OpenPulse - Open source code for numerical modelling of low-frequency acoustically induced vibration in gas pipeline systems

Theory Reference E: Notes on Geometric Stiffness Matrix for Stress Stiffening for the *OpenPulse* PIPE element

V1.0

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*Manuscript without citations and references. "Copy-paste" of some books.
*Geometric stiffness obtained from Przemieniecki. To be tested.

From ANSYS Theory Reference:

"Stress stiffening (also called geometric stiffening, incremental stiffening, initial stress stiffening, or differential stiffening by other authors) is the stiffening (or weakening) of a structure due to its stress state. This stiffening effect normally needs to be considered for thin structures with bending stiffness very small compared to axial stiffness, such as beams. This effect also augments the regular nonlinear stiffness matrix produced by large-strain or large-deflection effects. The effect of stress stiffening is accounted for by generating and then using an additional stiffness matrix, hereinafter called the "stress stiffness matrix". The stress stiffness matrix is added to the regular stiffness matrix in order to give the total stiffness. Stress stiffening may be used for harmonic analyses. Working with the stress stiffness matrix is the pressure load stiffness.

In some linear analyses, the static (or initial) stress state may be large enough that the additional stiffness effects must be included for accuracy. Modal and harmonic analyses are linear analyses for which the prestressing effects can be requested to be included. Note that in these cases the stress stiffness matrix is constant, so that the stresses computed in the analysis are assumed small compared to the prestress stress."

From Nonlin. F.E. for Cont. and Struct., Belytschko et al. Wiley. 2d Ed. 2014:

"**Linearization** of the nonlinear constitutive equation is carried out in two ways:

- with the continuum tangent moduli, which does not account for the actual constitutive update algorithm; the resulting material tangent stiffness matrix is called the *material tangent stiffness matrix*.
- with the algorithmic tangent moduli, which gives rise to the so-called *consistent tangent stiffness*.

The development of material tangent stiffness matrix is done by relating rates of the internal nodes $\dot{\mathbf{f}}_{int}$ and the nodal velocities $\dot{\mathbf{d}}$. The rate internal nodal forces in the total Lagrangian form consists of two parts:

- the first part involves the rate of stress ($\dot{\mathbf{S}}$) and this depends on the material response and leads to what is called material tangent stiffness which is denoted by $\mathbf{K}^{mat}$.
- the second part involves the current stat of stress $\mathbf{S}$, and accounts for geometric effects of the deformation (including rotation and stretching). This term is called the geometric stiffness. It is also called the initial stress matrix to indicate the role of the existing state of stress. It is denoted by $\mathbf{K}^{geo}$."

As a result of this linearization, we have the "total stiffness":

$$\mathbf{K} = \mathbf{K}^{mat} + \mathbf{K}^{geo}. \tag{1}$$

Przemieniecki: Geometric Stiffness Matrix - Timoshenko Beam

Including only axial effects:

$$\mathbf{K}_e^{geo} = \frac{T_e}{L_e}\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6/5 & 0 & 0 & 0 & L_e/10 & 0 & -6/5 & 0 & 0 & 0 & L_e/10 \\
0 & 0 & 6/5 & 0 & -L_e/10 & 0 & 0 & 0 & -6/5 & 0 & -L_e/10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -L_e/10 & 0 & 2L_e^2/15 & 0 & 0 & 0 & L_e/10 & 0 & -L_e^2/30 & 0 \\
0 & L_e/10 & 0 & 0 & 0 & 2L_e^2/15 & 0 & -L_e/10 & 0 & 0 & 0 & -L_e^2/30 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -6/5 & 0 & 0 & 0 & -L_e/10 & 0 & 6/5 & 0 & 0 & 0 & -L_e/10 \\
0 & 0 & -6/5 & 0 & L_e/10 & 0 & 0 & 0 & 6/5 & 0 & L_e/10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -L_e/10 & 0 & -L_e^2/30 & 0 & 0 & 0 & L_e/10 & 0 & 2L_e^2/15 & 0 \\
0 & L_e/10 & 0 & 0 & 0 & -L_e^2/30 & 0 & -L_e/10 & 0 & 0 & 0 & 2L_e^2/15
\end{bmatrix}, \tag{2}$$

where T_e is the axial element force and depends on the type of preload applied on the system.

Internal pressure effect - quasi-static preload

In the approach used in this work, an equivalent load that "induces" the stress field caused by the pressure effect is applied on each element as an external axial load, considering a static load case as "prestress".

If the **capped end condition** is considered, the **axial stress** is resultant from the difference between internal and external pressure acting on the both ends of the duct. This axial stress is given by (simply using the relation F/A with the areas of the caps and the section):

$$\sigma_1 = \frac{\overline{p}_i D_i^2 - p_o D_o^2}{D_o^2 - D_i^2}, \tag{3}$$

where $\overline{p}_i$ is the internal static pressure; D_i is the internal diameter; D_o is the external diameter; and p_o is the external pressure, considered constant in this work. The equivalent internal element force to produce this stress is given by:

$$\boxed{F_p^a = A_e \sigma_1}. \tag{4}$$

The axial stress σ_1 exists only in the capped end condition and the superposition of the equivalent element forces leads to a self-equilibrated external force (stretching or compressing the pipe in its axial direction).

