ANNALS OF PHYSICS 83, 491–529 (1974)

A Theory of Highly Condensed Matter*

J. D. WALECKA

Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California 94305

Received September 17, 1973

To discuss properties of cold, condensed stellar objects such as neutron stars, it is necessary to know the stress tensor $T_{\mu\nu}$, the source in Einstein's field equations, from nuclear matter densities upwards. To overcome some of the difficulties with the conventional many-body approach to this problem, a model relativistic, many-body, quantum field theory composed of a baryon field, a neutral scalar meson field coupled to the scalar density $\bar{\psi}\psi$, and a neutral vector meson field coupled to the conserved baryon current $i\bar{\psi}\gamma_\lambda\psi$ is developed. For a uniform system of given baryon density ρ_B, the linearized theory obtained by replacing the scalar and vector fields by their expectation values, $\phi \to \phi_0$, $V_\lambda \to i\delta_{\lambda 4} V_0$ can be solved exactly. The resulting equation of state for nuclear matter exhibits nuclear saturation, and if the two dimensionless coupling constants in this theory are matched to the binding energy and density of nuclear matter, predictions are obtained for all other systems at all densities. In particular, neutron matter is unbound and the equation of state for neutron matter at all densities is presented; it extrapolates smoothly into the relativistic form $P = \epsilon$. Comparison is made with some conventional many-body calculations.

The full field theory is developed by expanding the fields about the condensed values ϕ_0, V_0, and the unperturbed hamiltonian is shown to correspond to the linearized theory. The energy shift due to these quantum fluctuations in the fields is related to the baryon Green's function. V_0 is related directly to ρ_B; ϕ_0, however, must be determined through a self-consistency relation involving the baryon Green's function. The Feynman rules for this theory are developed. Expressions for the lowest-order contributions of the quantum fluctuations to the energy shift and ϕ_0 are derived. It is shown that the terms $q_\mu q_\nu$ in the vector-meson propagator do not contribute to these expressions, and a prescription involving assumptions on the limiting form of the theory as $\rho_B \to 0$ is presented which ensures that these lowest-order quantum fluctuations will yield finite results.

1. INTRODUCTION

In order to discuss the bulk properties of cold, condensed stellar objects such as neutron stars [1–7], it is necessary to have an equation of state which describes matter at all densities, in particular, from the densities observed for terrestrial

* Research supported in part by the National Science Foundation GP-39518.

nuclear matter upwards. There is only a very small experimental window on this equation of state under existing laboratory conditions, and while the use of semiempirical mass formulae allows one to go some distance from observed nuclei, a far-ranging theoretical extrapolation is generally required in this analysis. The most reliable existing theoretical extensions which attempt to include the interactions between the particles [8–23] generally involve the use of static, or near-static, nucleon–nucleon potentials chosen to describe the scattering of two free nucleons, and the solution to the nonrelativistic many-body Schrödinger equation through the summation of a selected class of diagrams as in the Brueckner–Bethe–Goldstone approach or through a cluster expansion for the energy [24]. Several objections to this approach can be raised:

(1) While the meson degrees of freedom may evidently be successfully eliminated in favor of static or near-static, two-body, nucleon–nucleon potentials at observed nuclear matter densities, a consistent theory should include these degrees of freedom *explicitly* as one proceeds to higher densities. For example, the meson field can, in principle, be grossly distorted when the nucleons are forced to overlap and many-body forces may be present. In addition, one should allow for the possibility that *real* mesons may eventually be present when the nucleon Fermi momentum is large enough.

(2) As the density and nucleon Fermi momentum are increased, the relativistic propagation of the nucleons as well as the retarded propagation of the virtual meson fields giving rise to the nuclear force must eventually be taken into account.

(3) Rather than obtaining a solution to the nonrelativistic Schrödinger equation, the aim of the microscopic many-body theory is to eventually compute the stress tensor $T_{\mu\nu}$ which serves as the source in Einstein's equation for the curvature of the metric tensor. Recall [7] that in general relativity there exists a local freely-falling frame in which gravitational effects are completely compensated by inertial forces and in which the laws of special relativity are assumed to hold. For a uniform fluid at rest in this frame, the stress tensor has the general form

$$T_{\mu\nu} = P\delta_{\mu\nu} + (\rho + P/c^2)\, u_\mu u_\nu\,, \tag{1.1}$$

with[1]

$$u_\mu \equiv (\mathbf{0}, ic). \tag{1.2}$$

[1] In this paper, we use the notation $v_\mu = (\mathbf{v}, v_4) = (\mathbf{v}, iv_0)$ for a four-vector, and repeated Greek indices are summed from 1 to 4, thus $v_\mu v_\mu = \mathbf{v}^2 + v_4{}^2 = \mathbf{v}^2 - v_0{}^2$. Repeated Latin indices are summed from 1 to 3. The gamma matrices are hermitian and satisfy $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$. We also employ the standard Dirac matrices $\boldsymbol{\gamma} = i\boldsymbol{\alpha}\beta$, $\gamma_4 = \beta$ and notation $\bar\psi = \psi^\dagger\gamma_4$, $\not{V} = V_\mu\gamma_\mu$. A carat over a symbol is generally used to denote an operator in the abstract occupation-number Hilbert space except when this is obvious as, for example, in the case of the creation and destruction operators.

This expression defines the pressure P and mass density ρ. Once the stress tensor is known in this frame it can be put into a generally covariant form and written in any other coordinate system. The Tolman, Oppenheimer, Volkoff (TOV) equations [4, 5, 7] for the metric in the presence of a static, spherically-symmetric mass distribution can then be integrated to find the bulk properties of a condensed star. Knowledge of the stress tensor for such a system is equivalent to knowledge of the *equation of state* $P(\rho)$ through Eq. (1.1). (Recall that we are discussing cold, condensed objects so that the temperature is effectively zero.) The task of the microscopic theory is thus to calculate the special-relativistic stress tensor $T_{\mu\nu}$. Such an approach is *fully relativistic* in both the special and general sense, and neglects only the effects of gravitational gradients on the equation of state itself.

2. Model

The only consistent approach that the author is aware of which meets these objections is the utilization of a local, relativistic, many-body quantum field theory [25–33].[2] One of the simplest lagrangian densities which describes the main features of the nucleon–nucleon interaction and which has not yet been applied to this problem is the following

$$\mathscr{L} = -\hbar c\left[\bar{\psi}\left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} + M\right)\psi\right] - \frac{c^2}{2}\left[\left(\frac{\partial\phi}{\partial x_\lambda}\right)^2 + \mu^2\phi^2\right] - \frac{1}{4}F_{\lambda\rho}F_{\lambda\rho} - \frac{m^2}{2}V_\lambda V_\lambda + ig_v\bar{\psi}\gamma_\lambda\psi V_\lambda + g_s\bar{\psi}\psi\phi, \tag{2.1}$$

where ψ is a baryon field of mass m_b, ϕ is a neutral scalar meson field of mass m_s, and V_λ is a neutral vector meson field of mass m_v. The quantities M, μ, m are inverse Compton wavelengths

$$M \equiv \frac{m_b c}{\hbar}; \qquad \mu \equiv \frac{m_s c}{\hbar}; \qquad m \equiv \frac{m_v c}{\hbar}, \tag{2.2}$$

and the neutral vector meson couples to the baryon current

$$B_\lambda \equiv i\bar{\psi}\gamma_\lambda\psi \tag{2.3}$$

while the scalar field couples to the scalar density $\bar{\psi}\psi$. The field tensor $F_{\lambda\rho}$ is given by the familiar expression

$$F_{\lambda\rho} = \frac{\partial V_\rho}{\partial x_\lambda} - \frac{\partial V_\lambda}{\partial x_\rho}. \tag{2.4}$$

[2] Many of the results in this paper are similar in spirit to those recently presented by Bowers *et al.* [33] although there is substantial difference in content and detail. The latter authors consider the interaction of baryons with pseudoscalar pions and work with a finite-temperature many-body theory.

One important feature of this lagrangian is that the baryon current is conserved[3]

$$\frac{\partial}{\partial x_\lambda} B_\lambda = 0. \tag{2.5}$$

This implies that derivative terms in the vector meson propagators will not contribute to the S-matrix because the vector meson couples only to a conserved current,[4] and the relativistic quantum field theory will be renormalizable. In fact, the theory is equivalent to massive quantum electrodynamics with an additional scalar meson.

To motivate the particular choice of lagrangian (2.1), consider the scattering of two free baryons computed in the ladder approximation to the Bethe–Salpeter equation. In this case, the interaction can be replaced by an equivalent momentum-space potential[5]

$$V(q)_{eq} = g_v{}^2 \frac{\gamma_\lambda^{(1)} \cdot \gamma_\lambda^{(2)}}{q^2 + m^2 - i\eta} - \frac{g_s{}^2}{c^2} \frac{1^{(1)} \cdot 1^{(2)}}{q^2 + \mu^2 - i\eta}, \tag{2.6}$$

where $q_\lambda = (\mathbf{q}, iq_0)$ is the four-momentum transfer and $\gamma_\lambda^{(1,2)}$ refer to the first and second particles respectively. Note particularly the signs in this expression. If the baryons are heavy and moving nonrelativistically then we can make the approximation $\gamma_\lambda^{(1)} \cdot \gamma_\lambda^{(2)} \to 1^{(1)} \cdot 1^{(2)}$, $|q_0| \ll |\mathbf{q}|$ in which case the equivalent potential becomes instantaneous and spin-independent and can be written in coordinate space as

$$V(x) = \delta(x_0) \left[g_v{}^2 \frac{e^{-mr}}{r} - \frac{g_s{}^2}{c^2} \frac{e^{-\mu r}}{r} \right] \cdot \frac{1}{4\pi}. \tag{2.7}$$

Now, if $g_v{}^2 > g_s{}^2/c^2$, this potential is repulsive at short distances. Furthermore, if $m > \mu$, the potential will be attractive at large distances. Thus this interaction contains the main features of the nucleon–nucleon force; it is repulsive at short distances (due to the exchange of a neutral vector meson coupled to the baryon current) and attractive at large distances (due to the exchange of a neutral scalar meson). These are also *just those features of the nucleon–nucleon force which are responsible for the saturation properties of nuclear matter* [24]. The reader might object to the fact that there is no one-pion-exchange tail in this interaction; however, the strong spin and isospin dependence of the potential arising from the exchange of an isovector, pseudoscalar pion [35] implies that the contribution of

[3] This follows either directly from the equations of motion [Eqs. (3.1)–(3.4)] or from the invariance of the lagrangian density under the phase transformation $\psi \to \psi e^{i\chi}$.

[4] The proof of this statement is given in Ref. [34, pp. 197–202].

[5] The argument is identical to that given in Ref. [24, pp. 401–402]. Current conservation has been used to eliminate the derivative terms in the vector meson propagator.[4]

the one-pion-exchange potential to the bulk properties of nuclear matter largely averages to zero.[6]

These arguments in terms of potentials are given only to motivate the choice of lagrangian. The philosophy of the present work will be the following: It is assumed that highly-condensed matter from nuclear matter densities upwards can be described with the model lagrangian density (2.1). The parameters in the lagrangian will in principle be chosen to reproduce the observed properties of nuclear matter. The theory will then be used to extrapolate away from the observed nuclear matter window with the hope that in this way meaningful *changes* in the stress tensor $T_{\mu\nu}$ can be computed up to very high density.

3. Linearized Theory

The equations of motion for the fields follow as the Euler–Lagrange equations of the lagrangian density (2.1) [37]. The meson fields satisfy

$$(\square - \mu^2)\,\phi = -\frac{g_s}{c^2}\,\bar{\psi}\psi, \tag{3.1}$$

$$\frac{\partial}{\partial x_\rho} F_{\lambda\rho} = -m^2 V_\lambda + i g_v \bar{\psi}\gamma_\lambda\psi, \tag{3.2}$$

while the baryon field develops according to

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} - \frac{i g_v}{\hbar c}\,\gamma_\mu V_\mu + M - \frac{g_s}{\hbar c}\,\phi\right)\psi = 0, \tag{3.3}$$

$$\bar{\psi}\left(-\gamma_\mu \frac{\overleftarrow{\partial}}{\partial x_\mu} - \frac{i g_v}{\hbar c}\,\gamma_\mu V_\mu + M - \frac{g_s}{\hbar c}\,\phi\right) = 0. \tag{3.4}$$

These equations are fully relativistic and Lorentz covariant. The exact coupled relativistic quantum field theory is, of course, extremely complicated. It has all the complexities of massive quantum electrodynamics without the benefit of a small coupling constant and by simply writing down a model lagrangian density, we have as yet accomplished nothing. Before proceeding to quantize the fields, however, consider first the classical field equations (3.1) and (3.2). We are interested in obtaining a solution to these equations for a system with uniform baryon density $\rho_B = \psi^\dagger\psi$ and corresponding uniform scalar density $\rho_s = \bar{\psi}\psi$. In fact, we are most interested in the case where the baryon density becomes large, $\rho_B \to \infty$. *In the case*

[6] The behavior of the pion degrees of freedom as a function of density may provide some modification of the bulk properties of highly-condensed matter, and may give rise to superfluid behavior [36].

that the baryon and scalar densities are given constants independent of spatial position and time, we can obtain an exact solution to the classical field equations (3.1) *and* (3.2). The solution will take the form

$$\phi = \phi_0 , \tag{3.5}$$

$$V_\lambda = i\delta_{\lambda 4} V_0 , \tag{3.6}$$

where ϕ_0 and V_0 are real constants independent of $\mathbf{x}$ and t satisfying

$$\phi_0 = \frac{g_s}{\mu^2 c^2} \bar{\psi}\psi \equiv \frac{g_s}{\mu^2 c^2} \rho_s , \tag{3.7}$$

$$V_0 = \frac{g_v}{m^2} \psi^\dagger \psi \equiv \frac{g_v}{m^2} \rho_B . \tag{3.8}$$

The baryon field correspondingly satisfies the *linearized* field equations

$$\left[\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{g_v}{\hbar c} \gamma_4 V_0 + \left(M - \frac{g_s}{\hbar c} \phi_0\right)\right] \psi = 0, \tag{3.9a}$$

$$\bar{\psi} \left[-\gamma_\mu \frac{\overleftarrow{\partial}}{\partial x_\mu} + \frac{g_v}{\hbar c} \gamma_4 V_0 + \left(M - \frac{g_s}{\hbar c} \phi_0\right)\right] = 0, \tag{3.9b}$$

which may also be solved exactly.

