $a_0, a_1, \ldots a_{n-1}$ for all integers $n \geq 1$

Eg: $a_n - 3a_{n-1} + 2a_{n-2} = 0$ — Homogeneous recurrence relation

Eg. $a_n - 5a_{n-1} + 6a_{n-2} = n^2+1$ — in homogeneous recurrence relation

The Fibonacci Recurrence relation

The recurrence relation $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$ with initial conditions $F_0 = F_1 = 1$ is known as Fibonacci recurrence relation

Solutions of recurrence relations

A sequence $\{a_n\}_{n=0}^{\infty}$ is said to be a solution of a recurrence relation of each value a_n (ie,) $a_0, a_1 \ldots a_n \ldots$ satisfies the recurrence relation.

Ex $\{a_n\}_{n=0}^{\infty}$ when $a_n = 2^n$ is the solution of the recurrence relation $a_n = 2a_{n-1}$, $n \geq 1$

Methods of solving recurrence relation:

Substitution method

1. Solve the recurrence relation $a_n = a_{n-1} + f(n)$ for $n \geq 1$ by substitution method

Solution

Here $a_1 = a_0 + f(1)$

$a_2 = a_1 + f(2) = a_0 + f(1) + f(2)$

$a_3 = a_2 + f(3) = a_0 + f(1) + f(2) + f(3)$

$\vdots$

$a_n = a_{n-1} + f(n) = a_0 + f(1) + f(2) \dots + f(n)$

$= a_0 + \sum_{k=1}^{n} f(k)$

Q. solve the recurrence relation

$a_n = a_{n-1} + n^2$, where $a_0 = 7$ by substitution method

Solution

given

$a_0 = 7$

$a_1 = a_0 + 1^2 = 7 + 1^2$

$a_2 = a_1 + 2^2 = 7 + 1^2 + 2^2$

$a_3 = a_2 + 3^2 = 7 + 1^2 + 2^2 + 3^2$

$\vdots$

$a_n = 7 + (1^2 + 2^2 + 3^2 \dots + n^2)$

$= 7 + \frac{n(n+1)(2n+1)}{6}$

3. solve

$a_n = a_{n-1} + 3^n$ where $a_0 = 1$

solution

$a_0 = 1$

$a_1 = a_0 + 3^1 = 1 + 3$

$a_2 = a_1 + 3^2 = 1 + 3 + 3^2$

$a_3 = a_2 + 3^3 = 1 + 3 + 3^2 + 3^3$ $\left(S_n = a\left(\frac{r^{n+1}-1}{r-1}\right)\right)$

$\vdots$

$a_n = 1 + 3 + 3^2 \cdots + 3^n$

$a_n = \frac{3^{n+1}-1}{3-1} = \frac{3^{n+1}-1}{2}$ is the solution

4. If a_n is a solution of $a_{n+1} = k a_n$ for $n \geq 0$ and $a_3 = \frac{153}{49}$ and $a_5 = \frac{1377}{2401}$ find the value of k.

solution

The general solution is $a_{n+1} = k a_n$ is

That is $a_n = k^n a_0$; $n \geq 1$

Here $a_3 = k^3 a_0 \Rightarrow k^3 a_0 = \frac{153}{49}$

$a_5 = k^5 a_0 \Rightarrow k^5 a_0 = \frac{1377}{2401}$

$\frac{k^5 a_0}{k^3 a_0} = \frac{1377}{2401} \times \frac{49}{153} = \frac{9}{49}$

$$K^2 = \frac{9}{49} \Rightarrow k = \pm \frac{3}{7}$$

Method - 2

Method of Generating functions

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

1. $\sum_{n=k}^{\infty} a_n x^n = A(x) - a_0 - a_1 x \; - - - \; - a_{k-1} x^{k-1}$

2. $\sum_{n=k}^{\infty} a_{n-1} x^n = x\left[A(x) - a_0 - a_1 x \; - - \; - a_{k-2} x^{k-2}\right]$

3. $\sum_{n=k}^{\infty} a_{n-2} x^n = x^2\left[A(x) - a_0 - a_1 x \; - - \; - a_{k-3} x^{k-3}\right]$

4. $\sum_{n=k}^{\infty} a_{n-3} x^n = x^3\left[A(x) - a_0 - a_1 x \; - - \; - a_{k-4} x^{k-4}\right]$

. . . .

5. $\sum_{n=k}^{\infty} a_{n-k} x^n = x^K\left[A(x)\right]$

where $A(x)$ is called a generating function for a given recurrence relation.

Q(1) solve the recurrence relation using generating function

$$a_n - 7a_{n-1} + 10a_{n-2} = 0 \quad \text{for } n \geq 2$$

$$a_0 = 10, \; a_1 = 41$$

Solution

step 1. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

consider the given recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$

step 2. Next multiply each term in the recurrence relation by x^n and sum from 2 to ∞

$$\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1} x^n + 10\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

step 3. Replace each infinite sum by an expression from the equivalent expressions:

$$\left[A(x) - a_0 - a_1 x\right] - 7x\left[A(x) - a_0\right] + 10x^2\left[A(x)\right] = 0$$

step 4. Simplify

$$A(x)\left[1 - 7x + 10x^2\right) = a_0 + a_1 x - 7a_0 x$$

$$A(x) = \frac{a_0 + a_1 x - 7a_0 x}{1 - 7x + 10x} = \frac{a_0 + x(a_1 - 7a_0)}{(1-2x)(1-5x)}$$

Given $a_0 = 10$
$a_1 = 41$

$$= \frac{10 + x(41 - 70)}{(1-2x)(1-5x)}$$

step 5. Decompose $A(x)$ as a sum of partial fractions:

$$A(x) = \frac{10 - x(29)}{(1-2x)(1-5x)}$$

using :

$$\frac{10 - 29x}{(1-2x)(1-5x)} = \frac{A}{1-2x} + \frac{B}{1-5x}$$

$$10 - 29x = A(1-5x) + B(1-2x) \quad \text{—— (1)}$$

put $x = \frac{1}{5}$ in (1)

$$10 - 29\left(\frac{1}{5}\right) = A(0) + B\left(1 - 2\left(\frac{1}{5}\right)\right)$$

$$\frac{50 - 29}{5} = \frac{3}{5}B$$

$$21 = 3B$$

$$B = \underline{\underline{7}}$$

put $x = \frac{1}{2}$ in (1)

$$10 - 29\left(\frac{1}{2}\right) = A\left(1 - \frac{5}{2}\right) + B(0)$$

$$-\frac{9}{2} = \frac{-3}{2}A$$

$$9 = 3A \qquad A = \underline{\underline{3}}$$

step 6 Express $A(x)$ as a sum of series

$$A(x) = \frac{3}{1-2x} + \frac{7}{1-5x}$$

$$= 3\sum_{n=0}^{\infty} 2^n x^n + 7\sum_{n=0}^{\infty} 5^n x^n$$

21

step 7 Express a_n as the coefficient of x^n in $A(x)$ ~~and in the sum of the other series~~:

$$a_n = (3)2^n + (-1)5^n$$

Q2 Using generating functions solve the recurrence relation $a_n - 4a_{n-1} + 3a_{n-2} = 0, n \geq 2$ with initial conditions $a_0 = 2$ and $a_1 = 4$

Solution

step 1. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

step 2

consider the given recurrence relation

$$a_n - 4a_{n-1} + 3a_{n-2} = 0 \quad \text{—} (1)$$

Multiply each term in (1) by x^n and sum from 2 to ∞, we get

$$\sum_{n=2}^{\infty} a_n x^n - 4\sum_{n=2}^{\infty} a_{n-1} x^n + 3\sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

step 3 Replace each infinite sum by an expression from the equivalent expressions

$$[A(x) - a_0 - a_1 x] - 4x[A(x) - a_0] + 3x^2[A(x)] = 0$$

step 4 simplify

$$A(x)[1 - 4x + 3x^2] = a_0 + a_1 x - 4a_0 x$$

$$A(x) = \frac{a_0 + a_1 x - 4a_0 x}{1 - 4x + 3x^2}$$

Given $a_0 = 2$, $a_1 = 4$

$$\therefore A(x) = \frac{2 + 4x - 8x}{1 - 4x + 3x^2} = \frac{2 - 4x}{(1-x)(1-3x)}$$

Step 5 Decompose $A(x)$ as a sum of partial fractions

$$A(x) = \frac{1}{1-x} + \frac{1}{1-3x}$$

Step 6 Express $A(x)$ as sum of infinite series

$$A(x) = (1-x)^{-1} + (1-3x)^{-1}$$

$$= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (3x)^n$$

Step 7 ~~A~~ Express a_n as the coefficient of x^n

$$a_n = 1^n + 3^n$$

Q.1 Solve $a_n - 3a_{n-1} \overline{\cancel{+}} 2 = 0$, $n \geq 1$ with $a_0 = 1$ by generating functions

Solution

Given recurrence relation is

$\cancel{a_n - 3a}$ $a_n - 3a_{n-1} \overline{\cancel{+}} 2 = 0$ ——— (1)

