Unit: V

MSR

Graph Theory & Applications

Introduction: Graph theory was born in 1736 with Euler's paper in which he solved the Konisberg bridge problem. For the next 100 years nothing more was done in the field. In 1847, Kirchoff developed the theory of trees for their applications in electrical networks.

Many situations that occur in Computer Science, Physical Science, Communication Science, Economics and many other areas can be analysed by using techiniques found in a relatively new area of mathematics called "Graph theory".

Basic Concepts:

→ A graph G is a pair of sets (V, E) where $V = \{v_1, v_2, \cdots\}$ called set of vertices, $E = \{e_1, e_2, \cdots\}$ called set of edges such that each edge $e_i \in E$ is associated with an unordered pair of vertices $\{v_i, v_j\}$

→ If an edge $e \in E$ is associated with (u, v) then e is said to 'connect' u and v. Here u and v are called end points of e.

→ The most common representation of a graph is by means of a diagram, in which the vertices are represented as points and each edge as a line segment joining its end vertices.

→ An edge having the same vertex as both its end vertices is called a loop (or) self-loop.

→ The edges which have the same end vertices are called parallel edges.

→ A graph that has neither self-loops nor parallel edges is called a simple graph.

→ A graph that has self loops (or) parallel edges (or) both is called a general graph.

→ A graph with a finite number of vertices as well as a finite number of edges is called a finite graph; otherwise it is an infinite graph.

→ If a vertex 'v' is an end vertex of some edge 'e' then we say that v & e are incident to each other.

→ Two non parallel edges are said to be adjacent if they are incident on a common vertex.

→ Two vertices are said to be adjacent vertices if they are the end vertices of same edge.

→ The number of vertices in a graph is called the order of the graph.

→ The number of edges in a graph is called the size of the graph.

→ A graph that contains parallel edges (or multiple edges) but no loops is called a multi graph.

→ The number of edges incident on a vertex 'v' with self-loops counted twice, is called the degree of vertex v. It is denoted by $deg(v)$ (or) $d(v)$.

→ A vertex whose degree is '0' is called an isolated vertex.

→ A vertex of degree '1' is called a pendant vertex.

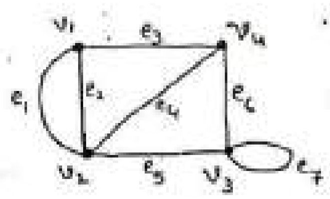
→ A pendant edge is an edge which is incident on a pendant vertex.

→ In a graph if all the vertices are of same degree 'K' then it is called a K-Regular graph (or) Regular graph of degree K.

→ Degree sequence of the graph is the arrangement of degrees of the vertices in non-decreasing order.

→ Degree of the graph is the minimum of the degrees of vertices of a graph. It is denoted by $\delta(G)$. Thus $\delta(G) = Min\{ d(v) \mid v \in V\}$.

Example: Consider the graph G :



Here : e_7 is a self loop.

: e_1, e_2 are parallel edges.

: e_3, v_4 are incident.

: e_3, e_6 are adjacent edges.

: v_2, v_4 are the adjacent vertices.

: Order of graph = no of vertices = 4

: Size of graph = no of edges = 7

: Given graph containing self-loops and parallel edges so it is a general graph.

: degrees of vertices : $d(v_1) = 3$, $d(v_2) = 4$, $d(v_3) = 4$, $d(v_4) = 3$

: In the given graph there are no isolated vertices, no pendant vertices.

: Given graph is not a regular graph.

: The degree sequence of the given graph is 3,3,4,4.

: Degree of the graph = Min {3,3,4,4} = 3.

Problems: 1. The sum of degrees of all the vertices in an undirected graph is twice the number of edges.

Sol: Since the degree of a vertex is the number of edges incident that vertex, and the sum of the degree counts the total number of times an edge is incident with a vertex, and every edge is incident with exactly two vertices and each edge gets counted twice, once at each end. i.e each edge contributes two degree.

Thus the sum of degrees equal twice the number of edges. i.e; $\sum_{i=1}^{n} \deg(v_i) = 2e$

* The problem(1) is called Handshaking theorem *

2. In a non-directed graph, the total number of odd degree vertices is even.

Sol: Let $G = (V, E)$ be a graph.

Let U: Set of even degree vertices, W = Set of odd degree vertices in G.

$$\therefore \sum_{v_i \in V} \deg(v_i) = \underset{(\text{even})}{\sum_{v_i \in U} \deg v_i} + \underset{(\text{odd})}{\sum_{v_i \in W} \deg v_i}$$

$$\Rightarrow 2e =$$

$$\Rightarrow \sum_{v_i \in W} \deg v_i = 2e - \sum_{v_i \in V} \deg v_i \longrightarrow (1)$$

Since $\sum_{v_i \in V} \deg(v_i)$ is always even, [Sum of degrees of even degree vertices is even]

So from (1), $\sum_{v_i \in W} \deg v_i$ is even.

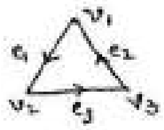
Directed graph:

→ A directed graph (or) a digraph G consists of a set V of vertices and a set E of edges such that $e \in E$ is associated with an ordered pair of vertices.

ie: if each edge of the graph G has a direction then the graph is called a directed graph.

→ In the diagram of directed graph, each edge $e = (u, v)$ is represented by an arrow from initial point u to the terminal point v.

Example: Consider the directed graph G :



Here $e_1 = (v_1, v_2)$ is an edge

$e_2 = (v_3, v_1)$ " "

$e_3 = (v_2, v_3)$ " "

→ If $e = (u, v)$ is a directed edge in a digraph, then

: u is called initial vertex, v is called terminal vertex of e

: 'e' is said to be incident from u and incident to v

: u is adjacent to v and v is adjacent from u

→ In a digraph G, the number of edges begining at v is called the out-degree of v, denoted by $deg^+(v)$.

→ The number of edges ending at v is called the In-degree of v, denoted by $deg^-(v)$.

→ The sum of the indegree and out-degree of a vertex is called the total degree of the vertex.

→ A vertex with zero indegree is called a Source and a vertex with zero outdegree is called a Sink.

Problems ① If $G=(V,E)$ be a directed graph with 'e' edges then

$$\sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v) = e.$$

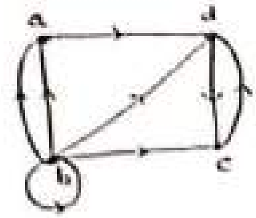
i.e. Sum of outdegrees of vertices = Sum of indegrees of vertices = no of edges.

Sol: W.K.T, any directed edge (u,v) contributes 1 to the indegree of u and 1 to the outdegree of v.

Further, a loop at v contributes 1 to the indegree and 1 to the outdegree of v.

∴ The Sum of indegrees = Sum of out-degrees = no of edges.

② Find the indegrees and outdegrees of the digraph G:



Sol:

$\deg^{-}(a) = 2$ $\qquad$ $\deg^{+}(a) = 1$

$\deg^{-}(b) = 1$ $\qquad$ $\deg^{+}(b) = 5$

$\deg^{-}(c) = 2$ $\qquad$ $\deg^{+}(c) = 1$

$\deg^{-}(d) = 3$ $\qquad$ $\deg^{+}(d) = 1$

③ Can there be a graph consisting of the vertices A, B, C, D with $\deg(A)=2$, $\deg(B)=3$, $\deg(C)=2$, $\deg(D)=2$?

Sol: W.K.T, in every graph the sum of degrees of vertices is Even. Here this sum $= 2+3+2+2 = 9$, odd number.

∴ there doesnot exist a graph of given kind.

④ Does there exist a graph with 12 vertices such that two of vertices have degree 3 and the remaining vertices have degree 4 (each)?

Sol: Here two vertices have degree 3,

Ten " " " 4.

So $\sum_{v \in V} d(v) = (2 \times 3) + (10 \times 4) = 46 = 2(23) =$ Even

∴ A graph of desired type exist.

Size of graph = no of edges = 23.

(5) Show that the degree of a vertex of a simple graph G having n vertices cannot exceed n-1.

Sol: Let v be a vertex in G.

Since G is simple, so G has no loops, no parallel edges.

$\therefore$ v can be adjacent to atmost all the remaining (n-1) vertices of G.

Thus the maximum degree of $v = n-1$.

(6) Show that the maximum number of edges in a simple graph G with n vertices is $\frac{n(n-1)}{2}$.

Sol: By the Handshaking theorem, $\sum_{i=1}^{n} d(v_i) = 2e$ $\boxed{e = \text{no of edges in } G}$

$\Rightarrow d(v_1) + d(v_2) + \cdots + d(v_n) = 2e$ — (1)

But the maximum degree of each vertex in G is $n-1$.

So (1) $\rightarrow$ $(n-1) + (n-1) + \cdots + (n-1) \ldots n$ times $= 2e$

$\Rightarrow n(n-1) = 2e$

$\Rightarrow e = \frac{n(n-1)}{2}$

$\therefore$ The maximum no of edges in G $= \frac{n(n-1)}{2}$.

(7) How many vertices are needed to construct a graph with 7 edges in which each vertex is of degree 2.