Quantum mechanically, the situation is more complicated. We may still assume, however, that we are discussing a system with uniform baryon density ρ_B, and to make the problem more concrete, we consider the system to be contained in a large box of volume $\Omega \equiv L^3$ and apply periodic boundary conditions whenever necessary. The exact size of the box Ω will drop out of any final physical results and, just as in standard nuclear matter discussions, we can compute local properties in the limit $\Omega \to \infty$ [24]. The many-body system will have a ground state $| \Psi_0 \rangle$ which is an eigenstate of four-momentum with vanishing three-momentum and energy E, $P_\mu = (\mathbf{0}, iE/c)$. For such a system, it follows from rotational invariance that the stress tensor operator has an expectation value of the form (1.1).

$$\langle \Psi_0 | \hat{T}_{\mu\nu} | \Psi_0 \rangle = P\delta_{\mu\nu} + (\rho + P/c^2)\, u_\mu u_\nu , \tag{3.10}$$

with

$$u_\mu \equiv (\mathbf{0}, ic). \tag{3.11}$$

By looking at the spatial and 4–4 components of this relation, the operators corresponding to the pressure and energy density may be identified as

$$\hat{P} = \tfrac{1}{3} \hat{T}_{ii} , \tag{3.12}$$

$$\hat{\epsilon} \equiv \hat{\rho} c^2 = -\hat{T}_{44} . \tag{3.13}$$

The hamiltonian for the system is canonically given by

$$\hat{H} = -\int \hat{T}_{44}\, d\mathbf{x} = \int \hat{\epsilon}\, d\mathbf{x}. \tag{3.14}$$

The space–time dependence of a Heisenberg operator in the theory is given by

$$\hat{O}_H(x_\mu) = e^{-(i/\hbar)\hat{\mathbb{P}}_\mu x_\mu}\hat{O}(0)\, e^{(i/\hbar)\hat{\mathbb{P}}_\mu x_\mu} \tag{3.15}$$

A diagonal matrix element of such an operator between eigenstates of four-momentum is independent of spatial position and time. This translational invariance of the theory implies that the energy density and baryon density can be written in terms of the total energy E and total baryon number B of the system of volume Ω as

$$E/\Omega = \epsilon, \tag{3.16a}$$

$$B/\Omega = \rho_B\,. \tag{3.16b}$$

The quantum mechanical statements corresponding to relations (3.7) and (3.8) are obtained from the ground-state matrix elements of Eqs. (3.1) and (3.2) using translational invariance. It follows that ϕ_0 and V_0 can be identified with the expectation values

$$\phi_0 = \langle \Psi_0 \mid \hat{\phi} \mid \Psi_0 \rangle = \frac{g_s}{\mu^2 c^2} \langle \Psi_0 \mid \bar{\psi}\psi \mid \Psi_0 \rangle \equiv \frac{g_s}{\mu^2 c^2}\, \rho_s\,, \tag{3.17}$$

$$i\delta_{\lambda 4} V_0 = \langle \Psi_0 \mid \hat{V}_\lambda \mid \Psi_0 \rangle = i\delta_{\lambda 4} \frac{g_v}{m^2} \langle \Psi_0 \mid \psi^\dagger \psi \mid \Psi_0 \rangle = i\delta_{\lambda 4} \frac{g_v}{m^2}\, \rho_B\,. \tag{3.18}$$

Note that the expectation value

$$\langle \Psi_0 \mid \bar{\psi}\gamma\psi \mid \Psi_0 \rangle = 0 \tag{3.19}$$

vanishes by rotational invariance. Since the total baryon number

$$\hat{B} \equiv \int \psi^\dagger(\mathbf{x})\, \psi(\mathbf{x})\, d\mathbf{x} \tag{3.20}$$

is a constant of the motion, it follows from translational invariance that ρ_B is also a constant of the motion and unchanged by the interactions.

We note that according to Eqs. (3.17) *and* (3.18) *the expectation values of the meson field operators become larger and larger as the baryon density increases. This is just the situation where we might expect to be able to replace a quantum field operator by the corresponding classical field.* To produce a real theory, the quantum fluctuations about these large expectation values must be examined. The formulation of this problem is a major concern of the present paper. First, however, we investigate the simplified theory where the meson fields are replaced by their corresponding, constant expectation values

$$\phi \to \phi_0 ; \qquad V_\lambda \to i\delta_{\lambda 4} V_0 \tag{3.21}$$

in the original lagrangian density (2.1).[7] Under the substitution (3.21), the lagrangian density (2.1) becomes

$$\mathscr{L}^0 = -\hbar c \left[\bar{\psi} \left(\gamma_\lambda \frac{\partial}{\partial x_\lambda} + \frac{g_v}{\hbar c} \gamma_4 V_0 + \left(M - \frac{g_s}{\hbar c} \phi_0\right)\right) \psi\right] - \frac{c^2\mu^2}{2} \phi_0{}^2 + \frac{m^2}{2} V_0{}^2. \tag{3.22}$$

Note that only the mass terms remain from the meson parts of the original lagrangian since ϕ_0 and V_0 are constants independent of $\mathbf{x}$ and t and that the scalar and vector terms enter with opposite signs because of the negative metric in the scalar product $V_\lambda V_\lambda = \mathbf{V}^2 - V_0{}^2$. The only derivative in this lagrangian is $\partial\psi/\partial x_\lambda$ and the canonical stress tensor is given by [37]

$$\begin{aligned} T^0_{\lambda\rho} &= \mathscr{L}^0 \delta_{\lambda\rho} - \frac{\partial \psi_\alpha}{\partial x_\rho} \frac{\partial \mathscr{L}^0}{\partial(\partial\psi_\alpha/\partial x_\lambda)} \\ &= \hbar c \bar{\psi} \gamma_\lambda \frac{\partial}{\partial x_\rho} \psi + \left(\frac{m^2}{2} V_0{}^2 - \frac{\mu^2 c^2}{2} \phi_0{}^2\right) \delta_{\lambda\rho}, \end{aligned} \tag{3.23}$$

where use has been made of the baryon field equations (3.9). The field equations (3.9) guarantee that the stress tensor is conserved

$$\frac{\partial}{\partial x_\lambda} T^0_{\lambda\rho} = 0. \tag{3.24}$$

The energy density, hamiltonian, and pressure can now be identified through Eqs. (3.12–3.14) to be

$$\begin{aligned} \epsilon = -T_{44} &= \psi^\dagger \left[c\boldsymbol{\alpha} \cdot \frac{\hbar}{i} \boldsymbol{\nabla} + \beta(m_b c^2 - g_s \phi_0) + g_v V_0\right] \psi \\ &\quad - \frac{m^2}{2} V_0{}^2 + \frac{\mu^2 c^2}{2} \phi_0{}^2, \end{aligned} \tag{3.25}$$

$$H = \int \epsilon \, d\mathbf{x}, \tag{3.26}$$

$$P = \frac{1}{3} T_{ii} = \frac{1}{3} \psi^\dagger \left[c\boldsymbol{\alpha} \cdot \frac{\hbar}{i} \boldsymbol{\nabla}\right] \psi + \frac{m^2}{2} V_0{}^2 - \frac{\mu^2 c^2}{2} \phi_0{}^2. \tag{3.27}$$

[7] The problem of baryons interacting with a classical scalar field of zero range has been studied by Marx [27] who pointed out that such an interaction by itself, if treated relativistically, can lead to nuclear saturation. This model was then applied to high-density matter by Marx and Nemeth [28]. The problem of stationary baryons interacting with a classical vector meson field was first studied by Zel'dovich [25].

To quantize the system, the baryon field operator will be expanded in normal-mode solutions to the classical field equation (3.9). We look for wave-function solutions of the form $\psi = \mathscr{U} e^{-(i/\hbar)Et} e^{i\mathbf{k}\cdot\mathbf{x}}$ where the $\mathbf{k}$'s are chosen to satisfy the periodic boundary conditions and arrive at the modified Dirac equation

$$\hbar c[\boldsymbol{\alpha} \cdot \mathbf{k} + \beta M^*]\mathscr{U} = (E_k - g_v V_0)\mathscr{U}. \tag{3.28}$$

The effective mass M^* is defined according to

$$M^* \equiv M - \frac{g_s}{\hbar c}\,\phi_0\,. \tag{3.29}$$

Upon squaring the relation (3.28), the eigenvalues are obtained as

$$E_k^{\pm} = g_v V_0 \pm [(\hbar \mathbf{k} c)^2 + (m_b{}^* c^2)^2]^{1/2}. \tag{3.30}$$

A complete set of Dirac wavefunctions for given $\mathbf{k}$ is then obtained from the solutions

$$\hbar c[\boldsymbol{\alpha} \cdot \mathbf{k} + \beta M^*]\, \mathscr{U}(\mathbf{k}\lambda) = (E_k{}^{+} - g_v V_0)\, \mathscr{U}(\mathbf{k}\lambda), \tag{3.31a}$$

$$\hbar c[\boldsymbol{\alpha} \cdot \mathbf{k} + \beta M^*]\, \mathscr{V}(\mathbf{k}\lambda) = (E_k{}^{-} - g_v V_0)\, \mathscr{V}(\mathbf{k}\lambda), \tag{3.31b}$$

where λ represents the baryon helicity and any other quantum numbers, such as third component of isotopic spin, necessary to specify the baryon states. The Dirac wavefunctions are normalized according to

$$\mathscr{U}^{\dagger}\mathscr{U} = \mathscr{V}^{\dagger}\mathscr{V} = 1 \tag{3.32}$$

The Dirac field operator in the Schrödinger representation may now be expanded as

$$\psi(\mathbf{x}) = \frac{1}{\Omega^{1/2}} \sum_{\mathbf{k}\lambda} [A_{\mathbf{k}\lambda}\mathscr{U}(\mathbf{k}\lambda)\, e^{i\mathbf{k}\cdot\mathbf{x}} + B_{\mathbf{k}\lambda}^{\dagger}\mathscr{V}(-\mathbf{k}\lambda)\, e^{-i\mathbf{k}\cdot\mathbf{x}}] \tag{3.33}$$

and the corresponding expressions ϵ, H, P obtained by inserting this expansion in Eqs. (3.25)–(3.27). The canonical quantization procedure for the Dirac field [37] leads us to interpret the normal-mode amplitudes $A_{\mathbf{k}\lambda}^{\dagger}$, $A_{\mathbf{k}\lambda}$, $B_{\mathbf{k}\lambda}^{\dagger}$, $B_{\mathbf{k}\lambda}$ as baryon and antibaryon creation and destruction operators. There is one further problem, however, and that is in the determination of the *ordering* of these amplitudes in the expressions for ϵ, H, P. In the classical field theory, the ordering of these amplitudes is completely undefined. Once the canonical quantization prescription has been followed, however, the operators fail to commute and the ordering becomes important. To define the quantum field theory the ordering of the normal-mode amplitudes as well as their anticommutation relations must be defined. We choose to define the theory by normal-ordering all expressions with respect to the set of operators $\{A_{\mathbf{k}\lambda}, A^\dagger_{\mathbf{k}\lambda}, B_{\mathbf{k}\lambda}, B^\dagger_{\mathbf{k}\lambda}\}$. This means that when Eq. (3.33) is inserted into Eqs. (3.25), (3.27), and (3.20) all destruction operators are to be moved to the right of all creation operators in every term and a sign of (-1) raised to the number of interchanges of fermion operators affixed to that term. We use the notation $:\bar{\psi}\Gamma\psi:$ for this normal-ordering procedure. This prescription leads to the expressions

$$\frac{\hat{H}^0}{\Omega} = \frac{1}{\Omega}\int d\mathbf{x}\left\{:\psi^\dagger\left[c\boldsymbol{\alpha}\cdot\frac{\hbar}{i}\nabla + \beta(m_b c^2 - g_s\phi_0) + g_v V_0\right]\psi: - \frac{1}{2}m^2 V_0^2 + \frac{1}{2}\mu^2 c^2\phi_0^2\right\}$$
$$= \frac{1}{2}\mu^2 c^2\phi_0^2 - \frac{1}{2}m^2 V_0^2 + \frac{1}{\Omega}\sum_{\mathbf{k}\lambda}[E_k{}^+ A^\dagger_{\mathbf{k}\lambda}A_{\mathbf{k}\lambda} + E_k{}^-(-B^\dagger_{\mathbf{k}\lambda}B_{\mathbf{k}\lambda})], \qquad (3.34)$$

$$\frac{\hat{B}}{\Omega} = \frac{1}{\Omega}\int d\mathbf{x}:\psi^\dagger(\mathbf{x})\,\psi(\mathbf{x}):$$
$$= \frac{1}{\Omega}\sum_{\mathbf{k}\lambda}[A^\dagger_{\mathbf{k}\lambda}A_{\mathbf{k}\lambda} + (-B^\dagger_{\mathbf{k}\lambda}B_{\mathbf{k}\lambda})]. \qquad (3.35)$$

These results may be combined with the aid of Eq. (3.30) and rewritten as

$$\frac{\hat{H}^0}{\Omega} = \frac{1}{2}\mu^2 c^2\phi_0^2 - \frac{1}{2}m^2 V_0^2 + g_v V_0\left(\frac{\hat{B}}{\Omega}\right)$$
$$+ \frac{1}{\Omega}\sum_{\mathbf{k}\lambda}[(\hbar\mathbf{k}c)^2 + (m_b{}^* c^2)^2]^{1/2}[A^\dagger_{\mathbf{k}\lambda}A_{\mathbf{k}\lambda} + B^\dagger_{\mathbf{k}\lambda}B_{\mathbf{k}\lambda}], \qquad (3.36)$$

$$\frac{\hat{B}}{\Omega} = \frac{1}{\Omega}\sum_{\mathbf{k}\lambda}[A^\dagger_{\mathbf{k}\lambda}A_{\mathbf{k}\lambda} - B^\dagger_{\mathbf{k}\lambda}B_{\mathbf{k}\lambda}]. \qquad (3.37)$$

The normal-ordering prescription used here has the feature that it leads to a positive-definite fermion contribution to the energy [the last term in Eq. (3.36)]. Recall that for a system with a given number of baryons, $\hat{B}$ is a constant of the motion. Equations (3.36) and (3.37) also reduce to the proper expressions in the limit of vanishing baryon density and have the correct behavior under particle–antiparticle conjugation.

The ground state $|F\rangle$ of the model hamiltonian $\hat{H}^0$ corresponding to a baryon number B is obtained by filling the momentum states $\mathbf{k}$ up to a Fermi wave number k_F. It is assumed that each momentum state can accommodate γ baryons. For example, $\gamma = 4$ for nuclear matter with equal numbers of neutrons and protons and $\gamma = 2$ for pure neutron matter. The baryon density can now be immediately written down, for example, as

$$\rho_B = \langle F \mid \frac{\hat{B}}{\Omega} \mid F \rangle = \langle F \mid :\psi^\dagger \psi: \mid F \rangle = \frac{1}{\Omega} \sum_{\mathbf{k}\lambda}^{k_F} \mathscr{U}^\dagger \mathscr{U} = \frac{\gamma}{(2\pi)^3} \int_0^{k_F} d\mathbf{k} = \frac{\gamma k_F{}^3}{6\pi^2}, \quad (3.38)$$

where the sum has been converted to an integral in standard fashion [24].