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function

Multiply each term in (1) by x^n and sum from 1 to ∞, we get

$$\sum_{n=1}^{\infty} a_n x^n - 3 \sum_{n=1}^{\infty} a_{n-1} x^n \overline{\cancel{+}} 2 \sum_{n=1}^{\infty} x^n$$

Replace the infinite sums by equivalent expressions we get

$$\left[A(x) - a_0\right] - 3x\left[A(x)\right] \overline{\cancel{+}} 2x \sum_{n=1}^{\infty} x^{n-1}$$

$$\Rightarrow A(x)\left[1 - 3x\right] = a_0 + 2x(1-x)^{-1}$$

given $a_0 = 1$

$$A(x)(1-3x) = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}.$$

$$A(x) = \frac{1+x}{(1-x)(1-3x)} = \frac{-1}{1-x} + \frac{2}{1-3x}$$

$$= -(1-x)^{-1} + 2(1-3x)^{-1}$$

$$= -\sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty}(3x)^n$$

$$\therefore \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left[(-1) + 2\cdot 3^n\right]x^n$$

$$a_n = -1 + 2(3)^n \quad \text{required solution}$$

Q 2) Solve the recurrence relation

$a_n - 6a_{n-1} = 0$ for $n \geq 1$ and $a_0 = 1$

by using generating functions

Solution

Given recurrence relation is

$$a_n - 6a_{n-1} = 0 \quad \text{--- (1)}$$

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function

Multiply each term in (1) by x^n and sum from 1 to ∞

we get $\sum_{n=1}^{\infty} a_n x^n - 6\sum_{n=1}^{\infty} a_{n-1} x^n = 0$

Replace the infinite sum by equivalent expressions we get

$$\left[A(x) - a_0\right] - 6x\left[A(x)\right] = 0$$

$$A(x)\left[1-6x\right] = a_0$$

given $a_0 = 1$

$$A(x) = \frac{1}{1-6x} = (1-6x)^{-1}$$

$$= \sum_{n=0}^{\infty} (6x)^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 6^n x^n$$

$\therefore a_n = 6^n$ which is the required soluh

(Q3) solve the recurrence relation

$$a_n - 9a_{n-1} + 20a_{n-2} = 0 \text{ for } n \geq 2$$

given that $a_0 = -3$, $a_1 = -10$.

Ans $2 \cdot 5^n - 5 \cdot 4^n$

Method of characteristic roots

Defn

Let $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0$,

$n \geq k$, $c_k \neq 0$ ——— (1)

be a linear homogeneous recurrence relation of degree k. Then the equation $t^k + c_1 t^{k-1} + c_2 t^{k-2} \cdots + c_k = 0$ — (2)

is called the characteristic equation of the given recurrence relation (1).

Degree of (2) is k, therefore it has 'k' roots

Let $\alpha_1, \alpha_2 \cdots \alpha_k$ be the roots of the equation (2)

and the roots $\alpha_1, \alpha_2 \cdots \alpha_k$ are called characteristic roots.

There are two types of characteristic roots

1. If the characteristic equation of a linear homogeneous recurrence relation of degree k has 'k' distinct roots say $\alpha_1, \alpha_2 \cdots \alpha_k$ then

$a_n = c_1 \alpha_1^n + c_2 \alpha_2^n + \cdots + c_k \alpha_k^n$

Where $c_1, c_2 \cdots c_k$ are constants, is the general solution of the given recurrence relation

2. If the characteristic equation of a linear homogeneous recurrence relation of degree k has a root 'α' repeated k times then

$$a_n = (D_1 + D_2 n + D_3 n^2 \ldots D_k n^{k-1}) \alpha^n$$

where $D_1, D_2 \ldots D_k$ are constants, is the general solution of the given recurrence relation

Example 1 solve $a_n - 3a_{n-1} + 2a_{n-2} = 0$ for $n \geq 2$

Solution

The characteristic equation is

$t^2 - 3t + 2 = 0$

$(t-1)(t-2) = 0$

$t = 1, 2$

The general solution is $a_n = c_1 1^n + c_2 2^n$

Example 2

Solve ~~$a_n - 3a_n -$~~

$a_n - 6a_{n-1} + 9a_{n-2} = 0$

Solution

The characteristic equation is

$t^2 - 6t + 9 = 0$

$(t-3)^2 = 0$

$t = 3, 3$

The general solution is $a_n = (D_1 + D_2 n) 3^n$

Example

Write the general form of the solution to

$a_n - 3a_{n-1} + 3a_{n-2} - a_{n-3} = 0$

Solution characteristic equation is

$$t^3 - 3t^2 + 3t - 1 = 0$$

$$(t-1)^3 = 0$$

$$t = 1, 1, 1$$

The general solution is

$$a_n = \left(D_1 + D_2 n + D_3 n^2\right) 1^n$$

Example

solve the recurrence relation

$a_n - 7a_{n-1} + 16a_{n-2} - 12a_{n-3} = 0$ for

$n \geq 3$ with the initial conditions $a_0 = 1, a_1 = 4, a_2 = 8$

Solution The characteristic equation is

$$t^3 - 7t^2 + 16t - 12 = 0$$

$$f(2) = 0$$

$(t-2)$ is a factor of $f(t)$

$$\begin{array}{r|rrrr} 2 & 1 & -7 & 16 & -12 \\ & 0 & 2 & -10 & 12 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$f(t) = (t-2)(t^2 - 5t + 6)$$

$$(t-2)^2(t-3) \qquad t = 2, 2, 3$$

The general solution is

$$~~a_n = D_1 2^n + D_2 n 2^n~~$$

$$a_n = (D_1 + D_2 n) 2^n + D_3 3^n \longrightarrow (1)$$

put $n=0$ in (1)

$$a_0 = D_1 + D_3 \longrightarrow (2)$$

put $n=1$ $\quad a_1 = (D_1 + D_2) 2 + D_3 (3) \longrightarrow (3)$

put $n=2$

$$a_2 = (D_1 + 2D_2) 2^2 + D_3 (3^2)$$

$$= (D_1 + 2D_2) 4 + 9 D_3 \longrightarrow (4)$$

Solve (2), (3) & (4) we get

$$D_1 = 5, \quad D_2 = 3 \quad D_3 = -4$$

Hence the unique solution of the recurrence relation is

$$a_n = (5) 2^n + 3(n 2^n) - 4(3^n)$$

Example

Find the characteristic polynomial, characteristic equation for the homogeneous recurrence relations whose general solution has the form

1) $a_n = B_1 + n B_2$

2) $a_n = B_1 + n B_2 + n^2 B_3$

3) $a_n = B_1 2^n + B_2 3^n$

4) $a_n = B_1 2^n + B_2 n 2^n$

Solution

1) Given general solution is

$$a_n = B_1 + n B_2$$
$$= B_1 (1)^n + n B_2 (1)^n$$

characteristic polynomial is $(t-1)^2$

characteristic equation is $(t-1)^2 = 0$

2) Given general solution is

$$a_n = B_1 + n B_2 + n^2 B_n$$
$$= B_1 1^n + n B_2 1^n + n^2 B_n 1^n$$

Characteristic polynomial is $(t-1)^3$

Characteristic equation is $(t-1)^3 = 0$

3) Given general solution is $a_n = B_1 2^n + B_2 3^n$

characteristic polynomial is $(t-2)(t-3) = t^2 - 5t +$

characteristic equation is $t^2 - 5t + 6 = 0$

4) Given general solution is

$$a_n = B_1 2^n + B_2 n 2^n$$

characteristic polynomial is $(t-2)^2$

characteristic equation is $(t-2)^2=0$

Example

solve the Fibonacci relation

$a_n = a_{n-1} + a_{n-2}$ with $a_0 = 0$, $a_1 = 1$ as

initial conditions

Solution

The characteristic equation is

$$a_n - a_{n-1} - a_{n-2} = 0$$

$$t^2 - t - 1 = 0$$

$$t = \frac{1 \pm \sqrt{5}}{2}$$

The complete solution is

$$a_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{--- (1)}$$

put $n=0$ in (1)

$$a_0 = c_1 + c_2 \quad \text{--- (2)}$$

put $n=1$ in (1)

$$a_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) \quad \text{--- (3)}$$

solving (2) and (3) $c_1 = \frac{1}{\sqrt{5}}$, $c_2 = \frac{-1}{\sqrt{5}}$

"complete" solution of the Fibonacci relation is given by

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Example

solve $a_n = 2(a_{n-1} - a_{n-2})$ for $n \geq 2$

with $a_0 = 1$ and $a_1 = 2$

solution

The characteristic equation is

$$a_n - 2a_{n-1} + 2a_{n-2} = 0$$

$$t^2 - 2t + 2 = 0$$

$$t = 1 \pm i$$

The general solution is

$$a_n = r^n \left(c_1 \cos n\theta + c_2 \sin n\theta\right)$$

where c_1, c_2 are arbitrary constants

$$r = |1 \pm i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$a_n = (\sqrt{2})^n \left[c_1 \cos \frac{n\pi}{4} + c_2 \sin \frac{n\pi}{4}\right] \quad \text{--- (1)}$$

put $n=0$

$$a_0 = (\sqrt{2})^0 [c_1 \cos 0 + c_2 \sin 0$$

$$\underline{\underline{1 = c_1}}$$

put $n=1$

$$a_1 = (\sqrt{2})^1 \left[c_1 \cos \frac{\pi}{4} + c_2 \sin \frac{\pi}{4}\right]$$

$$2 = \sqrt{2}\left[1\left(\frac{1}{\sqrt{2}}\right) + c_2\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$2 = 1 + c_2 \quad \Rightarrow c_2 = 1$$

The general solution is

$$a_n = (\sqrt{2})^n \left[\cos \frac{n\pi}{4} + \sin \frac{n\pi}{4}\right] \qquad n \geq 2$$

Solution of in homogeneous linear recurrence relations

$a_0 + c_1 a_{n-1} + \cdots + c_k a_{n-k} = f(n)$ for $n \geq k$
where $c_k \neq 0$ and $f(n)$ is a specified function of n

~~Ex~~ ~~1~~ Characteristic roots Method

The solution of a linear inhomogeneous recurrence relation with constant coefficients is the sum of the two parts. These are homogeneous solution and particular solution.