Sol: Let the required no of vertices $= n$. given $d(v) = 2 \; \forall \, v \in G$

By Handshaking theorem, $\sum_{i=1}^{n} d(v_i) = 2e$

$\Rightarrow d(v_1) + d(v_2) + \cdots + d(v_n) = 2e$

$\Rightarrow 2 + 2 + \cdots \, n$ terms $= 2e = 2(7) = 14$

$\Rightarrow 2n = 14 \Rightarrow n = 7$.

(8) Is there a simple graph, corresponding to the following degree sequences?

(i) (1,1,2,3) (ii) (2,2,4,6). Here 4 degrees exist so $\exists$ 4 vertices.

<u>Sol</u>: (i) Here the sum of degrees = 1+1+2+3 = 7, odd. So there exist no graph corresponding to this degree sequence.

(ii) Given degree sequence = (2,2,4,6)

Here no. of vertices = 4

Max. degree of a vertex = 6

$\therefore$ max. degree of $v = 4-1 = 3 \neq 6$.

But this is not possible, because max. degree cannot exist one less than the no. of vertices. i.e., Max. degree of $v = n-1$

(9) Does there exist a simple graph with seven vertices having degrees (1,3,3,4,5,6,6)

<u>Sol</u>: Suppose there exist a simple graph G with seven vertices satisfying the given properties.

Here two vertices are of degree 6.

ie; each of these two vertices is adjacent to every other vertex.

i.e; the degree of each vertex is atleast 2. [$\because$ total vertices = 7]

i.e; The graph has no vertex of degree 1.

But this is absurd [$\because$ Here given one vertex is of degree 1]

$\therefore$ There does not exist a simple graph with the given properties.

<u>Types of graphs</u>:

$\rightarrow$ A graph which contains only isolated vertices is called a <u>Null graph</u>.

Null graph of n vertices is denoted by N_n.

<u>Eg</u>: N_4 is denoted by

v_1 . . v_2

v_3 . . v_4

$\rightarrow$ A simple graph G is said to be <u>Complete graph</u> if G contains exactly one edge between each pair of distinct vertices.

→ Complete graph of 'n' vertices is denoted by K_n.

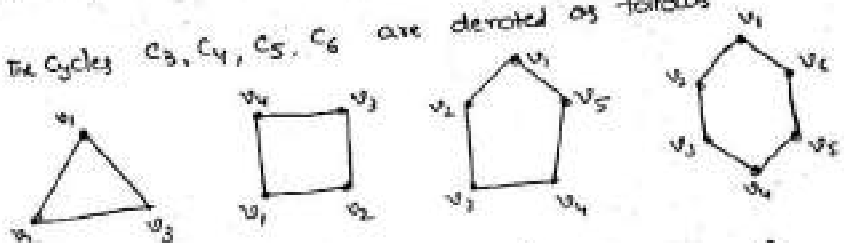
Eg:

K_2 K_3 K_4 K_5

→ K_n has exactly $\frac{n(n-1)}{2}$ edges.

→ The cycle graph C_n of length n $(n \geq 3)$ consists of n vertices $v_1, v_2, \dots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \dots, \{v_{n-1}, v_n\}$ and $\{v_n, v_1\}$.

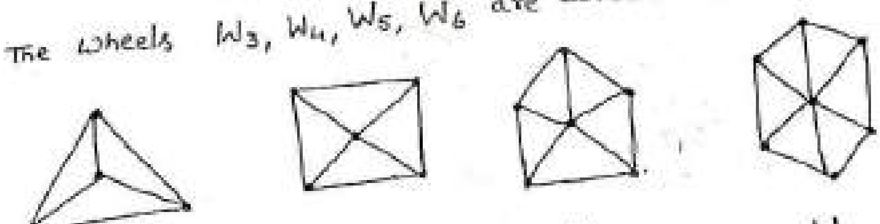
Eg: The cycles C_3, C_4, C_5, C_6 are denoted as follows.



C_3 C_4 C_5 C_6

→ The wheel graph W_n is obtained from C_n by adding a vertex v inside C_n and connecting it to every vertex in C_n.

Eg: The wheels W_3, W_4, W_5, W_6 are denoted as follows:



W_3 W_4 W_5 W_6

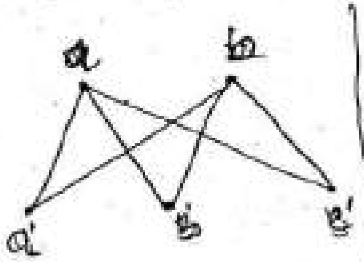
→ A graph $G = (V, E)$ is said to be Bipartite graph. if the vertex set V can be partitioned into two subsets (disjoint) V_1 and V_2 such that every edge (in E) connects a vertex in V_1 and a vertex in V_2. (no edge in G connects either two vertices in V_1 or in V_2).

→ The Complete bipartite graph on m and n vertices, denoted by $K_{m,n}$ is the graph, whose vertex set is partitioned into sets V_1 with m vertices and V_2 with n vertices in which there is an edge between each pair of vertices v_1 and v_2, where $v_1 \in V_1$ and $v_2 \in V_2$.

→ $K_{m,n}$ has $m+n$ vertices,
mn edges.

Eg: The complete bipartite graphs $K_{2,3}$, $K_{2,4}$, $K_{3,3}$ are as follows.

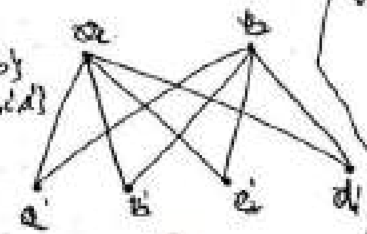
here $V_1=\{a,b\}$
$V_2=\{a',b',c'\}$



$K_{2,3}$

no of vertices $= 2+3=5$
no of edges $= 2\times3=6$

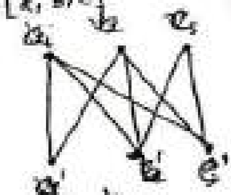
here
$V_1=\{a,b\}$
$V_2=\{a',b',c',d'\}$



$K_{2,4}$

no of vertices $= 6$
no of edges $= 8$

here $V_1=\{a,b,c\}$
$V_2=\{a',b',c'\}$



$K_{3,3}$

no of vertices $= 6$
no of edges $= 9$.

→ A complete bipartite graph $K_{m,n}$ is not Regular if $m \neq n$.

→ The maximum no of edges in a complete bipartite graph of n vertices is $\frac{n^2}{4}$.

Subgraphs and Isomorphic graphs:

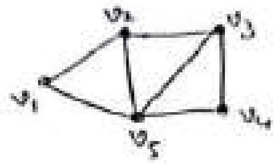
→ A graph G_1 is said to be a subgraph of graph G if the following conditions hold.

(i) All the vertices and edges of G_1 are in G
(ii) Every edge of G_1 has the same end vertices in G as in G_1.

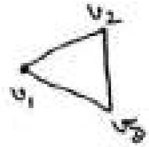
→ If all the vertices of G are in G' then the subgraph G' is called the <u>Spanning subgraph</u> of G.

→ If $G = (V, E)$ is a graph and $G_1 = (V_1, E_1)$ is a subgraph of G such that every edge $e = \{A, B\}$ of G, where $A, B \in V_1$ is an edge of G_1 also, then G_1 is called the <u>Induced Subgraph</u> of G

<u>Eg:</u> Consider the graph G :

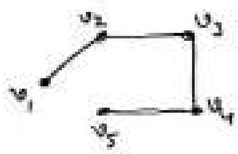


(i) Then the graph G_1 :



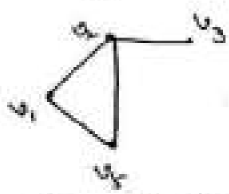
is a Subgraph of G.

(ii) The graph G_2 :



is a Spanning Subgraph of G.

(iii) The graph G_3 :



is an Induced subgraph of G.

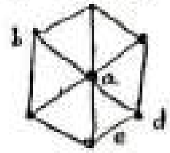
This is induced by set of vertices $V_1 = \{v_1, v_2, v_3, v_5\}$

→ Let G be a graph with m vertices & n edges.

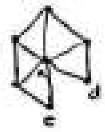
Then total <u>no</u> of subgraphs $= (2^m - 1) \times 2^n$.

<u>no</u> of Spanning subgraphs $= 2^n$

<u>Problem:</u> For the graph G shown below, draw the subgraphs (a) $G-e$ (b) $G-b$ (c) $G-c$

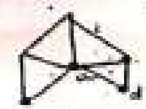


<u>Sol</u> : (a) $G-e$:



(b) $G-b$:

(c) $G-c$:



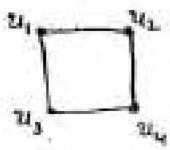
→ Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic if there exist a function $f : V_1 \to V_2$ such that

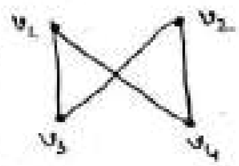
(i) f is 1-1 and onto (ie; f is bijection)

(ii) $\{a, b\}$ is an edge in E_1 iff $\{f(a), f(b)\}$ is an edge in E_2 for $a, b \in V_1$.

Note:
→ The number of vertices, number of edges, and the degrees of the vertices are all invariant under isomorphism. If any of these quantities are differ in two simple graphs, these graphs cannot be isomorphic.