It remains to specify the meson fields V_0 and ϕ_0. This will be done self-consistently by utilizing the original meson field equations as indicated in Eqs. (3.17) and (3.18).

$$V_0 = \frac{g_v}{m^2} \langle F \mid :\psi^\dagger \psi: \mid F \rangle = \frac{g_v}{m^2} \rho_B, \quad (3.39)$$

$$\phi_0 = \frac{g_s}{\mu^2 c^2} \langle F \mid :\bar{\psi}\psi: \mid F \rangle \equiv \frac{g_s}{\mu^2 c^2} \rho_s. \quad (3.40)$$

The densities have been normal ordered according to our prescription. Specification of ρ_B (a constant of the motion) specifies V_0. In contrast, Eq. (3.40) is a relation which determines the scalar density ρ_s self-consistently at a given baryon density. To see this, first rewrite the Dirac equation (3.31a) as

$$(\boldsymbol{\alpha} \cdot \mathbf{k} + \beta M^*) \mathscr{U}(\mathbf{k}\lambda) = (\mathbf{k}^2 + M^{*2})^{1/2} \mathscr{U}(\mathbf{k}\lambda), \quad (3.41a)$$

$$\mathscr{U}^\dagger(\mathbf{k}\lambda)(\boldsymbol{\alpha} \cdot \mathbf{k} + \beta M^*) = (\mathbf{k}^2 + M^{*2})^{1/2} \mathscr{U}^\dagger(\mathbf{k}\lambda). \quad (3.41b)$$

Multiplication of Eq. (3.41a) by $U^\dagger(\mathbf{k}\lambda)\beta$ from the left and Eq. (3.41b) by $\beta U(\mathbf{k}\lambda)$ from the right and addition of the results leads to

$$\mathscr{U}^\dagger \mathscr{U} \cdot M^* = \bar{\mathscr{U}} \mathscr{U} (\mathbf{k}^2 + M^{*2})^{1/2}, \quad (3.42)$$

where the properties of the Dirac matrices $\beta\boldsymbol{\alpha} + \boldsymbol{\alpha}\beta = 0$ and $\beta^2 = 1$ have been used. Explicit computation of the matrix element in Eq. (3.40) yields

$$\rho_s = \langle F \mid :\bar{\psi}\psi: \mid F \rangle = \frac{1}{\Omega} \sum_{\mathbf{k}\lambda}^{k_F} \bar{\mathscr{U}} \mathscr{U} = \frac{\gamma}{(2\pi)^3} \int_0^{k_F} \frac{M^*}{(\mathbf{k}^2 + M^{*2})^{1/2}} d\mathbf{k}, \quad (3.43)$$

while Eq. (3.29) leads to

$$M^* = M - \frac{g_s{}^2}{\hbar c^3} \frac{\rho_s}{\mu^2}. \quad (3.44)$$

For given ρ_B, Eq. (3.43) is an implicit equation for ρ_s. The energy density follows immediately from the ground-state matrix element of Eq. (3.36). Inserting the values of ϕ_0 and V_0 from Eqs. (3.39), (3.40) and converting a sum to an integral, we arrive at

$$\epsilon = \frac{g_s^2}{2\mu^2c^2}\,\rho_s^2 + \frac{g_v^2}{2m^2}\,\rho_B^2 + \hbar c\cdot\frac{\gamma}{(2\pi)^3}\int_0^{k_F}(k^2 + M^{*2})^{1/2}\,d\mathbf{k}. \tag{3.45}$$

Note that the third term on the right side of Eq. (3.36) serves to just change the sign of the second. The pressure may be obtained from Eq. (3.27) as

$$P = \langle F \mid \frac{1}{3}\,{:}\psi^\dagger\left(c\boldsymbol{\alpha}\cdot\frac{\hbar}{i}\,\nabla\right)\psi{:} + \frac{m^2}{2}\,V_0^2 - \frac{\mu^2c^2}{2}\,\phi_0^2 \mid F\rangle. \tag{3.46}$$

Utilization of the Dirac equation (3.41a), the relation (3.42), and a little algebra leads to the result

$$P = -\frac{g_s^2}{2\mu^2c^2}\,\rho_s^2 + \frac{g_v^2}{2m^2}\,\rho_B^2 + \frac{\hbar c}{3}\,\frac{\gamma}{(2\pi)^3}\int_0^{k_F}\frac{\mathbf{k}^2}{(\mathbf{k}^2 + M^{*2})^{1/2}}\,d\mathbf{k}. \tag{3.47}$$

Equations (3.38), (3.43), (3.44), (3.45), and (3.47) form the main result of this section. At a given baryon density ρ_B [Eq. (3.38)], Eqs. (3.43) and (3.44) can be solved self-consistently to determine the scalar density ρ_s. Equations (3.45) and (3.47) then yield the parametric equation of state $P(\rho_B)$, $\epsilon(\rho_B)$. The baryon density can be eliminated to give the equation of state $P(\epsilon)$ which determines the stress tensor $T_{\mu\nu}$ in Eq. (3.10) and is the desired result. (Recall $\epsilon \equiv \rho c^2$.)

To analyze these equations, we convert to a set of dimensionless coupling constants

$$c_s^2 \equiv \frac{g_s^2}{\hbar c^3}\cdot\frac{M^2}{\mu^2}, \tag{3.48a}$$

$$c_v^2 \equiv \frac{g_v^2}{\hbar c}\cdot\frac{M^2}{m^2}, \tag{3.48b}$$

and dimensionless variables

$$d_B \equiv \frac{\rho_B}{M^3}, \tag{3.49}$$

$$a \equiv \frac{m_b c^2 - (g_s^2/\mu^2c^2)\,\rho_s}{\hbar k_F c} = \frac{M^*}{k_F}. \tag{3.50}$$

The integrals in Eqs. (3.45) and (3.47) can be evaluated to yield

$$\frac{1}{m_b c^2}\left(\frac{\epsilon}{\rho_B}\right) = \frac{1}{2}\{c_v^2 + c_s^2[f(a)]^2\}\,d_B + 3g(a)\left(\frac{6\pi^2 d_B}{\gamma}\right)^{1/3}, \tag{3.51}$$

$$\frac{1}{m_b c^2}\left(\frac{P}{\rho_B}\right) = \frac{1}{2}\{c_v^2 - c_s^2[f(a)]^2\}\,d_B + \left[g(a) - \frac{a}{3}f(a)\right]\left(\frac{6\pi^2 d_B}{\gamma}\right)^{1/3}, \tag{3.52}$$

where

$$f(a) = 3a \int_0^1 \frac{x^2\,dx}{(x^2 + a^2)^{1/2}}$$
$$= \frac{3a}{2}\left[(1 + a^2)^{1/2} - \frac{a^2}{2} \ln \frac{(1 + a^2)^{1/2} + 1}{(1 + a^2)^{1/2} - 1}\right], \tag{3.53}$$

$$g(a) \equiv \int_0^1 x^2(x^2 + a^2)^{1/2}\,dx$$
$$= \frac{1}{8}\left[(1 + a^2)^{3/2} + (1 + a^2)^{1/2} - \frac{a^4}{2} \ln \frac{(1 + a^2)^{1/2} + 1}{(1 + a^2)^{1/2} - 1}\right]. \tag{3.54}$$

The self-consistency relations (3.43) and (3.44) read

$$\frac{\rho_s}{\rho_B} = f(a), \tag{3.55}$$

which can be converted with the aid of Eq. (3.50) into a transcendental equation for a

$$\frac{1}{c_s^2 d_B}\left[1 - \left(\frac{6\pi^2 d_B}{\gamma}\right)^{1/3} a\right] = f(a). \tag{3.56}$$

Examination of the two sides of this relation shows that there will always be one real, positive root. At low density ($d_B \to 0$, $a \to \infty$), the asymptotic form $f(a) \to_{a\to\infty} 1$ leads to $a \to (\gamma/6\pi^2\rho_B)^{1/3}$ and allows us to recover the result $\rho_s = \rho_B$. At high density ($d_B \to \infty$, $a \to 0$), the result $f(a) \to_{a\to 0} 3a/2$ implies the limiting values $a \to 2/3c_s^2 d_B$, $\rho_s/\rho_B \to 1/c_s^2 d_B$. At high densities, therefore, the *ratio* ρ_s/ρ_B goes to zero ([this is evident from a direct comparison of Eqs. (3.38) and (3.43)] while ρ_s itself approaches a finite asymptotic value).

Several features of this equation of state are of interest:

(1) By direct differentiation of Eq. (3.51) we can establish the result that

$$P = \rho_B^2 \frac{\partial}{\partial \rho_B}\left(\frac{\epsilon}{\rho_B}\right). \tag{3.57}$$

(Recall that a is an implicit function of ρ_B through Eq. (3.56).] This is just a rewriting of the first law of thermodynamics

$$P d\Omega = -dE, \tag{3.58}$$

since the volume Ω enters our result only through the variable $\rho_B = B/\Omega$. This result indicates that our *present set of approximations leads to a thermodynamically consistent theory.*

(2) In the *low-density limit* ($d_B \to 0, a \to \infty$) Eqs. (3.56) and (3.51) can be expanded in powers of $d_B^{1/3}$ to give

$$\frac{1}{m_b c^2}\left(\frac{\epsilon}{\rho_B}\right) \to 1 + \frac{3}{10}\left(\frac{6\pi^2 d_B}{\gamma}\right)^{2/3} + \frac{1}{2}c_v{}^2 d_B - \frac{1}{2}c_s{}^2 d_B - \frac{3}{56}\left(\frac{6\pi^2 d_B}{\gamma}\right)^{4/3}$$
$$+ \frac{3}{10}c_s{}^2 d_B\left(\frac{6\pi^2 d_B}{\gamma}\right)^{2/3} + \frac{3}{144}\left(\frac{6\pi^2 d_B}{\gamma}\right)^2 + O(d_B^{7/3}). \tag{3.59}$$

The first term is the rest mass of the nucleons. The second term is the Fermi energy of the nucleons and the fifth and seventh are relativistic corrections to this. The third term is a positive (repulsive) contribution coming from the vector meson field and the fourth is a negative (attractive) contribution from the scalar meson field. The sixth term indicates the *damping* of the scalar attraction at higher densities arising from the self-consistency relation (3.56).[7] Note that if $c_s{}^2$ is made large enough, the fourth term can be made to dominate and the *interaction energy per nucleon* $(\epsilon/\rho_B) - m_b c^2$ *will become negative as the density is increased. This indicates the system will be self-bound.*

(3) In the *high-density limit* ($d_B \to \infty, a \to 0$) a similar expansion yields

$$\frac{1}{m_b c^2}\left(\frac{\epsilon}{\rho_B}\right) \to \frac{1}{2}c_v{}^2 d_B + \frac{3}{4}\left(\frac{6\pi^2 d_B}{\gamma}\right)^{1/3} + \frac{1}{2c_s{}^2 d_B} + O(d_B^{-5/3}). \tag{3.60}$$

The first term is the vector repulsion which dominates at high density. The second term is the relativistic Fermi energy of the baryons, and the third term is the scalar contribution to the energy which now has also become repulsive. *Thus, at high densities the interaction energy per nucleon becomes large and positive. This indicates that the medium will always saturate*, i.e., there is a finite density at which the interaction energy per nucleon has a negative minimum.

(4) Combining Eqs. (3.57), (3.60) we observe that at high densities $P \to \epsilon$ because the vector repulsion is the dominating term. This implies that $v_s \to c$ where v_s is the thermodynamic speed of sound in the medium [7].[8] Such a high-density equation of state based on either repulsive potentials generated by a vector-meson field theory with stationary baryons and a Hartree approximation to the many-body problem, or on the classical solution to the field theory (see Eq. (3.8)), has been previously presented by Zel'dovich [25].

To proceed further numerical work is required. Donnelly has programmed a solution to Eq. (3.56) and computed ϵ and P. The resulting binding energy per nucleon is shown in Fig. 1. There are only two parameters in this theory $\{c_s{}^2, c_v{}^2\}$.

[8] We do not discuss here the type of sound waves which can actually propagate in this medium (see, for example, the discussion in Ref. [24, Section 16] and Ref. [38].)

These parameters have been chosen to reproduce the correct binding energy and density of nuclear matter [24] $\{\gamma = 4, \epsilon/\rho_B - m_b c^2 = -15.75 \text{ MeV}, k_F = 1.42 F^{-1}\}$. The resulting values of the dimensionless coupling constants are

$$c_s^2 = \frac{g_s^2}{\hbar c^3} \cdot \frac{M^2}{\mu^2} = 266.9, \tag{3.61a}$$

$$c_v^2 = \frac{g_v^2}{\hbar c} \cdot \frac{M^2}{m^2} = 195.7. \tag{3.61b}$$

For comparison, three curves from more standard nuclear-matter many-body theory are also shown in Fig. 1. Curves I and II are derived in detail in reference

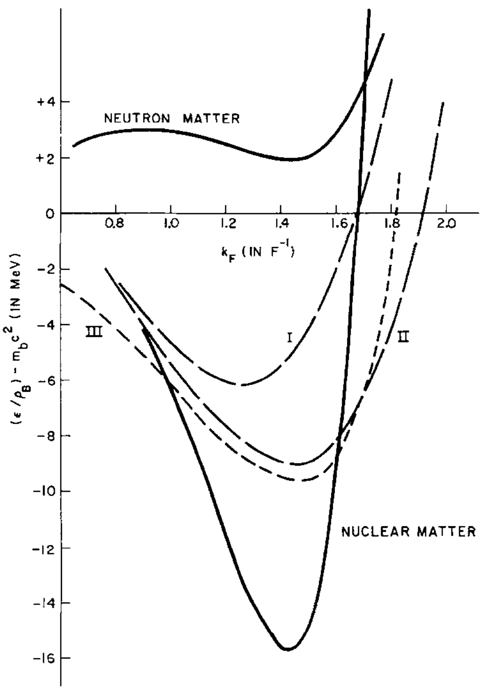

FIG. 1. Binding energy per nucleon vs. Fermi wave number for nuclear matter computed from the linearized theory, Eqs. (3.38) and (3.43)–(3.45). The two dimensionless coupling constants (Eqs. (3.61)) have been chosen to reproduce the binding energy and density of nuclear matter $\{\gamma = 4, \epsilon/\rho_B - m_b c^2 = -15.75 \text{ MeV}, k_F = 1.42 F^{-1}\}$ [24]. For comparison, three curves from more standard nuclear matter many-body theory are also shown. Curves I and II are derived in detail in Ref. [24] (see Ref. [24, Fig. 41.7]) while curve III is from the work of Pandharipande [22c] based on the nucleon–nucleon potentials of Reid. The prediction of the linearized theory for pure neutron matter, obtained by simply setting $\gamma = 2$, are also shown in this figure.

[24] while curve III is from the work of Pandharipande [22c] based on the nucleon–nucleon potentials of Reid.

Once the two parameters $\{c_s^2, c_v^2\}$ have been determined, all other quantities are predicted. For example, the equation of state for pure neutron matter is obtained by simply setting $\gamma = 2$. The binding energy per neutron in the vicinity of nuclear densities is also shown in Fig. 1. Note that neutron matter is unbound in this theory, in accord with most other calculations. Figure 2 compares the energy per nucleon as a function of k_F for pure neutron matter with the calculations of Pandharipande [22b] using the Reid and Hamada–Johnson potentials and a cluster expansion for the energy. The present equation of state is "stiffer" at high densities. The dip in the present result near nuclear matter wave numbers appears because the two parameters of the theory have been adjusted to give the correct binding energy per nucleon in nuclear matter. If the constants are adjusted to fit the theoretical results for nuclear matter (which are underbound), this dip disappears. Finally, in Fig. 3 we show the dimensionless equation of state $(1/M^3)(P/m_bc^2)$ vs$(1/M^3)(\epsilon/m_bc^2)$ for pure neutron matter from nuclear matter densities out to the smooth joining

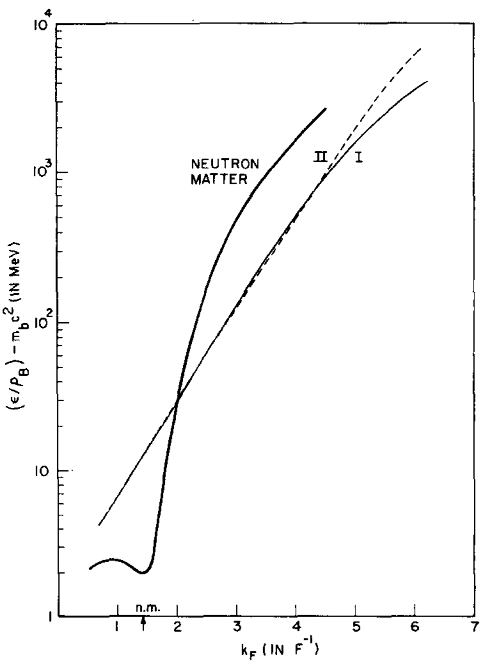

FIG. 2. Binding energy per nucleon vs Fermi wave number for neutron matter at high densities computed from the linearized theory (see Fig. 1). For comparison, calculations of Pandharipande [22b] using the Reid (I) and Hamada–Johnson [II] nucleon–nucleon potentials and a cluster expansion for the energy are also shown.

with the asymptotic equation of state $P = \epsilon$. Further theoretical consequences of this equation of state and the implications for condensed stellar structure will be discussed in a subsequent publication [39].