A solution, which satisfies the recurrence relation when the right-hand side of the ~~equation~~ relation is set to zero is called homogeneous solution.

A solution which satisfies the recurrence relation with $f(n)$ on the right hand side is called particular solution. The homogeneous solution is denoted by $a_n^{(h)}$ and the particular solution is denoted by $a_n^{(p)}$.

Follow the same procedure as in solving homogeneous recurrence relations for determining the homogeneous solution.

~~There~~ is To determine the particular solution use the following rules based on the nature of $f(n)$.

Rule 1 If $f(n)$ is of the form of a polynomial of degree m in n (i) $b_0 + b_1 n + b_2 n^2 - - + b_{m-1} n^{m-1} + b_m n^m$

Then particular solution will be of the form:

$$Q_0 + Q_1 n + Q_2 n^2 \ldots + Q_{m-1} n^{m-1} + Q_m n^m$$

provided one is not a characteristic root of the recurrence relation

Rule 2

If $f(n)$ is of the form

$$(b_0 + b_1 n + b_2 n^2 + \ldots + b_{m-1} n^{m-1} + b_m n^m) a^n$$

then particular solution is of the form

$$(Q_0 + Q_1 n + Q_2 n^2 \ldots + Q_{m-1} n^{m-1} + Q_m n^m) a^n$$

when 'a' is not a characteristic root of the recurrence relation

Rule 3

If 'a' is the characteristic root of the multiplicity $(r-1)$ when $f(n)$ is of the form $(b_0 + b_1 n + b_2 n^2 \ldots + b_{m-1} n^{m-1} + b_m n^m) a^n$, then particular solution is of the form

$$n^{r-1}(Q_0 + Q_1 n + Q_2 n^2 \ldots + Q_{m-1} n^{m-1} + Q_m n^m) a^n$$

→ General solution = Homogeneous solution + particular solution

$$a_n = a_n^{(h)} + a_n^{(p)}$$

Q) solve the recurrence relation

$$a_n - 9a_{n-1} + 20a_{n-2} = 1$$

Solution

Given recurrence relation is

$$a_n - 9a_{n-1} + 20a_{n-2} = 1 \quad \text{--- (1)}$$

Consider homogeneous relation

$$a_n - 9a_{n-1} + 20a_{n-2} = 0$$

The characteristic ~~eq~~ equation is $t^2 - 9t + 20 = 0$

$$t = 4, 5$$

$a_n^{(h)} = C_1 4^n + C_2 5^n$, Here one is not a characteristic root

Since the right side $f(n) = 1$ is a constant, by rule 1, the particular solution will also be a constant, say Q.

Substituting Q in (1) we get

$$Q - 9Q + 20Q = 1$$

$$Q = \frac{1}{12}$$

$$a_n^{(p)} = \frac{1}{12}$$

Hence the general solution

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = c_1 4^n + c_2 5^n + \frac{1}{12}$$

Q Solve $a_n - a_{n-1} - 6a_{n-2} = -30$ with $a_0 = 20$, $a_1 = -5$ —— (1)

Solution

consider homogeneous relation $a_n - a_{n-1} - 6a_{n-2} = 0$ —— (2)

The characteristic equation $t^2 - t - 6 = 0$

$$(t-3)(t+2) = 0 \quad t = -2, 3$$

$a_n^{(h)} = c_1(-2)^n + c_2(3)^n$, Here 1 is not a characteristic root

since the right side $f(n) = -30$ is a constant, by rule 1, the particular solution will also be a constant say Q.

~~Put $a_n = Q$~~ substituting Q in (1)

$$Q - Q - 6Q = -30$$

$$Q = 5$$

$$a_n^{(p)} = 5$$

General solution $a_n = a_n^{(h)} + a_n^{(p)}$

$$= c_1(-2)^n + c_2(3)^n + 5$$

—— (2)

put $n=0$ in ②

$$a_0 = c_1(-2)^0 + c_2(3)^0 + 5$$

$$20 = c_1 + c_2 + 5$$

$$c_1 + c_2 = 15 \longrightarrow ③$$

put $n=1$ in ②

$$a_1 = c_1(-2)^1 + c_2(3)^1 + 5$$

$$-5 = -2c_1 + 3c_2 + 5$$

$$-2c_1 + 3c_2 = -10 \longrightarrow ④$$

solving ③ and ④ $\quad c_1 = 11,\ c_2 = 4$

Complete solution is $a_n = 11(-2)^n + 4(3)^n + 5$

Problem

solve $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$

Solution

Given recurrence relation is

$$a_n - 7a_{n-1} + 10a_{n-2} = 4^n \longrightarrow ①$$

Homogeneous relation is $a_n - 7a_{n-1} + 10a_{n-2} = 0$

The characteristic equation is $t^2 - 7t + 10 = 0$

$$(t-2)(t-5) = 0$$

$$t = 2, 5$$

$$a_n^{(h)} = c_1 2^n + c_2 5^n$$

Since 4 is not a characteristic root

by Rule 2, the particular solution is $Q_0 4^n$

Substituting in ①, we get

$$Q_0 4^n - 7Q_0 4^{n-1} + 10 Q_0 4^{n-2} = 4^n$$

$$4^{n-2} Q_0 \left[4^2 - 7\times 4 + 10 \right] = 4^n$$

$$-2Q_0 4^{n-2} = 4^n$$

$$Q_0 = -8$$

$$a_n^{(p)} = -8 \, 4^n$$

Hence general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$c_1 2^n + c_2 5^n + (-8) 4^n$$

Problem

solve the recurrence relation

$$a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n \quad \text{with } a_0 = 1, a_1 = 2 \quad —①$$

Solution

Given recurrence relation is

$$a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n \quad — ①$$

Homogeneous relation is $a_n - 7a_{n-1} + 10a_{n-2} = 0$

The characteristic equation is $t^2 - 7t + 10 = 0$

$$(t-2)(t-5) = 0 \quad t = 2, 5$$

$$a_n^{(h)} = c_1 2^n + c_2 5^n$$

By Rule 1, the particular solution is of the form

$Q_0 + Q_1 n$

substituting this solution in (1), we get-

$$(Q_0 + Q_1 n) - 7(Q_0 + Q_1(n-1)) + 10(Q_0 + Q_1(n-2)) = 6 + 8n$$

$$\Rightarrow (4Q_0 - 13Q_1) + 4Q_1 n = 6 + 8n$$

By comparing like power coefficients on both sides, we get-

$$4Q_0 - 13Q_1 = 6$$

$$4Q_1 = 8$$

$$Q_1 = 2, Q_0 = 8$$

$$a_n^{(p)} = 8 + 2n$$

Then general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = C_1 2^n + C_2 5^n + 8 + 2n \quad -(2)$$

put $n = 0$ in (2) we get-

$$a_0 = C_1 + C_2 + 8$$

$$\Rightarrow 1 = C_1 + C_2 + 8$$

$$C_1 + C_2 = -7$$

put $n = 1$ in (2) we get-

$$a_1 = 2C_1 + 5C_2 + 10$$

$$\Rightarrow 2 = 2C_1 + 5C_2 + 10$$

$$2C_1 + 5C_2 = -8$$

By solving these two equations, we get-

$$C_1 = -9, \quad C_2 = 2$$

Hence Complete solution is $a_n = -9(2)^n + 2(5)^n + 8 + 2n$

Problem

Solve the recurring relation
$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$

Solution Given recurrence relation is

$$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1 \quad \text{---} (1)$$

The homogeneous relation is $a_n + 5a_{n-1} + 6a_{n-2} = 0$

The characteristic equation is $t^2 + 5t + 6 = 0$

$$(t+2)(t+3) = 0$$

$$t = -2, -3$$

$$\therefore a_n^{(h)} = c_1(-2)^n + c_2(-3)^n$$ Here '1' is not a

characteristic root.

