OpenPulse - Open source code for numerical modelling of low-frequency acoustically induced vibration in gas pipeline systems

Theory Reference E: Notes on Geometric Stiffness Matrix for Stress Stiffening for the *OpenPulse* PIPE element

V1.0

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21 May 2020

*Manuscript without citations and references. "Copy-paste" of some books.
*Geometric stiffness obtained by Olavo M. Silva. To be tested.

From ANSYS Theory Reference:

"Stress stiffening (also called geometric stiffening, incremental stiffening, initial stress stiffening, or differential stiffening by other authors) is the stiffening (or weakening) of a structure due to its stress state. This stiffening effect normally needs to be considered for thin structures with bending stiffness very small compared to axial stiffness, such as beams. This effect also augments the regular nonlinear stiffness matrix produced by large-strain or large-deflection effects. The effect of stress stiffening is accounted for by generating and then using an additional stiffness matrix, hereinafter called the "stress stiffness matrix". The stress stiffness matrix is added to the regular stiffness matrix in order to give the total stiffness. Stress stiffening may be used for harmonic analyses. Working with the stress stiffness matrix is the pressure load stiffness.

In some linear analyses, the static (or initial) stress state may be large enough that the additional stiffness effects must be included for accuracy. Modal and harmonic analyses are linear analyses for which the prestressing effects can be requested to be included. Note that in these cases the stress stiffness matrix is constant, so that the stresses computed in the analysis are assumed small compared to the prestress stress."

From Nonlin. F.E. for Cont. and Struct., Belytschko et al. Wiley. 2d Ed. 2014:

“Linearization of the nonlinear constitutive equation is carried out in two ways:

- with the continuum tangent moduli, which does not account for the actual constitutive update algorithm; the resulting material tangent stiffness matrix is called the *material tangent stiffness matrix.*
- with the algorithmic tangent moduli, which gives rise to the so-called *consistent tangent stiffness.*

The development of material tangent stiffness matrix is done by relating rates of the internal nodes $\dot{\mathbf{f}}_{int}$ and the nodal velocities $\dot{\mathbf{d}}$. The rate internal nodal forces in the total Lagrangian form consists of two parts:

- the first part involves the rate of stress ($\dot{\mathbf{S}}$) and this depends on the material response and leads to what is called material tangent stiffness which is denoted by $\mathbf{K}^{mat}$.
- the second part involves the current stat of stress $\mathbf{S}$, and accounts for geometric effects of the deformation (including rotation and stretching). This term is called the geometric stiffness. It is also called the initial stress matrix to indicate the role of the existing state of stress. It is denoted by $\mathbf{K}^{geo}$."

Belytschko+Hughes: Geometric Stiffness Matrix

General formulation (3D, continuum, FEM):

$$\mathbf{K}_{IJ}^{geo} = H_{IJ}\mathbf{I}, \tag{1}$$

where

$$\mathbf{H} = \int_{\Omega} \mathbf{B}^T\mathbf{S}\mathbf{B}d\Omega. \tag{2}$$

Considering the pipe element used in *OpenPulse* (Timoshenko beam, C_0):

$$\mathbf{K}_{ab}^{geo} = \int_{\Omega_e} \mathbf{B}_{ab}^T\mathbf{S}_{ab}\mathbf{B}_{ab}d\Omega_e, \tag{3}$$

$$\mathbf{K}_{ts}^{geo} = \int_{\Omega_e} \mathbf{B}_{ts}^T\mathbf{S}_{ts}\mathbf{B}_{ts}d\Omega_e. \tag{4}$$

Finally,

$$\mathbf{K}^{geo} = \mathbf{K}_{ab}^{geo} + \mathbf{K}_{ts}^{geo}. \tag{5}$$

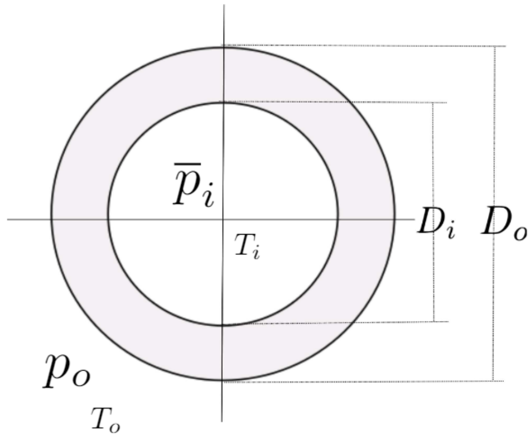

Figure 1: Pipe section.

