```
> with(tensor) :
> coords := [t, r, theta, phi];
```

$$coords := [t, r, \theta, \phi] \tag{1}$$

```
> g := array(symmetric, sparse, 1 ..4, 1 ..4);
```

$$g := array(symmetric, sparse, 1 ..4, 1 ..4, [\ ]) \tag{2}$$

```
> g[1, 1] :=-1;
```

$$g[2, 2] := \frac{(a(t))^2}{1 - k \cdot r^2};\ g[3, 3] := r^2;$$

$$g[4, 4] := r^2 \cdot (\sin(\text{theta}))^2;$$

$$g_{1,1} := -1$$

$$g_{2,2} := \frac{a(t)^2}{-k r^2 + 1}$$

$$g_{3,3} := r^2$$

$$g_{4,4} := r^2 \sin(\theta)^2 \tag{3}$$

```
> metric := create([-1,-1], eval(g));
```

$$metric := table\left(\left[index\_char = [-1, -1], compts = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a(t)^2}{-k r^2 + 1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin(\theta)^2 \end{bmatrix}\right]\right) \tag{4}$$

```
> tensorsGR(coords, metric, contra_metric, det_met, C1, C2, Rm, Rc, R, G, C);
> displayGR(Christoffel2, C2);
```

*The Christoffel Symbols of the Second Kind*

*non-zero components :*

$$\{1,22\} = -\frac{a(t)\left(\frac{d}{dt} a(t)\right)}{k r^2 - 1}$$

$$\{2,12\} = \frac{\frac{d}{dt} a(t)}{a(t)}$$

$$\{2,22\} = -\frac{k r}{k r^2 - 1}$$

$$\{2,33\} = \frac{(k r^2 - 1) r}{a(t)^2}$$

$$\{2,44\} = \frac{\left(k\, r^2 - 1\right) r \sin(\theta)^2}{a(t)^2}$$

$$\{3,23\} = \frac{1}{r}$$

$$\{3,44\} = -\sin(\theta)\cos(\theta)$$

$$\{4,24\} = \frac{1}{r}$$

$$\{4,34\} = \frac{\cos(\theta)}{\sin(\theta)} \qquad (5)$$

> *displayGR*(*Ricci*, *Rc*);

*The Ricci tensor*

*non-zero components :*

$$R11 = \frac{\frac{d^2}{dt^2}\, a(t)}{a(t)}$$

$$R12 = -\frac{2\left(\frac{d}{dt}\, a(t)\right)}{r\, a(t)}$$

$$R22 = \frac{a(t)\left(\frac{d^2}{dt^2}\, a(t)\right) + 2\, k}{k\, r^2 - 1}$$

$$R33 = -\frac{2\, k\, r^2 + a(t)^2 - 1}{a(t)^2}$$

$$R44 = \frac{2\cos(\theta)^2\, k\, r^2 + \cos(\theta)^2\, a(t)^2 - 2\, k\, r^2 - \cos(\theta)^2 - a(t)^2 + 1}{a(t)^2}$$

*character : [-1, -1]* **(6)**

> *displayGR*(*Ricciscalar*, *R* );

*The Ricci Scalar*

$$R = -\frac{2\left(\left(\frac{d^2}{dt^2}\, a(t)\right) r^2\, a(t) + 3\, k\, r^2 + a(t)^2 - 1\right)}{a(t)^2\, r^2} \qquad (7)$$

> *displayGR*(*Einstein*, *G*);

*The Einstein Tensor*

*non-zero components :*

$$G11 = -\frac{3\, k\, r^2 + a(t)^2 - 1}{a(t)^2\, r^2}$$

$$G12 = -\frac{2\left(\frac{\mathrm{d}}{\mathrm{d}t}\,a(t)\right)}{r\,a(t)}$$

$$G22 = -\frac{k\,r^2 + a(t)^2 - 1}{\left(k\,r^2 - 1\right)\,r^2}$$

$$G33 = \frac{r^2\left(a(t)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,a(t)\right) + k\right)}{a(t)^2}$$

$$G44 = -\frac{r^2\left(\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,a(t)\right)\cos(\theta)^2\,a(t) + \cos(\theta)^2\,k - a(t)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,a(t)\right) - k\right)}{a(t)^2}$$