Problems: ① Show that the graphs G and H are isomorphic, where

G : 

H : 

Sol: Here in both graphs no of vertices & no of edges are same, and also they have equal no of vertices of same degree.

The one-to-one correspondance between the vertices are:

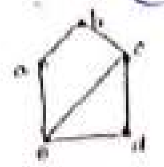
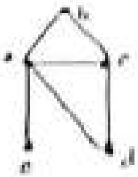
$u_1 \to v_1$
$u_2 \to v_3$
$u_3 \to v_4$
$u_4 \to v_2$

The correspondance between the edges are:

$\{u_1, u_2\} \to \{v_1, v_3\}$
$\{u_2, u_4\} \to \{v_3, v_2\}$
$\{u_3, u_4\} \to \{v_2, v_4\}$
$\{u_3, u_1\} \to \{v_1, v_4\}$

Thus the adjacent vertices in G are adjacent in H.

$\therefore$ G and H are isomorphic.

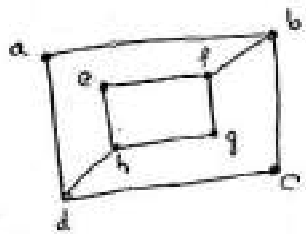
② Show that the graphs G  , H :  are not isomorphic

Sol:- Here Both G and H have 5 vertices, 6 edges.

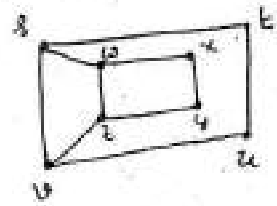
However, H has a vertex 'e' of degree 1, whereas G has no vertex of degree 1.

∴ G and H are not isomorphic.

③ Determine whether the following graphs are isomorphic.



G



H

Sol: Both the graphs have 8 vertices, 10 edges.

Also both have four vertices of degree 2 and four of degree 3.

But G & H are not isomorphic. Because in G, deg (a) = 2, So 'a' must correspond to either t (or) u (or) x (or) y in [redacted] H.

(Since these are the vertices of degree 2 in H)

here each of these four vertices t, u, x, y in H are adjacent to another vertex of degree 2 in H, which is not true for 'a' in G.

—

Connected Graphs :-

→ A walk is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices, such that each edge is incident with the vertices preceding and following it. [no edge appears more than once in a walk]

→ Vertices with which a walk begins and ends are called its terminal vertices.

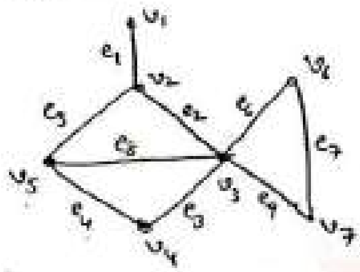
→ A walk begin and end at the same vertex is called a closed walk.

otherwise it is called an open walk.

→ An open walk in which no vertex appears more than once is called a path i.e; path does not intersect itself.

→ A closed walk in which no vertex appears more than once (except initial and final) is called a circuit.

Eg: Consider the following graph G:



Here : $v_1 e_1 v_2 e_2 v_3 e_6 v_6 e_7 v_7$ is a walk.

: Here v_1, v_7 are called terminal vertices.

: $v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_2$ is a closed walk.

: $v_3 e_6 v_6 e_7 v_7$ is a path.

: $v_2 e_5 v_5 e_4 v_4 e_3 v_3 e_2 v_2$ is a circuit.

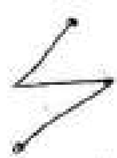
: $v_4 e_3 v_3 e_6 v_6 e_7 v_7 e_9 v_3 e_3 v_4 e_4 v_5$

is open walk

→ A graph G is said to be Connected if there is atleast one path between every pair of vertices in G. Otherwise G is disconnected.

→ A disconnected graph consists of two or more connected subgraphs each pair of which has no vertex in common. Each of these connected subgraphs is called a component.

Eg: The graph

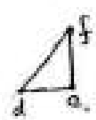


is connected

The graph

is disconnected. Here the subgraphs

and



are called Component

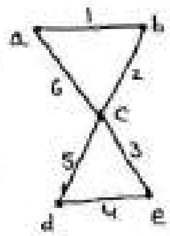
Euler circuits :

→ A circuit in a connected graph is called an Euler circuit if it contains every edge of the graph exactly once.

→ A connected graph with an Euler circuit is an Euler graph.

→ A closed walk in a graph contains all the edges of the graph then the walk is called an Euler line.

Eg: Consider the graph: G :



Here : 'a 1 b 2 c 3 e 4 d 5 c 6 a' is a circuit containing all the edges of G exactly once. So this is an Euler circuit and the graph G is an Euler graph.

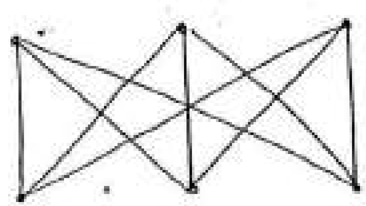
Note: → A connected graph G is an Euler graph iff all vertices of G are of even degree.

Problem: Show that the following graphs does not contain Euler circuit.

(i) G:

Sol: G is clearly connected but $\deg v_1 = 1 = \deg v_2$, odd, so G is not an Euler graph. So G does not contain an Euler circuit.

(ii)



: Here each vertex is of degree 3, odd so it does not contain an Euler circu-

(2) For what values of n is the graph of K_n Eulerian?

Sol: WKT K_n, the complete graph of n vertices is a connected graph in which ~~the vertices of~~ the degree of each vertex is $n-1$.

Since a graph is Euler graph if degree of every vertex is Even, So

$n-1$ is Even.

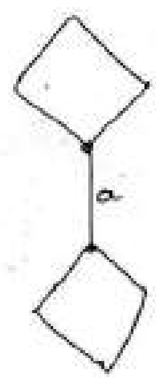
$\Rightarrow$ n is odd.

Thus K_n is Euler graph iff n is odd.

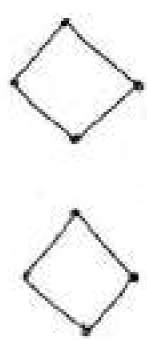
Note: The Complete graph $K_{m,n}$ is an Euler graph iff both m, n are Even.

2ks (3) Give an example of a connected graph G where removing any edge of G results in a disconnected graph.

Sol: Consider the following graph G.

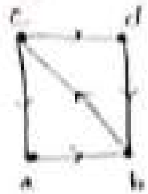


If we remove 'a' from G we get the disconnected graph G_1.



Note: A directed multigraph G has an Euler path if and only if it is connected and the indegree of each vertex is equal to its outdegree with the possible exception of two end vertices, for which the indegree of one is one larger than its outdegree and the indegree of the other is one less than its out degree.

Eg: Consider the directed multigraph G.



Here

Vertices	indegree	Outdegree
a	1	1
b	2	1
c	1	2
d	1	1

So, G has an Euler path : c – d – b – c – a – b.

Note: A directed multigraph G has an Euler circuit if and only if G is connected and indegree of every vertex = out degree of every vertex.

Eg: The above directed graph G has no Euler circuit, because indegree of every vertex ≠ outdegree of every vertex.

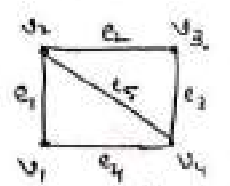
Hamiltonian graphs :

→ A cycle in a graph G that contain each vertex in G exactly once (except for the starting and end vertex) is called a Hamiltonian cycle.

→ A connected graph that contains a Hamiltonian cycle is called a Hamiltonian graph.

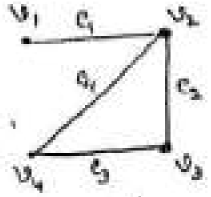
→ A Hamiltonian path is a simple path that contain all vertices of G.

Eg: Consider the graph G :



Here $v_2 e_2 v_3 e_3 v_4 e_4 v_1 e_1 v_2$ is a Hamiltonian cycle.

So G is a Hamiltonian graph.

Eg: 2 . The graph G : 

has no Hamiltonian Cycle. So This graph is not a Hamiltonian graph.

But here $v_1 e_1 v_2 e_2 v_3 e_3 v_4$ is a Hamiltonian path.

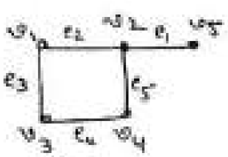
Some basic Rules for Constructing Hamiltonian paths and Cycles :

Rule (1): If G has n vertices, then a Hamiltonian path must Contain exactly $(n-1)$ edges and a Hamiltonian cycle must Contain exactly n edges.

Rule (2): If a vertex 'v' in G has degree 'n', then a Hamiltonian path must contain at least one edge incident on v and atmost two edges incident on v. A Hamiltonian cycle Contains exactly two edges incident on v. i.e; there cannot be 3 (or) more edges incident with one vertex in a Hamiltonian Cycle.

Rule (3): No cycle which does not contain all vertices of G can be formed when constructing a Hamiltonian path (or) Cycle.