Internal pressure effect - quasi-static preload

The internal pressure has a zero external resultant force acting on the pipe. However, in the approach used in this work, the load equivalent to the stress field induced by the pressure effect is applied on each element as an external axial load, considering static load case as "prestress".

The axial stress is resultant from the difference between internal and external pressure, and is given by:

$$\sigma_1 = \frac{\overline{p}_i D_i^2 - p_o D_o^2}{D_o^2 - D_i^2}, \tag{6}$$

where $\overline{p}_i$ is the internal static pressure; D_i is the internal diameter; D_o is the external diameter; and p_o is the external pressure, considered constant in this work.

The radial stress can be obtained using the Lamé stress distribution, using the internal and external pressures as a boundary condition:

$$\sigma_r(r) = \frac{\overline{p}_i D_i^2 - p_o D_o^2}{D_o^2 - D_i^2} - \frac{D_i^2 D_o^2 (\overline{p}_i - p_o)}{4r^2 (D_o^2 - D_i^2)}. \tag{7}$$

And the hoop (circumferential) stress is obtained using the same stress distribution considered in the radial case, resulting in:

$$\sigma_c(r) = \frac{\overline{p}_i D_i^2 - p_o D_o^2}{D_o^2 - D_i^2} + \frac{D_i^2 D_o^2 (\overline{p}_i - p_o)}{4r^2 (D_o^2 - D_i^2)}. \tag{8}$$

Consequently, the axial resultant force acting at each element can be found (see Theory Reference D):

$$F_p = A_e\big(\sigma_1 - \nu(\sigma_r + \sigma_c)\big), \tag{9}$$

The axial stress σ_1 is considered only in the "capped end" condition. The static problem is then solved using the assembled vector of element "pressure forces".

After the solution of the static problem, the resultant element stress state is obtained as follows:

$$\boldsymbol{\sigma}_{ab}^p = \mathbf{D}_{ab} \mathbf{B}_{ab} \mathbf{U}_e^{pstat}, \tag{10}$$

$$\boldsymbol{\sigma}_{ts}^p = \mathbf{D}_{ts} \mathbf{B}_{ts} \mathbf{U}_e^{pstat}. \tag{11}$$

Considering the strategy adopted for implementing the Timoshenko beam theory (see Theory Reference B):

$$\mathbf{S}^p_{ab} = \begin{bmatrix} \sigma^p_a & 0 & 0 \\ 0 & \sigma^p_{b2} & 0 \\ 0 & 0 & \sigma^p_{b3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}^p_{ab}(1) & 0 & 0 \\ 0 & \boldsymbol{\sigma}^p_{ab}(2) & 0 \\ 0 & 0 & \boldsymbol{\sigma}^p_{ab}(3) \end{bmatrix}, \tag{12}$$

and

$$\mathbf{S}^p_{ts} = \begin{bmatrix} \sigma^p_t & 0 & 0 \\ 0 & \sigma^p_{s2} & 0 \\ 0 & 0 & \sigma^p_{s3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}^p_{ts}(1) & 0 & 0 \\ 0 & \boldsymbol{\sigma}^p_{ts}(2) & 0 \\ 0 & 0 & \boldsymbol{\sigma}^p_{ts}(3) \end{bmatrix}. \tag{13}$$