By rule 1, The particular solution is form

$Q_0 + Q_1 n + Q_2 n^2$

substituting in (1)

$$(Q_0 + Q_1 n + Q_2 n^2) + 5(Q_0 + Q_1(n-1) + Q_2(n-1)^2) + 6(Q_0 + Q_1(n-2) + Q_2(n-2)^2) = 3n^2 - 2n + 1$$

simplifies

$$(12Q_0 - 17Q_1 + 29Q_2) + (12Q_1 - 34Q_2)n + 12Q_2 n^2 = 3n^2 - 2n + 1$$

Comparing like power coefficients on both sides we get

$\therefore \quad 12Q_2 = 3$

$12Q_1 - 34Q_2 = -2$

$12Q_0 - 17Q_1 + 29Q_2 = 1$

By solving we get $Q_1 = \frac{13}{24}$, $Q_2 = \frac{1}{4}$, $Q_0 = \frac{71}{288}$

$\therefore \quad a_n^{(p)} = \frac{71}{288} + \frac{13}{24} n + \frac{1}{4} n^2$

Hence complete solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$= c_1(-2)^n + c_2(-3)^n + \frac{71}{288} + \frac{13}{24}n + \frac{n^2}{4}$$

Q Solve the recurrence relation

$$a_n + a_{n-1} = 3n\, 2^n$$

Solution

Given relation is $a_n + a_{n-1} = 3n \cdot 2^n$

Homogeneous relation is $a_n + a_{n-1} = 0$

The characteristic equation $t + 1 = 0$

$t = -1$

$a_n^{(h)} = c_1 (-1)^n$

Since 2 is not a characteristic root, by Rule 2, the particular solution is $(Q_0 + Q_1 n)\, 2^n$.

43

Substituting this solution in ①, we obtain

$$(a_0 + a_1 n) 2^n + (a_0 + a_1(n-1)) 2^{n-1} = 3n2^n$$

which simplifies to $\left(\frac{3}{2}a_0 - \frac{1}{2}a_1\right)2^n + \frac{3}{2}a_1 n 2^n = 3n2^n$

Comparing both sides we get

$$\frac{3}{2}a_1 = 3 \quad \frac{3}{2}a_0 - \frac{1}{2}a_1 = 0$$

$$a_0 = \frac{2}{3}, \quad a_1 = 2$$

$$\Rightarrow a_n^{(p)} = \left(\frac{2}{3} + 2n\right)2^n$$

Hence complete solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$= c_1(-1)^n + \left(\frac{2}{3} + 2n\right)2^n$$

⑨ Solve $a_r - 10a_{r-1} + 25a_{r-2} = 2^r$

Solution

Given recurrence relation is

$$a_r - 10a_{r-1} + 25a_{r-2} = 2^r \quad — ①$$

Homogeneous ~~solution is~~ relation is

$$a_r - 10a_{r-1} + 25a_{r-2} = 0$$

The characteristic equation is $t^2 - 10t + 25 = 0$

$$(t-5)^2 = 0 \quad t = 5, 5$$

$$a_r^{(h)} = (c_1 + c_2 r)5^r$$

Hence since 2 is not characteristic root, by rule 2, the particular solution of the form

$a_r^{(p)} = Q\, 2^r$

substituting this solution in (1), we get

$$Q\, 2^r - 10Q\, 2^{r-1} + 25Q\, 2^{r-2} = 2^r$$

$$\Rightarrow 2^r\left[Q - \frac{10Q}{2} + \frac{25Q}{4}\right] = 2^r$$

$$\frac{4Q - 20Q + 25Q}{4} = 1$$

$$9Q = 4 \Rightarrow Q = \frac{4}{9}$$

$$a_r^{(p)} = \frac{4}{9}\, 2^r$$

Hence general solution is $a_r = a_r^{(h)} + a_r^{(p)}$

$$= (c_1 + c_2 r)\, 5^r + \frac{4}{9}\, 2^r$$

Problem

Find the general solution of

$$a_n - 3a_{n-1} - 4a_{n-2} = 4^n$$

Solution

Given recurrence relation is

$$a_n - 3a_{n-1} - 4a_{n-2} = 4^n$$

Homogeneous relation $a_n - 3a_{n-1} - 4a_{n-2} = 0$

The characteristic equation is $t^2 - 3t - 4 = 0$

$$(t+1)(t-4) = 0 \Rightarrow t = -1, 4$$

$$a_n^{(h)} = C_1(-1)^n + C_2(4)^n$$

Since 4 is a characteristic root, by Rule 3, the multiplicity 1 particular solution is $Q\, n\, 4^n$

Substituting this solution in (1), we get

$$Q n 4^n - 3Q(n-1)4^{n-1} - 4Q(n-2)4^{n-2} = 4^n$$

$$Q n 4^n - 3Q n 4^{n-1} + 3Q 4^{n-1} - 4Q n 4^{n-2} + 8Q 4^{n-2} = 4^n$$

$$(16Qn - 12Qn + 12Q - 4Qn + 8Q)4^{n-2} = 16 \cdot 4^{n-2}$$

$$20Q = 16$$

$$\Rightarrow Q = \frac{16}{20} = \frac{4}{5}$$

$$a_n^{(p)} = \frac{4}{5} n 4^n$$

The general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$= C_1(-1)^n + C_2(4)^n + \frac{4}{5} n 4^n$$

Problem

Solve $a_n - 6a_{n-1} + 8a_{n-2} = n4^n$

When $a_0 = 8$, $a_1 = 22$

Solution

Given recurrence relation is

$$a_n - 6a_{n-1} + 8a_{n-2} = n4^n \quad \text{—— (1)}$$

Homogeneous relation is $a_n - 6a_{n-1} + 8a_{n-2} =$

The characteristic equation is $t^2 - 6t + 8 = 0$

$$(t-2)(t-4) = 0$$

$$\Rightarrow t = 2, 4$$

$$a_n^{(h)} = c_1(2)^n + c_2(4)^n$$

Since 4 is the characteristic root with multiplicity 1

by rule 3, the particular solution is of the form

$n(Q_0 + Q_1 n)4^n$

substituting this solution in (1), we get

$$n(Q_0 + Q_1 n)4^n - 6\left[(n-1)(Q_0 + Q_1(n-1)).4^{n-1}\right] +$$

$$8\left[(n-2)(Q_0 + Q_1(n-2))4^{n-2}\right] = n4^n \quad \text{—— (2)}$$

The equation (2) holds for all values of n and in

particular when $n = 0$ $Q_0 + Q_1 = 0$ —— (3)

putting $n=1$ in (2), we get $P_0+3P_1=2$ — (4)

By solving (3), (4) we get

$$P_0=-1 \quad P_1=1$$

$$a_n^{(p)}=n(-1+n)4^n=n(n-1)4^n$$

The general solution is $a_n=a_n^{(h)}+a_n^{(p)}$

$$a_n=c_1 2^n+c_2 4^n+n(n-1)4^n$$

The initial conditions $a_0=8$, $a_1=22$

$$\Rightarrow c_1=5 \quad c_2=3$$

Hence complete solution is

$$a_n=5\cdot 2^n+3\cdot 4^n+n(n-1)4^n$$

Unit-4: Recurrence Relations

Numeric function:

- A function whose domain is the set of whole numbers and range is called a numeric function.

Eg: $3^0, 3^1, 3^2, \dots, 3^n, \dots$

Generating function: Let $\{a_0, a_1, a_2, \dots, a_n, \dots\}$ be a numeric function for 'a' then the infinite series $G(x)$ is said to be a generating function which is denoted by "$a_0x^0 + a_1x^1 + a_2x^2 + \dots$"

$$G(x) = \sum_{n=0}^{\infty} a_n x^n \text{ is called a generating function.}$$

Ex: for numeric function $\{3^n\}$ the generating function is given by,

$$G(x) = 3^0x^0 + 3^1x^1 + 3^2x^2 + \dots$$

$$= \sum_{n=0}^{\infty} a^n x^n$$

(2) $(1,1,1,1,1,1,1)$

$$G.F = 1\cdot x^0 + 1\cdot x^1 + 1\cdot x^2 + 1\cdot x^3 + 1\cdot x^4 + 1\cdot x^5 + 1\cdot x^6$$

$$= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \text{ in G.P}$$

$a=1, r=x$

$$= \frac{a(r^n-1)}{r-1} \text{ if } r>1$$

$$= \frac{1(x^7-1)}{x-1} \text{ if } (x>1)$$

3) write the generating function of $a_r = 2^r, r \geq 0$.

$$G(x) = 2^0x^0 + 2^1x^1 + 2^2x^2 + 2^3x^3 + \dots$$

$$= x^0 + 2x + 4x^2 + 8x^3 + \dots$$

$$= 2x + 4x^2 + 8x^3 + \dots$$

$$= (1-2x)^{-1}$$

$$= \frac{1}{1-2x}$$

4) write generating function $(1,2,3,4,\dots)$

$$G(x) = 1\cdot x^0 + 2\cdot x^1 + 3\cdot x^2 + 4\cdot x^3 + \dots$$

$$= 1+2x + 3x^2 + 4x^3 + \dots$$

$$= (1-x)^{-2}$$

$$= \frac{1}{(1-x)^2}$$

5) write G.F for $(1,-2,3,-4)$

$$G(x) = 1x^0 - 2x + 3x^2 - 4x^3 + \dots$$

$$= 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$= (1+x)^{-2}$$

6) Find the G.F of the seq $A = \{a_r\}_{r=0}^{\infty}$ where $a_r = \begin{cases} 1 & \text{if } 0 \le r \le 2 \\ 3 & \text{if } 3 \le r \le 5 \\ 0 & \text{if } r \ge 6 \end{cases}$

$G(x) = 1x^0 + 1x + 1x^2 + 3x^3 + 3x^4 + 3x^5 + 0x^6 + 0x^7 + \cdots$

$= 1 + x + x^2 + 3x^3 + 3x^4 + 3x^5$

Closed form expressions of generating functions:

- To express generating function in closed form in the closed form expression for the numeric geometric series, which is expression of the form.