Rule (4): once the Hamiltonian Cycle we are Constructing has passed through a vertex 'v', then all other unused edges incident on v Can be deleted

Problems: (1) Is the graph G : 

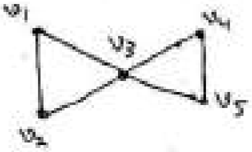
has HP, HC ?

Sol Here $v_5 e_1 v_2 e_2 v_1 e_3 v_3 e_4 v_5$ is a H.P.

but G has no HC, because $\deg(v_5) = 1$ (by Rule (2)).

HP: Hamiltonian path
HC: Hamiltonian Circuit

(2) Is the graph G:



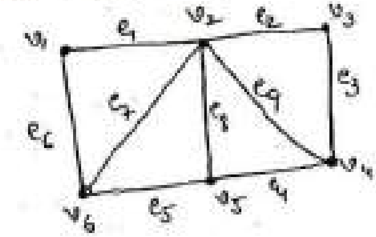
has H.P, HC?

Sol: Here $deg(v_3) = 4$, So G has no HC. (by Rule (2))

Also here $n = 5$, so H.P must contains $n-1 = 4$ edges, but here no of edges $= 6$. So G has no HP.

(3) Give an example of a graph which is Hamiltonian but not Eulerian and viceversa.

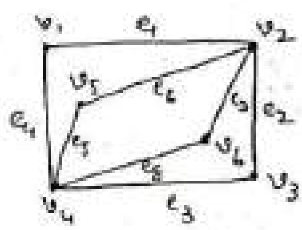
Sol: The following graph G_1 is Hamiltonian but not Eulerian.



In this graph $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_6 e_6 v_1$ is a HC So G_1 is Hamiltonian.

But G_1 is not Eulerian, because all vertices are not of an Even degree.

The following graph G_2:



is Eulerian but not Hamiltonian.

Reason: ∴ All vertices are of even degree So G_2 is Eulerian.

∴ G_2 does not contain a HC, because $deg(v_2) = 4$

Note:- (1) A simple connected graph with n vertices $(n \geq 3)$ is Hamiltonian if the sum of degrees of every pair of non-adjacent vertices is $\geq n$.

(2) A simple connected graph with n vertices $(n \geq 3)$ is Hamiltonian if degree of every vertex is $\geq \frac{n}{2}$.

(3) K_n is always a HG [∵ degree of every vertex $> n/2$]

Hamiltonian planar graphs:

→ Let G be a plane Hamiltonian graph with 'n' vertices and suppose C is a fixed HC in G. w.r.t. this cycle, a diagonal is an edge of G that does not lie on C.

→ Grinberg theorem: Let G be a simple plane graph with n vertices and let C is a fixed HC in G. Then w.r.t. this cycle C,

$$\sum_{i=3}^{n} (i-2)(r_i - s_i) = 0.$$

where r_i : number of internal regions in G of degree i.
s_i : " " external regions

Proof: Suppose there are 'd' diagonals of G in the interior of C.
Since G is planar, so no two of these diagonals cross each other.
Further each diagonal is an edge for exactly two regions in the interior of C.
Thus a diagonal splits the region through which passes into two parts.
If we consider drawing in these diagonals one at a time, after each one is drawn, we increase the number of regions of C by one.
ie: 'd' diagonals divide the interior of C into '$d+1$' regions.

$$\therefore \sum_{i=3}^{n} r_i = d+1 \Rightarrow d = \sum_{i=3}^{n} r_i - 1 \quad \text{—(1)}$$

Let R = The total sum of degrees of all regions interior to C.

Then $R = \sum_{i=3}^{n} i r_i$.

But in obtaining this total for R, each diagonal is counted twice and
(∵ it is an edge for exactly two of the regions)

each of the 'n' edges on C is counted once (∵ each bounds only one region interior to C)

So $k = \sum_{i=3}^{n} i r_i = 2d + n.$

$= 2\left(\sum_{i=3}^{n} r_i - 1\right) + n$ (by (1))

$= 2\sum_{i=3}^{n} r_i - 2 + n$

$\Rightarrow \sum_{i=3}^{n} i r_i - \sum_{i=3}^{n} 2 r_i = n-2$

$\Rightarrow \sum_{i=3}^{n} (i-2) r_i = n-2. \longrightarrow (2)$

Illy Considering the external regions of C, we get

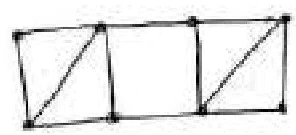
$\sum_{i=3}^{n} (i-2) s_i = n-2 \longrightarrow (3)$

$\therefore$ (2)−(3) $\Rightarrow \sum_{i=3}^{n} (i-2)(r_i - s_i) = 0.$

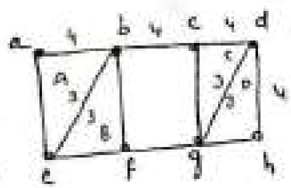
=

Problem: Use the Grinberg theorem, find the HC in the following graph G:

2K7, 2K5



Sol: Consider the given graph G:



Here there are four regions (A,B,C,D) of degree 3 and four regions of degree 4.

Suppose there is a HC in G.

Then by Grinberg theorem, $r_3 + s_3 = 4$, $r_4 + s_4 = 4$

$\therefore \sum_{i=3}^{4} (i-2)(r_i - s_i) = 0 \Rightarrow (3-2)(r_3 - s_3) + (4-2)(r_4 - s_4) = 0$

$\Rightarrow r_3 - s_3 + 2(r_4 - s_4) = 0$

$\Rightarrow r_3 - s_3 = -2(r_4 - s_4) \longrightarrow (1)$

Scanned with CamScanner

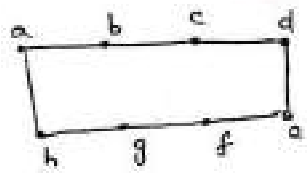
$\Rightarrow$ '$r_3 - s_3$ must be an even integer.

Since $r_3 + s_3 = 4$, So the only possible values for r_3 & s_3 are 4; 1 & 3; 2 and 2

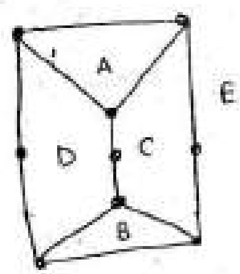
Clearly the difference of these gives an even number.

$\therefore$ our assumption that there exist a HC in G is true.

The HC for G is



② Use Grinberg theorem to establish that the following plane graph is not Hamiltonian.



Sol: Suppose there is a HC. Here there are two Cases.

Case (i): Out of two regions A, B; one is inner & other is exterior.

Let us take A as inner region.

Then A, C, D are inner regions with degrees 3, 6, 6 respectively.

$\therefore$ r_3 = No of inner regions of degree 3 = 1

r_4 = " " " 4 = 0

r_5 = " " " 5 = 0

r_6 = " " " 6 = 2

The region B and E (infinite region) are exterior regions with degrees 3 and 6 respectively.

$\therefore$ s_3 = no of exterior regions of degree 3 = 1

s_4 = " " " 4 = 0

s_5 = " " " 5 = 0

s_6 = " " " 6 = 1

$$\therefore \sum_{i=3}^{6} (r_i - s_i)(i-2) = 1(r_3 - s_3) + 2(r_4 - s_4) + 3(r_5 - s_5) + 4(r_6 - s_6)$$

$$= (1-1) + 2(0-0) + 3(0-0) + 4(2-1)$$

$$= 4 \neq 0$$

$\therefore$ By Grinberg theorem, there is no HC in G.

Hence G is not Hamiltonian.

Case (ii): Suppose both the regions A and B either inner (or) exterior.

Suppose both are inner.

$\therefore r_3 = 2, r_4 = 0, r_5 = 0, r_6 = 2$ &

$s_3 = 0, s_4 = 0, s_5 = 0, s_6 = 1$.

$$\therefore \sum_{i=3}^{6} (i-2)(r_i - s_i) = (2-0) + 2(0-0) + 3(0-0) + 4(2-1) = 2+4 = 6 \neq 0$$

$\therefore$ By Grinberg theorem, G is not Hamiltonian.

Note: (1) The complete graph K_n has always a HC.

(2) The complete bipartite graph $K_{m,n}$ is Hamiltonian $\Leftrightarrow$ m = n & n > 1

Graph Colouring & Chromatic Numbers:

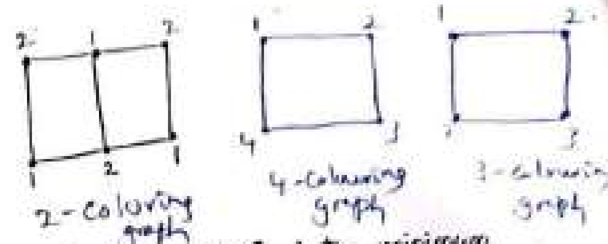
→ A coloring of a simple graph is an assignment of colours to its vertices such that no two adjacent vertices are assigned the same colour.

In, this case we say G is coloured.

→ The n-colouring of G is a colouring of G using n-colours.

→ If G has n-colouring, then G is said to be n-colourable.

Eg: The graph of 2-coloring is :



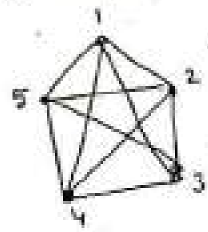
Chromatic Number: The chromatic number of a graph G is the minimum number of colors required for coloring of the graph.