*character : [-1, -1]* **(8)**

> *mixed* := *raise*(*contra_metric*, *G*, 2);

$$mixed := table\left(\left[index\_char = [-1, 1], compts = \left[\left[\frac{3\,k\,r^2 + a(t)^2 - 1}{a(t)^2\,r^2}, \frac{2\left(k\,r^2 - 1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\,a(t)\right)}{a(t)^3\,r}, 0, 0\right], \left[\frac{2\left(\frac{\mathrm{d}}{\mathrm{d}t}\,a(t)\right)}{r\,a(t)}, \frac{k\,r^2 + a(t)^2 - 1}{a(t)^2\,r^2}, 0, 0\right], \left[0, 0, \frac{a(t)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,a(t)\right) + k}{a(t)^2}, 0\right], \left[0, 0, 0, \frac{a(t)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,a(t)\right) + k}{a(t)^2}\right]\right]\right]\right) \quad \textbf{(9)}$$

> $\frac{3\left(\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,a(t)\right)^2 + k\right)}{(a(t))^2} = -8\cdot\pi\cdot\rho(t)$;

$$\frac{3\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\,a(t)\right)^2 + k\right)}{a(t)^2} = -8\,\pi\,\rho(t) \quad \textbf{(10)}$$

> $\rho(t) = -\frac{1}{8\cdot\pi}\cdot\frac{3\left(\left(\frac{\mathrm{d}}{\mathrm{d}\,t}\,a(t)\right)^2 + k\right)}{(a(t))^2}$;

$$\rho(t) = -\frac{3}{8}\,\frac{\left(\frac{d}{dt}\,a(t)\right)^2 + k}{\pi\,a(t)^2} \tag{11}$$

>

*Equation of state*

> $P(t) = \mu\cdot\rho(t);$

$$P(t) = \mu\,\rho(t) \tag{12}$$

> $P(t) = -\mu\cdot\frac{1}{8\cdot\pi}\cdot\frac{3\left(\left(\frac{d}{d\,t}\,a(t)\right)^2 + k\right)}{(a(t))^2};$

$$P(t) = -\frac{1}{8}\,\frac{\mu\left(3\left(\frac{d}{dt}\,a(t)\right)^2 + 3\,k\right)}{\pi\,a(t)^2} \tag{13}$$

>

# Conservation equation

>

>

>

> $\frac{d}{d\,t}\,\rho(t) + \frac{3}{a(t)}\cdot\frac{d}{d\,t}\,a(t)\cdot\left(P(t) + \rho(t)\right) = 0;$

$$\frac{d}{dt}\,\rho(t) + \frac{3\left(\frac{d}{dt}\,a(t)\right)\left(P(t) + \rho(t)\right)}{a(t)} = 0 \tag{14}$$

> $diff\left(-\frac{1}{8\cdot\pi}\cdot\frac{3\left(\left(\frac{d}{d\,t}\,a(t)\right)^2 + k\right)}{(a(t))^2}, t\right);$

$$-\frac{3}{4}\,\frac{\left(\frac{d}{dt}\,a(t)\right)\left(\frac{d^2}{dt^2}\,a(t)\right)}{\pi\,a(t)^2} + \frac{3}{4}\,\frac{\left(\left(\frac{d}{dt}\,a(t)\right)^2 + k\right)\left(\frac{d}{dt}\,a(t)\right)}{\pi\,a(t)^3} \tag{15}$$

> $-\frac{3}{4}\,\frac{\left(\frac{d}{dt}\,a(t)\right)\left(\frac{d^2}{dt^2}\,a(t)\right)}{\pi\,a(t)^2} + \frac{3}{4}\,\frac{\left(\left(\frac{d}{dt}\,a(t)\right)^2 + k\right)\left(\frac{d}{dt}\,a(t)\right)}{\pi\,a(t)^3} + \frac{3}{a(t)}\cdot\left(\frac{d}{d\,t}\,a(t)\right)\cdot\left(-\mu\cdot\frac{1}{8\cdot\pi}\cdot\frac{3\left(\left(\frac{d}{d\,t}\,a(t)\right)^2 + k\right)}{(a(t))^2} - \frac{1}{8\cdot\pi}\cdot\frac{3\left(\left(\frac{d}{d\,t}\,a(t)\right)^2 + k\right)}{(a(t))^2}\right) = 0;$

$$-\frac{3}{4}\,\frac{\left(\frac{d}{dt}\,a(t)\right)\left(\frac{d^2}{dt^2}\,a(t)\right)}{\pi\,a(t)^2} + \frac{3}{4}\,\frac{\left(\left(\frac{d}{dt}\,a(t)\right)^2 + k\right)\left(\frac{d}{dt}\,a(t)\right)}{\pi\,a(t)^3} \tag{16}$$