$a + ar + ar^2 + \cdots$

Note: It may be finite or infinite series

Q) $G.f = ?$

$a_n = 3^n,\ n \ge 0$ in closed form

$G(x) = 3x^0 + 3x^1 + 3x^2 + 3x^3 + \cdots$

$= 3(x^0 + x^1 + x^2 + x^3 + \cdots)$

$= 3(1-x)$

2) Find the G.F in closed form of fibinacci sequence $\{f_n\}$ defined by

$f_n = f_{n-1} + f_{n-2}$

Consider the generating function $f(x) = f_0 + f_1 x + f_2 x^2 \cdots$

$= \sum_{n=0}^{\infty} f_n x^n$ - ①

We have the fibinacci series $f_{n-1} + f_{n-2} = f_n$

multiply with x^n on both sides

$f_n x^n = f_{n-1} x^n + f_{n-2} x^n$

Taking sum on all $n = 2 \to \infty$ on both sides

$= \sum_{n=2}^{\infty} f_n x^n = \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n$ - ②

$= \sum_{n=2}^{\infty} f_n x^n = f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots + f$

$= (f_0 + f_1 x + f_2 x^2 + f_3 x^3 + \cdots) \cdots f_0 - f_1 x$

$\sum_{n=2}^{\infty} f_n(x) = G(x) - f_0 - f_1 \cdot x$ - ③ $[\because \text{from } ①]$

Now $\sum_{n=2}^{\infty} f_{n-1} x^n = f_1 x^2 + f_2 x^3 + f_3 x^4 + \cdots$

$= x(f_1 x + f_2 x^2 + f_3 x^3 + \cdots)$

add and sub f_0

$= x(G(x) - f_0)$ - ④

Now $\sum_{n=2}^{\infty} f_{n-2} x^n = f_0 x^2 + f_1 x^3 + f_2 x^4 + \cdots +$

$= x^2(G(x))$ - ⑤

Sub ③ ④ ⑤ ⑥ in ②

$G(x) - f_0 - f_1 x = x(G(x) - f_0) + G(x) x^2$

$G(x) - f_0 - f_1 x = x(G(x) - x f_0 + x^2 G(x))$

$G(x)[1 - x - x^2] = -x f_0 + f_0 + f_1 x$

Since $f_0 = 0,\ f_1 = 1$

$= -0 + 0 + x$

$G(x)(1 - x - x^2) = x$

$\therefore G(x) = \dfrac{x}{1 - x - x^2}$, is the seq G.F for the $\{f_n\}$

1) Find the sequence generated by the following functions:

i) $(2+x)^3$

ii) $2x^2(1-x)^{-1}$

iii) $3x^3 + e^{2x}$

Sol: $= 2^3(1+\frac{x}{2})^3$

$= 8\left[3C_0 + 3C_1 \frac{x}{2} + 3C_2(\frac{x}{2})^2 + 3C_3(\frac{x}{2})^3\right]$

$= 8\left[1 + \frac{3}{2}x + \frac{3}{4}x^2 + 1 \cdot \frac{x^3}{8}\right]$

$= 8 + 12x + 6x^2 + x^3$

$\therefore$ The seq generated by $(2+x)^3$ is $8, 12, 6, 1, 0, 0, 0, \ldots$

ii) $2x^2(1-x)^{-1}$

$= 2x^2(1 + x + x^2 + x^3 + \cdots)$

$= 2x^2 + 2x^3 + 2x^4 + 2x^5 + \cdots$

$\Rightarrow 0, 0, 2, 2, 2, \ldots$

iii) $3x^3 + 1 + \frac{2x}{1!} + \frac{4x^2}{2!} + \cdots$

$= 1 + 2x + 2x^2 + (\frac{4}{3} + 3)x^3 + \frac{2}{3}x^4$

$= 1 + 2x + 2x^2 + \frac{13}{3}x^3 + \frac{2}{3}x^4 + \cdots$

Seq: $1, 2, 2, \frac{13}{3}, \frac{2}{3}, \ldots$

ii) Sequence to general function:

a) $0, 1, 2, 3, \ldots$

$= 0x^0 + 1 \cdot x + 2x^2 + 3x^3 + \cdots$

$= x(1 + 2x + 3x^2 + \cdots)$

$= x(1-x)^{-2}$

$\dfrac{x}{(1-x)^2}$

$= \dfrac{(1-x)^2(1) - (x)(2(1}{(1-x)^4}$

$= \dfrac{(1-x)^2 + (2x - 2x^2)}{(1-x)^4}$

b) $1^2, 2^2, 3^2, \ldots$

$= 1^2x^0 + 2^2x + 3^2x^2 + \cdots$

$= 1 + 4x + 9x^2 + \cdots$

$=$

Extended binomial theorem:

Let x be real no. with $|x|<1$ and n be a real no., then

$$(1+x)^n = \sum_{r=0}^{\infty} {}^nC_r x^r$$

① ${}^{-n}C_r = (-1)^r \, {}^{(n+r-1)}C_r$

② $(1+x)^{-n} = \sum_{r=0}^{\infty} (-n)C_r x^r = \sum_{r=0}^{\infty} (-1)^r (n+r-1)C_r \cdot x^r$

③ $(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{(n+r-1)}C_r \cdot x^r$

Calculating coeff of General function:

Q) Find the coeff of x^{32} in ~~xxxxx~~ $(1+x^5+x^9)^{10}$

We want the coeff solutions of

$e_1+e_2+\dots+e_{10}=32$

$e_i = 0,5,9$ → Three 9's → 27

One 5 → 5

Six 0's → 0

32

$$\Rightarrow \frac{10!}{3!\,1!\,6!} = \frac{10\times9\times8\times7\times\cancel{6!}}{3!\times1!\times\cancel{6!}}$$

$$= \frac{10\times9\times8\times7}{3\times2\times1} = 15\times56 = 840$$

56
×15
280
56×
840

Q) Find coeff of x^{27} in $(x^4+x^5+x^6+\dots)^5$

Given, $(x^4+x^5+x^6+\dots)^5$

$= (x^4)^5(1+x+x^2+\dots)$

$= x^{20}[1-x]^{-5}$

$= x^{20}\sum_{r=0}^{\infty} {}^{(n+r-1)}C_r \cdot x^r \cdot \quad (n=5)$

$= \sum_{r=0}^{\infty} (4+r)C_r\, x^{20+r}$

∴ To get coeff of x^{27} put $20+r=27$

$r=7$

$= {}^{(4+7)}C_7 = {}^{11}C_7$

Q) x^{20} in $(x+x^2+2x^3+x^4)(x^2+x^3+x^4+\dots)^5$

$\Rightarrow x(1+x+2x^2+x^3)x^{10}(1-x)^{-5}$

$\Rightarrow x^{11}(1+x+2x^2+x^3)\sum_{r=0}^{\infty}(5+r-1)C_r x^r$

$\Rightarrow (1+x+2x^2+x^3)\sum (4+r)C_r x^{r+11}$

$r+11=20$ $r+12=20$ $r+13=20$ $r+14=20$

$r=9$ $r=8$ $r=7$ $r=6$

$\Rightarrow (4+9)C_9 + (4+8)C_8 + (4+7)C_7 + (4+6)C_6$

$= {}^{13}C_9 + {}^{12}C_8 + 2({}^{11}C_7) + {}^{10}C_6$

$= 2080$

Extended binomial theorem:

Let x be real no. with $|x|<1$ and n be a real no., then

$$(1+x)^n = \sum_{r=0}^{\infty} {}^nC_r x^r$$

① ${}^{-n}C_r = (-1)^r \, {}^{(n+r-1)}C_r$

② $(1+x)^{-n} = \sum_{r=0}^{\infty} (-n)C_r \, x^r = \sum_{r=0}^{\infty} (-1)^r (n+r-1)C_r \cdot x^r$

③ $(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{(n+r-1)}C_r \cdot x^r$

Calculating coeff of General function:

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$$e_1+e_2+\dots+e_{10} = 32$$

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one 5 → 5

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32

$$\Rightarrow \frac{10!}{3!\,1!\,6!} = \frac{10\times 9\times 8\times 7\times 6!}{3!\times 1!\times 6!}$$

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Given, $(x^4+x^5+x^6+\dots)^5$

