(1) Wormhole Metric problem

The line element of a static, spherically symmetric wormhole space-time is given by

$$ds^2 = -e^{\nu(r)}dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

Compute the non-zero components of the Christoffel symbols and Ricci tensor. Also obtain the Ricci scalar & the non-zero components of the Einstein tensor. Hence considering perfect fluid type of matter distribution write down the Einstein field equations.

Assume the pressure and energy density to be related by the linear equation of state $P(r) = -\mu \rho(r)$, where P is the pressure, ρ is the energy density and μ is the constant equation of state parameter. For Phantom matter $\mu < -1$. Use the conservation equation for stress-energy as $\frac{dP}{dr} + \frac{1}{2}\frac{d\nu}{dr}(p+\rho) = 0$, obtain the solution of $b(r)$ for phantom matter and plot $b(r)$ vs r. Boundary conditions: $b(r=0) = 0$, $b'(r=1) = 8\pi$

Assume ν to be constant.

(2) ~~Cosmology~~ Star problem

The density profile inside a star is given by

$$\rho = \rho_0\left(1 - \frac{r^2}{L^2}\right)$$

$\rho_0 = 450 \text{ MeV/fm}^3$, $L = 16$ km. The equation of state of the matter content inside the star has the form $P = \frac{1}{3}(\rho - K)$, $K = 225 \text{ MeV/fm}^3$

(i) Find the value of r at which pressure $P = 0$ ($r = R$)

(ii) Plot ρ vs r and P vs r from $r=0$ to $r=R$.

(iii) Plot P vs ρ from $r=0$ to $r=R$

(3) Cosmalogy Problem

The line element for an Isotropic, homogeneous, time evolving space-time is given as

$$ds^2 = -dt^2 + \frac{a^2(t)}{1-Kr^2} dr^2 + a^2(t) r^2 d\theta^2 + a^2(t) r^2 d\phi^2 \sin^2\theta$$

Compute the non-zero components of the Christoffel symbols and Ricci tensor. Also obtain the Ricci scalar and the non-zero components of the Einstein tensor. Hence considering perfect fluid type of matter distribution write down the Einstein field equations.

Assume the pressure and energy density related by the linear equation of state $P(r) = \mu\rho(r)$, where P is the pressure, ρ is the energy density and μ is the constant equation of state matter. Obtain the solution of $a(t)$ for $K = -1, 0, 1$. for baryonic matter $(\mu = 0)$ & plot $a(t)$ vs t

Use the conservation equation for stress energy as

$$\frac{d\rho}{dt} + \frac{3}{a}\frac{da}{dt}(\rho + P) = 0$$

(4) Deriving TOV equation

The Schwarzschild line element for an static, isotropic regions of space-time is given as

$$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

Compute the non-zero components of the Christoffel symbols and Ricci tensor. Also obtain the Ricci scalar and the non-zero components of the Einstein tensor. Hence considering perfect fluid type distribution of matter write down the Energy momentum tensor and Einstein field equations.

Show that conservation equation for stress energy in this case is $\cancel{\frac{dP}{dt} + \frac{1}{2}(P+\rho)\frac{d\nu}{dr} = 0}$ $\boxed{\frac{dP(r)}{dr} + \frac{dA(r)}{dr} \cdot \frac{1}{2A(r)} (P(r) + \rho(r)) = 0}$

Now using the Einstein field equations derive TOV equation.

$$\boxed{\frac{dP(r)}{dr} = -\frac{(P(r) + \rho(r))\,(M(r) + 4\pi P(r) r^3)}{r(r - 2M(r))}}$$

by replacing $B(r) = \left(1 - \frac{2M(r)}{r}\right)^{-1}$.