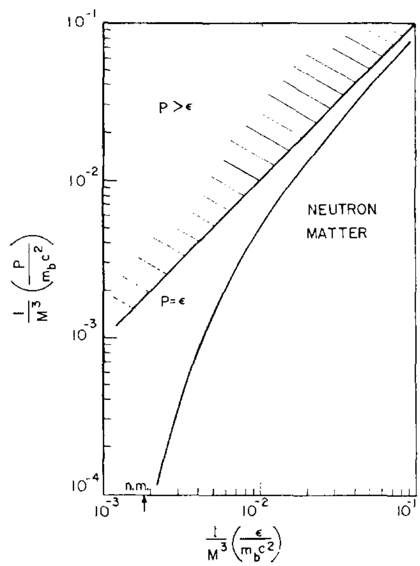

FIG. 3. Dimensionless equation of state $(1/M^3)(P/m_b c^2)$ vs $(1/M^3)(\epsilon/m_b c^2)$ for pure neutron matter from nuclear matter (n.m.) densities out to the smooth joining with the asymptotic, high-density equation of state $P = \epsilon$. These results are computed from the linearized theory (see Fig. 1).

As a final application of the present theory, we compute the fermion Green's function defined by [24]

$$i\bar{G}(x, x')^0_{\alpha\beta} = \langle F \mid P[\psi_\alpha(x), \bar{\psi}_\beta(x')] \mid F\rangle \tag{3.62a}$$

$$= \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')} i\bar{G}_{\alpha\beta}(k)^0. \tag{3.62b}$$

It is convenient to first make a canonical transformation to baryon particles above the Fermi sea and baryon holes below the Fermi sea

$$\begin{aligned} A_{\mathbf{k}\lambda} &\equiv b^\dagger_{-\mathbf{k}\lambda} \qquad |\,\mathbf{k}\,| < k_F\,, \\ &= A_{\mathbf{k}\lambda} \qquad |\,\mathbf{k}\,| > k_F\,. \end{aligned} \tag{3.63}$$

All the annihilation operators in the theory now annihilate the ground state

$$A_{\mathbf{k}\lambda} \mid F\rangle = b_{\mathbf{k}\lambda} \mid F\rangle = B_{\mathbf{k}\lambda} \mid F\rangle = 0. \tag{3.64}$$

Direct calculation then yields

$$\bar{G}(k)^0 = \frac{-1}{i\not{k} + M^*}, \tag{3.65}$$

which is exactly the same form as the usual Feynman Green's function; the only difference is that the singularity structure is modified from the free-particle case. In the free case, the singularities are located by the Feynman prescription $M^* \to M^* - i\eta$. The present prescription is

$$\begin{aligned} M^* &\to M^* - i\eta, \qquad \text{if} \quad |\mathbf{k}| > k_F, \\ k_0 &\to k_0 - i\eta, \qquad \text{if} \quad |\mathbf{k}| < k_F. \end{aligned} \tag{3.66}$$

For wave numbers below the Fermi level, the *frequency* is now given a small additional imaginary part instead of the mass. This singularity structure is illustrated in Fig. 4. Two equivalent representation of this Green's function are the following:

$$\bar{G}(k)^0 = \frac{-1}{[i\not{k} + M^*]_F} - 2\pi i \theta(k_0)\, \delta(k^2 + M^{*2})(i\not{k} - M^*)\, \theta(k_F - |\mathbf{k}|) \tag{3.67}$$

$$= (i\not{k} - M^*) \left[\frac{1}{\omega_k/c + k_0 - i\eta}\right]\left[\frac{1}{\omega_k/c - k_0 - i\eta\, \mathrm{sgn}(|\mathbf{k}| - k_F)}\right], \tag{3.68}$$

where $\omega_k/c \equiv (\mathbf{k}^2 + M^{*2})^{1/2}$, $k^2 = k_\mu k_\mu$, and the subscript F indicates the Feynman propagator.

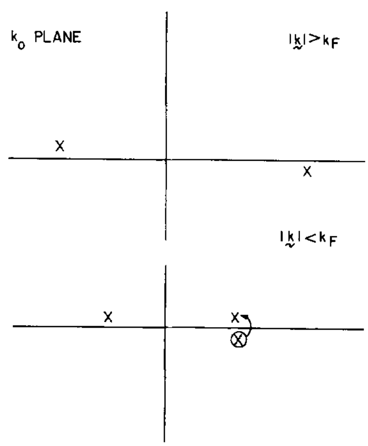

FIG. 4. Singularity structure of the baryon Green's function $\bar{G}(k)^0$ [Eqs. (3.65)–(3.66)].

4. Full Field Theory

Thus far we have examined the consequences of replacing the meson fields in the lagrangian density (2.1) by their constant expectation values $\phi \to \phi_0$, $V_\lambda \to i\delta_{\lambda 4} V_0$ determined self-consistently through the meson field equations (3.1) and (3.2) [see Eqs. (3.39), (3.40)]. The next step is to examine the full relativistic quantum field theory to attempt to evaluate the quantum fluctuations about the classical field values. The stress tensor corresponding to the lagrangian density (2.1) is canonically given by [37]

$$\begin{aligned}
T_{\mu\nu} = {} & \mathscr{L}\delta_{\mu\nu} - \frac{\partial\psi_\alpha}{\partial x_\nu}\frac{\partial\mathscr{L}}{\partial(\partial\psi_\alpha/\partial x_\mu)} - \frac{\partial\bar{\psi}_\alpha}{\partial x_\nu}\frac{\partial\mathscr{L}}{\partial(\partial\bar{\psi}_\alpha/\partial x_\mu)} \\
& - \frac{\partial\phi}{\partial x_\nu}\frac{\partial\mathscr{L}}{\partial(\partial\phi/\partial x_\mu)} - \frac{\partial V_\lambda}{\partial x_\nu}\frac{\partial\mathscr{L}}{\partial(\partial V_\lambda/\partial x_\mu)} \\
= {} & \left\{-\frac{1}{2}c^2\left[\left(\frac{\partial\phi}{\partial x_\lambda}\right)^2 + \mu^2\phi^2\right] - \frac{1}{4}F_{\lambda\rho}F_{\lambda\rho} - \frac{m^2}{2}V_\lambda V_\lambda\right\}\delta_{\mu\nu} + \hbar c\bar{\psi}\gamma_\mu\frac{\partial\psi}{\partial x_\nu} \\
& + c^2\left(\frac{\partial\phi}{\partial x_\nu}\right)\left(\frac{\partial\phi}{\partial x_\mu}\right) - \frac{\partial V_\lambda}{\partial x_\nu}F_{\lambda\mu},
\end{aligned} \tag{4.1}$$

where use has been made of the field equations (3.1)–(3.4). The field equations imply that this tensor is conserved

$$\frac{\partial}{\partial x_\mu}T_{\mu\nu} = 0, \tag{4.2}$$

which in turn implies that the four-momentum, defined by

$$P_\mu \equiv \frac{1}{ic}\int d\mathbf{x}\, T_{4\mu} = \left(\mathbf{P}, \frac{i}{c}H\right), \tag{4.3}$$

is a conserved four-vector. The stress tensor can be symmetrized using the procedure in Wentzel [37], but that is unnecessary for the present discussion. The canonical momenta are given by

$$\pi_s = \frac{\partial\mathscr{L}}{\partial(\partial\phi/\partial t)} = \frac{\partial\phi}{\partial t}, \tag{4.4a}$$

$$\pi_j^v = \frac{\partial\mathscr{L}}{\partial(\partial V_j/\partial t)} = \frac{1}{ic}F_{j4}, \tag{4.4b}$$

$$\pi_\psi = \frac{\partial\mathscr{L}}{\partial(\partial\psi/\partial t)} = i\hbar\psi^\dagger. \tag{4.4c}$$

The energy density can be identified as in Eq. (3.13) to be

$$
\begin{aligned}
\epsilon &= \rho c^2 = -T_{44} \\
&= \frac{1}{2}[\pi_s{}^2 + c^2(\nabla\phi^2) + \mu^2 c^2 \phi^2] + \frac{1}{2}\left[c^2(\boldsymbol{\pi}^v)^2 + \frac{c^2}{m^2}(\nabla\cdot\boldsymbol{\pi}^v)^2 + (\nabla_\wedge \mathbf{V})^2 + m^2\mathbf{V}^2\right] \\
&\quad + \hbar c \psi^\dagger\left[\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta M\right]\psi - i g_v \bar{\psi}\boldsymbol{\gamma}\psi\cdot\mathbf{V} - g_s\bar{\psi}\psi\phi + \frac{g_v c}{m^2}(\nabla\cdot\boldsymbol{\pi}^v)(\psi^\dagger\psi) \\
&\quad + \frac{g_v{}^2}{2m^2}(\psi^\dagger\psi)(\psi^\dagger\psi) - \frac{c^2}{m^2}\nabla\cdot\left[\boldsymbol{\pi}^v(\nabla\cdot\boldsymbol{\pi}^v) + \frac{g_v}{c}\boldsymbol{\pi}^v(\psi^\dagger\psi)\right].
\end{aligned} \tag{4.5}
$$

In arriving at this result, the equation of motion for the vector meson field [Eq. (3.2)] has been used in conjunction with Eq. (4.4b) to solve for the fourth component of the vector meson field

$$V_4 = \frac{ic}{m^2}\left[\nabla\cdot\boldsymbol{\pi}^v + \frac{g_v}{c}(\psi^\dagger\psi)\right]. \tag{4.6}$$

The total divergence in the last term in Eq. (4.5) will have no expectation value in the quantum theory since the diagonal matrix element of a total divergence between eigenstates of momentum vanishes by translational invariance

$$\langle\Psi_0 \mid \nabla\cdot\hat{\mathbf{O}}_H(x) \mid \Psi_0\rangle = [\nabla\cdot e^{-(i/\hbar)(P_f - P_i)\cdot x}\langle\Psi_0 \mid \hat{\mathbf{O}}(0) \mid \Psi_0\rangle]_{P_f = P_i} = 0. \tag{4.7}$$

The hamiltonian is given by Eq. (3.14)

$$
\begin{aligned}
H = \int \epsilon\, d\mathbf{x} = \int d\mathbf{x} \Big\{ & \frac{1}{2}[\pi_s{}^2 + c^2(\nabla\phi)^2 + \mu^2c^2\phi^2] \\
& + \frac{1}{2}\left[c^2(\boldsymbol{\pi}^v)^2 + \frac{c^2}{m^2}(\nabla\cdot\boldsymbol{\pi}^v)^2 + (\nabla_\wedge\mathbf{V})^2 + m^2\mathbf{V}^2\right] \\
& + \hbar c\psi^\dagger\left[\frac{1}{i}\boldsymbol{\alpha}\cdot\nabla + \beta M\right]\psi - i g_v\bar{\psi}\boldsymbol{\gamma}\psi\cdot\mathbf{V} - g_s\bar{\psi}\psi\phi \\
& + \frac{g_v c}{m^2}(\nabla\cdot\boldsymbol{\pi}^v)(\psi^\dagger\psi) + \frac{g_v{}^2}{2m^2}(\psi^\dagger\psi)(\psi^\dagger\psi)\Big\}.
\end{aligned} \tag{4.8}
$$

The total divergence in Eq. (4.5) also does not contribute to H since it can be converted to a surface integral which vanishes by the periodic boundary conditions.

In the following discussion, we assume there are B baryons in a volume Ω and concentrate on the hamiltonian H. Translational invariance can then be used to evaluate the baryon density ρ_B and energy density ϵ (Eq. (3.16)) and for the present purposes, we will use thermodynamic arguments to generate the pressure P

(Eq. (3.57)). In a thermodynamically consistent theory, this should yield the same P as generated from the stress tensor as long as the volume enters only through the baryon density. This was seen explicitly in the preceding section.

To take into account our observations on the condensed nature of the meson fields, we write them in the following fashion

$$\phi = \phi_0 + \sigma, \tag{4.9a}$$

$$V_\lambda = i\delta_{\lambda 4} V_0 + \eta_\lambda, \tag{4.9b}$$

and *it will be assumed that the uniform* $\mathbf{k} = 0$ *modes of* ϕ *and* V_4 *take constant, condensed, c-number values* just as in the treatment of a condensed boson system in standard many-body theory [24]. σ and η_λ will represent the quantum fluctuations about these values. By hypothesis, we have

$$\int \sigma(x)\, d\mathbf{x} = 0, \tag{4.10a}$$

$$\int \eta_4(x)\, d\mathbf{x} = 0, \tag{4.10b}$$

since they contain only $\mathbf{k} \neq 0$ components. Equation (4.6) now reads

$$\eta_4 = \frac{ic}{m^2}\left[\boldsymbol{\nabla}\cdot\boldsymbol{\pi}^v + \frac{g_v}{c}\,\delta(\psi^\dagger\psi)\right], \tag{4.11}$$

where $\delta(\psi^\dagger\psi)$ is defined by

$$\delta(\psi^\dagger\psi) \equiv \psi^\dagger\psi - \frac{m^2}{g_v} V_0 . \tag{4.12}$$

With the introduction of Eq. (4.9a) and (4.12), the classical field hamiltonian takes the following form

$$H = H_0 + H_1, \tag{4.13a}$$

$$H_0 = H_0{}^s + H_0{}^v + H_0{}^B, \tag{4.13b}$$

with

$$H_0{}^s = \int d\mathbf{x}\, \frac{1}{2}\,[\pi_s{}^2 + c^2(\boldsymbol{\nabla}\sigma)^2 + \mu^2c^2\sigma^2], \tag{4.14}$$

$$H_0{}^v = \int d\mathbf{x}\, \frac{1}{2}\left[c^2(\boldsymbol{\pi}^v)^2 + \frac{c^2}{m^2}(\boldsymbol{\nabla}\cdot\boldsymbol{\pi}^v)^2 + (\boldsymbol{\nabla}_\wedge\boldsymbol{\eta})^2 + m^2\boldsymbol{\eta}^2\right], \tag{4.15}$$

$$H_0{}^B = \int d\mathbf{x}\left\{\psi^\dagger\left[\frac{\hbar c}{i}\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + g_v V_0 + \beta(m_b c^2 - g_s\phi_0)\right]\psi + \frac{1}{2}\,\mu^2c^2\phi_0{}^2 - \frac{1}{2}\,m^2 V_0{}^2\right\}, \tag{4.16}$$

$$H_1 = \int d\mathbf{x} \left\{ -ig_v \bar{\psi} \boldsymbol{\gamma} \psi \cdot \boldsymbol{\eta} - g_s \bar{\psi} \psi \sigma \right.$$

$$\left. + \frac{g_v c}{m^2} (\nabla \cdot \boldsymbol{\pi}^v)\, \delta(\psi^\dagger \psi) + \frac{g_v^2}{2m^2}\, \delta(\psi^\dagger \psi)\, \delta(\psi^\dagger \psi) \right\}. \tag{4.17}$$

In arriving at Eq. (4.17), Eq. (4.10a) has been used, and a total divergence has again been integrated to zero. $H_0^{s,v}$ are the free hamiltonians for the scalar and vector fields σ and $\boldsymbol{\eta}$, respectively, since Eq. (4.4a,b) and Eq. (4.9) can be combined to give

$$\pi_s = \frac{\partial \sigma}{\partial t}, \tag{4.18a}$$

$$\pi_j{}^v = \frac{1}{ic} f_{j4}, \tag{4.18b}$$

with

$$f_{\mu\nu} \equiv \frac{\partial \eta_\nu}{\partial x_\mu} - \frac{\partial \eta_\mu}{\partial x_\nu}. \tag{4.19}$$

$H_0{}^B$ is precisely the hamiltonian H^0 which was diagonalized in the previous section (Eq. (3.36)). H_1 contains the interaction between the baryons and the scalar vector fluctuations σ and $\boldsymbol{\eta}$ in the condensed fields.