$= (x^4)^5(1+x+x^2+\dots)$

$= x^{20}[1-x]^{-5}$

$= x^{20}\sum_{r=0}^{\infty} {}^{(n+r-1)}C_r \cdot x^r \quad (n=5)$

$= \sum_{r=0}^{\infty} {}^{(4+r)}C_r \, x^{20+r}$

⇒ To get coeff of x^{27} put $20+r=27$

$r=7$

$= {}^{(4+7)}C_7 = {}^{11}C_7$

Q) x^{20} in $(x+x^2+2x^3+x^4)(x^2+x^3+x^4+\dots)^5$

$\Rightarrow x(1+x+2x^2+x^3)x^{10}(1-x)^{-5}$

$\Rightarrow x^{11}(1+x+2x^2+x^3)\sum_{r=0}^{\infty}(5+r-1)C_r \cdot x^r$

$\Rightarrow (1+x+2x^2+x^3)\sum (4+r)C_r \, x^{r+11}$

$r+11=20$, $r=9$ $\quad$ $r+12=20$, $r=8$ $\quad$ $r+13=20$, $r=7$ $\quad$ $r+14=20$, $r=6$

$\Rightarrow {}^{(4+9)}C_9 + {}^{(4+8)}C_8 + {}^{(4+7)}C_7 + {}^{(4+6)}C_6$

$= {}^{13}C_9 + {}^{12}C_8 + 2({}^{11}C_7) + {}^{10}C_6$

$= 2080$

in case of non negatives e^i can take values $0, 1, 2, \cdots$

$\exists f_i(x) = x^0 + x^1 + x^2 + \cdots$

The GF for non neg is $f(x) = f_1(x) f_2(x) f_3(x) f_4(x)$

$\Rightarrow (1+x+x^2+\cdots)(1+x+x^2+\cdots)(1+x+x^2+\cdots)(1+x+x^2+\cdots)$

$\Rightarrow (1+x+x^2+\cdots)^4 = ((1-x)^{-1})^4 = (1-x)^{-4}$

$\sum_{r=0}^{\infty} {}^{(4+r-1)}C_r x^r$

$\Rightarrow \sum_{r=0}^{\infty} {}^{(3+r)}C_r x^r$

put $r=25$

$\Rightarrow$ no. of non neg int sol is coeff of x^{25} in the expansion

$\Rightarrow {}^{28}C_{25}$

$\Rightarrow 3276$

ii) $f_i(x) = x^1 + x^2 + x^3 + \cdots$ for $i = 1, 2, 3, 4$

$f(x) = f_1 f_2 f_3 f_4$

$= (x+x^2+\cdots)(x+x^2+\cdots)(x+x^2+\cdots)(x+x^2+\cdots)$

$= (x+x^2+\cdots)^4$

$= x^4(1+x+\cdots)^4$

$= x^4(1-x)^{-4}$

$= x^4 \sum_{r=0}^{\infty} {}^{(4+r-1)}C_r x^r$

$= \sum_{r=0}^{\infty} {}^{(3+r)}C_r \cdot x^{4+r}$

put $r+4 = 25$

$r = 21$

${}^{(3+21)}C_{21} = 2024$

Generally recursive definition can be used to solve counting problems

→ sequence $\{a_n\}$ is an eqn that expresses $\{a_n\}$ in terms of one or more of the previous terms, of the sequence $\{a_n\}$ namely $a_0, a_1, a_2, \cdots a_{n-1}$, $\forall\ n \geq 1$

→ also reference as difference equation

i) $a_n - a_{n-1} = n \Rightarrow a_n = n + n a_{n-1}$

ii) $a_n - 3\underset{d=1}{a_{n-1}} + 2\underset{d=2}{a_{n-2}} = 0$

iii) $a_n - 5\underset{d=1}{a_{n-1}} + 6\underset{d=2}{a_{n-2}} = n^2+1$

} linear → deg 2, → deg 2

iv) $a_n^2 - a_{n-1}^2 = -1$ } not linear

* linear recurrence relation:

- let n, k be non neg int, a recurrence relation is of form

$c_0(n)a_n + c_1(n)a_{n-1} + c_2(n)a_{n-2} + \cdots + c_k(n)a_{n-k} = f(n)$ for $n \geq k$

where $c_0(n), c_1(n), \cdots f(n)$ are function of n

- If $c_0(n), c_k(n)$ are not 0 then k → degree of the above RR

- If $c_0(n), \cdots c_k(n)$ are constants then ① is called a linear LL with constant coeff

- If $f(x)$

A sequence $\{a_n\}_{n=0}^{\infty}$ is said to be a solution of RR if each value a_n satisfies the recurrence relation.

methods to solve recurrence relations

- method 1: Substitution
- method 2: Generating function
- method 3. characteristic root method

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Q) Solve the RR:

$a_n = a_{n-1} + n^2$ where $a_0 = 7$

If $n=1$, $a_1 = a_0 + n^2 = a_0 + 1^2$
$= 7 + 1^2$

If $n=2$, $a_2 = a_1 + 4 = 7 + (1^2 + 2^2)$

If $n=3$, $a_3 = a_2 + 3^2 = 7 + (1^2 + 2^2 + 3^2)$

$\vdots$

$a_n = a_{n-1} + n^2 = 7 + (1^2 + 2^2 + \cdots + n^2)$

$= 7 + \frac{n(n+1)(2n+1)}{6}$

$\therefore$ required sol of RR

Q) $a_n = a_{n-1} + \frac{1}{n(n+1)}$ where $a_0 = 1$

$n=1 \Rightarrow a_0 + \frac{1}{2} = 1 + \frac{1}{2} = 1 + \frac{1}{1\cdot 2}$

$n=2 \Rightarrow a_1 + \frac{1}{6} = 1 + \frac{1}{2} + \frac{1}{6} = 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3}$

$\vdots$

$a_n = 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$

Q) If a_n is sol of RR $a_{n+1} = k(a_n)$ for $n \geq 0$ and $a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{2401}$

find 'k'.

put $n = 2$

$a_{2+1} = a_3 = \frac{153}{49}$

$a_{3+1} = a_4 = k(a_3)$

$= a_4 = k\left(\frac{153}{49}\right)$

$a_5 = k(a_4) = k^2\left(\frac{153}{49}\right)$

given $a_5 = \frac{1377}{2401}$

$\Rightarrow k^2\left(\frac{153}{49}\right) = \frac{1377}{2401}$

$\Rightarrow k^2 = \frac{\overset{9}{\cancel{1377}}}{\underset{49}{\cancel{2401}}} \times \frac{\cancel{49}}{\cancel{153}}$

$k^2 = \frac{9}{49} \quad \therefore k = \pm\frac{3}{7}$

Method - 2: Using generating functions

① $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$ where $a_0 = 10$, $a_1 = 4$

Sol let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ —①

Given R.R,

$a_n - 7a_{n-1} + 10a_{n-2} = 0$

mul each term with ② x^n and sum overall $n = 2$ to ∞

② becomes $\sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1}x^n + 10\sum_{n=2}^{\infty} a_{n-2}x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 10x^2\sum_{n=0}^{\infty} a_{n-2}x^{n-2} = 0$

$\Rightarrow \left(\sum_{n=0}^{\infty} a_n x^n - a_2 - a_1 x\right) - 7x\left(\sum_{n=0}^{\infty} a_n x^n - a_0\right) + 10x^2\left(\sum_{n=1}^{\infty} a_n x^n\right)$

$\Rightarrow A(x) - a_0 - a_1 x - 7x A(x) + 7x a_0 + 10x^2 A(x) = 0$

here $a_0 = 10$ and $a_1 = 41$

$\Rightarrow A(x) - 7x A(x) + 10x^2 A(x) - a_0 - a_1 x + 7x a_0 = 0$

$A(x)[1 - 7x + 10x^2] - 10 - 41x + 7x(10) = 0$

$(10x^2 - 7x + 1)A(x) - 10 + 29x = 0$

$$A(x) = \frac{10 - 29x}{10x^2 - 7x + 1}$$

$\Rightarrow 10x^2 - 7x + 1$

$\Rightarrow 10x^2 - 5x - 2x + 1$

$= 5x(2x - 1) - 1(2x - 1)$

$= (2x - 2)(5x - 1)$

$= \dfrac{10 - 29x}{(2x-1)(5x-1)} = \dfrac{A}{2x - 1} + \dfrac{B}{5x - 1}$

$10 - 29x = A(5x - 1) + B(2x - 1)$

put $x = \frac{1}{5}$ to get B, $B = -7$

$\therefore A(x) = \dfrac{-3}{2x - 1} - \dfrac{7}{5x - 1}$

$= \sum_{n=0}^{\infty} a_n x^n = \dfrac{3}{1 - 2x} + \dfrac{7}{1 - 5x}$

$= 3\sum_{n=0}^{\infty} (2x)^n + 7\sum_{n=0}^{\infty} (5x)^n$

$= \sum 3(2^n)x^n + \sum_{n=0}^{\infty} 7(5^n)x^n = \sum(3\cdot 2^n + 7\cdot 5^n)x$

$\therefore a_n = 3 \cdot 2^n + 7 \cdot 5^n$

Q) using GF method, solve $a_n - 4a_{n-1} + 3a_{n-2} = 0$, $n \geq 2$ with condition $a_0 = 2$ and $a_1 = 4$

Sol $A(x) = \sum_{0}^{\infty} a_n x^n$

$a_n - 4a_{n-1} + 3a_{n-2} = 0$ —①

$\therefore$ mul x^n obs

$\sum a_n x^n - 4\sum a_{n-1}x^n + 3\sum a_{n-2}x^n = 0$

$= A_x - 4x[A(x) - a_0] + 3x^2[A(x) - a_0 - a_1 x] = 0$

$= A(x) - 4x(A(x))$ ~~...~~ $- 4x a_0 + 3x^2 A(x)$

....