It is denoted by $\chi(G)$. $\chi(G) = n$ means G is n-chromatic.

Eg: The chromatic number of above graph is 2.

Note → ① $\chi(K_n) = n$ where K_n is complete graph of n vertices.

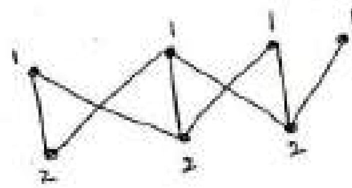
Eg: $\chi(K_5) = 5$:



② $\chi(K_{m,n}) = 2$, where $K_{m,n}$ is bipartite graph.

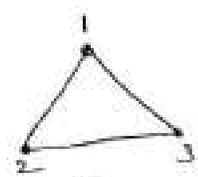
ie, the chromatic number of every bipartitegraph is 2.

Eg: $\chi(K_{4,3}) = 2$:

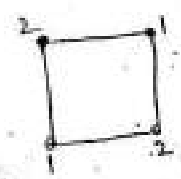


③ $\chi(C_n) = 2$ if n is even
$= 3$ if n is odd.

Eg: $\chi(C_3) = 3$:



$\chi(C_4) = 2$:



(4) A graph consisting of only isolated vertices is 1-chromatic

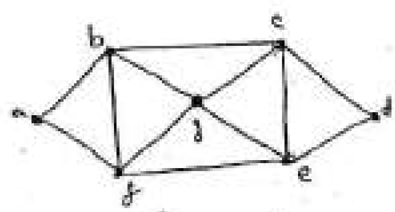
(5) $\chi(G) \leq$ no of vertices of G.

(6) Every tree with two (or) more vertices is 2-chromatic.

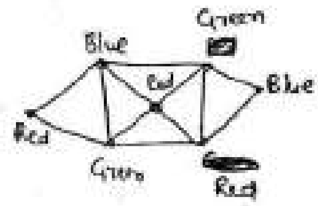
(7) Any planar graph is 4-colourable. This is called four color problem.

Problem: Find the chromatic numbers of the following graphs.

(i)

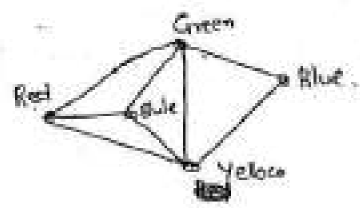


Sol: Consider the given graph. We assign the colours to each vertex so that no two adjacent vertices have the same color.



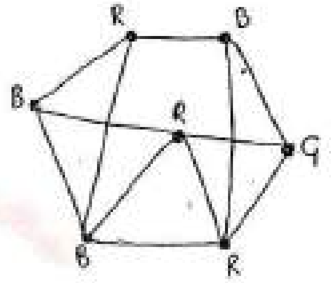
∴ The chromatic no = 3.

(ii) Chromatic no of



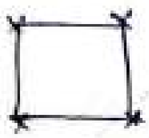
is 4

(2) Find the chromatic number of the following graph:

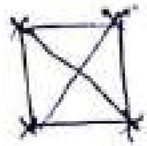


Sol: The chromatic number = 3

Regular graph: A graph in which all the vertices are of the same degree k is called a regular graph of degree k or a k regular graph.

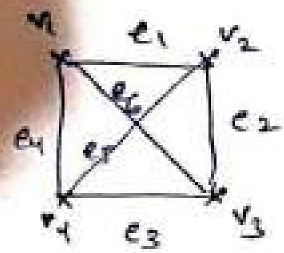


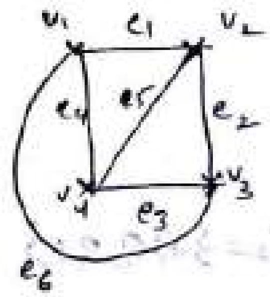
2-regular graph



3-regular graph

Planar graph: A graph G is said to be planar if it can be drawn in the plane without its edge crossing, otherwise G is non planar

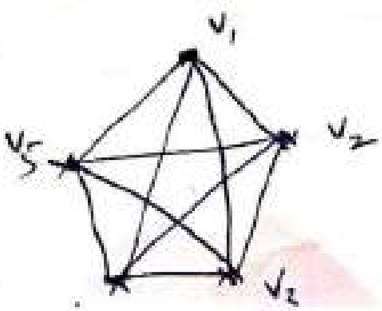


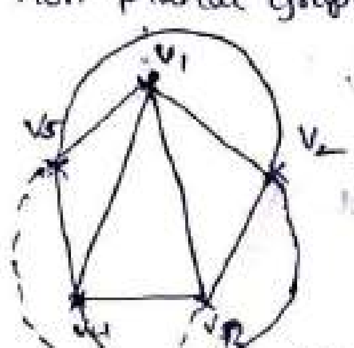


Above is a planar graph because no edge is crossing the another edge.

Q) Show that k_5 is a non planar graph.

Sol)





Fundamental theorem of graph theory (hand shaking property):

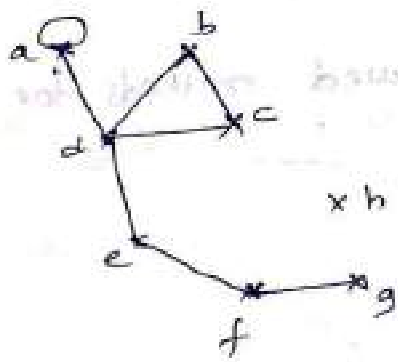
~~Let G be a graph with |E| edges and n vertices~~

Statement: Let G be an undirected graph with $|E|$ edges and $|V| = n$ vertices then $\sum_{i=1}^{n} d(v_i) = 2|E|$

Proof: Let G be a graph with $|E|$ edges and n vertices $v_1, v_2, \ldots, v_n$. When we sum over the degrees of all vertices we count each edge (v_i, v_j) twice.

Once when we count it as (v_i, v_j) in the degree v_i and again we count it as (v_j, v_i) in the degree v_j.

Then the total no. of degrees of v_i $\left(\sum_{i=1}^{n} d(v_i)\right) = 2|E|$



$d(a) = 3$
$d(b) = 2$
$d(c) = 2$
$d(d) = 4$
$d(e) = 2$
$d(f) = 2$
$d(g) = 1$
$d(h) = 0$

$\sum d(v_i) = 16$

no. of edges = 8

$\therefore \sum d(v_i) = 2|E|$

Q) Can there be graph consisting of the vertices a, b, c, d, e with degree $d(a) = 2$, $d(b) = 3$, $d(c) = 2$, $d(d) = 2$, $d(e) = 0$

Sol) $\sum_{i=1}^{n} d(v_i) = 9$, 9 cannot be expressed in the form $2|E|$

$\therefore$ Graph doesn't exist.

Q) Does there exist a graph with 12 vertices such that two of the vertices have degree 3 and remaining have degree 4. Find the edges of the graph

Sol) $\sum d(v_i) = 2\times3 + 4\times10$

$= 46$

By fundamental theorem of graph theory,

$\sum d(v_i) = 2|E|$

$46 = 2|E|$

$|E| = 23$

$\therefore$ No. of edges $= 23$

Representation of graphs:

There are two most commonly used methods for representing the graphs. they are

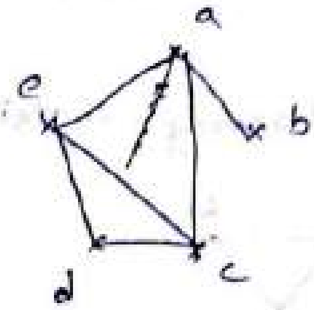
1) Adjacency list representation

2) Adjacency matrix representation

Adjacency list representation: - This type of representation of a graph G with n vertices stores an information only about the edges which are in a graph. It contains a list of vertices $v_1, v_2, \ldots, v_n$ together with a separate list for each vertex v_i.

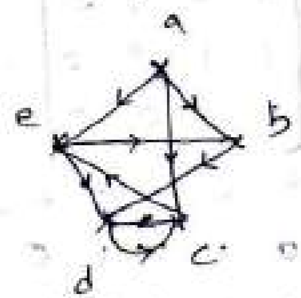
Ex: Draw a graph to the following adjacency list.

vertex	adjacent vertices
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, d, c



Draw a digraph

Vertex	adjacent vertices
a	b, c, e
b	d
c	e, d
d	c
e	b, d



Adjacency matrix representation:

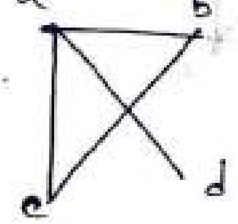
Suppose that G is a simple graph with $|V| = n$ and the vertices of G are listed arbitrarily as $1, 2, 3, \ldots, n$. The adjacency matrix A w.r.t this listing of the vertices is an $n \times n$ matrix with one as its entry when i and j are adjacent and 0 as the entry when they are not adjacent.