The normal-mode expansions of the baryon field presented in Eq. (3.33) and the canonical expansions for the spatial dependence of the scalar field [37]

$$\sigma = \frac{1}{\Omega^{1/2}} \sum_{\mathbf{k} \neq 0} \left(\frac{\hbar}{2\omega_k{}^s} \right)^{1/2} [c_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} + c_\mathbf{k}{}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{4.20a}$$

$$\pi_s = \frac{1}{\Omega^{1/2}} \sum_{\mathbf{k} \neq 0} \left(\frac{\hbar\omega_k{}^s}{2} \right)^{1/2} \frac{1}{i} [c_\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} - c_\mathbf{k}{}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{4.20b}$$

with $\omega_k{}^s/c = (\mathbf{k}^2 + \mu^2)^{1/2}$ and of the vector field $\boldsymbol{\eta} = \boldsymbol{\eta}_T + \boldsymbol{\eta}_L$, $\boldsymbol{\pi}^v = \boldsymbol{\pi}_T{}^v + \boldsymbol{\pi}_L{}^v$ [37]

$$\boldsymbol{\eta}_T = \frac{1}{\Omega^{1/2}} \sum_\mathbf{k} \sum_{\lambda=1}^{2} \left(\frac{\hbar c^2}{2\omega_k{}^v} \right)^{1/2} [d_{\mathbf{k}\lambda} \mathbf{e}_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{x}} + d_{\mathbf{k}\lambda}^\dagger \mathbf{e}_{\mathbf{k}\lambda}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{4.21a}$$

$$\boldsymbol{\eta}_L = \frac{1}{\Omega^{1/2}} \sum_\mathbf{k} \left(\frac{\hbar\omega_k{}^v}{2m^2} \right)^{1/2} [d_{\mathbf{k}0} \mathbf{e}_{\mathbf{k}0} e^{i\mathbf{k}\cdot\mathbf{x}} + d_{\mathbf{k}0}^\dagger \mathbf{e}_{\mathbf{k}0}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{4.21b}$$

$$\boldsymbol{\pi}_T{}^v = \frac{1}{\Omega^{1/2}} \sum_\mathbf{k} \sum_{\lambda=1}^{2} \left(\frac{\hbar\omega_k{}^v}{2c^2} \right)^{1/2} \frac{1}{i} [d_{\mathbf{k}\lambda} \mathbf{e}_{\mathbf{k}\lambda} e^{i\mathbf{k}\cdot\mathbf{x}} - d_{\mathbf{k}\lambda}^\dagger \mathbf{e}_{\mathbf{k}\lambda}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{4.21c}$$

$$\boldsymbol{\pi}_L{}^v = \frac{1}{\Omega^{1/2}} \sum_\mathbf{k} \left(\frac{\hbar m^2}{2\omega_k{}^v} \right)^{1/2} \frac{1}{i} [d_{\mathbf{k}0} \mathbf{e}_{\mathbf{k}0} e^{i\mathbf{k}\cdot\mathbf{x}} - d_{\mathbf{k}0}^\dagger \mathbf{e}_{\mathbf{k}0}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}], \tag{4.21d}$$

with $\omega_k^v/c = (\mathbf{k}^2 + m^2)^{1/2}$ may now be inserted in the hamiltonian. Here $\mathbf{e}_{\mathbf{k}1,2}$ are two orthogonal unit vectors perpendicular to $\mathbf{k}$ and $\mathbf{e}_{\mathbf{k}0}$ a unit vector along $\mathbf{k}$. We may now pass to the quantum field theory by imposing canonical commutation relations which leads to the interpretation of $\{c_\mathbf{k}^\dagger, c_\mathbf{k}, d_{\mathbf{k}\lambda}^\dagger, d_{\mathbf{k}\lambda}, A_{\mathbf{k}\lambda}^\dagger, A_{\mathbf{k}\lambda}, B_{\mathbf{k}\lambda}^\dagger, B_{\mathbf{k}\lambda}\}$ as creation and destruction operators for the scalar mesons, vector mesons, baryons, and antibaryons respectively. Equations (3.33), (4.20), and (4.21) are then the expansions of the fields in the Schrödinger representation.[9] Just as in the previous discussion, however, is also necessary to specify an *ordering* of the normal-mode amplitudes in passing from the classical to the quantum field theory. *We choose to specify the quantum field theory by normal ordering all bilinear products of field operators entering into currents and densities. Specifically, H_0 and the bilinear products of fermion field operators in H_1 are chosen to be normal ordered.* With this choice our model hamiltonian is

$$\hat{H}_0^s = \sum_{\mathbf{k}\neq 0} \hbar\omega_k^s c_\mathbf{k}^\dagger c_\mathbf{k}, \tag{4.22a}$$

$$\hat{H}_0^v = \sum_\mathbf{k} \sum_{\lambda=0}^{2} \hbar\omega_k^v d_{\mathbf{k}\lambda}^\dagger d_{\mathbf{k}\lambda}, \tag{4.22b}$$

$$\hat{H}_0^B = \int d\mathbf{x} \left[\frac{1}{2}\mu^2c^2\phi_0^2 - \frac{1}{2}m^2V_0^2\right] + g_vV_0\hat{B} + \sum_{\mathbf{k}\lambda} [(\hbar\mathbf{k}c)^2 + (m_b{}^*c^2)^2]^{1/2} (A_{\mathbf{k}\lambda}^\dagger A_{\mathbf{k}\lambda} + B_{\mathbf{k}\lambda}^\dagger B_{\mathbf{k}\lambda}), \tag{4.22c}$$

with

$$m_b{}^*c^2 \equiv m_bc^2 - g_s\phi_0. \tag{4.23}$$

The meson and baryon normal-ordered contributions to $\hat{H}_0$ are again positive definite. The ground state of $\hat{H}_0$ corresponding to a given number of baryons B is again the state $|F\rangle$ of the previous section. Note there are no real mesons present in this ground state so that

$$\hat{H}_0^s|F\rangle = \hat{H}_0^v|F\rangle = 0. \tag{4.24}$$

It is convenient to also make the canonical transformation of Eqs. (3.63) and (3.64) to baryon particles and holes. This means the fermion operators $\{A_{\mathbf{k}\lambda}, B_{\mathbf{k}\lambda}, b_{\mathbf{k}\lambda}, A_{\mathbf{k}\lambda}^\dagger, B_{\mathbf{k}\lambda}^\dagger, b_{\mathbf{k}\lambda}^\dagger\}$ now appear in the theory.

The next step is to make an appropriate choice of V_0. This can be done, as in the last section, by taking a diagonal matrix element of the equation of motion [Eq. (4.6)]

$$V_4 = \frac{ic}{m^2}\left[\nabla\cdot\boldsymbol{\pi}^v + \frac{g_v}{c}:\psi^\dagger\psi:\right], \tag{4.25}$$

[9] The relations (4.9a), (4.10a), and (4.20) can be understood as the result of replacing the operators c_0 and c_0^+ in the Schrödinger representation by the c-number $(N_0)^{1/2}$ where N_0 is the number of scalar mesons in the $\mathbf{k} = 0$ mode (see Ref. [24, Section 18]).

with respect to the exact ground state $|\Psi_0\rangle$ of $\hat{H}$ having four-momentum $P_\mu = (\mathbf{0}, (i/c)E)$. This yields [recall Eq. (4.7)]

$$V_0 = \frac{g_v}{m^2} \langle \Psi_0 \mid :\psi^\dagger\psi: \mid \Psi_0\rangle. \tag{4.26}$$

Using translational invariance and conservation of baryon number $\hat{B}$ we can write

$$\begin{aligned}\langle \Psi_0 \mid :\psi^\dagger\psi: \mid \Psi_0\rangle &= \frac{1}{\Omega} \langle \Psi_0 \mid \int d\mathbf{x}\, :\psi^\dagger\psi: \mid \Psi_0\rangle = \frac{1}{\Omega} \langle \Psi_0 \mid \hat{B} \mid \Psi_0\rangle \\ &= \frac{B}{\Omega} = \rho_B = \langle F \mid :\psi^\dagger\psi: \mid F\rangle \end{aligned} \tag{4.27}$$

so that

$$V_0 = \frac{g_v}{m^2} \rho_B \tag{4.28}$$

also in the interacting system. Inserting these results in Eq. (4.12) we find

$$\begin{aligned}\delta(:\psi^\dagger\psi:) &= :\psi^\dagger\psi: - \frac{m^2}{g_v} V_0 = :\psi^\dagger\psi: - \langle \Psi_0 \mid :\psi^\dagger\psi: \mid \Psi_0\rangle \\ &= :\psi^\dagger\psi: - \langle F \mid :\psi^\dagger\psi: \mid F\rangle \\ &= (:\psi^\dagger\psi:)_F ,\end{aligned} \tag{4.29}$$

where it follows that $(:\psi^\dagger\psi:)_F$ is normal-ordered with respect to the complete set of operators $\{A_{\mathbf{k}\lambda}, B_{\mathbf{k}\lambda}, b_{\mathbf{k}\lambda}, A^\dagger_{\mathbf{k}\lambda}, B^\dagger_{\mathbf{k}\lambda}, b^\dagger_{\mathbf{k}\lambda}\}$. If we also note that

$$\begin{aligned}(:\bar{\psi}\boldsymbol{\gamma}\psi:)_F &\equiv :\bar{\psi}\boldsymbol{\gamma}\psi: - \langle F \mid :\bar{\psi}\boldsymbol{\gamma}\psi: \mid F\rangle \\ &= :\bar{\psi}\boldsymbol{\gamma}\psi:\end{aligned} \tag{4.30}$$

and define

$$(:\bar{\psi}\psi:)_F \equiv :\bar{\psi}\psi: - \langle F \mid :\bar{\psi}\psi: \mid F\rangle \tag{4.31}$$

then the interaction hamiltonian $\hat{H}_1$ can finally be written

$$\begin{aligned}\hat{H}_1 = \int d\mathbf{x} \Big\{ &-ig_v(:\bar{\psi}\boldsymbol{\gamma}\psi:)_F \cdot \boldsymbol{\eta} - g_s(:\bar{\psi}\psi:)_F\, \sigma \\ &+ \frac{g_v c}{m^2} (\nabla \cdot \boldsymbol{\pi}^v)(:\psi^\dagger\psi:)_F + \frac{g_v^2}{2m^2} (:\psi^\dagger\psi:)_F\, (:\psi^\dagger\psi:)_F \Big\},\end{aligned} \tag{4.32}$$

where Eq. (4.10a) has again been utilized. The model hamiltonian (4.22) and (4.32), together with the Schrödinger expansions (4.20) and (4.21) and

$$\hat{\psi}(x) = \frac{1}{\Omega^{1/2}} \sum_{\mathbf{k}\lambda} \{A_{\mathbf{k}\lambda} \mathscr{U}(\mathbf{k}\lambda)\, e^{i\mathbf{k}\cdot\mathbf{x}} \theta(|\,\mathbf{k}\,| - k_F)$$
$$+ b^{\dagger}_{\mathbf{k}\lambda} \mathscr{U}(-\mathbf{k}\lambda)\, e^{-i\mathbf{k}\cdot\mathbf{x}} \theta(k_F - |\,\mathbf{k}\,|) + B^{\dagger}_{\mathbf{k}\lambda} \mathscr{V}(-\mathbf{k}\lambda)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\} \tag{4.33}$$

present an interacting quantum field theory. The problem at this point still depends parametrically on ρ_B and ϕ_0.