Even without the capped end condition, the internal pressure leads to **radial and hoop stresses at the pipe wall**. Considering thick wall formulation (section in plate stress state), the **radial stress** can be obtained using the Lamé stress distribution, using the internal and external pressures as a boundary condition:

$$\sigma_r(r) = \frac{\overline{p}_i D_i^2 - p_o D_o^2}{D_o^2 - D_i^2} - \frac{D_i^2 D_o^2 (\overline{p}_i - p_o)}{4r^2 (D_o^2 - D_i^2)}. \tag{5}$$

Otherwise, considering thin wall formulation (thin-walled vessel):

$$\sigma_r = \frac{\overline{p}_i D_o / 2}{t}, \tag{6}$$

where t is the wall thickness.

The **hoop (circumferential) stress** for thick wall (section in plane stress) is obtained using the same stress distribution considered in the radial case, resulting in:

$$\sigma_c(r) = \frac{\overline{p}_i D_i^2 - p_o D_o^2}{D_o^2 - D_i^2} + \frac{D_i^2 D_o^2 (\overline{p}_i - p_o)}{4r^2 (D_o^2 - D_i^2)}. \tag{7}$$

But, considering thin wall formulation (thin-walled vessel):

$$\sigma_c = -\overline{p}_i. \tag{8}$$

The axial equivalent force (acting at each element) that causes the radial and hoop stresses can be found by (see Boresi):

$$\boxed{F_p^{rc} = -A_e \nu (\sigma_r + \sigma_c)}. \tag{9}$$

However, different from the capped end situation (which leads to axial stress), the superposition of the equivalent element forces F_p^{rc} must result into a zero external resultant force acting on the pipe: there is stretching or compression in the radial

direction.

The static problem is then solved using the assembled vector of element "pressure forces":

$$\boxed{F_p = F_p^a + F_p^{rc} = A_e(\sigma_1 - \nu(\sigma_r + \sigma_c))}. \tag{10}$$

As the equivalent internal force F_p^{rc} has a zero <u>external resultant</u> force acting on the pipe, the true axial element force T_e is obtained by subtracting this equivalent force from the internal element force obtained from the result of the previous static problem. So,

$$\boxed{T_e = T_e^p = \frac{EA_e(u_7^e - u_1^e)}{L_e} - F_p^{rc}}, \tag{11}$$

with u_1^e and u_7^e considering local (element) coordinates.

With the obtained static solution, each element geometric stiffness matrix $\mathbf{K}_{e(p)}^{geo}$ can be obtained using (1).

Temperature effect: thermal gradient thru the thickness - quasi-static preload

OBS.: This is different from elongation due to the temperature, which does not cause significant prestress, but leads to a possible change of geometry. The equilibrium problem solved to obtain the total elongation is generally non-linear (large quasi-static displacements), and it is not possible to solve it in *OpenPulse*. So, this effect is neglected.

Considering an internal static temperature (averaged) Θ_i and external constant temperature Θ_o (**temperature of outside surface!!!**), the hoop (circumferential) stress is obtained by:

$$\sigma_c^t = -\frac{E\alpha(\Theta_o - \Theta_i)}{1 - \nu}, \tag{12}$$

where α coefficient of thermal expansion of the material of the pipe. In this special load case, considering thick wall and plane state stress, we have (see Ansys Theory Reference and Boresi):

$$\sigma_1^t = \sigma_c^t = -\frac{E\alpha(\Theta_o - \Theta_i)}{1 - \nu}, \tag{13}$$

and

$$\boxed{F_t = -A_e\frac{E\alpha(\Theta_o - \Theta_i)}{1 - \nu}}. \tag{14}$$

The static problem is then solved using the assembled vector of element "temperature forces" F_t. As the temperature effect has a zero <u>external resultant</u> force acting on the pipe, the true axial element force T_e is obtained by subtracting the equivalent force F_t from the internal element force obtained from the result of the previous static problem. So,

$$\boxed{T_e = T_e^t = \frac{EA_e(u_7^e - u_1^e)}{L_e} - F_t}, \tag{15}$$

with u_1^e and u_7^e considering local (element) coordinates.

With the obtained static solution, each element geometric stiffness matrix $\mathbf{K}_{e(t)}^{geo}$ can be obtained using (1).

Total element stiffness matrix

$$\mathbf{K}_e = \mathbf{K}_e^{mat} + \mathbf{K}_e^{geo}, \tag{16}$$

or

$$\mathbf{K}_e = \mathbf{K}_e^{mat} + \mathbf{K}_{e(p)}^{geo} + \mathbf{K}_{e(t)}^{geo}. \tag{17}$$

OBS.: $\mathbf{K}^{mat}$ is the "material" stiffness matrix obtained by linear FEM.