$\therefore a_{ij} = 1$ if i, j is an edge in G, 0 otherwise

$$a_{ij} = \begin{cases} 1, & \text{if } (i,j) \text{ is an edge in } G \\ 0, & \text{otherwise} \end{cases}$$

Q) Find adjacency matrix to the following graph

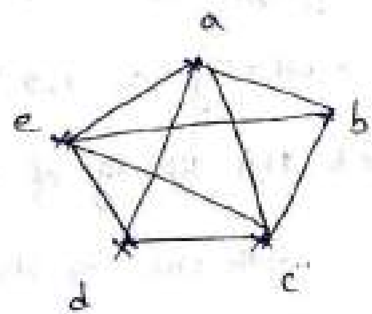
Sol)



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Q) If $A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$. Draw the graph

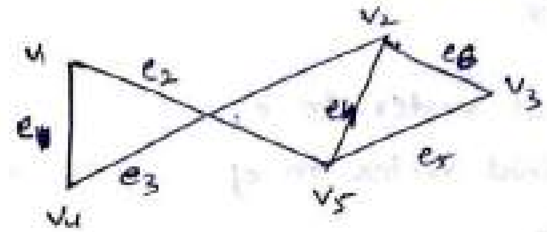
Sol)



Incidence matrix (Undirected graph):

Let G be an undirected graph. Suppose that 1,2,3,...,n are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G then incidence matrix w.r.t ordering of vertices and edges is an $n \times m$ matrix

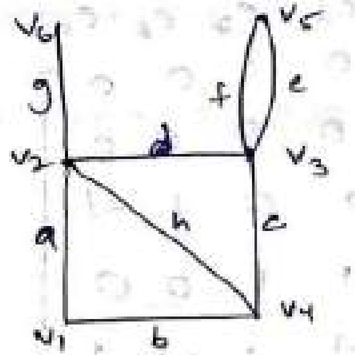
$$M = [m_{ij}] = \begin{cases} 1, & \text{when edge } e_j \text{ is incidence with } i \\ 0, & \text{otherwise} \end{cases}$$

1) 

Find incidence matrix . the following graph.

Sol)

$$\begin{array}{c c} & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2) 

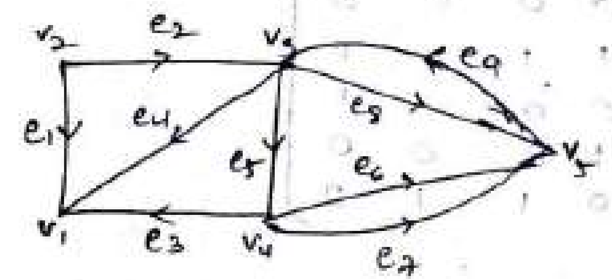
$$\begin{array}{c c} & \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Incidence matrix (Di-graph) :

The incidence matrix with the vertex set $\{v_1, v_2, \ldots, v_n\}$ and the edge set $\{e_1, e_2, \ldots, e_m\}$ and with no self loops is an $n \times m$ matrix

$$M = b_{ij} = \begin{cases} -1 & v_i \text{ is final vertex to } e_j \\ 1 & v_i \text{ is initial vertex to } e_j \\ 0 & \text{otherwise.} \end{cases}$$

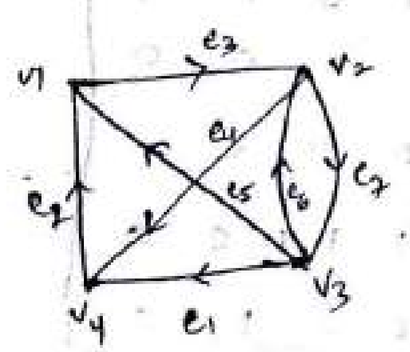
Q) Find incidence matrix to the following graph.



Sol)

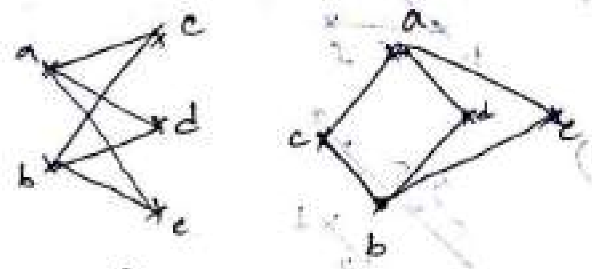
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array} \\ \begin{bmatrix} -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

Q) Find incidence matrix



Plane graphs: A plane graph G is said to be planar if it can be drawn in a plane so that its edges doesn't cross over. / A planar graph is a plane graph if it is already drawn in a plane by changing the place of the vertices so that no two edges crossover.

Ex: $K_{2,3}$ biparted graph

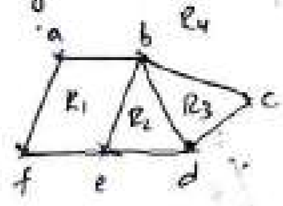


Region (or) ~~phase~~ face of a graph:

If a connected plane graph is drawn in the same plane so that the plane is divided into continuous regions called faces.

A face is characterised by the cycle that forms its boundary.

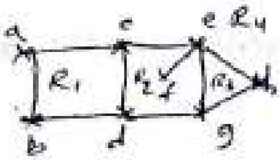
Ex:



R_1 is bounded by a–b–e–f–a
R_2 is bounded by b–d–e–b
R_3 is bounded by b–c–d–b
R_4 is bounded by a–b–c–d–e–f

Degree of a region: It is the length of the closed path boundaring the region.

Ex:

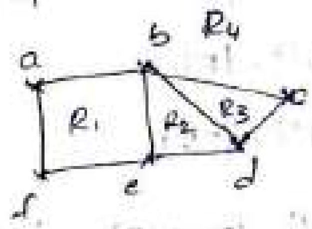


Region	Path	degree
R_1	a – c – d – b – a	4
R_2	c – e – f – e – g – d – c	6
R_3	e – h – g – e	3
R_4	a – c – e – h – g – d – b – a	7

Theorem ①: If G is a plane graph then the sum of degrees of the regions determined by G is twice the no. of edges of G.

Proof: Each edge of the plane graph is the boundary of two regions (or) is contained in a region and therefore it occurs twice in any path along the boundary of that region.

Thus every edge in a plane graph contribute twice in finding the degrees of regions of G.



Sum of degrees of the regions = 16

no. of edges = 8 = |E|

$\therefore \sum d(R_i) = 2|E|$

Eulers theorem: ~~Let~~ G be plane graph with R regions then $|V| - |E| + |R| = 2$ where $|V|, |E|$ are no. of vertices and no. of edges.

Scanned with CamScanner

Proof: We prove the theorem by using mathematical induction method on regions

Let $|R|=1$, then the graph is a spanning tree in the plane with edges $|V|=|E|-1$

i.e $|R|=1$

$$|V|-|E|+|R|=2$$

$$|V|-|E|+1=2$$

$$|E|=|V|-1$$

$\therefore$ the theorem is true for $|R|=1$

Let the theorem be true for $|R|=K$ and we prove that the theorem is true for $|R|=k-1$

From the graph G, delete one edge common to boundary of two regions then new graph G' will have one edge less than G and one region less than G

$$|V'|=|V|$$

$$|E'|=|E|-1$$

$$|R'|=|R|-1$$

$$\therefore \quad |V|-|E|+|R| = |V'|-\left[|E'|+1\right]+|R'|+1$$

$$= |V'|-|E'|+|R'|$$

Since $|V|-|E|+|R|=2 \Rightarrow |V'|-|E'|+|R'|=2$

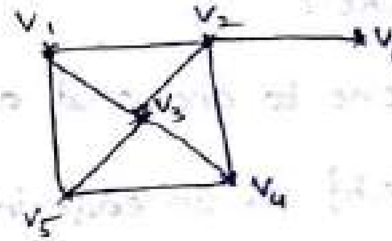
By the method of mathematical induction, it is t[rue] for all values of n.

Dual of a graph: A graph G has the dual implies that it must be a plane graph.

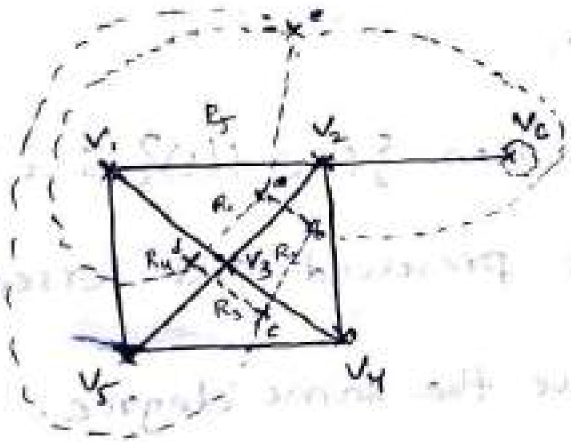
An edge forming a self loop in G yields a pendent edge in G'. Corresponding to each region of G, there is a vertex in G'.

Corresponding to each edge in G, there is an edge in G' then G' is said to be dual of G.

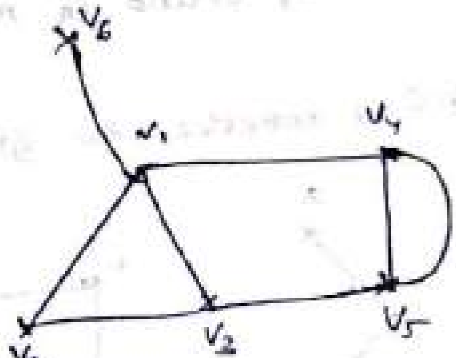
Q) Draw the geometrical representation of the dual of the following graph.