Thus, the geometric stiffness matrix can be obtained by:

$$\mathbf{K}^{geo}_{(p)} = \mathbf{K}^{geo}_{ab(p)} + \mathbf{K}^{geo}_{ts(p)} = \int_{\Omega_e} \mathbf{B}^T_{ab}\mathbf{S}^p_{ab}\mathbf{B}_{ab} d\Omega_e + \int_{\Omega_e} \mathbf{B}^T_{ts}\mathbf{S}^p_{ts}\mathbf{B}_{ts} d\Omega_e. \tag{14}$$

Temperature effect: thermal gradient thru the thickness - quasi-static preload

OBS.: This is different from elongation due to the temperature, which does not cause significant prestress, but leads to a possible change of geometry. The equilibrium problem solved to obtain the total elongation is generally non-linear (large quasi-static displacements), and it is not possible to solve it in *OpenPulse*. So, this effect is neglected.

Considering an internal static temperature (averaged) T_i and external constant temperature T_o (**temperature of outside surface!!!**), the hoop (circumferential) stress is obtained by:

$$\sigma^t_c = -\frac{E\alpha(T_o - T_i)}{1-\nu}, \tag{15}$$

where α coefficient of thermal expansion of the material of the pipe. In this special load case, we have (see Ansys Theory Reference):

$$\sigma^t_1 = \sigma^t_c = -\frac{E\alpha(T_o - T_i)}{1-\nu}, \tag{16}$$

and

$$F_t = -A_e\frac{E\alpha(T_o - T_i)}{1-\nu}. \tag{17}$$

The static problem is then solved using the assembled vector of element “temperature forces" F_t. The resultant element stress state is obtained as follows:

$$\boldsymbol{\sigma}^t_{ab} = \mathbf{D}_{ab}\mathbf{B}_{ab}\mathbf{U}^{tstat}_e, \tag{18}$$

$$\boldsymbol{\sigma}^t_{ts} = \mathbf{D}_{ts}\mathbf{B}_{ts}\mathbf{U}^{tstat}_e. \tag{19}$$

Considering the strategy adopted for implementing the Timoshenko beam theory (see Theory Reference B):

$$\mathbf{S}^t_{ab} = \begin{bmatrix} \sigma^t_a & 0 & 0 \\ 0 & \sigma^t_{b2} & 0 \\ 0 & 0 & \sigma^t_{b3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}^t_{ab}(1) & 0 & 0 \\ 0 & \boldsymbol{\sigma}^t_{ab}(2) & 0 \\ 0 & 0 & \boldsymbol{\sigma}^t_{ab}(3) \end{bmatrix}, \tag{20}$$

and

$$\mathbf{S}^t_{ts} = \begin{bmatrix} \sigma^t_t & 0 & 0 \\ 0 & \sigma^t_{s2} & 0 \\ 0 & 0 & \sigma^t_{s3} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\sigma}^t_{ts}(1) & 0 & 0 \\ 0 & \boldsymbol{\sigma}^t_{ts}(2) & 0 \\ 0 & 0 & \boldsymbol{\sigma}^t_{ts}(3) \end{bmatrix}. \tag{21}$$

Thus, the geometric stiffness matrix can be obtained by:

$$\mathbf{K}^{geo}_{(t)} = \mathbf{K}^{geo}_{ab(t)} + \mathbf{K}^{geo}_{ts(t)} = \int_{\Omega_e} \mathbf{B}^T_{ab}\mathbf{S}^t_{ab}\mathbf{B}_{ab} d\Omega_e + \int_{\Omega_e} \mathbf{B}^T_{ts}\mathbf{S}^t_{ts}\mathbf{B}_{ts} d\Omega_e. \tag{22}$$

Total element stiffness matrix

$$\mathbf{K} = \mathbf{K}^{mat} + \mathbf{K}^{geo}, \tag{23}$$

or

$$\mathbf{K} = \mathbf{K}^{mat} + \mathbf{K}^{geo}_{(p)} + \mathbf{K}^{geo}_{(t)}. \tag{24}$$

OBS.: $\mathbf{K}^{mat}$ is the “material" stiffness matrix obtained by linear FEM.