The next step is to develop a formula for the shift in the ground-state energy density due to the quantum fluctuations. The exact baryon Green's function is first introduced [24]

$$iG_{\alpha\beta}(x, x') \equiv \langle \Psi_0 \mid P[\hat{\psi}_{H\alpha}(x), \hat{\psi}^{\dagger}_{H\beta}(x')] \mid \Psi_0 \rangle, \tag{4.34}$$

where the operators are in the Heisenberg representation

$$\hat{\psi}_{H\alpha}(x) = e^{(i/\hbar)\hat{H}t} \hat{\psi}_{\alpha}(\mathbf{x})\, e^{-(i/\hbar)\hat{H}t}, \tag{4.35}$$

and $|\,\Psi_0\rangle$ is the normalized ground state. Translational invariance implies the Fourier representation

$$iG_{\alpha\beta}(x, x') = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')} iG_{\alpha\beta}(k) \tag{4.36}$$

with $d^4k \equiv d\mathbf{k}\, dk_0$. It is convenient to introduce a variable coupling constant in the interaction hamiltonian, and $\hat{H}_1(\lambda)$ is defined by replacing $g_v \to \lambda g_v$, $g_s \to \lambda g_s$ in Eq. (4.32). A standard result of quantum mechanics then states that the ground-state energy shift can be obtained as [24]

$$E - E_0 = \int_0^1 d\lambda\, \langle \Psi_0(\lambda) \mid \frac{d\hat{H}_1(\lambda)}{d\lambda} \mid \Psi_0(\lambda)\rangle, \tag{4.37}$$

where $|\,\Psi_0(\lambda)\rangle$ is the normalized ground state of the hamiltonian $\hat{H}_0 + \hat{H}_1(\lambda)$. We observe that

$$(:\bar{\psi} 0 \psi:)_F \equiv :\bar{\psi} 0 \psi: - \langle F \mid :\bar{\psi} 0 \psi: \mid F\rangle$$
$$= \bar{\psi} 0 \psi - \langle F \mid \bar{\psi} 0 \psi \mid F\rangle, \tag{4.38}$$

where 0 is an arbitrary Dirac matrix since any c-number term left over in the normal ordering cancels in the difference. Also, from Eq. (4.27)

$$\langle \Psi_0(\lambda) \mid (:\psi^{\dagger}\psi: - \langle F \mid :\psi^{\dagger}\psi: \mid F\rangle) \mid \Psi_0(\lambda)\rangle = 0$$
$$= \langle \Psi_0(\lambda) \mid \psi^{\dagger}\psi \mid \Psi_0(\lambda)\rangle - \langle F \mid \psi^{\dagger}\psi \mid F\rangle. \tag{4.39}$$

Starting with Eq. (4.32), the integrand in Eq. (4.37) can, therefore, be written

$$\langle \Psi_0(\lambda) \mid \lambda \frac{d}{d\lambda} \hat{H}_1(\lambda) \mid \Psi_0(\lambda)\rangle$$
$$= \int d\mathbf{x} \, \langle \Psi_0(\lambda) \mid \Big\{ -\lambda g_s \bar{\psi}\psi\sigma - i\lambda g_v \bar{\psi}\boldsymbol{\gamma} \cdot \boldsymbol{\eta}\psi + \lambda \frac{g_v c}{m^2} (\nabla \cdot \boldsymbol{\pi}^v)(\psi^\dagger\psi)$$
$$+ 2 \frac{\lambda^2 g_v{}^2}{2m^2} [(\psi^\dagger\psi)(\psi^\dagger\psi) - (\langle F \mid \psi^\dagger\psi \mid F\rangle)^2] \Big\} \mid \Psi_0(\lambda)\rangle, \tag{4.40}$$

where Eq. (4.10a) has also been used and a total divergence integrated to zero. Returning to the definition of the Green's function (3.44) and (4.35), it follows that

$$-\frac{\hbar}{i} \frac{\partial}{\partial t'} \, iG_{\alpha\beta}^{(\lambda)}(x, x')\Big|_{t' \to t^+} = \langle \Psi_0(\lambda) \mid [\hat{H}, \hat{\psi}_\beta{}^\dagger(\mathbf{x}')] \, \hat{\psi}_\alpha(\mathbf{x}) \mid \Psi_0(\lambda)\rangle, \tag{4.41}$$

This commutator can be evaluated in a straightforward fashion with the aid of the canonical anticommutation relations

$$\{\hat{\psi}_\alpha(\mathbf{x}), \hat{\psi}_\beta{}^\dagger(\mathbf{x}')\} = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x} - \mathbf{x}') \tag{4.42}$$

just as in conventional many-body theory [24]. After some algebra, the following identity can be established

$$\int d\mathbf{x} \lim_{\substack{t' \to t^+ \\ \mathbf{x}' \to \mathbf{x}}} \left[-\frac{\hbar}{i} \frac{\partial}{\partial t'} - \frac{\hbar c}{i} \boldsymbol{\alpha} \cdot \nabla_{x'} + \beta m_b{}^* c^2 \right] iG_{\alpha\beta}^{(\lambda)}(x, x')$$
$$= \langle \Psi_0(\lambda) \mid \lambda \frac{d}{d\lambda} \hat{H}_1(\lambda) \mid \Psi_0(\lambda)\rangle. \tag{4.43}$$

Note in particular that the last term in Eq. (4.40) with the correct factor of 2 is faithfully reproduced with this procedure. If the relativistic Green's function

$$i\bar{G}_{\alpha\beta} \equiv i[G_{\alpha\rho}(\gamma_4)_{\rho\beta}] \tag{4.44}$$

is introduced and translational invariance utilized, Eq. (4.43) takes the compact form

$$\hbar c \int d\mathbf{x} \lim_{\substack{t' \to t^+ \\ \mathbf{x}' \to \mathbf{x}}} \left[\gamma_\mu \frac{\partial}{\partial x_\mu} + M^* \right]_{\alpha\beta} i\bar{G}_{\beta\alpha}^{(\lambda)}(x, x') = \langle \Psi_0(\lambda) \mid \lambda \frac{d}{d\lambda} \hat{H}_1(\lambda) \mid \Psi_0(\lambda)\rangle. \tag{4.45}$$

Inserting the Fourier representation (4.36), the shift in the ground-state energy density due to the quantum fluctuations, Eq. (4.37), can be written

$$\epsilon - \epsilon_0 = \hbar c \lim_{\substack{\eta \to 0^+ \\ \boldsymbol{\rho} \to 0}} \int_0^1 \frac{d\lambda}{\lambda} \int \frac{d^4k}{(2\pi)^4} \, e^{ik_0\eta} e^{i\mathbf{k}\cdot\boldsymbol{\rho}} [i\not{k} + M^*]_{\beta\alpha} \, i\bar{G}_{\alpha\beta}^{(\lambda)}(k). \tag{4.46}$$

Note that only the baryon Green's function is needed in this expression. This result can be further simplified by invoking Dyson's equation for the baryon propagator

$$\bar{G}(k) = \bar{G}(k)^0 + \bar{G}(k)^0\, \Sigma^*(k)\, \bar{G}(k), \tag{4.47}$$

where $\Sigma^*(k)$ is the proper self-energy. It was shown in the preceding section (Eq. (3.65)) that

$$[i\not{k} + M^*]\, \bar{G}(k)^0 = -1, \tag{4.48}$$

and therefore we arrive at the very simple and familiar looking [24] result

$$\epsilon - \epsilon_0 = i\hbar c \lim_{\substack{\eta\to 0^+ \\ \mathbf{p}\to 0}} \int_0^1 \frac{d\lambda}{\lambda} \int \frac{d^4k}{(2\pi)^4}\, e^{ik_0\eta} e^{i\mathbf{k}\cdot\mathbf{p}} [-\Sigma^{*(\lambda)}_{\alpha\beta}(k)\, \bar{G}^{(\lambda)}_{\beta\alpha}(k)]. \tag{4.49}$$

There are two many-body parameters entering into this expression, the baryon density ρ_B and value of the condensed scalar field ϕ_0. *For a fixed value of the scalar field ϕ_0, we will assume that the energy density of the system reduces to that given by ϵ_0 in the limit $\rho_B \to 0$*

$$\epsilon \xrightarrow[\rho_B\to 0]{} \epsilon_0 = \tfrac{1}{2}\mu^2 c^2 \phi_0^2 + (m_b c^2 - g_s\phi_0)\, \rho_B + \cdots. \tag{4.50}$$

The first term is just the self-energy of the scalar field and the second contains the rest mass of the baryons and the interaction energy of the baryons with the scalar field. If the further limit $\phi_0 \to 0$ is now taken, the rest mass of the baryons can be identified according to

$$\epsilon \xrightarrow[\rho_B\to 0]{} m_b c^2 \rho_B. \tag{4.51}$$

Since the theory is only required to provide meaningful *changes* in the energy density for the many-body system, these limits will be built into the expression for the energy shift (4.49) by subtracting it twice at a fixed value of ϕ_0[10]

$$\begin{aligned}\epsilon - \epsilon_0 = -i\hbar c \lim_{\substack{\eta\to 0^+ \\ \mathbf{p}\to 0}} \int_0^1 \frac{d\lambda}{\lambda} \int \frac{d^4k}{(2\pi)^4}\, e^{ik_0\eta} e^{i\mathbf{k}\cdot\mathbf{p}} \{\Sigma^{*(\lambda)}_{\alpha\beta}(k)\, \bar{G}^{(\lambda)}_{\beta\alpha}(k) - [\ ]_{\rho_B=0}\} \\ - \rho_s \left[\frac{\partial}{\partial\rho_B}(\ )\right]_{\rho_B=0}.\end{aligned} \tag{4.52}$$

We choose to use ρ_s instead of ρ_B in the last quantity because of the way the mass

[10] The right sides of Eqs. (4.52) and (4.56) are at least of second and third order respectively in the coupling constants $\{g_s, g_v\}$. To this order we can replace $M^* \to M$ in the integrands and they become independent of ϕ_0.

term enters the hamiltonian. In the low-density limit, ρ_s and ρ_B are indistinguishable The limits (4.50) and (4.51) are now identically reproduced. Although the parameters ϕ_0 and ρ_B do indeed complicate the problem, their presence can be very useful at times, for example, as an aid in defining meaningful, finite expressions as in Eq. (4.52).

We must emphasize that we are making assumptions in imposing the limiting conditions (4.50) and (4.51). For example, it is assumed that the baryon system of interest is dense enough so that condensed meson fields can be introduced, yet can be made rare enough so that the limiting cases in Eqs. (4.50) and (4.51) are meaningful. In particular, it is assumed that nuclear and moderately subnuclear densities, where the elementary parameters in the theory (i.e., masses and coupling constants) are determined, is a meaningful limiting case of the theory. The exact conditions under which these requirements hold (indeed, whether there are conditions under which they can be met at all) can only be ascertained after the problem of including the quantum fluctuations in a finite, well-defined fashion has been formulated, and they have been explicitly evaluated.

Finally, the scalar field ϕ_0 is not arbitrary, but must be chosen self-consistently to satisfy the meson field equation (3.1)

$$\phi_0 \equiv \langle \Psi_0 \mid \hat{\phi} \mid \Psi_0 \rangle = \frac{g_s}{\mu^2 c^2} \langle \Psi_0 \mid :\bar{\psi}\psi: \mid \Psi_0 \rangle \tag{4.53}$$

(compare Eq. (3.17)). Subtracting the corresponding expectation value in the state $\mid F\rangle$ computed at the same values of $\{\rho_B, \phi_0\}$, we can write

$$\begin{aligned} \phi_0 &- \frac{g_s}{\mu^2 c^2} \langle F \mid :\bar{\psi}\psi: \mid F \rangle \\ &= \frac{g_s}{\mu^2 c^2} [\langle \Psi_0 \mid \bar{\psi}\psi \mid \Psi_0 \rangle - \langle F \mid \bar{\psi}\psi \mid F \rangle] \\ &= -\frac{g_s}{\mu^2 c^2} \lim_{\substack{\eta \to 0^+ \\ \boldsymbol{\rho} \to 0}} \int \frac{d^4k}{(2\pi)^4} e^{ik_0\eta} e^{i\mathbf{k}\cdot\boldsymbol{\rho}} [i\bar{G}_{\alpha\alpha}(k) - i\bar{G}_{\alpha\alpha}(k)^0]. \end{aligned} \tag{4.54}$$

At a given value of ρ_B, this is a self-consistency relation which is to be imposed at the end of the calculation. *We assume that at low baryon density ϕ_0 is given by the result of the previous section* (Eqs. (3.40) and (3.43))

$$\phi_0 \xrightarrow[\rho_B \to 0]{} \frac{g_s}{\mu^2 c^2} \langle F \mid :\bar{\psi}\psi: \mid F \rangle, \tag{4.55a}$$

$$\xrightarrow[\rho_B \to 0]{} \frac{g_s}{\mu^2 c^2} \rho_B \, . \tag{4.55b}$$

This condition is imposed by again subtracting the right side of Eq. (4.54) at a given value of ϕ_0[10,11]

$$\phi_0 - \frac{g_s}{\mu^2 c^2} \langle F \mid :\bar{\psi}\psi: \mid F \rangle$$
$$= -\frac{g_s}{\mu^2 c^2} \lim_{\substack{\eta \to 0^+ \\ \boldsymbol{\rho} \to 0}} \int \frac{d^4k}{(2\pi)^4} e^{ik_0\eta} e^{i\mathbf{k}\cdot\boldsymbol{\rho}} \{[i\bar{G}_{\alpha\alpha}(k) - i\bar{G}_{\alpha\alpha}(k)^0] - [\ ]_{\rho_B=0}\}$$
$$- \rho_s \left[\frac{\partial}{\partial \rho_B}(\ )\right]_{\rho_B=0} \tag{4.56}$$

(We again use ρ_s instead of ρ_B because of the form of Eq. (4.55a).)

Equations (4.52) and (4.56) are the principal results of the present analysis. Once the baryon Green's function has been determined, the right side of these relations can be calculated at a fixed value of ϕ_0 (and hence M^*). The appropriate value of ϕ_0 to be used is then obtained by solving Eq. (4.56). The theory has been constructed to reduce to that studied in the previous section as $\rho_B \to 0$, in particular, the physical mass of the baryons can be identified through Eq. (4.51). We shall now proceed to show that with this formulation, the quantum fluctuations, at least in lowest order, give rise to finite corrections to the equation of state $\epsilon(\rho_B)$, and that meaningful *changes* in the energy density and stress tensor can be calculated as the baryon density is changed even though we have a relativistic quantum field theory with all its accompanying divergences. In order to demonstrate this, we first discuss the Feynman rules for the baryon Green's function.

5. Feynman Rules

We proceed to discuss the Feynman rules for calculating the baryon Green's function in the theory governed by the hamiltonian in Eqs. (4.13) (4.22), and (4.32). Their derivation is precisely the same as that given in Ref. [24] in standard many-body theory so only the novel features of this problem will be presented here. The present notation is $d^4k = d\mathbf{k}\, dk_0 = d\mathbf{k}\, d\omega/c$ and $d^4x = d\mathbf{x}\, dx_0 = d\mathbf{x} \cdot c\, dt$. Thus if the nth term in the iterated series for the Green's function is written in terms of $\int \cdots \int d^4x_1 \cdots d^4x_n$, a factor of $(-i/\hbar c)^n$ appears in front.

(1) *There are no tadpole diagrams in the theory.* This follows because the expressions $(:\bar{\psi}0\psi:)_F$ are already normal-ordered with respect to all the fermion destruction operators in the theory. There are therefore no contractions within an

[11] Further subtractions may be necessary in this expression to yield unambiguous, finite results [See Section 6]. An alternative procedure for determining ϕ_0 is to first compute the energy $\epsilon(\phi_{0J}\rho_B)$ and then minimize with respect to ϕ_0.

F-ordered product when Wick's theorem is applied to the iterated series for the Green's function.

(2) There are interaction lines in the theory coming from both scalar and vector meson exchange.