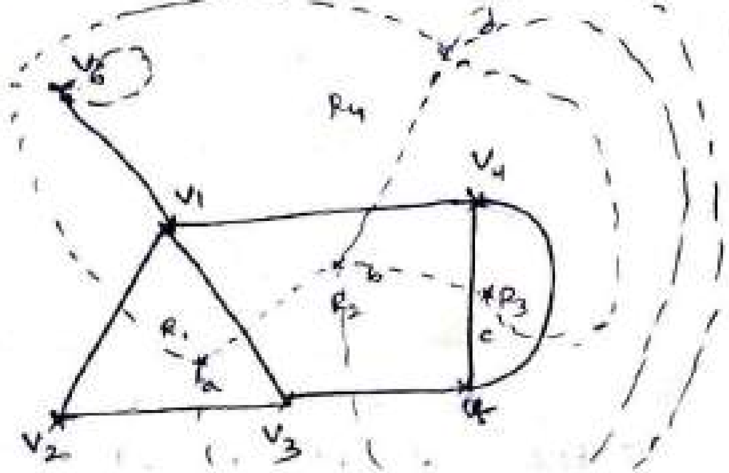


Sol)



(ii)





Hence ...
An Euler path or an ...

8.4 HAMILTONIAN GRAPHS

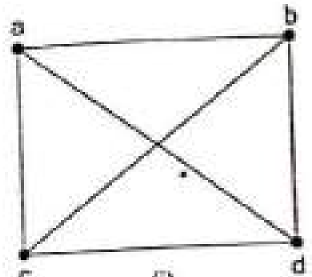
Definition : *A path in a graph G is called a* ***Hamiltonian path*** *if it contains every vertex of G.*

Definition : *A cycle in a graph G is called a* ***Hamiltonian cycle*** *if it contains every vertex of G.*

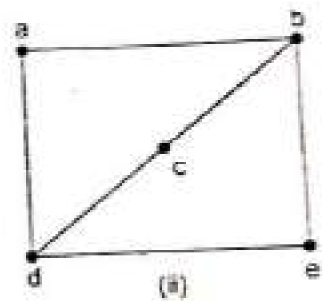
Definition : *A graph G is said to be a* ***Hamiltonian graph*** *if it contains a Hamiltonian cycle.*

Note : An Eulerian cycle uses every edge exactly once but may repeat vertices, while a Hamiltonian cycle uses each vertex exactly once (except for the first and last) but may skip edges.

Example 1. Determine which of the following are Hamiltonian graphs :



(i)



(ii)

(*i*) is Hamiltonian graph since it has an Hamiltonian cycle *a-b-c-d-a*.

(*ii*) is not Hamiltonian graph since it has no Hamiltonian cycle, but it has a Hamiltonian path *a-d-e-b-c*.

Rules for Constructing Hamiltonian Paths and Cycles in a graph G :

Rule 1. If G is a graph with 'n' vertices then a Hamiltonian cycle must contain exactly 'n' edges, and a Hamiltonian path must contain exactly '$n - 1$' edges.

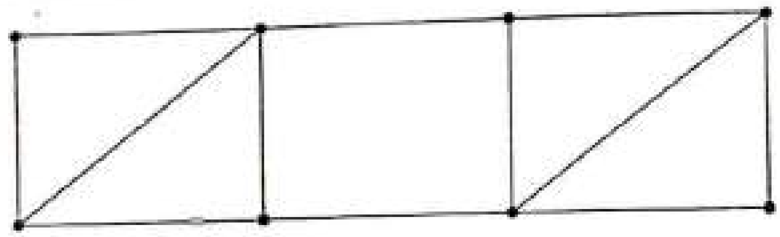
Rule 2. If 'v' is a vertex of degree 2 then both edges incident on 'v' should contain in every Hamiltonian cycle and every Hamiltonian path should contain at least one edge incident on v.

In general if degree of 'v' is 'k' then a Hamiltonian cycle contain exactly two edges incident on v and a Hamiltonian path must contain at least one edge incident on 'v' and at most two edges incident on 'v'.

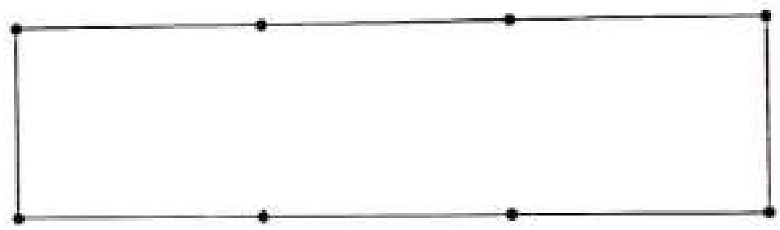
Rule 3. No cycle that does not contain all the vertices of G can be formed when building a Hamiltonian path or cycle.

Rule 4. Once a Hamiltonian cycle we are constructing has passed through a vertex 'v', then all the unused edges incident on v can be deleted.

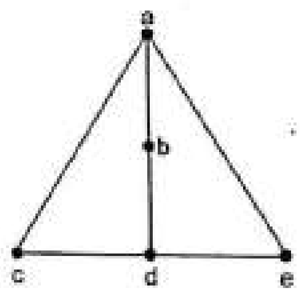
Example 2. Find the Hamiltonian cycle in the following graph.



The Hamiltonian cycle in the graph is



Example 3. Show that the following graph is not Hamiltonian.



Here deg $(b) = 2$, hence $\{b, a\}$, $\{b, d\}$ should be included in every Hamiltonian cycle. Also deg $(c) = 2$, therefore $\{c, a\}$ and $\{c, d\}$ should be in every Hamiltonian cycle. Thus a-b-d-c-a forms a cycle without containing the vertex 'e'. Hence it is not an Hamiltonian graph.

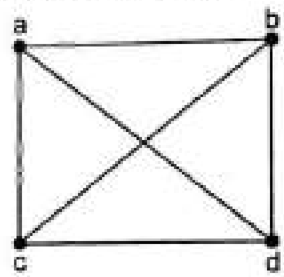
Example 4. Show that the following graph has no Hamiltonian cycle. But the graph has a Hamiltonian path.

7.5 PLANAR GRAPHS

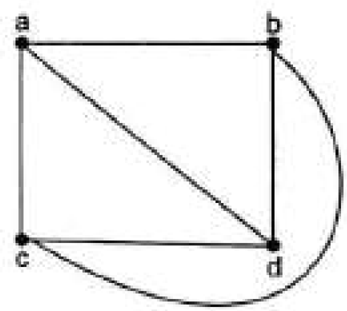
Definition. *A graph G is said to **be planar** if it can be drawn in a plane so that its edges do not cross over.*

Definition : *A planar graph is a **plane graph** if it is already drawn in the plane so that no two edges cross over.*

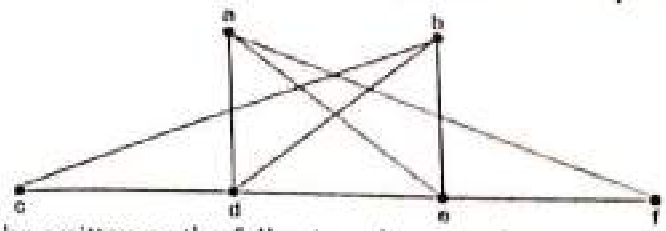
Example 1. The graph G shown below is planar.



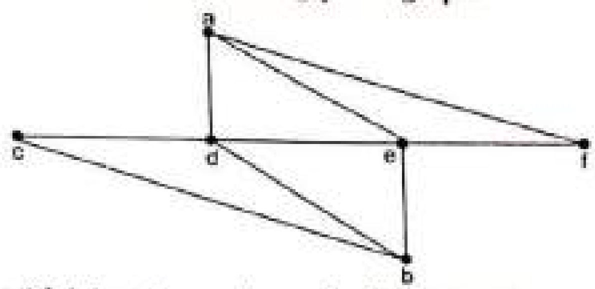
Since it can be written as



Example 2. Show that the following graph G is planar and draw plane graph of it.



It is planar. It can be written as the following plane graph.



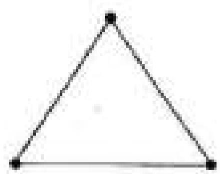
Definition : *A graph with 'n' vertices so that each of the 'n' vertices are adjacent to each of the other n-1vertices is called a* ***complete graph*** *and is denoted by* k_n.

Example 3. Show that k_n is planar for $n = 1, 2, 3, 4$.

