TOV problem

$G_{00} = T_{00}\, 8\pi$

$$\Rightarrow \frac{A}{B^2 r^2}\left(B^2 - B + r\frac{dB(r)}{dr}\right) = A(r)\,P(r)\,8\pi \quad \ldots(i)$$

$G_{11} = T_{11}\, 8\pi$

$$\Rightarrow \frac{A(r)(1-B(r)) + r\frac{dA(r)}{dr}}{A r^2} = 8\pi B(r) P(r)$$

$$\Rightarrow 8\pi B(r) P(r) = \frac{\frac{dA(r)}{dr}}{rA(r)} - \frac{B(r)}{r^2} + \frac{1}{r^2}$$

$$\Rightarrow 8\pi B(r) P(r) + \frac{B(r)}{r^2} - \frac{1}{r^2} = \frac{A'(r)}{rA(r)}$$

$$\Rightarrow \frac{A'(r)}{rA(r)} = \frac{8\pi r^2 B(r) P(r) + B(r) - 1}{r} \qquad \left[\because B(r) = \frac{1}{1-\frac{2M(r)}{r}}\right]$$

$$\Rightarrow \frac{A'(r)}{A(r)} = \frac{\frac{8\pi r^2 P(r)}{\left(1-\frac{2M(r)}{r}\right)} - 1 + \frac{1}{\left(1-\frac{2M(r)}{r}\right)}}{r}$$

Conservation equation :-

$$\frac{dP(r)}{dr} + \frac{P(r)\frac{d}{dr}A(r) + P(r)\frac{d}{dr}A(r)}{2A(r)} = 0$$

$$\Rightarrow \frac{dP(r)}{dr} = -\left(P(r) + P(r)\right)\frac{\frac{dA(r)}{dr}}{2A(r)}$$

$$\Rightarrow \frac{dP(r)}{dr} = -\frac{1}{2}\frac{\left(P(r)+P(r)\right)}{r}\left[\frac{8\pi r^2 P(r)}{1-\frac{2M(r)}{r}} - 1 + \frac{1}{1-\frac{2M(r)}{r}}\right]$$

$$= -\frac{1}{2}\frac{\left(P(r)+P(r)\right)}{r}\left[\frac{8\pi r^2 P(r) \cancel{-1} + \frac{2M(r)}{r} \cancel{+1}}{1-\frac{2M(r)}{r}}\right]$$

$$= -\frac{1}{2}\frac{\left(P(r)+P(r)\right)}{r}\frac{\left[8\pi r^3 P(r) + 2M(r)\right]}{\left[r - 2M(r)\right]}$$

$$\boxed{\frac{dP}{dr} = -\frac{\left(4\pi r^3 P(r) + M(r)\right)\left(P(r) + P(r)\right)}{r\left[r - 2M(r)\right]}}$$

$T_{\mu\nu;\mu} = 0 \Rightarrow \cancel{9T}$ (Conservation equation)

Einstein field equation

$$G_{\mu\nu} = K T_{\mu\nu} \quad [K = 8\pi]$$

$$G_{00} = 8\pi T_{00}, \quad G_{11} = 8\pi T_{11}, \quad G_{22} = 8\pi T_{22}, \quad G_{33} = 8\pi T_{33}$$

~~$[\Rightarrow G_{00} = 8\pi g_{00} T]$~~

For wormhole metric,

$$T_{00} = \rho(r) e^{\nu(r)}$$

$$T_{11} = \frac{P(r)}{1 - \frac{b(r)}{r}}$$

$$T_{22} = P(r) r^2$$

$$T_{33} = P(r) r^2 \sin^2\theta$$

Conservation equation :-

$$\frac{dP}{dr} + \frac{1}{2}(\rho + P)\frac{d\nu}{dr} = 0 \quad \ldots (1)$$

EoS (Equation of state) :- $P(r) = -2\rho(r) \quad [\mu = -2]$ ... (2)

Field equation - 1 $\rightarrow$ $G_{00} = 8\pi T_{00}$

$$\Rightarrow \frac{db}{dr}\frac{e^{\nu}}{r^2} = 8\pi \rho(r) e^{\nu(r)}$$

$$\Rightarrow \boxed{\rho(r) = \frac{1}{8\pi r^2}\frac{db(r)}{dr}} \quad \ldots (3)$$

$$\frac{dP}{dr} + \frac{1}{2}(\rho + P)\frac{d\nu}{dr} = 0 \quad [\because P(r) = -2\rho(r)]$$

$$\Rightarrow \frac{d}{dr}(-2\rho(r)) + \frac{1}{2}(-2\rho + \rho)\frac{d\nu}{dr} = 0$$

$$\Rightarrow -2\frac{d\rho(r)}{dr} - \frac{1}{2}\rho(r)\frac{d\nu}{dr} = 0$$

$$\Rightarrow -2\frac{d}{dr}\left(\frac{1}{8\pi r^2}\frac{db(r)}{dr}\right) - \frac{1}{16\pi r^2}\frac{db}{dr}\frac{d\nu}{dr} = 0$$

$$\Rightarrow -\frac{0.0625\frac{d}{dr}b(r)\frac{d}{dr}\nu(r)}{\pi r^2} - \frac{d^2 b(r)}{dr^2} \times \frac{1}{4\pi r^2} + \frac{db(r)}{dr}\frac{1}{2\pi r^3} = 0$$

As v is constant, above expression reduces to

$$\frac{-r\frac{d^2}{dr^2}b(r) + 2\cdot\frac{d}{dr}b(r)}{4\pi r^3} = 0$$

$$\Rightarrow \cancel{r}\ \frac{d^2b(r)}{dr^2} - \frac{2}{r}\frac{db(r)}{dr} = 0$$

$$\cancel{\Rightarrow \frac{d}{dr}\left[\frac{db(r)}{dr}\right.}$$

$$\Rightarrow -\frac{1}{4\pi r^2}\frac{d^2}{dr^2}b(r) + \frac{1}{2\pi r^3}\frac{db(r)}{dr} = 0$$

$$\Rightarrow \frac{d}{dr}\left[-\frac{1}{4\pi r^2}\frac{db(r)}{dr}\right] = 0$$

$$\Rightarrow -\frac{1}{4\pi r^2}\frac{db(r)}{dr} = C_2$$

$$\Rightarrow \frac{db(r)}{dr} = -4\pi C_2 r^2$$

$$\Rightarrow \frac{db(r)}{dr} = K_2 r^2$$

$$\Rightarrow b(r) = \frac{K_2 r^3}{3} + K_1$$

Normally we can write, $b(r) = C_1 + C_2 r^3$

Boundary condition :-

$b(r=0) = 0$

$b'(r=1) = 3\pi$

$$b(r=0) = C_1$$

$$\Rightarrow C_1 = 0$$

$$\therefore b(r) = C_2 r^3$$

$$\Rightarrow b'(r) = 3C_2 r^2$$

$$\Rightarrow b'(r=1) = 3C_2$$

$$\Rightarrow 3\pi = 3C_2$$

$$\Rightarrow C_2 = \frac{3\pi}{3}$$

$$\boxed{b(r) = \frac{3\pi}{3} r^3}$$

Now plot $b(r)$ vs r