(3) The scalar meson propagator is given by

$$\begin{aligned}\sigma_{I}{}^{\cdot}(x)\, \sigma_{I}{}^{\cdot}(x') &= \langle F \mid P[\hat{\sigma}_I(x), \hat{\sigma}_I(x')] \mid F\rangle \\ &= \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')} \left(\frac{\hbar}{ic}\, \frac{1}{k^2+\mu^2}\right),\end{aligned} \tag{5.1}$$

where the fields are expressed in the interaction representation

$$\hat{\sigma}_I(x) = e^{(i/\hbar)\hat{H}_0 t}\hat{\sigma}(\mathbf{x})\, e^{-(i/\hbar)\hat{H}_0 t} \tag{5.2}$$

and the mass is given a small negative imaginary part to locate the singularities in the propagator. Here $k^2 = k_\mu k_\mu$. The scalar meson propagator in momentum space is thus

$$\Delta(k) = \frac{1}{k^2+\mu^2}\,. \tag{5.3}$$

The scalar interaction hamiltonian is of the form

$$\hat{H}_\sigma{}' = -g_s(:\bar{\psi}\psi:)_F\, \sigma. \tag{5.4}$$

(4) There are several possible types of propagators involving vector mesons coming from the hamiltonian $\hat{H}_1$ in Eq. (4.32). Using the field expansions in Eq. (4.21), the following results are readily established

$$\begin{aligned}&\langle F \mid P[\hat{\eta}_{Ii}(x), \hat{\eta}_{Ij}(x')] \mid F\rangle \\ &\quad= \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')} \left[\frac{\hbar c}{i}\, \frac{1}{k^2+m^2} \left(\delta_{ij} + \frac{k_i k_j}{m^2}\right)\right], \\ &\langle F \mid P[\nabla\cdot\hat{\boldsymbol{\pi}}_I{}^v(x), \hat{\eta}_{Ij}(x')] \mid F\rangle \\ &\quad= \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')} \left[\frac{\hbar c}{i}\, \frac{1}{k^2+m^2} \left(\frac{k_0 k_j}{c^2}\right)\right], \\ &\langle F \mid P[\nabla\cdot\hat{\boldsymbol{\pi}}_I{}^v(x), \nabla\cdot\hat{\boldsymbol{\pi}}_I{}^v(x')] \mid F\rangle \\ &\quad= \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')} \left[\frac{\hbar c}{i}\, \frac{1}{k^2+m^2} \left(\frac{m^2\mathbf{k}^2}{c^2}\right)\right],\end{aligned} \tag{5.5}$$

where the fields are in the interaction representation. By combining all the different types of vector meson interactions, together with the point four-fermion interaction terms generated by the hamiltonian in Eq. (4.32), the following rule can be readily established: *The contribution of the vector mesons to the Feynman series for the* *baryon Green's function is the same as that obtained from the series generated by using a single vector meson interaction of the form*

$$\hat{H}_{\eta}' = -ig_v(:\bar{\psi}\gamma_\mu\psi:)_F\, \eta_\mu\,, \tag{5.6}$$

where the contractions of the vector fields η_μ are replaced by

$$\eta_\mu^{\cdot}\eta_\nu^{\cdot} \equiv \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')} \left[\frac{\hbar c}{i}\, \frac{1}{k^2+m^2} \left(\delta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}\right)\right]. \tag{5.7}$$

The proof is essentially identical to that used in showing the equivalence of the Coulomb and radiation gauges for computing the S-matrix in quantum electrodynamics [34]. Thus we need consider only a single vector meson propagator

$$D_{\mu\nu}(k) \equiv \frac{1}{k^2+m^2}\left(\delta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}\right) \tag{5.8}$$

and an effective interaction hamiltonian $\hat{H}' = \hat{H}_\sigma' + \hat{H}_\eta'$.

(5) Because the vector meson couples only to a conserved current, the $k_\mu k_\nu$ terms in the vector meson propagator are not expected to contribute to physical quantities. The proof for the S-matrix, as well as for any closed baryon loop, is the same as that used to show the gauge invariance of the S-matrix in quantum electrodynamics [34]. We shall explicitly demonstrate in the next section that to lowest order, these terms do not contribute to Eqs. (4.52) and (4.56). Thus we can replace $D_{\mu\nu}(k) \to \delta_{\mu\nu}(k^2+m^2)^{-1}$ to this order. The Feynman rules for the baryon Green's function have thus been reduced to an extremely simple form.

6. First Quantum Fluctuations

In this section we examine the first quantum fluctuations in the theory, and show that we have a prescription for making them well defined and finite. The lowest-order contributions to the baryon self-energy are shown in Fig. 5. Using the Feynman rules in the last section, the lowest order contribution to $\epsilon - \epsilon_0$ can be written in the form [we define $k \equiv t + q/2$]

$$\begin{aligned}
\epsilon - \epsilon_0 = {} & -i\hbar c \lim_{\substack{\eta\to 0^+ \\ \mathbf{p}\to 0}} \int_0^1 \frac{d\lambda}{\lambda} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4t}{(2\pi)^4}\, e^{i(t_0+q_0/2)\eta} e^{i(\mathbf{t}+\mathbf{q}/2)\cdot\mathbf{p}} \\
& \times \Big\{ i\lambda^2 \Big[\frac{g_v^2}{\hbar c} D_{\mu\nu}(q)\, \mathrm{Tr}\left(\gamma_\mu \frac{1}{i(\not{t}+\not{q}/2)+M^*}\, \gamma_\nu \frac{1}{i(\not{t}-\not{q}/2)+M^*}\right) \\
& - \frac{g_s^2}{\hbar c^3} \Delta(q)\, \mathrm{Tr}\left(\frac{1}{i(\not{t}+\not{q}/2)+M^*} \cdot \frac{1}{i(\not{t}-\not{q}/2)+M^*}\right)\Big] \\
& - [\;]_{\rho_B=0} \Big\} - \rho_s \left[\frac{\partial}{\partial\rho_B}(\;)\right]_{\rho_B=0}.
\end{aligned} \tag{6.1}$$

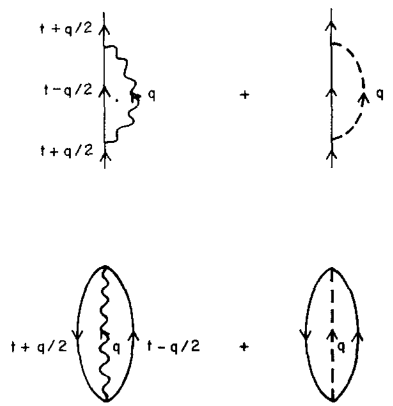

FIG. 5. Lowest order quantum fluctuation contribution to Σ^* (top) and $\epsilon - \epsilon_0$ (bottom). In these figures —>— denotes a baryon, ~~>~~ a vector meson, and - - -> - - a scalar meson.

Since $D_{\mu\nu}(q)$ and $\Delta(q)$ [Eqs. (5.3) and (5.8)] contain no dependence on the density, this is identical to the expression

$$\begin{aligned}\epsilon - \epsilon_0 = i\hbar c \lim_{\substack{\eta\to 0^+\\ \mathbf{p}\to 0}} \int_0^1 \frac{d\lambda}{\lambda} \int \frac{d^4q}{(2\pi)^4}\, e^{i\eta q_0/2} e^{i\mathbf{p}\cdot\mathbf{q}/2} \\ \times \{D_{\mu\nu}(q)[\Pi^*_{\mu\nu}(q)^{(1)} - (\;)_{\rho_B=0}] \\ + \Delta(q)[\Pi_s{}^*(q)^{(1)} - (\;)_{\rho_B=0}]\} - \rho_S \left[\frac{\partial}{\partial\rho_B}(\;)\right]_{\rho_B=0},\end{aligned} \tag{6.2}$$

where

$$\begin{aligned}\Pi^*_{\mu\nu}(q)^{(1)} \equiv \frac{\lambda^2 g_v{}^2}{\hbar c} \cdot \frac{1}{i} \int \frac{d^4t}{(2\pi)^4}\, e^{it_0\eta} e^{it\cdot\mathbf{p}} \\ \times \operatorname{Tr}\left(\gamma_\mu \frac{1}{i(\not{t} + \not{q}/2) + M^*} \gamma_\nu \frac{1}{i(\not{t} - \not{q}/2) + M^*}\right),\end{aligned} \tag{6.3}$$

$$\begin{aligned}\Pi_s{}^*(q)^{(1)} \equiv -\frac{\lambda^2 g_s{}^2}{\hbar c^3} \cdot \frac{1}{i} \int \frac{d^4t}{(2\pi)^4}\, e^{it_0\eta} e^{it\cdot\mathbf{p}} \\ \times \operatorname{Tr}\left(\frac{1}{i(\not{t} + \not{q}/2) + M^*} \cdot \frac{1}{i(\not{t} - \not{q}/2) + M^*}\right)\end{aligned} \tag{6.4}$$

are the lowest order vector meson and scalar meson self energies. As the expressions (6.3) and (6.4) stand, they are quadratically divergent at large t, reflecting the divergence of the self-mass of the two mesons in the zero-density theory. It is important to note, however, that the baryon Green's function in the present theory

(Eq. (3.65)) reduces to the Feynman Green's function when the magnitude of the three-momentum lies outside of Fermi sphere

$$\frac{1}{i\not{k} + M^*} = \frac{1}{[i\not{k} + M^*]_F} \qquad |\mathbf{k}| > k_F, \tag{6.5}$$

and that at $\rho_B = 0$, the baryon propagator also reduces identically to the Feynman propagator

$$\left[\frac{1}{i\not{k} + M^*}\right]_{\rho_B=0} = \frac{1}{[i\not{k} + M^*]_F}. \tag{6.6}$$

Here it has been assumed, in line with our previous discussion, that ϕ_0 and hence M^* are to be kept fixed.[10] *Thus for fixed q, the integrands in the expressions* $\Pi_{\mu\nu}^{*(1)}(q) - (\ )_{\rho_B=0}$ *and* $\Pi_s{}^*(q)^{(1)} - (\ )_{\rho_B=0}$ *vanish identically when* $|\mathbf{t}|$ *is large enough so that both three-momenta in the integrand lie outside the Fermi sphere. Thus the integrals in these expressions are actually finite, definite integrals with finite upper limits.* Since the integrals are now finite and well-defined, the limits $\eta, \rho \to 0$ can be taken inside the integrals (6.3) and (6.4).

We now proceed to show that the terms $q_\mu q_\nu/m^2$ in the vector meson propagator (Eq. (5.8)) vanish when contracted in the integrand in Eq. (6.2). We examine $q_\mu q_\nu \Pi_{\mu\nu}^*(q)$, making use of the fact that in the differences in Eq. (6.2), the integral in Eq. (6.3) is reduced to a well defined integral over a finite volume. Employing the identity

$$\begin{aligned}\operatorname{Tr}&\left[\not{q}\,\frac{1}{i(\not{t} + \not{q}/2) + M^*}\,\not{q}\,\frac{1}{i(\not{t} - \not{q}/2) + M^*}\right]\\ &= \frac{1}{i}\operatorname{Tr}\left[\not{q}\left(\frac{1}{i(\not{t} - \not{q}/2) + M^*} - \frac{1}{i(\not{t} + \not{q}/2) + M^*}\right)\right]\end{aligned} \tag{6.7}$$

on the integrand in Eq. (6.3) and changing dummy variables $t + q/2 \equiv l - q/2$ in the second term (a well-defined procedure now), one finds an identical cancellation from these two terms. Thus it follows that

$$q_\mu q_\nu [\Pi_{\mu\nu}^*(q)^{(1)} - (\ )_{\rho_B=0}] = 0 \tag{6.8}$$

and our assertion is proved.[12]

[12] The prescription giving rise to finite expressions for the polarization propagators in Eq. (6.2), in particular for $\Pi_{\mu\nu}^*(q)^{(1)}$ is somewhat similar to the gauge-invariant (i.e. preserving $q_\mu \Pi_{\mu\nu}^* = \Pi_{\mu\nu}^* q_\nu = 0$) regularization procedure of Pauli and Villars [40] where the contribution of another loop with a different (fictitious) baryon mass is subtracted away. Here, however, the gauge invariant regularization procedure follows rather naturally from our discussion of the many-body problem at finite baryon density.

It remains to establish the convergence of the integral over d^4q in Eq. (6.1). Again, for a fixed $\mathbf{t}$, the integrand vanishes identically for large enough q [i.e., $|\mathbf{t} \pm \mathbf{q}/2| > k_F$]. The only possible difficulty in doing the integral is therefore the region where t and q go off to infinity together, so that one of the fermion propagators in Eq. (6.1) cannot be replaced by the Feynman value. Let us investigate this possibility for the vector term (the scalar term can be handled in a similar fashion). Defining a new variable $t + q/2 \equiv k$, we suppose $|\mathbf{k}| < k_F$. Thus we wish to examine

$$\frac{i\lambda^2 g_v^2}{\hbar c} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2} \times \int \frac{d^4k}{(2\pi)^4} e^{ik_0\eta} e^{i\mathbf{k}\cdot\boldsymbol{\rho}} \operatorname{Tr} \left[\frac{1}{i\not{k} + M^*}\right] \gamma_\mu \left[\frac{1}{i(\not{k} - \not{q}) + M^*}\right]_F \gamma_\mu . \tag{6.9}$$

Equation (3.67) states that

$$\frac{1}{i\not{k} + M^*} = \frac{1}{[i\not{k} + M^*]_F} + 2\pi i \theta(k_0)\, \delta(k^2 + M^{*2})(i\not{k} - M^*)\, \theta(k_F - |\mathbf{k}|). \tag{6.10}$$

The first term leads to a density-independent result and is subtracted away in the expression (6.2). The coefficient of the second term

$$-\frac{i\lambda^2 g_v^2}{\hbar c} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2} \gamma_\mu \frac{1}{[i(\not{k} - \not{q}) + M^*]_F} \gamma_\mu \equiv \Sigma_v{}^*(k) \tag{6.11}$$

is just the contribution of the vector meson to the self-mass of a free baryon. The invariant form of this expression is [41]

$$\Sigma_v{}^*(k) = A + B(i\not{k} + M^*) + (i\not{k} + M^*)\, \Sigma_f(i\not{k} + M^*), \tag{6.12}$$

where A and B are logarithmically infinite constants. The last two terms in $\Sigma_v{}^*(k)$ do not contribute when combined with the second term in Eq. (6.10) which satisfies the Dirac equation

$$(i\not{k} + M^*)(i\not{k} - M^*)\, \delta(k^2 + M^{*2}) = -(k^2 + M^{*2})\, \delta(k^2 + M^{*2}) = 0.$$

Thus in Eq. (6.9) we are left with the expression

$$-A \int \frac{d^4k}{(2\pi)^4} e^{ik_0\eta} e^{i\mathbf{k}\cdot\boldsymbol{\rho}} \cdot 2\pi i \theta(k_0)\, \delta(k^2 + M^{*2})\, \theta(k_F - |\mathbf{k}|) \operatorname{Tr}(i\not{k} - M^*) = +iA\rho_s , \tag{6.13}$$

where the previous result, Eq. (3.43), has been used for ρ_s (assuming $\gamma = 2$). This term is subtracted away identically in the expression (6.2) since at fixed ϕ_0[10]

$$\rho_s \left[\frac{\partial}{\partial \rho_B} (A\rho_s)\right]_{\rho_B=0} = A\rho_s . \tag{6.14}$$

Thus we have established the result that the lowest-order quantum fluctuation corrections to $\epsilon - \epsilon_0$ are indeed finite and well defined.