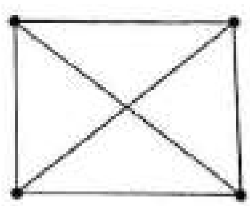
k_1 is planar.

k_2 is planar

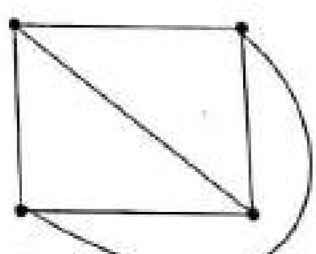
k_3 is planar



k_4 :

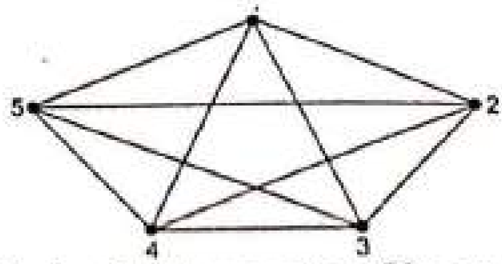


k_4 is planar since it can be written as

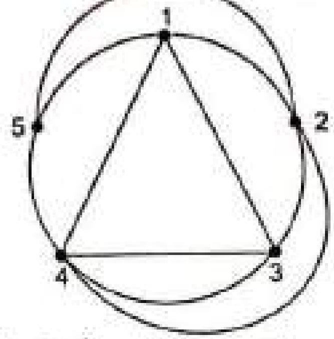


Example 4. Show that k_5 is not planar.

The graph k_5 is



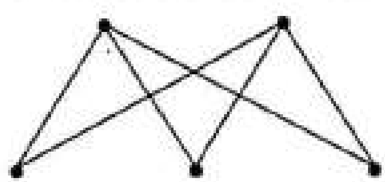
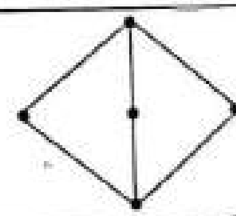
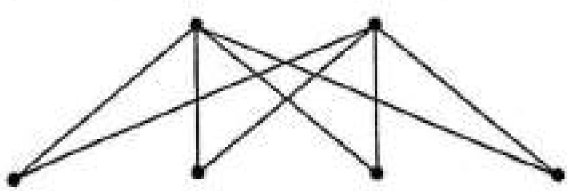
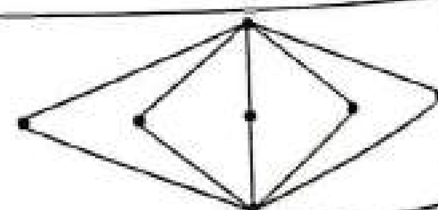
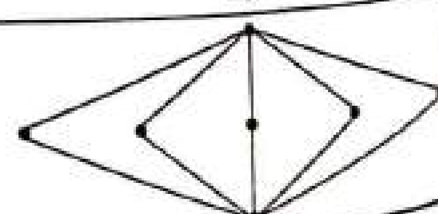
Now we attempt to write this k_5 without cross overs. Observe the vertices are symmetric.



We remain to draw the edge {3, 5}. Since all the vertices forms a circle, the edge {3, 5} should be drawn entirely inside the circle or outside the circle, otherwise it crosses the circle. It is not possible to draw {3, 5} inside the circle since it crosses {1, 4}. Also it is not possible to draw {3, 5} outside the circle. Since it crosses the edge {2, 4}. Thus, it is impossible to draw the edge {3, 5} without cross over. Thus k_5 is non-planar.

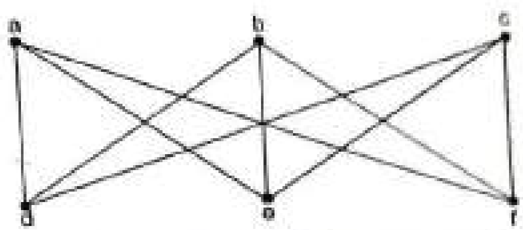
Definition : *A graph with $m + n$ vertices so that each of the first m vertices are adjacent to each of the second n vertices and there are no edges between the first m vertices also there are no edges between the second n vertices is called a* ***complete bipartite graph*** *and is denoted by $k_{m, n}$.*

Example : Show that $k_{m, n}$ is planar if either $m \leq 2$ or $n \leq 2$. (JNTU, MCA, Feb 2007)

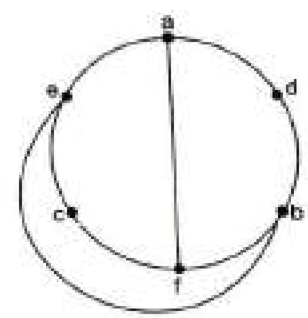
Name	*Graph*	*Plane Graph*
$k_{2,3}$		
$k_{2,4}$		
$k_{2,5}$		

We can write $k_{2,n}$ as n vertices in a row and the 2 vertices, one vertex above the row and the other below the row. Thus, $k_{m,n}$ is planar if either $m \leq 2$ or $n \leq 2$.

Example 5. Show that $k_{3,3}$ is non planar.



We form a circle containing all the vertices and then we try to draw remaining edges without cross over. Observe the vertices are symmetric :

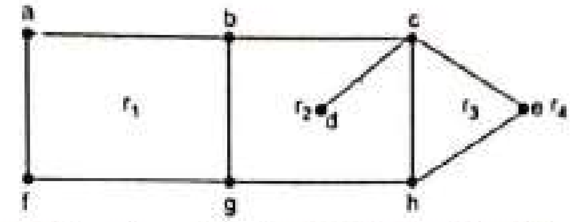


We remain to draw the edge $\{c, d\}$. It should be drawn either entirely inside the circle or outside the circle. When we attempt to draw $\{c, d\}$ inside the circle it crosses $\{a, f\}$. To draw $\{c, d\}$ outside the circle it crosses $\{b, e\}$. Thus, it is not possible to draw the edge $\{c, d\}$ without cross over. Therefore, $k_{3,3}$ is non-planar.

the 3 utilities to each of the 3 houses without the lines crossing.

Definition : *A plane graph divides the plane into connected areas called* ***regions*** *or* ***faces****. Each plane graph G determines a region of infinite area called the exterior region of G.*

Example 7. The following plane graph G has 4 regions r_1, r_2, r_3, r_4 in which r_4 is the exterior region.



Definition : ***The degree of a region is the length of the closed path bordering the region.***

Example 8. Find the degrees of the regions of the graph shown in example 4.8.

Region	border	degree
r_1	$a-b-g-f-a$	4
r_2	$b-c-d-c-h-g-b$	6
r_3	$c-e-h-c$	3
r_4	$a-b-c-e-h-g-f-a$	7

Note. V (G) = The set of all vertices of G

E (G) = The set of all edges of G

R (G) = The set of all regions of plane graph G.

Theorem : If G is a plane graph, then the sum of the degrees of the regions determined by G is twice the number of edges of G.

i.e.,
$$\sum_{r \in R(G)} \deg(r) = 2\,|E|$$

where | E | is the number of edges of G.

Proof : Each edge of the plane graph is a border of two regions or is contained in a region and will therefore occur twice in any path along the border of that region. Thus, every edge in the plane graph contribute two in determining the degrees of the regions of G.

Example 9. Verify sum of degrees theorem for the plane graph shown in example 4.8.

Number of edges in G is 10.

i.e.,
$$|E| = 10.$$

$$\sum_{r \in R(G)} \deg(r) = \deg(r_1) + \deg(r_2) + \deg(r_3) + \deg(r_4)$$
$$= 4 + 6 + 3 + 7$$
$$= 20 = 2 \times 10$$
$$= 2\,|E|$$

Example 10. Find the number of edges in a plane graph G containing 4 regions each of degree 3.

By sum of degrees theorem

$$\sum \deg(r) = 2\,(E)$$
$$4 \times 3 = 2\,|E| \;\Rightarrow\; |E| = 6.$$

∴

Theorem. (Euler's Formula) : If G is a connected plane graph, then $|V| - |E| + |R| = 2$, where $|V|$, $|E|$ and $|R|$ are respectively the number of vertices, edges and regions of G.

(JNTU, November 2006, Set 3)

Proof : We prove the theorem by mathematical induction on number of regions determined by G.

Let $|R| = 1$, then G is a tree, hence we know that $|E| = |V| - 1$

$$\therefore \quad |V| - |E| + |R| = |V| - (|V| - 1) + 1 = 2$$

$\therefore$ The result is true for $|R| = 1$.

Let us assume that the result is true for $|R| = k$ for $k \geq 1$.

Now suppose G is a connected plane graph that determines $k + 1$ regions. Delete an edge common to the boundary of two separate regions. The resulting graph G′ has the same number of vertices, one fewer edge, but also one fewer region, since two previous regions have been merged by the removal of the edge. Thus, we have $|V'| = |V|$, $|E'| = |E| - 1$ and $|R'| = |R| - 1$

$$\begin{aligned} \therefore \quad |V| - |E| + |R| &= |V'| - (|E'| + 1) + (|R'| + 1) \\ &= |V'| - |E'| + |R'| \\ &= 2 \quad \because \text{ The result is true for G}' \text{ which contains } k \text{ regions.} \end{aligned}$$

Thus, the theorem is proved by the mathematical induction.

Example 11. Suppose G is a connected planar graph with 14 regions each of degree 4, find the number of vertices in G.

By sum of degrees theorem

$$\sum_{r \in R(G)} \deg(r) = 2|E|$$

$\therefore$

$\therefore$ $$14 \times 4 = 2|E|$$

$$|E| = 28$$

By Eulers formula,

$\therefore$ $$|V| - |E| + |R| = 2$$

$$|V| = 2 + |E| - |R|$$

$$= 2 + 28 - 14$$

$$= 16.$$