$$+\frac{3\left(\frac{d}{dt}a(t)\right)\left(-\frac{1}{8}\,\frac{\mu\left(3\left(\frac{d}{dt}a(t)\right)^2+3\,k\right)}{\pi\,a(t)^2}-\frac{3}{8}\,\frac{\left(\frac{d}{dt}a(t)\right)^2+k}{\pi\,a(t)^2}\right)}{a(t)}=0$$

```
> eval( (16), [mu = 0]);
```

$$-\frac{3}{4}\,\frac{\left(\frac{d}{dt}a(t)\right)\left(\frac{d^2}{dt^2}a(t)\right)}{\pi\,a(t)^2}-\frac{3}{8}\,\frac{\left(\left(\frac{d}{dt}a(t)\right)^2+k\right)\left(\frac{d}{dt}a(t)\right)}{\pi\,a(t)^3}=0 \qquad \textbf{(17)}$$

```
> simplify( (17) );
```

$$-\frac{3}{8}\,\frac{\left(\frac{d}{dt}a(t)\right)\left(\left(\frac{d}{dt}a(t)\right)^2+2\,a(t)\left(\frac{d^2}{dt^2}a(t)\right)+k\right)}{\pi\,a(t)^3}=0 \qquad \textbf{(18)}$$

$$> ode := -\frac{3\cdot\left(\left(\frac{d}{dt}a(t)\right)\cdot\left(\frac{d^2}{dt^2}a(t)\right)\right)}{4\cdot\pi\cdot(a(t))^2}-\frac{3\cdot\left(\left(\frac{d}{dt}a(t)\right)\cdot\left(\left(\frac{d}{dt}a(t)\right)^2+k\right)\right)}{8\cdot\pi\cdot(a(t))^3}=0;$$

\>

$$ode := -\frac{3}{4}\,\frac{\left(\frac{d}{dt}a(t)\right)\left(\frac{d^2}{dt^2}a(t)\right)}{\pi\,a(t)^2}-\frac{3}{8}\,\frac{\left(\left(\frac{d}{dt}a(t)\right)^2+k\right)\left(\frac{d}{dt}a(t)\right)}{\pi\,a(t)^3}=0 \qquad \textbf{(19)}$$

```
> dsolve( (19), { a(t) } );
```

$$-\frac{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}{k}+\frac{1}{2}\,\frac{\_C1\arctan\left(\frac{\sqrt{k}\left(a(t)-\frac{1}{2}\,\frac{\_C1}{k}\right)}{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}\right)}{k^{3/2}}-t-\_C2=0, \qquad \textbf{(20)}$$

$$\frac{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}{k}-\frac{1}{2}\,\frac{\_C1\arctan\left(\frac{\sqrt{k}\left(a(t)-\frac{1}{2}\,\frac{\_C1}{k}\right)}{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}\right)}{k^{3/2}}-t-\_C2=0,$$

$$a(t)=\_C1$$

```
> dsolve( (19), { a(t) } );
```

$$-\frac{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}{k}+\frac{1}{2}\,\frac{\_C1\arctan\left(\frac{\sqrt{k}\left(a(t)-\frac{1}{2}\,\frac{\_C1}{k}\right)}{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}\right)}{k^{3/2}}-t-\_C2=0, \qquad \textbf{(21)}$$

$$\frac{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}{k}-\frac{1}{2}\,\frac{\_C1\arctan\left(\frac{\sqrt{k}\left(a(t)-\frac{1}{2}\,\frac{\_C1}{k}\right)}{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}\right)}{k^{3/2}}-t-\_C2=0,$$

$$a(t)=\_C1$$

```
> isolate( (19), a(t) );
```

$$\frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\,a(t)\right)\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\,a(t)\right)^2+2\,a(t)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\,a(t)\right)+k\right)}{a(t)^3}=0 \tag{22}$$

> *dsolve*(*ode*);

$$-\frac{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}{k}+\frac{1}{2}\,\frac{\_C1\,\arctan\left(\frac{\sqrt{k}\left(a(t)-\frac{1}{2}\,\frac{\_C1}{k}\right)}{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}\right)}{k^{3/2}}-t-\_C2=0, \tag{23}$$

$$\frac{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}{k}-\frac{1}{2}\,\frac{\_C1\,\arctan\left(\frac{\sqrt{k}\left(a(t)-\frac{1}{2}\,\frac{\_C1}{k}\right)}{\sqrt{-a(t)^2\,k+\_C1\,a(t)}}\right)}{k^{3/2}}-t-\_C2=0,$$