Q) Solve R.R $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$

★ Method 2: characteristics roots method

let $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0$ — ①

when $c_k \neq 0$ be a linear homogenous RR of degree k,

the characteristic eqn of ① is of the form.

$t^k + t_1 t^{k-1}$.

Degree of equation ② is k so it has 2 roots, $\alpha_1, \alpha_2, \cdots, \alpha_k$

these are called the characteristic roots.

- Two cases may rise:

case i) If $\alpha_1, \alpha_2, \cdots \alpha_k$ are distinct roots then the gen sol of R.R is

$a_n = c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_k\alpha_k^n$

where $c_1, c_2, \cdots c_k$ are constants

case ii) If root α repeated K then gen sol of RR is,

$a_n = (c_1 + c_2 n + c_3 n^2 + c_4 n^3 + \cdots + c_k n^{k-1})\alpha^n$

where $c_1, c_2, \cdots c_n$ constants.

① Solve R.R: $a_n - 3a_{n-1} + 2a_{n-2} = 0$

Sol The characteristic eqn of following R.R is.

$t^2 - 3t + 2 = 0$

$t^2 - 2t - t + 2 = 0$

$t(t-2) - 1(t-2) = 0$

$(t-1)(t-2) = 0$

∴ roots are real and distinct

General solution: $c_1 2^n + c_2 \cdot 1^n$

② Solve: $a_n - 6a_{n-1} + 9a_{n-2} = 0$

$t^2 - 6t + 9 = 0$

$t^2 - 3t - 3t + 9 = 0$

$t(t-3) - 3(t-3) = 0$

$(t-3)(t-3) = 0$

∴ roots are real and same

General Solution: $(c_1 + c_2 n)3^n$

③ Find the characteristic equation given homogenous RR takes form:

i) $a_n = \beta_1 + n\beta_2$

ii) $a_n = \beta_1 2^n + \beta_2 3^n$

$\Big\}$ β_1, β_2 are constants

i) roots are $t = 1, 1$

$\therefore$ char eq is $(t-1)^2 = 0$

$t^2 - 2t + 1 = 0$

ii) $a_n = B_1 2^n + B_2 3^n$

roots are $t = 2, 3$

$\Rightarrow (t-2)(t-3) = 0$

$\Rightarrow t^2 - 5t + 6 = 0$

5) $a_n = a_{n-1} + a_{n-2}$

with $a_0 = 0, a_1 = 1$

Given R.R $a_n - a_{n-1} - a_{n-2} = 0$

char eq $t^2 - t - 1 = 0$

$t = \frac{1 \pm \sqrt{5}}{2}$

$\Rightarrow t = \frac{1+\sqrt{5}}{2}$

$\Rightarrow t = \frac{1-\sqrt{5}}{2}$

GS of fibinacci relations is

$a_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$ -①

given initial conditions are,

$a_0 = 0, a_1 = 1$

put $n=0$ in ①

$a_0 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = 0$

$c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$ -②

put $n = 1$ in ①

$a_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$\Rightarrow c_1(1+\sqrt{5}) - c_1(1-\sqrt{5}) = 2$ (from ②)

$\Rightarrow c_1 + \sqrt{5}c_1 - c_1 + \sqrt{5}c_1 = 2$

$2\sqrt{5}c_1 = 2$

$c_1 = \frac{1}{\sqrt{5}} \Rightarrow c_2 = \frac{-1}{\sqrt{5}}$ $(\because c_2 = -c_1)$

$\therefore$ GS of RR

$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right) - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)$

★ Solution in-homogenous linear recurrence relation:

- general form is of form: $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = f(n)$

(for $n \geq k$)

where $f(n)$ is some specified function interms of n

To solve IHR we have 2 methods:

- characteristic root method
- Generating function

- The solution of a linear in-homogeneous recurrence relation with constant coeff is sum of two parts
- one is homogeneous solution and 2nd one is particular solution
- A sol, which satisfies RR when RHS is 0 is called homogeneous solution i.e a sol which satisfies RR is of form

$$a_n = a_n^{(h)} + a_n^{(p)}$$

- procedure :

① depends on nature of particular sol

Rule 1: If $f(n)$ is a polynomial of degree 'm' in 'n' it can be expressed as

$$(b_0 + b_1 n + b_2 n^2 + \dots + b_m n^m)$$

then PS can be expressed as

$$(Q_0 + Q_1 n + Q_2 n^2 + \dots + Q_m n^m)$$

Where '1' is not a characteristic root of a recurrence relation

Rule 2: If $f(n) = (b_0 + b_1 n + b_2 n^2 + \dots + b_m n^m)$ then the

P.S

Rule 3: If 'a' is the characteristic root with multiplicity 's' when $f(n)$ is in form:

$f(n) = (b_0 + b_1 n + b_2 n^2 + b_3 n^3 + \dots + b_m n^m)$ then

PS is in form: $n^s(Q_0 + Q_1 n + Q_2 n^2 + Q_3 n^3 + \dots + Q_m n^m) a^n$

Problems:

① Solve the recurrence relation : $a_n - a_{n-1} - 6a_{n-2} = -30$ — ①

with $a_0 = 20, a_1 = -5$

Sol $a_n = a_n^{(h)} + a_n^{(p)}$

h → homogeneous
P → particular integral

To find $a_n^{(h)}$:

Here RR from ①, $a_n - a_{n-1} - 6a_{n-2} = 0$ — ②

Char eq ② is $t^2 - t - 6 = 0$

$$t^2 - 3t + 2t - 6 = 0$$

$$(t-3)(t+2) = 0$$

$$t = 3, -2$$

To find $a_n^{(p)}$:

Since RHS of ① is:

$$f(n) = -30$$

Tak $Q = -30$ using rule ①

$$a_n = a_{n-1} = a_{n-2} = Q$$

from ①

$Q - Q - 6Q = -30$

$-6Q = -30$

$Q = 5$

$a_n^{(p)} = 5$

Hence the G.S is

$a_n = C_1(3)^n + C_2(-2)^n + 5$ -③

To find C_1, C_2

Given $a_0 = 20$ and $a_1 = -5$

from ③ $a_n = C_1 3^n + C_2(-2)^n + 5$

put $n = 0$

$a_0 = C_1 + C_2 + 5 = 20$

$= C_1 + C_2 = 15$ -④

put $n = 1$

$a_1 = C_1(-3) + C_2(-2) + 5 = -5$

$3C_1 - 2C_2 = -10$ -⑤

$a_n = C_1 2^n + C_2 5^n$

Q) Solve $a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n$ -①

G.S $\Rightarrow a_n = a_n^{(h)} + a_n^{(p)}$

To find $a_n^{(h)}$:

$a_n - 7a_{n-1} + 10a_{n-2} = 0$

$t^2 - 7t + 10 = 0$

$t^2 - 5t - 2t + 10 = 0$

$t(t-5) - 2(t-5) = 0$

$(t-5)(t-2) = 0$

$a_n^{(h)} = C_1 5^n + C_2 2^n$

$\therefore$ 1 is not a charac. root

$\Rightarrow 4Q_0 + 4Q_1 n - 13Q_1 = 6 + 8n$

$\therefore$ compare with RHS

$= (4Q_0 - 13Q_1) + n(4Q_1) = 6 + 8n$

$= 4Q_0 - 13Q_1 = 6$ -Ⓐ

$(4Q_1)n = 8n$ -Ⓑ

$= Q_1 = 2$, put in ①

$\Rightarrow 4Q_0 - 26 = 6$

$\Rightarrow Q_0 = \frac{32}{4} 8$

To find $a_n^{(p)}$:

Given $f(n) = 6 + 8n$

by rule 1,

P.S is of form $Q_0 + Q_1 n$

i.e, $a_n = Q_0 + Q_1 n$

$a_{n-1} = Q_0 + Q_1(n-1)$

$a_{n-2} = Q_0 + Q_1(n-2)$

$= (Q_0 + Q_1 n) - 7(Q_0 + Q_1(n-1)) + 10[Q_0 + Q_1(n-2)]$
$= 6 + 8n$

$= Q_0 + Q_1 n - 7Q_0 + 7Q_1(n-1) + 10Q_0 + 10Q_1(n-2)$
$= 6 + 8n$

$= Q_0 - 7Q_0 + 10Q_0 \overset{+Q_1 n}{-} 7nQ_1 + 7Q_1 + 10Q_1 n - 20Q_1$
$= 6 + 8n$