Employing Dyson's equation (4.47) and the Feynman rules for the self-energies in Fig. 5, the lowest-order correction to the value of the condensed scalar field can be written as

$$\begin{aligned}
&\phi_0 - \frac{g_s}{\mu^2 c^2} \langle F \mid :\bar{\psi}\psi: \mid F \rangle \\
&= \frac{g_s}{\mu^2 c^2} \lim_{\substack{\eta \to 0^+ \\ \mathbf{p} \to 0}} \int \frac{d^4 q}{(2\pi)^4} e^{i\eta q_0/2} e^{i\mathbf{p}\cdot\mathbf{q}/2} \Big\{ D_{\mu\nu}(q) \Big[-\frac{g_v^2}{\hbar c} \int \frac{d^4 t}{(2\pi)^4} e^{it_0\eta} e^{it\cdot\mathbf{p}} \\
&\quad \times \operatorname{Tr} \left(\frac{1}{i(\not{t} + \not{q}/2) + M^*} \gamma_\mu \frac{1}{i(\not{t} - \not{q}/2) + M^*} \gamma_\nu \frac{1}{i(\not{t} + \not{q}/2) + M^*} \right) \\
&\quad - (\ )_{\rho_B=0} \Big] + \Delta(q) \Big[\frac{g_s^2}{\hbar c^3} \int \frac{d^4 t}{(2\pi)^4} e^{it_0\eta} e^{it\cdot\mathbf{p}} \\
&\quad \times \operatorname{Tr} \left(\frac{1}{i(\not{t} + \not{q}/2) + M^*} \cdot \frac{1}{i(\not{t} - \not{q}/2) + M^*} \cdot \frac{1}{i(\not{t} + \not{q}/2) + M^*} \right) \\
&\quad - (\ )_{\rho_B=0} \Big] \Big\} - \rho_s \left[\frac{\partial}{\partial \rho_B} (\ ) \right]_{\rho_B=0} .
\end{aligned} \tag{6.15}$$

For fixed q, the subtraction procedure leads to finite integrals just as in the preceding section. We can again show that the $q_\mu q_\nu$ terms in $D_{\mu\nu}(q)$ vanish when contracted into the integrand in Eq. (6.15). We first make use of the fact that $D_{\mu\nu}(q)$ is an even function of q to rewrite the integrand, and then do a little algebra to establish the result

$$\begin{aligned}
&\frac{1}{2} \left[\operatorname{Tr} \left(\frac{1}{i(\not{t} + \not{q}/2) + M^*} \not{q} \frac{1}{i(\not{t} - \not{q}/2) + M^*} \not{q} \frac{1}{i(\not{t} + \not{q}/2) + M^*} \right) + (q \rightleftharpoons -q) \right] \\
&= \frac{1}{2i} \operatorname{Tr} \left[-\frac{1}{i(\not{t} + \not{q}/2) + M^*} \not{q} \frac{1}{i(\not{t} + \not{q}/2) + M^*} \right. \\
&\quad \left. + \frac{1}{i(\not{t} - \not{q}/2) + M^*} \not{q} \frac{1}{i(\not{t} - \not{q}/2) + M^*} \right] .
\end{aligned} \tag{6.16}$$

A change of variables $t - q/2 \equiv l + 2/q$ in the second integral shows the identical cancellation of these two terms.

For a fixed value of $\mathbf{t}$, the integrand in Eq. (6.15) vanishes identically for large $\mathbf{q}$, and the remaining integral over q is obviously convergent. Again, the only difficulty can arise when the momenta t and q go off to infinity together. A change of variables to $t - q/2 \equiv k$, $t + q/2 \equiv k + q$ with $|\mathbf{k}| < k_F$ produces a situation analogous to that studied in Eqs. (6.9)–(6.14), and Eq. (6.15) again leads to a finite result. The case $t + q/2 \equiv k$, $t - q/2 \equiv k - q$ with $|\mathbf{k}| < k_F$ is more complex since now two Green's functions propagate inside the Fermi sphere. After inserting the invariant form (6.12) and performing the remaining integrations, the following two densities appear in the result

$$\rho_s = 2\int_0^{k_F} \frac{d\mathbf{k}}{(2\pi)^3} \cdot \frac{M^*}{(\mathbf{k}^2 + M^{*2})^{1/2}} \xrightarrow[\rho_B \to 0]{} \rho_B, \tag{6.17a}$$

$$\frac{1}{2} M^* \frac{\partial}{\partial M^*} \rho_s = \int_0^{k_F} \frac{d\mathbf{k}}{(2\pi)^3} \cdot \frac{M^*}{(\mathbf{k}^2 + M^{*2})^{1/2}} \cdot \left(\frac{\mathbf{k}^2}{M^{*2} + \mathbf{k}^2}\right) \xrightarrow[\rho_B \to 0]{} \frac{\epsilon_F{}^0}{m_b{}^* c^2}, \tag{6.17b}$$

where $\epsilon_F{}^0 = (3/5)(\hbar^2 k_F{}^2/2m_b{}^*)\,\rho_B$ is the free Fermi energy density. Both will be multiplied by logarithmically divergent constants. The first term is removed in Eq. (6.15) by our assumption that the expression $(g_s/\mu^2c^2)\langle F \mid :\bar{\psi}\psi: \mid F\rangle$ yields the correct coefficient of ρ_B at low density, or more generally of ρ_s. The second can be similarly removed with the additional assumption that this expression correctly yields the coefficient of $(\epsilon_F{}^0/m_b{}^*c^2)$ at low densities, or more generally of $M^*/2(\partial\rho_s/\partial M^*)$. Thus if the right side of Eq. (6.15) is replaced by the expression

$$(\text{rhs}) \to (\text{rhs}) - \frac{1}{2} M^* \frac{\partial \rho_s}{\partial M^*} \left[m_b{}^* c^2 \frac{\partial}{\partial \epsilon_F{}^0} (\text{rhs})\right]_{\rho_B = 0} \tag{6.18}$$

then the resulting expression for ϕ_0 in Eq. (6.15) is finite and well defined when the lowest-order quantum fluctuations are included.

The soundness of these arguments on convergence, as well as the determination of the size of the lowest-order quantum fluctuation contributions and corresponding estimate of the range of validity of the linearized theory presented in Section 3, can only be ascertainted after a detailed calculation of the integrals in Eqs. (6.1) and (6.15). This will be discussed in a subsequent publication. The extension of these results to the higher-order quantum fluctuations is still an open problem.

7. Summary

In this paper we have attempted a microscopic calculation of the stress tensor, the source in Einstein's field equations, and the corresponding equation of state from nuclear matter densities upwards by working through a relativistic quantum field theory. A model problem of a baryon field coupled to a neutral scalar field through the scalar density $\bar{\psi}\psi$ and to a neutral vector meson field through the conserved baryon current $i\bar{\psi}\gamma_\lambda\psi$ is introduced to reproduce those features of the nucleon–nucleon interaction, a long-range attraction and short-range repulsion, which are predominantly responsible for nuclear saturation. It is argued that for a uniform system of high baryon density ρ_B, it should be a good approximation to replace the scalar and vector fields by their expectation values $\phi \to \phi_0$, $V_\lambda \to i\delta_{\lambda 4}V_0$. The resulting linearized theory can be solved exactly and yields an equation of state for nuclear matter which exhibits nuclear saturation. If the two dimensionless coupling constants in the theory are matched to the binding energy and density of nuclear matter, predictions are obtained for all other systems at all densities. In particular, neutron matter is unbound and the resulting equation of state for neutron matter is shown in Figs. 1–3. It extrapolates smoothly into the relation $P = \epsilon$ originally proposed by Zel'dovich [25]. This equation of state, though simple, has a great deal of physical content and its further consequences will be discussed in another publication.

The quantum fluctuations of the meson fields are investigated by expanding about the condensed, constant, c-number values ϕ_0 and V_0, and an interacting quantum field theory is then developed. A normal-ordering assumption is invoked in passing from the classical to the quantum field theory. The unperturbed hamiltonian in this problem is shown to correspond to the linearized theory. The energy shift $\epsilon - \epsilon_0$ due to these quantum fluctuations is related to the baryon Green's function. The field V_0 is related directly to ρ_B. In contrast, ϕ_0 is only determined at the end through a self-consistency relation involving the baryon Green's function.

The Feynman rules for this theory are developed. Expressions for the lowest-order contribution of the quantum fluctuations to $\epsilon - \epsilon_0$ and ϕ_0 are derived. It is shown that the terms $q_\mu q_\nu$ in the vector meson propagator do not contribute to these expressions and a prescription involving assumptions on the limiting form of the theory as $\rho_B \to 0$ is presented which ensures that these lowest-order quantum fluctuations will yield finite results. The explicit evaluation of the integrals is left for a subsequent publication.

The problem of extending these results to higher order and defining the theory through a set of Feynman rules for the baryon Green's function and manifestly finite expressions for $\epsilon - \epsilon_0$ and ϕ_0 is still an open one.[13] It may be essential to extend the analysis to higher order before meaningful estimates of the effect of quantum fluctuations on the linearized theory can be obtained. For example, if the main effect of the quantum fluctuations comes from the zero-point oscillations of

[13] This will probably necessitate explicitly including counter terms in the interaction hamiltonian $\hat{H}_I$ which we have so far tried to avoid by only computing changes relative to low-density values.

the sound, or zero-sound, modes of excitation of the medium [25, 38], the appropriate graphs generating this collective mode in the many-body system will first have to be summed.

Many fine details of the nuclear force are, of course, left out of this model. Thus, even if one could solve the resulting quantum field theory exactly, it would be difficult to claim any quantitative accuracy as compared with existing nuclear- and neutron-matter calculations starting from detailed nucleon–nucleon potentials and the many-body Schrödinger equation. In extrapolating from the nuclear window on the equation of state to very high densities, however, it is certainly more satisfying to work within the framework of a completely relativistic, quantum mechanical model.

Note added in proof: Equations (3.45) and (3.47) can alternatively be derived by summation of the tadpole diagrams and retention of the disconnected contributions to the meson propagators using fully relativistic Green's functions to compute the stress tensor for the many-particle sytem. The renormalizable quantum field theory requires counter terms of the form $\delta\mathscr{L} = \alpha\phi + (\beta/2!)\phi^2 + (\gamma/3!)\phi^3 + (\delta/4!)\phi^4$ whose parameters may be specified by considering vacuum amplitudes. In this fashion, a finite stress tensor for the many-body system can be calculated and no normal-ordering assumption is required. If the tadpole diagrams are summed in this manner, there are additional higher-order (in g_s^2) terms in the stress tensor after renormalization coming from a baryon–antibaryon loop which are not present in Eqs. (3.45) and (3.47). These and other higher-order effects will be discussed in a subsequent publication. The author is greatly indebted to Professor W. Bardeen for instruction on this aspect of the problem, and to S. Chin for his collaboration in this investigation.

Acknowledgments

The author would like to thank A. Bohr and the Niels Bohr Institute in Copenhagen for their hospitality during the spring of 1973 when the major part of this work was carried out. He would also like to thank L. E. Lundberg of the Bohr Institute and W. Bardeen, F. Bloch, S. Chin, and A. Fetter of Stanford for valuable comments, and T. W. Donnelly for the numerical results shown in Figs. 1–3.

References

1. L. Landau, *Physik. Zeit. Sowjetunion* **1** (1932), 285; *Nature* **141** (1938), 133.
2. S. Chandrasekhar, *Roy. Astronom. Soc. Monthly Notices* **95** (1935), 207; **95** (1935), 227.
3. J. R. Oppenheimer and R. Serber, *Phys. Rev.* **54** (1938), 540; J. R. Oppenheimer and H. Snyder, *Phys. Rev.* **56** (1939), 455.
4. J. R. Oppenheimer and G. M. Volkoff, *Phys. Rev.* **55** (1939), 374.
5. R. C. Tolman, *Phys. Rev.* **55** (1939), 364.
6. B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, "Gravitation Theory and Gravitational Collapse," Univ. of Chicago Press, Chicago, IL, 1965.
7. S. Weinberg, "Gravitation and Cosmology," John Wiley and Sons, Inc., New York, 1972.

8. A. G. W. CAMERON, *Astrophys. J.* **130** (1959), 884.
9. E. E. SALPETER, *Ann. of Phys.* **11** (1960), 393.
10. G. S. SAHAKIAN AND YU L. VARTANIAN, *Nuovo Cimento* **30** (1963), 82; *Soviet Astron.* **8** (1964), 147.
11. S. TSURUTA AND A. G. W. CAMERON, *Canad. J. Phys.* **44** (1966), 1895.
12. K. A. BRUECKNER, J. L. GAMMEL, AND J. T. KUBIS, *Phys. Rev.* **118** (1960), 1095.
13. J. S. LEVINGER AND L. M. SIMMONS, *Phys. Rev.* **124** (1961), 916.
14. J. NEMETH AND D. W. L. SPRUNG, *Phys. Rev.* **176** (1968), 1496.
15. C. T. P. WANG, *Phys. Rev.* **177** (1969), 1452.
16. M. BINDER, R. H. PIERCE, AND M. RAZAVY, *Canad. J. Phys.* **47** (1969), 2101.
17. C. G. WANG, W. K. ROSE, AND S. L. SCHLENKER, *Astrophys. J.* **160** (1970), L17.
18. E. ØSTGAARD, *Nucl. Phys.* **A154** (1970), 202.
19. P. J. SIEMANS AND V. R. PANDHARIPANDE, *Nucl. Phys.* **A173** (1971), 561.
20. G. BAYM, H. A. BETHE, AND C. J. PETHICK, *Nucl. Phys.* **A175** (1971), 225.
21. G. BAYM, C. PETHICK, AND P. SUTHERLAND, *Astrophys. J.* **170** (1971), 299.
22. (a) V. R. PANDHARIPANDE, *Nucl. Phys.* **A174** (1971), 641; (b) *Nucl. Phys.* **A178** (1971), 123; (c) *Nucl. Phys.* **A181** (1972), 33.
23. N. MILLER, C. W. WOO, J. W. CLARK, AND W. J. TER LOUW, *Nucl. Phys.* **A184** (1972), 1.
24. A. L. FETTER AND J. D. WALECKA, "Quantum Theory of Many-Particle Systems," McGraw-Hill Book Company, New York, 1971.
25. YA. B. ZEL'DOVICH, *Soviet Physics JETP* **14** (1962), 1143.
26. E. R. HARRISON, *Astrophys. J.* **142** (1965), 1643.
27. G. MARX, *Nucl. Phys.* **1** (1956), 660.
28. G. MARX AND J. NEMETH, *Act. Phys. Acad. Sci. Hungar.* **18** (1964), 77.
29. S. A. BLUDMAN AND M. A. RUDERMAN, *Phys. Rev.* **170** (1968), 1176.
30. M. RUDERMAN, *Phys. Rev.* **172** (1968), 1286.
31. C. B. DOVER AND R. H. LEMMER, *Phys. Rev.* **165** (1968), 1105; C. B. DOVER, *Ann. of Phys.* **50** (1968), 449.
32. W. D. BROWN, R. D. PUFF, AND L. WILETS, *Phys. Rev. C* **2** (1970), 331.
33. R. L. BOWERS, J. A. CAMPBELL, AND R. L. ZIMMERMAN, *Phys. Rev. D* 7 (1973);, 2278; *Phys. Rev. D* 7 (1973), 2289.
34. J. D. BJORKEN AND S. D. DRELL, "Relativistic Quantum Field Theory," McGraw-Hill Book Company, New York, 1965.
35. H. A. BETHE AND P. MORRISON, "Elementary Nuclear Theory," 2nd ed., John Wiley and Sons. Inc., New York, 1956.
36. R. F. SAWYER, *Phys. Rev. Lett.* **29** (1972), 382; D. J. SCALAPINO, *Phys. Rev. Lett.* **29** (1972), 386.
37. G. WENTZEL, "Quantum Theory of Fields," Interscience Publishers, New York, 1949.
38. G. KALMAN, *Phys. Rev.* **158** (1967), 144.
39. S. CHIN AND J. D. WALECKA, to be published.
40. W. PAULI AND F. VILLARS, *Rev. Mod. Phys.* **21** (1949), 434.
41. J. M. JAUCH AND F. ROHRLICH, "The Theory of Photons and Electrons," p. 181, Addison-Wesley Publishing Company, Reading, MA 1955.