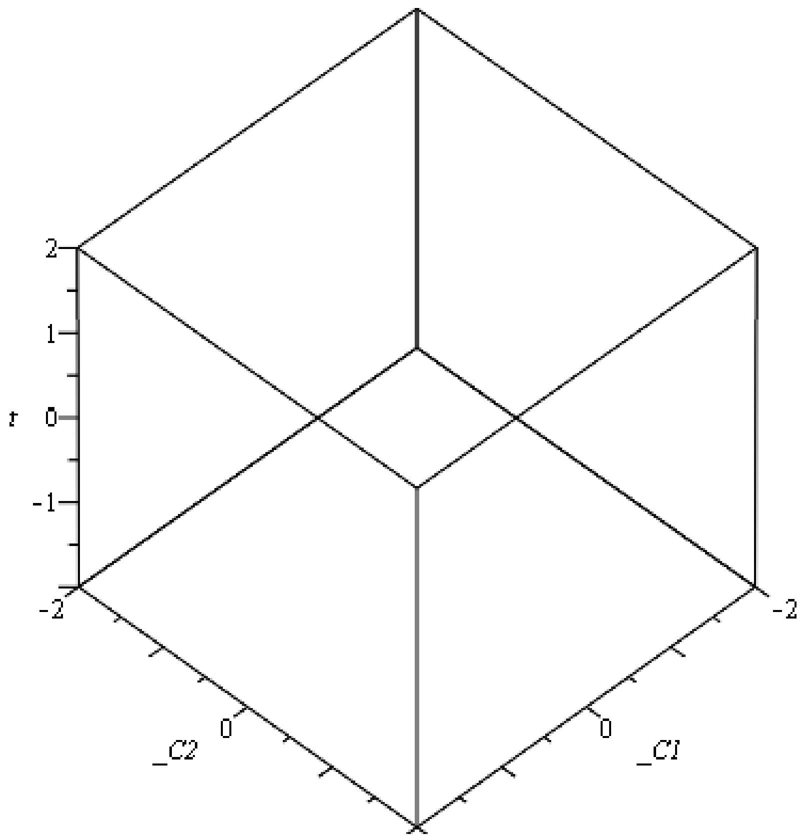
$$a(t)=\_C1$$

```
> op( eval( [(23)], [k = -1]));
```

$$\sqrt{a(t)^2+\_C1\,a(t)}-\frac{1}{2}\,\_C1\,\mathrm{arctanh}\left(\frac{a(t)+\frac{1}{2}\,\_C1}{\sqrt{a(t)^2+\_C1\,a(t)}}\right)-t-\_C2=0, \tag{24}$$

$$-\sqrt{a(t)^2+\_C1\,a(t)}+\frac{1}{2}\,\_C1\,\mathrm{arctanh}\left(\frac{a(t)+\frac{1}{2}\,\_C1}{\sqrt{a(t)^2+\_C1\,a(t)}}\right)-t-\_C2=0,\ a(t)=\_C1$$

```
> plots[:-display](plots[:-implicitplot3d]((a(t)^2+_C1*a(t))^(1/2)
  -1/2*_C1*arctanh((a(t)+1/2*_C1)/(a(t)^2+_C1*a(t))^(1/2))-t-_C2 =
  0, _C1 = -2 .. 2, _C2 = -2 .. 2, t = -2 .. 2), plots[:-
  implicitplot3d](-(a(t)^2+_C1*a(t))^(1/2)+1/2*_C1*arctanh((a(t)
  +1/2*_C1)/(a(t)^2+_C1*a(t))^(1/2))-t-_C2 = 0, _C1 = -2 .. 2, _C2
  = -2 .. 2, t = -2 .. 2), plots[:-implicitplot3d](a(t) = _C1, _C1
  = -2 .. 2, _C2 = -2 .. 2, t = -2 .. 2));
```




```
> op( eval( [(23)], [k = 1]));
```

$$-\sqrt{-a(t)^2 + \_C1\, a(t)} + \frac{1}{2}\,\_C1 \arctan\left(\frac{a(t) - \frac{1}{2}\,\_C1}{\sqrt{-a(t)^2 + \_C1\, a(t)}}\right) - t - \_C2 = 0,\quad \sqrt{-a(t)^2 + \_C1\, a(t)} - \frac{1}{2}\,\_C1 \arctan\left(\frac{a(t) - \frac{1}{2}\,\_C1}{\sqrt{-a(t)^2 + \_C1\, a(t)}}\right) - t - \_C2 = 0,\ a(t) = \_C1 \tag{25}$$

```
>
>
```