G.S: $a_n = C_1 5^n + C_2 2^n + (8 + 2n)$

③ Solve $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$

$a_n^{(h)} = a_n - 7a_{n-1} + 10a_{n-2} = 0$

$= t^2 - 7t + 10 = 0$

$= t^2 - 5t - 2t + 10 = 0$

$= t(t-5) - 2(t-5) = 0$

$= (t-2)(t-5)$

$\Rightarrow c_1 5^n + c_2 2^n$

$f(n) = 4^n$

particular solution for $a_n = Q4^n$

$a_n = Q4^n$

$a_{n-1} = Q4^{n-1}$

$a_{n-2} = Q4^{(n-2)}$

Sub in ①

$= Q4^n - 7Q_4^{n-1} + 10Q4^{n-2} = 4^n$

$= Q4^n - 7Q4^{n-1} + 10Q4^{n-2} = 4^n$

$= Q4^{n-2}(16 - 7(4) + 10) = 4^n$

$= Q(16 - 28 + 10) = \frac{4^n}{4^n \cdot 4^{-2}}$

$= Q = \frac{16}{-2} = -8$

$= a_n^{(P)} = Q4^n = (-8)(4)^n$

GS : $a_n = c_1(2)^n + c_2(5)^n - 8(4)^n$

Q) solve $a_n + a_{n-1} = (3n)2^n$

$a_n^{(h)}: \; t + 1 = 0$

$t = -1$

$f(n) = (3n)2^n$

∴ Solution is of form

~~[illegible]~~

~~[illegible]~~

$= a_n : (Q_0 + Q_1 n)2^n$

$a_{n-1} : (Q_0 + Q_1(n-1))2^{n-1}$

$= 2(Q_0 + Q_1 n) + \frac{2(Q_0 + Q_1(n-1))}{2} = (3n)(2^n)\,2$

$= 2(Q_0 + Q_1 n) + (Q_0 + nQ_1 - Q_1) = 2(3n)$

$= Q_0 + Q_0 + nQ_1 + nQ_1 - Q_1 = 6n$

$= 2Q_0 + 2nQ_1 - Q_1$

X

$= 2(Q_0 + Q_1 n) + Q_0 + (Q_1(n-1))2 = 6n$

$= \cancel{2Q_0 + Q_0 + 2Q_1 n \ldots = 6n}$ $\quad 2Q_0 + Q_0 + 2Q_1 n + Q_1 n - Q_1 = 6n$

$= 3Q_0 + 3Q_1 n - Q_1$

$= 3Q_0 +$

$= 3Q_0 - Q_1 = 0$ —①

$3Q_1 n = 6n$ —②

$= Q_1 = 2$ —③

$3Q_0 = 2$

$Q_0 = \frac{2}{3}$

$a_n = c_1(-1)^n + \frac{2}{3} + 2n$

Q) $a_n - 6a_{n-1} + 8a_{n-2} = n \cdot 4^n$

$a_n^{(h)} : t^2 - 6t + 8 = 0$

$= t^2 - 2t - 4t + 8 = 0$

$= t(t-2) - 4(t-2) = 0$

$= (t-4)(t-2) = 0$

$= t = 4, 2$

$= c_1 2^n + c_2 4^n$

∴ sol is of form [rule 3]

$n(a_0 + a_1 n)4^n$

=

S.no	Sequence	Generating function
1	$(n+1)$	$\frac{1}{(1-x)^2}$
2	n	$\frac{x}{(1-x)^2}$
3	$(n+2)(n+1)$	$\frac{2}{(1-x)^3}$
4	$(n+1)n$	$\frac{2x}{(1-x)^3}$
5	$\frac{n(n-1)}{2}$	$\frac{x^2}{(1-x)^3}$
6	n^2	$\frac{x(1+x)}{(1-x)^3}$
7	na^n	$\frac{ax}{(1-ax)^2}$
8	n^2a^n	$\frac{ax(1+ax)}{(1-ax)^3}$

Method -2: Solving IHR by GF method

Q) Solve : $a_n - a_{n-1} = 3(n-1)$ where $n \geq 1$, $a_0 = 2$.

. let $A(x) = \sum_{n=0}^{\infty} a_n x^n$

Given recurrence relation : $a_n - a_{n-1} = 3(n-1)$ —①

multiply equation ① with x^n and summing from $n=1$ and $n=\infty$

obs.

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 3\sum_{n=1}^{\infty} (n-1) x^n$$

$$\Rightarrow (A(x) - a_0) - x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$$

$$\Rightarrow A(x) - 2 - x A(x)$$

$$\Rightarrow 3\sum_{n=1}^{\infty} (n-1)x^n = \frac{x^2}{(1-x)^2}$$

$$\therefore A(x)(1-x) - 2 = 3 \frac{x^2}{(1-x)^2}$$

$$A(x)(1-x) = 2 + \frac{3x^2}{(1-x)^2}$$

$$A(x) = \frac{2}{(1-x)} + \frac{3x^2}{(1-x)^3}$$

$$\sum a_n x^n = 2\sum_{n=0}^{\infty} x^n + 3\sum_{n=0}^{\infty} \frac{n(n-1)}{2} x^n$$

$\therefore a_n = 2 + \frac{3}{2}(n(n-1))$ is the required solution.

(2) $a_n - 5a_{n-1} + 6a_{n-2} = n(n-1)$ for $n \geq 2$ $a_0=1, a_1=5$

Sol) $\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 5\sum_{n=2}^{\infty} a_{n-1} x^n + 6\sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2}^{\infty} n(n-1)x^n$

$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 5x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + 6x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = \sum_{n=2}^{\infty} n(n-1)x^n$

$\Rightarrow (A(x) - a_0 - a_1 x) - 5x(A(x) - a_0) + 6x^2 A(x) = \sum_{n=2}^{\infty} n(n-1)x^n$

$\Rightarrow A(x)[1-5x+6x^2] - a_0 - a_1 x + 5x a_0 = \sum_{n=2}^{\infty} n(n-1)x^n$

$\Rightarrow A(x)[1-5x+6x^2] - 1 - \cancel{5x} + \cancel{5x} = \frac{2x^2}{(1-x)^3}$

$\Rightarrow A(x)[1-5x+6x^2] = \frac{2x^2 - (1-x)^3}{(1-x)^3}$

$\Rightarrow A(x)(6x^2-2x-3x+1)$

$(3x(2x-1)-1(2x-1))$

$(3x-1)(2x-1) = \frac{2x^2-(1-x)^3}{(1-x)^3}$

$\Rightarrow A(x) = \frac{2x^2}{(1-x)^3(3x-1)(2x-1)} - \frac{1}{(3x-1)(2x-1)}$

$\Rightarrow A(x) = \frac{1-3x+5x^2-x^3}{(1-x)^3(1-2x)(1-3x)}$

$\Rightarrow A(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{1-2x} + \frac{E}{1-3x}$

On simplification,

$A = \frac{39}{12}$ $B = \frac{3}{2}$ $C=1$ $D=-10$ $E = \frac{21}{4}$

$\therefore A(x) = \frac{39}{12}(1-x)^{-1} + \frac{3}{2}(1-x)^{-2} + (1-x)^{-3} - 10(1-2x)^{-1}$
$+ \frac{21}{4}(1-3x)^{-1}$

$\sum_{n=0}^{\infty} a_n x^n = \frac{39}{12}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty}(n+1)x^n + \sum_{n=0}^{\infty}\frac{(n+2)(n+1)}{2}x^n$

Comparing the coff of x^n on both sides $-10\sum_{n=0}^{\infty} 2^n x^n$
$+ \frac{21}{4}\sum_{n=0}^{\infty} 3^n x^n$

$a_n = \frac{39}{12} + \frac{3}{2}(n+1) + \frac{(n+2)(n+1)}{2} - 10(2)^n + \frac{21}{4}(3)^n$

Solving non linear Recurrence relations

Q) Solve $a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0$ for $n \geq 0$

Given $a_0 = 4, a_1 = 13$

Sol) Let $b_n = a_n^2$

Eq ① can be written as

$a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0$ —— ①

$\Rightarrow b_{n+2} - 5b_{n+1} + 4b_n = 0$ for $n \geq 0$

$\Rightarrow b_n - 5b_{n-1} + 4b_{n-2} = 0$ for $n \geq 2$ put $n = n-2$

$\Rightarrow$ By characteristic root method

$t^2 - 5t + 4 = 0$

$t^2 - 4t - t + 4 = 0$

$t(t-4) - 1(t-4) = 0$

$t = 1, 4$

Sol of ② is $b_n = c_1 4^n + c_2 1^n$ —— ③

To find c_1, c_2

Give $a_0 = 4$

$\Rightarrow b_0 = a_0^2 = 4^2 = 16$

$a_1 = 13 \quad b_1 = a_1^2 = 13^2 = 169$

Put $n=0$ in (3)

$b_0 = c_1 4^0 + c_2 1^0 = 16$

$c_1 + c_2 = 16$

Put $n=1$ in (3) - (4)

$b_1 = c_1 4 + c_2 = 169$

$4c_1 + c_2 = 169$ — (5)

Solving (4) and (5)

$c_1 = 51 \quad c_2 = -35$

$\therefore$ GS sol of eq (1) is

$b_n = a_n^2 = 51(4)^n - 35(1)^n$

$a_n = \sqrt{51(4)^n - 35(1)^n}$